NBER WORKING PAPER SERIES

# MONEY IN A HETEROGENEOUS AGENT MODEL

Roger E.A. Farmer

Working Paper 32836
http://www.nber.org/papers/w32836

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2024

I have no sources of funding to declare. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

Money in a Heterogeneous Agent Model
Roger E.A. Farmer
NBER Working Paper No. 32836
August 2024
JEL No. D52,E30

**ABSTRACT**

I introduce money into an incomplete markets model with heterogeneous agents and uninsurable income risk. I show that the model exhibits both non-monetary and monetary equilibria, with the latter existing when income risk is sufficiently high. Using numerical methods, I characterize the properties of these equilibria and analyze their stability. I find that for a range of realistic parameter values, the non-monetary equilibrium is dynamically inefficient and indeterminate, and there is a second determinate monetary equilibrium with positive valued fiat money.

Roger E.A. Farmer
UCLA
Department of Economics
Box 951477
Los Angeles, CA 90095-1477
and CEPR
and also NBER
rfarmer@econ.ucla.edu

# Money in a Heterogeneous Agent Model

*By* Roger E. A. Farmer*

*I introduce money into an incomplete markets model with heterogeneous agents and uninsurable income risk. I show that the model exhibits both non-monetary and monetary equilibria, with the latter existing when income risk is sufficiently high. Using numerical methods, I characterize the properties of these equilibria and analyze their stability. I find that for a range of realistic parameter values, the non-monetary equilibrium is dynamically inefficient and indeterminate, and there is a second determinate monetary equilibrium with positive valued fiat money.*

## I. Introduction

In recent decades, many developed economies have experienced persistently low interest rates, often below the growth rate of GDP. This phenomenon is inconsistent with representative agent macroeconomic models and has generated interest in alternative models of which the leading explanations are the overlapping generations model (Weil, 1989; Gârleanu and Panageas, 2014; Farmer, 2018), long-run risk (Bansal and Yaron, 2004), rare disasters (Reitz, 1988; Barro, 2006) and uninsurable income risk (Huggett, 1993). This paper follows the path of uninsurable income risk by developing a version of the Huggett (1993) model with money.

Early contributions to the individual agent problem with uninsurable risk include Schectman (1976) and Schectman and Escudero (1977) who studied the agent's problem and Bewley (1977), Aiyagari (1994) and Huggett (1993) who developed general equilibrium models. Krusell and Smith (1998), in a model with production and aggregate capital, show that uninsurable risk does not have first-order effects on the properties of the model. The Krusell-Smith approach has been standard for at least three decades.

More recently, Reiter (2009, 2010) developed state-space methods that retain non-linearity in individual variables but linearize the model in aggregate shocks. Reiter (2023) extends this approach to second-order perturbations. An alternative linearization method developed by Boppart et al. (2018) and Auclert et al. (2021) linearizes the model in a space of sequences that converge locally to a steady state. To date, the models studied in this literature have assumed that the steady state around which the linearization takes place is determinate, an assumption which requires the steady-state equilibrium to be locally unique.

In this paper I contribute to our understanding of the issue of a low safe rate

* Department of Economics, University of Warwick, r.farmer.1@warwick.ac.uk and Department of Economics, UCLA, rfarmer@econ.ucla.edu.

of return by introducing money into an incomplete markets model with heterogeneous agents and uninsurable income risk. Building on the pure trade model of Huggett (1993), I show that idiosyncratic risk can lead to equilibria with interest rates below the growth rate, and I show how the introduction of fiat money affects the properties of the low interest rate equilibrium.

The introduction of fiat money to the Huggett model leads to two possible types of steady-state equilibria: a non-monetary equilibrium similar to that in Huggett's original model, and a new monetary equilibrium in which fiat money has positive value. Using numerical methods, I characterize these equilibria and analyze their stability properties for a range of parameter values using the state-space approach introduced by Reiter (2009, 2010).

The key contribution of my paper is to demonstrate how income uncertainty can simultaneously explain low interest rates and create a role for fiat money in incomplete market economies. I find that when income risk is sufficiently high, the non-monetary equilibrium becomes dynamically inefficient and indeterminate and a second, monetary equilibrium emerges. The monetary equilibrium is both locally unique and Pareto superior to the non-monetary equilibrium.

## II. Model Equations

There is a continuum of agents each of whom solves the problem

$$\max u_t = \sum_{t=1}^{\infty} \beta^{t-1} \mathbb{E}_t \left[ u(c_t) \right], \tag{1}$$

$$c_t + a_{t+1} = y_t, \tag{2}$$

$$y_t = R_t a_{t-1} + s_t, \tag{3}$$

$$R_t a_{t-1} \geq \underline{a}, \quad a_0 = a_{init}, \tag{4}$$

where $c_t$ is consumption of a unique perishable commodity, $u(c_t)$ is a concave twice differentiable utility function, $a_{init}$ is the agents initial asset holdings and $s_t$ is a non-negative random endowment with support $\mathcal{S} = \{s_1, s_2, \ldots s_{n_s}\}$ and probabilities $\pi = \{\pi_1, \pi_2, , \ldots, \pi_{n_s}\}$. I assume further that agents discount the future at rate $\beta \in (0,1)$ and utility is concave and represented by constant relative risk aversion preferences with coefficient of risk aversion $\rho$.

$$u(c_t) = \frac{{c_t}^{1-\rho}}{1-\rho}, \quad \text{and} \quad \rho \geq 1. \tag{5}$$

The realizations of $s_t$ are uncorrelated across individuals and $\mathbb{E}_{t-1}[s_t] = 1$. These assumptions imply that the aggregate endowment is non-stochastic and equal to 1.

Individuals trade consumption loans with each other at gross real interest rate $R_t$. Contracts cannot be indexed to individual income realizations and agents must remain solvent in each state. This assumption imposes the natural borrowing limit

$$R_t a_{t-1} \geq \underline{a} = -s_1. \tag{6}$$

The individual's asset holding is an element of $[\underline{a}, \bar{a}]$ where $\bar{a}$ is a finite upper bound, determined in equilibrium. Define an individual's *position* as a pair $\varepsilon_t = \{a_t, s_t\}$ where $a_t$ is his asset holding and $s_t$ is his income shock. Define the set $\mathcal{E} = [\underline{a}, \bar{a}] \times \mathcal{S}$, let $\mathcal{B}$ be the Borel $\sigma$-algebra and define a probability measure $\psi$ on $\mathcal{E}$. Then, for $B \in \mathcal{B}$, $\psi(B)$ is the mass of agents with position $\varepsilon \in \mathcal{B}$.

