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EQUILIBRIUM MODELS OF ENDOGENOUS FLUCTUATIONS: AN INTRODUCTION

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ABSTRACT

These lectures comment upon recent theoretical models of endogenous fluctuations in economic dynamics, including both the literature on nonlinear deterministic cycles and the literature on "sunspot equilibria". Two important themes include (1) reasons to be interested in models of purely endogenous fluctuations, even though actual economies are admittedly subject to exogenous stochastic shocks; and (2) the importance of market imperfections in making possible equilibria characterized by endogenous fluctuations of either of the two types.

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Recently there has been a revival of interest in endogenous models of economic fluctuations -- in models according to which fluctuations could continue to occur even in the absence of exogenous fluctuations in any of the external determinants of the economic environment, such as consumer tastes or the state of technology. One of the most important features of this recent literature has been careful attention to the consequences of optimizing behavior on the part of economic agents, and of a state of competitive equilibrium between the various producers and consumers in the economy, for the possibility of endogenous fluctuations. It is this question of the possibility of endogenous *equilibrium* fluctuations with which I am concerned here. (For a more broadly ranging survey of economic models characterized by endogenous fluctuations, see the lecture in this volume by Richard Day.)

I will begin, in section I, by clarifying what is meant by endogenous fluctuations, and contrasting two rather different kinds of models of endogenous fluctuations -- models in which equilibrium is *determinate but unstable* on the one hand, and *"sunspot"* models on the other. In section II, I discuss some of the reasons why purely exogenous models of the source of economic fluctuations have been so popular in modern theorizing, and argue that the grounds for dismissal of endogenous models *a priori* are not so strong as the tenacity of this methodological prejudice might suggest. Finally, in section III, I consider the extent to which methodological commitments to explaining fluctuations in terms of optimizing behavior and competitive equilibrium can justify an exclusive emphasis upon exogenous-shock models. I argue that the most plausible models of endogenous fluctuations depend crucially upon the existence of "market failures" of one kind or another, but that this need

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not imply that the endogenous cycle hypothesis is inconsistent with a belief in the rationality of economic actors.

I. Endogenous versus Exogenous Explanations of Economic Fluctuations: Preliminary Distinctions and Examples

By *exogenous shock models* of economic fluctuations I mean models in which equilibrium is *determinate* and *intrinsically stable,* so that in the absence of continuing exogenous shocks, the economy would tend toward a steady state, but because of exogenous shocks a stationary pattern of fluctuations is observed. Such models of economic fluctuations are quite familiar -- so familiar, indeed, that many readers may wonder how an economic explanation of fluctuations could be of any other kind. Typologies of business cycle theories, for example, that classify theories according to the dominant type of "impulse" assumed on the one hand, and the nature of the "propagation mechanism" posited on the other, take for granted this general structure of explanation. All of the leading current equilibrium business cycle theories are of this kind, but the textbook "Keynesian" and "monetarist" models are of this kind as well, as are the econometric models inspired by them, if one accepts the conventional identification of residuals in the various equations with exogenous shocks to fundamentals (of perhaps an unspecified nature). Models of this kind include both theories that attribute aggregate instability mainly to variations in government policy and theories that attribute it mainly to variations in private sector behavior (interpreted as originating in variations in tastes or technological possibilities).

But this structure of explanation is not the only logical possibility, and I argue below that it is not the only possibility consistent with economic theory. I will also argue that if aggregate fluctuations could be shown to be endogenous to some important degree, this would be of considerable importance. Before addressing the reasons for interest in the endogenous cycle hypothesis, however, it is useful to clarify exactly what is meant by it.

A simple definition might characterize endogenous models of economic fluctuations as ones in which persistent fluctuations occur despite an absence of variation in exogenous economic "fundamentals" such as tastes and technology and government policies. This is indeed a property of the theoretical models that are usually thought of as examples of endogenous cycle theories. But it is not an adequate definition from the point of view of explaining what is thought to be interesting about such theoretical examples. For the point of contention between proponents of exogenous and endogenous theories cannot be whether or not exogenous shocks in fact occur. Anyone must recognize that the aggregate economy is significantly affected by events that must be viewed as largely exogenous to the economic process itself -- political events, for example, but also the random timing of technological innovations. Hence we must be able to define endogenous fluctuations in such a way that the hypothesis does not deny this obvious fact.

A more careful definition requires a slightly more formal description. Let an economic model consist of a description of the evolution over time of variables of four sorts: exogenous "fundamental" variables to be denoted x_t, predetermined endogenous variables k_t, non-predetermined endogenous variables p_t, and "sunspot" variables s_t. All of these may be supposed to be vector quantities, taking values in sets X, K, P, and S respectively. (A given model might not involve state variables of all these types, but that can be dealt with by assuming, for example, that the set K consists of a single point.) The "fundamental" variables are exogenous variables (i.e., the evolution of x_t is described by a stationary Markov process on X, independent of the histories of the other state variables, and regarded as external to the economic process that is being modeled), whose values affect the economic relations determining the endogenous variables. The "sunspot" variables are also exogenous, but their values do not have any effect upon the economic relations determining the endogenous variables. As is discussed further below, the possibility that endogenous variables might nonetheless take different values depending upon the state s_t may be taken to represent the role of arbitrary revisions of expectations as an autonomous source of instability in the economy. The predetermined endogenous variables k_t are determined completely by period t-1, so that their values must be unaffected by the realization of either x_t (period t changes in fundamentals) or s_t (period t revisions of expectations). The values of the non-predetermined variables p_t may depend upon the realization of either x_t or s_t.

A description of the economy's evolution consists of a specification of the histories of the endogenous variables, given any possible history of realizations of the exogenous variables, i.e., a sequence of functions

$$p_t = \pi_t(x_t, s_t, x_{t-1}, s_{t-1}, \ldots, x_0, s_0, k_0)$$
$$k_{t+1} = \kappa_{t+1}(x_t, s_t, x_{t-1}, s_{t-1}, \ldots, x_0, s_0, k_0)$$

for t = 0, 1, 2, ... I will suppose that the *equilibrium conditions* take the form

$$k_{t+1} = g(x_t, k_t, p_t, \mu_t(p_{t+1})) \tag{1}$$

$$f(x_t, k_t, p_t, \mu_t(p_{t+1})) = 0 \tag{2}$$

where f is a vector-valued function of the same dimension as p_t, g is a vector-valued function of the same dimension as k_t, and $\mu_t(p_{t+1})$ denotes the probability distribution of possible values for p_{t+1},

given information available at time t. Equation (1) indicates the determination of the predetermined variables for period t+1 as a function of the economy's state in period t, including period t expectations regarding the future. An example of such a relation might be the determination of the capital stock in period t+1 by an investment decision in period t, that depends upon period t prices, interest rates, and so on, as well as expectations regarding the future values of such variables. A probability distribution for k_{t+1} is not included among the arguments of g, because the value of k_{t+1} can be known with certainty at time t, and because in writing the condition in this form I have solved for k_{t+1}. A probability distribution for x_{t+1} is not included among the arguments of g, because given the Markov process on X, this probability distribution is completely determined by the value of x_t. Equation (2) represents the determination of the non-predetermined variables p_t as a function of the predetermined period t state variables, the current state of exogenous fundamentals, and expectations. The variables p_t are only defined implicitly by (2), because the sort of equilibrium condition one has in mind (e.g., an equation stating that supply equals demand, to determine period t prices) may in general have multiple solutions for given values of the other variables; the values of predetermined variables are by contrast necessarily uniquely defined, if the set of state variables is made large enough (e.g., in the case of determination of the period t+1 capital stock, if the list of period t endogenous variables includes the level of investment in period t). A probability distribution for k_{t+1} is not included among the arguments of f, because this can be written as a function of the other arguments using (1), and again a probability distribution for x_{t+1} is not included because it can be written as a function of x_t. Neither current nor future values of the "sunspot" variables enter as arguments of either f or g -- this is what makes them "sunspot" variables rather than "fundamentals".

We will be concerned with the set of sequences of functions $\{\pi_t, \kappa_{t+1}\}$, for t = 0, 1, 2, ..., that satisfy (1) and (2) for all possibile histories of realizations of the exogenous variables $\{x_t, s_t\}$ for t = 0, 1, 2, ..., given an *initial condition* k_0 for the predetermined endogenous variables. Let this be referred to as the *equilibrium set* $E(k_0)$. In the case of greatest interest, the equilibrium set is non-empty for all k_0 in the set K; this would generally be considered a minimal requirement for an internally consistent model. But equilibrium need not be unique.

Now the point of view that underlies the conventional methodological preference for exogenous shock models of fluctuations can be stated, I believe, in terms of two general propositions regarding the structure of the equilibrium set. The two propositions are related, and I will refer to both of them as determinacy theses, but it is important to realize that they are logically distinct, and indeed the literature concerned with alternatives to exogenous shock models can on the whole be divided into two parts, depending upon which determinacy thesis is being challenged.

> <u>The Global Weak Determinacy Thesis.</u> The equilibrium values of (k_{t+1}, p_t) at any point in time t depend upon the history of realizations of the exogenous states up to that time, and upon the initial condition, only insofar as these affect the equilibrium conditions (1) and (2) for periods t or later. As a consequence, any equilibrium can be described by a pair of functions
>
> (3) $\quad p_t = \pi(x_t, k_t)$
>
> (4) $\quad k_{t+1} = \kappa(x_t, k_t)$
>
> which furthermore have the property that if (x^1, k^1) and (x^2, k^2) are such that $f(x^1, k^1, p, \mu) = f(x^2, k^2, p, \mu)$ and $g(x^1, k^1, p, \mu) = g(x^2, k^2, p, \mu)$ for all p in P and for μ any probability measure on P, and such that the distribution of possible values for x_{t+1} conditional upon x_t is the same when $x_t = x^1$ as when $x_t = x^2$, then $\pi(x^1, k^1) = \pi(x^2, k^2)$ and $\kappa(x^1, k^1) = \kappa(x^2, k^2)$. That is, there is no dependence upon "irrelevant" components of either exogenous state variables or of predetermined endogenous state variables.

According to this proposition, the realizations of "sunspot" states can have no effect upon the equilibrium evolution of the endogenous variables, because the "sunspot" states have no effect upon the equilibrium conditions (1) or (2); and the history of realizations $(x_0, x_1, ..., x_t)$ and the initial condition k_0 affect the determination of (k_{t+1}, p_t) only through the sufficient statistics (x_t, k_t), which quantities may affect (k_{t+1}, p_t) because they may enter the equilibrium conditions (1) and (2) for period t, and because x_t may give information about the future values of fundamentals as well. Note also that the proposition implies that π and κ are time-invariant functions; for dependence upon time would be like dependence upon a sunspot state, given the time-invariance of conditions (1) and (2).

The reason for the appeal of this determinacy thesis is simple: these properties of the solution follow immediately from our assumptions regarding the form of the equilibrium conditions if one supposes *that equilibrium is unique* for each k_0. The reasoning is simple. If equilibrium is unique, there exist unique equilibrium functions $\pi_0(x_0, s_0, k_0)$ and $\kappa_1(x_0, s_0, k_0)$. Furthermore, since the restrictions upon the sequences $\{k_{t+1}, p_t\}$ implied by equilibrium conditions (1)-(2) are the same regardless of the value of

s_0, the unique equilibrium values for p_0 and k_1 must be independent of s_0, allowing us to define functions

$$\pi(x, k) = \pi_0(x, s, k)$$

$$\kappa(x, k) = \kappa_1(x, s, k)$$

And the recursive form of the equilibrium conditions (1)-(2) implies that if the sequences $\{k_{t+1}, p_t\}$ for $t \geq 0$ are an equilibrium for initial condition k_0 and exogenous shocks $\{x_t\}$, then the sequences $\{k'_{t+1}, p'_t\}$ must be an equilibrium for initial condition k'_0 and exogenous shocks $\{x'_t\}$, where

$$k'_{t+1} = k_{t+2}$$

$$p'_t = p_{t+1}$$

$$x'_t = x_{t+1}$$

$$k'_0 = k_1$$

It then follows from the uniqueness of the equilibrium with initial condition k'_0 that one must have

$$\pi_1(x_1, s_1, x_0, s_0, k_0) = \pi(x_1, \kappa(x_0, k_0))$$

$$\kappa_2(x_1, s_1, x_0, s_0, k_0) = \kappa(x_1, \kappa(x_0, k_0))$$

The arguments proceeds in the same way for subsequent periods, so that the equilibrium must be of the form (3)-(4).

Uniqueness of equilibrium is often assumed, even though as noted above this need not follow from the definition given of equilibrum.[1] But the determinacy thesis is often assumed even when equilibrium is not assumed to be unique, and indeed even in the case of models where non-uniqueness can be shown to be possible. This is revealed by the fact that many authors choose a formalism with which to describe equilibria in their model that

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[1] To be more precise, it is often argued that if equilibrium is not unique, the definition of equilibrium must be supplemented by an additional "selection criterion" that selects one equilibrium for each model specification and initial condition. The argument just given then explains why the principle of global weak determinacy often plays a central role in proposed selection criteria, such as McCallum's (1983) concept of the "minimum state variable solution" for linear rational expectations models, or the "Markov equilibrium" refinement for stationary dynamic games (Maskin and Tirole (1989)).

assumes the existence of a representation of the form (3)-(4); see, e.g., Lucas and Stokey (1987).

