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ABSTRACT

This paper develops a novel method to estimate production functions. Earlier papers rely on special assumptions about the functional form of production functions. Our approach efficiently estimates all parameters of any production functions with Hicks-neutral productivity without additional exogenous variables or sources of variation in flexible input demand. We provide Monte Carlo Simulation evidence of our method's performance and test our approach on empirical data from Chilean and Colombian manufacturing industries.

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1 Introduction

Production functions are integral to economic models, and identifying their parameters has long been a major focus of applied economics research. Despite advancements improving estimation methods, robustly identifying all parameters of production functions remains largely unresolved (Olley and Pakes, 1996; Bond and Söderbom, 2005; Gandhi et al., 2020).

Similar to this paper, most of the recent production function estimation literature focuses on solving an endogeneity problem arising from productivity not being observed by the econometrician. If firms know their contemporaneous productivity before making their production decisions, both production inputs and outputs will generally be correlated with their productivity. Omitting productivity in estimations results in biased production function coefficient estimates, a problem known as the transmission or simultaneity bias. Even though there are many other potential concerns, such as non-random entry or exit when estimating production functions (Olley and Pakes, 1996), the transmission bias stands out as particularly challenging.

The control function approach, pioneered by Olley and Pakes (1996) and further developed by Petrin and Levinsohn (2012) and Ackerberg et al. (2015), aims at resolving the simultaneity problem by substituting unobserved productivity in the production function with an observable term in a two-stage estimation procedure. Its apparent simplicity and modest data requirements have made it popular in empirical microeconomics. However, over the past two decades, several scholars have identified both practical problems in its implementation and functional dependency issues leading to identification problems of production function parameters in various settings. Most strikingly, Bond and Söderbom (2005) and Gandhi et al. (2020) show that the commonly applied methodology cannot generally identify all parameters of gross-output production functions. Absent time-series variation in prices, the control function approach is only capable of identifying parameters of production functions excluding the proxy variable. Ackerberg et al. (2015), therefore, restrict the aim of their approach to reduced production functions excluding intermediate inputs.

In this paper, we develop a novel approach to the parametric estimation of production functions that is able to identify all parameters of gross-output production functions, including flexible inputs and their output elasticities. Our approach relies on the same general assumptions as the control function approach and does not need additional exogenous instruments, variation in a flexible input's demand, or additional data. Most notably, our approach does not rely on time-series variation in prices, which Gandhi et al. (2020) found to be a relatively weak source of identifying variation. Instead, we use the flexible input's demand function already implied by the model's assumptions to identify the parameters unidentifiable by the control function approach. We propose a comprehensive framework for estimating parametric production functions while demonstrating how to combine all moment conditions into a single constraint GMM estimator. The single-stage GMM procedure makes estimation more efficient and allows for the calculation of asymptotics-based standard errors. Our approach avoids regression-based polynomial approximations of unknown functions, as these approximations can reduce efficiency and lead to imprecise estimates in the presence of noisy data. We further show how our general approach can be applied to estimating parameters of the most commonly used production function forms such as Cobb-Douglas, CES, Translog, or the often assumed combination of Cobb-Douglas in capital and labor and Leontief in intermediate inputs. We further show how our approach extends to multiple flexible production inputs.

We provide evidence of our approach's performance in various scenarios based on simulated and real-world data. Monte Carlo simulations show that our approach is able to identify all production function and productivity's law of motion parameters with and without time-series variation in prices and price variation over firms. Furthermore, we demonstrate that the control function approach produces heavily biased estimates if the production function is falsely assumed to exclude the proxy variable but is instead of a Cobb-Douglas functional form. Applying our approach to real-world data from Chilean and Colombian manufacturing industries results in precise estimates of gross-output production function parameters.

This paper starts by formulating all necessary assumptions in Section 2 while revisiting the control function approach to production function estimation in Section 2.1. We proceed by presenting our alternative estimation approach in Section 2.2, followed by applications of the procedure to both simulated data in Section 3 and real-world data in Section 4. We then demonstrate how to extend the framework to multiple flexible inputs in Section 5. Section 6 relates our approach to previously developed methods in the literature. We discuss our findings and conclude in Section 7.

2 Two Estimation Approaches

A production function describes the technology a firm uses to transform production inputs, such as labor or intermediate inputs, into output. The most commonly used functional forms are Cobb-Douglas, constant elasticity of substitution (CES), or Translog functional form specifications. However, each functional form includes a general productivity or total factor productivity term (TFP) besides production inputs and parameters. As in most of the literature, we assume TFP to be Hicks-neutral, i.e., changes in TFP over time do not affect the relative composition of production inputs.[1] Assumption 1 summarizes this in general terms.

Assumption 1 *In each period t, firm j chooses to produce output $Q_{j,t}$ using capital $K_{j,t}$, labor $L_{j,t}$, and intermediate inputs $M_{j,t}$ under the production function*

$$F(K_{j,t}, L_{j,t}, M_{j,t}; \beta), \tag{1}$$

such that

$$Q_{j,t} = F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)\Omega_{j,t}, \tag{2}$$

with $F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)$ being differentiable in $M_{j,t}$. β is a vector of parameters describing output and possibly substitution elasticities, and $\Omega_{j,t}$ represents TFP. The econometrician does not observe $Q_{j,t}$ but

$$Y_{j,t} = Q_{j,t} e^{\epsilon_{j,t}}, \tag{3}$$

where $\epsilon_{j,t}$ is an i.i.d. random measurement error unanticipated by the firm, with $E[\epsilon_{j,t}] = 0$.

Firms observe their contemporary level of TFP, $\Omega_{j,t}$, and make their input and production decisions based on this information. However, we do not observe $\Omega_{j,t}$ in the data. Estimating the production

[1] See Doraszelski and Jaumandreu (2018) for an example of modelling approaches with non-Hicks-neutral or factor-augmenting productivity shocks.

function parameters while ignoring the connection between firms' decisions and TFP would cause the above-mentioned transmission bias (see Levinsohn and Petrin, 2003, for a derivation of the bias in the Cobb-Douglas case). Note that firms aim at producing output $Q_{j,t}$, not taking the unanticipated error $\epsilon_{j,t}$ into account, which is unobserved by both the firm and the econometrician. The produced output $Y_{j,t}$, observed in the data, includes the realization of this error term. Additionally, we presume that real values of input variables are present in the data.

We allow total factor productivity $\Omega_{j,t}$ to differ over firms and time. However, we assume productivity to be imperfectly persistent over time and that productivity shocks are exogenous and i.i.d. Assumption 2 formalizes this with lowercase letters representing logarithms of their uppercase versions.[2]

Assumption 2 *$\omega_{j,t}$ follows a first-order Markov process*

$$\omega_{j,t} = g(\omega_{j,t-1}, \rho_\omega) + \xi_{j,t}, \;\; \textit{with } E[\xi_{j,t}|\omega_{j,t-1}] = 0, \tag{4}$$

where $\xi_{j,t}$ represents a random exogenous productivity shock for firm j in period t. Neither $\omega_{j,t}$ nor $\xi_{j,t}$ are observed in the data, but they are known to the firm.

Inputs are typically divided into flexible and fixed inputs. We regard any input a firm can freely choose after observing its contemporary productivity level and whose choice is realized in the same period as *free inputs.* Their choices do not depend on their values from previous periods. Examples of such inputs likely to fulfill these assumptions include intermediate inputs or electricity. We define *fixed inputs* as inputs that the firm cannot choose freely after observing its current productivity level. This includes inputs we assume to be chosen in the same period but before observing contemporary productivity, such as labor, and inputs whose levels are chosen in prior periods but only materialize in the current one, such as fixed capital or machinery.[3] For comparability to previous approaches, we follow Gandhi et al. (2020) and assume intermediate inputs to be fully flexible, labor to be fixed, chosen in the current period but without knowledge of contemporary productivity, and capital to be fixed and chosen in the previous period. Assumptions 3-5 formally summarize the types of inputs and choice timings.

Assumption 3 *Capital input $K_{j,t}$ is fixed in the current period. It depreciates over time. Firms choose investment $I_{j,t}$ which affects the firm's capital input in period $t+1$.*

Assumption 4 *Labor is a fixed input. Firms can freely choose its level in the current period which is realized instantly. However, they do so before knowing their current period's productivity.*

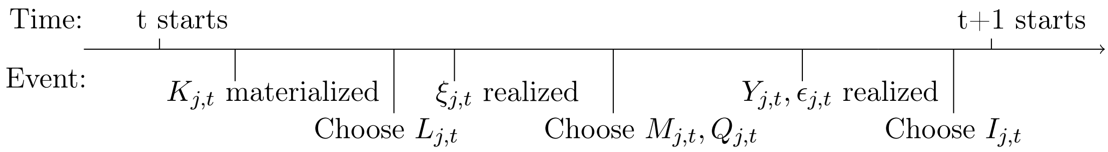
Assumption 5 *Intermediate inputs $M_{j,t}$ are flexible. Firms can freely choose their level in the current period after observing their current period's productivity, and their choice is immediately realized.*

[2]This assumption can be weakened by allowing $g(.)$ to include additional lagged values of $\omega_{j,t}$ or other additional variables. However, we stay with the simple first-order Markov process because most of the methodological and applied literature does not use higher orders. This should not impact the generality of the arguments behind identifying parameters in any of the presented estimators.

[3]Even though, depending on the context, one might assume labor to be flexible. However, especially in the European context, labor levels can often not be entirely freely chosen, as labor protection laws regulate the extent to which firms can lay off workers. We examine the case with fully flexible labor in the extensions in Section 5.

Assumptions 3-5 result in the firm's decision timing as depicted in Figure 1. A firm chooses the level of investment into fixed capital already in the period $t-1$. The resulting capital level is realized immediately at the beginning of the current period t. Based on this, the firm makes its labor input choice before the realization of its productivity shock $\xi_{j,t}$. Therefore, labor input depends on the previous period's productivity level and only on the expectation of the current period's productivity shock and not on its realization. After the contemporary productivity shock $\xi_{j,t}$ is realized and observed by the firm, it makes its intermediate inputs and output decisions at the same time before choosing its investment into the subsequent period's fixed capital.

Figure 1: Decision Timing



Our estimation framework can account for modifications in these assumptions as we show in Section 5 for the case of two flexible inputs chosen with full knowledge of $\xi_{j,t}$.

We summarize all relevant information the firm possesses at the time it makes its flexible input choices and the output choice in an information set denoted as $\mathbb{I}_{j,t}$, formally defined in the following assumption.

Assumption 6 *Set $\mathbb{I}_{j,t}$ describes firm j's information set in period t after the productivity shock $\xi_{j,t}$ is realized. It includes all previous realizations of the firm's state variables, but it does not contain any information about future realizations of any variable.*

Note that the realization of the random error $\epsilon_{j,t}$ in period t is not in the information set of the same period, but $\Omega_{j,t}$ and $\xi_{j,t}$ are included.

The last two assumptions needed in both approaches presented in this paper are that firms are profit maximizers when choosing the flexible input $M_{j,t}$ (Assumption 7) and that firms cannot influence prices in either input or output markets (Assumption 8). We presume that the econometrician observes output prices and flexible input prices for each firm in each period.[4] Note that firms choose the levels of capital and labor before choosing intermediate inputs. Therefore, firms can only choose intermediate inputs when choosing contemporary profits and $K_{j,t}, L_{j,t} \in \mathbb{I}_{j,t}$, meaning that firms make their intermediate inputs choices conditional on $K_{j,t}$ and $L_{j,t}$.