Huggett (1993) shows that when $R$ is constant, there exists a transition function $P : \mathcal{E} \times \mathcal{B} \to [0, 1]$ where $P(\varepsilon_t, B)$ is the probability that an agent with position $\varepsilon_t$ will have a position $\varepsilon_{t+1} \in \mathcal{B}$ and

$$\psi_{t+1}(B) = \int_{\mathcal{E}} P(\varepsilon_t, B) d\psi_t, \quad \text{for all} \in \mathcal{B}. \tag{7}$$

The following definition of a stationary equilibrium is from Huggett (1993, pp 947).

DEFINITION 1 (Huggett Equilibrium): *A Huggett equilibrium is a four-tuple* $\{c(\varepsilon), a(\varepsilon), R, \psi\}$ *satisfying*

*1) $c(\varepsilon)$ and $a(\varepsilon)$ are optimal decision rules given $R$.*

*2) Markets clear.*

$$(i) \int_{\mathcal{E}} c(\varepsilon) d\psi = 1, \qquad (ii) \int_{\mathcal{E}} a(\varepsilon) d\psi = 0.$$

*3) $\psi$ is a stationary probability measure:*

$$\psi(B) = \int_{\mathcal{E}} P(\varepsilon, B) d\psi \quad \textit{for all} \quad B \in \mathcal{B}.$$

Huggett (1993) gives conditions for existence of a solution to the agent's decision problem and existence of a stationary equilibrium and he shows, in a calibrated example, that the model is consistent with stationary equilibria in which $R < 1$. These equilibria are dynamically inefficient and they cannot exist in a model with complete markets. They occur in the Huggett model because the process that describes the evolution of the agents' assets is a regenerative process; that is, a stochastic process with time points at which, from a probabilistic point of view, the process restarts itself (Ross, 2007). Every sequence of assets returns almost surely to the lower bound, $\underline{a}$, and the agent has a stochastic, finite horizon. In this sense, the income fluctuations problem is similar to the optimization problem of the agents that populate an overlapping generations model.

## III. Money

In the paper that introduced the overlapping generations model to the English speaking world, Samuelson (1958) pointed out that when a competitive equilibrium is dynamically inefficient there is a role for fiat money to restore dynamic efficiency.[1] The analogy with the overlapping generations model suggests that there is a role for money in the Huggett model.

Let $p_t \in \mathbb{R}_+ \cup +\infty$ represent the price of a commodity in terms of money and let there be a fixed quantity, $M \in \mathbb{R}_+$ of money. If money is held in equilibrium, consumption loans and money must pay the same rate of return:

$$R_t = \frac{p_{t-1}}{p_t}. \tag{8}$$

Let $m_t = \frac{M}{p_t}$ be real money balances and notice that

$$\frac{M}{p_t} = \left(\frac{p_{t-1}}{p_t}\right)\frac{M}{p_{t-1}}, \tag{9}$$

$$m_t = R_t m_{t-1}, \tag{10}$$

where equation (10) follows from the definition of real balances and the no arbitrage assumption, Eqn. (8). This leads to the following modified definition of stationary equilibrium.

DEFINITION 2 (Stationary Monetary Equilibrium): *A stationary monetary equilibrium is a five-tuple $\{m, c(\varepsilon), a(\varepsilon), R, \psi\}$ satisfying*

*1) $c(\varepsilon)$ and $a(\varepsilon)$ are optimal decision rules given $R$.*

*2) Markets clear.*

$$(i) \int_{\mathcal{E}} c(\varepsilon) d\psi = 1, \qquad (ii) \int_{\mathcal{E}} a(\varepsilon) d\psi = m.$$

*3) $\psi$ is a stationary probability measure:*

$$\psi(B) = \int_{\mathcal{E}} P(\varepsilon, B) d\psi \quad \text{for all} \quad B \in \mathcal{B}.$$

*4) No arbitrage:*

$$m(1 - R) = 0.$$

[1] Allais (1947) published an earlier version of the overlapping generations model in French.

It follows from the fact that the Huggett model has an equilibrium (Huggett, 1993, Theorem 1 page 947) that the Huggett model with money has an equilibrium in which $p = \infty$, and $m = 0$. Huggett provides calibrated examples in which $\hat{R} < 1$, where $\hat{R}$ is the gross equilibrium interest rate. Define

$$a = \int_{\mathcal{E}} a(\varepsilon) d\psi$$

and let $a = f(R)$ be the steady state equilibrium asset demand function. It follows from the properties of individuals' demand functions that $f$ is increasing in $R$ and thus $f(1) > 0$ and the monetary Huggett model has a second equilibrium where $R = 1$, $m = f(1)$ and $p > 0$. I will refer to the equilibrium where $\hat{R}$ and $m = 0$ as the non-monetary steady state and the equilibrium where $R = 1$ as the monetary steady state.

## IV. Dynamic Equilibria

Huggett (1993) restricts himself to stationary equilibria. This section extends the Huggett equilibrium concept to dynamic equilibria and in Section V I study the determinacy properties of each of the steady-state equilibria using a discrete approximation to the monetary Huggett model based on the state space approach of Reiter (2009).

I introduce $m_t$ as an aggregate state variable and I define the concept of a dynamic monetary equilibrium in which the sequences $\{R_{t+1}\}_{t=1}^{\infty}$ and $\{m_t\}_{t=1}^{\infty}$ are jointly determined by the equations

$$\int_{\mathcal{E}} a_t(\varepsilon) d\psi = m_{t+1}, \tag{11}$$

$$R_{t+1} = \frac{m_{t+1}}{m_t}. \tag{12}$$

DEFINITION 3 (Dynamic Monetary Equilibrium ): *A dynamic monetary equilibrium is a set of sequences* $\{\psi_t, m_t, P_t, c_t(\varepsilon), a_t(\varepsilon), R_{t+1}\}_{t=1}^{\infty}$, *satisfying*

*1)* $c_t(\varepsilon)$ *and* $a_t(\varepsilon)$ *are optimal decision rules given* $\{R_{t+1}\}_{t=1}^{\infty}$.

