NBER WORKING PAPER SERIES

OVERLAPPING GENERATIONS MODELS, MULTIPLICITY OF STEADY STATES AND MOMENTARY EQUILIBRIA, AND ECONOMIC FLUCTUATIONS

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Working Paper 34193
http://www.nber.org/papers/w34193

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2025

Stiglitz gratefully acknowledges financial support from the Hewlett and Sloan Foundations. We are also indebted to Catalina Gomez Colomer, Ricardo Pommer Muñoz, and Hannah Kris for research assistance. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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JEL No. C61, E32

ABSTRACT

This paper examines the simplest OLG models with capital accumulation, demonstrating three results that stand in marked contrast to those of the standard model: first, the possibility of multiple steady states; second, the possibility of multiple momentary equilibria under rational expectations; third, one of implications of multiple momentary equilibria is that dynamics may be marked by complex fluctuations (lacking even periodicity), but still within well-defined bounds. We provide quite general conditions (with general utility and production functions) under which in the simplest of OLG models, there can be multiple steady states, multiple momentary equilibria, and complex dynamics. Furthermore, we present a simple illustration of wobbly growth by incorporating credit friction.

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Section 1. Introduction

For the past three-quarters of a century, there have been two contrasting foundations for macroeconomics: one derived from Ramsey's work, with infinitely lived individuals maximizing their utility, the other derived from Samuelson's overlapping generations (OLG). In recent years, the former has largely dominated. While all would recognize that individuals do have finite lives, the hope (or presumption) was that the infinitely lived model provided a good approximation—good enough that the benefits of that model's tractability prevailed.[2] To be sure, many economists recognized that the mathematical differences between the two models were sufficiently great that there was no assurance that that would be the case. And it was also evident that there were important policy differences: in an OLG model, for instance, there is no basis for Ricardian equivalence.[3]

This paper examines the simplest OLG models with capital accumulation[4], demonstrating three results that stand in marked contrast to those of the standard model: first, the possibility of multiple steady states; second, the possibility of multiple momentary equilibria under rational expectations; third, one of implications of multiple momentary equilibria is that dynamics may be marked by complex fluctuations (lacking even periodicity), but still within well-defined

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[2] One strand of work, while recognizing that individuals are finitely lived, assumed that they cared for their children as much as they cared for themselves, generating "dynastic" utility functions. Theory and empirics have cast doubt on that model: many individuals do not have children, and their behavior cannot be so described. Bequest behavior appears inconsistent with that hypothesis for a large proportion of those with children. And aggregate behavior, say in response to a tax cut, does not appear consistent with that hypothesis. See, for instance, Wilhelm (1996), Hurd and Smith (2002) and Wolff and Gittleman (2014).

The OLG model can easily be generalized, e.g. to the case where individuals die exponentially, as in Blanchard (1985); and as we note below, even if individuals were infinitely-long lived, if there are episodically binding capital constraints, the OLG model provides a better description of the economy than the dynastic utility model.

[3] The standard model predicts that it makes no difference whether government finances its expenditures by debt or lump sum taxes at any particularly moment; public debt substitutes for private debt, and individuals, taking into account future tax liabilities, leave consumption unchanged. See Barro (1974, 1996). For a more general formulation, which identifies crucial limitations, including that noted here in intergenerational altruism, see Stiglitz (1983, 1988), who also notes that this result can be viewed as an application to public finance of the generalized Modigliani-Miller theorem (Stiglitz, 1974), suggesting the irrelevance of corporate financial policy. There is a large literature showing that Ricardian equivalence (in any of its multiple forms) does not hold. Indeed, the debate in the literature is now about the magnitude of the debt-financed expenditure multiplier, whether it is less than or greater than unity, not whether it is zero. (See Auclert et al. 2024, Auerbach and Gorodnichenko 2012, Auerbach et al. 2022, Johnson, Parker and Souleles 2006; Parker et al. 2013, Morrison and Taubinsky 2023, Nakamura and Steinsson 2014, Ramey and Zubairy 2018.) The literature has now recognized that capital constraints and a host of other market failures undermines Ricardian equivalence—implicitly undermining the dynastic model itself. We note below Woodford's result that with capital constraints, such models are formally equivalent to an OLG model. See Woodford (1986).

[4] Diamond (1965) extended Samuelson's pure consumption loan model to include capital accumulation.

bounds.[5] By contrast, in what we shall refer to as the Ramsey model, that where a representative agent maximizes discounted utility over an infinite lifetime,[6] the transversality condition in combination of the Euler equation pin down both the momentary equilibrium and the entire trajectory, and so long as the pure rate of time preference is fixed (as it is in the standard formulations) ensure that the economy always approaches the same steady state and does so smoothly. In the OLG model, individuals maximize their lifetime utility, but even with rational expectations multiple momentary equilibria can arise, generating expectations-driven dynamics. We provide quite general conditions (with general utility and production functions) under which in the simplest of OLG models, there can be multiple steady states, multiple momentary equilibria, and complex dynamics.

There are multiple important economic implications arising from the differences we have just noted. In the standard Ramsey model, because there is a unique rational expectations equilibrium trajectory of capital accumulation, an economy experiencing a large shock simply moves "back in time" along that trajectory—eventually going to where it would have otherwise gone. Thus, while in the Ramsey models, history fades out, that may not be so in OLG models. In the OLG models with multiple steady states, a large enough shock will result in the economy moving to a different steady state.

Moreover, because in Ramsey models the equilibrium is approached along a saddle point trajectory, in the absence of futures markets extending *infinitely* far into the future (which they obviously don't—future markets are notoriously of limited duration) individuals have to form expectations extending infinitely far into the future—and those expectations have to be precisely right. If they aren't, the economy will set off on a trajectory in which the Euler conditions are satisfied every period, but eventually—and only eventually—it becomes apparent that the transversality conditions are not satisfied, and there will have to be a (possibly large) path

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[5] The presence of multiple momentary equilibria in a variety of contexts has been known in the literature, including earlier papers such as Stiglitz (1973) and Uzawa (1961, 1963). In the literature, the analysis is limited to mentioning the possibility of multiple momentary equilibria. They don't fully explore their implications for global dynamics.

[6] Ramsey actually analysed the problem of how a society *should* choose its savings rate to maximize societal welfare across generations, and he strongly argued *against* discounting, so it is perhaps a misnomer to refer to the standard macroeconomic model which purports to show how societal actually behaves, i.e. as if it maximized discounted utility over an infinite horizon as the Ramsey model, but we shall do so nonetheless.

correction.[7] By contrast, in the simple OLG models without non-reproducible assets like land, individuals only need to form expectations one period ahead. Thus, while expectations play a significant role in both OLG and Ramsey models, the demands put on those expectations (if they are to be "rational") in Ramsey stretch credulity.[8]

In our analysis, crucial in the existence of multiple momentary equilibria are assumptions about expectations formation. If expectations are myopic—today's interest rate is expected tomorrow—we show that there never exist multiple momentary equilibria. But if expectations are fully rational, with no stickiness, then under not implausible conditions, there will be multiple momentary equilibria, i.e. different sets of beliefs such that when individuals act according to those expectations, those expectations are realized. With a natural formulation of expectations stickiness, there is a critical level of stickiness such that if expectations are sticky enough, momentary equilibria are unique.

Our paper—both the short-run and long-run analysis—can be seen as a reflection of Keynes' animal spirits: if individuals expect, say, a boom, they act in such a way as to generate that boom. But while Keynes' animal spirits has often been interpreted as reflecting irrationality, here we show that such animal spirits can be fully consistent with rational expectations under certain plausible circumstances.

An important contribution of this paper is to extend the analysis of OLG's beyond steady states to dynamics. With multiple steady states, initial conditions matter in determining the long run—in contrast to the standard Ramsey model where, regardless of the initial value of state variables (here, the capital stock), the economy converges to the same steady state.[9] While

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[7] The fragility of saddle point trajectories in the absence of futures markets extending infinitely far into the future, especially in the presence of multiple capital goods, where expectations have to be formed about the returns on *each* of the assets, was central to the critique of the Ramsey model as a description of equilibrium trajectories in market economies by Hahn (1966) and Shell and Stiglitz (1967). Interestingly, this seeming fragility has been turned into a virtue by the advocates of these models: rational expectations, the Euler equations, the capital arbitrage equations, and the transversality conditions tie down the equilibrium precisely.

[8] If there are non-reproducible assets like land, agents need to form expectations infinitely far into the future, i.e., whether land price dynamics are sustainable infinitely far into the future, even though they live only for two periods. Usually, this dynamic path corresponds to a unique saddle path. But when there are multiple momentary equilibria, rational economic agents know that future generations, as long as they are rational, will surely change course in the middle of unsustainable dynamic paths. Hence, rational economic agents do not need to form expectations infinitely far into the future and hence the macroeconomy can be on an unsustainable path temporarily. Hirano and Stiglitz (2022b) develop such a theory with land speculation.

[9] We hasten to add that that is a feature not of Ramsey optimization, but of the parameterization of the representative individual's utility function. In the standard formulation, the rate of discount is fixed, and that ties down the long run

incremental policies, say a tax on the return to capital, will only have incremental effects on the steady state, a "large policy," a one-time large tax on the return to capital with proceeds redistributed to the young, for example, can jolt the economy from one steady state to another.

Matters are even more complicated with rational expectations (low levels of stickiness). "Expectations-driven dynamics" can easily arise[10]: There may be an infinity of paths consistent with rational expectations. Nonetheless, the common criticism of such models that "anything can happen" is not valid. Rather, there exist upper and lower bounds within which the macroeconomy wobbles. We suggest that the often-noted sudden reversal of confidence,[11] the Keynesian animal spirits referred to earlier, is consistent with expectations, at least at times, exhibiting at most limited stickiness. Our model captures what may be happening at such moments.[12] More generally, our analysis suggests that the pursuit of rationality, i.e., full human rationality entailing rational expectations, generates macroeconomic instability, while human irrationality can contribute to stability. This insight is in sharp contrast with the view in Akerlof and Shiller (2009), who emphasize that human irrationality, i.e., what the former chairman of the Federal Reserve Alan Greenspan referred to as irrational exuberance, is the key source of macroeconomic instability.

While this paper shows that there are a rich range of specifications of technology and preferences for which there can exist multiple momentary equilibria and multiple steady states, such multiplicity can particularly easily arise when the intertemporal elasticities of consumption and the elasticity of substitution in production are less than unity. There is strong empirical evidence on each. For instance, a recent meta-analysis on the intertemporal elasticity of substitution in consumption (IES) shows that the mean estimate of
the elasticity of intertemporal substitution reported in empirical studies is 0.5, and with many reputable studies showing much lower values. (Havranek *et al* 2015). Other studies suggest an

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interest rate, and hence the long run value of capital and consumption. In more general formulations (such as that of Koopmans (1960) and Iwai (1972)), the marginal rate of substitution between consumption at two different dates in steady state can depend on the level of consumption, and in that case, there can be multiple steady states even in a Ramsey model.

[10] See Cass and Shell (1983). See below for a fuller discussion of sunspot equilibria.

[11] A characteristic of the sudden stops to which Calvo (1998) drew attention, and evidenced in the 1997 East Asia crisis and the 2008 financial crisis. See Stiglitz (2010).

[12] We show that economies can experience the effects of animal spirits with rational expectations. It is obvious that is even easier for economies not so disciplined to exhibit animal spirits.

even lower value.[13][14] Similarly, the overwhelming consensus is that the elasticity of substitution in production is less than unity, and probably substantially so.[15][16]

The paper is divided into 9 sections, beyond this introduction. In section 2, we provide the basic intuition for our results, using a variant of the IS LM framework that is familiar to all economists. In section 3, we introduce the basic model, and in section 4 we analyze the existence of multiple steady states. Section 5 looks at the conditions for the existence of multiple momentary equilibria. Sections 6 and 7 then explore some of the implications of our analysis for dynamics. In section 8, we present a simple illustration of wobbly growth by incorporating credit friction. In section 9, we discuss the issue of the selection of equilibrium in the presence of multiple momentary equilibria, asking whether there are policies which might result in the selection of a particular equilibrium, and if so, could they be designed to enhance long run societal well-being. In section 10, we conclude with some general observations about the implications of this analysis.

Section 2. Basic intuitions

Traditional macroeconomics based on Hicks' interpretation of Keynes centered around an equilibrium defined by the intersection of the I-S curves, where investment and savings were both given as functions of the interest rate.[17] Investment (or, in the formal model presented

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[13] Taking into account, for instance, selective reporting as Havranek (2015) shows. Hall (1988) finds that the IES is unlikely to exceed 0.1. Best et al. (2020) show that the average IES is small, around 0.1.

[14] Further, with the standard separable constant elasticity of consumption utility function, whether the IES is less than, equal to, or greater than unity depends on the rate at which marginal utility diminishes with consumption. With logarithmic utility functions, the IES is unity. Standard estimates (e.g. based on behavior towards risk) suggest that marginal utility diminishes at a faster rate, i.e. that the IES is less than unity.

[15] While traditionally, most analyses took $\sigma < 1$, confirmed by more recent studies (Antras (2004), Oberfield and Raval (2014), Chirinko and Mallick (2017)), Piketty and Zucman (2014)'s analysis implies $\sigma > 1$. But Piketty and Zucman's results partially arise out of a confusion between wealth and capital. The difference is the capitalized value of rents, which arguably increased significantly in recent decades, so much so that in some countries arguably the capital output ratio has been declining even as the wealth output ratio has been increasing. See Stiglitz (2015).
A recent paper by Gechert et al. (2019) shows that the elasticity of substitution between capital and labor is around 0.3.

[16] Readers of this paper as well as our companion paper in this issue (Hirano and Stiglitz (2025)) will notice a certain tension between the assumptions made there (Cobb Douglas production and utility functions) and those that are at the center of this paper. In the absence of credit frictions, the former assumptions preclude multiple steady states and momentary equilibria; we invoke those assumptions, as we note, to derive the benchmark cases with balanced growth. In section 8, we present a simple illustration that shows that multiplicity of momentary equilibria can arise in the presence of credit frictions, even with these restrictive parameterizations.

[17] While in principle, in standard micro-founded economics, both should be a function of the real interest rate, they are often articulated as functions of nominal interest rates. Of course, if inflationary expectations are fixed, then the

below, the equilibrium desired level of capital next period) decreases with an increase in the interest rate. In the usual formulation, savings increase, so there is a unique equilibrium. But in the standard life-cycle model presented below, savings may decrease when the interest rate increases. This is the standard result with "target" savings—where individuals are saving to buy a house, to provide education for their children, and to make sure that they have sufficient funds for a comfortable retirement. The higher the return on their savings, the less they have to put away to meet their goals. Indeed, in a standard life cycle model, if the intertemporal elasticity of substitution between consumption while an individual is working and his retirement is less than unity (and the evidence presented earlier suggests that that is the case), then savings decreases with an increase in the interest rate.

With both the I and S curves being downward sloping, it is easy to see that there might be multiple intersections, as in Figure 1—i.e. multiple equilibria. The curvature of the I curve depends on properties of the production function, that of the S curve on properties of preferences. This kind of reasoning applies equally to the long run, where wages decrease as the long-run interest rate increases because of the adverse effects on the capital stock, and the short run, where wages are fixed, though clearly, the conditions under which there may be multiple intersections in the two situations will differ. (This is the crucial difference between the short-run and long-run IS curves: one takes today's wages as given—of course determined by the inherited capital stock—and the other tales it as endogenous, determined by the capital stock (investment), itself a function of the interest rate.)

To see how this plays out, consider any steady state, and what happens if the interest rate were to fall, and the wage rate to increase. The latter would lead to more savings (under our assumptions), and if the fall in interest rates were to increase the savings rate enough (or not diminish the rate too much), there would be an increase in aggregate savings—enough to sustain higher wages and lead to lower returns to capital. The smaller the elasticity of substitution in production, the larger the wage increase, and with a small intertemporal elasticity of substitution, a small decrease in the return to capital will induce a large increase in savings. Thus, the combination of a low elasticity of substitution in production and a small intertemporal elasticity

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real interest rate is just the nominal interest minus the expected rate of inflation, and the specification makes no difference.

of substitution can generate multiple steady states. These effects are strengthened if (a) there is a positive labor supply elasticity, so that the increase in wages generates even higher per capita incomes and savings; or if (b) there are non-homothetic preferences, so that when individuals get richer (or when wages increase), they want to disproportionately increase consumption in retirement.

Figure 1 Here

Figure 1: Steady state investment and savings as functions of the interest rate.
There may be multiple values of the (real) interest rate at which (steady state) investment equals savings.

Though all of this seems intuitive enough, it remains to show the conditions under which the increase in savings suffices in fact to sustain the necessary wage increases/interest rate decreases. The increase in savings will, as we have already noted, depend on properties of intertemporal preferences; the magnitude of the increase in wages and decrease in interest rates from an increase in capital depends on properties of the production functions. The question is, are there plausible combinations of preferences and technologies that give rise to multiple steady states, and if so, what combinations do so. While we do not provide a complete set of necessary and sufficient conditions, we do show that multiple steady states can arise in the simplest OLG formulations under plausible conditions.

The analysis of multiple momentary equilibria is parallel with one difference: given today's level of capital, wages today are determined.[18] Still, there may be multiple values of capital tomorrow *consistent with rational expectations.* As we have already noted, the model thus makes precise (in this context) the notion of Keynes' animal spirits—but shows that animal spirits do not need to be irrational. If markets expect interest rates to be high, they may indeed be high; and if they expect interest rates to be low, they may indeed be low.

Again, what matters is the combination of preferences and technology: the belief that there will be high interest rates may lead to lower savings rates—individuals need to save less for retirement to smooth consumption—and the lower savings rate may lead to high interest rates. As we'll see, the conditions for what we can think of as the short-run I and S curves to intersect

[18] At least in the full employment equilibria on which we focus here.

multiple times (i.e. multiple momentary equilibria) are more restrictive than the conditions for the long-run I and S curves to intersect multiple times (i.e. multiple steady states).

Section 3. The basic model and the basic analytical results

Consider a simple overlapping generations model in which everyone in each generation is identical. In each period young agents are born and live for two periods. There is no population growth rate.[19] Each person is endowed with a fixed amount of labor when young, and supplies it inelastically, receiving wage income, w_t.[20] For simplicity, we normalize the aggregate labor supply at unity. Each young person saves a fraction s_t of his income, generating the first and the second period consumption of

(3.1) $c_{1t} = (1 - s_t)w_t,$ and $c_{2t} = s_t(1 + r_t)\, w_t,$

where $1 + r_t$ is the interest rate between period t and $t + 1$. The holdings of capital by the young at time t becomes the capital stock at $t + 1$. This generates the dynamic equation of aggregate capital stock, k_t[21] i.e.,

(3.2) $k_t = s_t w_t$

The savings rate is chosen to maximize utility $u_t = u(c_{1t}.c_{2t})$ subject to their budget constraint, yielding the standard first order condition that the intertemporal marginal rate of substitution φ equals the expected return to savings, r_{t+1}^e:

(3.3) $\frac{\partial u(c_{1t},c_{2t})/\partial c_{1t}}{\partial u(c_{1t},c_{2t})/\partial c_{2t}} = 1 + r_{t+1}^e$

Competitive firms produce output by using capital and labor. Each firm has a standard neoclassical constant return to scale production function. Output per capita, y_t, is a function of capital per capita k_t,

(3.4) $y_t = f(k_t) = f(K_t/L_t),$

where K_t and $L_t = 1$ are aggregate capital and labor inputs, and given our normalization, $k_t = K_t$. We assume a constant rate of depreciation of capital, $\delta \in [0,1]$. Rental rates R_t and wages are given by

[19] Hirano and Stiglitz (2022a) study the case with population growth.
[20] As we noted in the introduction, this formulation makes it harder for there to exist multiple steady states and momentary equilibria.
[21] Because of our normalization, this is also the per capita level of capital.