Assumption 7 *Firms choose their flexible production input $M_{j,t}$ to maximize their contemporary profits $\Pi_{j,t}$, based on $\mathbb{I}_{j,t}$,*

$$\Pi_{j,t} = P^Y_{j,t} Q_{j,t} - P^M_{j,t} M_{j,t}, \tag{5}$$

with $P^Y_{j,t}$ representing the price for the firm's output and $P^M_{j,t}$ being the price for intermediate inputs for firm j in period t.

[4] If prices are not available, one could replace them with industry input and output price indices if common prices across firms is a reasonable assumption in the specific setting.

Assumption 8 *Firms cannot influence prices in input- and output markets and take their levels as given.*

Assumption 8 is necessary because if firms could influence input or output prices, their input and output decisions would depend on price elasticities and potentially competitors' decisions. We narrow our focus by making Assumption 8 to allow for relatively low data requirements and a concise exhibition of the core mechanisms of the approaches.

2.1 The Control Function Approach

The control function approach, as described in Ackerberg et al. (2015), does not directly look at the production function as described above. The control function approach instead uses the demand function for one perfectly flexible input targeted as the proxy variable in the first estimation stage and makes a monotonicity assumption:

Assumption 9 *The intermediate inputs demand function is given by*

$$M_{j,t} = D_t^M(K_{j,t}, L_{j,t}, \Omega_{j,t}; \beta). \tag{6}$$

and is strictly monotone in its only unobservable argument, productivity $\Omega_{j,t}$.

The strict monotonicity assumption implies that there is a unique $\Omega_{j,t}$ given $M_{j,t}, K_{j,t}$ and $L_{j,t}$.

We define the production function to be a function that has all inputs as arguments. Instead the control function approach assumes a particular class of functional forms for the production function, which allows output $Q_{j,t}$ to also be represented as a function solely containing labor $L_{j,t}$, capital $K_{j,t}$, and a vector of parameters β as arguments. In order to describe the control function approach we need to add Assumption 10.

Assumption 10 *Firm output* $Q_{j,t}$ *can be expressed in a reduced form,*

$$Q_{j,t} = \Psi(K_{j,t}, L_{j,t}; \beta)\Omega_{j,t}, \tag{7}$$

with β *being a vector of parameters describing output and possibly substitution elasticities and* $\Omega_{j,t}$ *representing TFP.*

We call Ψ the reduced production function. The relation between Ψ, F, and $M_{j,t}$ is

$$Q_{j,t} = F(K_{j,t}, L_{j,t}, D_t^M(K_{j,t}, L_{j,t}, \Omega_{j,t}; \beta); \beta)\Omega_{j,t}. \tag{8}$$

The existence of a reduced production function Ψ requires some kind of separability in F which allows one to separate the part of the function involving the fixed inputs from the flexible one and to only focus on the first part. An often used example of this is to assume that the underlying production function is Leontief in the flexible input and of any other functional form for the remaining inputs. We follow Ackerberg et al. (2015) in our description of the control function approach, while the approach has been implemented in slightly different ways by other scholars with mostly the same key assumptions (Olley and Pakes, 1996; Levinsohn and Petrin, 2003; De Loecker and Warzynski, 2012). However, Bond and Söderbom (2005) and Gandhi et al. (2020) show that none of the implementations of the control function approach can identify all parameters of production functions when intermediate inputs cannot be separated from Ψ.[5]

Assumption 9 is generally redundant in presence of assumptions 10, 3, 4, 5, 7, and 8, because they already imply the functional form of any flexible input demand function. However, if the application does not allow to assume that firms choose at least one flexible input in order to maximize profits, and Assumption 7 has to be dropped, it states the crucial condition (monotonicity and scalar unobservability) for the control function approach to work.

First Stage

The control function approach's first estimation stage incorporates a proxy variable for the unobserved productivity term. This idea is the key feature of the control function approach and has been implemented in various modified settings (Olley and Pakes, 1996; Levinsohn and Petrin, 2003; De Loecker and Warzynski, 2012; Ackerberg et al., 2015).

Ackerberg et al. (2015) use the definition of the intermediate demand function in Equation 9 and invert it to get an expression for unobserved productivity as a function of observed variables and parameters, i.e.

$$\Omega_{j,t} = D_t^{M^{-1}}(K_{j,t}, L_{j,t}, M_{j,t}; \beta). \tag{9}$$

Substituting Equation 9 into the production function from Equation 7 and taking the natural logarithm leads to an expression that only contains observable variables, target parameters, and the i.i.d error term,

$$y_{j,t} = \psi(k_{j,t}, l_{j,t}; \beta) + d_t^{M^{-1}}(k_{j,t}, l_{j,t}, m_{j,t}; \beta) + \epsilon_{j,t} \equiv \Theta_{j,t}(k_{j,t}, l_{j,t}, m_{j,t}; \beta) + \epsilon_{j,t}, \tag{10}$$

where lowercase letters represent the natural logarithm of their uppercase counterparts. Ackerberg et al. (2015) argue that none of the parameters of interest can generally be identified from equation 10 because there is no variation in the data that allows to separately identify the inverse demand function from the production function as the same variables are present in both functions. However, approximating the composite term $\Theta_{j,t}$ in Equation 10 allows to estimate $\epsilon_{j,t}$ and to form an expression that includes information about $\omega_{j,t}$. Ackerberg et al. (2015) do so by regressing the logarithm of output $y_{j,t}$ onto a low-degree polynomial of the natural logarithms of production inputs.[6] Differencing the logarithm of the production function from the estimate of $\Theta_{j,t}(k_{j,t}, l_{j,t}, m_{j,t}; \beta)$ allows them to find an expression of $\omega_{j,t}$ that only contains observables and the parameters of interest,

$$\omega_{j,t} = \hat{\Theta}(k_{j,t}, l_{j,t}, m_{j,t}; \beta) - \psi(k_{j,t}, l_{j,t}; \beta). \tag{11}$$

Note that we cannot calculate $\omega_{j,t}$ from this expression alone because we do not yet have estimates for β. Instead, we have an expression which we can use to calculate $\omega_{j,t}$ for any vector of values for β. This is an important part for the GMM estimator in the second stage.

[5] See Gandhi et al. (2017) for a detailed examination.

[6] Levinsohn and Petrin (2003) use a locally weighted non-parametric regression approach instead of the more often implemented polynomial regression approach described in Olley and Pakes (1996) or Ackerberg et al. (2015).

Second Stage

The second estimation stage is a general methods of moments (GMM) estimator that identifies the production function parameter vector β and the parameters governing the law of motion of productivity $\omega_{j,t}$. It aims to identify all structural parameters of the production function using the expression for $\omega_{j,t}$ derived in the previous step and the law of motion over time for $\omega_{j,t}$ from Assumption 2. By Assumption 2 contemporary productivity shocks $\xi_{j,t}$ are independent from previous realizations of any variable and, therefore, $E[\xi_{j,t}|\mathbb{I}_{j,t-1}] = 0$ should hold. This implies a set of orthogonality conditions which Ackerberg et al. (2015) use to form moments conditions. Because a firm's choice of a fixed input has been made prior to the realization of $\xi_{j,t}$ and by assumption $\xi_{j,t}$ is independent from any previously realized variable, $E[x_{fixed;j,t} \cdot \xi_{j,t}] = 0$ should be a valid moment. However, firms' choices of flexible inputs take the contemporary productivity shock $\xi_{j,t}$ into account. Therefore, the same orthogonality condition should not hold for flexible inputs. Ackerberg et al. (2015) propose to use lagged choices of flexible inputs to form the moment condition $E[x_{flex;j,t-1} \cdot \xi_{j,t}] = 0$ instead. The last set of moments concerns the law of motion for productivity. Assumption 2 states that $\xi_{j,t}$ is not just independent from previous input choices but also from previous realizations of $\omega_{j,t}$. Therefore, $E[\omega_{j,t-1} \cdot \xi_{j,t}] = 0$ should be a valid moment condition. For a Cobb-Douglas functional form, the moment conditions to estimate the production function and productivity law of motion parameters would be

$$\mathbb{E}\begin{bmatrix} \xi_{j,t} \\ K_{j,t}\xi_{j,t} \\ L_{j,t}\xi_{j,t} \\ \omega_{j,t-1}\xi_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

This framework produces consistent results if all assumptions are valid. However, its estimates can become severely biased, if the true functional form in the data generating process (DGP) does not allow the reduced-form representation as we show in Section 3. Additionally, the polynomial approximation in the first estimation step can introduce substantial approximation errors if a non-suitable degree of the polynomial approximation is chosen, if data is noisy, or not properly distributed.

2.2 A New Approach to Gross Output Production Function Estimation

This section lays out a new approach for the parametric estimation of gross output production functions (TJ approach hereafter), which is the main contribution of this paper. The approach relies on the same assumptions as the control function approach with the only exception that instead of restricting the procedure to the class of production functions in Assumption 10, our approach allows for any production function fulfilling Assumption 1. Furthermore, it does not depend on the availability of additional data. Instead, we exploit that firms choose flexible inputs in each period conditional on values of fixed inputs and exogenous variables. The choices of flexible input levels do not depend on previous periods' choices of the same input and, therefore, have no dynamic implications. This property leads to a particularly simple demand function which we can always solve for unobserved productivity under Assumptions 1-8.

The level of intermediate inputs $M_{j,t}$ is the only choice variable in the firm's profit maximization problem after the firm observes its contemporary productivity shock because by Assumptions 3 and 4 the firm already chose the current period's levels of both labor $L_{j,t}$ and capital $K_{j,t}$. We show in

Section 5 how this timing assumption can be altered to allow simultaneous choices of intermediate inputs and labor after observing productivity. With these timing assumptions in mind, we can write the firm's profit maximization problem after the current period's productivity has been realized as

$$\max_{M_{j,t}} \pi_{j,t} = P^Y_{j,t} Q_{j,t} - P^M_{j,t} M_{j,t}. \tag{13}$$

Substituting the production function from Equation 2 in Equation 13 and taking the partial derivative with respect to $M_{j,t}$ gives us the FOC for the profit maximization problem,

$$\frac{\partial \pi(K_{j,t}, L_{j,t}, M_{j,t}, \Omega_{j,t}, P^Y_{j,t}, P^M_{j,t}; \beta)}{\partial M_{j,t}} = P^Y_{j,t} \Omega_{j,t} \frac{\partial F(K_{j,t}, L_{j,t}, M_{j,t}, P^Y_{j,t}, P^M_{j,t}; \beta)}{\partial M_{j,t}} - P^M_{j,t} = 0. \tag{14}$$

Using this FOC, we can express unobserved productivity as a known function of observed variables and parameters we aim to estimate, i.e.,

$$\Omega_{j,t} = \frac{P^M_{j,t}}{P^Y_{j,t} \frac{\partial F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)}{\partial M_{j,t}}}. \tag{15}$$

Note that the control function approach in Section 2.1 does not include prices in its definition of the intermediate inputs demand function in Equation 6 (and, therefore, also in its inverse in Equation 9). Omitting prices in this demand function is generally only valid if the production function is Leontief in intermediate inputs, which leads to the firm always setting intermediate inputs levels as stated in Equation 42 in Section 2.2.4 no matter the price for intermediate inputs. We can evaluate the expression in Equation 15 with any vector of values for β for any specified functional form of the production function included in Assumption 1 that has a defined partial derivative for intermediate inputs. This allows us to use it as a constraint in the constrained GMM estimation procedure. Even if the production function had an assumed functional form that included $\Omega_{j,t}$ in F, therefore violating Assumption 1 and additionally would not allow for solving the first order condition in Equation 14 for $\Omega_{j,t}$ analytically, one could still calculate $\Omega_{j,t}$ by solving this implicit equation numerically as long as it is monotone in productivity.