*2) Markets clear in every period.*

$$(i) \int_{\mathcal{E}} c_t(\varepsilon) d\psi = 1, \qquad (ii) \int_{\mathcal{E}} a_t(\varepsilon) d\psi = m_{t+1}, \qquad t = 1, \dots, \infty.$$

*3) The sequence of probability measures* $\{\psi_t\}_{t=1}^{\infty}$ *is a generated by the equation*

$$\psi_{t+1}(B) = \int_{\mathcal{E}} P_t(\varepsilon, B, m_t) d\psi_t \quad \textit{for all} \quad B \in \mathcal{B},$$
$$\psi_1 = \bar{\psi}_1.$$

*4) The sequence of money balances is generated by the difference equation*

$$\begin{aligned} m_{t+1} &= R_{t+1} m_t, \quad t = 1, \dots, \infty, \\ m_1 &= \frac{M}{p_1}. \end{aligned}$$

Schectman (1976, Theorem 1.5 page 223) proves that the optimal decision of an individual is characterized by a non-decreasing function $g : \mathbb{R}_+ \to [\underline{a}, \bar{a}]$ where

$$\begin{aligned} a_{t+1} &= g(y_t), \\ c_t &= y_t - g(y_t), \\ y_{t+1} &= R_{t+1} a_t + s_{t+1}. \end{aligned}$$

The individual enters the period with cash $y_t$ which he may consume, $c_t$, or save, $a_{t+1}$. In the subsequent period his cash position is equal to $R_{t+1}a_t + s_{t+1}$ where $s_{t+1}$ is a random variable with support $\mathcal{S}$. He faces the constrained optimization problem defined in equations (1) – (4). The Kuhn-Tucker conditions for this problem have two parts. For asset values and states in which the borrowing constraint is non-binding the Euler equation holds with equality,

$$u_c(y_t - a_t) = \mathbb{E}_t \left[ u_c(R_{t+1} a_t + s_{t+1} - a_{t+1}) \right], \quad R_{t+1} a_t \geq \underline{a}. \tag{13}$$

For states in which the Euler equation would lead to a value of $a_t$ that violates the constraint $R_{t+1}a_t \geq \underline{a}$ the Euler equation holds as an inequality and the asset constraint is binding,

$$u_c(y_t - a_t) \geq \mathbb{E}_t \left[ u_c(R_{t+1} a_t + s_{t+1} - a_{t+1}) \right], \quad R_{t+1} a_t = \underline{a}. \tag{14}$$

Section V shows how to construct a discrete approximation to the sequence of functions $\{g_t\}$, where $a_t = g_t(y_t)$, and how to construct discrete approximations to the sequences of transition functions $P_t$ and measures $\psi_t$ using the methods of Reiter (2009).

## V. A Discrete Approximation to the Monetary Huggett Model

The definition of a dynamic equilibrium contains three infinite dimensional objects. The first is the mapping $g_t : \mathbb{R}_+ \to [\underline{a}, \bar{a}]$ which is a function of the current cash balance. The second is the function $\psi_t$ which keeps track of the measure of agents for each $s_t \in B$ and the third is the transition function $P_t$ which determines the evolution of $\psi_t$. In the Reiter method, $g_t$ and $\psi_t$ are approximated by finite dimensional vectors and $P_t$ is approximated by a matrix. Sections V.A and V.B explain the computational procedure used to generate these approximations.

### A. Approximating the function $g(\cdot)$

To solve the agent's optimization problem I use the method of endogenous gridpoints (Carroll, 2006). To implement this method, following Reiter (2009), I approximate the function $g_t$ by a cubic spline that holds exactly at a set of $n_x$ knot points. The method is described in detail in the following paragraphs.

Let $\{x_0, x_1, \ldots, x_{n_x-1}\}$ be a grid of values for cash, $x_t$, and let $\{a_0, a_1, \ldots, a_{n_x-1}\}$ be the asset choices that satisfy the function $a_j = g_t(x_j)$. I choose a predetermined set of positive numbers $\{dx_1, \ldots, dx_{n_x-1}\}$, where $n_x$ is a positive integer and I define an endogenous grid $\{x_0, x_0 + dx_1, \ldots x_0 + dx_{n_x-1}\}$. In this construction, $x_0$ is the cash value at which the borrowing constraint binds and it is solved for endogenously. Given $x_0$, the rest of the grid is constructed by adding the fixed numbers $\{dx_1, \ldots, dx_{n_x-1}\}$ to the initial point. This is the sense in which the grid points are endogenous.

To complete the construction I solve a set of $n_x$ Euler equations to find

$$x_0, \quad \text{and} \quad \{a_1, \ldots, a_{n_x-1}\},$$

as functions of the fixed numbers

$$\underline{a}, \quad \text{and} \quad \{dx_1 \ldots, dx_{n_x-1}\}.$$

The pairs,

$$\{(x_0, \underline{a}), (x_1, a_1), \ldots, (x_{n_x-1}, a_{n_x-1})\}$$

are connected by the equations

$$\underline{a} = g(x_0), \quad a_1 = g(x_1), \quad \ldots \quad a_{n_x-1} = g(x_{n_x-1}).$$

For any vector $\xi_t = [x_{t,0}, a_{t,1}, \ldots, a_{t,n_x-1}]^\top$ I approximate the function $g_t(x_t)$ by the cubic spline which ensures that the Kuhn Tucker conditions hold exactly at the $n_x$ knot points $\{x_{t,0}, \underline{a}\}$, $\{x_{t,1}, a_{t,1}\}$, $\ldots$, $\{x_{t,n_x-1}, a_{t,n_x-1}\}$. For values of $x < x_0$, I set $g(x) = \underline{a}$ and for values of $x > x_{n_x-1}$ I approximate $g(x)$ by a linear function with the same slope as the cubic spline at the last knot point.

$$g(x) = \begin{cases} \underline{a} & \text{if} \quad x \leq x_0 \\ CS(x; \xi) & \text{if} \quad x_0 \leq x \leq x_{n_x-1} \\ a_{n-1} + g_x|_{x_{n_x-1}} (x - x_{n_x-1}) & \text{if} \quad x \geq x_{n_x-1}. \end{cases}$$

Here, $CS$ is the cubic spline generated by $\xi$, and $g_x|_{n_x-1}$ is the slope of the cubic spline at the last knot point.

Define the set $\mathcal{D} = \mathbb{R}_+ \times [\underline{a}, \bar{a}]^{n_x - 1}$ and a function $F : \mathcal{D}^2 \to \mathbb{R}^{n_x}$.

$$F(\xi_t, \xi_{t+1}; R_{t+1}) = \begin{bmatrix} 1 & - & \dfrac{\sum_k \pi_k u_c[R_{t+1} a_{t,1} + s_k - g_{t+1}(R_{t+1} a_{t,1} + s_k)]}{u_c[x_{t,1} - a_{t,1}]} \\ 1 & - & \dfrac{\sum_k \pi_k u_c[R_{t+1} a_{t,2} + s_k - g_{t+1}(R_{t+1} a_{t,2} + s_k)]}{u_c[x_{t,2} - a_{t,2}]} \\ & & \vdots \\ 1 & - & \dfrac{\sum_k \pi_k u_c[R_{t+1} a_{t,n_x} + s_k - g_{t+1}(R_{t+1} a_{n_x} + s_k)]}{u_c[x_{t,n_x} - a_{t,n_x}]} \\ 1 & - & \dfrac{\sum_k \pi_k u_c(R_{t+1} \underline{a} + s_k - g_{t+1}(R_{t+1} \underline{a} + s_k)]}{u_c(x_0 - \underline{a})} \end{bmatrix}.$$

For fixed $R$, the steady solution to this problem is the vector $\xi$ that solves the equation

$$F(\xi, \xi; R) = 0. \tag{15}$$

The equations defined by setting $F(\xi, \xi; R) = 0$ represent the Euler equation residuals measured in units of marginal utility and the cubic spline generated by $\xi$ is a finite representation of the function $a = g(x)$ which solves the consumers problem at the steady state.

