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ABSTRACT

We characterize a planner's optimal allocation of consumption and capital in an overlapping generations model with exogenous government purchases, privately-observed idiosyncratic shocks to the depreciation rate of capital, and a proportional cost of reversing investment to transform used capital to consumption. We show how a package of various taxes and government bonds can finance government purchases and support the same balanced growth path as in the planner's optimum. The optimal tax rate on capital income implements the planner's optimal (but incomplete) sharing of idiosyncratic depreciation risks, while respecting the private nature of these risks.

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How should a government finance an exogenous stream of purchases of goods and services? We address this fundamental question in public finance in two steps. First, we analyze the allocations of consumption, leisure, and capital chosen by a planner to maximize a utilitarian social welfare function that is a weighted sum of the utilities of all generations, with geometrically declining weights on future generations. Second, we characterize a basket of fiscal parameters that induce a competitive economy to attain the optimal allocations of consumption, leisure, and capital, while also paying for the stream of exogenous government purchases. Along a balanced growth path in a competitive economy, the optimal allocation is attained by constant values of five fiscal parameters, τ_K, τ_C, τ_L, τ_S, and b, which are, respectively, the capital income tax rate, the consumption tax rate, the income tax rate on labor income, a social security tax rate on labor income used to pay social security benefits to old consumers in a pay-as-you-go social security system, and the ratio of government bonds to the aggregate capital stock. We derive five linear equations in the five fiscal parameters that characterize fiscal policies under which the equilibrium along a balanced growth path in the competitive economy matches the allocations chosen by the planner along a balanced growth path.

We develop and analyze an overlapping generations model that extends the seminal Diamond (1965) model in three ways. First, the depreciation rate of physical capital is subject to owner-specific idiosyncratic shocks that are observable only to the individual owners of capital; there is no aggregate uncertainty. Because the owner-specific realizations of the depreciation shocks are not publicly observable, there is no scope for an agent–or a planner–to fully pool capital depreciation risks and offer a riskfree rate of a return on a bundle of capital. In the competitive economy that we analyze, the fiscal authority issues riskfree government bonds, and in equilibrium the riskfree interest rate on government bonds will be lower than the economy-wide average rate of return on capital, which is non-stochastic.

Second, when a unit of output is produced, it can be used either for one unit of consumption or one unit of capital. After a unit of capital is used in production, the remaining undepreciated portion of that unit of capital, which we call *used capital*, can either be used in production in the following period or it can be transformed into consumption goods at a cost of q per unit of used capital. That is, each unit of used capital can be transformed into $1-q$ units of consumption. We will show that the cost of reversing capital investment, q, introduces an opportunity for the planner to spread a fraction of the idiosyncratic depreciation risks among the owners of capital in a way that does not require direct observability of the privately observable idiosyncratic shocks. In a competitive economy, the reversibility cost also pins down the optimal value of τ_K, the tax rate on capital income; taxable capital income is gross capital income minus gross investment. We show that the optimal value of τ_K equals q. Importantly, by specifying taxable capital income as gross capital income minus gross investment, the effective tax rate on capital is zero, so that there is no wedge between the expected social and expected private rates of return on capital in the competitive economy. In the existing macroeconomics literature on optimal taxation, taxable capital income is typically defined as gross capital income minus depreciation, so the effective tax rate on capital is not zero and there is a wedge between expected social and expected private rates of return on capital.

Third, as in Diamond (1965), labor is supplied only in the first period of the two-period lifetimes of consumers. However, unlike in Diamond, labor supply is endogenous. With endogenous labor supply, we show that in period t, the optimal configurations of the consumption tax rate ($\tau_{C,t}$), the labor income tax rate ($\tau_{L,t}$), and the social security tax rate ($\tau_{S,t}$), which is also levied on wage income, satisfy $\tau_{C,t}+\tau_{L,t}+\tau_{S,t}=0$, which has a straightforward interpretation: workers face a total tax rate of $\tau_{L,t}+\tau_{S,t}=-\tau_{C,t}$ on their wage income, which implies that labor supply is subsidized at rate τ_C. Equivalently, leisure is taxed at rate τ_C. With leisure taxed at the same rate as consumption, there is no distortion between leisure and consumption.

The two features of optimal tax policy described above–that $\tau_K=q$ and that the tax rates $\tau_{C,t}$, $\tau_{L,t}$, and $\tau_{S,t}$ sum to zero–hold in every period, regardless of whether the economy is on a balanced growth path. Along a balanced growth path, the optimal values of $\tau_{K,t}$, $\tau_{C,t}$, $\tau_{L,t}$, and $\tau_{S,t}$ are constant and we derive these constant values. In order for our overlapping generations model to accommodate a balanced growth path, the utility function of individual consumers must be homogeneous in consumption when young and consumption when old.

Section 2 presents the preferences and technology, including the structure of private information about idiosyncratic depreciation shocks, in our overlapping generations model. The social welfare function maximized by the planner is introduced in Section 3, with four separate subsections devoted to, respectively, the optimal sharing of idiosyncratic depreciation risks, optimal allocation of consumption over the life of individual consumers, optimal intratemporal intergenerational allocation of consumption, and optimal allocation of young consumers' time to labor and leisure. The characterizations of the planner's optimum in subsections 3.1 to 3.4 apply in every period, regardless of whether the economy is along a balanced growth. In subsection 3.5, we introduce preferences that allow the economy to have balanced growth paths, and characterize the planner's optimal conditions along a balanced growth path. Taking the planner's optimal allocations of consumption, leisure, and capital along a balanced growth path as given, Section 4 derives the implementation of these allocations in a competitive economy using the five fiscal parameters $\tau_K, \tau_C, \tau_L, \tau_S$, and b described above. Subsections 4.1 to 4.5 describe, respectively, the optimal value of τ_K to implement the planner's optimal sharing of depreciation risks; consumers' optimal holding of riskfree government bonds; the optimal consumption of young consumers faced with tax rates τ_C, τ_L, and τ_S; the optimal tradeoff between leisure and consumption of young consumers facing these tax rates; and the government budget constraint. The results of these five subsections are combined in Section 5 to present the combination of the fiscal parameters that implements the planner's optimum along a balanced growth path. Concluding remarks are presented in Section 6.

1 Literature Review

This paper builds on two related literatures with a long tradition in macroeconomics and public finance: one literature analyzes positive and normative questions related to the size of government debt and another literature analyzes positive and normative questions related to the taxation of capital. Since government debt and the taxation of capital are alternative means of collecting funds to pay for government purchases of goods and services, these two literatures overlap and can be viewed as part of the broader question of how to finance an exogenous stream of government purchases. In a standard neoclassical growth model with infinitely-lived forward looking consumers, changes in the level of government bonds arising from changes in lump-sum taxes would have no effect–an idea known as Ricardian Equivalence. Diamond (1965) developed and analyzed government bonds in a neoclassical growth model in which the economy is populated by overlapping generations of finitely-lived consumers. In Diamond's model, an increase in government bonds crowds out capital and drives up the rate of return on capital; that is, Ricardian Equivalence does not hold. However, Barro (1974) showed that if the finitely-lived consumers have operative altruistic bequest motives toward their children, then government bonds do not crowd out capital and Ricardian Equivalence is restored. Since Barro emphasized Ricardian Equivalence, analyses of government bonds have focussed on "non-Ricardian" models. Diamond's seminal overlapping generations model, with various extensions, is a popular framework for these analyses.

Samuelson (1968) used a version of Diamond's overlapping generations model to characterize the equilibrium growth path that maximizes a utilitarian social welfare function that is a weighted average of the lifetime utilities of the infinite sequence of generations, with geometrically declining weights. Samuelson's analysis implies that along an optimal balanced growth path, the equilibrium rate of return capital is pinned down by the rate at which the planner discounts the utility of generation $t + 1$ relative to the utility of generation t. Notably, that rate of return is independent of the rate of time preference in individual utility functions. This result was surprising in light of the "modified Golden Rule," according to which the balanced growth path that is optimal from the point of view of infinitely-lived consumers is characterized by the equality of the rate of return on capital and the rate of time preference, as in Cass (1965). Acikgoz et al. (2023) specifies infinitely-lived consumers and finds "that it is optimal to equalize the pre-tax return on capital and the rate of time preference in the long-run, i.e. the capital stock satisfies the modified golden rule." (p.1) However, in the current paper, we adopt Samuelson's social welfare function and exploit the fact that in an overlapping generations model with risky capital, the riskfree interest rate on government bonds along a balanced growth path depends on the planner's rate of intergenerational preference and is independent of consumers' rate of time preference.

Papers that use overlapping generations models to assess the welfare impact of government bonds typically focus on the utility of a single generation along a balanced growth path. Blanchard (2019), Falkenheim (2022), and Ball and Mankiw (2023) examine the impact of government bonds on the utility of a generation along a balanced growth path, but they do not derive the optimal amount of government bonds. Abel and Panageas (2022) also focuses on the utility of a generation along a balanced growth path, and characterizes the ratio of government bonds to the capital stock that maximizes utility without any primary government budget surpluses. If that ratio is positive, the level of government bonds is such that the riskfree interest rate on bonds equals the growth rate of the economy. The optimal level of debt is the highest value of debt that can be rolled over forever without any primary surpluses. Similarly, Kocherlakota (2023) examines an economy without growth and finds that when government debt can be rolled over forever, ex ante welfare is maximized when the interest rate is zero (which equals the growth rate). Aiyagari and McGrattan (1998) also computes the optimal amount of government bonds, but that paper uses infinitely-lived consumers rather than overlapping generations of consumers with finite lives. They find that the optimal ratio of government bonds to GDP is about 2/3.

In the Diamond model, individuals hold both capital and government bonds in their portfolios, but, if, as in Diamond (1965), there is no uncertainty, the rates of return on capital and government bonds are riskfree and always equal; therefore, there is no meaningful portfolio allocation decision for individual consumers. To introduce a meaningful portfolio allocation decision, we assume that individual portfolio holders optimally allocate their portfolios to riskfree government bonds and to risky capital. To use the terminology in Bulow and Summers (1984), the components of risk faced by holders of capital can be categorized as "income risk," which is the risk associated with the marginal product of capital, and "capital risk," which is associated with changes in the price of the capital. They argue that capital risk is quantitatively more important than income risk, and their Table 1 presents data on the standard deviations of depreciation rates of various specific models of cars and trucks. Consistent with Bulow and Summers, we model capital risk as arising from idiosyncratic shocks to the depreciation rate of capital held by different consumers. Stochastic depreciation has been used in general equilibrium models by Barro (2023) and Abel and Panageas (2022). Blanchard and Weil (2001a, footnote 11) proposes an extension of their model to include stochastic depreciation. In all three of these papers, the shock to the depreciation rate is an aggregate shock. By contrast, in the current paper, there are no aggregate shocks; the depreciation shock is a privately-observed idiosyncratic shock, which is uninsurable. Since there are no aggregate shocks, the evolution of macroeconomic aggregates is deterministic, yet the presence of uninsurable idiosyncratic shocks to the rate of return on capital drives a wedge between the aggregate rate of return on capital and the interest rate on riskfree government bonds.

A foundational result in the literature on optimal taxation of capital income is that the optimal tax rate is zero in the long run. This result is widely-known as the Chamley-Judd result based on Chamley (1986) and Judd (1985, 1999). Chari and Kehoe (1999) illustrates this result in overlapping generations models as well as infinite-horizon models and Atkeson et al. (1999) promotes this result with the provocative title "Taxing Capital Income: A Bad Idea." Jones et al. (1997) extends this result from the taxation of physical capital to the taxation of human capital and concludes that the optimal tax rate on labor income in the long run is also zero. Aguiar et al. (2021) focuses on Pareto-improving tax policies rather than on policies that maximize a social welfare function and provides "sufficient conditions for the feasibility of Pareto-improving fiscal policies when the risk-free interest rate on government bonds is below the growth rate ($r < g$) in the Bewley-Hugget-Aiyagari model." (p. 33) It also finds that Pareto-improving fiscal policies can be feasible in dynamically efficient as well as dynamically inefficient economies.

Analyses of optimal taxation of capital income, including all of the papers cited in the paragraph immediately above, typically examine linear taxes on taxable capital income and define taxable capital income as gross capital income less economic depreciation. However, if the cost of funds for firms is not tax-deductible, as would be the case if firms are financed completely by equity (which is assumed in these papers implicitly, if not explicitly), then the effective tax rate on capital would equal the statutory tax rate on capital income.[1] A positive effective tax rate on capital income distorts the optimal capital investment of firms by driving a wedge between the social rate of return on capital and the private after-tax rate of return faced by owners of capital in a competitive economy. In a deterministic framework, an optimal capital income tax *system* would have a zero effective tax rate on capital income, so that the social and private rates of return on capital are equalized. Hall and Jorgenson (1971), building on the seminal work of Hall and Jorgenson (1967) and Jorgenson (1963), shows that the capital income tax is not distortionary if taxable capital income is defined as gross capital income minus investment expenditure. That is, the effective tax rate on capital income is zero in a tax system that allows immediate expensing of capital expenditure. As emphasized in Abel (2007), even with a zero effective tax rate on capital, the government can collect substantial tax revenue from a tax on gross capital income minus gross investment–up to about 1/6 of GDP in the United States. The inclusion of privately-observed idiosyncratic risks to the depreciation rate of capital introduces additional nuances, which we explore in this paper. Nevertheless, even in the presence of these idiosyncratic shocks, the capital income tax does not drive a wedge between the expected values of the social rate of return and the private rate of return on capital.