Another tacit assumption of much analysis is the following.

The Asymptotic Strong Determinacy Thesis. Eventually, the equilibrium values of (k_{t+1}, p_t) depend only upon the realizations of the exogenous fundamentals $\{x_s\}$ for $s \leq t$; that is, the initial condition k_0 will have no effect upon the equilibrium values far enough in the future. Formally, for any initial condition k_0 and for any sequence of exogenous shocks $\{x_0, x_{-1}, x_{-2}, \ldots\}$, the sequences $\{k'_s, p'_s\}$ defined recursively by

$$
\begin{aligned}
k'_s(0) &= k_0 \\
k'_s(n) &= \kappa(x_{n-s}, k'_s(n-1)), & 1 \leq n \leq s \\
k'_s &= k'_s(s), & s \geq 1 \\
p'_s &= \pi(x_0, k'_s(s-1)), & s \geq 1
\end{aligned}
$$

converge to values (k^*, p^*) as s goes to infinity, which limiting values may depend upon the sequence $\{x_0, x_{-1}, x_{-2}, \ldots\}$ but are independent of k_0.[2]

The idea behind this proposition is that recursive substitution of expression (4) into itself ought to yield a sequence of representations

$$
\begin{aligned}
k_{t+1} &= \kappa(x_t, k_t) \\
&= \kappa(x_t, \kappa(x_{t-1}, k_{t-1})) \\
&= \kappa(x_t, \kappa(x_{t-1}, \kappa(x_{t-2}, k_{t-2}))) \\
&= \ldots
\end{aligned}
$$

that converges eventually to a representation

$$k_{t+1} = \kappa^*(x_t, x_{t-1}, x_{t-2}, \ldots)$$

in which k_{t+1} is written as a function of a possibly infinite history of previous exogenous shocks, with lagged endogenous vartiables completely eliminated. The existence of such a representation also allows p_t to be written as a function of a possibly infinite history of previous exogenous shocks alone, i.e.,

[2] In this statement I am assuming the validity of the representation (3)-(4), since I am mainly interested in the consequences of asymptotic strong determinacy when conjoined with global weak determinacy, but strictly speaking it is logically possible to assert asymptotic strong determinacy while denying the global weak determinacy thesis, in which case one would allow (k^*, p^*) to depend upon the history of realizations of all exogenous state variables, including sunspot variables.

$$
\begin{aligned}
p_t &= \pi^*(x_t, x_{t-1}, x_{t-2}, \ldots) \\
&= \pi(x_t, \kappa^*(x_{t-1}, x_{t-2}, \ldots))
\end{aligned}
$$

These functions κ^* and π^* then describe the limits (k^*, p^*) referred to in the above statement.

One reason for the intuitive appeal of such a postulate is probably familiarity with the properties of linear systems (which are often assumed to provide an adequate approximation for purposes of quantitative investigations of equilibrium models of economic fluctuations). For suppose that the function κ is linear, i.e., that one can write $\kappa(x, k) = Ax + Bk$, for some matrices A and B. Suppose furthermore that the exogenous shocks to fundamentals satisfy some uniform bound $|x_t| \leq \bar{x} < \infty$, and that the equilibrium dynamics are such that a bounded sequence of exogenous shocks to fundamentals implies bounded fluctuations in the endogenous variables, i.e., that the matrix B has all eigenvalues with modulus less than one. Then recursive substitution of the kind described is possible and yields

$$
\kappa^*(x_t, x_{t-1}, x_{t-2}, \ldots) = \Sigma_{j=0}^{\infty} B^j A x_{t-j}
$$

Now the theses of global weak determinacy and asymptotic strong determinacy together clearly imply that persistent fluctuations in the endogenous variables must be explained by fluctuations in the exogenous fundamentals. For suppose that there were no variation in fundamentals. Then any equilibium would have to be described by a pair of functions

$$
p_t = \pi(k_t) \tag{5}
$$

$$
k_{t+1} = \kappa(k_t) \tag{6}
$$

and the function κ would have to have the property that the limit

$$
\lim_{n \to \infty} \kappa^n(k_0) = k^*
$$

exists and is independent of k_0. But this would imply that k_t must asymptotically converge to the constant value k^*, and that p_t must asymptotically converge to the constant value $p^* = \pi(k^*)$. Hence fluctuations in the endogenous state variables must eventually die out. It follows that a continuing pattern of fluctuations must be explained in terms of continuing shocks to fundamentals.

On the other hand, it should be clear at this point that one might wish to deny this conclusion -- to assert the possibility of endogenous fluctuations -- without one's being committed to a belief that there are in fact no exogenous variations in fundamentals. For one might wish to challenge one or both of the determinacy theses, both of which remain strong restrictions (and the denial of either of which is logically possible) even in the case of stochastic variation in fundamentals. And an obvious way of demonstrating that one or the other of these propositions need not be true of well-posed economic models is to exhibit examples of well-posed models with the property that even if fundamentals are assumed to be constant, equilibria characterized by persistent fluctuations exist. This is what I take to be the point of the theoretical literature on endogenous cycle models. The challenge posed to orthodox business cycle theory is not the suggestion that perhaps fluctuations in fundamentals are unimportant, but rather the suggestion that one or both of the determinacy theses might be too restrictive. And one might refer broadly to all models of fluctuations inconsistent with either of these theses as "endogenous" explanations.

The isolation of two distinct determinacy theses indicates that models of fluctuations might be "endogenous" in either of two distinct senses; a model might contradict the thesis of asymptotic strong determinacy while possibly remaining consistent with the thesis of global weak determinacy, or it might contradict the latter while possibly remaining consistent with the former. And indeed there exist to some extent two distinct classes of endogenous models of fluctuations, insofar as most examples in the literature are constructed in order to challenge of the determinacy theses or the other.

On the one hand there are models of *determinate but unstable dynamics.* In such models, equilibria have a representation of the form (3)-(4), but differing initial conditions k_0 will imply different dynamic paths that fail to converge even far in the future. For example, in the absence of shocks to fundamentals, a model may imply deterministic equilibrium dynamics of the form (6), but it may be the case that all of the steady state equilibria, i.e., all of the vectors k^* that are fixed points of the map κ in (6), are *unstable,* because the derivative matrix $D\kappa(k^*)$ in each case has one or more eigenvalues of modulus greater than one. In such a case, the dynamics will not converge asymptotically to a steady state, for almost all initial conditions. Instead, the dynamics may converge to a deterministic periodic orbit, or even to a chaotic attractor, in which case bounded but aperiodic fluctuations continue forever. Such models are consistent with the thesis of global weak determinacy, but contradict asymptotic strong determinacy.

Examples of this kind include the optimal growth models considered by Benhabib and Nishimura (1979, 1985) and Boldrin and Montrucchio (1986), among others. The simplest example in which complex dynamics are possible is a two-sector growth model (capital and exogenously supplied labor are used to produce both the capital good and a consumption good) with a population of identical infinite lived consumers with stationary recursive preferences. In such a model, because of the first welfare theorem, an intertemporal competitive equilibrium must maximize the utility of the representative consumer subject to the constraints imposed by the technology and the initial capital stock. As a consequence, there must be a unique equilibrium for each value of the initial capital stock (k_0, if we let the scalar k_t denote the quantity of capital brought into period t), and this equilibrium will be independent of the realizations of any extraneous "sunspot" variables, since the equivalent optimal planning problem does not involve them. Hence the global weak determinacy thesis is valid for this kind of model. The equilibrium dynamics for the capital stock can be described by a first-order difference equation of the form (6), where $\kappa(k)$ denotes the optimal production of capital goods in a period, plus undepreciated capital goods remaining at the end of that period, as a function of the capital stock carried into that period. But whether or not global asymptotic determinacy obtains depends upon the form of the map κ.

It can be shown, under relatively weak assumptions concerning the decreasing returns to additional capital inputs given the fixed exogenous labor supply, that the map κ must be such that for any k_t above a finite critical value $\bar{k}$, $\kappa(k_t) < k_t$. As aresult, the dynamics for the capital stock will eventually be confined to the bounded interval $[0, \bar{k}]$. But this alone does not imply convergence. Under additional assumptions -- for example, a one-sector technology (i.e., the production technologies for the capital good and the consumption good are identical) and preferences for the representative consumer that are additively separable over time -- one can show that κ must be a monotonically increasing function, with a unique fixed point $k^* > 0$, and $\kappa(k)$ greater or less than k according to whether k is less than or greater than k^*. In this case, k_t must converge asymptotically to k^* for all initial conditions $k_0 > 0$. But as was first shown by Benhabib and Nishimura (1985), and as is further developed by Boldrin (1989), in the case of a general two-sector technology, κ need not be a monotonically increasing function. If for some range of capital stocks k_t, the optimal production program involves a sufficiently higher capital-labor ratio in the consumption good sector, κ will be a decreasing function of k_t, and if it is sharply enough decreasing near the steady state capital stock (specifically, if $\kappa'(k^*) < -1$), the dynamics near the steady state will be unstable, so that except under fortuitous conditions, k_t will not remain near k^* asymptotically. If the relative capital intensity of the consumption goods sector increases as the overall ratio of capital to labor in the economy increases, then κ can be a hump-shaped function of the kind shown in Figure 1. If the hump is steep enough, quite complicated dynamics are possible, and indeed the asymptotic dynamics may be "chaotic".[3]

As a contrasting case, there are also models of fluctuations due to *self-fulfilling expectations*, often referred to as "sunspot" equilibria.[4] Probably the best-known example is the overlapping generations model of fiat money, studied by Azariadis (1981), Azariadis and Guesnerie (1982, 1986), and many subsequent authors. In this model, there is no predetermined state variable k_t. The level of money prices p_t in each period is determined by an equilibrium condition of the form

$$f(p_t, \mu(p_{t+1})) = 0 \tag{7}$$

Here I have taken as given a non-stochastic path for the money supply, and assumed that there exist no other exogenous shocks to fundamentals either. Equation (7) indicates that expectations regarding the period t+1 price level (and hence the real returns to holding money) affect desired money holdings in period t, and so the period t price level that equates the supply of and demand for real balances. Any stochastic process for $\{p_t\}$ that satisfies (7) represents a rational expectations equilibrium for such an economy.

Even if we restrict attention to deterministic equilibria ("perfect foresight" equilibria), i.e., sequences of prices $\{p_t\}$ satisfying

$$f(p_t, p_{t+1}) = 0 \tag{8}$$

equilibrium may be *indeterminate*. For there is nothing to determine what the initial price level p_0 must be, except expectations about p_1. The price level p_0 might be anything in a certain interval of values, if appropriate expectations regarding p_1 exist. The equilibrium value of p_1 could similarly be anything in a certain interval of values, given appropriate

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[3]For a simple discussion of how a hump-shaped map can have this consequence, see May (1976). For further examples of economic models resulting in dynamics of this sort, see those sources or Boldrin and Woodford (1990).

[4]The term is due to Cass and Shell (1983). The first general equilibrium example is due to Shell (1977), although the indeterminacy of intertemporal equilibrium as pointing to the possibility of purely "speculative" fluctuations was discussed as early as Samuelson (1957), and the indeterminacy of rational expectations equilibrium in *ad hoc* macroeconomic models was much discussed in the 1970's (see, e.g., Shiller (1978)).