(3.5a) $R_t = f'(k_t)$, and (3.5b) $w_t = f(k_t) - f'(k_t)k_t = w(k_t)$,

with $f'(k_t) < 0$ and $w'(k_t) > 0$.[22] The interest rate equals the (net) return to holding capital.[23]

(3.5c) $1 + r_{t+1} = f'(k_{t+1}) + 1 - \delta$.

Savings is critically dependent on expectations, as (3.3) makes clear. Much of modern macroeconomics has focused on the case of rational expectations, where individuals at time t, in making their savings decisions, expect the return that is actually realized, i.e. $1 + r_{t+1}^e = 1 + r_{t+1}$. The other polar case is myopic expectations, where individuals expect tomorrow's return to be the same as today, i.e. $1 + r_{t+1}^e = f'(k_t) + 1 - \delta$. We'll see that the dynamics in these two cases is markedly different, and later we will consider a natural generalization of these two cases, where $1 + r_t^e = \lambda f'(k_t) + (1 - \lambda)f'(k_{t+1}) + 1 - \delta$, i.e. a linear combination of myopic and rational expectations, with λ representing the degree of stickiness, and $\lambda = 1$ representing perfectly sticky expectations. We write $r_t^e = r_t^e(k_t, k_{t+1})$. Then, in its most general form, the central dynamic equation to be investigated in this paper is

(3.6) $k_{t+1} = s_t(r_{t+1}^e(k_t, k_{t+1}), w_t(k_t))\, w(k_t)$

relating k_t and k_{t+1}. This formulation recognizes that preferences may be non-homothetic, and the savings rate could depend on an individual's income. What is important is that given k_0, the only future-thinking aspect of the dynamics are expectations about returns, which depend only (in our formulation) on the levels of capital stock at $t + 1$ and (outside of rational expectations) on t.

The rest of the paper looks at special cases of (3.6)—steady states and out of steady state dynamics. Even under highly restrictive assumptions about technology and preferences, there are a rich set of dynamics associated with (3.6), far richer than in the standard model.

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[22] We impose standard restrictions on f. While we focus on the more realistic case of an elasticity of substitution (in production) less than unity, we also consider cases where the elasticity of substitution is greater than unity and where it varies with k itself. When the elasticity of substitution is less than unity, we assume $\lim(1 + r) = lim f'(k)$ as $k \to \infty$ is zero; $lim f(k)$ as $k \to \infty$ is bounded at $\bar{f}$, so of course $lim\, w(k)$ as $k \to \infty \leq \bar{f}$; $lim\, f'(k)$ as $k \to 0$ is infinite; and ; $lim\, f(k)$ as $k \to 0$ is 0. By contrast, when the elasticity of substitution is greater than unity, we assume $\lim(1 + r) = lim f'(k)$ as $k \to \infty$ is bounded; $lim\, f'(k)$ as $k \to 0$ is also bounded.

[23] With $\delta = 1$, $1 + r_{t+1} = f'(k_{t+1})$; with $\delta = 0$, $r_{t+1} = f'(k_{t+1})$.

Section 4. Steady states

A steady state is a value of k_t such that $k_t = k_{t+1}$, with k_t given by (3.2) and where wages and interest rates are given by (3.5b) and (3.5c), and trivially, because the economy is in steady state, expectations are rational. Dropping the subscript t, any value of k solving

(4.1) $k = s\big(w(k), r(k)\big)w(k)$

is a steady state. Both sides may be an increasing functions of k—k increases wages and possibly, on net, the savings rate—and if so it is *possible* that there could be multiple values of k solving (4.1). To see how easy it is for that to be the case, below we explore several parametric models, each of which brings out a particular dimension of the problem.

In the previous section, we cast the problem into a standard I-S framework, where equilibrium investment, k, is a function of r given by

(4.2) $k = f'^{-1}(1 + r)$.

so

(4.3) $\frac{dln\, k}{d \ln 1+r} = -\frac{\sigma}{S_L}$,

where S_L is the share of labor and σ is the elasticity of substitution. With a Cobb-Douglas production function, of course, where the share of labor is fixed, the logarithmic derivative is fixed at $1/S_L$. With a constant elasticity of substitution production function, with elasticity less than unity, for small k (large r), the share of labor is small, so the slope is large, while for large k, the slope converges to 1. Conversely if the elasticity of substitution is greater than 1. The different cases are plotted in Figure 2. (Unlike Figure 1, we put $1 + r$ on the horizontal axis, and k on the vertical axis, since this is the more natural formulation—we are asking what is the equilibrium value of k corresponding to any given interest rate—even though it differs from the more common formulation in economics. We continue this convention in the subsequent figures.)

Figure 2 Here

Figure 2: Steady state "Investment" as a function of the interest rate.

The steady state value of k decreases as r increases, with the logarithmic slope equaling $-\frac{\sigma}{S_L}$. The figure illustrates three cases with the elasticity of substitution equal to, greater than, and less than unity. Because we plot $ln(k)$ vs. $ln(1+r)$, with $r \geq 0$, $ln(1 + r) \geq 0$. $ln(k)$ can be positive or negative,

To simplify the exposition of the effect of a change of r on savings, we assume that the savings *rate* depends just on r. Total savings, S, are then just $s(r)w$ (since in the life cycle model, the only savings are those of workers for retirement), so, taking into account the effect of a change in r on k and that on wages, we have

(4.4) $\frac{dln\, S}{d\ln 1+r} = \frac{s_{1+r}(1+r)}{s} - \frac{dln\, w}{dln\, k}\frac{dlnk}{dln(1+r)} = \eta - \frac{S_K}{S_L}$.

where $\frac{s_{1+r}(1+r)}{s} \equiv \eta$, the interest elasticity of savings. Total savings will be a declining function of r if $\eta - \frac{S_K}{S_L} < 0$, a sufficient condition for which is that $\eta < 0$.[24]

4.1. Homothetic preferences

With homotheticity, the savings rate is just a function of the rate of interest. To this, note that the equilibrium ratio of consumption of an individual at the two dates is just a *function* of $1 + r$, that is, since utility maximizing individuals set their marginal rate of substitution between consumption in the two dates equal to $1 + r$, the relative price of consumption in the two dates, if we know the slope of the indifference curve (the marginal rate of substitution), we know the value of relative consumption. All Engle curves are straight lines through the origin. We write

(4.5) $\frac{c_{1t}}{c_{2t}} \equiv \rho[1 + r]$.

with $\rho' < 0$, and there are no a priori restrictions on ρ''.

Then, dropping the subscript *t*, it follows directly from the budget constraint

$$w - c_1 = (1 + r)c_2$$

that

(4.6a) $c_1 = \frac{zw}{1+z}$, and (4.6b) $k = sw = \frac{w}{1+z}$,

[24] In the general case, a change in r changes k, and that changes w, and both effects have to be taken into account.

where $z = (1+r)\rho$. That is, the savings rate is just a function of r $(=\frac{1}{1+z})$, as hypothesized earlier.

A steady state requires

(4.7) $\frac{k}{w} = s = \frac{1}{1+z}$.

Showing that there are multiple steady states consists then of showing that a large change in k induces a change in $\frac{k}{w}$ (because w changes, in the long run with k), and a change in the interest rate, and therefore savings rate, which is self-sustaining, i.e. for which (4.7) holds. The left-hand side is just a function of technology, the RHS of preferences.

Taking first the logarithmic derivative of the LHS with respect to k, we have

$\frac{dln\ k/w}{dln\ k} = 1 - \frac{S_K}{\sigma}$. Using (4.3) we then have

(4.8) $\frac{dln\ k/w}{dln\ 1+r} = \frac{S_K - \sigma}{S_L}$.

Even if the elasticity of substitution is fixed, unless $\sigma = 1$, the shares will depend on k, and thus so will the logarithmic derivative (4.8). Moreover, σ itself varies with k. In the next subsection, for instance, we consider a case where, depending on k, the elasticity of substitution may be either infinite or zero. If, however, σ is always less than 1, the slope is positive for small k (large r), negative for large k (small r); while if σ is always greater than 1, the sign is always negative. The cases are illustrated in the different panels of Figure 3.

Figure 3 Here

Figure 3 k/w as a function of the (real) interest rate.

If $\sigma \geq 1$, $ln(k/w)$ decreases with r; if $\sigma = 1$, $\ln(k/w)$ is linear in $ln(1+r)$. But if $\sigma \leq 1$, k/w increases as r increases for small r and conversely for large r. Because we plot $ln(k/w)$ vs. $ln(1+r)$, with $r \geq 0$, $\ln(1+r) \geq 0$. $ln(k/w)$ can be positive or negative.

Steady state savings

One the other hand, straightforward differentiation yields

(4.9) $\frac{dln\, s}{d\ln 1+r} \equiv \eta = -(1-s)(1-\zeta)$

where $\zeta \equiv \frac{\rho\prime(1+r)}{\rho}$, the intertemporal elasticity of substitution (with respect to the intertemporal price, $1+\ r$). If the intertemporal elasticity is unity (the logarithmic utility function), then s is fixed ($\eta = 0$). If the intertemporal elasticity is less than unity—the empirically relevant case—the savings elasticity is negative. Individuals care a lot about income smoothing, so that when the interest rate falls, they have to save more.

Thus, when the intertemporal elasticity of substitution ζ is unity, then the slope of the LHS is zero, and for there to be multiplicity of steady states, k/w has to be non-monotonic—which will always be the case with a constant elasticity of substitution in production not equal to unity. When the intertemporal elasticity is small (less than unity), the savings rate decreases with an increase in the interest rate and when the intertemporal elasticity greater than unity, in which case savings increases with the interest rate. (See Figure 4)

Even with a constant intertemporal elasticity, it is apparent that η is not constant (unless $\zeta = 1$), because s varies. When the intertemporal elasticity is less than 1, the limiting savings rates (when $1+r$ is zero and infinity) are 1 and zero, respectively; and conversely when the intertemporal elasticity is greater than 1.

Figure 4 Here

Savings rate as a function of the interest rate.

The savings rate increases or decreases with r as the intertemporal elasticity of substitution is greater than, equal to, or less than unity. Because we plot $ln(s)$ vs. $ln(1+r)$, with $r \geq 0$, $ln(1+r) \geq 0$. Because $s \leq 1$, $ln(s)$ is negative,

Putting the results together, we can see multiple situations where there may be multiple steady states, i.e. multiple solutions to (4.1). This is illustrated in Figures 5a, b, and c, where we have superimposed the curve giving $ln\ s$ as a function of $ln(1+r)$ onto the earlier derived curve giving $ln(k/w)$ as a function of $ln(1+r)$. In each of these situations, we see there exists multiple equilibria, i.e. multiple values of $1+\ r$ (corresponding to multiple values of k) for which savings equals investment in steady state.

The most interesting case is that where the production and intertemporal elasticities are both less than 1. Then an increase in r reduces k, lowers wages, and so long as it doesn't fall too much, k/w falls. At the same time, the increase in r means that the savings rate can fall to ensure consumption smoothing. With both k/w and s falling as r increases, it is possible that there can be multiple steady states, as illustrated in Figure 5b.

There is, of course, no reason to limit ourselves to preferences with a constant elasticity of consumption or constant elasticities of substitution in production, and indeed, these elasticities may be less than unity for some values of r and greater than unity for others, implying that aggregate savings (as a function of r) may not be monotonic, and giving rise to the possibility of even more steady states. For instance, in Figure 5d we consider a case where when r is very high or very low (so the ratio of the consumption in the two periods is very high or very low) there is a low intertemporal elasticity of substitution, but for intermediate values of the interest rate, the intertemporal elasticity of substitution is very high. This implies that for very low and high rates of interest, the savings rate declines with an increase in r, but for intermediate values of r, it increases. There can be multiple (three) steady states, even when the production function has constant elasticity of substitution, indeed, even in the production function is Cobb-Douglas.

Figure 5 Here

Figure 5 There can exist multiple values of $ln(1 + r)$ for which $k/w = s$.

Figures 5a, b, and c illustrate just three of the possibilities, superimposing Figures 3 and 4 on top of each other. In Figure 5a we have a constant savings rate (corresponding to an intertemporal elasticity of substitution of unity-- $\zeta = 1$, intersecting twice with k/w for the case of $\sigma < 1$. In 5b we again have k/w for $\sigma < 1$, but this time with an intertemporal elasticity of substitution less than unity. Figure 5c shows that one can multiplicity of steady states with $\sigma < 1$ but a positive intertemporal elasticity of substitution. Figure 5d illustrates a case with variable intertemporal elasticities of substitution. (Because we plot $ln(k/w)$ and s vs. $ln(1 + r)$, with $r \geq 0$, $ln\ (1\ +\ r) \geq 0$. $ln(k/2)$ can be positive or negative, but $ln(s)$ is always negative.)

4.2. Two technologies: an analytic example

To show further the possibility of multiple steady states, we consider an economy with two linear (infinite elasticity of substitution) technologies, defined by

(4.10) $Y_i = \mu_i L + \vartheta_i K$

respectively, with the aggregate production function being $Y = max\ Y_i$. Assume

(4.11) $\frac{\vartheta_2}{\mu_2} > \frac{\vartheta_1}{\mu_1}, \vartheta_2 > \vartheta_1, \mu_1 > \mu_2$

so that technology 2 is used when

(4.12) $k > [\vartheta_2 - \vartheta_1]/\{\mu_1 - \mu_2\} \equiv \hat{k}$

Then the economy has three regimes

a) $k < \hat{k},\ 1 + r = \vartheta_1, w = \mu_1,$
b) $k > \hat{k},\ 1 + r = \vartheta_2, w = \mu_2,$
c) $k = \hat{k},\ \vartheta_2 > 1 + r > \vartheta_1,\ \mu_2 < w < \mu_1.$

In each of these regimes, for any savings function, and for any value of k, we can calculate sw, e.g. in regime (a), $sw = s(\vartheta_1)\mu_1$.[25] sw increases or decreases discretely at $k = \hat{k}$.

Now k/w is a simple function of k, increasing linearly from 0 to $\hat{k}$, then falls discretely, after which it again increases linearly. Similarly, s is constant from $k = 0$ to $k = \hat{k}$, then jumps up or down (depending on the sign of η, the savings elasticity). As Figures 6a and 6b illustrate, there can exist multiple steady states[26] whether the savings elasticity is positive or negative. Figure 6b is a case where the savings elasticity is negative, and there are multiple steady states so long as

(4.13) $s(\vartheta_1) < \frac{\hat{k}}{\mu_1}$

In the case of a positive savings elasticity, there are multiple steady states so long as[27]

(4.14) $s(\vartheta_1) < \frac{\hat{k}}{\mu_1}$ and $s(\vartheta_2) > \frac{\hat{k}}{\mu_2}$

[25] For simplicity, we continue to assume homotheticity of preferences, so that the savings rate only depends on the interest rate, as discussed in the previous subsection. It is easy to extend the analysis to the more general case.
[26] In addition to the low k and high k steady states, there is a steady state at $k = \hat{k}$, where there is a particular distribution of income consistent with $\hat{k}$ that supports $\hat{k}$.
[27] We have identified two of the three steady states, i.e. where $s(\vartheta_1)\mu_1 = k < \hat{k}$ and $s(\vartheta_2)\mu_2 = k > \hat{k}$. The distribution of income at $\hat{k}$ is indeterminate, with the wage ranging from μ_1 to μ_2, and the return to capita ranging from ϑ_2 to ϑ_1. There is a third steady state with $\mu_i + \vartheta_i\hat{k} = \hat{y}$, where $y \equiv \frac{Y}{L}$, output per worker (so $\hat{y}$ is output per worker at $\hat{k}$), with $s(1 + r^*)w^* = \hat{k}$, for some $\{w^*, r^*\}$ such that $w^* + (1 + r^*)\hat{k} = \hat{y}$, $\mu_1 < w^* < \mu_2$, and $\vartheta_2 < 1 + r^* < \vartheta_1$.

With $\hat{k}$ just determined by technological parameters, and s being determined just by preferences, it should be clear that there may well be multiple steady states. Note, in particular, that we have had to say very little about the details of preferences. For instance, if the savings elasticity is negative, all that we require is that the savings rate associated with the interest rate associated with the labor-intensive technology satisfy (4.13), and multiplicity of steady states can arise even with logarithmic preferences, where the savings rate is fixed, so long as the saving rate satisfies (4.13).

In on-line appendix A, we provide conditions for the existence of multiple steady states for the special case of Leontief preferences.

Figure 6 Here

Figure 6 Two technique Economy There are two technologies, each with infinite elasticity of substitution. When there is capital scarcity, the labor-intensive technology is used. Wages and interest rates are fixed within each regime. Figure 6a plots k/w as a function of k, showing that there may be multiple equilibrium when the savings elasticity is negative. (This will be the case as long as $s(\vartheta_1) < \frac{\hat{k}}{\mu_1}$) Figure 6b shows multiplicity of steady states when the savings elasticity is positive. Figure 6c illustrates the case where the savings rate is fixed.

Section 5. Dynamics with unique momentary equilibrium

The dynamics of the economy was described earlier in equation (3.6). As we comment later, given k_t there may or may not be a unique solution of (3.6) for k_{t+1}, given k_t. If there is, we can rewrite (3.6) as simply

(5.1) $k_{t+1} = \gamma(k_t)$,

depicted in Figure 7. In the Figure, we depict two cases, one in which there are myopic expectations, and

(5.1a) $k_{t+1}=s_t(f'(k_t))\, w(k_t)$,

While with rational expectations

(5.1b) $k_{t+1}/s_t(f'(k_{t+1})) = w(k_t)$.

We can write the two dynamic functions as $k_{t+1} = \gamma_M(k_t)$ and $k_{t+1} = \gamma_{RE}(k_t)$, respectively, and illustrated by the small dashed and solid lines in Figure 7.

A steady state can now be expressed as any solution to

(5.2) $k = \gamma(k)$,

and represented diagrammatically as any point at which the γ curve crosses the 45-degree line, as depicted.

Figure 7 Here

Figure 7 Multiple steady states.