Angeletos (2007) presents a model of infinitely-lived consumers who face uninsurable idiosyncratic capital risk. The main analytic finding is that under certain conditions the introduction of this uninsurable capital risk reduces the steady-state interest rate and the steady-state capital stock relative to the case with complete markets. There is no role for government bonds in this framework because Ricardian Equivalence holds. Optimal taxation is left as an open question. Evans (2014) takes up the open question of optimal taxation in a version of the Angeletos model extended to allow the idiosyncratic shocks to be persistent. Evans finds that with i.i.d. shocks, optimal tax policy subsidizes capital investment to overcome the reduction in the capital stock highlighted in Angeletos. However, when the shocks are persistent, the uninsured idiosyncratic shocks spread out the distribution of wealth, leading the fiscal authority to want to tax capital income as a way to share risks

[1] Auerbach (2005, p.3) presents a simple explanation of this feature of capital income taxes.

across owners of capital. Panousi and Reis (2022) analyzes a continuous-time model of infinitely-lived agents. Since Ricardian Equivalence holds in this model, government bonds are not analyzed. Similar to Evans, the optimal tax rate on capital is negative when the idiosyncratic capital risk is sufficiently small that the incentive to use a positive tax rate to reduce wealth inequality is dominated by the incentive to use a negative tax rate to stimulate capital accumulation. Alternatively, the optimal tax rate on capital is positive when the idiosyncratic capital risk is sufficiently large that the incentive to use a positive tax rate to reduce wealth inequality dominates the incentive to use a negative tax rate to stimulate capital accumulation.

Panousi and Reis (2021) is well described by its title: "A Unified Framework for Optimal Taxation with Undiversifiable Risk," where "unified" refers to including idiosyncratic undiversifiable risks to both labor income and capital income. The fiscal financing instruments in Panousi and Reis are limited to capital income taxes and lump-sum taxes and do not include government bonds. Krueger et al. (2021) analyzes optimal taxes on capital income in the presence of uninsurable idiosyncratic labor income risk, but unlike the "unified" framework of Panousi and Reis (2021), there is no capital income risk. However, Krueger, Ludwig and Villalvazo examines a broader range of fiscal tools than in Panousi and Reis, including, in a section on discussion and robustness, endogenous labor supply and labor income taxation, pay-as-you-go social security, and government bonds. Davila et al. (2012) considers the role of capital income taxes in implementing a planner's optimum in the presence of uninsurable idiosyncratic shocks to labor income. It points to different situations in which the laissez-faire economy can either overaccumulate or underaccumulate capital, and examines the implications for the optimal capital income tax. As commented above about a different set of papers, none of the papers in this paragraph or the preceding paragraph allow for immediate expensing of capital expenditures. Therefore, these papers do not exploit the opportunity to impose a zero effective tax rate on capital even while the government can collect positive capital income tax revenue.

2 An Overlapping Generations Model

Consider an overlapping generations model in which consumers live for two periods. At the beginning period t, a continuum of unit measure of ex ante identical young consumers are born. During the first period of their lives, which is period t, we refer to them as young consumers. These consumers are indexed by i, which is uniformly distributed on the unit interval $[0, 1]$. During period t, when they are young, all consumers, regardless of their type i, behave identically. Each young consumer supplies $l_t \in [0, 1]$ units of labor. Labor-augmenting productivity growth increases the *effective* units of labor supplied by each young consumer by a factor $1+g$ per period. Old consumers do not supply labor, so $L_t \equiv (1 + g)^t l_t$ is the amount of effective units of labor used in production in period t. Aggregate output in period t is $Y_t = F(K_t, L_t)$, where K_t is the aggregate amount of capital owned by old consumers at the beginning of period t and $F(K_t, L_t)$ is strictly increasing, strictly concave in K_t and L_t and linearly homogeneous in K_t and L_t with $F_K(0, L_t) = \infty$, $F_L(K_t, 0) = \infty$, $F_K(\infty, L_t) = 0$ and $F_K(K_t, \infty) = 0$. Each unit of output can be consumed or held as new capital to be used in production in the following period; the technological terms of trade between consumption and new capital is one unit of new capital can be acquired by giving up one unit of consumption.

In the second period of their lives, old consumers use their capital, K_{t+1}, in production. After it is used in production, the capital owned by type-i consumers depreciates at rate $\delta + \varepsilon_{i,t+1}$, where $0 < \delta < 1$ and $\varepsilon_{i,t+1}$ is a mean-zero idiosyncratic depreciation shock that is i.i.d. across i and across t and has a strictly positive variance. The support of $\varepsilon_{i,t+1}$ is $(-\delta, 1 - \delta)$ so that $0 < 1 - \delta - \varepsilon_{i,t+1} < 1$. Importantly, $\varepsilon_{i,t+1}$ is privately observed only by old type-i consumers at time $t + 1$. All units of used capital are identical to each other, regardless of the type i of the old owner of the capital; furthermore, all units of used capital are identical to units of new capital. The idiosyncratic depreciation shock, $\varepsilon_{i,t+1}$, is specific to the owner (type-i) rather than specific to the physical unit of capital. That is, a given unit of capital has a depreciation shock $\varepsilon_{i,t+1}$ if operated by a type-i old consumer but $\varepsilon_{j,t+1}$ if operated by a type-j consumer, reflecting random idiosyncratic care, or simply luck, in maintaining the capital. The aggregate amount of used capital after production is completed in period $t+1$ is $(1-\delta) K_{t+1}$, which is the average of the used capital $(1-\delta-\varepsilon_{i,t+1}) K_{t+1}$ held by old type-i consumers. Old type-i consumers can dispose of their used capital to acquire consumption goods in two different ways: (1) old consumers can exchange used capital in return for consumption goods–either with young consumers at a market price that we discuss in subsection 4.1 or with a planner on terms that we discuss in subsection 3.1; (2) each unit of used capital can be physically transformed into $1-q$ units of consumption, where $0 \leq q < 1$ is a fixed parameter. If $q = 0$, then capital investment is freely and completely reversible as is standard in aggregate models of capital accumulation. For values of q that are positive but less than one, capital investment is characterized by "costly reversibility," as in Abel and Eberly (1996).[2]

All type-i consumers born at the beginning of period t have the utility function

$$u_{i,t} = (1-\beta) \ln C_t^y + h(1-l_t) + \frac{\beta}{1-\gamma} \ln \left(E \left\{ C_{i,t+1}^{o\,1-\gamma} \right\} \right), \tag{1}$$

where $E\left\{ C_{i,t+1}^{o\,1-\gamma} \right\} \equiv \int_0^1 C_{i,t+1}^{o\,1-\gamma} di$, $0 < \beta < 1$, $\gamma > 0$, and, for $x \geq 0$, $h'(x) > 0$, $h''(x) \leq 0$, $h'(1) < \infty$, and $\lim_{x \to 0} h'(x) = \infty$. This utility function combines the two-period version of the Epstein-Zin-Weil (Epstein and Zin (1989) and Weil (1990)) parameterization of Kreps and Porteus (1978) preferences used in Blanchard and Weil (2001b) and Abel and Panageas (2022) with an additively separable utility function of leisure. The intertemporal elasticity of substitution between consumption when young and consumption when old is equal to one and the coefficient of relative risk aversion equals γ. For the case in which $\gamma = 1$, the utility function when old is simply $\beta E\left\{ \ln C_{i,t+1}^o \right\}$. The utility function in equation (1) satisfies the conditions in King et al. (1988) that allow for a balanced growth path.

In addition to the purchases of output by individuals to use for consumption or capital, the government purchases an exogenous amount of output, G_t, in period t. The government consumes all of its purchases, which implies the capital stock in the economy is owned entirely

[2]If $q = 1$, then investment is completely irreversible as in the Sargent (1980) general equilibrium model with irreversible investment. We do not consider $q = 1$ in this paper because it would lead to complete risk sharing that would be implemented in the competitive economy with $\tau_K = 1$. With immediate expensing of purchases of new capital, the net after-tax price of new capital, $1 - \tau_K$, would be zero.

by the private sector. The economy is a closed economy so the GDP expenditure identity in this economy can be written as

$$C_t^y + C_t^o + (g_{K,t+1} + \delta) K_t + G_t = Y_t, \tag{2}$$

where

$$C_t^o \equiv \int_0^1 C_{i,t} di \tag{3}$$

is the aggregate amount of consumption by old consumers during period t, and $g_{K,t+1} \equiv \frac{K_{t+1} - K_t}{K_t}$ so aggregate gross investment during period t is $K_{t+1} - (1 - \delta) K_t = (g_{K,t+1} + \delta) K_t$.

Finally, note that the idiosyncratic depreciation shocks are the only source of uncertainty, so there is no aggregate uncertainty. Therefore, macroeconomic aggregates such as C_t^y, C_t^o, K_t, and Y_t evolve deterministically. Nevertheless, because individuals cannot insure their privately-observed depreciation risk, they bear some risk and are willing to pay a premium to hold riskfree assets. That is, the riskfree rate of return in a competitive economy, or the shadow riskfree rate of return in a planned economy, is lower than the economy-wide average rate of return on capital, $F_K(K_t, L_t) - \delta$.

3 Planner

Now consider a planner who maximizes the following social welfare function from Samuelson (1968), which is a geometrically weighted average of the utility of all generations born from the beginning of period 0 onward, plus the weighted remaining utility of old consumers in period 0, who were born in period $t-1$,

$$\mathcal{W} = \sum_{t=0}^{\infty} (1+\eta)^{-t} U_t + (1+\eta) \frac{\beta}{1-\gamma} \int_0^1 \ln \left(E \left\{ C_{i,0}^{o^{1-\gamma}} \right\} \right) di, \tag{4}$$

where $\eta > 0$, $U_t \equiv \int_0^1 u_{i,t} di$, and $u_{i,t}$ is, from equation (1), the ex ante utility of type-i consumers born at the beginning of period t, which is deterministic. However, $C_{i,t}^o$ is stochastic because old consumers hold capital with an idiosyncratic risky rate of return. Because the idiosyncratic shock, $\varepsilon_{i,t}$, is privately observed by type-i consumers, there is no scope for private or social insurance arrangements that *directly* condition on the realization of the idiosyncratic shock, $\varepsilon_{i,t}$.[3] As shown in Appendix A, along a balanced growth path with growth rate g per period, $\sum_{t=0}^{\infty} (1+\eta)^{-t} U_t$ converges if and only if $\eta > 0$.

We solve the planner's problem in two stages. First, in each period we show the planner's optimal sharing of depreciation risks among ex ante identical old consumers with different ex post realizations of the depreciation shocks. The solution of this risk sharing problem requires only that the planner equally weights the utility of ex ante identical consumers and that the utility function is concave in consumption when old. Second, given the planner's optimal sharing of idiosyncratic risks, in each period the planner chooses the optimal amount of labor provided by young consumers, the aggregate allocation of consumption to the young cohort (equally divided among all young consumers), the aggregate allocation of consumption of old cohort, and the optimal aggregate allocation of capital to young consumers at the end of the period (equally divided among all young consumers). A formal statement of the second stage of the planner's problem and the associated first-order conditions are contained in Appendix C. Here we present the first-order conditions by considering perturbations around a candidate solution of the problem.

Given a candidate allocation of consumption and leisure to young consumers and capital to old consumers, the planner can evaluate social welfare, $\mathcal{W}$, in equation (4). In the following four subsections, we examine whether $\mathcal{W}$ can be increased by, respectively, (1) risk sharing among old consumers of different types i facilitated by the planner using a scheme that elicits truthful revelation by old consumers of their type-i depreciation shocks; (2) intertemporal transfers between consumption when young and consumption when old for any given consumer; (3) intratemporal intergenerational transfers of consumption between young consumers and old consumers (without conditioning on the type i of old consumers) in a given period; or (4) reallocation of a young consumer's time between labor and leisure. In

[3] In subsection 3.1, we show how a planner can devise a scheme that induces old consumers to truthfully reveal their realizations of the idiosyncratic shock, $\varepsilon_{i,t}$.

subsection 3.5, we focus on the balanced growth path consistent with the planner's optimum.

3.1 Optimal Sharing of Idiosyncratic Risks

Consider the idiosyncratic depreciation risk associated with holding capital. At the beginning of period t, each old consumer holds K_t units of capital and uses this capital to earn $F_K(K_t, L_t) K_t$ of capital income in the form of consumption goods. Then the capital held by type-i consumers depreciates at rate $\delta + \varepsilon_{i,t}$, so the remaining amount of used capital is $X_{i,t} \equiv (1 - \delta - \varepsilon_{i,t}) K_t > 0$, which can be transformed into $(1-q) X_{i,t}$ consumption goods. Because the utility function is concave in the consumption of old consumers, the planner would like to minimize the cross-type variation in consumption financed by $X_{i,t}$, while holding mean consumption financed by $X_{i,t}$ fixed. Moreover, the planner would like to exchange the remaining used capital held by old consumers for consumption goods allocated to young consumers in order to avoid the socially wasteful cost of transforming used capital goods into consumption. As long as $K_{t+1} > (1-\delta) K_t$, the allocation of capital to young consumers at the end of period t will be large enough to absorb all of the used capital of old consumers. In facilitating these intergenerational exchanges of used capital for consumption goods, the planner cannot directly observe the idiosyncratic shocks $\varepsilon_{i,t}$ and hence cannot directly observe $X_{i,t}$.

To facilitate the exchange of used capital, the planner can ask old consumers to declare their values of $X_{i,t}$. Each old consumer is allowed to make only one declaration.[4] In response to a declared value $\widehat{X}_{i,t}$, the planner offers to give an old type-i consumer an amount $p\left(\widehat{X}_{i,t}\right)$ of consumption goods in exchange for $\widehat{X}_{i,t}$ units of used capital. In order for the type i consumer to be *able* to participate in this exchange, $\widehat{X}_{i,t} \leq X_{i,t}$; otherwise, the old consumer would not be able to deliver the declared amount $\widehat{X}_{i,t}$ to the planner. Will the type-i consumer *choose* to participate in such an exchange, and, if so, will the consumer truthfully declare $\widehat{X}_{i,t} = X_{i,t}$? To derive the function $p\left(\widehat{X}_{i,t}\right)$ for which the answers to

[4] Thus we rule out the possibility that an old type-i consumer could pose as two different people and falsely declare $\widehat{X}_{i,t} = \frac{X_{i,t}}{2}$ twice in an attempt to obtain $2p\left(\frac{\widehat{X}_{i,t}}{2}\right)$, which, from equation (6) equals $2\left[q\overline{X}_t + (1-q)\frac{\widehat{X}_{i,t}}{2}\right] = 2q\overline{X}_t + (1-q)\widehat{X}_{i,t} > q\overline{X}_t + (1-q)\widehat{X}_{i,t} = p\left(\widehat{X}_{i,t}\right)$.

these two questions are both affirmative, we derive the participation constraint (PC) and the truthtelling (TT) condition.