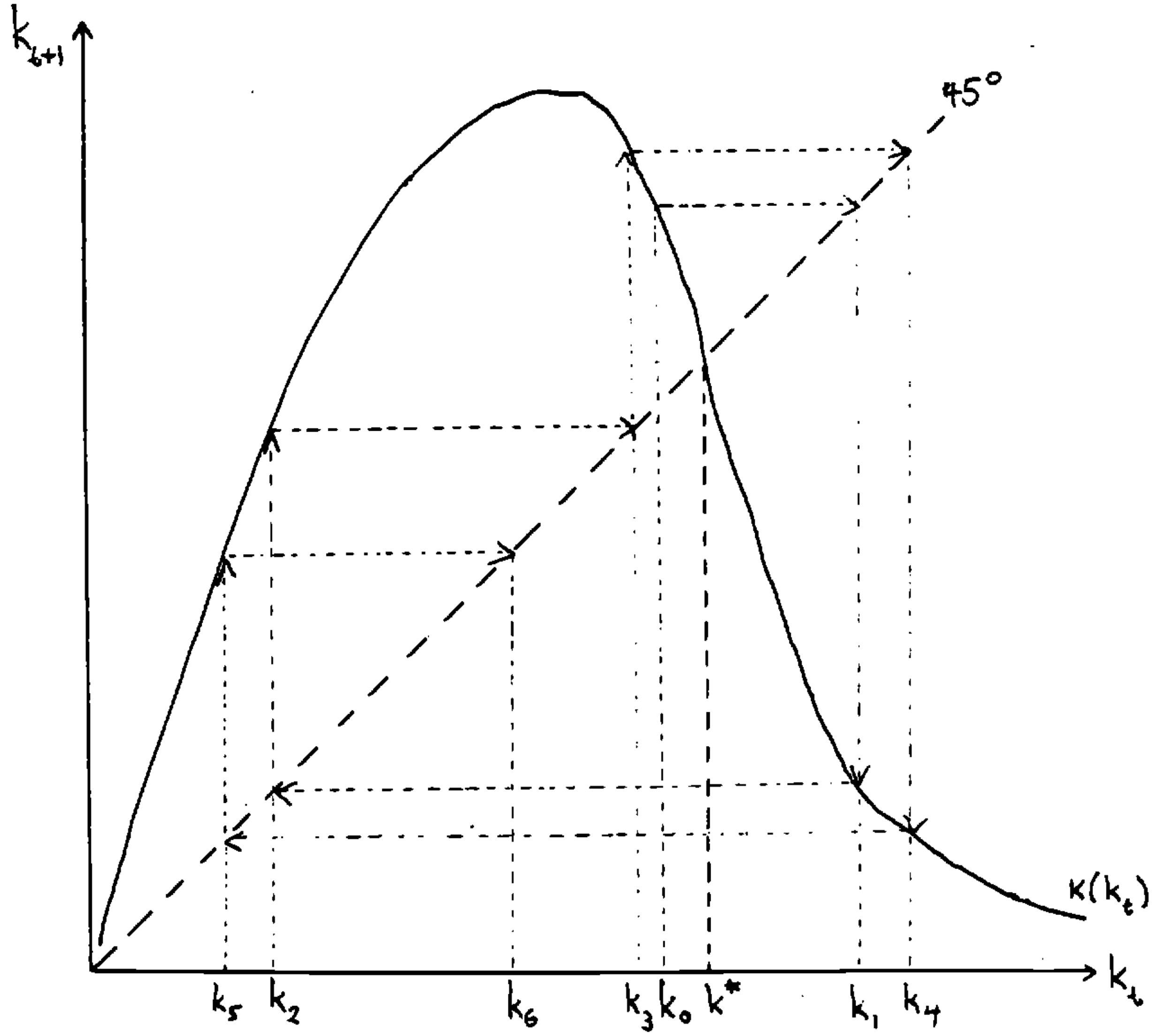


Figure 1.

expectations regarding p_2, and so on. To verify that many different values for p_0 are equally consistent with perfect foresight equilibrium, one must show that each of them can be justified by a sequence of expected future price levels extending into the indefinite future. This can easily occur. For example, in the overlapping generations model, money is the only way in which wealth can be transferred from the first to the second period of life. As a result, an increase in the expected price level in the second period of life may increase rather than decrease the real money balances desired by the young agents, if the income effect of changes in expected return on savings outweighs the substitution effect. If the income effect is sufficiently strong, the graph of pairs (p_t, p_{t+1}) satisfying (8) may be sharply backward-bending, as shown in Figure 2. [5] The case drawn is that in which

(9) $$f_2(p^*, p^*)/f_1(p^*, p^*) \; > \; 1$$

A sequence of values $\{p_t\}$ of the kind shown is a perfect foresight equilibrium, as is the steady state $p_t = p^*$ for all t. It will be observed that a similar construction is possible starting from any p_0 close enough to p^*. In this sense perfect foresight equilibrium is indeterminate; each such sequence represents an equilibrium that can occur if only it is expected to.

Such indeterminacy also creates the possibility of equilibrium fluctuations in response to events ("sunspots") that do not change economic fundamentals. Consider a sunspot variable s_t that follows a two-state Markov chain, where $0 < q_{ij} < 1$ is the probability that state i is followed by state j, for i, j = 1, 2, and consider the possibility of an equilibrium in which $p_t = p_i$ whenever $s_t = i$, for i = 1, 2. The numbers (p_1, p_2) describe a rational expectations equilibrium if and only if the induced stochastic process for $\{p_t\}$ satisfies (7), i.e., if

(10) $$f(p_i, \{p_1, p_2; q_{i1}\}) = 0, \qquad i = 1, 2$$

[5] Here I have graphed this function with the inverse price level on the axes, so that the graph indicates the demand for real balances (on the horizontal axis) as a function of the expected real value of money in the following period (on the vertical axis).

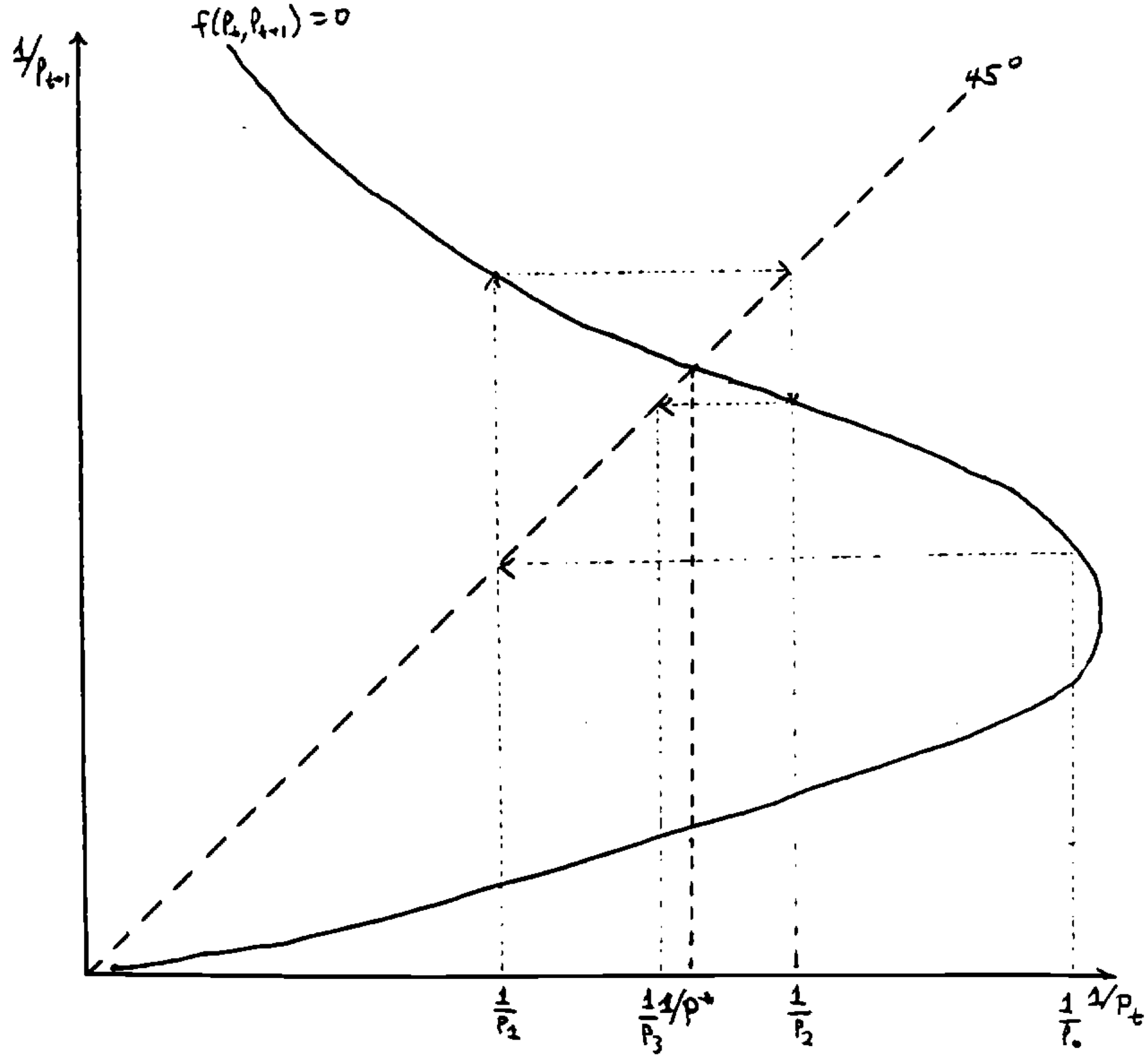


Figure 2.

where $\{p, p'; q\}$ denotes the probability distribution in which the value p occurs with probability q and the value p' with probability 1-q. Given the transition probabilities $\{q_{ij}\}$, (10) is a set of two equations for the two variables (p_1, p_2). One solution is $p_1 = p_2 = p^*$, the deterministic steady state equilibrium, but there may also be solutions with $p_1 \neq p_2$, in which case the price level depends upon the realization of the "sunspot" variable. In such a case the sunspot realization affects p_t through its effect upon expectations regarding the distribution $\mu(p_{t+1})$, which change in expectations is rational if people will continue to change their expectations in response to the sunspot realizations in the future. Thus the belief that the sunspot variable indicates something that makes it appropriate to change one's expectations is self-fulfilling. The possibility of self-fulfilling revisions of expectations of this sort is clearly closely related to the indeterminacy of equilibrium just demonstrated for the deterministic case.[6]

The formal possibility of sunspot equilibria as solutions to (10) is illustrated by Figure 3. Here the two equations in (10) are graphed; the intersections of the two curves represent rational expectations equilibria. The figure is drawn for the case of preferences and endowments like those that give rise to Figure 2, and a sunspot process with q_{11} and q_{22} both small positive quantities. Because q_{12} is near 1, the first equation in (10) gives p_1 as a function of p_2, where the function is similar to the one that gives p_t as a function of p_{t+1} in Figure 2. (In Figure 3, this graph is labeled "$p_1(p_2)$".) Because q_{21} is near 1, the second equation in (10) gives p_2 as a similar function of p_1. (In Figure 3, this graph is labeled "$p_2(p_1)$".) The same condition discussed earlier -- the demand for real balances a sufficiently sharply decreasing function of the real return on money -- makes the two curves cross at (p^{*-1}, p^{*-1}) in the directions shown. This crossing condition, together with the fact that desired real balances do not grow without bound as the expected return on

[6] For further discussion of this relationship, see my (1984, 1986c).

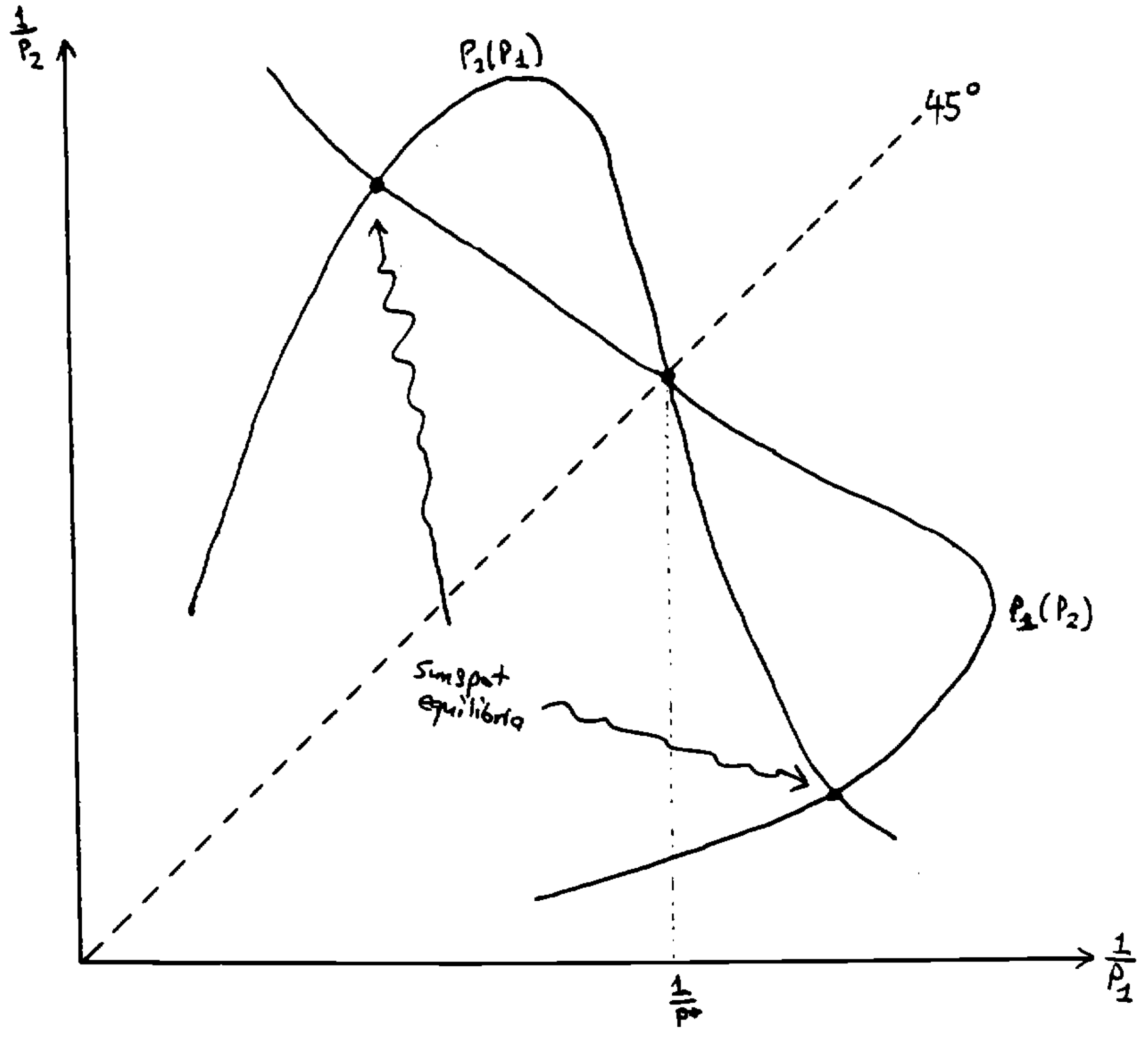


Figure 3.

money is made lower, then guarantees the existence of at least two off-diagonal intersections between the two curves, as shown.