Furthermore, the TJ approach replaces $\Omega_{j,t}$ in Equation 2 with Equation 15 to express the production function for an optimal choice of $M_{j,t}$ without unobserved productivity

$$Y_{j,t} = F(K_{j,t}, L_{j,t}, M_{j,t}; \beta) \frac{P^M_{j,t}}{P^Y_{j,t} \frac{\partial F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)}{\partial M_{j,t}}} e^{\epsilon_{j,t}}. \tag{16}$$

The only unobserved term in this optimized production function is $\epsilon_{j,t}$. Taking the natural logarithm and rearranging terms allows us to solve the expression in Equation 16 for the unobserved random error,

$$\epsilon_{j,t} = y_{j,t} - p^M_{j,t} + p^Y_{j,t} + ln \left(\frac{\partial F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)}{\partial M_{j,t}} \right) f(k_{j,t}, l_{j,t}, m_{j,t}; \beta), \tag{17}$$

which the TJ approach utilizes as a second constraint in the GMM optimization problem.

We need at least as many moment conditions as parameters for our GMM estimator to identify all parameters. The exact set of moment conditions used in the estimation depends on the functional form of the production function. However, we can use $\epsilon_{j,t}$'s zero mean property in a first set of moment conditions for flexible inputs as $\epsilon_{j,t}$ is not part of the information set $\mathbb{I}_{j,t}$ when the firm chooses inputs. A second set of moment conditions relies, similarly to the second stage of the control function approach, on timing assumptions of fixed inputs and the development of productivity. Using the law of motion for productivity, we can utilize the orthogonality of $\xi_{j,t}$ and fixed inputs in a second moment conditions set. $\xi_{j,t}$ is by assumption orthogonal to any fixed input because its realization is not in the information set when the firm chooses fixed inputs. The third set uses the assumption that $\xi_{j,t}$ is independent of prior realizations of productivity to identify $\Omega_{j,t}$'s law of motion parameters.

Let $A(X;\theta)$ represent the empirical equivalents of the sets of moment conditions with θ being the vector of target parameters, and X being a matrix of data. The general constrained GMM problem of the TJ approach for J firms and T periods with weighting matrix W is then

$$\begin{aligned}
\underset{\theta}{\arg\min} \quad & J(X,W;\theta) = (T\cdot J)A(X;\theta)^T W A(X;\theta) \\
s.t. \quad & \Omega_{j,t} = \frac{P^M_{j,t}}{P^Y_{j,t}\frac{\partial F(K_{j,t},L_{j,t},M_{j,t};\beta)}{\partial M_{j,t}}}, \\
& \xi_{j,t} = \omega_{j,t} - g(\omega_{j,t-1};\rho_\omega), \\
& \epsilon_{j,t} = y_{j,t} - p^M_{j,t} + p^Y_{j,t} + ln\left(\frac{\partial F(K_{j,t},L_{j,t},M_{j,t};\beta)}{\partial M_{j,t}}\right) f(k_{j,t},l_{j,t},m_{j,t};\beta), \quad \begin{matrix}\forall t=1,...,T, \\ \forall j = 1,...,J\end{matrix}.
\end{aligned} \tag{18}$$

The following sections describe our approach in detail for specific functional forms of the production function.

2.2.1 Case: Cobb-Douglas Production Function

In this section, we take our general estimation approach laid out in the previous section and describe in detail how it can be applied to the special case of a gross output production function with a Cobb-Douglas functional form. The Cobb-Douglas production function is of particular interest because it is one of the most widely used representation of a production function in all fields of empirical economics (Mankiw et al., 1992; Romer, 1990; Hall and Jones, 1999; Hsieh and Klenow, 2009; De Loecker et al., 2020) and is often employed in the description of new estimation techniques (Olley and Pakes, 1996; Levinsohn and Petrin, 2003; Ackerberg et al., 2015). The Cobb-Douglas production function takes the following form.

Cobb-Douglas Case: *In each period t, firm j chooses to produce output $Q_{j,t}$ using production inputs capital $K_{j,t}$, labor $L_{j,t}$, and intermediate inputs $M_{j,t}$ under the production function*

$$F(K_{j,t},L_{j,t},M_{j,t};\beta) = \beta_0 K^{\beta_k}_{j,t} L^{\beta_l}_{j,t} M^{\beta_m}_{j,t}. \tag{19}$$

Replacing the partial derivative of the production function in Equation 15 with the derivative of the Cobb-Douglas production function leads to the following expression for $\Omega_{j,t}$ as a function of observable variables and production function parameters

$$\Omega_{j,t} = \frac{P^M_{j,t}}{P^Y_{j,t}\beta_0\beta_m K^{\beta_k}_{j,t} L^{\beta_l}_{j,t} M^{\beta_m - 1}_{j,t}}. \tag{20}$$

We then replace $\Omega_{j,t}$ in the Cobb-Douglas production function from Assumption 2.2.1 with its expression from Equation 20, take the natural logarithm, and rearrange terms. This leads to a simple expression for the unobserved unanticipated error term $\epsilon_{j,t}$ which we can use as the Cobb-Douglas version of the general constraint in Equation 17 above for the GMM minization problem,

$$\epsilon_{j,t} = ln(\beta_m) - ln\left(\frac{P_M M_{j,t}}{P_Y Y_{j,t}}\right). \tag{21}$$

We need to define a set of moment conditions for the GMM estimator. In total, we have five parameters to estimate if we assume that the law of motion for productivity has the following linear form $\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}$. We, therefore, need at least five moment conditions for the model to be identified. The first moment condition relies on the unanticipated error term in the production function $\epsilon_{j,t}$, which has by assumption an expected value of zero. We can, therefore, use $E[\epsilon_{j,t}] = 0$ as the first moment condition. The rest of the moments rely on the productivity shock $\xi_{j,t}$. The obvious candidate for a moment condition is $E[\xi_{j,t}] = 0$ because the productivity shock has an expectation of zero by assumption. The level of the firm's capital is fixed in the current period and should be orthogonal to the contemporary realization of $\xi_{j,t}$ as $\xi_{j,t}$ is not part of the firm's information set $\mathbb{I}_{j,t-1}$ when choosing $K_{j,t}$ through investments in $t-1$. This leads to the moment condition $E[K_{j,t}\xi_{j,t}] = 0$. The choice of the labor input should also be independent from the contemporary productivity shock because firms choose labor before $\xi_{j,t}$ is realized excluding it from the information set $\mathbb{I}_{j,t-1}$. We can use this argument to form the third moment condition $E[L_{j,t-1}\xi_{j,t}] = 0$. The last moment condition utilizes the independence of a prior realization of productivity $\omega_{j,t-1}$ and the contemporary productivity shock $\xi_{j,t}$ to form the last moment condition $E[\omega_{j,t-1}\xi_{j,t}] = 0$. Combined we have the moment vector

$$\mathbb{E}\begin{bmatrix} \epsilon_{j,t} \\ \xi_{j,t} \\ K_{j,t}\xi_{j,t} \\ L_{j,t}\xi_{j,t} \\ \omega_{j,t-1}\xi_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

with the empirical moment equivalents

$$A(X;\theta) = \begin{pmatrix} \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} \epsilon_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} \xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} K_{j,t}\xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} L_{j,t}\xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} \omega_{j,t-1}\xi_{j,t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

where $\theta = (\beta_0, \beta_k, \beta_l, \beta_m, \rho_\omega)$. We can use the empirical equivalents of the moment conditions to define the constrained GMM minimization problem as

$$\begin{aligned} \arg\min_{\theta} \quad & J(X,W;\theta) = (T \cdot J)A(X;\theta)^T W A(X;\theta) \\ s.t. \quad & \omega_{j,t} = p^M_{j,t} - p^Y_{j,t} - ln(\beta_0) - ln(\beta_m) - \beta_k k_{j,t} - \beta_l l_{j,t} - (\beta_m - 1)m_{j,t}, \\ & \xi_{j,t} = \omega_{j,t} - \rho_\omega \omega_{j,t-1}, \\ & \epsilon_{j,t} = ln(\beta_m) - ln\left(\frac{P_M M_{j,t}}{P_Y Y_{j,t}}\right) \qquad \begin{matrix} \forall t = 1, ..., T, \\ \forall j = 1, ..., J \end{matrix}, \end{aligned} \tag{24}$$

where W represents a weighting matrix and the constraints are the functional restrictions defined by the model above.

One can implement the constrained GMM either by giving the minimzation problem and the constraints to a constrained optimizer available for most statistical software or by substituting in the constrains in the formulation of the criterion function itself, which is done more often in microeconomic applications.

2.2.2 Case: Translog Production Function

The Transcendental Logarithmic (Translog) production function is another popular functional form because it can be interpreted as a polynomial approximation of any underlying production function and it nests other widely used functional forms such as the Cobb-Douglas production function as special cases (Kmenta, 1967; Griliches and Ringstad, 1971). The number of parameters depends on the assumed degree of the polynomial and it is formally defined as

Translog Case: *In each period t, firm j chooses to produce output $Q_{j,t}$ using production inputs capital $K_{j,t}$, labor $L_{j,t}$, and intermediate inputs $M_{j,t}$ under the production function*

$$F(K_{j,t}, L_{j,t}, M_{j,t}; \beta, \iota) = exp\left(\sum_{0 \leq k+l+m \leq \iota} \beta_{k,l,m} k_{j,t}^{k} l_{j,t}^{l} m_{j,t}^{m}\right), \tag{25}$$

with ι representing the degree of the polynomial.

To keep notation concise we focus in the derivations on a Translog production function of degree two, which is in its logarithmic form,

$$\begin{aligned} F(K_{j,t}, L_{j,t}, M_{j,t}; \beta, \iota = 2) =& exp(\beta_{1,0,0}k_{j,t} + \beta_{0,1,0}l_{j,t} + \beta_{0,0,1}m_{j,t} + \beta_{2,0,0}k_{j,t}^2 + \beta_{0,2,0}l_{j,t}^2 + \beta_{0,0,2}m_{j,t}^2 \\ &+ \beta_{1,1,0}k_{j,t}l_{j,t} + \beta_{1,0,1}k_{j,t}m_{j,t} + \beta_{0,1,1}l_{j,t}m_{j,t}). \end{aligned} \tag{26}$$

Analogously to the previous case, we use the first order conditions of intermediate inputs from the firm's profit maximization problem after it observes its current period's productivity shock. Solving the first order condition for $\Omega_{j,t}$ results in an expression for productivity that only includes observed variables and parameters,

$$\begin{aligned} \Omega_{j,t} =& \frac{P_{j,t}^{M}(\beta_{0,0,1} + \beta_{1,0,1}k_{j,t} + \beta_{0,1,1}l_{j,t} + 2\beta_{0,0,2}m_{j,t})^{-1}}{exp(\beta_{1,0,0}k_{j,t} + \beta_{0,1,0}l_{j,t} + \beta_{0,0,1}m_{j,t} + \beta_{2,0,0}k_{j,t}^2 + \beta_{0,2,0}l_{j,t}^2 + \beta_{0,0,2}m_{j,t}^2)} \\ & \overline{exp(\beta_{1,1,0}k_{j,t}l_{j,t} + \beta_{1,0,1}k_{j,t}m_{j,t} + \beta_{0,1,1}l_{j,t}m_{j,t})}. \end{aligned} \tag{27}$$

