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Paul Beaudry
Dana S. Galizia
Franck Portier

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ABSTRACT

Macroeconomic dynamics are shaped by how individual incentives to spend and accumulate interact with the decisions of others. The goal of this paper is to identify—within a simple large-game-theoretic structure—which types of agent interactions favor which types of dynamic equilibrium outcomes. In particular, we extend the static analysis of Cooper and John 1988 to a dynamic setting to clarify the role of strategic complementarity and substitutability in delivering dynamics such as monotonic convergence to a unique steady state, hysteresis, endogenous cycles, and indeterminacy.

Paul Beaudry
Vancouver School of Economics,
University of British Columbia
6000 Iona Drive
Vancouver, V6T 1Z1
Canada
and NBER
Paul.Beaudry@ubc.ca

Franck Portier
Department of Economics
University College London
30 Gordon Street
WC1H 0AX London
United Kingdom
and CEPR
f.portier@ucl.ac.uk

Dana S. Galizia
Department of Economics, Carleton University
Room B-845, Loeb Building
1125 Colonel By Drive
Ottawa, ON
K1S 5B6
Canada
dana.galizia@gmail.com

Introduction

Standard business cycle models are typically populated with agents that fundamentally dislike fluctuations in their economic conditions. If one of these agents were isolated on a desert island, they would not make decisions that would endogenously induce booms and busts or create hysteresis. Instead, in the absence of any shocks, this desert-island economy would most likely feature determinate, monotonic convergence to a unique (saddle-path) stable steady state. We refer to such dynamics as "standard dynamics".

Of course, agents in macro models are not isolated. Instead, they typically interact with one another through various channels. Nonetheless, in many models, the standard dynamics favored at the individual agent level are often maintained in the model's aggregate equilibrium; that is, the endogenous interactions in the model often support—and in many cases actually reinforce—standard dynamics.

While this is true for a large class of macro models in the literature, there is another group of models for which the equilibrium features qualitatively different dynamic behavior—such as endogenous oscillations, local instability, or indeterminacy—despite the fact that, again, individual agents have an innate preference for standard dynamics. Our goal in this paper is to provide a framework to help understand when and why this occurs. Our basic approach is to adapt and extend the analysis of Cooper and John [1988], which was done in a purely static environment, to a richer dynamic environment of the type often encountered in macro models.

Specifically, we frame our analysis around two fundamental economic forces that are present in almost all modern mainstream business cycle models. The first such force is that of *accumulation*: agents accumulate stocks of various kinds, and these stocks are the primary mechanisms by which economic decisions are transmitted through time. The second fundamental force is that of *strategic interaction*: agents' optimal decisions for how much to accumulate affect and are affected by the decisions of other agents. Following Cooper and John [1988], we refer to strategic interactions as characterized either by substitutability, where a higher action by one agent favors a lower level by others, or complementarity, where a higher action by one agent favors a higher action by others. Similar to the approach used by Cooper and John [1988], in order to make the analysis as widely applicable as possible we do not make explicit the precise microeconomic structures that give rise to these two fundamental economic forces.

The starting point of our analysis is to suppose, as is typically the case in macro models, that in the absence of any strategic interaction standard dynamics would prevail. We then aim to understand whether and when the forces of strategic interaction would render the equilibrium dynamics non-standard. We focus broadly on three different types of (not necessarily mutually exclusive) non-standard dynamics that could in principle emerge as the type and strength of strategic interactions vary. The first, which we call *cyclicality*, is when interactions cause the aggregate system to naturally oscillate (even in the absence of any shocks) around some point (e.g., a steady state).

The second type of non-standard dynamics we consider is *local instability*, whereby interactions cause the equilibrium system to no longer exhibit convergence to a unique steady state. Following the emergence of local instability, the system may become globally explosive, or it may converge to some other state such as indefinite but bounded fluctuations (i.e., limit cycles or chaos) or, depending on initial conditions, one of a multiplicity of steady states (in which case the system exhibits hysteresis).

The final type of non-standard dynamics we consider is *dynamic indeterminacy*, defined as a situation where there exists a continuum of equilibrium paths satisfying the boundary (transversality) conditions of the model (see, e.g., Chamley [1993], Benhabib and Farmer [1994], Farmer and Guo [1994], and Galí [1994], Benhabib and Farmer [1999]). In this case, equilibrium outcomes can be subject to self-fulfilling beliefs, despite the fact that, taking the actions of all others as given, the optimal choices of any one individual agent are entirely determinate.