As a prelude to deriving the PC and TT conditions, note that an old type-i consumer receives $p\left(\widehat{X}_{i,t}\right)$ units of consumption from the planner in exchange for $\widehat{X}_{i,t}$ units of used capital plus $(1-q)\left(X_{i,t}-\widehat{X}_{i,t}\right)$ units of consumption from directly transforming $X_{i,t}-\widehat{X}_{i,t}$ units of capital into consumption. Thus, the total amount of consumption the old type-i consumer obtains from their $X_{i,t}$ units of used capital when declaring $\widehat{X}_{i,t}$ units of used capital is

$$\zeta\left(\widehat{X}_{i,t}, X_{i,t}\right) \equiv p\left(\widehat{X}_{i,t}\right)+(1-q)\left(X_{i,t}-\widehat{X}_{i,t}\right). \tag{5}$$

An old type-i consumer with capital $X_{i.t}$ will choose to accept $p\left(\widehat{X}_{i,t}\right)$ units of consumption in exchange for giving $\widehat{X}_{i,t}$ units of used capital to the planner, rather than simply transforming $X_{i,t}$ units of capital into $(1-q)\,X_{i,t}$ units of consumption, if and only if $\zeta\left(\widehat{X}_{i,t}, X_{i,t}\right) \geq (1-q)\,X_{i,t}$, so the participation constraint can be written as

$$p\left(\widehat{X}_{i,t}\right) \geq (1-q)\,\widehat{X}_{i,t}. \tag{PC}$$

An old type-i consumer will truthfully declare $\widehat{X}_{i,t} = X_{i,t}$ if and only if $\zeta\left(X_{i,t}, X_{i,t}\right) \geq \zeta\left(\widehat{X}_{i,t}, X_{i,t}\right)$ for all $\widehat{X}_{i,t} \leq X_{i,t}$, which is equivalent to $p\left(X_{i,t}\right)-p\left(\widehat{X}_{i,t}\right) \geq (1-q)\left(X_{i,t}-\widehat{X}_{i,t}\right)$, so the truthtelling condition can be written as

$$\frac{p\left(X_{i,t}\right)-p\left(\widehat{X}_{i,t}\right)}{X_{i,t}-\widehat{X}_{i,t}} \geq 1-q. \tag{TT}$$

If the function $p\left(\widehat{X}_{i,t}\right)$ satisfies the PC and TT conditions, then all old consumers willingly and truthfully participate in exchanges with the planner. Thus, old consumers transfer all of their used capital, which totals $\overline{X}_t \equiv \int_0^1 X_{i,t}di$ in the aggregate, to the planner. The planner transfers this aggregate amount of used capital to the generation of young consumers in exchange for an equal amount of consumption, which is then transferred to the generation of old consumers in the aggregate amount $\int_0^1 p\left(\widehat{X}_{i,t}\right) di$. Thus the planner satisfies the materials balance (MB) equation, which is

$$\int_0^1 p\left(\widehat{X}_{i,t}\right) di = \overline{X}_t = (1-\delta) K_t. \tag{MB}$$

Since consumers have concave utility functions, a planner who places equal weight on the utilities of all old consumers will want to minimize the variance of $p\left(\widehat{X}_{i,t}\right)$ across types i. The following proposition provides an explicit specification of the transfer function that minimizes the variance of $p\left(\widehat{X}_{i,t}\right)$ subject to satisfying the participation constraint (PC), the truth-telling condition (TT), and the materials balance equation (MB).

Proposition 1 *Define $g(z) \equiv q\overline{X}_t + (1-q) z$ for $z > 0$. The function $p\left(\widehat{X}_{i,t}\right) \equiv g\left(\widehat{X}_{i,t}\right)$ minimizes the variance of $p\left(\widehat{X}_{i,t}\right)$ across all types i, subject to the participation constraint (PC), the truth-telling condition (TT), and the materials balance equation (MB).*

Proposition 1 implies that

$$p\left(\widehat{X}_{i,t}\right) \equiv q\overline{X}_t + (1-q) \widehat{X}_{i,t} \tag{6}$$

and truthtelling, along with $\overline{X}_t = (1-\delta) K_t$, implies

$$p\left(\widehat{X}_{i,t}\right) = p(X_{i,t}) = [1-\delta-(1-q)\varepsilon_{i,t}] K_t. \tag{7}$$

Remarkably, the linear exchange function $p\left(\widehat{X}_{i,t}\right)$ does not depend on the particular specification of the utility function. It only depends on the concavity of utility of consumption when old.

The consumption of old type-i consumers at time t consists of $p(X_{i,t}) \equiv q\overline{X}_t + (1-q) X_{i,t}$ plus or minus any intratemporal transfers between young consumers and old consumers at time t. Since these intratemporal transfers do not condition on type i, we can write $C^o_{i,t+1}$ as

$$C^o_{i,t+1} = C^o_{t+1} - (1-q) K_{t+1} \varepsilon_{i,t+1}, \tag{8}$$

where the component of consumption when old that depends on type-i is captured by $-(1-q)K_{t+1}\varepsilon_{i,t+1}$.

3.1.1. An aside on the social value of the reversibility cost

We have shown that the reversibility cost, q, enables the planner to shrink the exposure of old consumers to depreciation risk by a factor $1-q$. In this subsection, we consider two economies, H and L, on balanced growth paths with $K_{t+1} \geq K_t > (1-\delta)K_t$ so that gross investment is positive in every period. Both economies have identical preferences, identical social welfare functions, and identical technologies except that they have different constant values of q, denoted $q_H > q_L$. When the highest attainable value of social welfare, $\mathcal{W}_L$, is achieved in economy L, the optimal linear exchange function $p\left(\widehat{X}_{i,t}\right)$ is, from equation (6), $p_L\left(\widehat{X}_{i,t}\right) = q_L\overline{X}_t + (1-q_L)\widehat{X}_{i,t}$. The planner can use the same payoff function in economy H, because $p_L\left(\widehat{X}_{i,t}\right) \geq (1-q_L)\widehat{X}_{i,t} > (1-q_H)\widehat{X}_{i,t}$, which satisfies the participation constraint (PC) and $\frac{p_L(X_{i,t})-p_L\left(\widehat{X}_{i,t}\right)}{X_{i,t}-\widehat{X}_{i,t}} = 1-q_L > 1-q_H$, which satisfies the truthtelling (TT) condition in economy H, and $p_L\left(\widehat{X}_{i,t}\right)$ satisfies the materials balance (MB) condition. In addition, the planner can attain exactly the same path of allocations of C_t^y, C_t^o, and K_t in economy H as in economy L. Therefore, the planner can achieve a level of social welfare in economy H as least as large as $\mathcal{W}_L$. In fact, the planner can achieve a higher level of social welfare in economy H simply using the linear exchange function $p_H\left(\widehat{X}_{i,t}\right) = q_H\overline{X}_t + (1-q_H)\widehat{X}_{i,t}$, while maintaining the same path of allocations of C_t^y, C_t^o, and K_t in economy H as in economy L. The increase in social welfare arises from the increase in risk sharing that reduces the share of idiosyncratic capital depreciation risk borne by old consumers to $1-q_H$ from $1-q_L$. Economy H is able to attain a higher level of social welfare, despite the fact that it faces a less favorable technology, specifically, a higher reversibility cost, q, than does economy L because old consumers never choose to physically transform used capital to consumption goods. That is, the linear exchange function is constructed to make such transformations inoperative and to provide more risk mitigation in economy H than in economy L.

3.2 Intertemporal Reallocation of Consumption over the Life of an Individual Consumer

Consider an intertemporal reallocation of consumption over the two periods of a given consumer's lifetime that reduces C_t^y by one unit and increases the consumer's capital carried into period $t+1$ by one unit, thereby increasing $C_{i,t+1}^o$ by $1 + F_K(K_{t+1}, L_{t+1}) - \delta - (1-q)\varepsilon_{i,t+1}$ units under the optimal risk-sharing arrangement described in Subsection 3.1. The impact of this intertemporal rearrangement of resources on social welfare, $\mathcal{W}$, in equation (4), taking account of the planner's risk mitigation using the scheme based on $p\left(\widehat{X}_{i,t}\right)$, is $d\mathcal{W} = -\frac{1-\beta}{C_t^y} + E\left\{[1 + F_K(K_{t+1}, L_{t+1}) - \delta - (1-q)\varepsilon_{i,t+1}]\,\beta \frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}$. Setting $d\mathcal{W} = 0$ yields the first-order condition

$$\frac{\beta}{1-\beta} E\left\{[1 + F_K(K_{t+1}, L_{t+1}) - \delta - (1-q)\varepsilon_{i,t+1}] \frac{C_{i,t+1}^{o^{1-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} \frac{C_t^y}{C_{i,t+1}^o}\right\} = 1. \tag{9}$$

3.3 Intratemporal Intergenerational Transfers

Consider an intratemporal transfer in period $t+1$ of consumption from each young consumer to each old consumer in the preceding generation. The impact on social welfare, $\mathcal{W}$, in equation (4) is $d\mathcal{W} = -\frac{1-\beta}{C_{t+1}^y} + (1+\eta)\,\beta E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}$. Setting $d\mathcal{W} = 0$ yields the first-order condition

$$\frac{1-\beta}{C_{t+1}^y} = (1+\eta)\,\beta E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}. \tag{10}$$

The shadow riskfree interest rate from period t to period $t+1$, $r_{f,t+1}$, satisfies

$$\frac{1-\beta}{C_t^y} = (1+r_{f,t+1})\,\beta E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}. \tag{11}$$

Divide each side of equation (11) by the corresponding side of equation (10), and rearrange to obtain

$$1 + r_{f,t+1} = (1 + \eta) \frac{C_{t+1}^y}{C_t^y}. \tag{12}$$

As shown by Samuelson (1968) in a deterministic overlapping generations model, the equilibrium rate of return on riskless assets depends on the planner's rate of intergenerational preference, η, but is independent of β, which captures the rate of time preference of individual consumers.

3.4 Optimal Allocation of Time to Labor When Young

Consider increasing the fraction of young consumers' time devoted to labor, l_t, by dl_t, which reduces the fraction of their time devoted to leisure, $1 - l_t$, by dl_t in period t. The amount of effective labor supplied increases by $(1+g)^t \, dl_t$ and the amount of output in period t increases by $F_L(K_t, L_t)(1+g)^t \, dl_t$. The impact on social welfare, $\mathcal{W}$, is $d\mathcal{W} = (1-\beta) F_L(K_t, L_t)(1+g)^t \frac{1}{C_t^y} dl_t - h'(1-l_t) dl_t$. Setting $d\mathcal{W} = 0$ implies

$$(1-\beta) F_L(K_t, L_t) \frac{(1+g)^t}{C_t^y} = h'(1-l_t). \tag{13}$$

The left hand side of equation (13) is the increase in utility associated with the additional consumption made possible by an increase in the amount of time devoted to labor; the right hand side is the loss in utility associated with the reduction in leisure when more time is devoted to labor.

3.5 Optimal Balanced Growth Paths

We now present simple conditions capturing optimal risk sharing, optimal intertemporal allocation of consumption over the life of an individual consumer, optimal intratemporal intergenerational reallocation of consumption, and optimal allocation of young consumers' time to labor and leisure along a balanced growth path.

First, the function $p\left(\widehat{X}_{i,t}\right)$ in equation (6) implements optimal risk sharing among old consumers in every period. Thus, consumption of old type-i consumers in equation (8) holds along the optimal balanced growth path.

Second, along a balanced growth path, the expected rate of return on capital is constant and denoted as $r \equiv F_K\left(K_{t+1}, L_{t+1}\right) - \delta$. The planner's optimal intertemporal allocation of consumption over an individual's lifetime in equation (9) can written as

$$E\left\{\left[1 + r - (1-q)\,\varepsilon_{i,t+1}\right]\frac{\beta}{1-\beta}\frac{1}{E\left\{C_{i,t+1}^{o\,1-\gamma}\right\}}\frac{C_t^y}{C_{i,t+1}^{o\gamma}}\right\} = 1. \tag{14}$$

Third, along a balanced growth path, $\frac{C_{t+1}^y}{C_t^y} = 1+g$, so the planner's optimal intratemporal intergenerational allocation in equation (12)can be written as

$$1 + r_f = (1+\eta)(1+g) > 1+g, \tag{15}$$

where the inequality follows from $\eta > 0$. Note that since the economy-wide average rate of return on idiosyncratically risky capital, r, is greater than or equal to r_f, equation (15) implies that, consistent with dynamic efficiency, $r > g$, which must be case for a socially optimal balanced growth path.

Fourth, we repeat, for convenience, the planner's optimal tradeoff between leisure and consumption when young in equation (13) as

$$(1-\beta)\frac{F_L\left(K_t, L_t\right)(1+g)^t}{C_t^y} = h'\left(1-l_t\right). \tag{16}$$

Along a balanced growth path, $F_L\left(K_t, L_t\right)$ is constant and C_t^y grows at rate g per period, so the left hand side of equation (16) is constant over time. As shown in Appendix B, there is a value of $l \in (0,1)$ for which the right hand side of equation (16) equals the constant value on the left hand side. It is constant along a balanced growth path.