Inserting the expression from Equation 27 into the production function from Equation 25 leads to

$$Y_{j,t} = \frac{M_{j,t}P_{j,t}^{M}}{(\beta_{0,0,1} + \beta_{1,0,1}k_{j,t} + \beta_{0,1,1}l_{j,t} + 2\beta_{0,0,2}m_{j,t})P_{j,t}^{Y}} e^{\epsilon_{j,t}}, \tag{28}$$

a function of observables and the error term $\epsilon_{j,t}$ as the only unobservable variable. Similar to the previous case, we can rearrange Equation 28 to get the following expression for the error term that only depends on observables and parameters,

$$\epsilon_{j,t} = ln\left(\frac{Y_{j,t}P_{j,t}^{Y}(\beta_{0,0,1} + \beta_{1,0,1}k_{j,t} + 2\beta_{0,0,2}m_{j,t} + \beta_{0,1,1}l_{j,t})}{P_{j,t}^{M}M_{j,t}}\right). \tag{29}$$

The TJ approach again uses this expression to calculate $\epsilon_{j,t}$ for any vector of parameter values and, therefore, use it as a constraint and utilize the property $E[\epsilon_{j,t}] = 0$ from Assumption 1 in the moment conditions. We have in total nine production function parameters and as many parameters as the law of motion for productivity involves. We, therefore, need at least the same amount of moment conditions for our estimator to identify all parameters. The first set of moments involves the unanticipated error term $\epsilon_{j,t}$, which is realized after the firm makes all of its contemporary input and output decisions excluding it from the firm's information set $\mathbb{I}_{j,t}$. $\epsilon_{j,t}$ should, therefore, be orthogonal to any of the production inputs allowing us to use the following moment conditions

$$\mathbb{E}\begin{bmatrix} M_{j,t}\epsilon_{j,t} \\ M_{j,t}^2\epsilon_{j,t} \\ M_{j,t}K_{j,t}\epsilon_{j,t} \\ M_{j,t}L_{j,t}\epsilon_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

Following the same procedure as in the previous case, we assume the law of motion for $\omega_{j,t}$ to be the following AR(1) process, $\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}$. Using timing assumptions for capital, and labor we can again utilize orthogonality conditions for $\xi_{j,t}$ with fixed production inputs arising from $\xi_{j,t}$ not being present in the relevant information set $\mathbb{I}_{j,t-1}$ when choosing flexible inputs. Additionally, $\xi_{j,t}$ is by assumption orthogonal to previous realizations of productivity. The additional moment conditions are

$$\mathbb{E}\begin{bmatrix} K_{j,t}\xi_{j,t} \\ L_{j,t}\xi_{j,t} \\ K_{j,t}^2\xi_{j,t} \\ L_{j,t}^2\xi_{j,t} \\ K_{j,t}L_{j,t}\xi_{j,t} \\ \omega_{j,t}\xi_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{31}$$

Both moment vectors combined give us the minimum number of moment conditions necessary to identify all parameters.[7] The rest follows analogously to the previous case. The empirical equivalent to all combined moment conditions is

$$A(X;\theta) = \begin{pmatrix} \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} K_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} L_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} K_{j,t}^2\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} L_{j,t}^2\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} K_{j,t}L_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} M_{j,t}\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} M_{j,t}^2\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} M_{j,t}K_{j,t}\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} M_{j,t}L_{j,t}\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} \omega_{j,t}\xi_{j,t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

Using these moments, the full constrained GMM minimization problem for the degree two Translog pro-

[7] If one would like to include higher order terms for past productivity or additional variables the law of motion for productivity, the orthogonality between them and $\xi_{j,t}$ can be used to form further necessary moment conditions.

duction function estimation procedure with $\theta = (\beta_{1,0,0}, \beta_{0,1,0}, \beta_{0,0,1}, \beta_{2,0,0}, \beta_{0,2,0}, \beta_{0,0,2}, \beta_{1,1,0}, \beta_{1,0,1}, \beta_{0,1,1}, \rho_\omega)$ is

$$\begin{aligned}
\arg\min_{\theta} \quad & J(X,W;\theta) = (T\cdot J)A(X;\theta)^T W A(X;\theta) \\
s.t. \quad & \Omega_{j,t} = \frac{P^M_{j,t}(\beta_{0,0,1} + \beta_{1,0,1}k_{j,t} + \beta_{0,1,1}l_{j,t} + 2\beta_{0,0,2}m_{j,t})^{-1}}{exp(\beta_{1,0,0}k_{j,t} + \beta_{0,1,0}l_{j,t} + \beta_{0,0,1}m_{j,t} + \beta_{2,0,0}k^2_{j,t} + \beta_{0,2,0}l^2_{j,t} + \beta_{0,0,2}m^2_{j,t})} \\
& \overline{exp(\beta_{1,1,0}k_{j,t}l_{j,t} + \beta_{1,0,1}k_{j,t}m_{j,t} + \beta_{0,1,1}l_{j,t}m_{j,t})} \\
& \xi_{j,t} = \omega_{j,t} - \rho_\omega \omega_{j,t-1}, \\
& \epsilon_{j,t} = ln\left(\frac{Y_{j,t}P^Y_{j,t}(\beta_{0,0,1} + \beta_{1,0,1}k_{j,t} + 2\beta_{0,0,2}m_{j,t} + \beta_{0,1,1}l_{j,t})}{P^M_{j,t}M_{j,t}}\right), \qquad \begin{matrix}\forall t = 1,...,T, \\ \forall j = 1,...,J\end{matrix}.
\end{aligned} \tag{33}$$

2.2.3 Case: Constant Elasticity of Substitution Production Function

The third often used functional form of a production function is the constant elasticity of substitution (CES) form. The CES production function incorporates the Cobb-Douglas and Leontief forms as a special case and introduces the additional parameter σ describing the elasticity of substitution between the inputs. The β parameters describe the shares of the inputs on production and sum up to one. We define the CES production function as

Constant Elasticity of Substitution Case: *In each period t, firm j chooses to produce output $Y_{j,t}$ using production inputs capital $K_{j,t}$, labor $L_{j,t}$, and intermediate inputs $M_{j,t}$ under the production function*

$$Y_{j,t} = \left[\beta_k K^\sigma_{j,t} + \beta_l L^\sigma_{j,t} + (1-\beta_k-\beta_l)M^\sigma_{j,t}\right]^{\frac{1}{\sigma}} \Omega_{j,t} e^{\epsilon_{j,t}}, \tag{34}$$

where β_k and β_l are the input shares of capital and labor, and σ is a measure of the substitutability of capital, labor, and intermediate inputs.

Similar to the previous cases, we can replace the general version of the partial derivative of the production function for intermediate inputs $M_{j,t}$ in Equation 15 with the partial derivative of the CES production function and solve for productivity to get $\Omega_{j,t}$ as a function of observables and the parameters we aim to identify,

$$\Omega_{j,t} = \frac{M^{1-\sigma}_{j,t}(\beta_k K^\sigma_{j,t} + \beta_l L^\sigma_{j,t} - (\beta_k+\beta_l-1)M^\sigma_{j,t})^{\frac{\sigma-1}{\sigma}} P^M_{j,t}}{P^Y_{j,t}(\beta_k+\beta_l-1)}. \tag{35}$$

We then replace unobserved productivity in the production function with its expression from Equation 35. Rearranging terms leads to an expression for observed output that only includes one unobservable $\epsilon_{j,t}$, observed variables, and parameters,

$$Y_{j,t} = \frac{M^{1-\sigma}_{j,t}(\beta_k(M^\sigma_{j,t} - K^\sigma_{j,t}) + \beta_l(M^\sigma_{j,t} - L^\sigma_{j,t}) - M^\sigma_{j,t})P^M_{j,t}}{P^Y_{j,t}(\beta_k+\beta_l-1)} e^{\epsilon_{j,t}}. \tag{36}$$

We can use this expression to form a moment condition based on the orthogonality assumption of the unanticipated error term $\epsilon_{j,t}$. Solving Equation 36 for $\epsilon_{j,t}$ gives us an expression for the error term based on observables and parameters,

$$\epsilon_{j,t} = ln\left(\frac{P^Y_{j,t}(\beta_k + \beta_l - 1)}{M^{1-\sigma}_{j,t}(\beta_k(M^\sigma_{j,t} - K^\sigma_{j,t}) + \beta_l(M^\sigma_{j,t} - L^\sigma_{j,t}) - M^\sigma_{j,t})P^M_{j,t}}\right). \tag{37}$$

This expression allows us to calculate the error term for any parameter vector, again enabling us to use it as a constraint and to utilize it in combination with $E[\epsilon_{j,t}] = 0$ from Assumption 1 to form moment conditions in our GMM procedure.

Apart from the obvious moment condition choice of $E[\epsilon_{j,t}] = 0$, we need at least two more moment conditions to estimate all production function parameters and as many more as there are parameters in the law of motion for productivity. Defining the parametric form of the law of motion for $\omega_{j,t}$ as before to be $\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}$, we can use the same moment conditions as in the Cobb-Douglas case in Section 2.2.1 above. Combining all moment conditions we use the following moment vector for the CES production function GMM estimator

$$\mathbb{E}\begin{bmatrix}\epsilon_{j,t}\\ K_{j,t}\xi_{j,t}\\ L_{j,t}\xi_{j,t}\\ \omega_{j,t-1}\xi_{j,t}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{38}$$

with the same empirical equivalents as in the Cobb-Douglas case, i.e.

$$A(X;\theta) = \begin{pmatrix} \frac{1}{J\cdot T}\sum\limits_{\substack{t=1\\j=1}}^{\substack{T\\J}} \epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum\limits_{\substack{t=1\\j=1}}^{\substack{T\\J}} \xi_{j,t} \\ \frac{1}{J\cdot T}\sum\limits_{\substack{t=1\\j=1}}^{\substack{T\\J}} K_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum\limits_{\substack{t=1\\j=1}}^{\substack{T\\J}} L_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum\limits_{\substack{t=1\\j=1}}^{\substack{T\\J}} \omega_{j,t-1}\xi_{j,t} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \tag{39}$$

where $\theta = (\beta_k, \beta_l, \sigma, \rho_\omega)$. The constrained GMM problem follows analogously to the previous cases with the CES specific expression for $\Omega_{j,t}, \xi_{j,t},$ and $\epsilon_{j,t}$ as

$$
\begin{aligned}
\arg\min_{\theta} \quad & J(X,W;\theta) = (T\cdot J)A(X;\theta)^T W A(X;\theta) \\
s.t. \quad & \Omega_{j,t} = \frac{M_{j,t}^{1-\sigma}(\beta_k K_{j,t}^{\sigma} + \beta_l L_{j,t}^{\sigma} - (\beta_k+\beta_l-1)M_{j,t}^{\sigma})^{\frac{\sigma-1}{\sigma}} P_{j,t}^{M}}{P_{j,t}^{Y}(\beta_k+\beta_l-1)}, \\
& \xi_{j,t} = \omega_{j,t} - \rho_\omega \omega_{j,t-1}, \\
& \epsilon_{j,t} = ln\left(\frac{P_{j,t}^{Y}(\beta_k+\beta_l-1)}{M_{j,t}^{1-\sigma}(\beta_k(M_{j,t}^{\sigma}-K_{j,t}^{\sigma}) + \beta_l(M_{j,t}^{\sigma}-L_{j,t}^{\sigma}) - M_{j,t}^{\sigma})P_{j,t}^{M}}\right) \quad \begin{matrix}\forall t = 1,...,T, \\ \forall j = 1,...,J\end{matrix}.
\end{aligned} \tag{40}
$$

2.2.4 Case: Partially Leontief Production Function

A large part of the literature developing and applying the control function approach, as described in Section 2.1, has focused on a reduced production function that does not include intermediate inputs, even though the firm utilizes intermediate inputs in its production. One functional form of the production function often used as a motivation for the reduced representation is a production function that is Leontief in intermediate inputs but Cobb-Douglas in capital and labor. We follow Ackerberg et al. (2015) and define it as

Partially Leontief Case: In each period t, firm j chooses to produce output $Q_{j,t}$ using production inputs capital $K_{j,t}$, labor $L_{j,t}$, and intermediate inputs $M_{j,t}$ under the production function

$$
Q_{j,t} = F(K_{j,t}, L_{j,t}, M_{j,t}, \Omega_{j,t}; \beta, \tau) = \min\{\beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} \Omega_{j,t}, \tau M_{j,t}\}, \tag{41}
$$

with β_0 representing a constant, and β_k, and β_l being output elasticities of the respective inputs. $\Omega_{j,t}$ represents TFP and τ is a scaling parameter for intermediate inputs.