The Azariadis-Guesnerie construction, just discussed, depends upon strong income effects, so that the demand for real balances can be a sharply decreasing function of the expected return to holding money. While this is a theoretical possibility in the overlapping generations model, it is much more difficult for this to occur when the possibility of substituting between money and other assets is admitted, and it is not consistent with observed experience with the effects of inflation on money demand. It is accordingly perhaps useful to point out that even in the sort of simple model just considered, the existence of sunspot equilibria does not depend upon this effect. Even when desired real balances are monotonically increasing in expected return (monotonically decreasing in expected inflation), it is still true that many values of p_t are consistent with equilibrium, given appropriate expectations regarding p_{t+1}, and so on, and sunspot equilibria are often possible. Chiappori and Guesnerie (1988) consider sunspot equilibria of the following form. Suppose that the sunspot variable is a countably infinite Markov chain, with a state space corresponding to the (positive and negative) integers, and suppose that if the sunspot state is i in period t, in period t+1 it will be i-1 with probability 1/2 and i+1 with probability 1/2. Consider again the possibility of equilibria in which $p_t = p_i$ whenever the sunspot state is i, for some fixed sequence of price levels (p_i) where i ranges over the positive and negative integers. A sequence (p_i) represents a rational expectations equilibrium if and only if

$$f(p_i, \{p_{i-1}, p_{i+1}; 1/2\}) = 0 \tag{11}$$

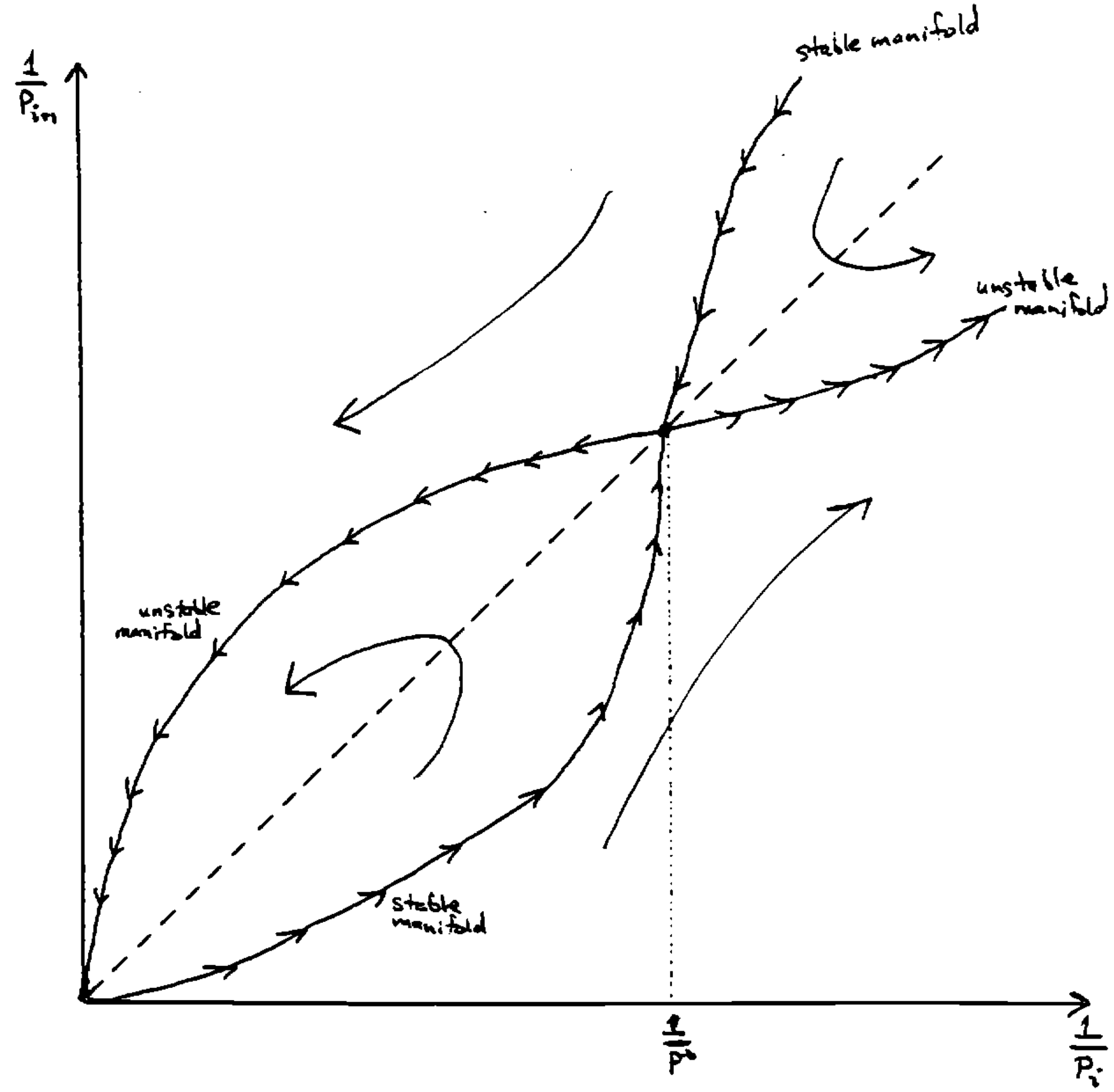
for i = ..., -2, -1, 0, 1, 2, ... Solutions to (11) can be analyzed in the same fashion as the trajectories of a discrete time dynamical system. If the left hand side is monotonic in all three (say, decreasing in p_i and increasing in both p_{i-1} and p_{i+1}, as occurs if substitution effects outweigh income effects), then one can solve for p_{i+1} as a function of (p_{i-1}, p_i), and for p_{i-1} as a function of (p_i, p_{i+1}). Then given any point (p_i, p_{i+1}) in the domain on which these maps are defined, we can define a "forward" mapping that takes such a point to (p_{i+1}, p_{i+2}), and a "backward" mapping that takes it to (p_{i-1}, p_i). We then wish to study the itineraries of points in the plane under repeated applications of these mappings. If one is able to apply both mappings an unlimited number of times (so as to define a complete "trajectory") without reaching a point where prices become negative or where the mapping ceases to be defined, then one obtains a sequence (p_i) that represents a rational expectations equilibrium. One such solution is the sequence $p_i = p^*$ for all i; this fixed point of the "dynamical" system defined by (11) corresponds to the monetary steady state. But there may be "trajectories" other than fixed points that can be continued forever as well, and these correspond to sunspot equilibria.

Chiappori and Guesnerie show that the dynamics in the plane induced by the "forward" mapping can easily look like those shown in Figure 4. (Again I have graphed the inverse price level on the two axes.) Here the solid lines with arrows superimposed represent the stable and unstable manifolds of the fixed point (p^{*-1}, p^{*-1}).[7] Now consider a point (p_1^{-1}, p_2^{-1}) somewhere on the segment of the stable manifold that connects (0,0) to (p^{*-1}, p^{*-1}). Applying the "forward" and "backward" maps repeatedly to this point, one generates a sequence of values (p_i), such that for all i, (p_i^{-1}, p_{i+1}^{-1}) lies on that same segment of the stable manifold, and such that

$$p^* < \ldots < p_{i+2} < p_{i+1} < p_i < p_{i-1} < p_{i-2} < \ldots < \infty$$

This describes a sunspot equilibrium in which the price level fluctuates forever between p^* and infinity; the fluctuations are not transient, in the sense that every state i is eventually visited infinitely often, with probability one. When the sunspot state changes from i to i+1, the price level falls, because, because the expected price level in the future falls, increasing the current demand for real money balances through the standard Cagan-Bresciani-Turroni

[7]Note that (0, 0) is another fixed point, representing the deterministic equilibrium in which money is not valued in any period because it is not expected to be valued in the future.



$\frac{1}{P_{in}}$
stable manifold
unstable manifold
unstable manifold
stable manifold
$\frac{1}{P^*}$
$\frac{1}{P_i}$

Figure 4.

effect of inflation expectations on money demand. Again, the belief that the sunspot realizations will affect the price level in this way is self-fulfilling.[8]

These two types of models of endogenous fluctuations are different in some important respects. One is that sunspot equilibria are inherently stochastic. Hence the attempt to distinguish empirically whether economic time series are "genuinely random" or not (see, e.g., Brock (1986)) can not be relevant to distinguishing between exogenous shock theories and endogenous theories of economic fluctuations.[9]

Another difference is the extent to which nonlinearities are essential to the possibility of endogenous fluctuations of the two sorts. As indicated above, the asymptotic strong determinacy thesis is very generally true of linear systems of equilibrium conditions, assuming that global weak determinacy obtains and that equilibrium fluctuations do not grow without bound. Hence nonlinearities in the equilibrium conditions are crucial to the possibility of endogenous fluctuations of the kind that I have called "determinate but unstable", and in the case of most examples in the literature (in particular, the more interesting examples, which are those of chaotic dynamics), the nonlinearities must be quite severe. For example, in the case of one-dimensional dynamics of the kind illustrated in Figure 1, it is necessary for the function κ to go from having a slope greater than 1 for low values of k to having a slope less than -1 for high values of k. Global weak determinacy is not, on the contrary, an especially general property even of linear models, and the features of the equilibrium conditions associated with the Azariadis-Guesnerie model that allow sunspot equilibria to exist have mostly to do with the derivatives of the functions involved

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[8]A similar construction is possible using points on the unstable manifold, or, indeed, many other points in the plane that lie on "trajectories" that can be extended arbitrarily in both directions, as is discussed by Chiappori and Guesnerie.

[9]For reasons explained above, the distinction between random and deterministic time series is not even an appropriate way of trying to distinguish what I have called "determinate but unstable" dynamics from the dynamics associated with a pure exogenous shock model.

(in particular, of the function f in (7)) near the steady-state values of the arguments.[10] For suppose that (7) were exactly linear, i.e., that it took the form

$$(12) \qquad f_1\,(p_t - p^*) \;+\; f_2\,E_t(p_{t+1} - p^*) \;=\; 0$$

and suppose also that, in accordance with (9), the constant coefficients are such that $f_2/f_1 > 1$. Then if $\{s_t\}$ is a sequence of independent random variables with mean zero, one class of solutions to (12) in which the sunspot realizations affect the price level would be

$$p_t \;=\; p^* \;+\; c\,\Sigma_{j=0}^{\infty}\,(f_1/f_2)^j\, s_{t-j}$$

where $c \neq 0$ is an arbitrary constant. (If the random variables $\{s_t\}$ are uniformly bounded, then the infinite sum in this expression is always well-defined.)

Because sunspot equilibria are perfectly consistent with linearity, it will often be convenient to use linear methods, just as in the case of the familiar exogenous shock models, both in theoretical analyses of the predicted character of economic fluctuations, and in empirical testing of the implications of the models. This is a great advantage over the models of determinate but unstable dynamics, as it allows for the analysis and testing of more sophisticated (and so, possibly, more realistic) versions of the models. Finally, it should be observed that tests for nonlinearity in the laws governing the evolution of observed economic time series (see references below at footnote 13) have no clear connection with the issue of whether observed fluctuations are exogenous or endogenous in character, any more than do the tests for genuine randomness, insofar as one important class of models of endogenous fluctuations does not depend upon nonlinearity.

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[10]This may seem paradoxical, given that Azariadis and Guesnerie (1986) establish that the existence of a two-period deterministic cycle is necessary and sufficient for the existence of the two-state Markov equilibria, and the existence of deterministic cycles is dependent upon nonlinearity in the generic case, as just argued. Nonetheless a *sufficient* condition for the existence of the two-state Markov equilibria is (9), as shown by Azariadis (1981), and this condition does not involve any nonlinear aspects of the function f; (9) is both necessary and sufficient for the existence of sunspot equilibria that remain forever near the steady state equilibrium, as shown by my (1986c). Furthermore, together with the boundary assumptions on the behavior of f made by Azariadis and Guesnerie (which imply nonlinearity, at least at extreme values of the arguments), (9) is also a sufficient condition for the existence of a period-two deterministic cycle. On the other hand, (9) is sufficient for the existence of two-state Markov equilibria even in the absence of the boundary conditions just referred to.