### B. *Approximating the transition function $P$ and the measure $\psi$*

The construction in this section is based on Reiter (2010, Page 6). I approximate the wealth distribution by a discrete grid of $n_y$ points on the support $[y_0, \bar{y}]$ where $\bar{y}$ is chosen so that the mass of individuals at $\bar{y}$ is approximately zero.[2] I divide the interval $[y_0, \bar{y}]$ into $n_y - 1$ equal sized intervals and I define a vector $\psi_t = [\psi_0, \psi_1, \dots, \psi_{n_y - 1}]^\top$ where $\psi_j$ is the mass of individuals with initial cash position in the interval $(y_j, y_{j+1})$ and $\psi_0$ is the mass of agents at $y_0$, the point at which the borrowing constraint binds.

An individual at cash grid point $y_{i,t}$ demands assets $a_{i,t+1} = g_t(y_{i,t})$. Asset demand by all people at grid point $i$ is $g_t(y_{i,t})\psi_{i,t}$ where $\psi_{i,t}$ is the mass of people with cash $y_{i,t}$. To compute aggregate asset demand I approximate the integral $\int_{\mathcal{E}} a_{t+1}(\varepsilon)$ by the sum

$$a_{t+1} = \int_{\mathcal{E}} a_{t+1}(\varepsilon) \approx \sum_{i=0}^{n_y - 1} \psi_{i,t} g_t(y_{i,t}).$$

[2]Note that $n_y$, the number of grid points I use to approximate the wealth distribution, is distinct from $n_x$, the number of knots I use to generate the cubic spline that approximates the function $a = g(x)$.

To approximate the transition function $P_t$, I first compute the measure of agents who move from grid point $i$ to grid point $j$ for all $\{i, j\} \in \{y_0, y_1, \ldots, y_{n_y-1}\}$. Because individuals receive different income shocks, the mass of people in any give interval at date $t$ will be dispersed across different intervals at date $t+1$. For each $s_k \in \mathcal{S}$, a fraction $\pi_k$ of agents will arrive at the cash position $y_{k,t+1}$,

$$y_{k,t+1} = R_{t+1} g_t(y_{i,t}) + s_k.$$

I assign the mass of agents that arrives at position $y_{k,t+1}$ to the two adjacent points that bracket it in proportion to the distance from each point. This assignment is achieved by the function $\Lambda(j, y)$

$$\Lambda(j, y) = \begin{cases} \dfrac{y_{j+1} - y}{y_{j+1} - y_j} & \text{if} \quad y_j \leq y \leq y_{j+1} \\[2ex] \dfrac{y - y_{j-1}}{y_j - y_{j-1}} & \text{if} \quad y_{j-1} \leq y \leq y_j \\[2ex] 0 & \text{otherwise}, \end{cases} \tag{16}$$

where $\Lambda(j, y)$ is non-negative and strictly positive at at most two grid points. Further, by construction,

$$\sum_{j=0}^{n_y-1} \Lambda(j, y) = 1, \quad \text{for all } y_0 \leq y \leq \bar{y}.$$

A person with cash $y_{i,t}$ who saves $a_{t+1} = g_t(y_{i,t})$ and receives income realization $s_k$ has cash $y_{t+1} = R_{t+1} g_t(y_{i,t}) + s_k$ at date $t+1$. There is an initial mass of $\psi_{i,t}$ of these people who are distributed randomly across the income states $s_k \in \mathcal{S}$ with probabilities $\pi_k$. These people are allocated to the income grid by the function $\Lambda(j, y_{t+1})$.

The $\{i, j\}$ element of the matrix $P$ is defined as

$$P_t(i, j) = \sum_{k=0}^{n_y-1} \pi_k \Lambda(j, R_{t+1} g_t(y_{i,t}) + s_k).$$

This construction leads to a *stochastic matrix* $P_t$ (Cox and Miller, 1965, page 85) which maps measures of agents at date $t$ to measures of agents at date $t+1$. The elements of $P_t$ are all non-negative and the columns of $P_t$ sum to 1. $P_t$ is a function of $\xi_t$ – through the dependence of $g_t$ on $\xi_t$ – and of $R_{t+1}$,

$$\psi_{t+1} = P_t(\xi_t, R_{t+1}) \psi_t. \tag{17}$$

Equation (17) allows me to compute the sequence of asset distributions $\{g_t \psi_t\}_{t=1}^{\infty}$

from an initial distribution $g_1\psi_1$.

## VI. Solving for Dynamic Equilibria in the Discrete Model

In Section VI I characterize approximate dynamic equilibria as the solution to a linear expectational difference equation in the neighborhood of a steady state.

### A. A Linear Approximation to a Rational Expectations Equilibrium

Let $X_t$ be a vector of endogenous variables and let $V_t$ be a vector of shocks. Sims (2001) points out that an expectational linear difference equation $AX_t = F\mathbb{E}_t[X_{t+1}] + BX_{t-1} + \Gamma V_t$ can be written as a purely backward looking equation in an expanded vector $Z_t = [X_t, \mathbb{E}_{t-1}(X_{t+1})]^\top$ by introducing the date $t$ expectations of date $t+1$ variables as auxiliary variables and a set of non-fundamental shocks, $W_t$ defined as

$$W_t = X_t - \mathbb{E}_{t-1}[X_t].$$

In the spirit of this approach, define vectors $Z_t$, $V_t$ and $W_t$ and a function $G : \mathbf{Z} \times \mathbf{Z} \to \mathbb{R}^n$,[3]

$$G(Z_t, Z_{t-1}, V_t, W_t) = \begin{bmatrix} F\left(\xi_t, \mathbb{E}_t[\xi_{t+1}]; \mathbb{E}_t[R_{t+1}]\right) \\ \psi_{t+1} - P\left(\xi_t, \mathbb{E}_t[R_{t+1}]\right)\psi_t \\ \sum\limits_{i=0}^{n_y-1} \psi_{i,t} g_t(x_{i,t}) - m_{t+1} \\ m_{t+1} - \mathbb{E}_t[R_{t+1}]m_t - W_{R,t} - V_t \\ \xi_t - \mathbb{E}_{t-1}(\xi_t) - W_{\xi,t} \\ \psi_t - \mathbb{E}_{t-1}(\psi_t) \\ m_t - \mathbb{E}_{t-1}(m_t) - W_{m,t} \end{bmatrix}. \tag{18}$$

The vector $Z_t$ is defined as

$$Z_t = [\xi_t, \psi_t, m_t, \mathbb{E}_t(\xi_{t+1}), \mathbb{E}_t(\psi_{t+1}), \mathbb{E}_t(m_{t+1}), \mathbb{E}_t(R_{t+1})]^\top \in \mathbf{Z},$$

where, $\xi_t$ and $\mathbb{E}_t[\xi_{t+1}]$ are elements of $\mathcal{D}$, $\psi$ and $\mathbb{E}_t[\psi_{t+1}]$ are elements of $\mathbb{R}^{n_y}$, and $m_t$, $\mathbb{E}_t[m_{t+1}]$ and $\mathbb{E}_t[R_{t+1}]$ are elements of $\mathbb{R}_+$. The vector $Z_t \in \mathbf{Z} \equiv \mathcal{D}^2 \times [0,1]^{2n_y} \times \mathbb{R}_+^3$ has dimension $n = 2n_x + 2n_y + 3$.