We may summarize the key take-aways from our analysis as follows. First, when it occurs, non-standard dynamics can generally be thought of as arising in situations where the economy becomes sufficiently sensitive to economic conditions. The precise form of any non-standard dynamics depends on what in particular the economy is sensitive to. Local instability occurs when the economy is sufficiently sensitive to the *current* state (the current values of the stock variables in our model), while indeterminacy occurs when the economy is sufficiently sensitive to the anticipated *future* state (the anticipated level of future accumulation in our model). Furthermore, as clarified in Cooper and John [1988], complementarity tends to amplify the response of the economy to changes in economic conditions, while substitutability tends to dampen it. As a result, as we show, substitutability typically does not tend to produce the kind of sensitivity required for non-standard dynamics to emerge. On the other hand, enough complementarity always produces non-standard dynamics.

Second, note that an increase in the current level of a given stock could in general make an agent want to increase or decrease their current accumulation of that stock. We show that a necessary condition for cyclicality to arise is that a stock exerting a negative effect on current accumulation must play an important role. When this is the case, beginning from steady state, an increase in the current level of that stock initially reduces agents' desire to accumulate, and with enough complementarity this gets amplified so much in equilibrium that the future level of the stock actually ends up overshooting the steady state level, with the result being cyclicality. If the economy has only one stock variable, then this cyclicality first emerges in the form of damped two-period oscillations. In the case of multiple stock variables, however, the damped oscillations that emerge could in principle be of any length.

Third, the precise behavior of the system immediately following the emergence of local instability depends on whether the dynamics were monotonic or cyclical just beforehand. In the cyclical dynamics case the emergence of instability would generally be associated with the appearance of a limit cycle, while in the monotonic dynamics case the emergence of instability would typically associated with the emergence of multiple saddle-path stable steady states.

Finally, *"static" interaction*—defined as a situation where the individual desire to accumulate in the current period is directly affected by the current aggregate level of accumulation—does not tend to produce indeterminacy. The reasons for this are somewhat subtle and are discussed in Section 3. In contrast, indeterminacy will generally emerge in situations of *"dynamic" interaction*, defined as a situation where the anticipated *future* level of aggregate accumulation affects the anticipated *future* value of an individual agent's stock, and this in turn affects that agent's *current* desire to accumulate that stock. We will drop the quotation marks for "static" and "dynamic" interactions in the rest of the paper, but one should keep in mind that in either case the agents in our model are generally assumed to be forward-looking.

The rest of the paper is organized as follows. In Section 1, we set up the basic agent interaction framework we employ throughout the paper. In Section 2, we fully analyze the case of static interactions. In Section 3, we discuss in some detail the case of dynamic interactions and the conditions that would produce indeterminacy. In Section 4, we discuss the occurence of determinate smooth oscillations. In Section 5, we put together static and dynamic interactions and numerically show that the main intuitions we have previously obtained analytically carry though. In the conclusion, we discuss how our framework helps understanding the dynamics of some well-known macroeconomic models.

1 Modeling Environment

In this section, we set up a general modeling environment that we will rely on throughout the paper. Our goal in setting up this environment is to capture in a simple and general way what we see as some key fundamental forces that are present in most dynamic macro models. As such, rather than focusing on particular economic mechanisms (e.g., consumption-smoothing, sticky prices, etc.), we instead express our model in terms of *classes* of mechanisms, as defined by the static and dynamic forces they exert. We should emphasize that any given particular mechanism in a macro model may have features belonging to more than one of these categories, and therefore mapping a particular mechanism into our environment is not always straightforward. Nonetheless, we will show that these categorizations are quite helpful in clarifying which features of a mechanism are likely to be responsible for driving aggregate dynamics.

1.1 Primitives

Time is discrete and runs from 0 to infinity. The economy is deterministic and is populated with $N \to \infty$ identical agents. Each period, an agent chooses a strategy variable (or "action") $e_t \in [\underline{e}, \infty)$, which ultimately accumulates into a private stock X_t. Agents attempt to maximize

$$\sum_{t=0}^{\infty} \beta^t V\left(e_t, X_t, \bar{e}_t\right) , \tag{1}$$

where $\beta \in [0, 1)$ is the discount factor, $\bar{e}_t$ denotes the aggregate (average) level of activity in the economy (which each agent takes as given), and X accumulates according to

$$X_{t+1} = \left(1 - \delta_X\right) X_t + e_t \, , \tag{2}$$

with $0 < \delta_X \leq 1$ and X_0 given. We assume the period payoff function V is three times continuously differentiable in all arguments, and strictly concave in (e, X) for all $\bar{e}$.