Even though partial derivatives for the only flexible input, intermediate inputs, are not well behaved, we can still under Assumptions 3, 4, 5, 7, and 8, use its first order condition to find an analytical expression for productivity, $\Omega_{j,t}$. The FOC for intermediate inputs implies that firms always[8] choose to set intermediate inputs such that

$$
\beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} \Omega_{j,t} = \tau M_{j,t}. \tag{42}
$$

Solving Equation 42 for $\Omega_{j,t}$ gives us the following expression for unobserved productivity

$$
\Omega_{j,t} = \frac{\tau M_{j,t}}{\beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l}}. \tag{43}
$$

We can use this result to calculate $\Omega_{j,t}$ for every value of the production function parameters, and in addition use it as a constraint in the estimation procedure.

Furthermore, the FOC for intermediate inputs in Equation 42 allows us to express observed output $Y_{j,t}$ as

$$
Y_{j,t} \equiv \beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} \Omega_{j,t} e^{\epsilon_{j,t}}. \tag{44}
$$

[8] Gandhi et al. (2017) point out that if we assumed $\Lambda(M_{j,t}, \tau)$ instead of $\tau M_{j,t}$ and if $\Lambda(M_{j,t}, \tau)$ was nonlinear in $M_{j,t}$, firms might in some special cases choose not to set inputs such that Equation 42 holds. However, for the purpose of exposition, we follow Ackerberg et al. (2015)'s linear specification.

Replacing $\Omega_{j,t}$ with its expression in Equation 43, we can represent the unanticipated error term in observed output as

$$\epsilon_{j,t} = ln(Y_{j,t}) - ln(\tau M_{j,t}). \tag{45}$$

All terms but τ and intermediate inputs drop out of the expression in Equation 45. However, we can still use it as a constraint in the GMM estimation. Because τ and intermediate inputs remain in the expression, we can also use a moment condition on $\epsilon_{j,t}$ to estimate τ. On the other hand, this expression is clearly uninformative on any of the β parameters and we cannot use it to form moment conditions on any of the β parameters. We, therefore, use the law of motion for $\Omega_{j,t}$ in combination with the firm's information set $\mathbb{I}_{j,t}$ to find moment conditions for β_0, β_l, β_k, and the parameters of the law of motion for $\Omega_{j,t}$. For simplicity, we again assume this law of motion to take the same linear form as above: $\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}$.

The contemporary realization of the productivity shock, $\xi_{j,t}$, is not in the firm's information set when it makes the input decisions for labor and capital. Therefore, $K_{j,t}$ and $L_{j,t}$ should be orthogonal to $\xi_{j,t}$ giving us two more moment conditions. The last moment condition involves the orthogonally assumption between productivity shocks and prior realizations of productivity. All moment conditions combined result in the moment vector

$$\mathbb{E}\begin{bmatrix} M_{j,t}\epsilon_{j,t} \\ \xi_{j,t} \\ K_{j,t}\xi_{j,t} \\ L_{j,t}\xi_{j,t} \\ \omega_{j,t-1}\xi_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{46}$$

with their empirical equivalents

$$A(X;\theta) = \begin{pmatrix} \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} M_{j,t}\epsilon_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} \xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} K_{j,t}\xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} L_{j,t}\xi_{j,t} \\ \frac{1}{J \cdot T} \sum_{\substack{t=1 \\ j=1}}^{\substack{T \\ J}} \omega_{j,t-1}\xi_{j,t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{47}$$

where $\theta = (\beta_0, \beta_k, \beta_l, \tau, \rho_\omega)$. Combining all parts of the model and the empirical moment conditions, we can write the constrained GMM estimation problem as,

$$\begin{aligned}
\underset{\theta}{\arg\min} \quad & J(X,W;\theta) = (T \cdot J)A(X;\theta)^T W A(X;\theta) \\
s.t. \quad & \Omega_{j,t} = \frac{\tau M_{j,t}}{\beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l}}, \\
& \xi_{j,t} = \omega_{j,t} - \rho_\omega \omega_{j,t-1}, \\
& \epsilon_{j,t} = ln(Y_{j,t}) - ln(\tau M_{j,t}) \qquad \begin{matrix} \forall t = 1,...,T, \\ \forall j = 1,...,J \end{matrix},
\end{aligned} \tag{48}$$

where W again represents a weighting matrix and the constraints are the functional restrictions defined by the model.

3 Simulation Based Evidence of Performance

This section analyzes the performance of the TJ approach we developed in Section 2.2 in various scenarios. We first show that the TJ approach can retrieve unbiased estimates of production function parameters in the most simple scenario of common and constant input and output prices. We then analyze how our approach performs in the presence of variation in prices over firms, time, or both. Furthermore, we examine how the classical control function approach performs compared to the TJ approach if applied to a DGP based on a Cobb-Douglas production function incorporating intermediate inputs.

For all exercises, we assume a Cobb-Douglas production function of the form

$$F(K_{j,t}, L_{j,t}, M_{j,t}; \beta) = \beta_0 K_{j,t}^{\beta_K} L_{j,t}^{\beta_L} M_{j,t}^{\beta_M}, \tag{49}$$

with $\beta_0 = 0.8$, $\beta_K = 0.2$, $\beta_L = 0.3$, and $\beta_M = 0.5$. The firm produces observed output such that

$$Y_{j,t} = F(K_{j,t}, L_{j,t}, M_{j,t}; \beta)\Omega_{j,t} e^{\epsilon_{j,t}}. \tag{50}$$

Productivity develops over time as an AR(1) process with $\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}$, where $\rho_\omega = 0.7$. Consistent with Assumptions 3-5, the level of capital input is fixed, while firms can invest in next period's capital. Firms choose their labor inputs in the contemporary period, however, before observing $\xi_{j,t}$. In contrast, firms choose intermediate inputs after observing $\xi_{j,t}$. We describe the exact data generation procedure in more detail in Appendix A. Some of the current discussion in the production function estimation literature has been centered around utilizing price variation over time to identify production function parameters (Bond and Söderbom, 2005; Gandhi et al., 2020). The TJ approach should be able to identify all parameters disregarding the presence or absence of price variation over time or firms. We test this by simulating data with (i) common prices for all firms that are also constant over time, (ii) common prices for all firms that vary across firms, (iii) prices that vary across firms but are constant over time, and (iv) prices varying over firms and time. For the cases in which we allow time-varying prices, we model them as a first-order auto-regressive process (AR(1)). For each case, we generate 1000 times input and output decision data for 2000 firms for 20 years and estimate production function and productivity law of motion parameters using the TJ approach for each simulated dataset.

The results in Table 1 confirm that the TJ approach can identify all production function parameters reliably disregarding the presence or absence of input and output price variation over time or firms. The upper left panel shows the results for the first case in which both input prices and output prices are common across firms and constant over time. Average parameter estimates over all simulation runs are virtually indistinguishable from the true values. The mean error representing the average difference between the true value and the estimate is for all cases below 0.01 with the constant having the highest average error of 0.0065. The maximum error over all runs is always below 0.04 with the constant having the highest and the intermediate inputs output elasticity having the lowest maximum error. Introducing time variation in prices does not change the general picture. The average estimate values in the upper right panel are again nearly identical to the true parameters, however, both average and maximum errors over all runs increase slightly. Keeping prices constant over time but allowing them to vary between firms in the lower left panel again slightly increases both error measures marginally but the average parameter estimates stay within 0.006 of their true values. Allowing prices to vary over both dimensions leaves both error statistics almost unchanged while average parameter estimates are again virtually undistinguishable from their true values.

Table 1: Results - TJ Approach - Price Variation Cobb-Douglas

Parameter	True Value	Constant, Common Prices			Non-constant, Common Prices		
		Mean	Mean Error	Max Error	Mean	Mean Error	Max Error
β_0	0.8000	0.7981	0.0065	0.0344	0.7963	0.0122	0.0475
β_L	0.3000	0.2996	0.0007	0.0044	0.2990	0.0028	0.0101
β_K	0.2000	0.2003	0.0030	0.0124	0.2007	0.0041	0.0168
β_M	0.5000	0.5000	0.0002	0.0009	0.5000	0.0002	0.0009
ρ_ω	0.7000	0.7009	0.0036	0.0151	0.7023	0.0075	0.0265
Parameter	True Value	Constant, Non-common Prices			Non-constant, Non-common Prices		
		Mean	Mean Error	Max Error	Mean	Mean Error	Max Error
β_0	0.8000	0.7949	0.0148	0.0529	0.7966	0.0124	0.0495
β_L	0.3000	0.2989	0.0035	0.0122	0.2993	0.0028	0.0100
β_K	0.2000	0.2012	0.0046	0.0202	0.2008	0.0043	0.0175
β_M	0.5000	0.5000	0.0002	0.0009	0.5000	0.0002	0.0009
ρ_ω	0.7000	0.7026	0.0094	0.0325	0.7017	0.0074	0.0290

Notes: Parameter estimates for 1000 simulated dataset with 2000 firms over 20 periods. The estimation sample for each run consists of 38000 (2000 firms over 19 years) data points because we need lagged values in the estimation procedure. Estimation is based on simulated data with common and constant prices (upper left panel), common prices that vary over time (upper right panel), non-common but constant prices (lower left panel), and non-common and time-varying prices (lower right panel). See Appendix A for an description of the exact data generation procedure. 'Mean' represents the average of all estimates over all simulation runs. 'Mean Error' is the average absolute difference between the true parameter and the estimate. 'Max Error' represents the highest absolute difference between the estimate and the true value over all runs.

After demonstrating that the TJ approach successfully identifies all production function parameters in different pricing environments, we now examine how the control function approach performs in comparison to the TJ approach in recovering the parameters of a Cobb-Douglas production function with capital, labor, and intermediate inputs, as defined above in Equation 49. This assumption clearly favors the proposed TJ approach as our model assumes the correct DGP while the control function approach leaves out intermediate inputs. However, it is ex-ante unclear how large the bias will be. For this experiment, we return to the basic setting with fixed prices over time and firms while using the same parameters as before. We estimate both models with the same simulated data from the same

DGP.
The results in Table 2 show that the TJ approach performs as expected and again recovers the correct parameter values. As before, the mean estimates are almost indistinguishable from their true parameters. The mean errors are all below 0.007 and maximum errors do not exceed 0.03. The control function approach on the other hand is not able to recover most of the true parameters. All but one estimate are, on average, substantially biased and both error statistics show that it consistently fails to get close to the true parameters. It severely overestimates the output elasticity of labor with an average estimate of 0.6, double the size of the true value. Its average error is close to the maximum error, suggesting that it constantly overestimates the parameter and not just struggles to recover the true parameters in a few runs. The control function approach also overestimates the output elasticity of capital with a mean estimate of about 0.4 instead of the true vale of 0.2. The mean error over all runs is close to the maximum error further constituting that the parameter is not just imprecisely estimated but strongly upward biased. The control function approach gets relatively close to the scale parameter β_0 with an average estimate of 0.71, only being about 0.09 too low. Surprisingly, even though all production function parameters are severely biased, it manages to get an unbiased estimate of the law of motion parameter for productivity ρ_ω.