$V_t$ is a vector of fundamental shocks that affect the aggregate variable $m_t$. Although I am dealing with a model with no aggregate shocks I will retain $V_t$ as a variable to allow me to study the impact of a once and for all change in the initial money supply. I assume $V_1 \in \mathbb{R}$ and $V_t = 0$ for all $t = 2, \ldots \infty$.

The vector of non fundamental shocks, $W = \{W_\xi, W_R, W_m\}$, has three pieces. The first piece, $W_\xi$, has $n_x$ elements. These elements are associated with the Euler equation residuals and their role is to ensure boundedness of the solution by

[3] I have taken a short cut here. Because I will be dealing with linear approximations to the function $G$, I have replaced $\mathbb{E}[G(X)]$ by $G(\mathbb{E}[X])$.

imposing the transversality condition on each individuals' optimization problem. The solution algorithm solves for each of them to keep the sequence $\{Z_t\}$ on the stable manifold in the neighborhood of the steady state.

The second two pieces, $W_m$ and $W_R$ are associated with the real value of money, $m_t$ and the interest rate $R_{t+1}$. Neither of these variables is pinned down by an initial condition and $W_{1,m}$ and $W_{1,R}$ are free. In the case where rational expectations is unique, there is a unique pair $\{W_{1,m}, W_{1,R}\}$ that places the dynamical system on the stable manifold in the neighborhood of the steady state. In the case of an indeterminate steady state, there may be many pairs of $\{W_{1,m}, W_{1,R}\}$ each of which is associated with a continuation sequence $\{Z_t\}_{t=2}^{\infty}$ that converges back to the steady state.

Although $\psi_t$ and $\mathbb{E}_t[\psi_{t+1}]$ both appear in the model, the assumption that there is a continuum of agents means that the evolution of $\psi_t$ is non-stochastic and $\mathbb{E}_t[\psi_{t+1}] = \psi_{t+1}$. Hence, the variables $W_{t,\psi}$ are identically zero and I have omitted them from the model.

PROPOSITION 1: *The model is characterized by a bounded sequence* $\{Z_t\}_{t=1}^{\infty}$ *that satisfies the following conditions*

1) *For each* $t = 1, \ldots, \infty$,

$$G(Z_t, Z_{t-1}, V_t, W_t) = 0.$$

2) $\{Z_t\}_{t=1}^{\infty}$ *satisfies the boundary conditions*

$$\psi_1 = \bar{\psi}_1, \qquad \lim_{t\to\infty} Z_t < \infty,$$

3) *Let* $Z_t = \bar{Z}$ *be a steady state such that* $G(\bar{Z}, \bar{Z}, 0, 0) = 0$ *and define the matrices*

$$J_1 = \left.\frac{\partial G(Z_t, Z_{t-1}, V_t, W_t)}{\partial Z_t}\right|_{\bar{Z},\bar{Z},0,0}, \qquad J_2 = -\left.\frac{\partial G(Z_t, Z_{t-1}, V_t, W_t)}{\partial Z_{t-1}}\right|_{\bar{Z},\bar{Z},0,0},$$

$$\Psi = -\left.\frac{\partial G(Z_t, Z_{t-1}, V_t, W_t)}{\partial V_t}\right|_{\bar{Z},\bar{Z},0,0}, \qquad \Pi = -\left.\frac{\partial G(Z_t, Z_{t-1}, V_t, W_t)}{\partial W_t}\right|_{\bar{Z},\bar{Z},0,0}.$$

*In the neighborhood of* $\bar{Z}$, *equilibrium sequences are approximated by the linear equation*

$$J_1 \tilde{Z}_t = J_2 \tilde{Z}_{t-1} + \Psi V_t + \Pi W_t, \tag{19}$$

*where* $\tilde{Z}_t \equiv Z_t - \bar{Z}$.

### B. Conditions for Existence and Uniqueness

We seek to construct a bounded sequence $\{\tilde{Z}_t\}_{t=1}^{\infty}$ that satisfies the equation

$$J_1\tilde{Z}_t = J_2\tilde{Z}_{t-1} + \Psi V_t + \Pi W_t, \quad t = 2, \dots, \infty, \tag{20}$$

with the initial conditions

$$\psi_1 = \bar{\psi}_1, \qquad V_1 = 1, \quad V_t = 0,\ t = 2, \dots, \infty. \tag{21}$$

Eqn. (20) is a difference equation in a vector $\tilde{Z}_t$ of $n = 2n_x + 2n_y + 3$ variables with $n_y$ initial conditions. The *stable manifold* is the linear subspace $\mathbf{L}_S$ with the property that all sequences $\{\tilde{Z}_t\}_{t=1}^{\infty}$ that satisfy equation (20) with initial condition $\tilde{Z}_1 \in \mathbf{L}_S$ converge, asymptotically to $\bar{Z}$. The local stability of sequences $\{\tilde{Z}_t\}$ depends on the relationship between the dimension of $W$ and the number of unstable generalized eigenvalues of $\{J_1, J_2\}$.[4] If a generalized eigenvalue lies outside the unit circle I will refer to it as an *unstable root.* If it lies on or inside the unit circle I will refer to it as a *stable root.*

For most practical problems, a rule of thumb is that there is a unique solution to a linear rational expectations model when the number of non-fundamental shocks is equal to the number of unstable roots. For the problem defined by Eqn. (20) the number of non-fundamental shocks is equal to $n_x + 2$ and the condition for a unique solution is that there are $n_x + 2$ unstable roots.

## VII. Calibrated Examples of Dynamic Equilibria

This section describes the properties of a calibrated example written in MatLab and accessible in the online appendix. For the calibration I chose $n_x = 20$, $n_y = 50$ and $n_s = 500$. I assume that the income shocks are i.i.d. and represented by a discrete uniform distribution with support $\{s_1, \dots, s_{n_s}\}$ and mean $\mu = 1$ and I set the discount parameter $\beta = 0.97$.