The appearance of the aggregate variable $\bar{e}$ in the objective function (1) is the central mechanism by which individual agents' actions affect each other. Using subscripts to denote partial derivatives, the case where $V_{\bar{e}}(e, X, \bar{e}) = 0$ for all $e, X, \bar{e}$ corresponds to autarky. To use Cooper and John's [1988] terminology, there will be spillovers whenever we instead have $V_{\bar{e}}(e, X, \bar{e}) \neq 0$, and strategic interactions if, in addition, we have $V_{j\bar{e}}(e, X, \bar{e}) \neq 0$ for at least one $j \in \{e, X\}$.

To maintain as much generality as possible, we will not be specific about the exact economic mechanism by which activity e translates into the stock X, nor about their precise economic content, although we will provide some fully micro-founded examples. The only conceptual distinction we wish to make is that, for a given level of e_{t+1}, and taking the path of aggregate activity as given, X will be referred to as a "durables" stock (D-stock) if, when an individual's level of X_t unexpectedly increases, it favors a lower level of current activity e_t by that individual. In contrast, X will be referred to as a "capital" stock (K-stock) if an unexpected increase in X_t favors *more* e_t. While we will be more precise about the mathematical meaning of this distinction in our analysis below, the following two examples illustrate why we chose this terminology. To make things as clear as possible, assume in both cases that there are no interactions (i.e., agents live in autarky). First, consider a model where households get concave utility from consumption of a durable goods stock X, and new durables e are produced by household labor effort, which incurs disutility. Because utility is concave, an unexpected increase in the stock of durables X_t lowers the marginal the value of creating new durables, which in turn lowers the optimal amount of durables production e_t. Thus, X is a D-stock in this case. Alternatively, consider a model where X is productive capital, e is investment, output is produced using only X as an input, households get concave utility from consumption, and output is split between consumption and investment. Then an unexpected increase in X_t increases output, relaxing the household's budget constraint, in turn lowering the marginal utility cost of investment, thereby favoring a higher level of e_t.

Our principal focus in this paper is on how the type and direction of strategic interactions influence the equilibrium outcomes of the model.[1] Different macroeconomic mechanisms typically generate different such interactions. For example, suppose individual households accumulate productive capital X, for which there is a centralized Walrasian market where households can rent capital to each other. Households produce output using capital (whether their own or rented). In this case we will have $V_{e\bar{e}} < 0$ and $V_{X\bar{e}} > 0$: an increase in desired investment $\bar{e}$ by other households raises the market demand for output and, in turn, existing productive capital, which raises the market prices of both. Facing these price increases, an individual household will optimally reduce their own investment ($V_{e\bar{e}} < 0$), while the higher rental rate increases the rental value of their own capital stock ($V_{X\bar{e}} > 0$).

In the above example, as in any purely Walrasian environment, all strategic interactions between agents were mediated by prices (namely, the capital rental rate). In addition to price-mediated ones, strategic interactions could also come from, for example, thick-market externalities (as in search models, see e.g. Diamond [1982]), direct productive externalities (as in endogenous growth models, see e.g. Aghion and Howitt [1992] or Benhabib and Farmer [1994]), pecuniary externality (as in models with credit constraints, see e.g. Kiyotaki and Moore [1997]) or aggregate demand externalities (as in many models featuring monopolistic competition, see e.g. Blanchard and Kiyotaki [1987]).

Taking X_0 and the sequence $\{\bar{e}_t\}_{t=0}^{\infty}$ of aggregate activity as given, agents maximize (1) subject to (2) and the transversality condition (TVC)

$$\lim_{T\to\infty} \beta^T \lambda_{X,T} X_T = 0\,, \tag{3}$$

where λ_X is the Lagrange multipliers on (2). This yields the following first-order conditions

[1] Our set-up is not well suited to addressing questions of welfare, and therefore we will not have anything interesting to say about cases where there are spillovers without strategic interaction.

(FOCs):

$$V_e(e_t, X_t, \bar{e}_t) + \lambda_{X,t} = 0\,, \tag{4}$$

$$\lambda_{X,t} = \beta\left[V_X(e_{t+1}, X_{t+1}, \bar{e}_{t+1}) + (1-\delta_X)\,\lambda_{X,t+1}\right]\,, \tag{5}$$

1.2 Equilibrium

We will always restrict the analysis to symmetric equilibrium allocations. Thus, the equilibrium is characterized by equations (2)-(5), plus the symmetry condition $\bar{e}_t = e_t$.