The degree to which these two types of models of endogenous fluctuations are qualitatively dissimilar has probably been obscured by the fact that the overlapping generations model of fiat money, with a backward-bending demand for real money balances as shown in Figure 2, has been much discussed as a leading example both of sunspot equilibria (Azariadis and Guesnerie) and of deterministic equilibrium cycles (Grandmont (1985)). It will be observed that the mapping in Figure 2 is the same as that in Figure 1, but with the axes reversed. That is, the function that gives the demand for real balances as a function of the expected real value of money in the next period is the same sort of hump-shaped function that gives k_{t+1} as a function of k_t in the growth model of Benhabib and Nishimura. Accordingly, similar methods can be used (see Grandmont) to show that if the hump is steep enough, the "backward perfect foresight dynamics" of the overlapping generations model will be characterized by an unstable steady state, the existence of deterministic equilibrium cycles, and even the existence of chaotic equilibrium trajectories.

But this should not be taken to mean that the mechanisms giving rise to sunspot equilibria on the one hand, and the sort of endogenous deterministic cycles studied by authors such as Benhabib and Nishimura on the other, are essentially the same. For in fact the Grandmont example has little in common with the main literature on deterministic cycles.[11] The "backward perfect foresight dynamics" are simply not the dynamics of interest, and the Grandmont example does not really have dynamical properties similar to those of models where a hump-shaped map describes the evolution of a predetermined state variable. In the overlapping generations model, there is not anything that fixes the price level at some future date (the way that an initial capital stock is given by history in the Benhabib-Nishimura model), so that one would be interested in deriving the consequences of that expectation for the price level in previous periods. And even if there were, one

[11] For further discussion, see Boldrin and Woodford (1990).

would not be interested in tracing the consequences of that expectation into the indefinite past (in the way that one solves a growth model forward into the indefinite future), so that one would not be interested in the asymptotic consequences of repeated iteration of the "backward perfect foresight dynamics" map (such as whether the dynamics asymptotically approach a fixed point or a cycle, or instead are forever aperiodic). In the case of the forward dynamics of a predetermined state variable, the existence of "chaotic" dynamics is interesting because it indicates that the time series generated could be very irregular even in the absence of exogenous shocks, indeed could closely resemble a "truly stochastic" time series. But the existence of chaotic "backward perfect foresight dynamics" in the overlapping generations model is not of similar interest. No such construction is needed to demonstrate that irregular or apparently stochastic equilibrium dynamics are possible in that model. Even setting aside the possibility of sunspot equilibria, it will be observed from Figure 2 that a given equilibrium price level p_t can often be equally well justified by two different expectations regarding p_{t+1}, and so the *forward* perfect foresight dynamics, being often not uniquely defined, plainly allow for very irregular trajectories, since a very complex rule may be used to determine which value of p_{t+1} occurs following each time that p_t takes such a value.

The Grandmont analysis does suffice to demonstrate the possibility of deterministic cycles of all periods, since the existence of deterministic cycles in the backward dynamics is equivalent to the existence of such cycles in the forward dynamics. But as I have argued above, it is not really the possibility of deterministic cycles in the absence of exogenous shocks that is the important feature of models such as that of Benhabib and Nishimura, but rather the fact that the equilibrium dynamics are determinate but unstable, a property not true of the Grandmont example. In my view the overlapping generations model illustrates the possibility of endogenous equilibrium fluctuations as a result of the indeterminacy of rational expectations equilibrium, and the deterministic cycles studied by Grandmont are best understood as simply degenerate, limiting cases of finite-state Markovian sunspot equilibria of the kind studied by Azariadis and Guesnerie. The sort of determinate cycles exhibited by Benhabib and Nishimura represent a distinct type of endogenous fluctuations.

Still, some may ask, why should the question of whether observed aggregate fluctuations are to some extent endogenous matter? Of course, all will agree that a more accurate model of economic fluctuations would be useful, and methodological blinders that prevent one from discovering the true structure are obviously undesirable. But this does not explain why the endogenous or exogenous origin of fluctuations should be a question of interest in itself, apart from the interest that there might be in arguing for some particular model that happens to explain fluctuations as endogenous. Indeed, it might be argued that endogenous and exogenous explanations considered as general categories cannot be regarded as different for any practical purposes, due to the substantial continuity that exists between the two categories. Not much can follow from the claim that an economy fluctuates in response to random events that are true "sunspot" variables -- in the sense of having *no* effect whatsoever on fundamentals -- as opposed to its fluctuating in response to events that represent changes of negligible size in fundamentals.[12] Nor can much follow from the claim that global asymptotic determinacy does not hold for an economy, and so that the effect of initial conditions on the endogenous state variables remains non-negligible forever -- as opposed to the effect eventually dying out, as claimed by the determinacy thesis, but with an extremely slow rate of decay.

But these are not really adequate reasons for ignoring the possibility of endogenous fluctuations. The simple fact that the boundary between the different categories of explanations that would be most relevant is not susceptible of clear definition does not mean that models that represent "ideal types" of the endogenous category are not useful in

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[12] Indeed, one can show that when sunspot equilibria exist, the equilibrium response to small shocks to fundamentals is also indeterminate, and that among the possible rational expectations equilibria are equilibria in which the endogenous state variables respond very strongly despite the fact that the change in fundamentals is very small. Such "over-response" to a change in fundamentals would be observationally indistinguishable from a "sunspot" equilibrium. See Farmer and Woodford (1984), my (1986c, theorem 2), and Chiappori and Guesnerie (1988).

demonstrating what would be meant by an explanation of that kind and under what circumstances it would be possible. And it is possible to speak of relatively general implications of the hypothesis of endogenous instability that make the question of interest even when framed so generally.

For one, it can be stated with reasonable generality that an endogenous explanation of aggregate fluctuations implies that they are inefficient and so undesirable. The general argument is that, under the usual modeling assumptions of strictly concave production sets and utility functions, fluctuations in the allocation of resources that occur other than as a response to fluctuations in either tastes or technological possibilities must reduce expected utility compared to a steadier growth path. More precise results along this line are discussed in the next section. It is shown there that the claim just made is subject to a number of qualifications; one can construct theoretical examples of both sunspot equilibria and of determinate but unstable equilibrium dynamics in which the equilibria are Pareto optimal. Nonetheless it is argued that the cases of most likely practical relevance under which endogenous fluctuations of either sort are possible are conditions under which the fluctuating equilibria are inefficient (and can occur only because of some kind of "market failure"). Nor is it by any means the case that exogenous shock models must imply that fluctuations are not a problem; the mere fact that exogenous shocks to fundamentals imply that *some* response would be efficient does not mean that the one that actually occurs must be. Nonetheless this is a property of at least the currently most popular class of exogenous shock models ("real business cycle" models), and this is not surprising, given the general predilection of economists to be led from a basic commitment to explanation in terms of optimization and equilibrium into the assumption of a perfect system of competitive markets, except in cases where the phenomenon to be explained is clearly incompatible with such an assumption.

Secondly, endogenous explanations as a class result in a presumption that policy interventions ought to exist in principle that can suppress or at least significantly reduce the fluctuations, without requiring a radical alteration of the structure of the economy, and, in particular, without having to cure the underlying "market failures" (due to private information, say, or increasing returns) that allow the inefficient fluctuations to occur in equilibrium in the first place. In the case of exogenous shocks, policy interventions can affect the nature of the fluctuations that occur in response to the shocks, but it is hard to prevent fluctuations of one sort or another from occurring. If the fluctuations are purely endogenous then there is no reason why the economy could not follow a completely steady path, and the modifications required to get this to happen might be minimal. For example, in the case of sunspot equilibria, there will typically also exist equilibria in which no fluctuations in response to the sunspot variables; one simply needs to design a policy regime that prevents the occurrence of the sunspot equilibria and leaves the non-fluctuating equilibrium or equilibria as the only possibility. The type of intervention needed may only be a credible commitment to intervene if fluctuations were to arise, which will never have to be acted upon in equilibrium.[13] In the case of determinate but unstable dynamics, elimination of the endogenous cycles requires only that the feedback loop that sustains them be weakened to the point that the cycles cease to be self-sustaining, not that the nature of any of the causal links in the chain that creates the cycles be completely changed. Thus, in the case of the dynamics represented by Figure 1, an intervention that changed the shape of the hump to make it a bit less steep would succeed in rendering stable the deterministic steady state; it is not necessary to transform the dynamics to the extent that they are no longer described by a hump-shaped map. The issue of stabilization policy will not be discussed in the case of any of the examples of endogenous cycle models that sketched here, both because of space limitations and because of the foolishness of talking too much about the policy prescriptions that might be drawn from models whose empirical relevance has not yet been established. But the fact that models of the general class discussed here

[13] For an example of stabilization policy of this kind, see my (1986a).

could well have consequences for policy analysis that are different from those associated with more conventional models remains an important reason for being interested in the question of those models' logical coherence and empirical adequacy.

II. The Consistency of Endogenous Fluctuations with Optimizing Behavior

In discussing reasons for neglect of the hypothesis of endogenous fluctuations in the previous section, I have set aside what is perhaps the most serious objection to this general class of explanations. This is the view that the possibility of endogenous fluctuations can be ignored, not because of a special methodological commitment to the determinacy theses, but as a consequence of more basic methodological commitments -- specifically, commitments to explaining economic phenomena in terms of optimizing behavior and competitive equilibrium. If it can be shown that economic models founded upon these postulates necessarily satisfy the determinacy theses, then there is no need to argue for them as independent modeling principles.

The examples presented in section I already have demonstrated that no really strong claim of this kind is tenable, since the economies described are ones in which all agents maximize their expected utility, all agents have rational expectations, and all markets are perfectly competitive and clear at all times. Nonetheless, sufficiently restricted versions of this claim are actually true. Some may feel that these suffice to create a presumption against the empirical relevance of the endogenous cycle hypothesis. I wish instead to emphasize that these results show to what extent endogenous fluctuations, if they do occur, are likely to be connected with the failure of an ideal system of competitive markets to exist.

One important general result of this kind is the following.

<u>The Sunspot Irrelevance Theorem.</u> Suppose that the economy is perfectly competitive and that the standard conditions required to prove the efficiency of competitive equilibrium (no externalities, no distorting taxes, etc.) are satisfied. In particular, suppose that there exist only a finite number of distinct consumer types,

> and suppose that there exists a complete set of Arrow-Debreu contingent claims markets, including markets for securities contingent upon all possible realizations of the "sunspot" variables. Finally, suppose that production sets are convex (no ncreasing returns) and that consumers von Neumann-Morgenstern utility functions re strictly concave (consumers are risk averse). Then no rational expectations equilibria involve fluctuations in the allocation of resources, or fluctuations in the relative prices of any goods, in response to the realization of the "sunspot" variables.

The basic idea behind this result was first demonstrated by Cass and Shell (1983). The result has been extended by Balasko (1984), and a thorough discussion is given in Guesnerie and Laffont (1988). The basic idea is that under the conditions assumed, a rational expectations equilibrium is equivalent to an Arrow-Debreu equilibrium, and involves a Pareto optimal allocation of resources. But no allocation of resources which depends upon the sunspot state can be Pareto optimal (given that preferences, technology, and endowments are independent of the sunspot state). For if an allocation that fluctuates in response to the sunspot state is feasible, then there exists another allocation that is not contingent upon the sunspot state (e.g., take in each state the allocation that is the probability-weighted average of the allocations previously specified for the various sunspot states) that is also feasible (because of convexity of the production sets) and gives a higher expected utility to all consumers whose allocation previously depended upon the sunspot state (because of strict concavity of the utility functions). Hence no sunspot-contingent allocation can be an equilibrium allocation. But then relative prices of goods cannot differ across sunspot states either, insofar as in a competitive equilibrium these relative prices must correspond to marginal rates of substitution in consumption and marginal rates of transformation in production, which will not differ if the allocation of resources does not.

This strong result might appear to justify the view described above, according to which sunspot fluctuations are simply inconsistent with rational expectations equilibrium, when the full consequences of optimization and equilibrium are properly taken into account. But the irrelevance theorem contains many qualifications, which indicate ways in which self-fulfilling expectations may be a source of economic fluctuations, even granting the postulates of optimization, rational expectations, and equilibrium.

For one, the "averaged" allocation referred to above need not be feasible if there are indivisibilities, non-convex adjustment costs, or increasing returns to scale; in such cases, a randomized allocation (where the randomization is independent of any variation in fundamentals) can be efficient,[14] and as a result might be associated with a competitive equilibrium even under circumstances under which equilibirum would have to be efficient. Second, if consumers' utility functions are not strictly concave, then the "averaged" allocation need not be a Pareto improvement over the sunspot allocation, and so again there might be a Pareto optimal sunspot equilibrium. Guesnerie and Laffont (1988) exhibit an example of this kind, based upon locally "risk-loving" behavior of the Friedman-Savage sort, in which the sunspot equilibrium Pareto-dominates the unique non-sunspot equilibrium. And third, an equilibrium may not be Pareto optimal, because of any of a variety of sorts of violations of the conditions under which it is possible to prove the First Welfare Theorem. As a result an equilibrium allocation might involve fluctuations in response to sunspot realizations.