### A. Interest Rates and Money Demand For Two Different Values of the Shock Variance

In Figure 1 I graph money demand for values of $R$ in the interval $[0.96, 1.02]$ for risk aversion of $\rho = 1$ and two different values of $\sigma$. When $\sigma = 0.4$, the lower bound of the support of the shock distribution is 0.31; I call this the *low variance case.* When $\sigma = 0.48$, the lower bound of the support is 0.17; I call this the *high variance case.*

For the low variance case, the autarkic interest rate is equal to 1% which is lower than the discount rate of 3% but still positive. For the high variance case, the agent acts as if he was more risk averse in the sense that the autarkic interest

[4]The *generalized eigenvalues* of a pair of square matrices $\{J_1, J_2\}$ are values of $\lambda \in \mathbb{C}^n$ that solve the problem $|J_1 - \lambda J_2| = 0$.

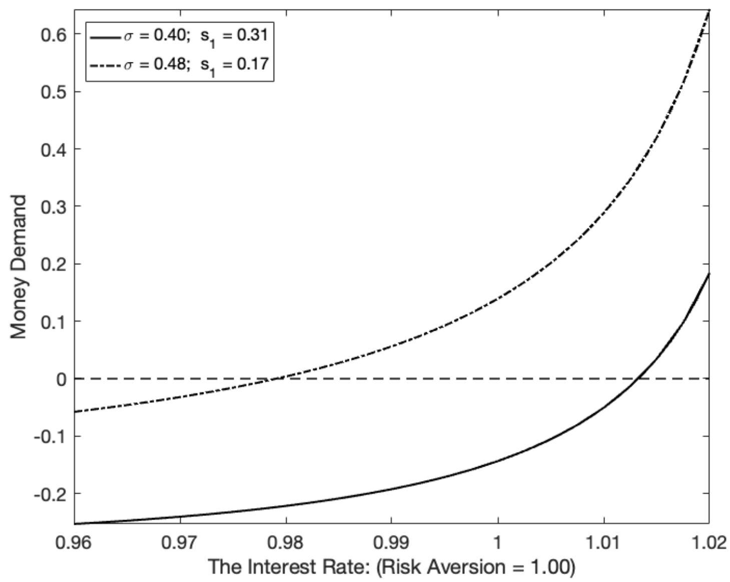


Figure 1: Money Demand

rate becomes negative and is equal to $-2\%$. This is the phenomenon noted by Huggett (1993) who pointed out that uninsurable income risk can explain why the rate of return on a safe asset has been consistently lower than the growth rate of GDP.

Figure 1 shows that, in the high variance case, the monetary equilibrium, $R = 1$, is consistent with positive valued fiat money and a non-negative price level. In contrast, for the low variance case, the monetary equilibrium is inconsistent with a positive price level. This is apparent from Figure 1 which shows that, when $\sigma = 0.40$, money demand is negative at $R = 1$, implying that this equilibrium does not exist.

### *B. Interest Rates and Money Demand at the Two Steady States*

Table 1 reports the results of a sensitivity analysis to alternative values of the shock standard deviation and the coefficient of risk aversion. For a grid of five values for each of these variables, I calculate the steady interest rate at the non-monetary steady-state and the value of money demanded at the monetary steady-state.

Table 1A, reports the value of the gross non-monetary interest rate for values of risk aversion between 1.0 to 3.0 – these are the rows of the table – and values of the standard deviation of the shocks between 0.4 and 0.48 – these are the columns of the table. A stationary allocation in general equilibrium is *dynamically efficient*

| | | $\sigma$ | | | | | $\sigma$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.40 | 0.42 | 0.44 | 0.46 | 0.48 | 0.40 | 0.42 | 0.44 | 0.46 | 0.48 |
| | 1.0 | **1.01** | **1.01** | **1.01** | 0.99 | 0.98 | **-14** | **-8.0** | **-1.0** | 6.0 | 14.0 |
| | 1.5 | **1.01** | 0.99 | 0.98 | 0.97 | 0.94 | **-3.0** | 5.0 | 14.0 | 23.0 | 33.0 |
| $\rho$ | 2.0 | 0.99 | 0.98 | 0.96 | 0.94 | 0.90 | 8.0 | 18.0 | 29.0 | 41.0 | 53.0 |
| | 2.5 | 0.98 | 0.96 | 0.94 | 0.90 | 0.85 | 20.0 | 32.0 | 44.0 | 58.0 | 73.0 |
| | 3.0 | 0.97 | 0.94 | 0.91 | 0.86 | 0.78 | 32.0 | 45.0 | 60.0 | 76.0 | 93.0 |
| A: $R$: (Non-monetary Equilibrium) | | | | | | | B: $m$: (Monetary Equilibrium.) | | | | |

Table 1: Some Properties of the Two Steady States

if and only if the gross interest rate is greater than or equal to 1 (Malinvaud, 1953; Cass, 1972). Table 1A illustrates that for low risk aversion and low shock variance, the non-monetary equilibrium is dynamically efficient. This is the case for the four values in the top left of the table where the gross interest rate is greater than 1. I have reported these values in bold typeface.

For higher values of the shock variance and higher values of risk aversion the gross equilibrium non-monetary interest rate is less than one, implying that the steady state is dynamically inefficient. The gross interest rate varies from 0.99, when $\sigma = 0.42$ and $\rho = 1.5$ to 0.78, when $\sigma = 0.48$ and $\rho$, is 3.0. These values correspond to net real interest rates of $-1\%$ and $-22\%$.

Table 1B, reports the value of money demand at the monetary steady state for the same five values of $\sigma$ and $\rho$. Notice that for the three cases when the non-monetary steady state is dynamically efficient, money demand in the monetary steady-state is negative. These are the four values in bold type in the top left corner of the table. In contrast, when the non-monetary equilibrium is dynamically inefficient, money has positive value in the monetary steady state ranging from a low value of 5% of GDP when $\sigma = 0.42$ and $\rho = 1.5$ to a high of 93% of GDP for the case when $\sigma = 0.48$ and $\rho = 3.0$.

### *C. The Determinacy Properties of the Two Steady States*

To compute the dynamic properties of equilibria, the Matlab code *dynamics.m* – available in the online appendix – constructs the matrices $J_1$, $J_2$, $\Psi$ and $\Pi$ from Eqn. (20) using numerical derivatives and, in the file *Maincode.m*, I feed these matrices into Sims' program *Gensys* (Sims, 2001).[5] In my numerical analysis of the model for different parameter values I found that $\{J_1, J_2\}$ always has at least $n_x + 1$ unstable roots but there is no guarantee that there will be $n_x + 2$, which is the rule of thumb for there to be a determinate steady-state.