With k_{t+1} depending on k_t, any value of k crossing the 45-degree line is a steady state, and any crossing from above is stable. There may be multiple steady states, with different levels of consumption and welfare, but history matters: the economy cannot just jump to the "preferred" steady state. The set of steady states is the same whether expectations are myopic or rational, and the stability of the steady state is unaffected; but out of steady state behaviour differs, i.e. the pace of convergence in general differs, with convergence for rational expectations faster. The solid line represents the dynamics with rational expectations ($\gamma_{RE}(k_t)$) while the small dashed line represents the dynamics with myopic ($\gamma_M(k_t)$).

The analysis of section 4 showed that there will in general be multiple such crossings. Since in steady states, there is no difference between rational expectations and myopic expectations, the set of steady states (the set of values of k_t for which $\gamma(k_t)= k_t$), is obviously the same; but the dynamics (e.g. the pace of convergence) may differ. To see this more clearly, we continue with our focus on the case where the savings rate depends just on the interest rate (as is the case with homothetic preferences). Consider first myopic expectations:

(5.3) $k_{t+1} = s(r(k_t))w_t(k_t)$.

Plotting savings sw as a function of k_t, as in our earlier analysis of steady states, we can easily identify stable vs. unstable steady states. Whether the steady state is stable with myopic expectations depends on whether sw crosses the 45-degree line from above or below, or analytically, taking logarithmic derivatives, on whether(5.5) $\eta \frac{S_L}{\sigma} + \frac{1-S_L}{\sigma} < 1 \; or > 1$;

Of the three steady states in the Figure 7, two are stable and one unstable.

With rational expectations, matters are more complicated. k_{t+1} is now the solution to

(5.6) $\frac{k_{t+1}}{s(r(k_{t+1}))} = w(k_t)$;

so assuming for the moment that there is a unique solution for k_{t+1} for any given k_t, stability now depends on whether

(5.7) $1 - \eta \frac{S_L}{\sigma} < or > \frac{1-S_L}{\sigma}$,

i.e. it is precisely the same condition. Still, the speed of convergence (divergence) is different, since with myopic expectations

(5.8a) $\frac{dln\, k_{t+1}}{dln\, k_t} = \eta \frac{S_L}{\sigma} + \frac{1-S_L}{\sigma}$.

while with rational expectations

(5.8b) $\frac{dln\, k_{t+1}}{dln\, k_t} = \frac{\frac{1-S_L}{\sigma}}{1 - \eta \frac{S_L}{\sigma}}$.

Consider the lower steady state in Figure 7. Note that for values of $k_t < k^*$, on the rational expectations trajectory, k_{t+1} is higher than on the myopic expectations trajectory. The reason is simple: with rational expectations, workers realize that k will be higher next period than this, and therefore the return on the capital they invest will be lower than it was in the current period; and therefore (if the interest elasticity of the savings rate is negative) they will need to save more for their retirement. Thus, convergence to the steady state is faster.

Section 6. Multiple momentary equilibria

Most of modern macroeconomics has been based on the assumption that individuals have rational expectations. Behavior is forward-looking. Typically, this introduces enormous complexity, since what happens tomorrow depends on what is expected the year after. But luckily, the dynamics for this standard OLG model without non-reproducible assets such as land is much simpler: the return to capital invested at time t depends only on the amount of capital at time $t + 1$. Individuals must, of course, form expectations of what is happening at time t (that is, how much others are simultaneously investing; and, if technology were changing, or if there were wage rigidities, of how these might play out). While there could be a wide range of such expectations formation, much of modern macroeconomics has tried to short-circuit such a behavioral analysis, focusing instead on the hypothesis of rational expectations. In the remainder of this section, we will follow in that tradition.

The savings equals investment equation can then be written in the case where savings depends just on the (rationally expected) return to capital as

(6.1) $\Omega(k_{t+1}) \equiv \frac{k_{t+1}}{s(k_{t+1})} = w(k_t)$.

The denominator of the left-hand side (as we have already noted) may well be increasing in k_{t+1}, so much so that the LHS may, for a range of values of k_{t+1}, be decreasing in k_{t+1}. If $\Omega(k_{t+1})$ is not monotonic, there will exist a range of values of k_t (or more precisely, of $w(k_t)$) for which there are multiple momentary equilibria—multiple solutions to (6.1) as illustrated in Figure 8. Define $\underline{\Omega}$ as the minimum value of Ω for which there are multiple values of k solving $\underline{\Omega} = \Omega(k)$; and similarly, $\overline{\Omega}$ as the maximum value of Ω for which there are multiple value of k. Define $\underline{k}$ as the solution to $w(k_t) = \underline{\Omega}$ and similarly for $\overline{k}$. Then so long as for some value of k_t,

(6.2) $\underline{\Omega} < w(k_t) < \overline{\Omega}$,

there will be indeterminacy in the momentary equilibrium, and accordingly, as we shall shortly show, in the dynamic trajectory of the economy. (See Figure 8). Since $w'(k_t) > 0$ under standard assumptions on production functions, $w(k_t = 0) = 0$, and there exists values of $w(k_t)$ for which, for some value of k_t, there exist multiple values of k_{t+1} which satisfies (4.2). That is, *if* $\Omega(k_{t+1})$ *is not monotonic, there will be multiple momentary equilibria*. Monotonicity of $\Omega(k_{t+1})$ is a necessary and sufficient condition for a unique momentary equilibrium.

Figure 8 Here

Figure 8 Non-monotonicity of Ω and the existence of multiple momentary equilibria

If $\Omega \equiv \frac{k}{s(r(k))}$ is not monotonic, there will exist multiple momentary equilibria, i.e. multiple values of k at which savings equals investments with rational expectations.

The rest of this section is devoted to showing that under plausible restrictions on preferences and technology, $\Omega(k_{t+1})$ will not be monotonic over some range of values of k_{t+1}, implying that there is a range of values of k_t for which there are multiple momentary equilibria.

First, however, we note the similarity and difference between the analysis of section 4 on multiple steady states, and section 6, on multiple momentary equilibria. Both focus on the fundamental equation of OLG models, $k_{t+1} = s_t w_t$. In the case of steady states, we drop the subscript *t*, and the analysis focuses on *separating* impacts of technology and preferences, so formulating investment and savings as functions of the interest rates, and wages as a function of investment, we look for the circumstances in which $\frac{k}{w} = s$. Here, we are looking at the equilibrium at a moment of time, where k_t and hence w_t is given. But the savings rate and therefore next period capital stock depends on expectations of *r*, and we ask, with rational expectations, are there multiple solutions to $\frac{k_{t+1}}{s(r(k_{t+1}))}$. Here the focus is on *interactions* between savings and production (i.e. the returns to savings).

6.1. General formulation

Differentiating $\Omega(k_{t+1})$ with respect to k_{t+1} yields

(6.3) $\Omega'(k_{t+1}) = \frac{1}{s_t}\left(1 - \frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}\right) = \frac{1}{s_t}\left(1 + \eta\frac{S_L}{\sigma}\right)$

where $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is the interest rate elasticity of savings. $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} < 0$ is the elasticity of the interest rate with respect to the capital stock. As we have already noted, these elasticities depend on the intertemporal elasticity of substitution in consumption (IES) and the elasticity of substitution between capital and labor (ES), respectively.

A sufficient condition for $\Omega'(k_{t+1}) > 0$ is that $\frac{dlog(s_t)}{dlog(1+r_{t+1})} \geq 0$. That is, if the saving rate is a monotonically increasing function of the interest rate, there is a unique momentary equilibrium. If, however, for some values of k_{t+1}, $\frac{dlog(s_t)}{dlog(1+r_{t+1})} < 0$, Ω may not be invertible, i.e., for some values of k_t, there may be multiple values of k_{t+1}, all consistent with rational expectations.

In particular, assuming homothetic preferences, we have

(6.4) $s_t\Omega'(k_{t+1}) = 1 + \eta\frac{S_L}{\sigma} = 1 - (1-s)(1-\zeta)\frac{S_L}{\sigma}$

Even if there is a constant intertemporal elasticity of substitution (i.e. ζ is constant) and constant elasticity of substitution in production, $\Omega'(k_{t+1})$ can change signs, because s varies, from approximately 1, when the interest rate is near zero (when k approaches infinity, assuming the intertemporal elasticity of substitution is less than unity)—implying that $\Omega' > 0$ — to approximately zero, when the interest rate approaches infinity; but if the elasticity of substitution in production is less than one, when the interest rate approaches infinity $S_L \to 0$, so again $\Omega' > 0$. But there can exist a value of k in between for which

(6.5) $(1-s)(1-\zeta)\frac{S_L}{\sigma} = 1,$

if σ is sufficiently small. It is even easier for Ω' to switch signs if σ or ζ are not constant.

6.2. Leontief preferences

Consider first Leontief preferences, $u_t = min\left(\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2}\right)$ for which it can be shown that $s_t = \frac{1}{1+\frac{a_1}{a_2}(1+r_{t+1})} = \frac{1}{1+a(1+r_{t+1})}$, where now $a \equiv \frac{a_1}{a_2}$. a_1 and a_2 are weights on consumption in working (young) and retirement (old) periods, respectively, and a large $a > 1$ means that individuals put more weight on consumption when young rather than when old. Then

(6.6) $\Omega = k(1 + a(1 + r(k))$

So

(6.7) $\frac{dln\,\Omega}{dln\,k} = 1 - \frac{a(1+r)}{1+a(1+r)}\frac{S_L}{\sigma}$

All the parameters in (6.7) are technological, except a, which represents preference for consumption today relative to the future. For every $\{\sigma, r, S_L\}$, there exists an a large enough, i.e.

(6.8) $a \geq \frac{1}{(1+r)(\frac{S_L}{\sigma}-1)}$

such that $\frac{dln\,\Omega}{dln\,k} \leq 0$, so that Ω looks as depicted in Figure 8. In other words, assuming $\sigma < 1$, for any given elasticity of substitution production function, there is a k (and an associated $r\ and\ S_L$), and an a such that Ω is not monotonic.

6.3. Two techniques of production.

Now, consider the two-technique technology discussed above. Here, it proves convenient to recall how the interest rate depends on k, jumping down as k goes from below $\hat{k}$ to above $\hat{k}$, with corresponding jumps in s and k/s, as depicted in Figure 6. Figure 6 depicted k/w as a function of k and s as a function of k. r, of course changes discretely at $k = \hat{k}$. Figure 9 plots

k/s as a function of k. $k/s = k/s(\vartheta_1)$ for small k (i.e. increases linearly with k) until $\hat{k}$, then it falls (if the savings interest elasticity is negative) and increases linearly above that--$k/s(\vartheta_2)$. *There always exists multiple momentary equilibrium when* $k_t < \hat{k}$ *if*

(6.9a) $\frac{\hat{k}}{s(\vartheta_1)} > \mu_1 > \frac{\hat{k}}{s(\vartheta_2)}$.

Similarly, *there always exists multiple momentary equilibrium when* $k_t > \hat{k}$ *if*

(6.9b) $\frac{\hat{k}}{s(\vartheta_1)} > \mu_2 > \frac{\hat{k}}{s(\vartheta_2)}$.

Note that when (6.9a) is satisfied, so is (4.14a), so there are multiple steady states, but even when there exist multiple steady states, the momentary equilibrium can be unique. With four technology parameters and two taste parameters, it is easy for these inequalities to be satisfied. They are the conditions under which there can be animal spirits in a world with rational expectations, i.e. where pessimism about the rates of return on capital leads to so much savings that the pessimism becomes justified.

Note that the two-technology economy of this subsection simply reflects a production function in which the elasticity of substitution for $k \neq \hat{k}$ is infinite, but at $k = \hat{k}$, the elasticity of substitution is zero. Similar results would, of course, obtain if the elasticity of substitution were large for both high and low k but of intermediate value in between. The general formulation presented earlier makes it clear how easy it is for there to be multiplicity of momentary equilibria.

Figure 9 Here

Figure 9. There always exists multiple momentary equilibria if $\frac{\hat{k}}{s(\vartheta_1)} > \mu_1 > \frac{\hat{k}}{s(\vartheta_2)}$ and $\frac{\hat{k}}{s(\vartheta_1)} > \mu_2 > \frac{\hat{k}}{s(\vartheta_2)}$.

6.4. Multiple momentary equilibria and multiple steady states

We can use a slight extension of Figure 8 to illustrate that when there are multiple momentary equilibria, there are likely to be multiple steady states. A steady state can be described as any value of k for which $\Omega(k) = w(k)$. In Figure 10, we superimpose the curve $w(k)$ onto Figure 7, and whenever the two curves intersect, there is a steady state. In the Figure, there are three such intersections. Since the $\Omega(k)$ curve shifts up and down as savings changes, for a given technology, *there are intertemporal preferences (weighting future vs. current consumption) such that there are multiple steady states whenever $\Omega(k)$ is not monotonic.*

Figure 10 Here

Figure 10 Multiplicity of momentary equilibria and multiplicity of steady states.

The Figure shows that when there is a multiplicity of momentary equilibria, it is likely that there is a multiplicity of steady states. Any value of k such that $\Omega\ (k) = w(k)$ is a steady state.

These formal results describing steady states can be better understood if we expressly look at the dynamics.

Section 7. Wobbly Dynamics

Earlier, we analyzed the dynamics of this simple OLG model in the case where momentary equilibrium was unique, both for the case of myopic expectations and rational expectations. When, however, there are multiple momentary equilibria—as can arise with rational expectations—then the economy may "wobble," choosing variously one or the other momentary equilibria, randomly, or according to some convention (such as in sunspot equilibria).[28] There can in fact be an infinity of trajectories consistent with rational expectations.

To see this, we consider a simple extension of Figure 7 for the case of multiple momentary equilibria, noting that when there are, for a range of values of k_t, multiple equilibrium values of k_{t+1}, as depicted in Figure 11. Let $\underline{k}$ be the lowest value of k_t for which there exists multiple momentary equilibria (here 3) and $\overline{k}$. Then for any $\underline{k} < k_t < \overline{k}$, there are three possible values of k_{t+1} on a possible rational expectations trajectory.

Figure 11 here

Dynamics where there are multiple momentary equilibria

When there are multiple momentary equilibria for $\underline{k} < k_t < \overline{k}$, the economy can wobble around, so long as k_t remains within the bounds of $\underline{k}$ and $\overline{k}$, If $k^H < \overline{k}$, even if the upper steady state value of k is locally stable, if there are expectations of a low value of k next period, and a

[28] We should emphasize that wobbly dynamics are quite different from chaotic dynamics. The former arises only when there are multiple momentary equilibria; the latter arises in well-defined but complex deterministic systems, where the trajectories exhibit seemingly random and unpredictable behavior.

correspondingly have value of r, then savings will be low, and those expectations can be realized: the steady state is globally unstable.

In some conditions, there are sequences of "choices" which may lead to a (globally or locally) stable steady state[29], but there are other conditions under which there is no globally stable steady state. The critical condition rests on where the steady states lie vs. the bounds on multiple equilibria (remember, that where there are multiple equilibria depends on the value of k). Consider, in particular the case where there are three steady states with $0 < \underline{k} < k^L < k^H < \overline{k}$, where k^H denotes the upper steady state and k^L, the lower. Even when the economy is at say k^H, the economy may suddenly jump in a fully rational expectations equilibrium to a smaller value of k. Nothing in the theory ensures that it will remain at k^H.[30] The economy can bounce around infinitely without converging. There can be an infinite number of rational expectations trajectories, though there are three steady states, where the *correspondence* giving k_{t+1} for any value of k_t crosses the 45 degree line (i.e. $k_{t+1} = k_t$).

In the case depicted in Figure 11, there are a rich set of *possible* dynamics, if we do not constrain in any way the equilibrium selection. (By contrast, if we have very sticky expectations, i.e. myopia, then there is no equilibrium selection problem, since given k_t there is a unique value of k_{t+1}) There are many possible "rules" that might constrain momentary equilibrium selection. Consider, for instance, a slight variant of myopia: if the economy has remained at the same value of k_t for two periods, then individuals believe it will remain there. Then, once one of the steady states has been established (by remaining there long enough), it "sticks." But the economy

[29] In particular, we define a phase transition as a change in the economy from a situation where the number of momentary equilibria changes. A phase transition from a state with a unique momentary equilibrium to a state with multiple momentary equilibria occurs when the economy initially starts from the outer region of $\underline{k}$ or $\overline{k}$ and moves in. If the economy ever moves outside the bounds $\{\underline{k}, \overline{k}\}$, its trajectory is determinate, until it moves back into those bounds. In fully stable trajectories, that is impossible.

With $k_{t+1} = s(\vartheta_2)\mu_1$, expectations of a high interest rate can return, so $k_{t+2} = s(\vartheta_1)\mu_2$. Then expectations could shift to a lower r, so $k_{t+3} = s(\vartheta_2)\mu_1$. In this simple model, then, the values of k can randomly move among the four values, $s(\vartheta_1)\mu_1, s(\vartheta_1)\mu_2, s(\vartheta_2)\mu_1, and\ s(\vartheta_2)\mu_2$. (There is a fifth value of k to which the economy can move at any time, $\hat{k}$, but for the moment we ignore this, to simplify the discussion.) That is, referring to the four states as A, B, C, D, and noting that an economy in A can go to A or C, in C can go to B or D, in D can remain at D or go to B, any sequence of pairs (AA,AC, CD, CB, DD, DB, BA, BC) is a rational expectation equilibrium, e.g. AAACDDDBACDDB….. . There are an infinite number of such sequences and they may have no periodicity.

could be in the steady state for one period, and move out of it, even on a rational expectations trajectory. Rational animal spirits can be tamed.

Alternatively, consider a consistently pessimistic economy, such that when there is a choice of equilibria, always chooses the lowest value of k_t. There is then a well-defined trajectory, with a discontinuity in behavior at $k_t = \overline{k}$, where there is an upward jump in k_{t+1}. Or a consistently optimistic economy, such that when there is a choice of equilibria, always chooses the highest value of k_t. There is then an upward jump in k_{t+1} at $k_t = \underline{k}$. In all of these cases, the rule for momentary equilibrium selection determines the long run trajectory, given the initial value of k, but there can still be multiple steady states.

But none of these formulations really accounts for the role of (rational or irrational) animal spirits—a role for unpredictable expectations shaping economic trajectories. In the real world, of course, there are "shocks" to the economic system, innovations, political events, pandemics, wars, etc. than can unleash animal spirits, and though there is no presumption that individuals will, in responding to these shocks do so in ways consistent with rational expectations, there is a presumption that they will not respond myopically: the world of the future *will be* different. None of this is, of course, captured in our simplistic model.

Welfare decreasing innovations

Interestingly, changes in technology which increase income per capita in, say, the high steady state k^H may simultaneously change its stability properties, moving the economy from a situation where $k^H > \overline{k}$, so that at k^H, there is a unique momentary equilibrium, to one in which $k^H < \overline{k}$, so there are multiple momentary equilibria. (See Hirano and Stiglitz (2022a)). In that case, while incomes initially increase, animal spirits may move the economy to a worse equilibrium—and even eventually into a steady state with a permanently lower per capita income, with low savings and high interest rates.