Table 2: Results - Misspecification Cobb-Douglas

Parameter	True Value	Mean ACF	Mean TJ	Mean Error ACF	Mean Error TJ	Max Error ACF	Max Error TJ
β_0	0.8000	0.7107	0.7988	0.0893	0.0065	0.1037	0.0284
β_L	0.3000	0.6001	0.2997	0.3001	0.0007	0.3109	0.0054
β_K	0.2000	0.3998	0.2003	0.1998	0.0031	0.2259	0.0141
β_M	0.5000	NaN	0.5000	NaN	0.0002	NaN	0.0010
ρ_ω	0.7000	0.7000	0.7010	0.0040	0.0036	0.0169	0.0202

Notes: Parameter estimates for 1000 simulated dataset with 2000 firms over 20 periods. 38000 data points are used in the estimation because of the use of lagged values. See Appendix A for an description of the exact data generation procedure. 'Mean ACF' and 'Mean TJ represent the respective average of all estimates over all simulation runs. 'Mean Error ACF' and 'Mean Error TJ' are the respective average absolute differences between the true parameters and the estimates. 'Max Error ACF' and 'Max Error TJ' represent the highest absolute differences between the respective estimates and their true values over all runs.

4 Application to Chilean and Colombian Manufacturing Industries

The estimator from the TJ approach performs well in the perfect conditions our simulated data provide. However, empirical datasets often have less favorable conditions as they can contain random measurement error, missing observations, and data generating processes might not exactly fit model assumptions. We, therefore, apply our estimation procedure to publicly available data from both Chilean and Colombian manufacturing firms. Previous papers developing production function estimation methods already used the same datasets to evaluate their estimators' performances which allows a comparison between approaches' results.

The Data

The data we use in this section comes from two sources. The first one, covering Chilean firms, is the Chilean manufacturing census conducted by the Instituto Nacional de Estadística. It covers all firms above 10 employees from 1979 until 1996. The second dataset, covering Colombian firms, stems from the Colombian manufacturing census also including all firms above 10 employees between 1981 and 1991, collected by Colombia's Departamento Administrativo Nacional de Estadística. Gandhi et al. (2020) provide both prepared datasets within their replication package. We adopt these datasets to make comparisons particularly straightforward. Gandhi et al. (2020) follow Greenstreet (2007)'s convention in their variable construction, using price deflators provided by Pombo (1999) for the Colombian data, and by Bergoeing et al. (2003) for Chilean industries. We restrict our estimation sample to a subset of five manufacturing industries used by both Gandhi et al. (2020) and Levinsohn and Petrin (2003), with ISIC codes 311 (Food Products), 321 (Textiles), 322 (Apparel), 331 (Wood Products), and 381 (Metals).[9]

Deflated revenues serve as a measure for gross output $Y_{j,t}$, a sum of raw materials, energy, and service expenditures represents intermediate inputs $M_{j,t}$, and labor $L_{j,t}$ is a sum of blue- and white-collar employees weighted by the ratio of the mean blue collar-wage to the mean white-collar wage. Gandhi et al. (2020) calculate a firm's capital stock using the perpetual inventory method by adding investments in capital to the depreciated capital stock of the previous period. We provide summary statistics of the key variables in the cleaned sample in Table 3.

Table 3: Summary Statistics - Chilean and Colombian Data

	Colombia				Chile			
Variable	Mean	Median	Min	Max	Mean	Median	Min	Max
Output	17227.40	2573.60	10.56	1411053.90	17227.40	2573.60	10.56	1411053.90
Labor	103.40	35.25	0.39	8135.26	103.40	35.25	0.39	8135.26
Capital	5173.19	574.05	0.22	1702810.60	5173.19	574.05	0.22	1702810.60
Intermediate Inputs	11549.45	1529.66	1.49	865833.88	11549.45	1529.66	1.49	865833.88
ln(Share)	-0.55	-0.50	-4.63	3.35	-0.55	-0.50	-4.63	3.35

Notes: These summary statistics are based on the data provided in Gandhi et al. (2020)'s replication package. We restrict the sample to the same manufacturing industries with the following ISIC codes: 311 (Food Products), 321 (Textiles), 322 (Apparel), 331 (Wood Products), 381 (Metals). 'ln(Share)' is the logarithm of the nominal share of intermediate inputs from revenue.

Estimation Procedure

We estimate parameters of a Cobb-Douglas production function as described in Section 2.2 separately for each of the five industries of each country. As before, we assume the law of motion of productivity to follow an AR(1) process. Keeping the same timing assumptions as in Section 2, we have intermediate inputs as the only flexible input chosen after observing the contemporary productivity level. Its demand function serves to calculate productivity for given parameter guesses and data, as derived above in Equation 14. However, the Chilean and Colombian data does not include price information. Fortunately, Gandhi et al. (2020) state that the share of intermediate inputs to output in the data is measured in nominal terms and, therefore, includes prices. Even though we cannot recover prices from this one equation alone, we can rearrange terms to calculate the ratio of intermediate inputs prices to output prices, as

$$share_{j,t} = \frac{P^M_{j,t} M_{j,t}}{P^Y_{j,t} Y_{j,t}} \Leftrightarrow \frac{P^M_{j,t}}{P^Y_{j,t}} = \frac{share_{j,t} \cdot Y_{j,t}}{M_{j,t}}. \tag{51}$$

We replace in the estimation procedure the price ratio in the intermediate inputs' demand function with the expression above. Furthermore, the log-linear form of the law of motion for productivity allows us to reduce the dimension of the numerical optimization of the criterion function by estimating the parameters of the law of motion using OLS within each evaluation of the criterion function. For

[9] Levinsohn and Petrin (2003) additionally apply their estimation framework to industries 313 (Beverages), 342 (Printing and Publishing), and 352 (Other Chemicals).

comparability, we calculate block bootstrapped standard errors for all point estimates using 250 iterations, even though we could use standard errors based on asymptotic properties of GMM estimators as our approach fits in the standard GMM framework.

We compare the resulting estimates of our approach to a naive linear regression of the logarithm of output onto the three production inputs and a constant, estimated with OLS. The linear regression serves as a baseline that is simple to estimate but disregards the transmission bias through unobserved productivity. It does not include calculating productivity or estimating its law of motion. We, therefore, leave the corresponding cells in the results tables (Tables 4 and 5) blank.

Results

The results for each of the industries in Table 4 for Chilean firms and Table 5 for Colombian firms show that the TJ approach is able to retrieve sensible production function parameter estimates and that there are substantial differences between the estimates from the TJ approach and the naive OLS estimates. The standard error estimates of the TJ approach are extremely small suggesting that estimates are precise while rendering all estimates significantly different from zero at the 0.1% level.

We find strong differences between the parameter estimates of different industries, even though they paint the same qualitative picture. The intermediate input's output elasticity is with estimates between 0.5 and 0.7 in all industries higher than the output elasticities of both labor and capital. Labor's output elasticity is with estimates ranging from 0.15 to 0.4 in all but one industry located above the output elasticity for capital (0.08-0.2). This ordering is generally consistent with results from previous approaches using the same data even though exact point estimates differ (Levinsohn and Petrin, 2003; Gandhi et al., 2020). We do not impose constant returns to scale, however, the sum of all inputs' output elasticities is always close to one suggesting that firms in all covered industries have constant returns to scale. The estimates for the parameter of productivity's law of motion ρ_ω is in every industry between 0.77 and 0.91 demonstrating that productivity is highly persistent over time in all industries.

Furthermore, we find substantial differences between the results from our proposed estimator from the TJ approach and the estimates from naive OLS regressions. The output elasticity of intermediate inputs is for all industries upwards biased in comparison to our proposed estimator. Capital's output elasticities from the OLS regressions are always downward biased being as small as 1/8th the size of our estimator. The OLS estimates of the output elasticities of labor tend to be downward biased, though less drastically. We estimate β_0 with the proposed estimator to be up to seven times higher than the OLS estimate.

Table 4: Results - Chile - Cobb-Douglas

Parameter	311 TJ	311 OLS	321 TJ	321 OLS	322 TJ	322 OLS	331 TJ	331 OLS	381 TJ	381 OLS
β_0	3.9237	0.7009	10.1560	1.4107	8.7901	1.5052	5.7485	1.0314	10.6379	1.5305
	(0.1169)	(0.03)	(2.5565)	(0.082)	(2.2638)	(0.0616)	(0.3938)	(0.0554)	(1.1895)	(0.0608)
β_M	0.6856	0.8596	0.5582	0.7504	0.5675	0.7176	0.6142	0.8142	0.5126	0.7173
	(0.0029)	(0.0072)	(0.0063)	(0.0143)	(0.0071)	(0.0113)	(0.0058)	(0.0106)	(0.0064)	(0.0123)
β_K	0.1306	0.0476	0.1140	0.0567	0.0872	0.0342	0.0827	0.0266	0.1377	0.0658
	(0.0052)	(0.0037)	(0.0318)	(0.0074)	(0.0221)	(0.0085)	(0.0115)	(0.0069)	(0.0149)	(0.0101)
β_L	0.1966	0.1474	0.3149	0.2288	0.3361	0.3087	0.3352	0.2010	0.4030	0.2958
	(0.009)	(0.0078)	(0.025)	(0.0185)	(0.0333)	(0.0177)	(0.0188)	(0.0132)	(0.0249)	(0.0185)
ρ_ω	0.8355	NA	0.8111	NA	0.8108	NA	0.7657	NA	0.8177	NA
	(0.0084)	NA	(0.0492)	NA	(0.0494)	NA	(0.0129)	NA	(0.0118)	NA
Obs	16210	19374	4589	5592	3811	4830	4137	5148	4662	5942

Notes: This table shows the point estimates of a Cobb-Douglas production function using the proposed new estimator and corresponding OLS estimates. Block bootstrapped standard errors with 250 iterations in parentheses below point estimates.

Table 5: Results - Colombia - Cobb-Douglas

Parameter	311 TJ	311 OLS	321 TJ	321 OLS	322 TJ	322 OLS	331 TJ	331 OLS	381 TJ	381 OLS
β_0	2.9729	0.9817	5.1919	1.1862	6.8940	1.7715	6.7971	1.6759	4.6117	1.2493
	(0.8111)	(0.0355)	(1.4244)	(0.0631)	(1.0692)	(0.0551)	(1.4026)	(0.1059)	(0.4241)	(0.0353)
β_M	0.6948	0.8302	0.5508	0.7778	0.5024	0.6029	0.5119	0.6447	0.5363	0.7517
	(0.0069)	(0.0102)	(0.0078)	(0.0165)	(0.009)	(0.0179)	(0.016)	(0.023)	(0.0057)	(0.0108)
β_K	0.1622	0.0423	0.1968	0.0420	0.1686	0.0206	0.0913	0.0374	0.1797	0.0262
	(0.0317)	(0.0074)	(0.0527)	(0.011)	(0.0239)	(0.0084)	(0.0258)	(0.0124)	(0.0204)	(0.0087)
β_L	0.1525	0.1376	0.2470	0.2090	0.2601	0.4047	0.4028	0.3442	0.3303	0.2720
	(0.021)	(0.0107)	(0.0461)	(0.0199)	(0.0265)	(0.027)	(0.0452)	(0.0271)	(0.0261)	(0.0147)
ρ_ω	0.8973	NA	0.8935	NA	0.9087	NA	0.8613	NA	0.8361	NA
	(0.0293)	NA	(0.031)	NA	(0.0129)	NA	(0.0281)	NA	(0.0149)	NA
Obs	5244	6187	2345	2857	3683	4478	786	968	3006	3699

Notes: This table shows the point estimates of a Cobb-Douglas production function using the proposed new estimator and corresponding OLS estimates. Block bootstrapped standard errors with 250 iterations in parentheses below point estimates.