One reason that a competitive equilibrium might not be Pareto optimal is absence of a complete set of Arrow-Debreu markets for contingent commodities. Here it is important to note that if one allows for equilibria in which prices and supply and demand behavior may be contingent upon sunspot variables, then the First Welfare Theorem requires, among other things, a complete set of markets for securities contingent upon all possible realizations of the sunspot variables. Even if there exists a complete set of frictionless markets in all other senses, so that an equilibrium not contingent on the sunspot states would necessarily be Pareto optimal, if there do not exist markets for insurance against sunspot risk, in which all consumers who will ever exist can trade prior to the realization of any of the sunspot states, then there might also exist inefficient sunspot equilibria. Given

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[14]Hence the use of "lotteries" to support efficient allocations in generalizations of the notion of Arrow-Debreu equilibrium to economies with non-convexities, e.g., Rogerson (1988). The relation of this idea to the literature on "sunspot equilibria" is developed explicitly in recent work by Karl Shell and Randy Wright.

the large number of types of random signals that might conceivably serve to coordinate shifts in people's expectations, it is not implausible to suppose that a complete set of such markets do not in fact exist, in which case the irrelevance theorem is no practical significance.

The sunspot equilibria in the Azariadis-Guesnerie model referred to in section I are of this kind.[15] They cease to be possible if markets for insurance against sunspot realizations are introduced. To see why, it is necessary to describe in greater detail the microeconomic foundations of the demand for money in the model considered by Azariadis and Guesnerie. In this model, each consumer lives for two consecutive periods, and acquires money (the only asset) by selling goods in the first period of life, holds the money until the second period of life, and then uses it to purchase goods. Each consumer's preferences over consumption are additively separable between the two periods of life. The demand for real balances in a given period is then just the desired saving by young consumers in that period; this depends upon the expected real return on savings, which in turn depends upon the rate of inflation. Now suppose that there were also a complete set of markets for securities that paid off in the event of different sunspot histories, which securities are all in zero net supply. Suppose that all consumers who will ever consume in any period have an opportunity to trade in these securities before the realization of any of the sunspot states, and that they trade so as to maximize their expected utility, taking into account what their consumption will be in the event of each of the possible sunspot histories, and with common (correct) expectations regarding the probability with which the

[15] Strictly speaking, this is not a model in which rational expectations equilibrium, even when not contingent upon sunspot states, must be Pareto optimal, as discussed below. But monetary equilibria in which the value of money is forever bounded away from zero are necessarily Pareto optimal (Balasko and Shell (1980)) if not contingent upon sunspot states, and the Azariadis-Guesnerie sunspot equilibria have this property. Furthermore, as is shown in my (1984), sunspot equilibria of the same kind exist under exactly analogous conditions in an overlapping generations model in which the store of value ("land") pays a constant positive real dividend, unlike the fiat money considered by Azariadis and Guesnerie. In the case of the economy with "land", dynamic inefficiency is impossible, and competitive equilibrium is necessarily Pareto optimal if not contingent upon sunspot states. Nonetheless, inefficient sunspot equilibria can exist, if markets do not exist in which all agents can insure against the sunspot realizations.

different sunspot histories will occur and what the consequences of each will be for market prices. It is then impossible that a rational expectations equilibrium could involve a different allocation of resources in a given period in the case of different sunspot states. For if it did, then in one state the consumption of the old would have to be higher than in another, while the consumption of the young would have to be correspondingly lower in that state, given that endowments cannot depend upon the sunspot state.[16] But then, given strict concavity of the consumers' von Neumann-Morgenstern utility functions, the marginal utility of consumption by the old must be lower in the first state than in the second, while the marginal utility of the young is higher in that state. This is inconsistent with rational expectations equilibrium in the case of insurance markets against sunspot risk, since that would require the ratio of marginal utilities between the two states to be the same for all consumers who consume in both states, i.e., equal to the relative price of contingent claims to consumption in the two states divided by the relative probability of the two states occurring (upon which probability both types of consumers must agree at the time of the trading in the contingent claims).

There are several possible defenses of the relevance of the Azariadis-Guesnerie sunspot equilibria despite this result. One is to observe that all sorts of random events could play the role of the sunspot states, so that even if there were trading in claims contingent upon some of them, one could still have sunspot equilibria in which the allocation of resources fluctuated in response to other events, against which insurance was not possible. This line of argument is developed by Azariadis (1981) and Azariadis and Guesnerie (1982). It is not entirely convincing, however. After all, as noted in my (1989b), the existence of a sunspot equilibrium requires a great degree of coordination by agents as to

[16] The argument here assumes an exchange economy where the total available supply is simply given by consumers' aggregate endowment. But a similar argument is possible if one allows for variation in endogenous labor supply across sunspot states, as in Azariadis (1981); then instead of the *goods* consumed by the young being lower in the first state, it is their consumption of leisure, but the argument proceeds in the same way.

what signals will be interpreted in what way. As a result, the fact that the existence of an Azariadis-Guesnerie sunspot equilibrium would create a profit opportunity for those who were to introduce a new type of contingent security, the introduction of which would then prevent the equilibrium fluctuations, is no trivial problem; for it is not clear how easily coordination upon some new, as yet uninsurable, sunspot event could arise. Furthermore, it is not true that a large number of types of securities must be traded in order to rule out the Azariadis-Guesnerie sunspot equilibria; the existence of securities contingent upon the price level would suffice.

Another defence of these equilibria is to point out that trading in securities contingent upon the sunspot states is consistent with the existence of the sunspot equilibria, if trading is possible only by consumers who are "alive" at the time of the trading (i.e., who consume in that period). For in the Azariadis-Guesnerie model, each consumer lives for only two consecutive periods. Thus the only types of consumers who would care to insure against sunspot risk in period t are the consumers who consume in periods t-1 and t, and those who consume in periods t and t+1; and of these, only the former are "alive" in any period prior to the realization of the period t sunspot state. But if all consumers in a given generation have the same preferences and endowments (as assumed by Azariadis and Guesnerie), then trading in period t-1 in securities contingent upon the period t sunspot state by members of the generation that consumes in periods t-1 and t will result in market clearing securities prices such that no consumer's consumption allocation is any different due to the existence of the markets for contingent securities.[17] This resolution, however, is

[17] Tom Sargent has suggested the following interpretation of this result. One may suppose that consumers do not insure against sunspot risk by trading in contingent securities at any time before their first period of "life" because consumers "born" in the same period but subsequent to different histories of sunspot realizations are distinct individuals who do not desire to pool their risk *ex ante*. Under this interpretation, an alternative allocation of resources should be considered to be "Pareto improving" only if the expected utility of no consumer type is reduced, where consumers "born" subsequent to distinct sunspot histories are treated as distinct types, and where the expected utility of each type is measured as of the first period in which that consumer type consumes. With this considerably weakened criterion for Pareto optimality, the argument sketched above according to which sunspot fluctuations cannot be Pareto optimal does not work, and indeed the Azariadis-Guesnerie sunspot equilibria are Pareto optimal. This interpretation is of some relevance to the issue discussed at the end of section II, of whether sunspot fluctuations are necessarily inefficient and

convincing only under the literal interpretation of the Azariadis-Guesnerie model as referring to consumers whose actual lifespans are only two periods. An interpretation of this model of fiat money (and the associated possibility of sunspot equilibria) that I prefer is one according to which it represents an economy of long-lived consumers who are, however, constrained in their ability to borrow every other period, due to periodic fluctuations in either their endowments or their taste for consumption.[18] In this interpretation, the motivation for the restriction upon trading in the sunspot-contingent securities by the other type of consumers disappears.

The best response to the objection that introduction of trading in contingent claims removes the possibility of sunspot equilibria in the Azariadis-Guesnerie model is to observe that this result is rather special to that model; many other examples of sunspot equilibria do not depend upon the non-existence of or restrictions upon participation in markets for insurance against sunspot risk. For sunspot equilibria can fail to be Pareto optimal for many reasons other than the absence of opportunity for trade of that sort. There are many kinds of models in which equilibrium is not, or at least need not be, Pareto optimal, even when sunspot equilibria are not considered. In these same kinds of models, inefficient sunspot equilibria may be possible, as conditions assumed by the irrelevance theorem do not hold. In some of these cases, the introduction of markets for insurance against sunspot risk has no effect upon the existence or character of the sunspot equilibria at all.

For example, equilibrium need not be Pareto optimal in overlapping generations models, despite the absence of any restrictions upon trading by any consumer types, due to

hence undesirable. Since, however, the Azariadis-Guesnerie example of sunspot equilibria does not seem to me the type of example of greatest potential macroeconomic relevance, I will not discuss further the desirability of this interpretation of that example.

[18]See Townsend (1980), Woodford (1986b), or Sargent (1987, ch. 6). This interpretation has the advantage of providing a theory of the demand for fiat money that does not require assumptions that imply the possibility of a dynamically inefficient equilibrium even in the presence of perfect intertemporal markets, regarding which see below. For example, this sort of monetary theory is consistent with an assumption of Barro-type bequest links between generations, or the existence of "land", as long as the use of "land" as a means of payment is assumed to involve sufficiently large transactions costs as to continue to allow the borrowing constraints to bind periodically. On the possibility of sunspot equilibria in a monetary economy of this kind, see my (1988a).

the existence of an infinite sequence of distinct consumer types.[19] And it can be shown (Cass and Shell (1989)) that sunspot equilibria are possible in overlapping generations models, even with *ex ante* trading in claims contingent upon the sunspot history by all consumer types who will ever consume, if one allows for preferences that are not additively separable between periods, unlike Azariadis and Guesnerie, so that the above impossibility argument does not go through, and if one considers equilibria with a "dynamically inefficient" allocation of resources (due to a real rate of return that is on average lower than the growth rate of endowments), unlike the allocations associated with the Azariadis-Guesnerie equilibria. There are many reasons, however, for doubting that the theoretical possibility of dynamic inefficiency in economies with a complete set of perfectly competitive markets is empirically realistic. These may relate either to a belief that at least some positive fraction of the economy's endowment at all times is under the control of "dynastic" families who because of bequest linkages behave like infinite-horizon maximizers, as argued by Barro (1974), or to a belief that assets exist that are sufficiently productive that a finite equilibrium value for those assets implies a finite value for the economy's aggregate endowment as well. (One might think that actual land has the properties of the ideal "land" referred to above, or more plausibly, that physical capital in advanced economies is too productive to be consistent with a dynamically inefficient equilibrium allocation, as argued by Abel *et al* (1986).) Hence this does not seem to me the reason for the possible existence of sunspot equilibria that is of the greatest practical importance.

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[19]The existence in the case of equilibrium prices of a well-defined budget constraint for each consumer type need not imply a well-defined value for the economy's aggregate endowment, if the number of consumer types is not finite, so that the standard proof of the First Welfare Theorem is invalid. This explains the reference to a finite number of consumer types in the statement above of the sunspot irrelevance theorem. A First Welfare Theorem can, however, be proved for some classes of economies with an infinite sequence of consumer types -- for example, if there exists "land" of the kind mentioned above, or if some finite number of consumer types have a total endowment that is more than some positive fraction of the economy's aggregate endowment in all periods. The sunspot irrelevance theorem can be extended to all such cases.

More relevant, in my view, is the fact that price signals may fail to guide the economy to an efficient allocation of resources in the event of any of a number of types of "market failures". I will limit my attention here to some of the types of market failures that have been given frequent attention in macroeconomic modelling, for reasons unrelated to an interest in the possibility of sunspot equilibrium. Three broad classes of deviations from the idealized competitive model that are arguably important for macroeconomics come to mind. First, there are the types of imperfect financial intermediation that make possible a role for intrinsically valueless fiat money (even in a world where dynamic inefficiency would not exist in the case of perfect frictionless markets), or that may explain the apparent importance of disruptions of financial intermediation in generating recessions and depressions. Second, there are models of rigid money prices together with associated non-price rationing of goods in some markets, often invoked to explain the non-neutrality of variations in the money supply and the role of variations in aggregate demand more generally in generating temporary fluctuations in the level of economic activity. And third, there are increasing returns to scale in production, due either to externalities between firms who individually face decreasing returns (the "thick market" externalities discussed by Diamond (1982), Howitt and McAfee (1988), and Hall (1989), or the technological spillovers discussed by Murphy *et al* (1988)), or to increasing returns at the firm level combined (in order for equilibrium to exist) with imperfect competition between firms, of a kind that have often been invoked as an explanation of observed cyclical variations in productivity (Murphy *et al* (1989)), as well as of certain facts about long run growth (Romer (1986)). All three types of imperfections are known to be possible sources of sunspot equilibria. What is more, all three types of imperfections may result in sunspot equilibria even in representative consumer economies, i.e., in economies in which all consumers are assumed to be identical infinite lived Barro "dynasties". Examples of the latter sort are plainly not dependent upon assumptions about whether or not there is trading in securities contingent upon sunspot realizations. For given that all traders have identical circumstances, trading in such securities simply results in market-clearing at securities prices such that no trader wishes either to buy or sell the securities in question, and as a result equilibrium behavior is unaffected by the existence of the markets for these securities. I will emphasize here only representative-consumer examples of these types for this reason.