4 Implementing the Planner's Optimal Balanced Growth Path in a Competitive Economy

We now turn our attention to competitive economies, where consumption, labor supply, and asset accumulation decisions are made by individual consumers rather than by a planner. Workers earn a competitive wage so their labor income in period t is $W_t = F_L(K_t, L_t) L_t = F_L(K_t, L_t)(1+g)^t l_t$. Owners of capital earn gross capital income (before depreciation) in period t equal to $F_K(K_t, L_t) K_t$.

In this section, we explore the impact of five distinct fiscal tools and their potential roles in implementing the planner's optimum along a balanced growth path:

1. Capital income is taxed at rate τ_K. Taxable capital income in period t is gross capital income, $F_K(K_t, L_t) K_t$, less gross investment, which, along a balanced growth path, is $(g+\delta) K_t$;

2. The planner's optimal allocation implies that the shadow riskfree rate, r_f, will satisfy $1 + r_f = (1+\eta)(1+g)$ along a balanced growth path, as shown in equation (15). In the competitive economy, consumers have the opportunity to earn a gross riskfree interest rate of $(1+\eta)(1+g)$ on bonds and a gross risky rate of return on capital with expected value of $1 + F_K(K_{t+1}, L_{t+1}) - \delta$ and standard deviation proportional to the standard deviation of $\varepsilon_{i,t+1}$. Given the riskfree interest rate and the distribution of the risky rate of return on capital, their optimal portfolio allocation decision determines the amount of government bonds, B_{t+1}, that they want to hold;

3. Consumption by all consumers–young and old–in every period is taxed at rate τ_C. Because each consumer faces the same consumption tax rate in both periods of life, the consumption tax does not introduce an intertemporal distortion in the consumer's decision problem. Consumption goods obtained by old consumers from the disposition of their used capital–whether through transformation of used capital into consumption goods or through the sale of used capital to young consumers–is exempt from the

consumption tax. However, as described below, the foregone tax revenue resulting from this exemption is captured by an additional levy on old consumers.

4. There is a balanced-budget pay-as-you-go social security system that collects taxes $\tau_S W_t$ from each young consumer in period t and pays a social security benefit of $\tau_S W_t$ to each old consumer in period t. The role of the social security system is to intratemporally transfer consumption goods between the two generations alive in each period.

5. In addition to the social security tax on labor income, young consumers pay an income tax at rate τ_L on labor income, W_t. Since consumption is taxed at rate τ_C, leisure must also be taxed at rate τ_C to avoid distorting the consumption/leisure margin. If $\tau_C > 0$, leisure is taxed by subsidizing labor through negative values of τ_L or τ_S. We show that $\tau_C + \tau_L + \tau_S = 0$ prevents distortion of the consumption/leisure margin.

We focus our analysis on balanced growth paths, along which various macroeconomic aggregates, including K_t, W_t, C_t^y, C_t^o, and B_t, grow at the rate g per period, so $w_t \equiv \frac{W_t}{K_t}$, $c_t^y \equiv \frac{C_t^y}{K_t}$, $c_t^o \equiv \frac{C_t^o}{K_t}$, and $b_t \equiv \frac{B_t}{K_t}$ are constant and equal to w, c^y, c^o, and b, respectively. Also, all five of the fiscal parameters τ_K, τ_L, τ_S, τ_C, and b are constant over time along a balanced growth path.

As described above, in period t, young consumers earn labor income W_t and pay labor income tax $\tau_L W_t$, social security tax $\tau_S W_t$, and consumption tax $\tau_C C_t^y$. In period $t+1$, when these consumers are old, they receive a social security benefit $\tau_S W_{t+1}$ and pay a consumption tax $\tau_C C_{t+1}^o$ (see equation (22) and the discussion that precedes it for a more nuanced description of the tax revenue from the consumption tax).

Young consumers in period t have disposable income of $(1 - \tau_L - \tau_S) W_t$ and spend $(1 + \tau_C) C_t^y$ to purchase consumption in period t, so their saving in period t is

$$S_t = (1 - \tau_L - \tau_S) W_t - (1 + \tau_C) C_t^y. \tag{17}$$

Young consumers use their saving, S_t, to purchase one-period riskfree government bonds,

B_{t+1}, which pay an interest rate r_f, and risky capital, K_{t+1}. When young consumers purchase newly-produced capital at the end of period t, they receive an immediate payment from the government in the amount of τ_K per unit of capital expenditure. This payment can be interpreted as an investment tax credit on new capital at rate τ_K. Equivalently, it can be viewed as an immediate 100% depreciation allowance that reduces taxable capital income by the amount of expenditure on new capital and hence reduces the consumer's taxes by τ_K per dollar of expenditure on new capital. Thus, the after-tax price of a unit of new capital is $1-\tau_K$. An immediate 100% depreciation allowance is often called *immediate expensing*, and it has been in force for qualified equipment in the United States since 2017.

In period $t+1$, when the consumers born at the beginning of period t are old, they earn gross capital income, $F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}$, which is taxed at rate τ_K. Along a balanced growth path, aggregate capital income tax revenue in period $t+1$ is $\tau_K F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}-(g+\delta)K_{t+1}$, consisting of the tax on gross capital income, $\tau_K F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}$, less an investment tax credit on gross investment by young consumers in new (rather than used) capital, which is $K_{t+2}-(1-\delta)K_{t+1}=(1+g-(1-\delta))K_{t+1}=(g+\delta)K_{t+1}$.

Since the effective price of purchasing capital is $1-\tau_K$ per unit of capital,[5] young consumers in period t spend $(1-\tau_K)K_{t+1}$ to purchase capital at the end of the period. Thus, the uses of aggregate saving by young consumers in period t can be written as

$$S_t=(1-\tau_K)K_{t+1}+B_{t+1}. \tag{18}$$

In period $t+1$, consumers who were born at the beginning of period t are old consumers. Old type-i consumers in period $t+1$ consume

$$\begin{aligned} C^o_{i,t+1}=&\frac{1}{1+\tau_C}\left[(1-\tau_K)F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}+(1+r_f)B_{t+1}+\tau_S W_{t+1}\right]\\ &+\max\{1-q,p_U\}\times\left[(1-\delta-\varepsilon_{i,t+1})K_{t+1}-\frac{\tau_C}{1+\tau_C}(1-\delta)K_{t+1}\right], \end{aligned} \tag{19}$$

where p_U is the price at which old consumers call sell a unit of used capital to young con-

[5] As discussed above, the price of new capital is $1-\tau_K$ and, as discussed below, the equilibrium price of used capital is $1-\tau_K$.

sumers.

The term in square brackets in the first line on the right hand side of equation (19) consists of three terms, each multiplied by $\frac{1}{1+\tau_C}$: $(1-\tau_K) F_K(K_{t+1}, L_{t+1}) K_{t+1}$ is gross capital income after the payment of capital income taxes; $(1+r_f) B_{t+1}$ is the principal and interest received by old consumers on their holdings of B_{t+1} units of government bonds; and $\tau_S W_{t+1}$ is the social security benefit received by old consumers in period $t+1$. Taken together, these three items enable old consumers to purchase $\frac{1}{1+\tau_C}\left[(1-\tau_K) F_K(K_{t+1}, L_{t+1}) K_{t+1} + (1+r_f) B_{t+1} + \tau_S W_{t+1}\right]$ units of consumption in period $t+1$.

The second line on the right hand side of equation (19) represents the amount of consumption an old type-i consumer can obtain by optimally disposing of used capital. The first of the two terms on the second line, $\max\{1-q, p_U\}$, captures the two different opportunities for each old consumer to dispose of used capital in exchange for consumption goods. One opportunity is to physically transform each unit of used capital into $1-q$ units of consumption goods. The other opportunity is to sell the used capital to one or more young consumers. We assume that the component of consumption financed by the disposition of used capital is exempt from the consumption tax. Because the tax authorities cannot observe the amount of used capital held by any individual old consumer, old consumers can privately transform these goods into consumption without revealing to the tax authority the amount of such consumption; these consumption goods are effectively exempt from the consumption tax. For old consumers who sell their used capital to young consumers, the exemption can be administered by presenting the bill of sale for used capital. Old consumers will be willing to sell their used capital only if $p_U \geq 1-q$.[6]

Because young consumers can use some of their disposable income to buy newly-produced output, then costlessly convert it to capital and receive an immediate reduction in taxes equal to a fraction τ_K of their expenditure on new capital, young consumers are willing to pay as

[6] If $p_U = 1-q$, an old consumer would be indifferent between directly transforming their used capital into consumption goods, on the one hand, and selling their used capital to young consumers, on the other hand. We resolve this indifference by assuming that, in this situation, an old consumer prefers to sell the used capital rather than have society bear the deadweight cost of transforming used capital to consumption goods. Effectively we are assuming that when an action, such as transforming used capital into consumption, imposes a cost on society, the consumer will avoid that action if and only if the consumer does not bear any personal cost of avoiding that action.

much as $1-\tau_K$ to purchase a unit of used capital. (A unit of used capital is equivalent to a unit of new capital, except that it is not eligible for expensing.) Thus young consumers will be willing to buy used capital only if $p_U \leq 1-\tau_K$. Therefore, there will be nonzero sales of used capital from old consumers to young consumers only if

$$1-q \leq p_U \leq 1-\tau_K, \tag{20}$$

which requires $\tau_K \leq q$.

Along a balanced growth path $K_{t+2} = (1+g) K_{t+1} > (1-\delta) K_{t+1}$ so the demand for capital by young consumers in period $t+1$ will be large enough to absorb all of the used capital offered for sale by old consumers, and some young consumers will buy new capital, which qualifies for immediate expensing. Therefore, if equation (20) holds, old consumers will sell all of their used capital (rather than transform it into consumption) and the demand for capital by young consumers will drive the equilibrium value of p_U to equal $1-\tau_K$. Therefore,

$$\max\{1-q, p_U\} = 1-\tau_K, \quad \text{if } \tau_K \leq q. \tag{21}$$

The second term on the second line of equation (19) consists of (1) $(1-\delta-\varepsilon_{i,t+1}) K_{t+1}$, which is the amount of used capital owned by an old type-i consumer and is not publicly observable, minus (2) $\frac{\tau_C}{1+\tau_C}(1-\delta) K_{t+1}$, which is an additional levy on old consumers to replace the reduction in aggregate consumption tax revenue resulting from the consumption tax exemption described above. Now substitute equation (21) into equation (19) and rearrange to obtain

$$\begin{aligned} C^o_{i,t+1} = {} & \frac{1}{1+\tau_C}\left\{\left[F_K\left(K_{t+1}, L_{t+1}\right)+1-\delta\right](1-\tau_K) K_{t+1} + (1+r_f) B_{t+1} + \tau_S W_{t+1}\right\} \\ & -(1-\tau_K) K_{t+1}\varepsilon_{i,t+1}. \end{aligned} \tag{22}$$

Now we are prepared to explore the values of the fiscal parameters that can implement the planner's optimum along a balanced growth path. We organize the presentation into five subsections. Each subsection will produce a linear equation in one or more of the five fiscal parameters: τ_K, τ_C, τ_L, τ_S, and b. The coefficients and intercepts of these linear equations depend on the planner's optimal allocations of C_t^y, C_t^o, l_t, and K_t, as well as the values of r_f, $F_K(K_t, L_t) - \delta$, and $W_t = F_L(K_t, L_t) L_t$ implied by these allocations. We focus on finding the values of the fiscal parameters that will induce the equilibrium allocations along the balanced growth path in the competitive economy to match the allocations along the balanced growth path in the planner's optimum.

4.1 Optimal Sharing of Idiosyncratic Depreciation Risks

In this subsection, we show how τ_K, the tax rate on taxable capital income, can be used to implement the sharing of idiosyncratic depreciation risks that is achieved by the planner using the function $p\left(\widehat{X}_{i,t}\right)$ in equation (6). First, observe from equation (8) that the stochastic component of $C_{i,t+1}^o$ in the planner's optimal allocation is $(1-q) K_{t+1}\varepsilon_{i,t+1}$ and from equation (22) that the stochastic component of $C_{i,t+1}^o$ in the competitive economy is $(1-\tau_K) K_{t+1}\varepsilon_{i,t+1}$. Therefore, the planner's optimal sharing of idiosyncratic risks will be attained in the competitive economy if

$$\tau_K = q. \tag{23}$$

4.2 Optimal Holding Government Bonds by Consumers

In this subsection, we discuss the amount of riskfree government bonds that consumers hold, along with risky capital and the riskfree claim on social security benefits when old to pay for consumption when old.

Calculate the economy-wide average of both sides of equation (22), using $C_{t+1}^o \equiv \int_0^1 C_{i,t+1}^o di$ from equation (3) and $\int_0^1 \varepsilon_{i,t+1} di = 0$, and multiply both sides of the resulting equation by $1+\tau_C$ to obtain

$$(1+\tau_C) C_{t+1}^o = [F_K(K_{t+1}, L_{t+1}) + 1 - \delta](1-\tau_K) K_{t+1} + (1+r_f) B_{t+1} + \tau_S W_{t+1}. \tag{24}$$

Define the economy-wide average rate of return on capital as

$$r_{t+1} \equiv F_K\left(K_{t+1}, L_{t+1}\right) - \delta. \tag{25}$$

Divide both sides of equation (24) by K_{t+1} and use the definition of r_{t+1} to obtain

$$\left(1+\tau_C\right) c^o = \left(1+r\right)\left(1-\tau_K\right) + \tau_S w + \left(1+r_f\right) b. \tag{26}$$

along a balanced growth path.