7.2 Wobbly dynamics for the two-technology economy with homothetic preferences

We can explore the rich set of rational expectations trajectories in the two-technology economy of section 4.2, and in particular a situation where there is multiplicity of momentary equilibria both when $k_t < \hat{k}$ and $k_t > \hat{k}$, i.e. both (6.9a) and (6.9b) are satisfied. It follows that the correspondence between k_t and k_{t+1} appears as in Figure 12: when $k_t < \hat{k}$, $w = \mu_1$, and

k_{t+1} equals either $\{s(\vartheta_1)\mu_1, \hat{k}, or\ s(\vartheta_2)\mu_1\}$, as depicted, but when $k_t > \hat{k}, k_{t+1}$ equals either $\{s(\vartheta_2)\mu_2, \hat{k}, or\ s(\vartheta_1)\mu_2\}$. Hence the economy can be in the lower steady state (with $k_L = s(\vartheta_1)\mu_1$, but then suddenly jump (based on animal spirits—the belief that interest rates are going to fall) to $s(\vartheta_2)\mu_1$, and with the resulting high savings, interest rates do fall. At the high k and resulting high wages, savings (if beliefs remain that interest rates will remain low) rise to $s(\vartheta_2)\mu_2$.[31] The economy can remain there for a while, but then animal spirits can move the economy back—a (rationally expected) increase in interest rates lows savings, to $s(\vartheta_1)\mu_2$, and at that lower savings, interest rates do rise. Note that all that this pattern of wobbly dynamics requires is that $s(\vartheta_1)\mu_2 < \hat{k}$ and $s(\vartheta_2)\mu_1 > \hat{k}$. But the conditions for the existence of multiple momentary equilibria assure us that that is the case. In this case, *whenever there exists multiplicity of momentary equilibria, there exists wobbly dynamics*, i.e. the economy can wobble from one steady state to another. In more general cases, the wobbles can take on much more complex patterns. (See Hirano and Stiglitz (2022a) for a fuller description of these wobbles and the conditions in which they can occur.)

Figure 12 Here

Figure 12 Two technology economy: The correspondence between k_t and k_{t+1} ensures that the economy can wobble between $(\vartheta_1)\mu_1, s(\vartheta_1)\mu_2, s(\vartheta_2)\mu_1, and\ s(\vartheta_2)\mu_2$

7.3. Land speculation, credit frictions and wobbly dynamics.

The possibility of wobbly dynamics is further enhanced by the presence of a non-produced asset, like land, or credit frictions.[32]

With land, whether there are multiple momentary equilibria depends not just on the value of k, but also on the value of the price of land, P. A high price of land diverts savings from productive investment into land speculation, resulting in a higher return to capital. At the same time, the higher return to capital entails a faster increasing price of land, if the capital arbitrage equation (requiring that the return to holding land and capital must be the same), further raising the price of land. But if animal spirits are at play (as they can be even in our rational expectations model), a sudden change in expectations, from bull to bearish, results in a fall in land prices, negative (or low) returns to capital, and high savings (to meet retirement needs, given the lower return to

[32] See Hirano and Stiglitz (2022b).

capital, or in more general models, for precautionary purposes), which sustains the lower returns to capital. The low price of land sustains more capital accumulation for the further reason—fewer savings are diverted into land speculation. The downward movement in r is thus reinforced. But so long as the economy remains within the bounds in which there are multiple momentary equilibria, animal spirits can at some later date become bullish, with r increasing, the price of land increasing, and capital accumulation falling in tandem.

If the price of land gets too high and exceeds a critical value, expectations must turn bearish, and this endogenous change in expectations leads to an endogenous land price crash. Similarly, if the price of land gets too low and exceeds another critical value, expectations must turn bullish, leading to an endogenous land price boom. Between these critical values, animal spirits drive economic fluctuations.[33]

Section 8. A simple illustration of wobbly growth with credit frictions

A standard criticism of simple models such as those presented in the text is that they leave out real-life frictions. Here, we show that embodying those frictions may actually make it more likely that there be multiplicity of momentary equilibria—and consequently more likely that there will be wobbly dynamics. In particular, we extend the basic model to an economy with endogenous growth and credit frictions, and in which if individuals expect low interest rates, savings will be high, borrowing constraints become relaxed, which expands capital investment, leading to high economic growth; but conversely, if they expect high interest rates, there will be less incentive for capital investment, leading to high interest rates—confirming expectations--and low economic growth. In this way, whether the *growth rate* will be high or low will be determined by (self-fulfilling) beliefs regarding equilibrium interest rates and whether borrowing constraints are or are not binding.

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[33] Analytically, the existence of multiple steady states can be seen even more easily. In steady state, the value of land is D/r, where D is the value of land rents and r is the interest rate (ignoring the possibility that there might be a bubble, where land is not held for its rents, but to sell to the next generation.) Then in steady state $s(r)w(k) = k + D(r)/r$, where the production function (technology) defines the relationship between k and r and D and r. In standard technologies, an increase in capital leads to a decrease in the return on capital, but can lead to an increase or decrease in the return on land, depending on whether capital and land are complements or substitutes. Rewriting the above equation as $s(r)w(k) - D(r)/r = k$, as k increases, the LHS can increase if the savings rate doesn't decrease too fast (i.e. if the elasticity of substitution is not too small, so r falls a lot, and if s' isn't too positive) or if D decreases enough (that is, the fall in the interest rate increases, at any D, the value of land, but if land and capital are complements, the increase in k lowers land rents, so much so that the PDV os land rents is lowered.) We omit the precise calculations. Hirano and Stiglitz (2022b) show that there may exist, under quite natural conditions, many more steady states and momentary equilibria.

8.1. The environment

We assume within any generation a fraction η of the young are entrepreneurs who have capital investment opportunities. The remaining fraction $1-\eta$ are savers (workers) who don't have capital investment opportunities.

The budget constraint of an entrepreneur i is given by

(8.1) $c_t^i + k_{t+1}^i = w_t + b_t^i$ and $c_{t+1}^i = R_t^c, k_{t+1}^i - (1+r_t)b_t^i$,

where k_{t+1}^i is the entrepreneur's capital investment at date t. b_t^i is the amount of borrowing at date t if $b_t^i > 0$, and lending if $b_t^i < 0$. As earlier, $R_t^c = R_t + 1 - \delta$ is the rate of return from capital investment between date t and $t+1$, where R_t is the rental rate of capital and δ is the depreciation rate of capital.

The budget constraint of a saver i is given by

(8.2) $c_t^i = w_t + b_t^i$ and $c_{t+1}^i = -(1+r_t)b_t^i$,

Since savers don't have investment opportunities, they lend all savings to entrepreneurs at the interest rate r_t.

We assume credit frictions. The entrepreneurs cannot borrow unless they have collateral. Only a fraction $\theta \in (0,1]$ of the value of capital can be used as collateral, so the borrowing constraint is

(8.3) $b_t^i \leq \theta k_{t+1}^i$,

8.2. Utility and production functions

For simplicity, in this example, we assume that the utility function of each young entrepreneur is the log-utility function given in section 4.3.1, while that of each young saver is Leontief utility given by (4.16).[34]

The production sector exhibits standard Marshallian external increasing returns to scale in capital investment (Aoki 1970, 1971; Frankel 1962; Romer 1986.). Competitive firms produce

[34] These parameterizations simplify the analysis. What is important is that workers have an intertemporal elasticity of substitution less than unity.

the final consumption goods by using capital and labor. To keep things simple, we assume that the production function of each firm j is given by

(8.4) $y_{tj} = (k_{tj})^{\alpha}(\chi(K_t)l_{tj})^{1-\alpha}$,

where k_{tj} and l_{tj} are capital and labor inputs of firm j. $\chi(K_t)$ is labor productivity, with $\chi'(K_t) > 0$, where K_t is aggregate capital stock at date t. When we aggregate over all firms, we get aggregate level returns to scale. We assume $\chi(K_t)$ takes on a particular functional form:

(8.5) $\chi(K_t) = aK_t$.

This is a key (though conventional) simplifying assumption.

The individual firm ignores its tiny influence on the aggregate capital stock and thus on the productivity of its own worker. Thus, each firm employs capital and labor up to the point where its private marginal product equals the rental rate of capital and the wage rate, respectively. Using (8.5), factor prices and the aggregate production function become, for every t

(8.6a) $R_t = \alpha A$, (8.6b) $w_t = (1-\alpha)AK_t$; and (8.6c) $Y_t = AK_t$,

where $A \equiv a^{1-\alpha}$. In steady state, the wage rate and aggregate output grow at the same rate of aggregate capital stock, so $\frac{w_{t+1}}{w_t} = \frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t}$, and the rental rate of capital is constant, with the increasing labor productivity as aggregate capital stock increases canceling out the effect of capital deepening and the diminishing returns associated with it. $R^c = \alpha A + 1 - \delta$ is also constant, and we assume $R^c = \alpha A + 1 - \delta > 1$, i.e., the net return from capital investment is greater than unity.

8.3. The capital-investment function

Investment depends on whether the borrowing constraint binds or not. When the borrowing constraint does not bind, the equilibrium interest rate equals R^c and entrepreneurs invest up to the point where the marginal return equals R^c. When the borrowing constraint binds, i.e., $R^c > 1 + r_t$, we can derive the capital investment function of entrepreneurs by substituting (8.3) into (8.1), yielding

(8.7) $k_{t+1}^{i} = \frac{sw_t}{1-\theta}$, i.e., the investment function equals leverage, $1/1-\theta$, times savings sw_t. (Recall that s is the saving rate, which is fixed under our specification of entrepreneurs' preferences.)

8.4. Multiple momentary equilibria

Workers' savings are given by

(8.8) $-b_t = \frac{w_t}{1+a(1+r_{t+1})}$,

The downward sloping curve depicted in figure A2. The figure, which also draws the investment function derived earlier, illustrates the possibility of two values of r_t at which savings equals investment--two interest rates that clear credit markets.

Figure 13 Here

Figure 13 two equilibrium interest rates that clear credit markets

When the borrowing constraint binds, i.e., $R^c > 1 + r_t$, the equilibrium interest rate is determined according to

(8.9a) $\frac{\eta s w_t}{1-\theta} = \left[\eta s + \frac{1-\eta}{1+a(1+r_t)}\right] w_t$,

where the left-hand side is the aggregate investment demand, while the right-hand side is the aggregate savings of the economy. Solving for $1 + r_t$ yields

(8.9b) $1 + r_t = \frac{1}{a}\left(\frac{(1-\eta)(1-\theta)}{\eta s \theta} - 1\right)$.

On the other hand, when the borrowing constraint does not bind, the equilibrium interest rate equals R^c, as we have previously noted. But whether the borrowing constraint binds or not depends on expectations.[35]

8.5. Dynamics

The evolution of aggregate capital follows according to

[35] Certain technical conditions have to be satisfied to ensure that there are two intersections of the savings-investment curves. We focus on the case where $\underline{\theta} \equiv \frac{1-\eta}{1-\eta+(1+aR^c)\eta s} < \theta < \frac{1-\eta}{1-\eta+(1+a)\eta s} \equiv \overline{\theta}$, which ensures that $R^c > \frac{1}{a}\left(\frac{(1-\eta)(1-\theta)}{\eta s \theta} - 1\right)$, i.e. that R^c is greater than the solution (A.9b).

(8.10) $K_{t+1} = \left[\eta s + \frac{1-\eta}{1+a(1+r_t)}\right] w_t = \left[\eta s + \frac{1-\eta}{1+a(1+r_t)}\right] A(1-\alpha)K_t.$

Hence, the growth rate of the economy can be written as

(8.11) $\frac{K_{t+1}}{K_t} = \left[\eta s + \frac{1-\eta}{1+a(1+r_t)}\right] A(1-\alpha).$

From (8.11), it is straightforward that the growth rate of the economy will be high or low depending on expectations regarding the equilibrium interest rates. If individuals believe that the equilibrium interest rate will be low, savers (workers) save a lot, which leads to low interest rates. At that low interest rates, entrepreneurs invest in capital investment with maximum leverage, which generates high economic growth. Conversely, if they believe that the equilibrium interest rate will be high, investing and saving are indifferent even for entrepreneurs. In other words, entrepreneurs have no incentive to invest in capital by borrowing up to the limit. Moreover, savers save less. These lead to less capital investment in equilibrium, resulting in low economic growth.[36]

Section 9. Policy and Equilibrium Selection

As we have noted, when there are multiple momentary equilibria, the theory provides no answer to the question of which possible equilibrium is selected and how all workers coordinate their actions. If some, for instance, think that interest rates will be low (the economy will be in the high investment equilibrium) and others that interest rates will be high, the outcome that emerges will be different from that expected by everyone. There are, of course, many alternative coordinating mechanisms—as earlier literature explored, actions could be coordinated on the number of observed sunspots. This will give rise to complex trajectories, but if all believe that when there are a large number of sunspots, interest rates will be low, there will be a self-fulfilling equilibrium. Alternatively, all could coordinate on whether the central bank announces that monetary policy will be tight—interest rates high. Again, the central bank's prophecy will turn out to be correct, but not necessarily because of anything it has done. We could be living in a world where monetary policy has no effect—other than its role as a coordinating mechanism.

[36] As another model that shows multiplicity of momentary equilibria depending on expectations of interest rates, Greenwald and Stiglitz (2003) show that if the market believes that a firm is not likely to go bankrupt, interest rates will be low, and at the low interest rate, the probability of bankruptcy will be low. But if the market believes that the probability of bankruptcy is high, then there will be a high interest rate, and a correspondingly high probability of bankruptcy. Both can be rational expectations equilibria.

If there is enough stickiness (e.g. in expectations) then it is less likely that there will be multiplicity of momentary equilibria: the past determines the future. By the same token, the selection of one of the set of possible momentary equilibria today may be able to determine the future trajectory. For policy-making, this has a great advantage: decisions made today (together with time-consistent projections of interventions in the future) determine the trajectory, and the policy maker can choose among possible trajectories that which maximizes intertemporal welfare.

But if animal spirits are volatile, an equilibrium selection choice in the short run will not tie down the process of convergence to a particular long-run equilibrium, since, in future periods, the equilibrium-selection process may give rise to a choice of a different kind of equilibrium. We have noted that if within our world with no shocks, there is *enough* stickiness in expectations, as in the BDK model, then the problem of multiplicity of momentary equilibria simply doesn't arise. But we have also observed that if there is sufficiently little stickiness, in a broad range of models it does arise. And we have argued that in more realistic settings, where there are non-stationary shocks to the economy, there is likely room for highly volatile animal spirits—we would suggest consistent with what is observed.

The rational expectations trajectories which emerge do not, in general, maximize any long-run social welfare function—the average value of utility or the present discounted value of utility is not, in general, maximized. The fluctuations that occur are, in general, inefficient, and in some cases, there may be oversaving (i.e. the interest rate may be lower than the rate of growth),[37] and this can be true even in the presence of a non-produced asset like land.

That raises the question of whether there are government policies which might select one of the momentary equilibria—and indeed structure a sequence of such choices so as to maximize societal well-being. If so, that might seem to resolve the problem of indeterminacy of momentary equilibria and even of long run steady states. Can we, in other words, embed an OLG model within a Ramsey setting?

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[37] Obviously, oversaving cannot occur in our model with zero population growth, if $f' > 0$, since then the interest rate always exceeds the (zero) growth rate. But if we modify the model slightly to allow population growth at the rate n, then if the savings rate is higher enough, then the interest rate may be less than the growth rate, and the economy is dynamically inefficient. Then if the government can make a one period commitment, then the young can induce lower savings by promising to provide a lump sum social security payment financed by a tax on savings, (Without commitment powers, the next generation might renege on the provision of the social security payments, and knowing this, the younger generation might not reduce its savings.)

There is one setting in which the answer is yes: if each generation cares not only for itself, but for all future generations, choosing its behavior (savings rate) to maximize its utility—which corresponds to its view of social welfare—then, under the conditions under which time consistency problems do not arise, there is, given any initial condition (value of K_0), a well-defined trajectory of savings rates. (There still may be multiple steady states, with the economy converging to different steady states from different initial conditions.)

But, as Phelps and Pollack (1968) observed long ago, this is not the equilibrium which emerges with less altruistic individuals. Today's government will reflect today's citizens preferences, recognizing that subsequent generations will make their own decisions. If they know their descendants' preferences (including degree of altruism), they take that into account, and we have a standard intertemporal welfare problem without commitment (where we may have to worry about time consistency.) Assume, for example, that as in the model presented here, individuals are perfectly selfish, and that there is an infinitesimal growth rate of labor (not changing the dynamics presented here in any significant way, but giving majority control each period to the young). Their welfare is maximized when the interest rate is the highest possible, so that if the government can coordinate equilibrium selection (which is can through the use of tax policy, for instance, destroying the low interest rate equilibrium by imposing a lump-sum tax used to finance an interest rate subsidy, which induces individuals not to save), it will ensure that it is always the high interest rate equilibrium that is chosen. That implies that the next generation's welfare is lower than it would otherwise be.

It is obvious that the resulting equilibrium does not maximize intertemporal social welfare in general, and in particular, when the intergenerational discount rate is low (including the case of zero intergenerational discounting, as Ramsey argued for). Assume that there were a generation of young that realized this, and they themselves decided to act to maximize social welfare, not their own welfare, but knowing that their children would likely be as selfish as those in the past. They would then know if they saved a lot, their children would be better off, but their selfish children would save little, to maximize their own well-being. Succeeding generations may be better off, simply because while their savings rate was low, the benefits of the high wages generate a higher level of capital. Thus, their children would be wealthier than they: their sacrifice increased inequality, and it is possible that later descendants would be slightly better off as well. If the idealistic youth contemplating self-sacrifice for future generations are thus sufficiently inequality averse (i.e. maximizing social welfare with a sufficiently inequality averse social welfare function), they clearly will not undertake any sacrifice of their own well-being to enhance that of their selfish descendant(s). The seemingly socially inefficient equilibria will persist. (It is, of course, Pareto efficient.)

Of course, there are circumstances in which one generation can have a longer-term effect, e.g. when it knows that future generations will suffer from sticky expectations. (Depending on the set of instruments available to this generation, it can induce changes of behavior in future generations, as if it were at least partially forcing future generations in ways consistent with what it desires, and thus partially solving the commitment problem. (Korinek and Stiglitz (2008))). For instance, if the current generation believes future generations are myopic, believing that they will believe that whatever interest rate occurred last period will reoccur, then the current generation can choose the low interest rate equilibrium, ensuring that all future generations will do so—and thus *if they are altruistic enough, the economy will select the low interest rate equilibrium—it is unique, and the unique steady state is the low interest rate equilibrium.*

Section 10. Concluding comments

The OLG model with capital accumulation is an extraordinarily simple model, but remarkably, its full static and dynamic implications, with general production and preferences, have not been adequately explored, partly because, though simple to formulate, the analysis turns out to be somewhat complicated, with multiple steady states and even multiple momentary equilibria easily arising, even if they are precluded by some of the extreme assumptions (like Cobb-Douglas production functions and logarithmic preferences) commonly employed.[38]

The OLG models explored here have very different properties than those exhibited by the prevailing macroeconomic models, based on infinitely lived individuals (or dynastic families). We've highlighted here three, the existence of multiple steady states and of multiple momentary equilibria, and the presence of multiple complex equilibrium dynamics, which can occur even

[38] As we note in the introduction, our paper builds on the foundational work of Samuelson (1958) and Diamond (1965). There is a large literature on sunspot equilibria (multiple momentary equilibria) in pure consumption models or models with only a non-produced asset (money). See Cass and Shell (1983), Grandmont (1985), Matsuyama (1991), Golosov and Menzio (2020), Azariadis (1981), Azariadis and Guesnerie (1986), and Grandmont (1985) focuses on deterministic cycles and chaotic dynamics.

with rational expectations, where initial conditions matter and convergence to the steady state may not be monotonic—or may not even occur.