A representative consumer variant of an economy in which financial constraints create a role for valued fiat money (even under circumstances under which dynamic inefficiency would not exist in the case of perfect markets) is the cash-in-advance economy considered by Lucas and Stokey (1987), among others. Here again an equilibrium condition of the form (7) is obtained, although the micro-economic foundations of the condition are different. The demand for real money balances (and hence the equilibrium price level) in period t depends upon the extent to which consumers desire in that period to purchase "cash goods" as opposed to "credit goods"; the optimal tradeoff between the two kinds of purchases depends upon the expected price level in period t+1 as well as the price level in period t, insofar as credit goods will be purchased in period t to the point where the value of a marginal unit of currency in period t+1 is equal to the value of the marginal quantity of credit goods that could have been purchased in period t by promising to pay that amount at the beginning of period t+1. Again the sort of situation depicted in Figures 2 and 3 is possible, for the right kind of utility function. Indeed, my (1988a) shows that in the case that the representative consumer's utility function is additively separable between periods and between cash and credit goods within each period, there is an exact formal correspondence between the equilibrium conditions of the Azariadis-Guesnerie model and those of the Lucas-Stokey model, with "consumption of cash goods" (respectively, "credit goods") taking the place of "consumption by old consumers" (respectively, "young consumers") in each period, and with the endowment of the representative consumer each period (that can be transformed into either cash or credit goods) taking the place of the endowment of the young consumers each period (some of which is consumed by them and some by the old). As a result, the conditions known to allow the existence of sunspot equilibria in the Azariadis-Guesnerie model can be immediately translated into the context of the Lucas-Stokey model (roughly, that the utility function for cash goods consumption be sufficiently strongly concave near the level of consumption associated with the monetary steady state).[20]

In this case, however, the introduction of trading in securities contingent upon the history of sunspot realizations does not affect the conditions for the existence of sunspot equilibria. Following Lucas and Stokey, we can introduce securities trading by supposing that in each period there is first a securities market sub-period, then a goods market sub-period, with cash goods purchased in the second sub-period having to be paid for with money held at the end of the first sub-period (possibly acquired by selling securities), and with securities purchased in the first sub-period of period t paying off (or being able to be traded again) in the first sub-period of the period t+1. In equilibrium, the ratio of marginal utilities of cash goods consumption in two sunspot states in period t will have to equal the ratio of the money price level in the two states, times the relative price in the period t-1 securities market sub-period of securities paying off contingent claims to money (payable in the first sub-period of period t) in the two states, divided by the relative probability of occurrence of the two states (given information at the time of securities trading in period t-1). But the ratio of marginal utilities of credit goods need not equal this, if the cash-in-advance constraint binds in one or more of the states, since the shadow price on the cash-in-advance constraint may be different in the different sunspot states. Hence the previous argument for the impossibility of sunspot equilibria with insurance against sunspot risk does not go through. Indeed, because of the assumption of a representative consumer,

[20] For a more thorough analysis of the conditions under which sunspot equilibria exist in the Lucas-Stokey model, see my (1988c).

there is no change in the conditions for the existence of, or in the predicted character of, the sunspot equilibria at all.[21]

My (1988b) demonstrates the possibility of sunspot equilibria in an economy with perfect financial intermediation, but in which output prices are rigid in money terms, so that variations in aggregate demand induce a change in equilibrium supply, despite the fact that marginal costs must vary relative to price.[22] In this economy, the existence of the rigid price and rationing in the product market results in an equilibrium level of output y_t that (in the absence of shocks to fundamentals) will be given by

$$y_t = f(i_{t-1}, i_t) \tag{13}$$

where i_t denotes investment in period t. Here past investment (which equals the period t capital stock, assuming complete depreciation each period[23]) enters through the effect of the capital stock and hence capacity upon equilibrium supply, while the current investment enters through the "multiplier" effect of current investment spending upon aggregate demand. The returns r_t per unit of capital in period t depend upon the amount of capital in place and the level of output that ends up being produced. Substituting (13) into this relationship allows one to write

$$r_t = g(i_{t-1}, i_t) \tag{14}$$

Finally, due to the special form of preferences assumed for the representative consumer, it can be shown that in equilibrium the expected real return on all assets must always equal a constant r^*. As a result, the level of investment chosen in equilibrium in period t will be

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[21] Other examples of economies with infinite lived consumers in which financial constraints result in the possibility of sunspot equilibria are discussed in my (1988a). Indeterminacy of equilibrium, endogenous cycles, and sunspot equilibria in represenative consumer monetary economies of the Sidrauski-Brock variety are discussed in Matsuyama (1989a, 1989c).

[22] This example was originally introduced in the lecture notes for this workshop, but the presentation here has been greatly condensed due to space limitations.

[23] Complete depreciation is in no way essential to the logic of this example. The same analysis applies if in each equation one replaces i_t by the capital stock chosen for period t+1. I have used the notation i_t only to make it clear that this is a non-predetermined endogenous state variable determined in period t.

that level that results in an expected return on capital in the following period that is exactly equal to this desired return, so that

(15) $$E_t\,[g(i_t, i_{t+1})] = r^*$$

Condition (15) indicates how equilibrium investment in period t is determined by expectations regarding investment (and hence aggregate demand) in period t+1. This is an equilibrium condition of the same form as (7), and can be analyzed in a similar manner. Any stochastic process for $\{i_t\}$ that satisfies (15) at all times (and stays within certain bounds assumed in deriving (15)) represents a rational expectations equilibrium; a fluctuating solution represents an equilibrium in which investment spending fluctuates in response to self-fulfilling expectations, and results in fluctuations in economic activity through (13). Stationary fluctuating solutions to (15) exist, including finite-state Markov process equilibria of the kind discussed by Azariadis and Guesnerie, if both the "multiplier" effect of i_t on y_t and the "accelerator" effect of expectations regarding y_{t+1} on i_t are sufficiently strong.[24]

Indeterminacy of rational expectations equilibrium and the possibility of sunspot equilibria in dynamic models with increasing returns and/or externalities have been discussed by a number of recent authors (Hammour (1988), Spear (1988), Murphy *et al* (1988), Kehoe *et al* (1989), Matsuyama (1989b)). A slight modification of a standard one-sector growth model with a representative consumer can illustrate this possibility. Let Y(K) denote aggregate output, net of depreciation, when the aggregate capital stock is K, and let R(K) denote the gross real return per unit of capital under the same circumstances. Then if consumers are all identical and seek to maximize an infinite discounted sum of utilities $\sum_{t=0}^{\infty} \beta^t u(C_t)$, where $0 < \beta < 1$ and u is an increasing concave function, intertemporal optimization requires a consumption plan satisfying

[24] Other types of market imperfections which allow fluctuations in aggregate demand for produced goods to result in fluctuations in equilibrium labor demand, such as the oligopolistic model of Rotemberg and Woodford (1989), also allow sunspot equilibria to exist under certain conditions.

(16) $$u'(C_t) = \beta\, R(K_{t+1})\, E_t[u'(C_{t+1})]$$

together with a transversality condition. The resulting evolution of the aggregate capital stock will then be given by

(17) $$K_{t+1} = Y(K_t) - C_t$$

Any stochastic processes for $\{K_t, C_t\}$ satisfying (16) and (17) for given initial condition K_0, and with both variables forever bounded (so that the transversality condition is also satisfied) will constitute a rational expectations equilibrium.

Now there will typically exist a steady state equilibrium for such an economy, namely a pair (K^*, C^*) such that if $K_0 = K^*$, then a possible equilibrium is $C_t = C^*$, $K_t = K^*$, for all t. These quantities will satisfy the equations $R(K^*) = \beta^{-1}$, $C^* = Y(K^*) - K^*$. Under the assumptions that $R'(K^*) \neq 0$, $Y'(K^*) \neq 1$, it can be shown that in the case of any equilibrium in which C_t and K_t remain sufficiently near to the steady state values in all periods, the equilibrium is well approximated by a solution to the following linear approximation to the system (16) - (17):

(18) $$\begin{bmatrix} E_t[C_{t+1}-C^*] \\ K_{t+1} - K^* \end{bmatrix} = \begin{bmatrix} 1-BR' & BR'Y' \\ -1 & Y' \end{bmatrix} \begin{bmatrix} C_t - C^* \\ K_t - K^* \end{bmatrix}$$

Here $B = -\beta u'(C^*)/u''(C^*)$, and the derivatives R' and Y' are evaluated at K^*. It can be shown furthermore (Woodford (1986c)[25]) that

(i) in the case that one eigenvalue of the matrix in (18) is real with modulus less than one, and one is real with modulus greater than one -- the case of "saddlepoint stability" or "exact determinacy" -- then there is a unique rational expectations equilibrium in which the state variables remain forever near the steady state values, for each choice of K_0 sufficiently near K^*, and this equilibrium is in all cases described by the same pair of functions $C_t = \gamma(K_t)$, $K_{t+1} = \kappa(K_t)$, so that a representation of the form (5) - (6) exists. Furthermore, the functions γ and κ are such that in all cases this equilibrium converges asymptotically to the steady state values of the state variables. Hence both determinacy theses are valid for this class of equilibria, and no equilibria involving endogenous fluctuations are possible, at least near the steady state values of the state variables.

(ii) in the case that both eigenvalues of the matrix in (18) have modulus less than one -- the case of "indeterminacy" -- then there exists a large set of rational

[25] See also my (1984) for a more elementary discussion of this classification of local dynamics.

expectations equilibria for each choice of K_0 sufficiently near K^*, in which the values of the state variables remain forever near their steady state values, including a large set of stationary sunspot equilibria, where the stationarity of the latter stochasticprocesses implies that the sunspot fluctuations do not die down in amplitude asymptotically.

(iii) in the case that both eigenvalues of the matrix in (18) have modulus greater than one -- the case of "instability" -- then for each choice of K_0 sufficiently near K^*, with $K_0 \neq K^*$, there exists no equilibrium in which the state variables remain forever near their steady state values. It follows in addition that for most initial conditions there exist no equilibria converging asymptotically to the steady state values of the state variables, so that in this case the thesis of global asymptotic determinacy must be invalid.

Finally, it can be seen that in the present example, assuming that $Y' > 0$, case (i) occurs if $R' < 0$, case (ii) occurs if $R' > 0$ and $Y' < 1$, and case (iii) occurs if $R' > 0$ and $Y' > 1$.

Now in the standard neoclassical growth model, $Y(K) = F(K)$, the production function shared by all firms, and under the standard assumption of decreasing returns to scale, or equivalently constant returns with a fixed factor (inelastically supplied labor), one must have $F'' < 0$. The real rate of return in equilibrium is furthermore given by the marginal product of capital, $R(K) = F'(K)$. From this it follows that at the steady state equilibrium, $Y' = R = \beta^{-1} > 1$, and $R' = F'' < 0$, so that only case (i) is possible. (Since we have already shown in section I that in this model, both determinacy theses hold not just locally but globally, this must be the case.)

But increasing returns and externalities and/or imperfect competition allow other possibilities. I will discuss here the case of external increasing returns because of its simplicity, but Hammour (1988) shows that similar possibilities arise in the case of increasing returns internal to the firm combined with imperfect competition. Suppose that a given firm's production function is $y = F(k, K)$, where k is the capital used by that firm and K is the aggregate capital stock. Then the first order condition for optimal capital accumulation by each firm is $R = F_1(K, K)$, so that $R(K)$ is no longer the derivative of $Y(K) = F(K, K)$. If $F_2 < 0$ (which Hammour interprets as a congestion externality), it is possible to have $Y' < 1$ at the steady state despite the fact that one must have $R =$

$\beta^{-1} > 1$.[26] And the second order condition for optimal capital accumulation is $F_{11} < 0$, which no longer implies that one must have $R' < 0$. If one has external increasing returns despite decreasing returns for the individual firm, one may have $R' > 0$.[27] Hence all three cases are in general possible, and in particular case (ii) is possible, in which case sunspot equilibria exist in which fluctuations of the consumption and capital accumulation paths continue to occur forever in response to arbitrary random events.[28] Kehoe *et al* (1989) show that similar effects can result from the presence of distorting taxes in a growth model. In all of these examples, because they involve representative consumers, the existence of sunspot equilibria is independent of assumptions regarding the existence of contingent claims markets.