5 Extension to Simultaneous Choice of Multiple Flexible Inputs

In the derivation of our approach in Section 2 we assume labor to be quasi-fixed in the current period. Even though, we allowe firms to choose labor in the same period, we assume firms to do so before observing the contemporary productivity shock $\xi_{j,t}$. This is equivalent to assuming firms to choose labor in $t-1$ if state variables do not change.[10] However, depending on the application the researcher might not be comfortable assuming firms not to know their current period's productivity when choosing labor while knowing it when choosing intermediate inputs. We show in this section that the TJ approach can easily be modified to allow for two completely flexible inputs chosen at the same time.

Take Assumptions 1-8 and consider the following modification in the labor timing assumption (Assumption 4) to accommodate the new setup.

Assumption 4b *Labor is a flexible input. Firms can freely choose its level in the current period t*

[10] One case where this might matter is if prices change between periods. However, assuming that firms know the price of labor for t already in $t-1$ would again render the two cases equivalent.

after observing their contemporary productivity shock $\xi_{j,t}$ and productivity level $\Omega_{j,t}$. Their choices are immediately realized.

Firms now consider both labor and intermediate inputs in their static profit maximization problem after observing their productivity shock $\xi_{j,t}$ giving us two first order conditions; one for intermediate inputs and one for labor. We can use the second first order condition as an additional constraint in GMM estimation procedure. For the purpose of the demonstration, assume the production function to be of the same Cobb-Douglas functional form as in Section 2.2.1. The firm's profit maximization problem after observing contemporaneous productivity becomes then

$$\max_{M_{j,t},L_{j,t}} \pi_{j,t} = P^Y_{j,t}\beta_0 K^{\beta_k}_{j,t} L^{\beta_l}_{j,t} M^{\beta_m}_{j,t}\Omega_{j,t} - P^M_{j,t}M_{j,t} - P^L_{j,t}L_{j,t}, \tag{52}$$

with the associated FOCs being

$$\begin{aligned} \frac{\partial \pi_{j,t}}{\partial M_{j,t}} &= P^Y_{j,t}\beta_0 K^{\beta_k}_{j,t} L^{\beta_l}_{j,t}\beta_m M^{\beta_m - 1}_{j,t}\Omega_{j,t} - P^M_{j,t} = 0, \\ \frac{\partial \pi_{j,t}}{\partial L_{j,t}} &= P^Y_{j,t}\beta_0 K^{\beta_k}_{j,t}\beta_l L^{\beta_l - 1}_{j,t} M^{\beta_m}_{j,t}\Omega_{j,t} - P^L_{j,t} = 0 \end{aligned} \tag{53}$$

As in the case with one flexible input, we can solve the FOC for intermediate inputs for $\Omega_{j,t}$, which gives us the same expression for unobserved productivity as before,

$$\Omega_{j,t} = \frac{P^M_{j,t}}{P^Y_{j,t}\beta_0\beta_m K^{\beta_k}_{j,t} L^{\beta_l}_{j,t} M^{\beta_m - 1}_{j,t}}. \tag{54}$$

Replacing $\Omega_{j,t}$ in the production function and rearranging terms gives us the same expression as above for the unanticipated error term $\epsilon_{j,t}$ that only depends on output, output price, price for intermediate inputs, intermediate inputs, and its output elasticity

$$\epsilon_{j,t} = ln(\beta_m) - ln\left(\frac{P_M M_{j,t}}{P_Y Y_{j,t}}\right). \tag{55}$$

Additionally, we can get an analogous expression for $\epsilon_{j,t}$ from the second FOC using the same steps. This expression only includs the second flexible input $L_{j,t}$, its output elasticity, price of labor, output price, and output

$$\epsilon_{j,t} = ln(\beta_l) - ln\left(\frac{P_L L_{j,t}}{P_Y Y_{j,t}}\right). \tag{56}$$

We can utilize these both expressions for $\epsilon_{j,t}$ as constraints in the constrained GMM procedure. Furthermore, a moment condition involving the expression for the error term from Equation 55 is informative on β_m and a moment condition utilizing the error term from Equation 56 is informative on β_l. Using this result, the timing assumption of the unanticipated error $\epsilon_{j,t}$, and its distributional assumption, $E[\epsilon_{j,t}M_{j,t}] = 0$ and $E[\epsilon_{j,t}L_{j,t}] = 0$ should be valid moment conditions.

We need two more moment conditions and as many more as there are parameters in the law of motion for $\Omega_{j,t}$. Following the procedure in Section 2.2, we retrieve the rest of the moment conditions using timing assumptions in combination with $\Omega_{j,t}$'s law of motion. Capital input $K_{j,t}$ is chosen in $t-1$, meaning that $\xi_{j,t}$ is not in the firm's information set $\mathbb{I}_{j,t}$ when making the investment choice. $K_{j,t}$ should, therefore, be orthogonal to the productivity shock allowing us to use $E[K_{j,t}\xi_{j,t}] = 0$ as a moment condition. Prior levels of $\omega_{j,t}$ are, by assumption, independent from the productivity shock making $E[\omega_{j,t-1}\xi_{j,t}] = 0$ a valid moment condition. If the law of motion includes higher polynomials, one could use their independence from $\xi_{j,t}$ to form additional moment conditions analogously. The zero mean assumption of $\xi_{j,t}$ gives us the last moment condition, $E[\xi_{j,t}] = 0$.

Combining all moment conditions leads to the following vector of moments

$$\mathbb{E}\begin{bmatrix} L_{j,t}\epsilon_{j,t} \\ M_{j,t}\epsilon_{j,t} \\ \xi_{j,t} \\ K_{j,t}\xi_{j,t} \\ \omega_{j,t-1}\xi_{j,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

with their empirical counterparts

$$A(X;\theta) = \begin{pmatrix} \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} M_{j,t}\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} L_{j,t}\epsilon_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} \xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} K_{j,t}\xi_{j,t} \\ \frac{1}{J\cdot T}\sum_{\substack{t=1\\j=1}}^{\substack{T\\J}} \omega_{j,t-1}\xi_{j,t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{58}$$

where $\theta = (\beta_0, \beta_k, \beta_l, \beta_m, \rho_\omega)$. The constrained GMM minimization problem to estimate all parameters in the presence of two flexible inputs is then

$$\begin{aligned} \arg\min_{\theta} \quad & J(X,W;\theta) = (T\cdot J)A(X;\theta)^T W A(X;\theta) \\ s.t. \quad & \omega_{j,t} = p^M_{j,t} - p^Y_{j,t} - ln(\beta_0) - ln(\beta_m) - \beta_k k_{j,t} - \beta_l l_{j,t} - (\beta_m - 1)m_{j,t}, \\ & \xi_{j,t} = \omega_{j,t} - \rho_\omega \omega_{j,t-1}, \\ & \epsilon_{j,t} = ln(\beta_l) - ln\left(\frac{P_L L_{j,t}}{P_Y Y_{j,t}}\right), \\ & \epsilon_{j,t} = ln(\beta_m) - ln\left(\frac{P_M M_{j,t}}{P_Y Y_{j,t}}\right), \qquad \begin{matrix} \forall t = 1,...,T, \\ \forall j = 1,...,J \end{matrix}. \end{aligned} \tag{59}$$

6 Relation to Previous Approaches

This paper contributes and relates to an already extensive list of papers on the estimation of production functions. We particularly address challenges of identifying gross output production functions in the estimation of parametric production functions in the presence of unobserved productivity and flexible inputs. The TJ approach relates to, builds upon, and refines several key strands of the literature.

Control Function Approach and Its Limitations

A series of papers started by Olley and Pakes in 1996 including subsequent work by Levinsohn and Petrin (2003) and Ackerberg et al. (2015), and others, developed related methods building on the key idea of inverting demand functions of production inputs to solve the endogeneity issue of unobserved productivity. However, these methods make a highly restrictive functional form assumption and face identification problems when applied to gross output production functions with flexible inputs. We develop an approach that builds on the key idea in these papers of using an input's demand function while substantially expanding the scope of the approach to a much broader class of production functions and solving the problem of identifying gross output production function's parameters. Wooldridge (2009) showed how the two stages of the control function approach can be combined into one GMM estimator. However, Wooldridge (2009)'s approach still relies on treating the demand for the flexible input as an unknown function to be approximated with a polynomial function. The TJ approach also combines all moment conditions to a single GMM estimator resulting in a clear framework, more efficient estimates, and the ability to use analytical standard error formulas available for GMM estimators. However, the TJ approach further extends the scope of the approach to a much broader variety of production functions and does not rely on polynomial approximations of input demand functions.

Price Variation as Instruments and Dynamic Panel Approaches

Bond and Söderbom (2005) state that time-series variation could help identifying flexible inputs' parameters in the control function approach. However, Gandhi et al. (2020) show that this is a relatively weak source of identifying variation. Ackerberg et al. (2007) explain that using input prices as instruments requires strong variation in prices across firms while the price variation has to be uncorrelated with contemporary productivity shocks, which might not hold. Doraszelski and Jaumandreu (2013, 2018) propose using lagged prices as instruments arguing that previous prices are less likely to be correlated with contemporary productivity shocks. Lagged prices can also be included in our GMM procedure as additional instruments to improve the precision estimates. In contrast to approaches that rely on price variation, variation in prices is not essential for identifying parameters in the TJ approach. It is instead fully robust to price variation over firms or time.

Dynamic panel data methods, such as those developed by Anderson and Hsiao (1982); Arellano and Bond (1991); Arellano and Bover (1995); Blundell and Bond (1998) and Blundell and Bond (2000), offer an alternative approach to addressing the simultaneity issue by using lagged inputs as instruments. In contrast to methods built on first-order conditions or the inversion of the demand function of a flexible input, they allow for additional unobservable variables in the demand for flexible inputs. However, their derivation builds on a linear law of motion for productivity and cannot separate productivity from the unanticipated error term in the production function (Ackerberg et al., 2015). This is crucial when the aim of the application is to estimate productivity. Additionally, Gandhi et al. (2020) show, that dynamic panel approaches face similar challenges as the control function approach when estimating gross output production functions with flexible inputs.

Exploiting First-Order Conditions

Several papers have explored using first-order conditions directly to identify production function parameters. Griliches and Ringstad developed in 1971 an early approach relying on the first-order conditions to recover the output share of flexible inputs allowing them to identify parameters of a Cobb-Douglas production function. While the approach used in Doraszelski and Jaumandreu (2013) is probably the approach most closely related to ours, it explores a special case of a Cobb-Douglas production function in the context of endogenous technological change and uses lagged price information as instruments in identifying production function parameters. Grieco et al. (2016) develop an approach that relies on first-order conditions of two flexible inputs, while physical quantities of one of them and its price are unobserved to the econometrician. Gandhi et al. (2020) explores a similar approach as ours in a non-parametric setting. Their approach uses the flexible input's nominal share of output to estimate the flexible input's output elasticity and similar moment conditions as the TJ approach and the control function approach to estimate fixed inputs' output elasticities. However, it uses Sieve estimators in a two-stage setup. Even though the non-parametric nature of the approach allows to be agnostic about the production function's functional form, Gandhi et al. (2020) only identify inputs' output elasticities and do not estimate any additional structural parameters.