4.3 Optimal Consumption of Young Consumers

Because the utility function of each consumer is homogenous in the amount of consumption in the two periods of life, the optimal consumption of young consumers in period t is proportional to the present value of lifetime resources, $\left(1-\tau_L-\tau_S\right) W_t + \frac{1}{1+r_f}\tau_S W_{t+1}$. Since $W_{t+1} = \left(1+g\right) W_t$ along a balanced growth path, the present value of lifetime resources of a consumer born at the beginning of period t is

$$V_t = \left(1-\tau_L-\rho\tau_S\right) W_t, \tag{27}$$

where

$$0 < \rho \equiv \frac{r_f - g}{1+r_f} < 1, \tag{28}$$

and the inequality follows from equation (15). Equations (B.13) and (B.14) in Appendix B imply that along a balanced growth path, the optimal consumption of young consumers satisfies

$$\left(1+\tau_C\right) C_t^y = \psi V_t = \psi\left(1-\tau_L-\rho\tau_S\right) W_t, \tag{29}$$

where

$$0 < \psi \equiv 1 - \beta < 1. \tag{30}$$

Divide equation (29) by K_t to obtain

$$(1 + \tau_C)\, c^y = \psi\, (1 - \tau_L - \rho\tau_S)\, w \tag{31}$$

along a balanced growth path.

Lemma 1 *Along the planner's optimal balanced growth path,* $D \equiv (1 - \rho)\, c^o - \frac{\beta}{1-\beta} c^y = \frac{\pi}{1+\eta} > 0$*, where* $\pi \equiv r - r_f$ *is the ex ante risk premium on risky capital.*

4.4 Optimal Tradeoff between Leisure and Consumption When Young

From equation (B.16) in Appendix B, in a competitive economy along a balanced growth path, a consumer's optimal choices of consumption and leisure when young satisfy

$$(1 - \beta)\frac{1 - \tau_L - \tau_S}{1 + \tau_C}\frac{F_L\,(K_t, L_t)\,(1 + g)^t}{C_t^y} = h'\,(1 - l^*). \tag{32}$$

The left hand side of equation (32) is the amount of additional utility the consumer can obtain in the first period by giving up a unit of leisure to increase labor supply by one unit, earn additional after-tax wage income of $(1 - \tau_L - \tau_S)\, F_L\,(K_t, L_t)\,(1 + g)^t$, and use the additional income to purchase consumption goods at a tax-inclusive price of $1 + \tau_C$. The right hand side of equation (32) is the reduction in utility obtained from leisure.

Inspection of equation (32), which characterizes a young consumer's optimal tradeoff between consumption and leisure in a competitive economy, reveals that it is identical to the characterization of the planner's optimal allocation of leisure and consumption when young in equation (16) if and only if $\frac{1-\tau_L-\tau_S}{1+\tau_C}=1$. We write this condition more simply as[7]

$$\tau_L+\tau_S+\tau_C=0. \tag{33}$$

4.5 Government Budget Constraint

Along a balanced growth path, the government budget constraint in period t is

$$G_t=\tau_K\left[F_K-(\delta+g)\right]K_t+\tau_L W_t+\left(g-r_f\right)B_t+\tau_C\left(C_t^y+C_t^o\right). \tag{34}$$

The left hand side of equation (34) is government spending on the purchases of goods and the right hand side contains four components of government revenue: capital income tax revenue, $\tau_K\left[F_K-(\delta+g)\right]K_t$, which is the tax on gross capital income, $\tau_K F_K\left(K_t,L_t\right)K_t$, less the reduction in tax revenue resulting from the expensing of gross investment, $\tau_K\left(g+\delta\right)K_t$; labor income tax revenue, $\tau_L W_t$; bond seignorage, $\left(g-r_f\right)B_t$; and consumption tax revenue, $\tau_C\left(C_t^y+C_t^o\right)$, which requires some explanation because consumption acquired by old consumers using the proceeds of disposing of their old capital is exempt from the consumption tax. We assume that even though an old consumer's holding of used capital is private information, the transactions in which young consumers buy used capital from old consumers are publicly observable. Receipts documenting these sales, which in the aggregate total $\left(1-\tau_K\right)\left(1-\delta\right)K_t$, can be used to claim exemption from the consumption tax. If, contrary to our formulation, the consumption tax were levied on the purchases of consumption goods by old consumers using the proceeds from the sales of used capital, $\left(1-\tau_K\right)\left(1-\delta\right)K_t$, then these proceeds could be used to purchase $\frac{1}{1+\tau_C}\left(1-\tau_K\right)\left(1-\delta\right)K_t$ units of consumption, which would generate consumption tax revenue of $\frac{\tau_C}{1+\tau_C}\left(1-\tau_K\right)\left(1-\delta\right)K_t$. To replace this foregone tax revenue resulting from the exemption, the government levies an additional tax of $\frac{\tau_C}{1+\tau_C}\left(1-\tau_K\right)\left(1-\delta\right)K_t$ on old consumers, as shown in equation (19). The total tax revenue from the consumption taxes on young and old consumers, plus the levy of $\frac{\tau_C}{1+\tau_C}\left(1-\tau_K\right)\left(1-\delta\right)K_t$ on old consumers, equals $\tau_C\left(C_t^y+C_t^o\right)$. Effectively, all consumption

[7]With the balanced-budget pay-as-you-go social security system, the young consumer's anticipated social security benefit when old, $\tau_S W_{t+1}$, is independent of the consumer's labor supply when young in period t.

is taxed at rate τ_C, except for the consumption financed from the idiosyncratic depreciation shocks, $-(1-\tau_K) K_t \varepsilon_{i,t}$, which is zero in the aggregate.[8]

Two other comments about the government budget constraint are in order: First, equation (15) implies that along a balanced growth path in the planner's optimum, $r_f > g$, so that bond seignorage is negative. That is, the interest cost of government debt exceeds the increase in government debt so the government must run a primary surplus along the balanced growth path. Second, the social security tax rate, τ_S, does not enter the government budget constraint in equation (34) because the social security system is a pay-as-you-go system with a balanced budget in every period.

Now divide both sides of the government budget constraint in equation (34) by K_t and use $r = F_K(K_t, L_t) - \delta$ from equation (25), which is constant along a balanced growth path, to obtain

$$\frac{G_t}{K_t} = \tau_K (r-g) + \tau_C (c^y + c^o) + \tau_L w + (g - r_f) b. \tag{35}$$

It will be convenient to use the GDP expenditure identity from equation (2), rewritten along a balanced growth path as $G_t = Y_t - (C_t^y + C_t^o) - (g+\delta) K_t$, and the GDP income identity, $Y_t = (r+\delta) K_t + W_t$, to obtain

$$\frac{G_t}{K_t} = r - g + w - (c^y + c^o) \tag{36}$$

along a balanced growth path. Therefore, the government budget constraint in equation (35) can be written as

$$\tau_K (r-g) + \tau_C (c^y + c^o) + \tau_L w + (g - r_f) b = r - g + w - (c^y + c^o). \tag{37}$$

[8]If the idiosyncratic depreciation shocks were observable by the fiscal authority, it could simply tax all consumption at rate τ_C, with an exemption of $-(1-\tau_K) K_t \varepsilon_{i,t}$ for type-i consumers. However, such an exemption is not implementable when the depreciation shocks are private information. The scheme in the text, which exempts all consumption purchased with proceeds of used capital, and then levies an additional tax of $\frac{\tau_C}{1+\tau_C}(1-\tau_K)(1-\delta) K_t$, collects the same revenue as the unimplementable scheme that simply provides a consumption-tax exemption of $-(1-\tau_K) K_t \varepsilon_{i,t}$ for type-i consumers. We assume that the tax of $\frac{\tau_C}{1+\tau_C}(1-\tau_K)(1-\delta) K_t$ on all old consumers does not depend on the capital purchased by any individual consumer; this levy is essentially a lump-sum tax and we do not view this levy as part of the optimal capital income tax rate.

5 Fiscal Policies that Attain the Planner's Optimum

In Section 4, each of subsections 4.1 to 4.5 presents a linear equation in the five fiscal parameters τ_K, τ_C, τ_L, τ_S, and b. Each equation characterizes the values of these fiscal parameters for which a particular aspect of behavior in the competitive economy matches the corresponding aspect in the planner's optimum along a balanced growth path. Specifically, equation (23) presents the value of τ_K for which the competitive economy matches the degree of sharing of idiosyncratic risks in the planner's optimum; equation (26) is a linear equation in τ_K, τ_S, and b that describes a consumer's holding of riskfree government bonds, which along with the amount of capital in the planner's optimum and the riskfree claim on social security benefits when old, will allow the consumer to pay for the amount of consumption when old in the planner's optimum; equation (31) is a linear equation in τ_C, τ_L, and τ_S for which the optimal consumption of young consumers along a balanced growth path in the competitive economy matches the consumption of young consumers in the planner's optimum for a given level of $w = \frac{W_t}{K_t} = F_L(K_t, L_t)\frac{(1+g)^t}{K_t} l^*$; equation (33), $\tau_L + \tau_S + \tau_C = 0$, is a necessary and sufficient condition for the tradeoff between leisure and consumption in the competitive economy to match the corresponding condition in the planner's optimum; and equation (37), which expresses the government budget constraint as a linear function of τ_K, τ_L, τ_C, and b for given values of c^y, c^o, w, r, and r_f along the optimal growth path in the planner's economy. Taken together, these five linear equations in τ_K, τ_L, τ_S, τ_C, and b are a system of linear equations that can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1+r & c^o & 0 & -w & -(1+r_f) \\ 0 & c^y & \psi w & \rho\psi w & 0 \\ 0 & 1 & 1 & 1 & 0 \\ r-g & c^y+c^o & w & 0 & g-r_f \end{bmatrix} \begin{bmatrix} \tau_K \\ \tau_C \\ \tau_L \\ \tau_S \\ b \end{bmatrix} = \begin{bmatrix} q \\ 1+r-c^o \\ \psi w - c^y \\ 0 \\ r-g+w-(c^y+c^o) \end{bmatrix} \tag{38}$$

or more compactly as

$$A \times x = z, \tag{39}$$

where A is the 5×5 matrix on the left hand side of equation (38), x is the 5×1 vector of fiscal parameters $\begin{bmatrix} \tau_K & \tau_C & \tau_L & \tau_S & b \end{bmatrix}^T$, and z is the 5×1 vector on the right hand side of equation (38).

Lemma 2 *The determinant of the* 5×5 *matrix* A *on the left hand side of equation (38) is* $\det A = (1-\rho)(1+r_f)\,\psi w D > 0$*, where, from Lemma 1,* $D \equiv (1-\rho)\,c^o - \frac{\beta}{1-\beta} c^y > 0$*.*

Lemma 2 implies that the matrix A is invertible so there is a unique solution $x = A^{-1} \times z$ of the linear system in equation (39).[9]

The solution $x = A^{-1} \times z$ of the linear system will be economically meaningful only if implies that after-tax prices and after-tax wages are positive. Formally, economically meaningful values of x must lie in the set

$$P \equiv \{x\colon\ \tau_K < 1,\ \tau_C > -1,\ \tau_L + \tau_S < 1\}, \tag{40}$$

In the set P, the after-tax price of capital, $1 - \tau_K$, the after-tax price of consumption, $1 + \tau_C$, and the after-tax wage, $1 - \tau_L - \tau_S$ are positive.

Proposition 2 *The solution to the matrix equation (38) is*

$$\begin{bmatrix} \tau_K \\ \tau_C \\ \tau_L \\ \tau_S \\ b \end{bmatrix} = x^* \equiv \begin{bmatrix} q \\ -q \\ 1 - (1-q)\,\frac{c^y - \rho\psi w}{(1-\rho)\psi w} \\ (1-q)\,\frac{c^y - \psi w}{(1-\rho)\psi w} \\ -\frac{1}{1+r_f}\,(1-q)\left[(1+r) - \left(c^o - \frac{c^y - \psi w}{(1-\rho)\psi}\right)\right] \end{bmatrix}$$

[9] This uniqueness contrasts with the findings in Chari and Kehoe (1999) that "a tax system which includes any two of consumption, labor, and capital income taxes can decentralize the Ramsey allocations." (p. 1674) The introduction of privately-observed idiosyncratic depreciation shocks in the current paper adds a new dimension to the planner's optimum and requires an additional fiscal instrument. Therefore, unlike in Chari and Kehoe (1999), we cannot discard one of the three taxes on consumption, labor, and capital income.

and $x^* \in P$.

A particularly interesting feature of the optimal fiscal policy vector $x^* = A^{-1}z$ is that the optimal tax rate on taxable capital income, τ_K, is q. Even though the taxing authority cannot observe the idiosyncratic shocks, $\varepsilon_{i,t}$, the tax rate of capital income, τ_K, reduces the standard deviation of shocks to consumption of old consumers to a fraction $1 - \tau_K = 1 - q$ of the standard deviation of $K_t \varepsilon_{i,t}$. This reduction in standard deviation arises because consumers who buy used capital do not get the tax benefit associated with the purchase of new capital, which is effectively an investment tax credit at rate $\tau_K = q$. Therefore, the equilibrium price that old consumers receive when they sell their used capital is $1 - \tau_K$. It is as if the tax authority levies a tax on the sale of used capital, but instead, the reduction in the price of used capital is implemented by the equilibrium market price rather than through taxation of the sale of used capital. There is no need for the tax authority to observe the amount of used capital sold by any particular consumers.

Our finding that the optimal tax rate on capital income along a balanced growth path is $\tau_K = q$ is a major departure from the Chamley-Judd result (and others cited in the literature review in Section 1) that the optimal value of the capital income tax rate is zero in the long run. That departure reflects different definitions of taxable capital income and different economic environments. In the current paper, the capital income tax system is specified to include immediate expensing of spending on new capital, rather than spreading out the tax deductibility of capital expenditures over time through depreciation allowances equal to economic depreciation. With immediate expensing, taxable capital income is defined as gross capital income minus expenditure on the purchase of new capital. In the absence of uncertainty, this definition of taxable capital income implies that the effective tax rate on capital income equals zero and hence the after-tax private rate of return on capital, the pre-tax private rate of return on capital, and the social rate of return on capital are all equal. Thus τ_K does not distort intertemporal margins. With the inclusion of the idiosyncratic uncertainty specified in this paper, the economy-wide average rate of return on capital is still invariant to the tax rate τ_K. The role of τ_K is to provide insurance to reduce the exposure of individual old consumers to depreciation risk.