While we also showed that with sufficient stickiness in expectations, there is a unique momentary equilibrium, we also note that when the extreme strictures of rational expectations are relaxed—that today's beliefs are *precisely* satisfied tomorrow—and replaced with the hypothesis that today's beliefs are *approximately* satisfied tomorrow, the set of circumstances in which multiple momentary equilibria (and the consequent wobbly dynamics) is greatly expanded.[39] Moreover, when a number of the other simplifying assumptions we have made are lifted—if the labor force responds positively to wage increases and if there are non-homothetic preferences, such that when individuals get richer (or when wages increase), they want to disproportionately increase consumption in retirement, there exists a non-reproducible asset such as land, there are financial frictions—then there is even a greater likelihood of multiple steady states and multiple momentary equilibria[40] and convergence to the steady state, if it occurs, may not be monotonic.

The fact that results can differ so markedly between economies in which individuals have finite lives and those when they live infinitely long suggests that more attention should be paid to the former models. This is especially so because, as Woodford (1986) showed, the mathematical structure of the overlapping generations model is formally analogous to that of infinitely-lived agents models in which some agents are finance-constrained, even if others are not. The behaviour of economic agents that expect to be financially constrained is much like that of finite lived agents as described in the current paper. In this interpretation of our model, the "one

[39] Consider, for instance, "consistent expectations," whereby market participants make only rough forecasts; we only require, that if they expect interest rates to increase next period, and act accordingly, they do increase, and perhaps roughly commensurate with what they expected. It should be clear that it is easy for animal spirits to have wide play in economies with consistent expectations. This is important: some have argued that once one leaves the world of rational expectations, analysis is untethered; one can make irrational expectational assumptions that might generate all manner of behaviors. Our claim is simply that if there is scope for animal spirits under the extreme discipline of rational expectations, then there is even greater scope for animal spirits in economies with less severe strictures.

[40] There are a number of other simplifications which likely limit the extent of momentary indeterminancy in standard analyses. For instance, such indeterminacy easily arises in economies with heterogeneous individuals with different preferences and endowments, as Uzawa (1961, 1963) showed. Mueller and Woodford (1988) proved that the dimensionality of indeterminacy increases as the number of goods increases. Geanakoplos and Polemarchakis (1991) derived a similar result. These results suggest that with more complexity, the dimensional of multiplicity of equilibria may increase.

Financial frictions also can give rise to multiplicity of momentary equilibria. On-line appendix B shows multiplicity of momentary equilibria in a model with credit frictions.

period" in the overlapping generations model does not have to be the biological working life span and could be relatively short.

There are multiple other reasons why there should be more attention to OLG models, including that the dynamic properties of an economy with a mixture of dynastic families (acting according to the Ramsey model) and those with limited altruism (or finite lives with no altruism) are more in accord with those described here than with the Ramsey model.[41] [42]

We should be clear: The key properties of the *standard* Ramsey model to which we have called attention—the uniqueness of momentary equilibrium and the uniqueness of steady states are not general, even in models with infinitely lived individuals. Key are four sets of special assumptions—that all individuals are identical, that the intertemporal marginal rate of substitution, in steady state, does not depend on the level of consumption itself, that there are no financial frictions, and that there is limited capital heterogeneity. Under the first assumption, from any set of initial conditions, there is in general a unique trajectory maximizing utility—a unique momentary equilibrium. And under the second condition, asymptotically, regardless of the initial conditions, the interest rate equals the pure rate of time preference, which is fixed. But if the intertemporal marginal rate of substitution depends on the level of consumption, as with Koopmans preferences, then there can be multiple steady states, even in a Ramsey-like economy with a representative agent. And, as Uzawa (1961, 1963) showed, if there are heterogenous individuals with different preferences and endowment and different sectors with different factor intensities, there can be multiple momentary equilibria, giving rise to complex trajectories, with myopic or rational expectations, even with Ramsey individuals maximizing intertemporal preferences. Similarly, Woodford (1994) showed the existence of sunspot equilibria in a cash-in-advance model with just infinitely lived agents. And several studies show that in the presence of

[41] See Mattauch et al (2018) and Stiglitz (2018), and Mueller and Woodford (1988), who show, in particular, that local indeterminacy near the steady state continues to be possible even when infinite lived agents own a large fraction of total wealth, so long as their consumption is not too great a part of total consumption.

[42] We noted in footnote 2 the work of Blanchard (1985), who considers a model in which individuals work in every period and receive wages, but their labor-productivity may decrease and they face death shocks. So long as they expect to earn less income in the future due to a decline in labor productivity or retirement, individuals' behaviour becomes life-cycle, as in a two-period OLG model. The two-period OLG model we employ corresponds to the case where the second period is the (zero productivity) retirement period, whereas an infinitely-lived agent model corresponds to a case where individuals' labor-productivity never declines and there is no retirement period.

distributional effects and/or capital heterogeneity (as in putty clay models), convergence may be oscillatory—or there may even be limit cycles.[43]

Still, this paper should have made clear that it is much easier for multiplicity of momentary equilibria to arise in OLG models: it can occur even when there are no distributional issues. Everyone in the same generation is identical. This paper has also suggested that extending the analysis of this paper to make it more realistic—introducing another non-produced asset like land or credit frictions—may actually make multiplicity of momentary equilibria more likely to occur.

A recent strand of macroeconomics (Vines and Wills 2020) puts multiple equilibria at the center of macroeconomic analysis. While there has, for instance, been a large literature in macroeconomic models showing the existence of multiple equilibria in static or two-or-three-period models[44], there has been a dearth of literature in dynamic OLG models with capital accumulation, which provides the key link across generations.[45][46] This paper should be viewed as a prologue to a broader reconstruction of macroeconomics based on more general OLG models. For instance, here we have ignored all the central market failures—nominal and real wage rigidities, costly movements of factors across sectors, capital and financial market imperfections, and the fact that expectations are not in general rational—that are also central to

[43] See, e.g. Cass and Stiglitz (1969), Akerlof and Stiglitz (1969) and Stiglitz (1967). While these lessons from the earlier growth literature seem to have been lost, the much richer recent literature on complex dynamics has reinforced these earlier insights.

[44] See, e.g., Diamond (1982) and Cooper and John (1988) for static models, and see Neary and Stiglitz (1983) and Stiglitz (1994) and Lamont (1995) for two-period models, and Diamond and Dybvig (1983) for a three-period model. It is likely that had any of these works been extended into a formal dynamic model, they would naturally exhibit wobbly dynamics, suggesting that wobbly dynamics may arise in a wide variety of models. Earlier, Stiglitz (1967) had shown that in such models, simple savings behavior, consistent with OLG models, could give rise to cyclical dynamics.

[45] Some of the literature in OLG models show multiple steady states by introducing increasing returns to scale (externalities), or some frictions such as search frictions, frictions in wages or prices or a zero lower bound on the nominal interest rates (see a survey paper by Farmer (2020)). In contrast, as we show, in even the simplest overlapping generations models, a multiplicity of steady states can arise. Also, many of the papers that show multiple momentary equilibria in OLG models focus on pure consumption economies (see earlier footnote).

[46] As we have already noted, most of the literature in OLG, beginning with Samuelson and Diamond, focused only on steady states, typically simply overlooking the possibility of multiple steady states (or making special parametric assumptions in which that possibility cannot arise), and because they focused on steady states, they simply didn't address the question of multiplicity of momentary equilibria. Exceptions include Stiglitz (1973), who titled his paper, "The badly behaved economy with the well behaved production function," and Tirole (1985), who developed an OLG model with capital and pure bubble assets, he imposed a condition ensuring a unique momentary equilibrium. In that setting, he showed that there is a unique saddle path converging to a steady state with positive bubbles, and the presence of bubbles restores dynamic efficiency.

macroeconomic growth and fluctuations.[47] Incorporating these frictions will make it even easier to create "animal spirits" multiple momentary equilibria. For example, in the model we present in section 8, we show that they can arise when there is the possibility of credit rationing—with the credit rationing constraint binding when there are robust expectations, and not when there are not, and either can arise at any moment of time.

There are also important interactions between fluctuations and growth that should be explored: real estate booms may, for instance, have adverse effects on endogenous growth, and more broadly, the uncertainty created by fluctuations dampens incentives to invest, including in R&D. Thus, the models we presented here can be thought of as prototypes of how to analyse both short-run macroeconomic equilibria and global dynamics when the existence of multiplicity of momentary equilibrium depends on endogenous state variables and when there can be multiple steady states.

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[47] Some of these topics we take up in the companion paper (Hirano and Stiglitz, 2025a and 2025b).

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On line appendix A: Multiple steady states with two technology economy and Leontief preferences

In the limiting case where individuals want to strongly smooth their consumption over time[48]

$u_t = min\left(\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2}\right)$,

so the savings rate is given by

$s_t = \frac{1}{1+\frac{a_1}{a_2}(1+r_{t+1})} = \frac{1}{1+a(1+r_{t+1})}$, where now $a \equiv \frac{a_1}{a_2}$.

Then (4.13) becomes

$$\frac{1}{1+a(1+\vartheta_1)} < \frac{\hat{k}}{\mu_1}$$

For large enough a, this condition is clearly satisfied. Indeed, this inequality is an inequality involving 5 parameters, limited only by (4.11) and non-negativity constraints, so there are a wide range of parameters (even for given a, determining preferences) for which there are multiple steady states.

[48] Similar results obtain when individuals save for a particular target, such as buying a home or paying for their children's education. When the interest rate is low (high), to meet savings targets, people (don't) need to save more. References to target savings include Gourinchas and Parker (2002) and papers cited in the paper.

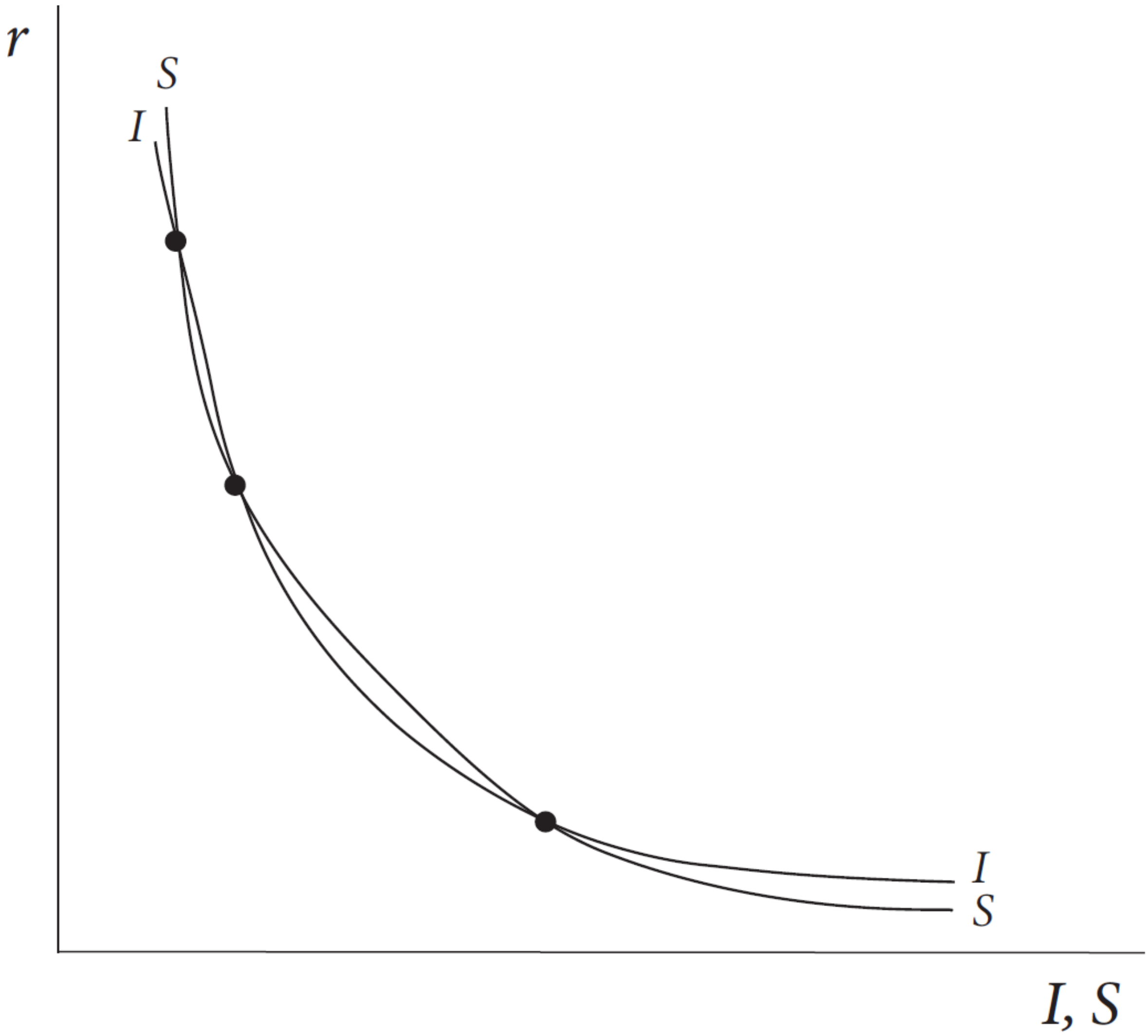


Figure 1

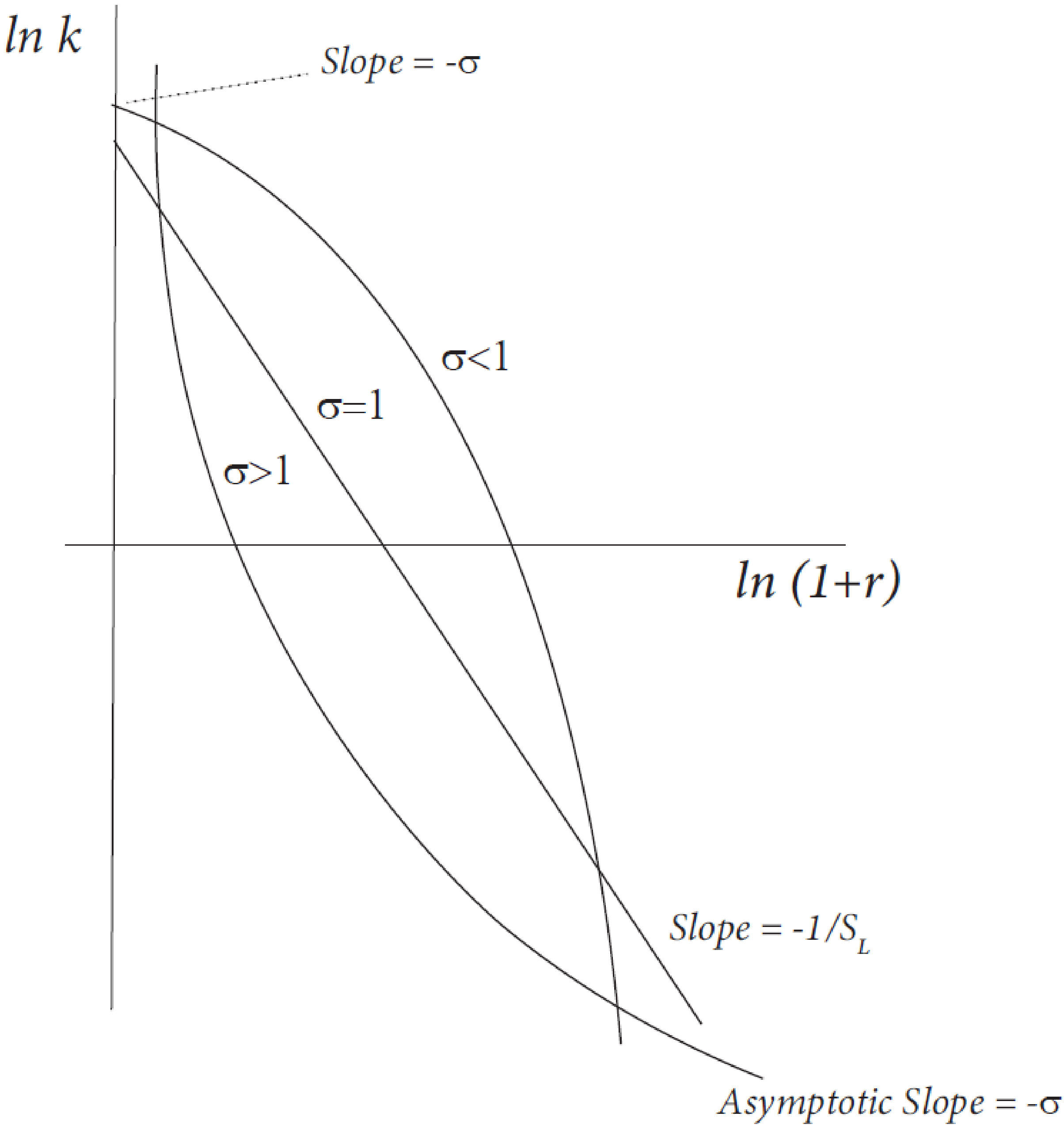
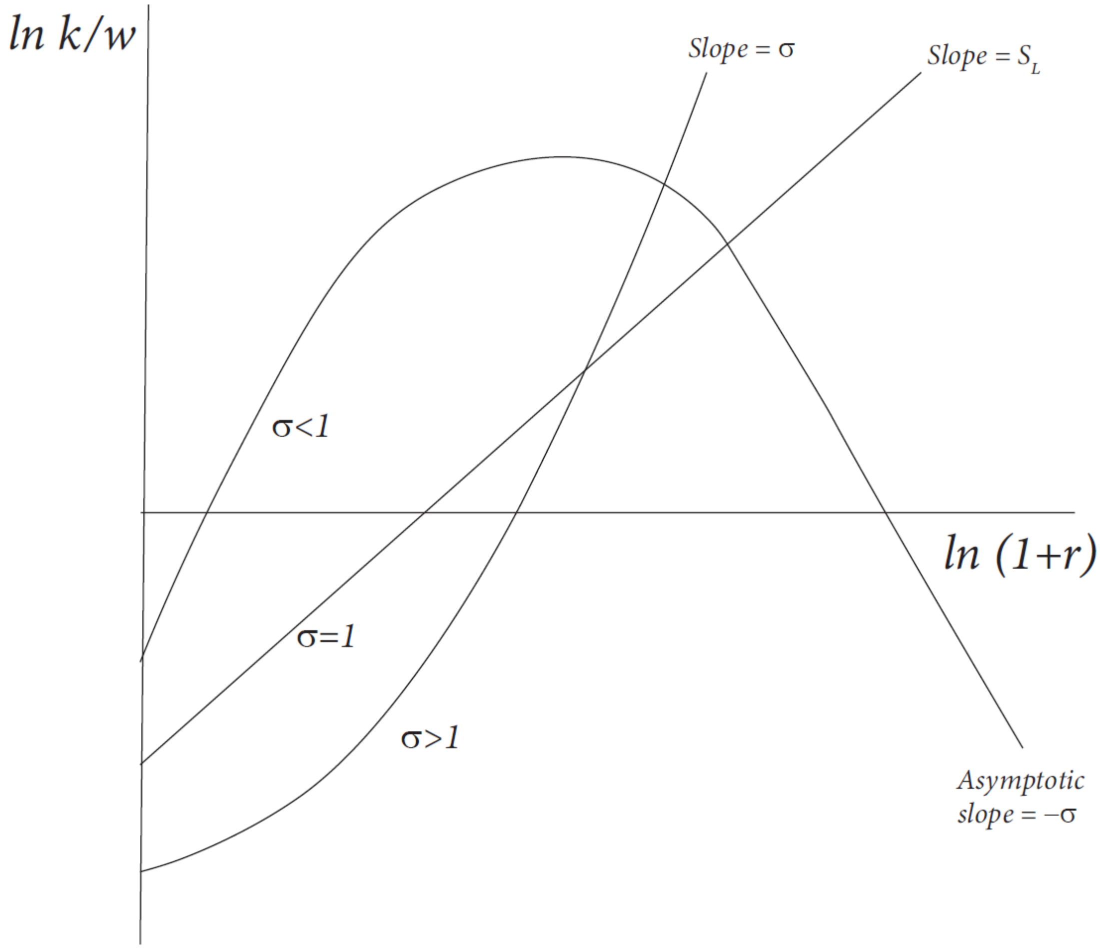