The sort of endogenous fluctuations that we have identified with determinate but unstable equilibrium dynamics are also inconsistent with assumptions of optimization and equilibrium, in at least certain special cases that are not so special as to be completely without interest. The case that has been most studied is that of deterministic optimal growth models, which is to say perfectly competitive representative consumer economies with decreasing returns to scale technologies. I have already shown in section I that in the case of a one-sector technology and additively separable preferences for the representative consumer, the thesis of global asymptotic determinacy is valid, ruling out determinate but

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[26] Imperfect competition can also drive a wedge between the real rate of return and the marginal product of capital. For example, excess capacity due to Chamberlinian competition between firms could also result in $R > Y'$.

[27] Alternatively, $R' > 0$ may result from increasing returns at the firm level, with an interior optimum existing for the capital accumulation decision of the firm despite $F_{11} > 0$ due to the fact that firms face downward-sloping demand curves.

[28] Equilibrium may be indeterminate and sunspot equilibria may exist even when the local dynamics are not "indeterminate" in the sense of case (ii). For example, even in the case of type (iii) local dynamics, all perfect foresight equilibria beginning near the steady state may diverge from the steady state and be attracted to an invariant circle that is "stable" in the sense of having a stable manifold that includes all points near it. In such a case, perfect foresight equilibrium is indeterminate and sunspot equilibria exist; see the discussion of the Diamond model in Woodford (1984). Hammour shows how examples of this kind can be constructed for a continuous time variant of this model using the Hopf bifurcation theorem; similar techniques (see, e.g., Reichlin (1986)) are available in the discrete time case. Similar cycles, with a similar implication for indeterminacy and sunspot equilibria, are shown to be a possible consequence of the externalities and increasing returns associated with a search technology for matching trading partners in Diamond and Fudenberg (1989).

unstable dynamics. I have also already indicated that on the other hand, deterministic equilibrium cycles and chaotic equilibrium dynamics are possible in the case of optimal growth models with multi-sector technologies. But even in the case of relatively general multi-sector technologies, stronmg conclusions are possible concerning asymptotic dynamics if the represenative consumer does not discount future consumption very much.

Consider an n-sector growth model, in which n distinct capital goods in addition to one or more consumption goods are produced using those same n capital goods together with fixed factors such as inelastically supplied labor. Let $V(k_t, k_{t+1})$ denote the maximum possible level of single period utility of consumption in period t by the representative consumer that is technologically feasible, given a vector of capital stocks k_t to use along with the fixed factors, and given that at least a vector k_{t+1} of capital goods must be produced. This function is defined on a set D of 2n-vectors (k_t, k_{t+1}) that represent technologically feasible possibilities for the evolution of the capital stocks. Then because competitive equilibrium must maximize the welfare of the representative consumer, given an initial vector of capital stocks k_0, the unique equilibrium allocation of resources corresponds to the sequence of capital stocks $\{k_t\}$ that maximizes $\sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1})$ subject to the constraint that $(k_t, k_{t+1}) \varepsilon D$ for all t, and given the initial condition k_0. The following result is of particular interest.

> <u>The Turnpike Theorem.</u> Let V be increasing in its first vector of arguments, decreasing in its second vector of arguments, and strictly concave, and let D be convex and compact. Then for given V and D, there exists a discount factor $\bar{\beta} < 1$ such that if the discount factor of the representative consumer lies in the interval $\bar{\beta} < \beta < 1$, the equilibrium dynamics are such that
>
> $$\lim_{t \to \infty} k_t = k^*$$
>
> where k^* is the steady state capital stock, regardless of the value of k_0.

This version of the theorem is due to Scheinkman (1976); related theorems were first proved for continuous time optimal growth models by McKenzie and Rockafeller (see

McKenzie (1986)). This result establishes a sense in which, if one believes that the case of greatest empirical relevance is that in which consumers do not discount the future very much, then the thesis of asymptotic global determinacy is a consequence of equilibrium theory, even for multi-sector economies. While the proof of the result involves many technicalities, the intuition is reasonably simple. A perpetually fluctuating path for the vector of capital stocks does not achieve as high an average value of the strictly concave objective function V, as does a constant vector of capital stocks which is near the long run average vector in the fluctuating case. Hence in order to maximize the long run average value of V, one would eventually move the vector of capital stocks to the value that maximizes V(k, k) and keep it there, regardless of the initial condition. In the case that one is maximizing a discounted sum, the initial condition matters, but if the discounting is sufficiently weak it is still optimal to move the vector of capital stocks asymptotically toward a constant vector, the value of which involves a correction for the value of β.

Despite the strength of this result, it is worth emphasizing that there are still many ways in which endogenous fluctuations can occur in an optimal growth model. One is to suppose that the rate of time discount is simply greater than is consistent with the turnpike property. Indeed, Boldrin and Montrucchio (1986) prove an "anti-turnpike" result, according to which any twice differentiable function κ mapping an n-dimensional compact, convex set into itself corresponds to the equilibrium dynamics generated by some n-sector optimal growth model satisfying the Scheinkman conditions, assuming that the discount factor can be chosen arbitrarily in the interval $0 < \beta < 1$. It should also be noted that the theorem says only that some $\bar{\beta} < 1$ exists for given V and D; this does not mean that for β arbitrarily close to 1, one cannot find a V and D satisfying the Scheinkman conditions for which the turnpike property would not hold. One simply needs to find a V and D for which $\bar{\beta}$ is even higher; for further discussion of why this is possible, see Boldrin and Woodford

(1990).[29] Finally, it should be noted that a $\bar{\beta} < 1$ need not exist if V is not strictly concave. While concavity is a standard assumption (albeit not the only case of possible empirical interest, as discussed above), strict concavity is a bit more special. In particular, in the case of a multi-sector technology with constant returns to scale, strict concavity does not hold in general, even though there are diminishing returns to capital in the sense that all sectors require a fixed factor, as long as the number of distinct capital goods exceeds the number of fixed factors, or the number of capital goods is as large as the number of fixed factors and utility is linear in consumption. This is the basis of a famous counter-example due to Weitzman (reported in Samuelson (1973)). Of course strict concavity can be achieved in any such example by introducing even a very small amount of decreasing returns to scale; but if the perturbation is small, the turnpike property will hold only for very low rates of time preference. Hence it is not clear how empirically unrealistic are the technology and preference specifications needed for endogenous cycles in optimal growth models.

Nonetheless, the known examples of really complex endogenous fluctuations in optimal growth models, i.e., the examples of chaotic dynamics[30], involve what seem to be extremely high rates of time preference. Furthermore, the reliance upon complications that become possible only in the case of multi-sector technologies may not be of practical relevance for business cycle theory, given that the kind of fluctuations one seeks to explain typically involve a large degree of co-movement between different types of investment, rather than cyclical variations in the type of capital goods that are accumulated. A more

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[29] Benhabib and Rustichini (1989) show how to construct explicit examples of continuous-time models with two capital goods sectors and a consumption sector (all Cobb-Douglas with constant returns) using the two capital goods and inelastically supplied labor, in which equilibrium limit cycles exist, for rates of time preference arbitrarily close to the rate of depreciation of capital. These examples do not satisfy the strict concavity condition needed for the turnpike theorem, for the reason discussed immediately below. Nonetheless perturbations of these examples that make V strictly concave should be possible that preserve the limit cycles while keeping the rate of time preference within a neighborhood of desired size of the depreciation rate.

[30] These are of greater interest than the examples of deterministic cycles, not only because observed aggregate fluctuations are not close to being exactly repetitive, but also because many of the examples of deterministic cycles simply establish the possibility of a bifurcation creating a deterministic cycle of very small amplitude in a neighborhood of the steady state. The examples of chaos necessarily involve fluctuations of a larger amplitude insofar as a greater degree of nonlinearity in the dynamics is required.

important qualification to the turnpike theorem may accordingly be that, like the sunspot irrelevance theorem, it has little relevance once one admits the existence of market imperfections of any of a variety of kinds.

All of the kinds of market imperfections just discussed in connection with sunspot equilibria are also conditions which can result in failure of the turnpike property.[31] I will here discuss only a very simple example, that shows how determinate but unstable dynamics, including possibly chaotic dynamics, can arise in a growth model with a simple one-sector technology, and even in the case of arbitrarily low rates of time preference on the part of all consumers, if one abandons the assumption of complete financial markets. Consider again the one-sector growth model of section I, but now suppose that there are two types of infinite lived consumers, "entrepreneurs" who alone are able to invest in capital, hire labor and organize production, and "workers" who alone supply labor. (Some sort of consumer heterogeneity must be introduced, or financial markets have no effect upon equilibrium.[32]) The assumption that consumers are heterogeneous does not in itself allow for more complex dynamics, if there are complete financial markets, for in that case a competitive equilibrium still must maximize a weighted sum of the utilities of consumption of the two consumer types.[33] But suppose also that entrepreneurs are unable to finance investment other than out of their own funds, owing to adverse selection or moral hazard problems.[34] Then the capital stock k_{t+1} carried into period t+1 can never be larger than the

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[31] See Boldrin and Woodford (1990). The models discussed above in which financial constraints result in a role for fiat money obviously allow for determinstic equilibria that cycle forever, given that they can generate price level dynamics that exactly replicate those of the Grandmont (1985) model. However, these are not examples of determinate but unstable dynamics; the equilibrium cycles are a manifestation of the indeterminacy of equilibrium, and equilibria converging to the steady state generally exist among others. The Hammour (1988a) example of deterministic motion on an invariant cycle, discussed in footnote 34 above, is also necessarily associated with indeterminacy of equilibrium, but this case does to some extent involve intrinsically unstable dynamics, insofar as for almost all initial conditions there are no equilibria converging to the steady state.

[32] Bewley (1986) shows how deterministic equilibrium cycles may be possible in a model with two infinite lived consumer types when no borrowing and lending are possible, where the heterogeneity has to do with the timing of the consumers' endowments.

[33] Bewley (1982) and Yano (1984) provide turnpike theorems for economies with multiple consumer types.

[34] Greenwald and Stiglitz (1988) argue for the importance of financial constraints of this kind in the generation and propagation of aggregate fluctuations. The generalized model discussed below, in which non-

wealth of the entrepreneurs, which in turn can never be larger than the gross returns to the existing capital stock, $k_t F'(k_t)$, where $F(k)$ denotes aggregate production given a capital stock k and the exogenous labor supply. Now if F is a sufficiently concave function (i.e., if there is not too much factor substitutability in production), $k F'(k)$ is decreasing in k, for large k. As a result, large values of k_t will result in conditions that force k_{t+1} to be low. One case that is particularly simple to analyze is that in which the entrepreneurs have logarithmic utility, in which case they consume exactly a fraction $(1-\beta)$ of their wealth each period, regardless of the expected return on savings. Then the capital stock evolves according to

$$k_{t+1} = \beta k_t F'(k_t)$$

In the event that F is sufficiently concave, this map has a graph of the kind shown in Figure 1, and can result in an unstable steady state and equilibrium paths that converge to a deterministic cycle or that are even chaotic. The conditions for this to occur in no way depend upon a low value of β for the entrepreneurs; indeed, for a given production function F, raising β makes the hump in Figure 1 steeper, making the steady state more unstable and allowing more complex fluctuations. Nor do they have anything to do with the rate of time preference of the workers.

In my (1989a) this example is extended to allow for endogenous labor supply and a competitive market for debt issued by entrpreneurs and held by workers. Complete financial markets still do not exist if one assumes the existence of firm-specific productivity shocks, that have no effect upon aggregate production possibilities because of the existence of a continuum of firms with independent shocks, and that cannot be insured against because their realization is private information. That is, entrepreneurs can finance investment by issuing straight debt securities, but cannot issue securities contingent upon the uncertain events that will affect the return upon that investment. Insolvency risk then limits the extent to which entrepreneurs are willing to leverage themselves in order to invest

contingent debt contracts are possible but not securities contingent upon firm-specific risk, coincides closely with their analysis.

in physical capital, even when the expected return on capital exceeds the real rate of return at which they can borrow. As a result entrepreneurial wealth continues to be an important determinant of the level of investment, making unstable dynamics possible in a similar way.

My overall conclusion about the two types of models of endogenous fluctuations is roughly the same. The most plausible conditions under which either sunspot equilibria or determinate but unstable dynamics can occur would seem to be conditions under which such equilibria are possible only because of market imperfections. As a result, endogenous fluctuations of these kinds, if they occur, will indicate an inefficient phenomenon and a flaw in the functioning of the competitive mechanism, even if it does not follow from that that an intervention that succeeds in eliminating the endogenous fluctuations must necessarily bring about an improvement. On the other hand, it is important to note that equilibria of these kinds can result from a variety of kinds of market imperfections that are often argued to be of importance for macroeconomics, and that -- once these imperfections are granted -- the existence of the endogenous fluctuations is fully consistent with optimizing behavior, rational expectations, and equilibrium. Whether these theoretical possibilities are of any practical importance in explaining aggregate fluctuations will, of course, depend upon the construction of examples that are not only logically coherent, but whose assumptions (including quantitative assumptions about parameter values) are empirically realistic and whose predictions match actual time series.[35]

[35]For crude but illustrative examples of discussion of the empirical realism of the parameter values required for endogenous fluctuations to exist, and comparison of quantitative properties of the predicted fluctuations to actual business cycles, see my (1988a, 1988b).

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