The most important differences in economic environments are that the current model includes (1) uninsurable idiosyncratic depreciation risk and (2) a reversibility cost, q, of transforming used capital into consumption. In fact, if the reversibility cost, q, is specified to equal zero, then the optimal tax rate on capital income along a balanced growth path equals zero, consistent with the Chamley-Judd result. However, when q is positive, it interacts with the uninsurable idiosyncratic depreciation risk by defining the scope of the planner's ability to spread idiosyncratic depreciation risks among old consumers in a way specified by the transfer function $p\left(\widehat{X}_{i,t}\right) \equiv q\overline{X}_t + (1-q)\widehat{X}_{i,t}$ in equation (6). The current paper provides a novel role for the capital income tax as a means of providing optimal (but partial) insurance against idiosyncratic capital depreciation risks that cannot be directly insured by the planner or a competitive economy because of the nature of the private information about the realizations of these risks.

The second element of the optimal fiscal policy vector in Proposition 2 is $\tau_C = -q$. In allocating consumption and capital to young consumers, the planner faces a one-to-one tradeoff between consumption and capital. In the competitive economy, a young consumer can reduce consumption by one unit, thereby reducing expenditure by $1+\tau_C$ units. Because the price of new capital facing young consumers is $1-\tau_K$, the consumer can acquire $(1+\tau_C)/(1-\tau_K)$ units of capital. With $\tau_K = q$ and $\tau_C = -q$, the tradeoff ratio, $(1+\tau_C)/(1-\tau_K)$, equals $(1-q)/(1-q) = 1$, which is the same as tradeoff ratio faced by the planner.

To interpret the optimal values of the remaining fiscal parameters, we present the following corollary to Proposition 2.

Corollary 1 *The vector of optimal fiscal parameters along the optimal balanced growth path is* $x^* = \left[\begin{array}{ccccc} q & -q & q-\tau_S & \tau_S & \frac{1}{1+r_f}\left\{(1-q)\left[c^o-(1+r)\right]-\tau_S w\right\}\end{array}\right]^T$, *where* $\tau_S = (1-q)\frac{c^y-\psi w}{(1-\rho)\psi w}$.

To interpret the optimal value of τ_S, rewrite equation (31) as

$$(1+\tau_C)\,c^y = \left[1-\tau_L-\tau_S+(1-\rho)\,\tau_S\right]\psi w. \tag{41}$$

Now set τ_C equal to its optimal value, $-q$, and set $-\tau_L-\tau_S = -q$ (because $\tau_C+\tau_L+\tau_S = 0$), then divide both sides by $1-q$ to obtain

$$c^y = \psi w + \frac{(1-\rho)\,\psi w}{1-q}\tau_S. \tag{42}$$

Observe that equation (42) directly implies the optimal value of τ_S, which is the fourth element of x^* in Proposition 2.

The role of the social security tax, τ_S, is to increase the present value of lifetime resources of young consumers, though the explanation is nuanced. Along the planner's optimal balanced growth path, the present value of the social security benefits when old in period $t+1$, net of the social security taxes paid when young in period t, is $-\tau_S W_t + \frac{1}{1+r_f}\tau_S W_{t+1}$ $= \left(\frac{1+g}{1+r_f} - 1\right)\tau_S W_t$. Since $r_f > g$ along the planner's optimal balanced growth path so that $\frac{1+g}{1+r_f} - 1 < 0$, it appears, at first glance, that an increase in τ_S reduces the present value of lifetime resources. However, in the context of optimal fiscal policy, an increase in τ_S must be accompanied by an equal-sized decrease in τ_L, since $\tau_L + \tau_S = -\tau_C = q$. With offsetting changes in τ_L and τ_S, after-tax wage income when young, $(1-\tau_L-\tau_S)\,W_t$, is unchanged, but an increase in τ_S increases the social security benefit when old, $\tau_S W_{t+1}$, thereby increasing the present value of lifetime resources and increasing c^y for given w. Thus, as indicated by equation (42), the balanced growth path in the competitive economy will match the balanced growth path in the planner's optimum if and only if τ_S has the same sign as $c^y - \psi w$ in the planner's optimum.

The optimal value of τ_L is shown as $q - \tau_S$ in Corollary 1. As discussed above, any change in τ_S must be matched by an equal-sized change in τ_L in the opposite direction.

Finally, consider the optimal value of b, which is shown in Corollary 1 as $b = \frac{1}{1+r_f}\left\{(1-q)\left[c^o - (1+r)\right] - \tau_S w\right\}$. Multiply both sides of this equation by K_{t+1} and use $\tau_K = q = -\tau_C$ to obtain

$$B_{t+1} = bK_{t+1} = \frac{1}{1+r_f}\left\{(1+\tau_C)\,C^o_{t+1} - \left[(1+r)\,(1-\tau_K)\,K_{t+1} + \tau_S W_{t+1}\right]\right\}. \tag{43}$$

The term in curly braces on the right hand side of equation (43) is the excess of aggregate expenditure on consumption by old consumers in period $t+1$ over the amount in square brackets, which is the aggregate amount of funds available from their holding of capital and the receipt of social security benefits in period $t+1$. Consumers use riskfree government bonds to pay for this excess amount.

6 Concluding Remarks

We have examined a package of fiscal tools, consisting of government bonds and a variety of taxes, that will finance an exogenous stream of government purchases of goods. The benchmark we use for assessing the optimality of the particular settings of the fiscal tools is the allocation chosen by a planner to maximize a social welfare function. The analytic framework we use is a version of the classic Diamond (1965) overlapping generations model extended to include added richness and specified in a way to be easily tractable. Specifically, the three major extensions are (1) capital is subject to idiosyncratic depreciation shocks that are observable only by the owners of the capital; (2) capital investment is subject to costly reversibility in the sense that one unit of consumption can be costlessly transformed into one unit of capital, but one unit of used capital can be transformed into only $1-q$ units of consumption; and (3) young consumers endogenously choose how much labor to supply. The private information about idiosyncratic depreciation shocks prevents complete sharing of these risks. The investment reversibility cost, q, plays an important role in shaping the mechanisms for sharing depreciation risks both in the planner's allocation and in the competitive economy. The endogeneity of labor supply adds an additional margin–the leisure-consumption margin–that must be addressed by the planner as well as the fiscal authority in a competitive economy.

Our analysis begins with planner's problem. Following Samuelson (1968), the social welfare function maximized by the planner is a weighted sum of the utilities of all generations with weights that decline geometrically at rate η for later generations. We find that along a balanced growth path consistent with the planner's allocation, the shadow riskfree interest rate, r_f, is constant. Similar to Samuelson's finding for the rate of return on capital, which is riskfree in his framework, the shadow riskfree interest rate along the balanced growth path depends only on the planner's rate of intergenerational preference, $\eta > 0$ and the exogenous growth rate of labor productivity, g. In particular, with Epstein-Zin-Weil preferences with an intertemporal elasticity of substitution equal to one, $1 + r_f = (1 + \eta)(1 + g)$. Because the idiosyncratic risks on capital cannot be completely shared, the average rate of return on risky capital exceeds the shadow riskfree interest rate by an endogenous risk premium.

Relative to Samuelson (1968) and relative to the literature on optimal taxation of capital income, the most novel feature of the model is the idiosyncratic depreciation risk. Because the planner cannot directly observe the amount of used capital owned by each old consumer, the planner develops a transfer scheme in which old consumers declare the amount of used capital they own and, based on that declaration by each old consumer, the planner offers to give that old consumer a specified amount of the consumption good in exchange for the declared amount of used capital. In order to elicit truthful declarations, the slope of this transfer function (that is, the amount of additional consumption the old consumer receives in exchange for a one-unit increase in the declared amount of used capital) must exceed or equal $1 - q$ because old consumers can directly transform one unit of used capital into $1 - q$ units of consumption. Since the planner wants to maximize social welfare and individual utility is concave in consumption at each age, the planner wants to minimize the cross-sectional variance of consumption of old consumers by minimizing the slope of the transfer function. To minimize this slope while inducing truthful revelation of idiosyncratic shocks, the slope of the transfer function must equal $1 - q$.

In addition to the condition describing the shadow riskfree interest rate and the condition setting the slope of the transfer function described above equal to $1 - q$, the planner's optimum satisfies an Euler equation characterizing the optimal allocation of consumption over the lifetimes of individual consumers, and a first-order condition characterizing the optimal tradeoff between leisure and consumption. Finally, the planner must make sure that the allocations of output to consumption and capital accumulation leave just enough output to satisfy the government's exogenous demand for goods. Altogether, there are five conditions on the planner's optimum.

After characterizing the planner's optimum, we move to the competitive economy where the fiscal authority has five tools–taxes on capital income, consumption, and labor income, a pay-as-you-go social security system, and government bonds–to pay for government purchases of goods; to make up for the consumption tax revenue lost to exemptions for consumption goods acquired by the disposal of used capital, these five fiscal tools are supplemented by a levy on old consumers. Our goal is to find settings of the five tools for which the balanced growth path in the competitive economy matches the balanced growth path in the planner's optimum. More precisely, we look for settings of the fiscal tools for which the five conditions in the planner's optimum are satisfied. In conducting this analysis, we take the planner's allocation as given, without, of course, being able to observe the values of the idiosyncratic depreciation shocks. There are five equations in the competitive economy that correspond to the five conditions in the planner's optimum. Each of these five equations is linear in fiscal tools, where the coefficients and intercepts depend on the planner's allocation. We choose the values of the fiscal tools so that these five equations in the competitive economy match the corresponding five equations in the planner's optimum.

We find that the optimal capital income tax rate, τ_K, equals q, which stands in sharp contrast to the literature on optimal capital income taxation, most notability, the Chamley-Judd result that the optimal tax rate on capital income is zero in the long run. In an important departure from the existing literature on optimal capital income taxation, the capital income tax system in this economy incorporates expensing of expenditures on new (but not used) capital, which is equivalent to an investment tax credit at rate τ_K on new capital. As we have shown, the economy-wide average rate of return on capital is invariant to the tax rate τ_K. The role of the capital income tax is to cushion the impact of idiosyncratic shocks on the consumption of old consumers. However, as discussed in Section 5, risk mitigation does not arise from the fiscal authority taxing the sale of used capital. Equilibrium in the market economy effectively imposes this tax by reducing the price of a unit of used capital to $1 - \tau_K = 1 - q$ and thereby reduces the cross-sectional standard deviation of the proceeds from the sale of used capital by a fraction q.

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Appendix

A Convergence of Infinite-Horizon Sum in Social Welfare Function

Proposition 3 *Assume that the utility function of individual consumers is given by equation (1). Along a balanced growth path,* $\sum_{t=0}^{\infty} (1+\eta)^{-t} U_t$ *is finite if and only if* $\eta > 0$.

Proof of Proposition 3. Along a balanced growth path l_t is constant over time, $C_t^y = (1+g)^t C_0^y$ and $\frac{1}{1-\gamma} E\left\{C_{i,t+1}^{o\,1-\gamma}\right\} = \left[(1+g)^t\right]^{1-\gamma} \frac{1}{1-\gamma} E\left\{C_{i,1}^{o\,1-\gamma}\right\}$ so $(1-\beta)\ln C_t^y = (1-\beta)\ln C_0^y + (1-\beta)\, t \ln(1+g)$ and $\frac{\beta}{1-\gamma} \ln\left(E\left\{C_{i,t+1}^{o\,1-\gamma}\right\}\right) = \frac{\beta}{1-\gamma} \ln\left(E\left\{C_{i,1}^{o\,1-\gamma}\right\}\right) + \beta t \ln(1+g)$.[10] Therefore, $U_t = U_0 + t\ln(1+g)$ so $\sum_{t=0}^{\infty} (1+\eta)^{-t} U_t = U_0 \sum_{t=0}^{\infty} (1+\eta)^{-t} + \ln(1+g) \sum_{t=0}^{\infty} t(1+\eta)^{-t}$. The first of the two summations on the right hand side is finite if and only if $\eta > 0$. Rewrite the second summation, $\sum_{t=0}^{\infty} t(1+\eta)^{-t}$, as $\sum_{t=0}^{\infty} t x^t$ where $x = \frac{1}{1+\eta} > 0$. Note that $\sum_{t=0}^{\infty} t x^t = x \sum_{t=0}^{\infty} t x^{t-1} = x \frac{d}{dx} \sum_{t=0}^{\infty} x^t = x \frac{d}{dx} \frac{1}{1-x} = x \frac{1}{(1-x)^2}$ is finite if and only if $x < 1$, equivalently, $\eta > 0$. ■

B Optimal Decisions of Individual Consumers

In this appendix, we derive optimal consumption in each period and optimal leisure when young for a consumer born at the beginning of period t. Rewrite equation (17), $S_t = (1-\tau_L-\tau_S) W_t - (1+\tau_C) C_t^y$, to obtain

$$(1+\tau_C) C_t^y = (1-\tau_L-\tau_S) W_t - S_t. \tag{B.1}$$

Define

$$Z_{t+1} \equiv (1-\tau_K) K_{t+1} + B_{t+1} + \frac{1}{1+r_f} \tau_S W_{t+1}, \tag{B.2}$$

[10] If $\gamma = 1$, then $\frac{\beta}{1-\gamma} \ln\left(E\left\{C_{i,t+1}^{o\,1-\gamma}\right\}\right)$ is replaced by $\beta \ln E\left\{\ln C_{i,t+1}^{o}\right\} = \beta \ln E\left\{\ln C_{i,1}^{o}\right\} + \beta t \ln(1+g)$.

which consists of the consumer's holding of tradeable assets from equation (18), $S_t = (1-\tau_K) K_{t+1} + B_{t+1}$, plus the present value of the non-tradeable claim on the future social security benefit when old, $\frac{1}{1+r_f}\tau_S W_{t+1}$. Therefore,

$$S_t = Z_{t+1} - \frac{1}{1+r_f}\tau_S W_{t+1}. \tag{B.3}$$

Substitute the expression for S_t from equation (B.3) into the right hand side of equation (B.1) to obtain

$$(1+\tau_C) C_t^y = (1-\tau_L-\tau_S) W_t + \frac{1}{1+r_f}\tau_S W_{t+1} - Z_{t+1}. \tag{B.4}$$