Figure2



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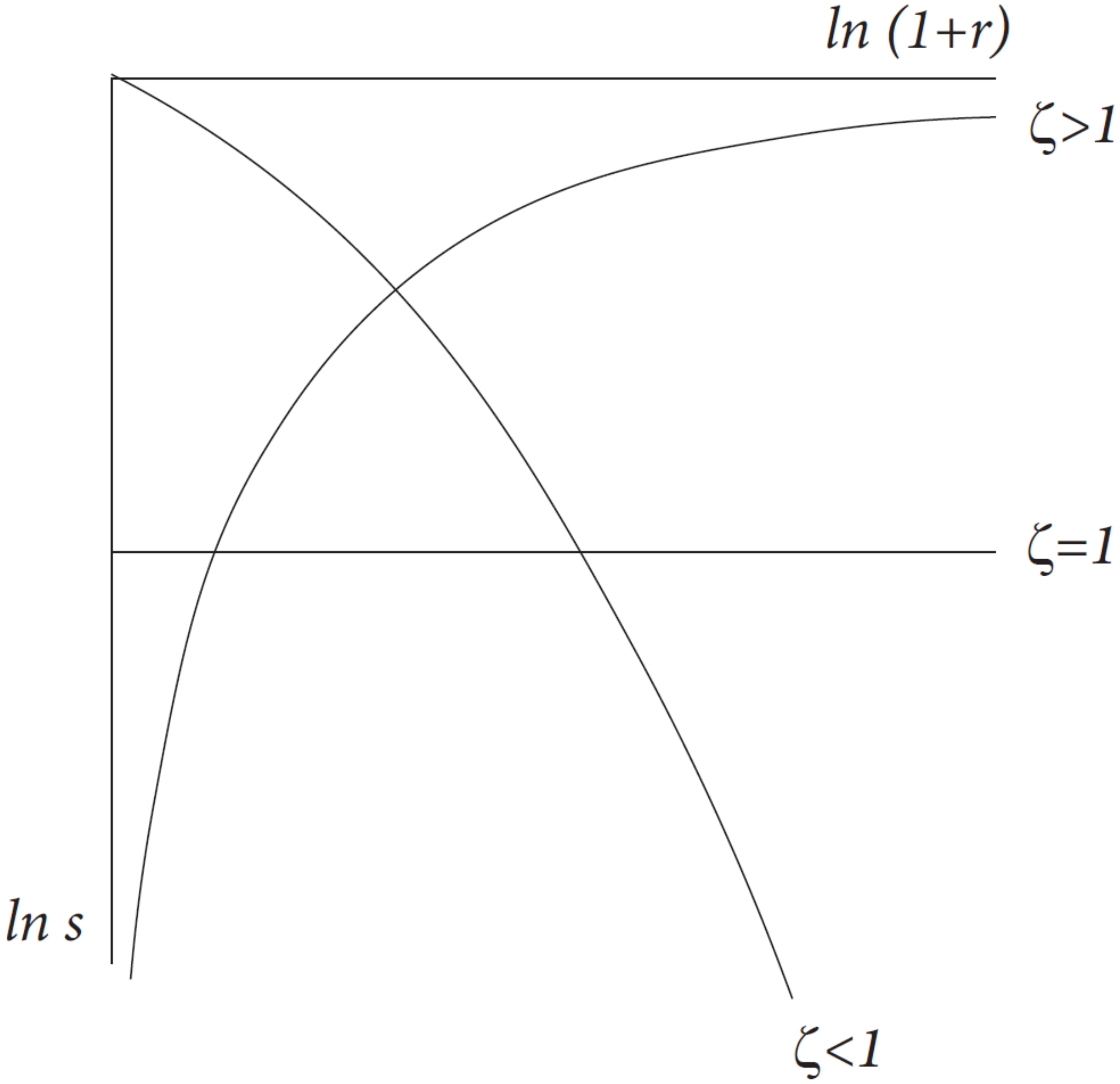

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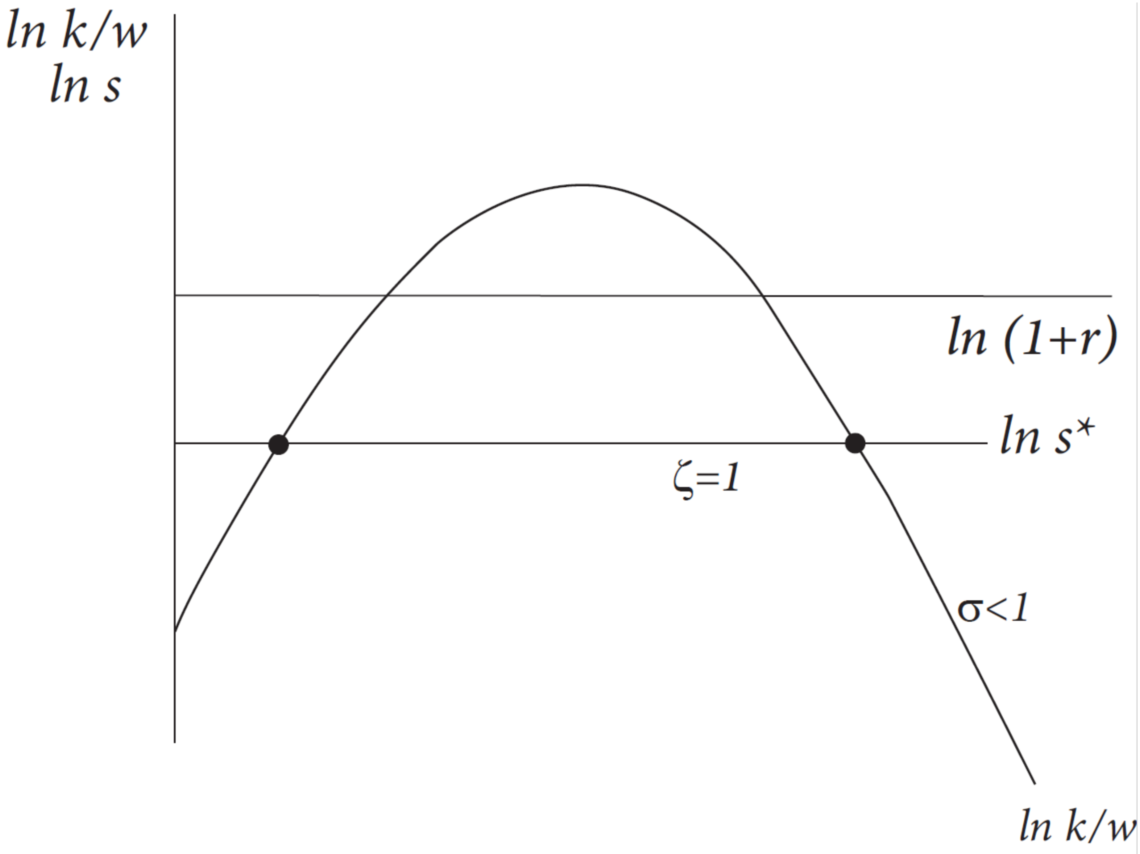