[5] *Gensys* takes as inputs, the matrices $J_1, J_2, \Psi$ and $\Pi$ and computes a solution of the form

$$\tilde{Z}_t = G\tilde{Z}_{t-1} + HV_t. \tag{22}$$

| | | $\sigma$ | | | | | $\sigma$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.40 | 0.42 | 0.44 | 0.46 | 0.48 | 0.40 | 0.42 | 0.44 | 0.46 | 0.48 |
| | 1.0 | **1.01** | **1.01** | **1.01** | 0.99 | 0.98 | **0.98** | **0.99** | **0.99** | 1.01 | 1.02 |
| | 1.5 | **1.01** | 0.99 | 0.99 | 0.97 | 0.95 | **0.99** | 1.01 | 1.01 | 1.02 | 1.03 |
| $\rho$ | 2.0 | 0.99 | 0.98 | 0.97 | 0.94 | 0.90 | 1.01 | 1.01 | 1.02 | 1.03 | 1.03 |
| | 2.5 | 0.98 | 0.96 | 0.94 | 0.90 | 0.85 | 1.01 | 1.02 | 1.03 | 1.03 | 1.04 |
| | 3.0 | 0.97 | 0.94 | 0.91 | 0.86 | 0.78 | 1.02 | 1.02 | 1.03 | 1.04 | 1.04 |
| A: Marginal Root: Non-Monetary | | | | | | | B: Marginal Root: Monetary | | | | |

Table 2: Determinacy at the Two Steady States

To explore the determinacy of the two steady-states I ordered the roots of $\{J_1, J_2\}$ from largest to the smallest and I pick out the root in position $n_x+2$ which I call the *marginal root.* When the marginal root is larger than 1 in absolute value, the equilibrium is locally unique and the steady-state is determinate. When the marginal root is less than one in absolute value there is a one dimensional manifold close to the steady state such that sequences that begin on this manifold converge back to the steady state. In this case the steady-state is indeterminate. In all cases when the rule of thumb suggests that the model is determinate (indeterminate), *Gensys* – which checks a complete set of necessary and sufficient conditions – reports existence and uniqueness (non-uniqueness) of equilibrium.

In Table 2A, I report the values of the marginal root for five values of risk aversion between 1.0 to 3.0 and five values of the shock standard deviation from 0.40 to 0.48, at the non-monetary steady-state. This table shows that the marginal root is greater than one for the four values in the upper left of the table which correspond to steady state values of the non-monetary steady-state in which the equilibrium is dynamically efficient. In all four of these steady states the non-monetary steady-state equilibrium is both dynamically efficient and determinate.

For all other values - corresponding to dynamically inefficient non-monetary steady states – the marginal root is less than one and the steady state is indeterminate. In all cases at the non-monetary steady-state, the marginal root is equal to the steady-state non-monetary interest rate (compare Table 1A with Table 2A). This is not an accident, it follows from the properties of linearizing the equation

$$m_{t+1} = R_{t+1} m_t$$

around a steady state in which $\bar{m} = 0$.

A similar picture arises in Table 2B, which reports the value of the marginal root at the monetary steady-state equilibrium. With the exception of the four values in the upper left corner – these are the values for which the monetary steady-state would require a negative price level – the marginal root is greater than one implying that the monetary equilibrium, when it exists, is locally unique and the steady-state is determinate.

### D. Impulse Responses at an Indeterminate Steady State

In Figure 2 I report the result of an exercise in which I set $V = 0.01$ and I fix $W_{1,m} = 0$. This is one of the possible responses of the economy to a fundamental shock around an indeterminate steady state. The top panel of Figure 2 shows that a money financed government purchase of goods equal to 1% of GDP generates an increase in the interest rate of 4.25% points on impact. In period 2 the interest drops back to approximately 80 basis points above its steady state value and declines slowly back to the steady state. After 25 periods $R$ is still approximately 20 basis points above its long-run value.

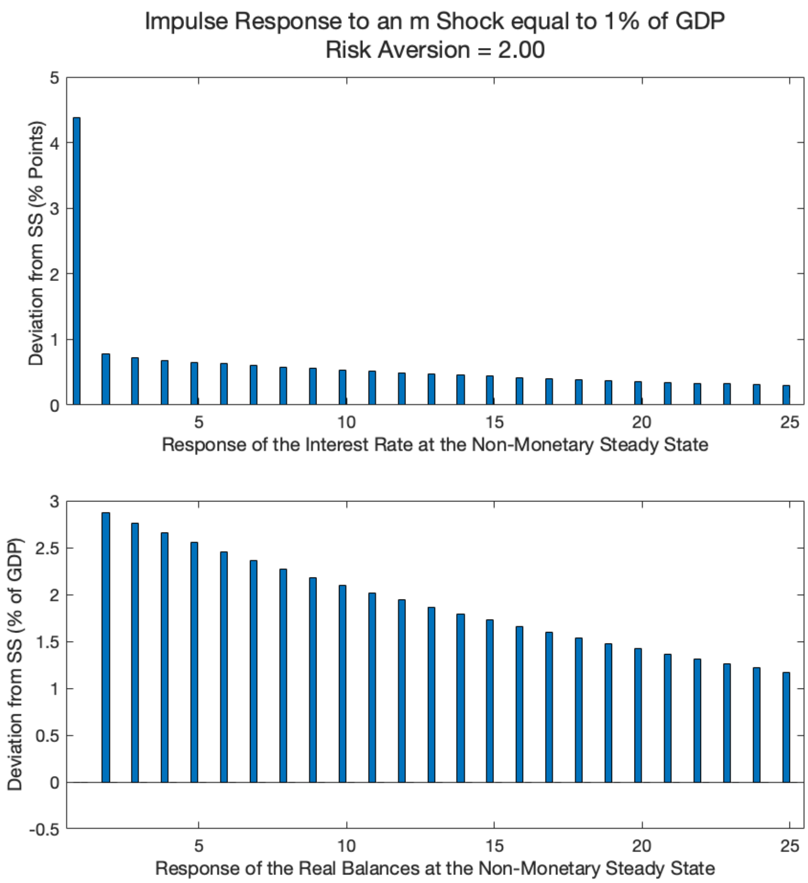


Figure 2: Dynamic Equilibria at the Non-Monetary Steady State

The lower panel of Figure 2 shows the response of $m$. In the first period I have constrained the response by picking the equilibrium in which $W_{1,m} = 0$ and hence

$m_1 = 0$, its steady state value. In period 2, $m_2$ increases by 2.9% of GDP and subsequently declines slowly back to zero.

The existence of indeterminacy in a perfect foresight model is important because it is associated with the existence of multiple *stationary* rational expectations equilibria in a model with aggregate shocks (Benhabib and Farmer, 1999). I have argued elsewhere that by making beliefs fundamental, equilibrium models with an indeterminate steady state can help understand a range of empirical phenomena including sticky prices, sticky inflation rates and excess asset price volatility (Farmer, 1999). This paper has demonstrated that models with uninsurable income risk support equilibria of this kind.