Now consider consumption when old. Rewrite equation (22),

$$C_{i,t+1}^o = \frac{1}{1+\tau_C}\left\{\left[F_K\left(K_{t+1}, L_{t+1}\right) + 1 - \delta\right](1-\tau_K) K_{t+1} + (1+r_f) B_{t+1} + \tau_S W_{t+1}\right\} - (1-\tau_K) K_{t+1}\varepsilon_{i,t+1},$$

using the expected rate of return on capital, which is constant along a balanced growth path, $r \equiv F_K\left(K_{t+1}, L_{t+1}\right) - \delta$, as

$$C_{i,t+1}^o = \frac{1}{1+\tau_C}\left\{(1+r)\frac{(1-\tau_K) K_{t+1}}{Z_{t+1}} + (1+r_f)\frac{B_{t+1} + \frac{1}{1+r_f}\tau_S W_{t+1}}{Z_{t+1}}\right\} Z_{t+1} - (1-\tau_K) K_{t+1}\varepsilon_{i,t+1}. \tag{B.5}$$

Equivalently,

$$C_{i,t+1}^o = \frac{1}{1+\tau_C}\left\{(1+r)\omega + (1+r_f)(1-\omega)\right\} Z_{t+1} - (1-\tau_K) K_{t+1}\varepsilon_{i,t+1}, \tag{B.6}$$

where $\omega \equiv \frac{(1-\tau_K)K_{t+1}}{Z_{t+1}}$ and $1-\omega = \frac{B_{t+1}+\frac{1}{1+r_f}\tau_S W_{t+1}}{Z_{t+1}}$. Define the expected rate of return on the portfolio of tradeable and non-tradeable claims as

$$R(\omega) \equiv (1+r)\omega + (1+r_f)(1-\omega). \tag{B.7}$$

Thus, the consumption of an old type-i consumer in period $t+1$ can be written as

$$C^o_{i,t+1} = \frac{1}{1+\tau_C}\left\{R(\omega) - (1+\tau_C)\,\omega\varepsilon_{i,t+1}\right\} Z_{t+1}. \tag{B.8}$$

The utility function of a type-i consumer born at the beginning of period t is, from equation (1),

$$u_{i,t} = (1-\beta)\ln C^y_t + h(1-l_t) + \frac{\beta}{1-\gamma}\ln\left(E\left\{C^{o\,1-\gamma}_{i,t+1}\right\}\right). \tag{B.9}$$

Differentiate the utility function in equation (B.9) with respect to Z_{t+1} and set the derivative equal to zero to obtain

$$\frac{1-\beta}{C^y_t}\frac{\partial C^y_t}{\partial Z_{t+1}} + E\left\{\beta\frac{C^{o\,-\gamma}_{i,t+1}}{E\left\{C^{o\,1-\gamma}_{i,t+1}\right\}}\frac{\partial C^o_{i,t+1}}{\partial Z_{t+1}}\right\} = 0. \tag{B.10}$$

Use equation (B.4) to obtain $\frac{\partial C^y_t}{\partial Z_{t+1}} = -\frac{1}{1+\tau_C}$ and equation (B.8) to obtain $\frac{\partial C^o_{i,t+1}}{\partial Z_{t+1}} = \frac{C^o_{i,t+1}}{Z_{t+1}}$ and substitute these partial derivatives into equation (B.10) to obtain

$$\frac{1}{1+\tau_C}\frac{1-\beta}{C^y_t} = E\left\{\beta\frac{C^{o\,-\gamma}_{i,t+1}}{E\left\{C^{o\,1-\gamma}_{i,t+1}\right\}}\frac{C^o_{i,t+1}}{Z_{t+1}}\right\} = \frac{\beta}{Z_{t+1}}, \tag{B.11}$$

where the second equality follows from the fact that Z_{t+1} is not random. Use equation (B.4) to substitute $(1-\tau_L-\tau_S)\,W_t + \frac{1}{1+r_f}\tau_S W_{t+1} - (1+\tau_C)\,C^y_t$ for Z_{t+1} in equation (B.11) and rearrange to obtain

$$(1+\tau_C)\,C^y_t = (1-\beta)\left[(1-\tau_L-\tau_S)\,W_t + \frac{1}{1+r_f}\tau_S W_{t+1}\right]. \tag{B.12}$$

Use $W_{t+1} = (1+g)\,W_t$ along a balanced growth, together with $\rho \equiv \frac{r_f-g}{1+r_f}$ from equation (28) and

$$\psi \equiv 1-\beta \tag{B.13}$$

to rewrite equation (B.12) as

$$(1+\tau_C)\, C_t^y = \psi\,(1-\tau_L-\rho\tau_S)\, W_t. \tag{B.14}$$

Differentiate the utility function in equation (B.9) with respect to l_t and set the derivative equal to zero to obtain

$$\frac{1-\beta}{C_t^y}\frac{\partial C_t^y}{\partial l_t} - h'(1-l_t) = 0 \tag{B.15}$$

and use $\frac{\partial C_t^y}{\partial l_t} = \frac{1-\tau_L-\tau_S}{1+\tau_C} F_L(K_t, L_t)(1+g)^t$ to obtain

$$(1-\beta)\,\frac{1-\tau_L-\tau_S}{1+\tau_C}\,\frac{F_L(K_t, L_t)(1+g)^t}{C_t^y} = h'(1-l_t). \tag{B.16}$$

Along a balanced growth path, C_t^y grows at the rate g per period, so the left hand side of equation (B.16) is constant. Observe that $\lim_{l\to 0}\frac{F_L(K_t,L_t)}{C_t^y} = \frac{F_L(K_t,0)}{\lim_{l\to 0} C_t^y} = \infty$ and $\lim_{l\to 0} h'(1-l_t) = h'(1) < \infty$ so the left hand side of equation (B.16) is greater than the right hand side of equation (B.16) in the limit as $l \to 0$. However, $\lim_{l\to 1}\frac{F_L(K_t,L_t)}{C_t^y} = \frac{F_L\left(K_t,(1+g)^t\right)}{\lim_{l\to 1} C_t^y} < \infty$ and $\lim_{l\to 1} h'(1-l_t) = \lim_{x\to 0} h'(x) = \infty$ so the left hand side of equation (B.16) is less than the right hand side of equation (B.16) in the limit as $l \to 1$. Therefore, equation (B.16) has a solution with $0 < l < 1$.

C The Planning Problem

As shown in Section 4.1 the optimal allocation of idiosyncratic depreciation risk results in $C_{i,t+1}^o = C_{t+1}^o - (1-q)\, K_{t+1}\varepsilon_{i,t+1}$, for some C_{t+1}^o to be determined as part of the planner's optimization problem. Attach a sequence of Lagrange multipliers μ_t to the feasibility constraints

$$C_t^y + C_t^o + K_{t+1} - (1-\delta)\, K_t + G_t = F(K_t, L_t)\,,$$

and rewrite the planner's problem as

$$\min_{\mu_t \geq 0} \max_{C_t^y, C_0^o, C_{t+1}^o, K_{t+1}, l_t} \left\{ \begin{array}{c} \sum_{t=0}^{\infty} (1+\eta)^{-t} \left[(1-\beta) \ln C_t^y + h(1-l_t) + \frac{\beta}{1-\gamma} \ln \left(E\left\{ C_{i,t+1}^{o^{1-\gamma}} \right\} \right) \right] \\ -\sum_{j=0}^{\infty} \mu_t \left[C_t^y + C_t^o + K_{t+1} - (1-\delta) K_t + G_t - F(K_t, L_t) \right] \\ +(1+\eta) \frac{\beta}{1-\gamma} \ln \left(E\left\{ C_{i,0}^{o^{1-\gamma}} \right\} \right) \end{array} \right\}$$

The resulting first order condition (FOC) for C_t^y gives

$$(1+\eta)^{-t} \frac{1-\beta}{C_t^y} = \mu_t. \tag{C.1}$$

Similarly, the FOC with respect to C_{t+1}^o yields

$$(1+\eta)^{-t} \beta \frac{E\left\{ C_{i,t+1}^{o^{-\gamma}} \right\}}{E\left\{ C_{i,t+1}^{o^{1-\gamma}} \right\}} = \mu_{t+1}. \tag{C.2}$$

The FOC with respect to K_{t+1} is

$$-\beta (1+\eta)^{-t} \frac{E\left\{ C_{i,t+1}^{o^{-\gamma}} (1-q) \varepsilon_{i,t+1} \right\}}{E\left\{ C_{i,t+1}^{o^{1-\gamma}} \right\}} - \mu_t + \mu_{t+1} \left[(1-\delta) + F_K(K_{t+1}, L_{t+1}) \right] = 0. \tag{C.3}$$

The FOC with respect to l_t is

$$(1+\eta)^{-t} h' (1-l_t) = \mu_t (1+g)^t F_L(K_t, L_t). \tag{C.4}$$

Evaluating (C.1) at $t+1$ and combining with (C.2) yields

$$(1-\beta) \frac{1}{C_{t+1}^y} = (1+\eta) \beta \frac{E\left\{ C_{i,t+1}^{o^{-\gamma}} \right\}}{E\left\{ C_{i,t+1}^{o^{1-\gamma}} \right\}}, \tag{C.5}$$

which is equation (10). (The first order condition with respect to C_0^o along with equation (C.1) evaluated at $t = 0$ imply that (C.5) also applies when the time-subscript $t+1$ in (C.5) is replaced with zero.)

Substituting μ_t from (C.1) and μ_{t+1} from (C.2) into (C.3), and dividing all terms by $(1+\eta)^{-t}$ gives

$$-\beta \frac{E\left\{C_{i,t+1}^{o^{-\gamma}}(1-q)\,\varepsilon_{i,t+1}\right\}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} - \frac{1-\beta}{C_t^y} + \beta \frac{E\left\{C_{i,t+1}^{o^{-\gamma}}\right\}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\left[(1-\delta)+F_K\left(K_{t+1},L_{t+1}\right)\right]=0,$$

which is equation (9), after dividing through by $\frac{1-\beta}{C_t^y}$ and collecting terms.

Finally, substituting (C.1) into (C.4) and re-arranging gives (16).

D Proofs

Proof of Proposition 1. Since $\overline{X}_t = (1-\delta)\,K_t > 0$ and $q \geq 0$, we have $g(z) \geq (1-q)\,z$, so $p\left(\widehat{X}_{i,t}\right) \geq (1-q)\,\widehat{X}_{i,t}$, which is the participation constraint (PC). Since $\frac{g(z_2)-g(z_1)}{z_2-z_1} = 1-q$, the function $p\left(\widehat{X}_{i,t}\right) \equiv g\left(\widehat{X}_{i,t}\right)$ satisfies the truthtelling condition (TT). Setting $p\left(\widehat{X}_{i,t}\right) \equiv g\left(\widehat{X}_{i,t}\right)$, satisfaction of the TT condition implies that $\widehat{X}_{i,t} = X_{i,t}$, so $\int_0^1 p\left(\widehat{X}_{i,t}\right) di = q\overline{X}_t + (1-q)\int_0^1 \widehat{X}_{i,t} di = q\overline{X}_t + (1-q)\int_0^1 X_{i,t} di = \overline{X}_t$, which is the materials balance (MB) condition.

Now consider an alternative candidate function $\widetilde{g}(z) \equiv g(z) + d(z)$ where $d(z)$ is simply the difference in value between $\widetilde{g}(z)$ and $g(z)$. It suffices to prove that if $\widetilde{g}(z)$ satisfies the truth-telling condition (TT), then $var(\widetilde{g}(z)) \geq var(g(z))$. Observe that $\frac{\widetilde{g}(z_2)-\widetilde{g}(z_1)}{z_2-z_1} = \frac{g(z_2)-g(z_1)}{z_2-z_1} + \frac{d(z_2)-d(z_1)}{z_2-z_1} = 1-q+\frac{d(z_2)-d(z_1)}{z_2-z_1}$. In order for $\widetilde{g}(z)$ to satisfy the truthtelling condition, $\frac{d(z_2)-d(z_1)}{z_2-z_1}$ must be non-negative. Therefore, $d(z)$ is non-decreasing in z, and hence $Cov(g(z), d(z)) \geq 0$. Therefore, $var(\widetilde{g}(z)) = var(g(z)) + var(d(z)) + 2Cov(g(z), d(z)) \geq var(g(z))$. ■

Proof of Lemma 1. Use $1-\rho = 1-\frac{r_f-g}{1+r_f} = \frac{1+g}{1+r_f}$ to rewrite $D \equiv (1-\rho)\,c^o - \frac{\beta}{1-\beta}c^y$ as $D = \frac{1}{K_t}\left[\frac{1+g}{1+r_f}c^o K_t - \frac{\beta}{1-\beta}c^y K_t\right] = \frac{1}{K_t}\left[\frac{1}{1+r_f}c^o K_{t+1} - \frac{\beta}{1-\beta}c^y K_t\right]$ so

$$D = \frac{1}{K_t}\left[\frac{1}{1+r_f}C_{t+1}^o - \frac{\beta}{1-\beta}C_t^y\right] \tag{D.1}$$

along a balanced growth path.