Figure 5a

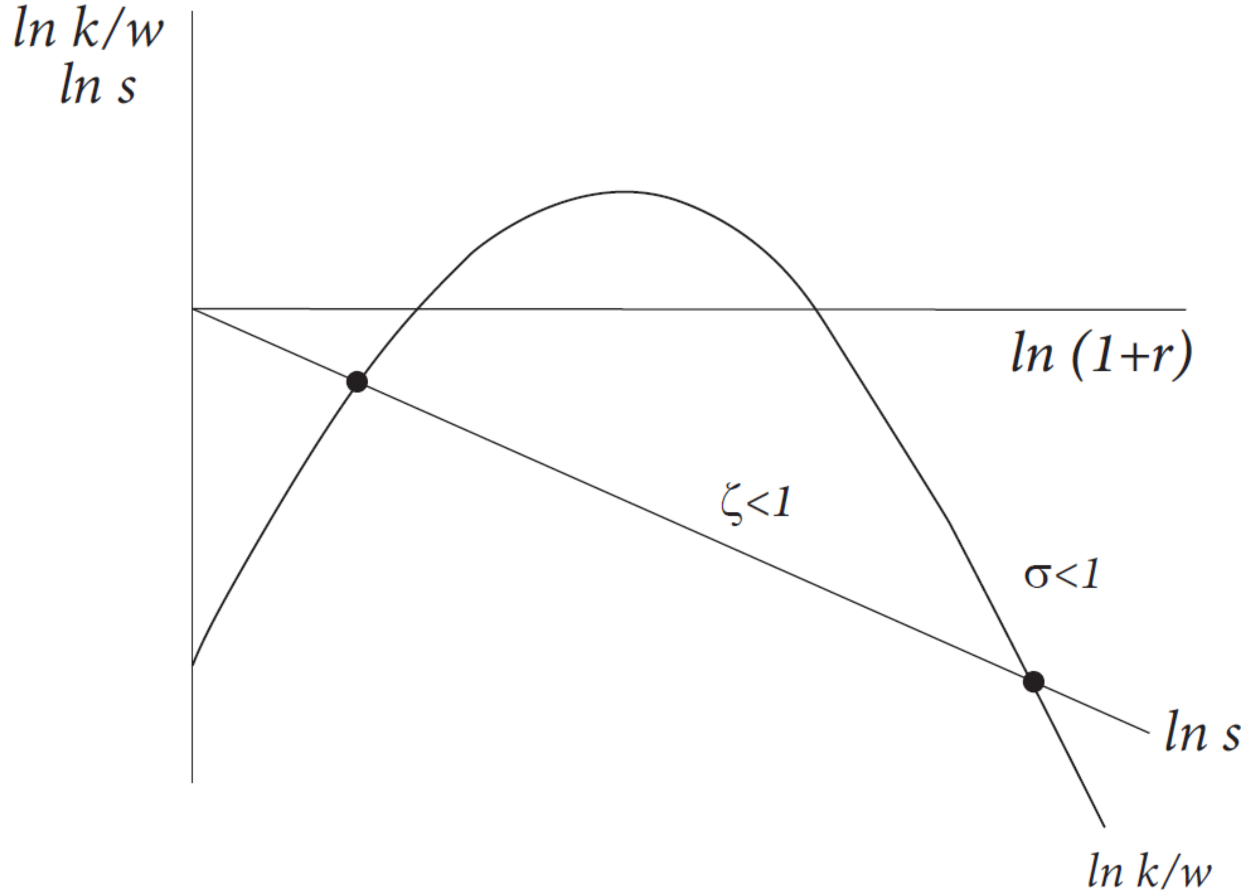


Figure 5b

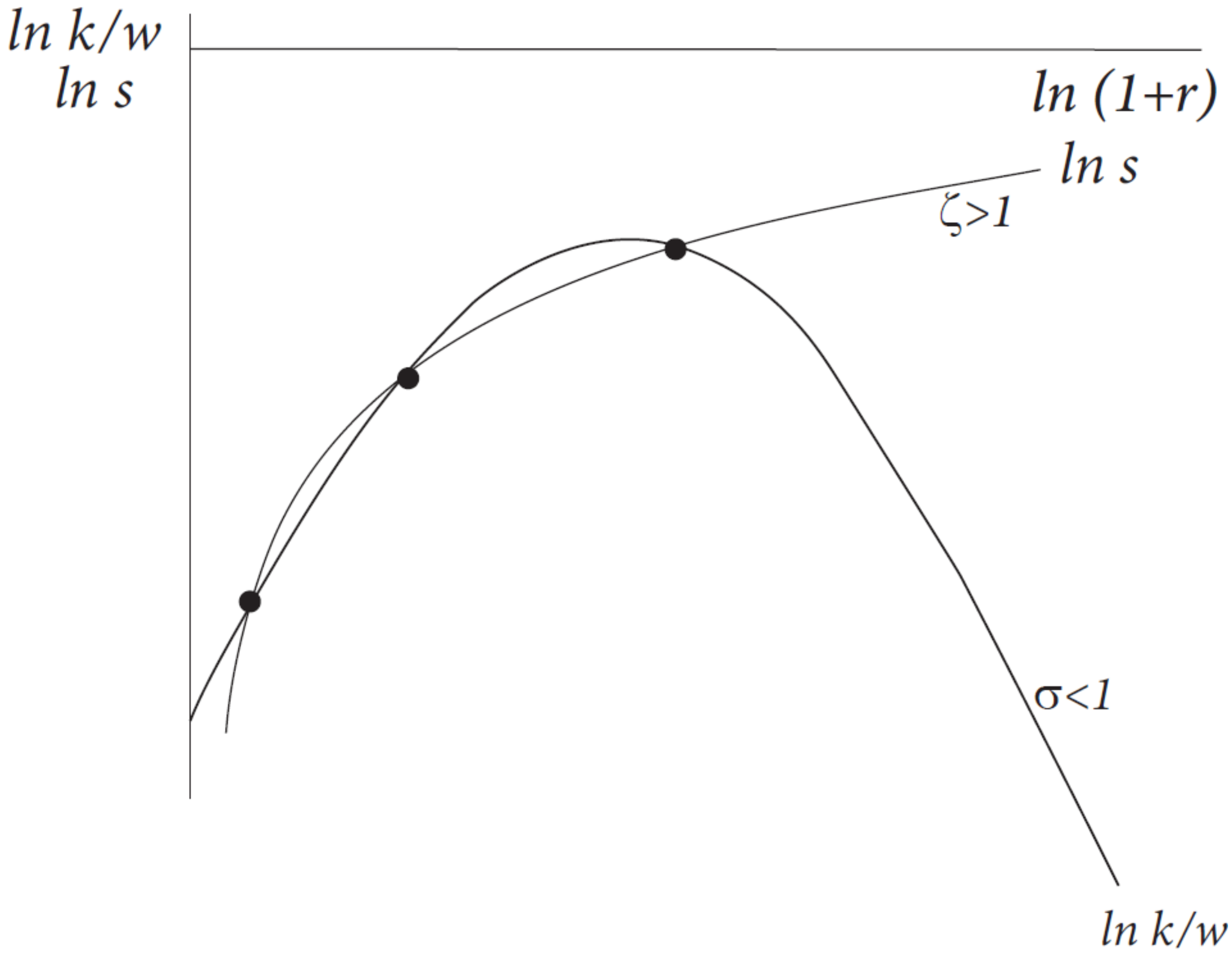


Figure 5c

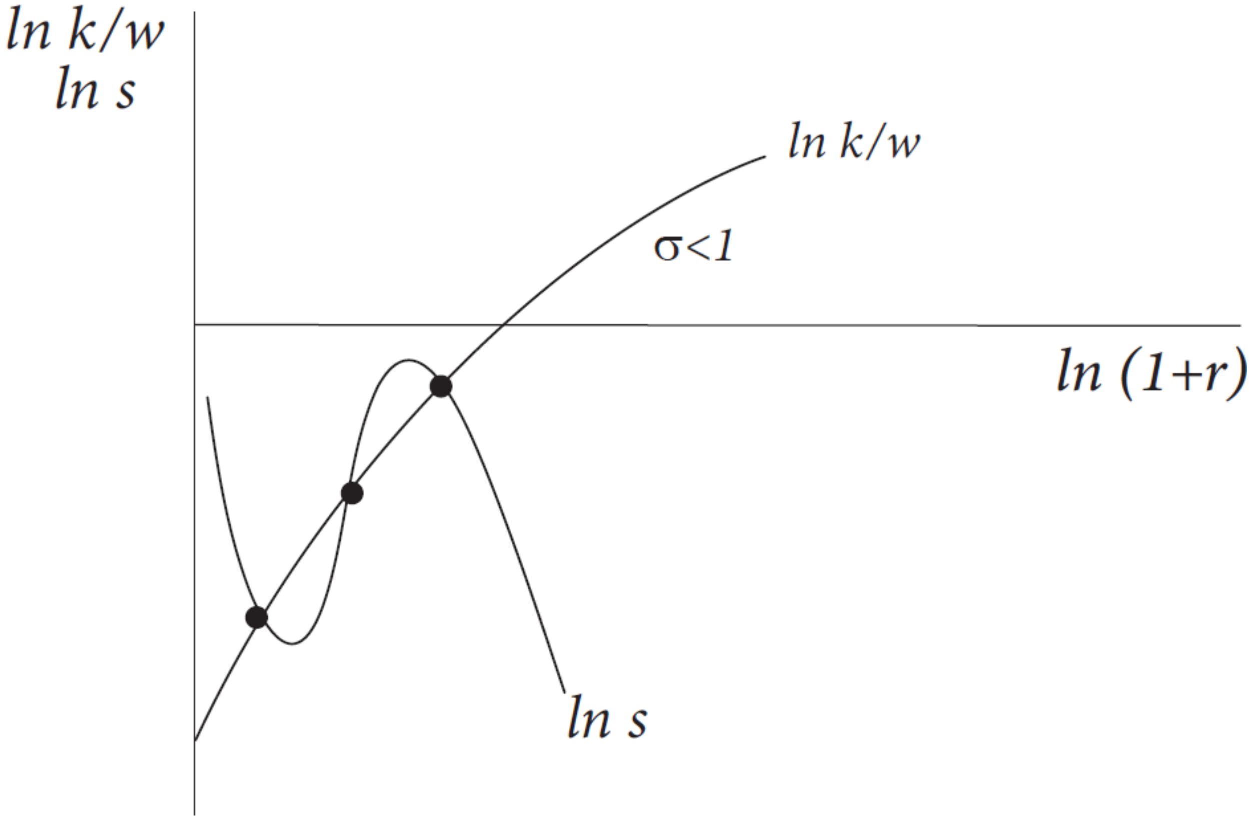


Figure 5d

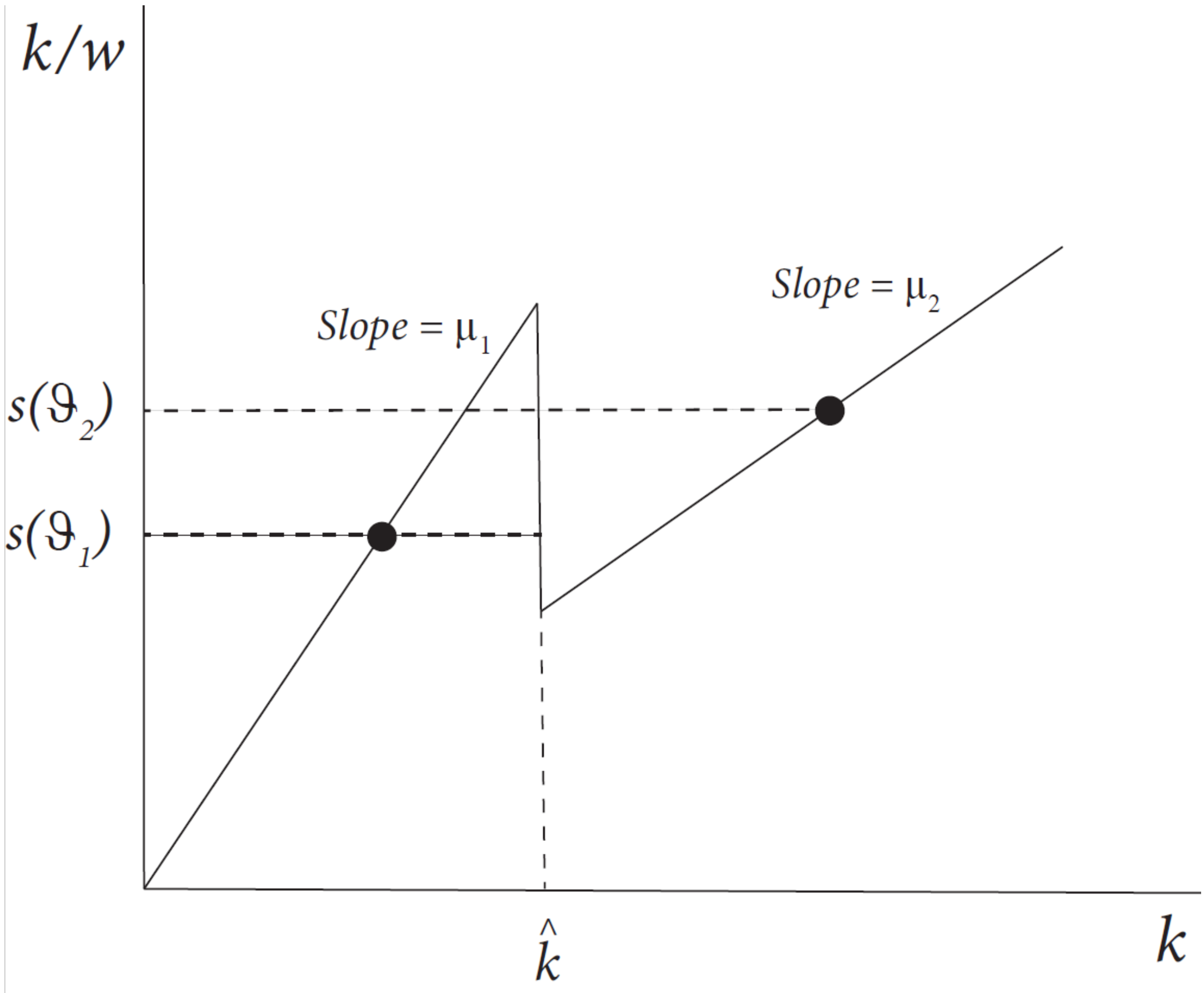


Figure 6a

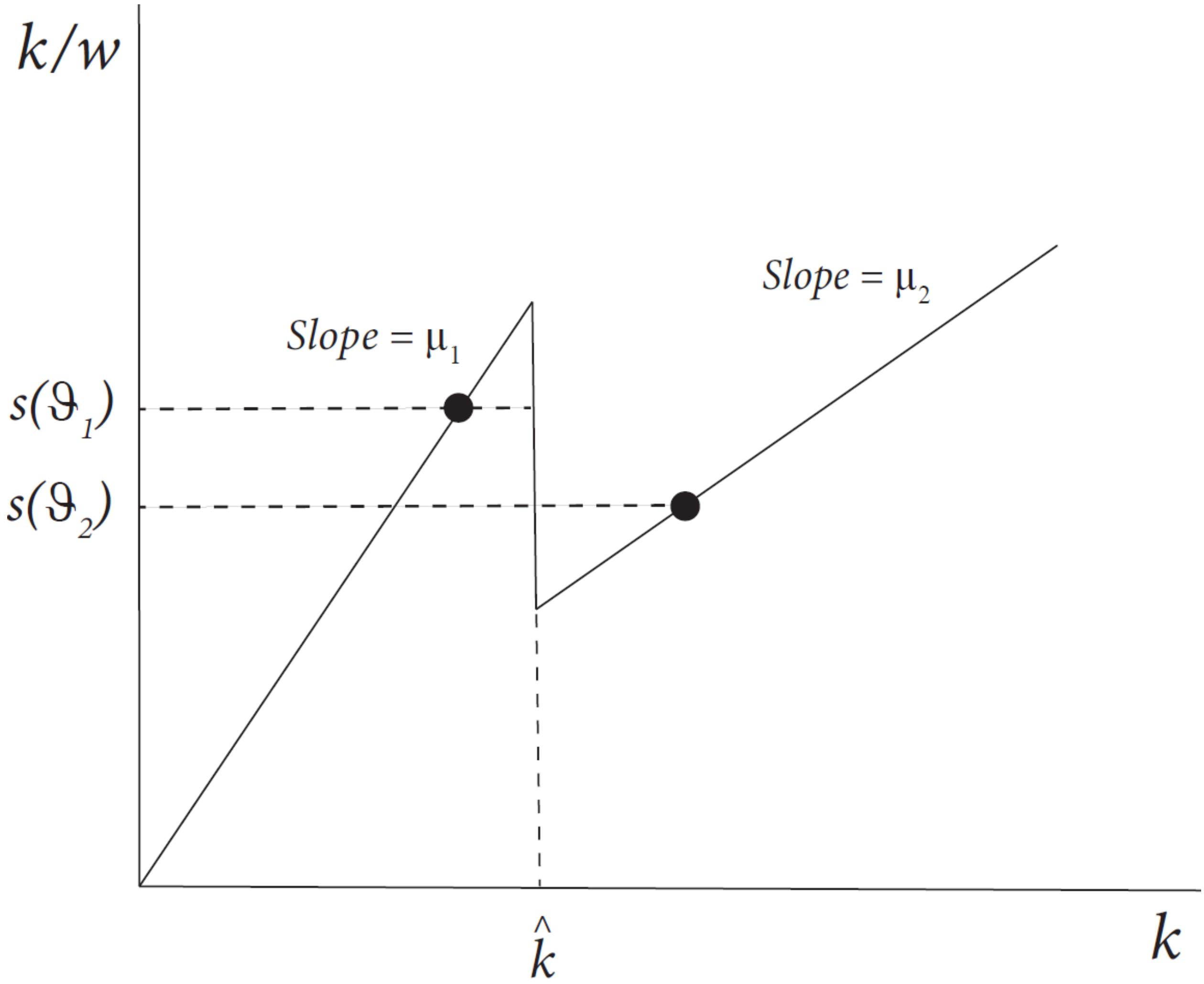


Figure 6b

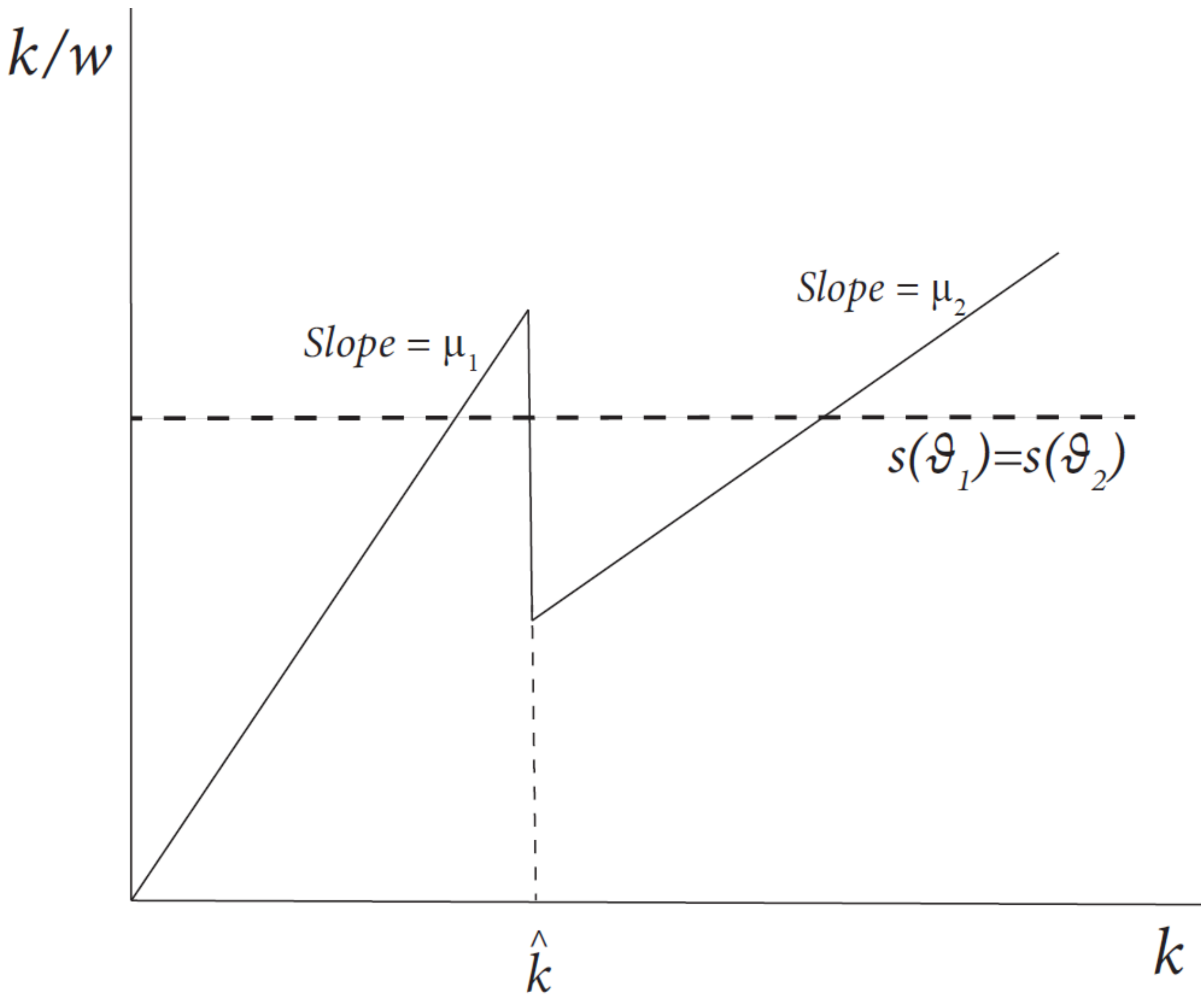


Figure 6c

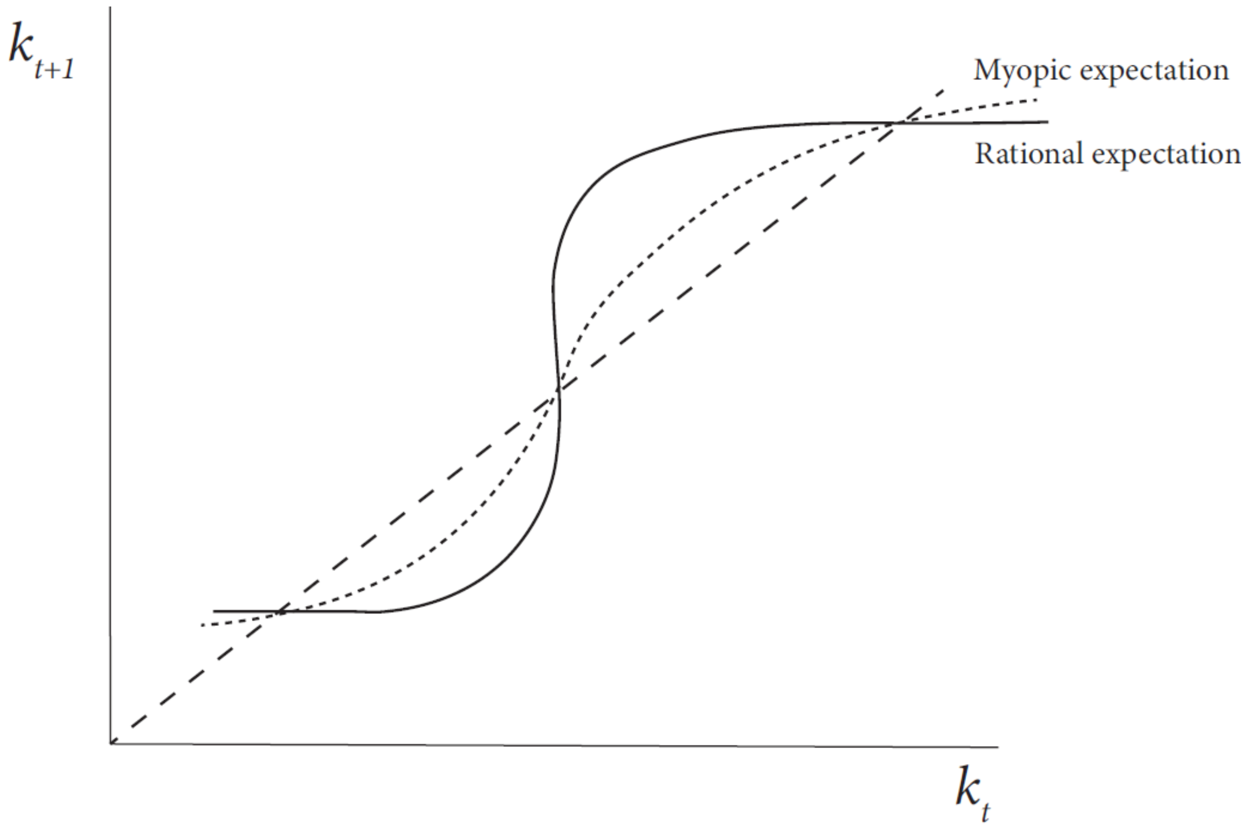


Figure 7

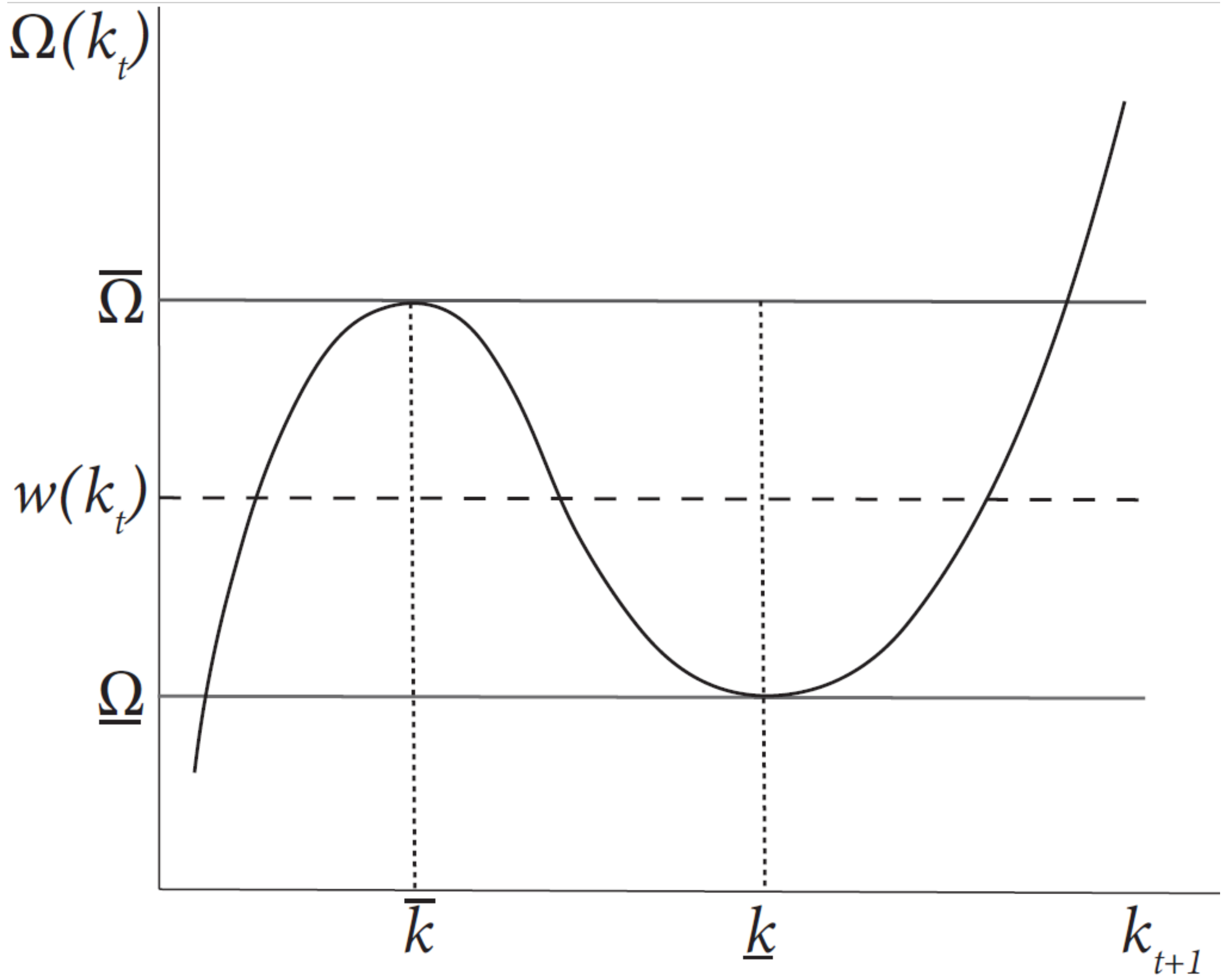
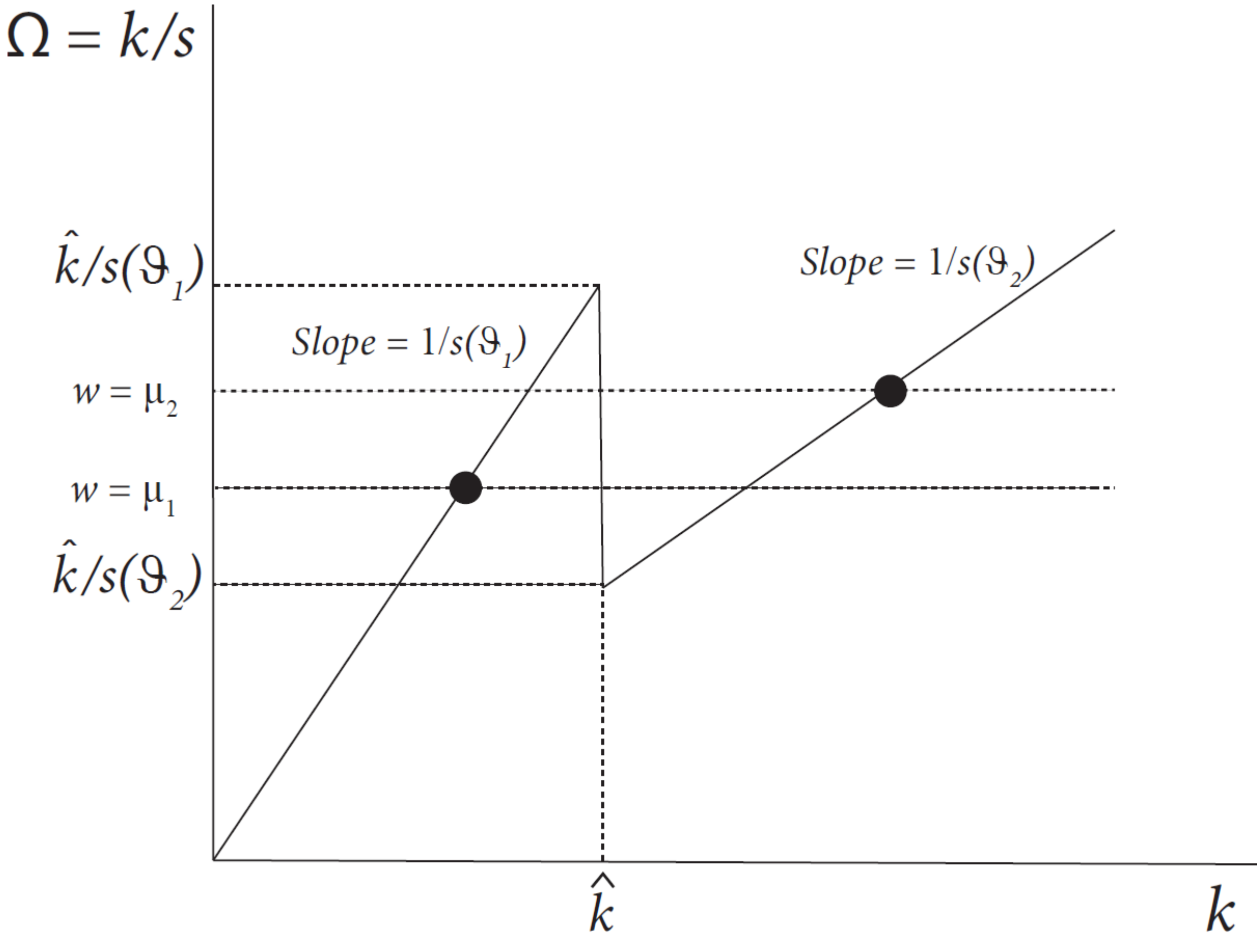