## VIII. Some Implications for Monetary and Fiscal Policy

The model I have described in this paper does not include a role for fiscal and monetary policy but it would not be difficult to add these features by expanding equations (8), (9) and (10) to allow for government purchases and the payment of interest on money. The following equations expand the model in this way:

$$R_t = R^N_{t-1}\left(\frac{p_{t-1}}{p_t}\right), \tag{23}$$

$$\frac{M_t}{p_t} = R^N_{t-1}\left(\frac{p_{t-1}}{p_t}\right)\left(\frac{M_{t-1}}{p_{t-1}}\right) + g_t, \tag{24}$$

$$m_t = R_t m_{t-1} + \delta_t. \tag{25}$$

Here, $R^N_{t-1}$ is the gross nominal interest rate on money held between periods $t-1$ and $t$ and $\delta_t$ is government purchases of real goods and services net of taxes. To complete the model we need to specify how these variables are determined.

Define the gross inflation rate, $\pi_t = p_t/p_{t-1}$, and consider the following monetary and fiscal rules:

$$R^N_t = \kappa\left(\frac{\pi_t}{\bar{\pi}}\right)^{\eta}, \quad \eta \geq 0, \tag{26}$$

$$\delta_t = \bar{\delta}\left(\frac{m_t}{\bar{m}}\right)^{-\mu}, \quad \mu \leq 0, \tag{27}$$

where $\kappa$, $\bar{m}$, $\eta$ and $\bar{\delta}$ are policy parameters. In representative agent economies if $\eta\beta > 1$ ($\eta\beta \leq 1$) monetary policy is said to be *active* (*passive*) and if $|\mu\beta| > 1$ ($|\mu\beta| \leq 1$), fiscal policy is said to be *passive* (*active*), (Leeper, 1991). Advocates of the fiscal theory of the price level argue that policy makers should pick an active-passive combination to ensure that the price level is uniquely defined.

The exchange economy studied in this paper is one where $\kappa = 1$, $\eta = 0$, and $\bar{\delta} = \mu = 0$. This is a passive monetary policy and an active fiscal policy in which the price level, in a representative agent economy, would be pinned down at the monetary equilibrium by the fiscal theory of the price level (Leeper and Leith, 2016). Further, in the representative agent economy, this equilibrium is unique.

In the heterogeneous economy, there is a second dynamically inefficient equilibrium and the policy conclusion, that governments should pick an active-passive policy combination is no longer clear cut. My analysis suggests the use of caution in transferring policy conclusions that apply in a representative agent environment to richer models that allow for the existence of multiple equilibria.

## IX. Concluding Comments

This paper has explored the role of money in an incomplete markets model with heterogeneous agents and uninsurable income risk. By introducing fiat money as a store of value, I have shown that the model can exhibit both non-monetary and monetary equilibria, with the latter existing when income risk is sufficiently high. By characterizing the determinacy properties of both the non-monetary and monetary steady states, I showed that when the the non-monetary equilibrium is dynamically inefficient, it is also indeterminate, opening the possibility that there may exist equilibria driven by non-fundamental shocks.

*

REFERENCES

Rao S. Aiyagari. Uninsured idiosyncratic income risk and aggregate saving. *Quarterly Journal of Economics*, 109(3):659–684, 1994.

Maurice Allais. *Economie et intérêt.* Imprimerie Nationale, Paris, 1947.

Adrien Auclert, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models. *Econometrica*, 89(5):2375–2408, 2021.

Ravi Bansal and Amir Yaron. Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance*, 59(4):1481–1509, 2004.

Robert J. Barro. Rare disasters and asset markets in the twentieth century. *Quartlerly Journal of Economics*, 121(3):823–866, 2006.

Jess Benhabib and Roger E. A. Farmer. Indeterminacy and sunspots in macroeconomics. In John B. Taylor and Michael Woodford, editors, *Handbook of Macroeconomics*. North-Holland, 1999.

Truman Bewley. The permanent income hypothesis: A theoretical formulation. *Journal of Economic Theory*, 16(2):252–292, 1977.

Timo Boppart, Per Krussell, and Kurt Mittman. Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control*, 89:68–92, 2018.

C. D. Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312 – 320, 2006.

David Cass. On capital overaccumulation in the aggregative neoclassical model of economic growth: a complete characterization. *Journal of Economic Theory*, 4 (2):200–223, 1972.

D. R. Cox and H. D. Miller. *Theory of Stochastic Processes*. Chapman and Hall, New York, N.Y., 1965.

Roger E. A. Farmer. *The Macroeconomics of Self-Fulfilling Prophecies.* MIT Press, Cambridge, MA, second edition, 1999.

Roger E. A. Farmer. Pricing Assets in a Perpetual Youth Model. *Review of Economic Dynamics*, 30:106–124, 2018.

Nicolae Gârleanu Gârleanu and Stavros Panageas. Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. *Journal of Political Economy*, 123(3):670–685, 2014.

Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969, 1993.

Per Krusell and Anthony A. Smith. Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998.

Eric M Leeper. Equilibria under 'active' and 'passive' monetary and fiscal policies. *Journal of Monetary Economics*, 27(1):129–147, 1991.

Eric M. Leeper and Campbell Leith. Understanding Inflation as a Joint Monetary-Fiscal Phenomenon. In John B. Taylor and Harald Uhlig, editors, *Handbook of Macroeconomics*, volume 2, chapter 30, pages 2305–2415. Elsevier, 2016.

Edmond Malinvaud. Capital accumulation and efficient allocation of resources. *Econometrica*, 21(2):233–268, 1953.

Michael Reiter. Solving Heterogeneous-Agent Models by Projection and Perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665, 2009.

Michael Reiter. Nonlinear solution of heterogeneous agent models by approximate aggregation. 2010. URL `https://myservice.ihs.ac.at/mreiter/research.html`.

Michael Reiter. State reduction and second-order perturbations ofheterogeneous agent models. 2023. URL `https://myservice.ihs.ac.at/mreiter/quadperturbha.pdf`.

Thomas A. Reitz. The equity risk premium a solution. *Journal of Monetary Economics*, 22(1):117–131, 1988.

Sheldon M. Ross. *Introduction to Probability Models*. Academic Press, New York, 2007.

Paul A. Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(December): 467–482, 1958.

Jack Schectman. An income fluctuations problem. *Journal of Economic Theory*, 12(2):218–241, 1976.

Jack Schectman and Vera Escudero. Some results on an income fluctuation problem. *Journal of Economic Theory*, 16(2):151–166, 1977.

Christopher A. Sims. Solving linear rational expectations models. *Journal of Computational Economics*, 20(1-2):1–20, 2001.

Philippe Weil. The equity premium puzzle and the risk free rate puzzle. *Journal of Monetary Economics*, 24(3):401–421, 1989.