We repeat, for convenience, the Euler equation for risky capital from equation (14), replacing $F_K(K_{t+1}, L_{t+1}) - \delta$ with r to obtain

$$\frac{\beta}{1-\beta} E\left\{[1 + r - (1-q)\varepsilon_{i,t+1}] \frac{1}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} \frac{C_t^y}{C_{i,t+1}^{o^{\gamma}}}\right\} = 1. \tag{D.2}$$

Similarly, the Euler equation for riskfree assets is

$$(1 + r_f) \frac{\beta}{1-\beta} E\left\{\frac{1}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} \frac{C_t^y}{C_{i,t+1}^{o^{\gamma}}}\right\} = 1. \tag{D.3}$$

Subtract equation (D.3) from equation (D.2) and rearrange to obtain

$$\pi \equiv r - r_f = \left[E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}\right]^{-1} E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} (1-q)\varepsilon_{i,t+1}\right\}. \tag{D.4}$$

Since $\frac{C_{t+1}^o - (1-q)K_{t+1}\varepsilon_{i,t+1}}{C_{i,t+1}^o} \equiv 1$ for all i,

$$E\left\{\frac{C_{i,t+1}^{o^{1-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} \frac{C_{t+1}^o - (1-q)K_{t+1}\varepsilon_{i,t+1}}{C_{i,t+1}^o}\right\} = 1 \tag{D.5}$$

so

$$C_{t+1}^o E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\} - E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} (1-q)\varepsilon_{i,t+1}\right\} K_{t+1} = 1 \tag{D.6}$$

Now divide both sides of equation (D.6) by $E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}$ and use the expression for π from equation (D.4) to obtain

$$C_{t+1}^o - \pi K_{t+1} = \left[E\left\{\frac{C_{i,t+1}^{o^{-\gamma}}}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}}\right\}\right]^{-1} \tag{D.7}$$

Next substitute the reciprocal of the left hand side of equation (D.7) for $E\left\{\frac{1}{E\left\{C_{i,t+1}^{o^{1-\gamma}}\right\}} \frac{1}{C_{i,t+1}^{o^{\gamma}}}\right\}$

in equation (D.3) and rearrange to obtain

$$C_t^y = \frac{1}{1+r_f}\frac{1-\beta}{\beta}\left(C_{t+1}^o - \pi K_{t+1}\right). \tag{D.8}$$

Divide both sides of equation (D.8) by $K_{t+1} = (1+g)\,K_t$ to obtain

$$c^y = (c^o - \pi)\,J \tag{D.9}$$

where $J \equiv \frac{1+g}{1+r_f}\frac{1-\beta}{\beta}$.

Next divide both sides of the expenditure identity for GDP from equation (2) by K_t to obtain

$$c^y + c^o + g + \delta + \frac{G_t}{K_t} = \frac{Y_t}{K_t}. \tag{D.10}$$

Define $\alpha \equiv \frac{K_t F_K(K_t, L_t)}{Y_t}$ as the elasticity of output, Y_t, with respect to K_t, along the balanced growth path in the planner's optimum, which is constant. Use $F_K(K_t, L_t) = r + \delta$ from equation (25) and the definition of π, which is $r - r_f$, along a balanced growth path, $\frac{Y_t}{K_t} = \frac{r_f+\pi+\delta}{\alpha}$, so equation (D.10) can be written as

$$c^y + c^o + g + \delta + \frac{G_t}{K_t} = \frac{r_f + \pi + \delta}{\alpha}. \tag{D.11}$$

Substitute equation (D.9) into equation (D.11) to obtain

$$c^o = \frac{1}{1+J}\left[\frac{r_f+\delta}{\alpha} - (g+\delta) - \frac{G_t}{K_t} + \left(J + \frac{1}{\alpha}\right)\pi\right]. \tag{D.12}$$

Substitute equation (D.12) into equation (D.9) to obtain

$$c^y = \frac{J}{1+J}\left[\frac{r_f+\delta}{\alpha} - (g+\delta) - \frac{G_t}{K_t} + \frac{1-\alpha}{\alpha}\pi\right]. \tag{D.13}$$

Use $D \equiv -\rho c^o + c^o + c^y - \frac{1}{\psi} c^y$, $1 - \rho = \frac{1+g}{1+r_f}$, $\psi \equiv 1 - \beta$, and $J \equiv \frac{1+g}{1+r_f}\frac{1-\beta}{\beta}$ to obtain

$$D = \frac{1+g}{1+r_f} c^o - \frac{\beta}{1-\beta} c^y = \frac{\beta}{1-\beta} \left(J c^o - c^y \right). \tag{D.14}$$

Now substitute equations (D.12) and (D.13) into equation (D.14) to obtain

$$D = \frac{\beta}{1-\beta} \frac{J}{1+J} \left[\left(J + \frac{1}{\alpha} \right) - \frac{1-\alpha}{\alpha} \right] \pi = \frac{\beta}{1-\beta} J \pi. \tag{D.15}$$

Substitute $J \equiv \frac{1+g}{1+r_f}\frac{1-\beta}{\beta}$ into equation (D.15) to obtain

$$D = \frac{1+g}{1+r_f} \pi = \frac{1}{1+\eta} \pi, \tag{D.16}$$

where the second equality follows from equation (15). ■

Proof of Lemma 2. Use a cofactor expansion along the first row of A to obtain

$$\det A = \det \begin{bmatrix} c^o & 0 & -w & -(1+r_f) \\ c^y & \psi w & \rho \psi w & 0 \\ 1 & 1 & 1 & 0 \\ c^y + c^o & w & 0 & g - r_f \end{bmatrix} \tag{D.17}$$

and then use a cofactor expansion along the fourth column of the 4×4 matrix in equation (D.17) to obtain

$$\det A = (1+r_f) \det \begin{bmatrix} c^y & \psi w & \rho \psi w \\ 1 & 1 & 1 \\ c^y + c^o & w & 0 \end{bmatrix} + (g - r_f) \det \begin{bmatrix} c^o & 0 & -w \\ c^y & \psi w & \rho \psi w \\ 1 & 1 & 1 \end{bmatrix}. \tag{D.18}$$

Compute the determinants of the two 3×3 matrices in equation (D.18) to obtain

$$\begin{aligned} \det A = {} & (1+r_f) \left[\rho \psi w^2 + \psi w \left(c^y + c^o \right) - \rho \psi w \left(c^y + c^o \right) - w c^y \right] \\ & + (g - r_f) \left[\psi w c^o - c^y w + \psi w^2 - \rho \psi w c^o \right] \end{aligned} \tag{D.19}$$

and use $\rho = \frac{r_f - g}{1+r_f}$ to obtain

$$\det A = (1+r_f)\left[\begin{array}{c} \rho\psi w^2 + \psi w\,(c^y + c^o) - \rho\psi w\,(c^y + c^o) - wc^y \\ -\rho\left[\psi w c^o - c^y w + \psi w^2 - \rho\psi w c^o\right] \end{array}\right]. \tag{D.20}$$

Rearrange the expression for $\det A$ to obtain

$$\det A = (1+r_f)\,\psi w\left[\begin{array}{c} (c^y + c^o) - \rho\,(c^y + c^o) - \frac{1}{\psi}c^y \\ -\rho c^o + \rho c^y \frac{1}{\psi} + \rho^2 c^o \end{array}\right] \tag{D.21}$$

and simplify to obtain

$$\det A = (1-\rho)\,(1+r_f)\,\psi w\left[(c^y + c^o) - \frac{1}{\psi}c^y - \rho c^o\right]. \tag{D.22}$$

Use $D \equiv -\rho c^o + c^o + c^y - \frac{1}{\psi}c^y > 0$ from Lemma 1 to obtain

$$\det A = (1-\rho)\,(1+r_f)\,\psi w D > 0. \tag{D.23}$$

■

The following lemma is useful in proving Proposition 2.

Lemma 3 *Define* $\zeta \equiv r - g - \rho\,(1+r)$ *along the planner's optimal growth path and recall* $\pi \equiv r - r_f$. *Then* $\zeta = \frac{1}{1+\eta}\pi$.

Proof of Lemma 3. Use $\rho \equiv \frac{r_f - g}{1+r_f}$ to obtain $\zeta \equiv r - g - \rho\,(1+r) = r - g - \frac{r_f-g}{1+r_f}(1+r) = r_f + \pi - g - \frac{r_f-g}{1+r_f}(1+r_f+\pi) = r_f - g - \frac{r_f-g}{1+r_f}(1+r_f) + \pi - \frac{r_f-g}{1+r_f}\pi = \left(1 - \frac{r_f-g}{1+r_f}\right)\pi = \left(\frac{1+g}{1+r_f}\right)\pi$ $= \frac{\pi}{1+\eta}$, where the final equality follows from equation (15). ■

Proof of Proposition 2. The inner product of the first row of the matrix and the solution vector is q, which matches the first element on the right hand side of the matrix equation. The inner product of the second row of the matrix and the solution vector is $(1+r)\,q - qc^o - w\,(1-q)\,\frac{c^y - \psi w}{(1-\rho)\psi w} + (1-q)\left[(1+r) - \left(c^o - \frac{c^y-\psi w}{(1-\rho)\psi}\right)\right] = (1+r)\,q + (1-q)\,(1+r) - qc^o - (1-q)\,c^o - (1-q)\,\frac{c^y-\psi w}{(1-\rho)\psi} + (1-q)\,\frac{c^y-\psi w}{(1-\rho)\psi} = 1 + r - c^o$, which matches the second element on the right hand side of the matrix equation. The inner product of the third row of the matrix and the solution vector is $-qc^y + \psi w \left[1 - (1-q)\frac{c^y - \rho\psi w}{(1-\rho)\psi w}\right] + \rho\psi w (1-q)\frac{c^y - \psi w}{(1-\rho)\psi w} = -qc^y + \psi w - (1-q)\frac{c^y}{1-\rho} + (1-q)\frac{\rho\psi w}{1-\rho} + \rho(1-q)\frac{c^y}{1-\rho} - \rho(1-q)\frac{\psi w}{1-\rho} = -qc^y + \psi w - (1-q)c^y = \psi w - c^y$, which matches the third element on the right hand side of the matrix equation. The inner product of the fourth row of the matrix and the solution vector is $-q + \left[1 - (1-q)\frac{c^y - \rho\psi w}{(1-\rho)\psi w}\right] + \left[(1-q)\frac{c^y - \psi w}{(1-\rho)\psi w}\right] = (1-q)\left[1 - \frac{c^y - \rho\psi w}{(1-\rho)\psi w} + \frac{c^y - \psi w}{(1-\rho)\psi w}\right] = (1-q)\left[1 + \frac{\rho}{1-\rho} - \frac{1}{1-\rho}\right] = 0$, which matches the fourth element on the right hand side of the matrix equation. The inner product of the fifth row of the matrix and the solution vector is $q(r-g) - q(c^y + c^o) + w\left[1 - (1-q)\frac{c^y - \rho\psi w}{(1-\rho)\psi w}\right] + \frac{r_f - g}{1+r_f}(1-q)\left[(1+r) - \left(c^o - \frac{c^y - \psi w}{(1-\rho)\psi}\right)\right]$ $= q(r-g) - q(c^y + c^o) + w - (1-q)\frac{c^y - \rho\psi w}{(1-\rho)\psi} + \rho(1-q)(1+r) - \rho(1-q)\left(c^o - \frac{c^y - \psi w}{(1-\rho)\psi}\right) = (1-q)(r-g) + q(r-g) - (1-q)(r-g) + (1-q)(c^y + c^o) - (1-q)(c^y + c^o) - q(c^y + c^o) + w - (1-q)\frac{c^y - \rho\psi w}{(1-\rho)\psi} + \rho(1-q)(1+r) - \rho(1-q)\left(c^o - \frac{c^y - \psi w}{(1-\rho)\psi}\right) = r - g - (c^y + c^o) + w - (1-q)\left[(r-g) - \rho(1+r) - (c^y + c^o) + \frac{c^y - \rho\psi w}{(1-\rho)\psi} + \rho\left(c^o - \frac{c^y - \psi w}{(1-\rho)\psi}\right)\right] = r - g - (c^y + c^o) + w - (1-q)\left[(r-g) - \rho(1+r) - (c^y + c^o) + \frac{c^y}{(1-\rho)\psi} + \rho c^o - \frac{\rho c^y}{(1-\rho)\psi}\right] = r - g - (c^y + c^o) + w - (1-q)\left[(r-g) - \rho(1+r) - (c^y + c^o) + \rho c^o + \frac{c^y}{\psi}\right]$. Use the definitions $D \equiv -\rho c^o + c^y + c^o - \frac{1}{\psi}c^y$ and $\zeta \equiv r - g - \rho(1+r)$ to obtain $r - g - (c^y + c^o) + w - (1-q)(\zeta - D) = r - g - (c^y + c^o) + w$ since Lemma 1 states that $D = \frac{\pi}{1+\eta}$ and Lemma 3 implies that $\zeta = \frac{\pi}{1+\eta}$. Therefore, the inner product of the fifth row of the matrix and the solution vector is $r - g - (c^y + c^o) + w$, which matches the fifth element on the right hand side of the matrix equation. ■

Now prove that $x^* \in P \equiv \{x\colon \tau_K < 1, \tau_C > -1, \tau_L + \tau_S < 1\}$. By inspection of x^*, $\tau_K = q < 1$, and $\tau_C = -q > -1$, which implies, since $\tau_C + \tau_L + \tau_S = 0$, that $\tau_L + \tau_S = q < 1$.

Proof. of Corollary 1 The first, second, and fourth elements of x^* follow directly from Proposition 2. From Proposition 2, the third element of x^* can be written as $1 - (1-q)\left[\frac{c^y - \psi w}{(1-\rho)\psi w} + \frac{\psi w - \rho\psi w}{(1-\rho)\psi w}\right]$ $= 1 - (1-q)\left[\frac{c^y - \psi w}{(1-\rho)\psi w} + 1\right] = q - (1-q)\frac{c^y - \psi w}{(1-\rho)\psi w} = q - \tau_S$, which is the third element of x^* in Corollary 1. From Proposition 2, the fifth element of x^* can be written as $\frac{1}{1+r_f}(1-q)\left[c^o - (1+r) - \frac{c^y - \psi w}{(1-\rho)\psi w}w\right] = \frac{1}{1+r_f}\left\{(1-q)\left[c^o - (1+r)\right] - (1-q)\frac{c^y - \psi w}{(1-\rho)\psi w}w\right\} = \frac{1}{1+r_f}\left\{(1-q)\left[c^o - (1+r)\right] - \tau_S w\right\}$. ■