$\Omega(k_t)$
$\overline{\Omega}$
$w(k_t)$
$\underline{\Omega}$
$\overline{k}$
$\underline{k}$
k_{t+1}

Figure 8



$\Omega = k/s$
$\hat{k}/s(\vartheta_1)$
$Slope = 1/s(\vartheta_2)$
$Slope = 1/s(\vartheta_1)$
$w = \mu_2$
$w = \mu_1$
$\hat{k}/s(\vartheta_2)$
$\hat{k}$
k

Figure 9

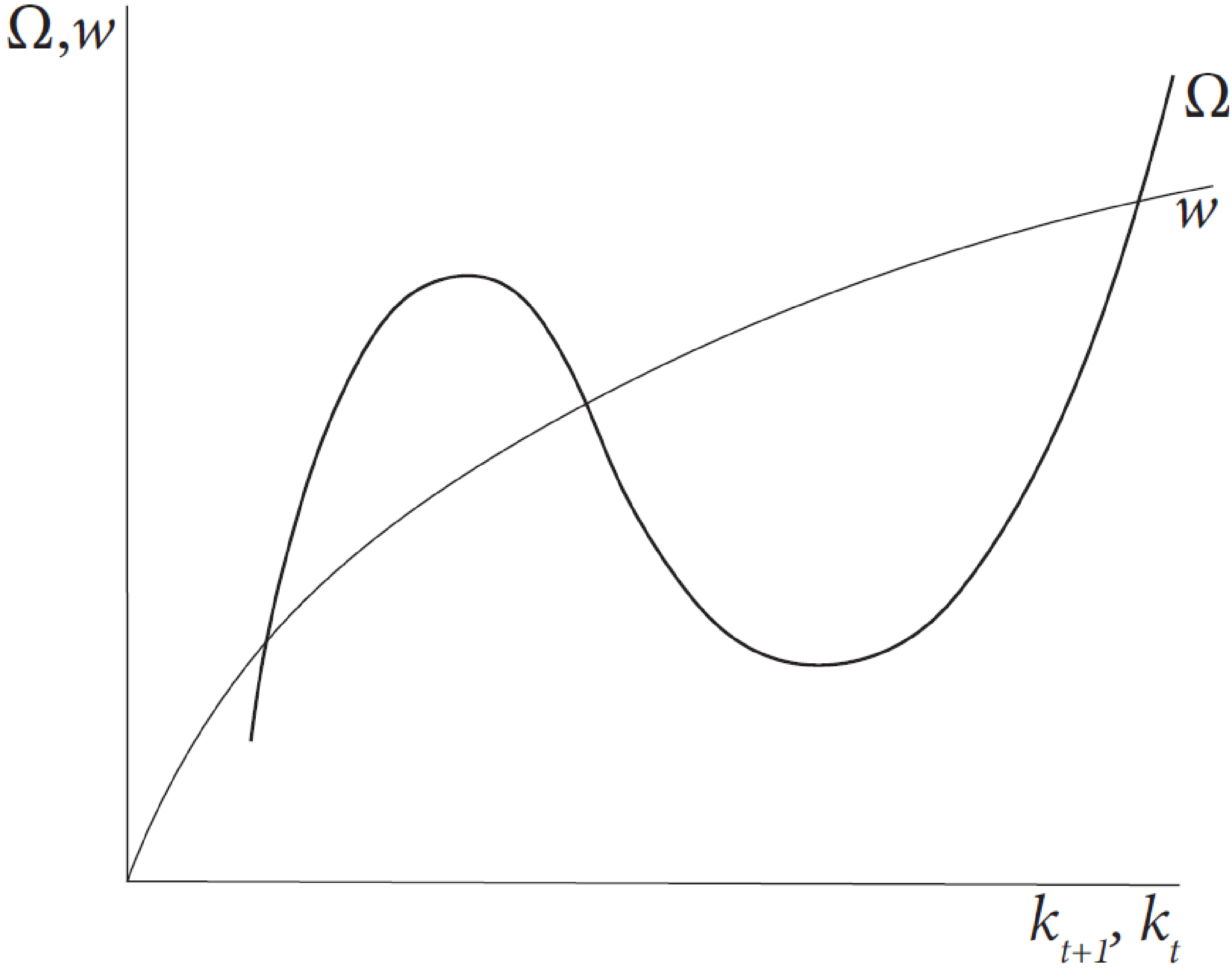


Figure 10

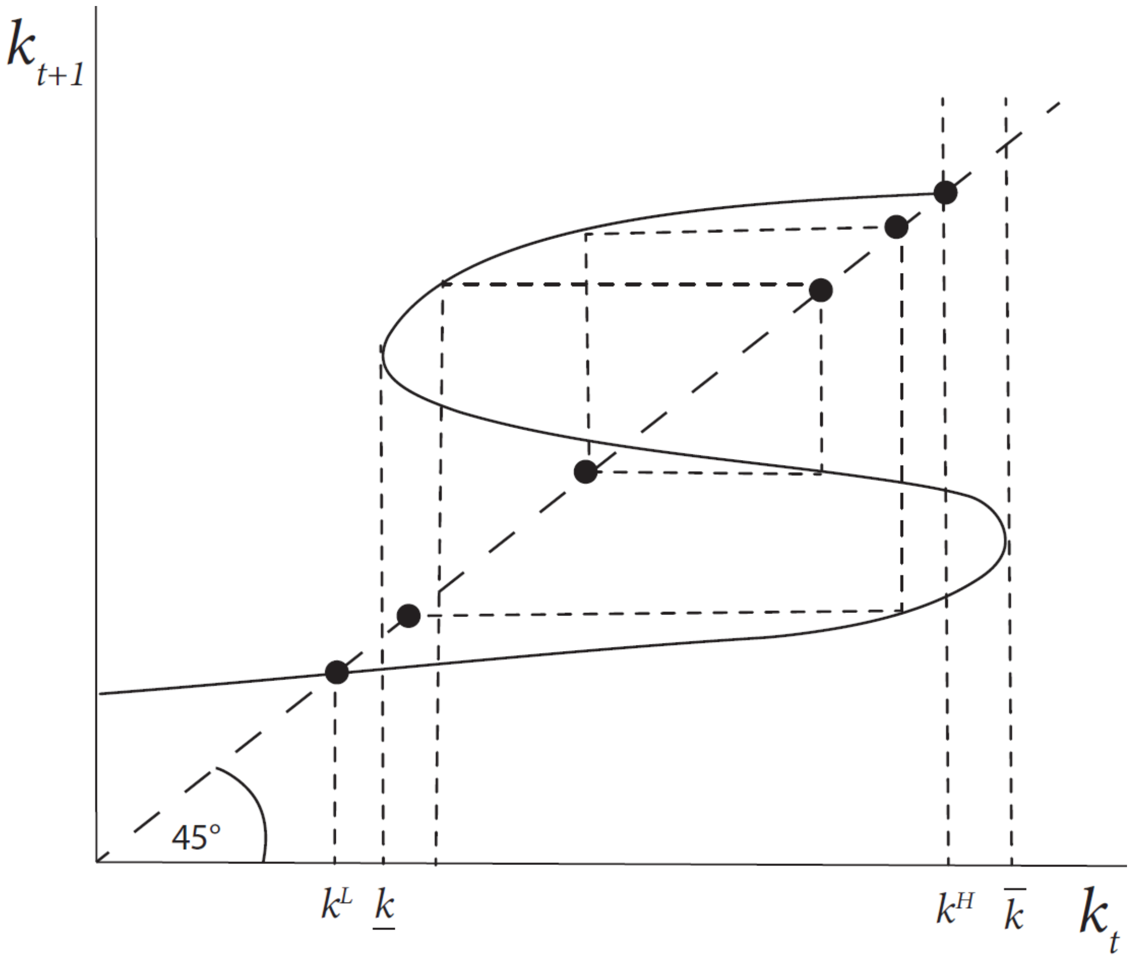
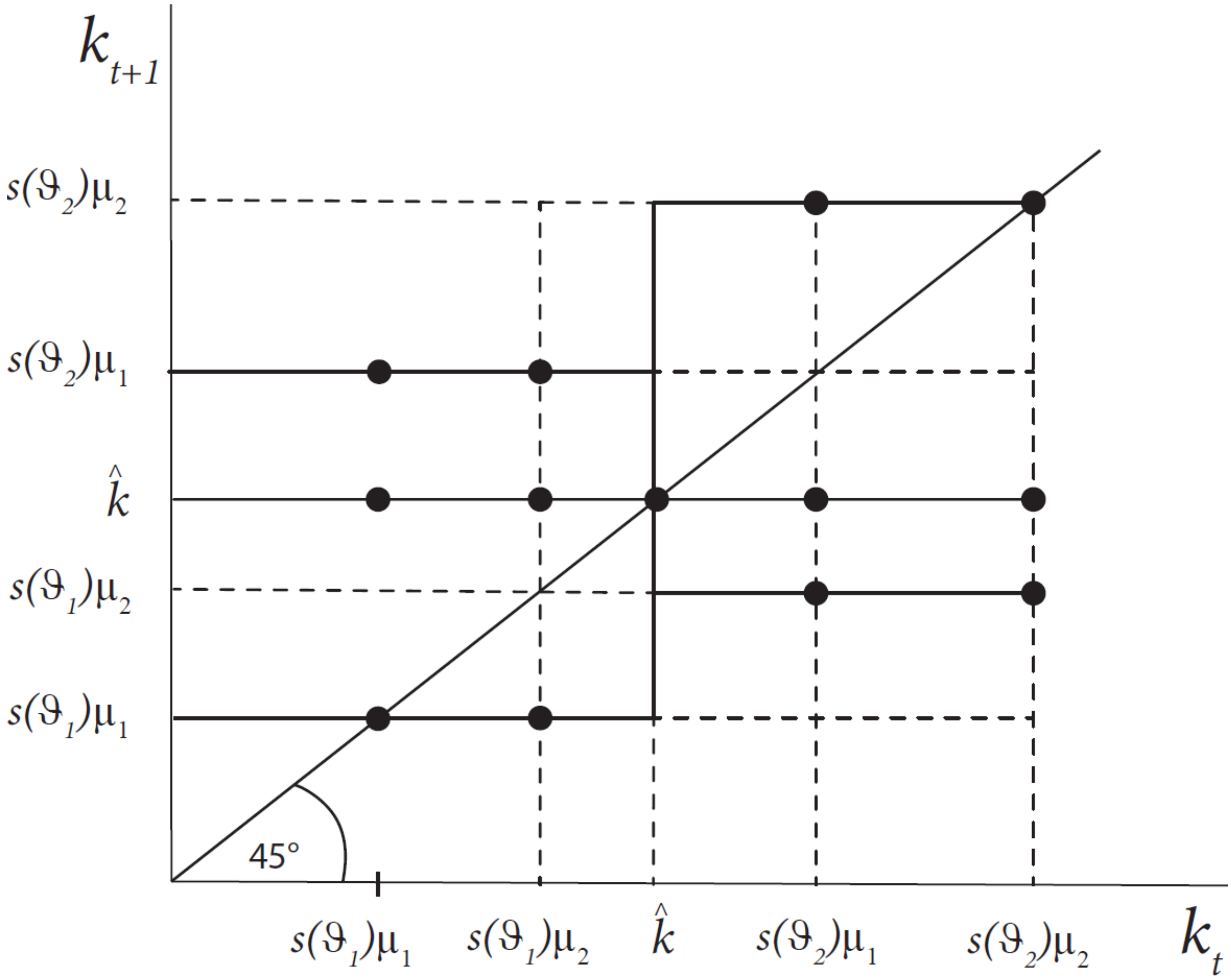
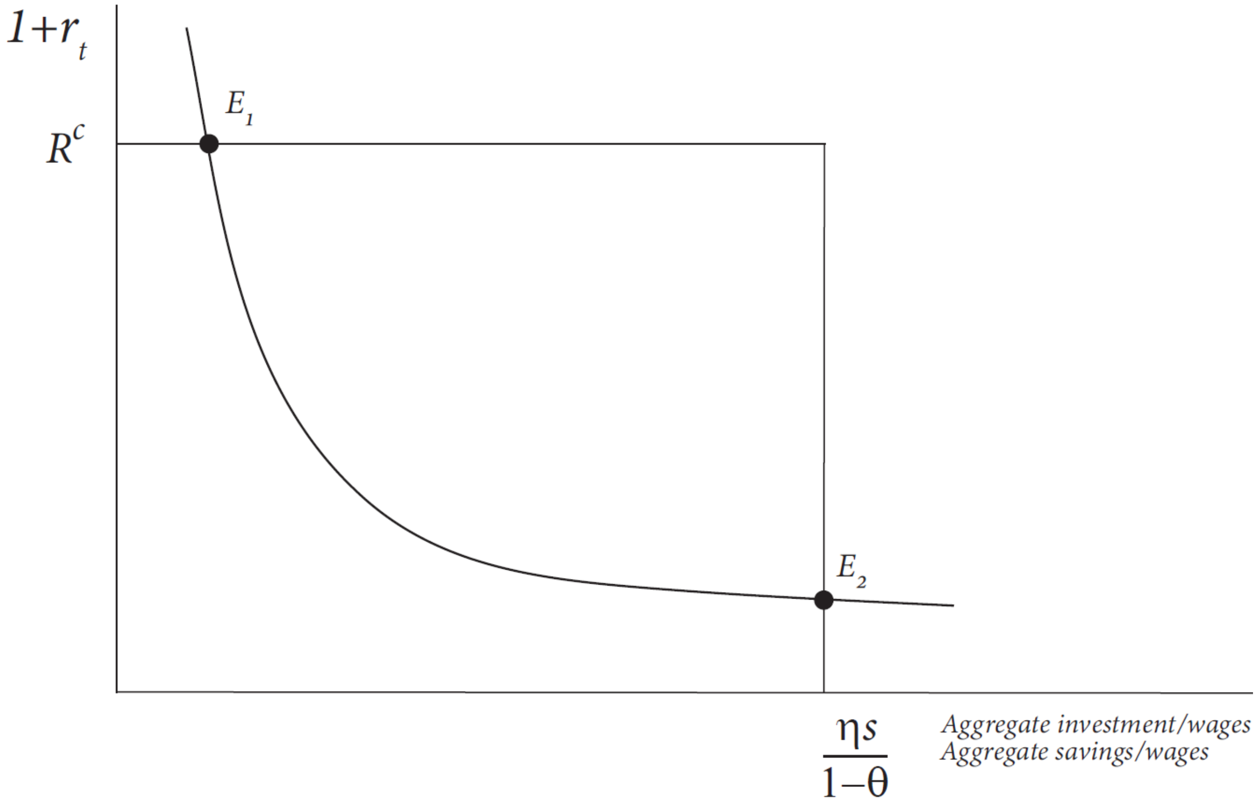


Figure 11



k_{t+1}
$s(\vartheta_2)\mu_2$
$s(\vartheta_2)\mu_1$
$\hat{k}$
$s(\vartheta_1)\mu_2$
$s(\vartheta_1)\mu_1$
45°
$s(\vartheta_1)\mu_1$
$s(\vartheta_1)\mu_2$
$\hat{k}$
$s(\vartheta_2)\mu_1$
$s(\vartheta_2)\mu_2$
k_t

Figure 12



1+r_t
R^c
E_1
E_2
ηs/(1−θ)
Aggregate investment/wages
Aggregate savings/wages

Figure 13