Production Functions with Factor Augmenting Productivity

An additional strand of literature focuses on weakening the main restriction our framework puts on the production function, Hicks-neutrality of productivity. If the productivity term is not Hicks-neutral, it cannot always easily be separated from the function of inputs. In this case, it depends on the functional form of F if one can solve the first order condition for unobserved productivity. However, if solving for productivity is not possible, one can still use numerical methods to solve this implicit function. Doraszelski and Jaumandreu (2018) examine a more challenging case in which they introduce an unobserved factor-augmenting productivity term for labor in addition a Hicks-neutral productivity term in a CES production function. Their approach uses first-order conditions for two flexible inputs to identify all structural parameters. Demirer (2020) generalizes the approach from Doraszelski and Jaumandreu (2018) in a non-parametric setup while developing a non-parametric estimation approach that does not make restrictions on competition in output markets. This is important when focusing on identifying markups. Furthermore, Raval (2019) focuses on a parametric value-added production function with labor-augmenting productivity, while Zhang (2019) focuses on material-augmenting productivity in a CES production function.

7 Discussion and Conclusion

In this paper, we contribute to the fast growing literature on production function estimation techniques by developing a novel approach to estimating parametric gross-output production functions. Our approach operates under the same assumptions and data requirements as the widely used control function approach. However, we are able to identify parameters of any gross output production function with Hicks-neutral productivity exploiting first-order conditions of flexible inputs and timing assumptions. Our approach substantially expands the scope of applications while including the reduced production function assumed in the canonical control function approach as a special case. We derive estimators from our general framework for the most commonly used production functions, Cobb-Douglas, CES, Translog, and the case of a partially Leontief, partially Cobb-Douglas production function often used as a motivation for the reduced form assumed in the control function approach. For each case, we derive explicit one-step GMM estimators, which do not rely on approximating unknown functions but use the existing structure of the model. We furthermore show how to extend our approach to situations with multiple flexible inputs.

We provide simulation based evidence that our estimator is able to efficiently recover all structural parameters of gross output production functions in absence or presence of time-series price variation or variation of prices over firms. We furthermore show that applying the commonly used control function approach to data from gross output production functions that do not adhere to the restrictive functional form assumption leads to heavily biased estimates, while our new estimator is able to retrieve unbiased estimates. Applying our estimator to data from Chilean and Colombian manufacturing industries, we are able to retrieve sensible and precise estimates of gross output Cobb-Douglas production functions that are substantially different from naive OLS estimates.

The main focus of this paper is to provide a comprehensive general framework to the estimation of parametric production functions. We improve upon existing approaches by showing how to estimate structural parameters of any Hicks-neutral production function with at least one flexible input from panel data with minimal assumptions. In contrast, the commonly applied control function approach relies on a restrictive functional assumption for the production function which allows to express output as a reduced function of fixed inputs alone, excluding intermediate inputs. Even though focusing on further potential threats to identification in more complex environments is beyond the scope of this paper, our framework is general enough, that it could be expanded to address further issues such as non-random firm exit or market power in inputs or output markets. Even in challenging cases where, e.g., non-Hicks-neutral productivity does not allow for solving the first-order conditions analytically for productivity, numerical implicit equation solvers can be applied. We hope that the simplicity, generality, and relatively low data requirements of the approach spur adoption and further development of the approach.

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A Data Generation Procedure

This Appendix describes the data generating process for the simulated data used in Section 3. We simulate input and output decisions of 2000 firms over 20 years. Firms operate in perfectly competitive environments in both input and output markets. Prices may vary over firms and time, depending on the application.

Production Function and Productivity

Firms produce output $Q_{j,t}$ using inputs capital $K_{j,t}$, $L_{j,t}$, and $M_{j,t}$ under a Cobb-Douglas production function, such that

$$Q_{j,t} = \beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} M_{j,t}^{\beta_m} e^{\omega_{j,t}}, \tag{A1}$$

with the scaling parameter $\beta_0 = 0.8$ and output elasticities $\beta_l = 0.5, \beta_k = 0.2,$ and $\beta_m = 0.3$. Observed output $Y_{j,t} = Q_{j,t}e^{\epsilon_{j,t}}$ where we draw the random error $\epsilon_{j,t}$ from a normal distribution with parameters $\mu_\epsilon = 0$ and $\sigma_\epsilon = 0.1$. Productivity develops over time under the following first-order auto-regressive process (AR(1))

$$\omega_{j,t} = \rho_\omega \omega_{j,t-1} + \xi_{j,t}, \tag{A2}$$

where $\rho_\omega = 0.7$ and $\xi_{j,t}$ is a random productivity i.i.d shock drawn from a normal distribution with parameters $\mu_\xi = 0$ and $\sigma_\xi = 0.5$. We draw the initial $\omega_{j,0}$ for each firm from a normal distribution with the same parameters.

Capital Development

Each firm's capital stock develops exogenously over time following the AR(1) process

$$K_{j,t} = \rho_K K_{j,t-1} + I_{j,t-1}, \tag{A3}$$

with a 15% depreciation rate of capital, i.e., $\rho_K = 0.85$. We draw firms' investment in capital from a Log-normal distribution with $\mu_I = 0.1$ and $\sigma_I = 1$. The initial capital stock for each firm is drawn from a Log-normal distribution with parameters $\mu_I = 0.5$ and $\sigma_I = 1$.[11]

Labor and Intermediate Inputs Decisions

We assume in the main part of the paper that firms do not observe their contemporary productivity level prior to choosing their labor input. Therefore, the firm chooses its labor input to maximize expected profit

$$\begin{aligned} \max_{L_{j,t}} \;\; & E\left[\pi_{j,t}\right] = \left[P_{j,t}^Y Q_{j,t} - P_{j,t}^L L_{j,t}\right] \\ s.t. \;\; & Q_{j,t} = \beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} M_{j,t}^{\beta_m} e^{\omega_{j,t}}. \end{aligned} \tag{A4}$$

The first order condition is

$$\frac{\partial E[\pi_{j,t}]}{\partial L_{j,t}} = E\left[P_{j,t}^Y \beta_0 \beta_l L_{j,t}^{\beta_l - 1} K_{j,t}^{\beta_k} M_{j,t}^{\beta_m} e^{\omega_{j,t}} - P_{j,t}^L\right] \stackrel{!}{=} 0. \tag{A5}$$

[11] This differs to previous papers simulating data for production function estimation methods which model the investment decision of capital in their data generating process. To keep the data generating process simple while focusing on the essential parts, we let capital develop exogenously because the investment decision is not a key part of the behavioral model of the firm.

Capital $K_{j,t}$ is fixed and only $M_{j,t}$ and $\omega_{j,t}$ depend on stochastic components. Taking all deterministic terms out of the expectation leads to

$$\begin{aligned} \beta_0 \beta_l L_{j,t}^{\beta_l - 1} K_{j,t}^{\beta_k} E[M_{j,t}]^{\beta_m} E[e^{\omega_{j,t}}] &= P_{j,t}^L \\ \Leftrightarrow L_{j,t} &= \left(\frac{P_{j,t}^L}{P_{j,t}^Y \beta_0 \beta_l K_{j,t}^{\beta_k} E[M_{j,t}]^{\beta_m} E[e^{\omega_{j,t}}]} \right)^{\frac{1}{\beta_l - 1}}. \end{aligned} \tag{A6}$$

Using the timing assumptions for input choices and the profit maximization assumption, we have an expression for intermediate inputs given levels of capital, labor, and productivity,

$$M_{j,t} = \left(\frac{P_{j,t}^M}{P_{j,t}^Y \beta_0 \beta_m K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} e^{\omega_{j,t}}} \right)^{\frac{1}{\beta_m - 1}}. \tag{A7}$$

Replacing $M_{j,t}$ in Equation A6 with Equation A7 and taking all deterministic terms out of the expectation leads to

$$L_{j,t} = \left(\frac{P_{j,t}^L}{P_{j,t}^Y \beta_0 \beta_l K_{j,t}^{\beta_k} \left(\left(\frac{P_{j,t}^M}{P_{j,t}^Y \beta_0 K_{j,t}^{\beta_k} L_{j,t}^{\beta_l} E[e^{\omega_{j,t}}]} \right)^{\frac{1}{\beta_m - 1}} \right)^{\beta_m} E[e^{\omega_{j,t}}]} \right)^{\frac{1}{\beta_l - 1}}. \tag{A8}$$

Rearranging terms, simplifying the expression, inserting the law of motion for $\omega_{j,t}$ and using the parameters of the distribution of $\xi_{j,t}$ results in the firms' optimal labor input decision,

$$\begin{aligned} L_{j,t} &= \left(\frac{\beta_l^{\beta_m - 1} {P_{j,t}^L}^{1-\beta_m} {P_{j,t}^M}^{\beta_m}}{\beta_0 \beta_m^{\beta_m} K_{j,t}^{\beta_k} P_{j,t}^Y E[e^{\omega_{j,t}}]} \right)^{\frac{1}{\beta_m + \beta_l - 1}} \\ L_{j,t} &= \left(\frac{\beta_l^{\beta_m - 1} {P_{j,t}^L}^{1-\beta_m} {P_{j,t}^M}^{\beta_m}}{\beta_0 \beta_m^{\beta_m} K_{j,t}^{\beta_k} P_{j,t}^Y e^{\omega_{j,t-1}} E[e^{\xi_{j,t}}]} \right)^{\frac{1}{\beta_m + \beta_l - 1}} \\ L_{j,t} &= \left(\frac{\beta_l^{\beta_m - 1} {P_{j,t}^L}^{1-\beta_m} {P_{j,t}^M}^{\beta_m}}{\beta_0 \beta_m^{\beta_m} K_{j,t}^{\beta_k} P_{j,t}^Y e^{\omega_{j,t-1}} e^{\mu_\xi + \frac{\sigma_\xi^2}{2}}} \right)^{\frac{1}{\beta_m + \beta_l - 1}}. \end{aligned} \tag{A9}$$

Finally, we compute the optimal intermediate inputs choice for each firm in each period given the labor choice and the realization of contemporary productivity from Equation A7. We set input and output prices for each scenario in Table 1 as shown in Table A1, with non-constant prices for good X following an AR(1) process $P_{j,t}^X = \rho_P P_{j,t-1}^X + \kappa_{j,t}^X$ with $\rho_P = 0.5$ for all prices and $P_{j,0}^X = P_0^X$ and $\kappa_{j,t}^X = \kappa_t^X$ if prices are common among all firms.

Table A1: Price Parameters

		Common: Yes	Common: No
Constant	Yes	$P^Y_{j,t} = 1$ $P^L_{j,t} = 1$ $P^M_{j,t} = 1$	$P^Y_j \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^L_j \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^M_j \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$
	No	$P^Y_t \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^L_t \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^M_t \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$	$P^Y_{j,0}, \kappa^Y_{j,t} \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^L_{j,0}, \kappa^L_{j,t} \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$ $P^M_{j,0}, \kappa^M_{j,t} \sim$ Log-normal$(\mu = 0, \sigma = 0.1)$