1.3 An Example

As an illustration of our framework, we provide here a general-equilibrium example of a model in which a V function of the type described above can be derived explicitly. There is a continuum of measure one of households and firms. Household preferences are given by

$$\sum_{t=0}^{\infty} \beta^t\left[u(c_t - \gamma s_t) - \phi(\ell_t)\right]\,,$$

where c is consumption, ℓ is labor supply, γ is a parameter, s is a stock of consumption habit (when $\gamma > 0$) or of durable goods (when $\gamma < 0$) that accumulates according to $s_{t+1} = (1-\delta)s_t + c_t$, u is a strictly increasing concave function satisfying $\lim_{x\to 0} u'(x) = \infty$, and ϕ is a strictly increasing convex function. We assume that $\delta > \gamma$, which is a necessary condition for a steady state of this model to exist. Households face the budget constraint

$$c_t = w_t \ell_t + \pi_t\,,$$

where w_t is the wage and π_t are lump-sum firm profits.

Production by each firm is given by

$$y_t = \chi_t \ell_t^{\alpha}\,.$$

The model features productive externalities due to the fact that

$$\chi_t = \chi \bar{\ell}_t^{\nu\alpha},$$

where $\bar{\ell}$ is the average level of labor employed by firms in the economy, and ν parameterizes the strength of the productive externalities. We impose the restriction $\nu > -1$ so that, in equilibrium, an increase in labor is always associated with an increase in output. Individual firms take both w_t and χ_t as given when choosing ℓ_t.

In equilibrium, firms' optimal choice of ℓ_t satisfies

$$w_t = \alpha\chi\ell_t^{(1+\nu)\alpha-1} = \alpha\chi^{\frac{1}{(1+\nu)\alpha}} y_t^{1-\frac{1}{(1+\nu)\alpha}},$$

and therefore firm profits are $\pi_t = (1-\alpha)y_t$. Further, aggregate consumption is given in equilibrium by $\bar{c}_t = y_t$. Thus, we may re-arrange the household's budget constraint to obtain

$$\ell_t = \alpha^{-1}\chi^{-\frac{1}{(1+\nu)\alpha}}\bar{c}_t^{\frac{1}{(1+\nu)\alpha}-1}\left[c_t - (1-\alpha)\,\bar{c}_t\right].$$

Substituting this into the household's period-t utility function, we can re-cast the HH problem as maximizing $\sum \beta^t V(c_t, s_t, \bar{c}_t)$ subject to $s_t = (1-\delta)s_{t-1} + c_t$, where

$$V(c, s, \bar{c}) \equiv u(c - \gamma s) - \phi\left(\alpha^{-1}\chi^{-\frac{1}{(1+\nu)\alpha}}\bar{c}_t^{\frac{1}{(1+\nu)\alpha}-1}\left[c - (1-\alpha)\,\bar{c}\right]\right).$$

It can be verified that, given our assumption that $\delta > \gamma$, an unexpected increase in a given household's s favors an *increase* in their demand for consumption if $\gamma > 0$. That is, consumption habit is an example of a type of K-stock: the larger the habit stock a household has, the more consumption they will want to purchase (all else equal). If $\gamma < 0$, an unexpected increase in a given household's s favors an *decrease* in their demand for consumption: since u is concave, an increase in the durables stock s lowers the marginal value of additional consumption. Thus, s is a D-stock in this version of the model.

1.4 Working Assumptions

Returning to our general framework, we start by assuming that, in the absence of interactions, the system features standard dynamics.

Assumption A1. (Autarky) *In the absence of any interactions, the equilibrium of the model is unique and features monotonic convergence to a unique, saddle-path stable steady state (SS), which we refer to as SS_0.*

Note that the assumption of saddle-path stability implies that the linearized system features one stable and one unstable eigenvalue, while the assumption of monotonic convergence implies that the stable eigenvalue is non-negative.

We also make the following assumption.

Assumption A2. (Non-Degeneracy) *In the absence of any interactions,* *(i) the stable eigenvalue is non-zero; and*

(ii) X_t locally affects the equilibrium choice of e_t.

Assumption A2 rules out, for the model with no interactions, certain uninteresting degenerate cases that would otherwise require special treatment throughout. In particular, part *(i)* ensures that the system does not always immediately return to SS_0 following any deviation, while part *(ii)* rules out the case where the level of accumulation e_t is always constant and equal to its SS level regardless of the value of X_t.

As we will see, because of the dynamic nature of our model, strategic interactions may ultimately affect equilibrium outcomes through a number of distinct channels. One such channel stems from the effect of a change in the aggregate action $\bar{e}_t$ on the contemporaneous private choice of e_t. If, all else equal, a higher level of $\bar{e}_t$ favors a lower level of e_t, then we have strategic substitutability. In this case, if all agents have an initial desire to accumulate more, then aggregate accumulation $\bar{e}_t$ will rise, which, because of the substitutability, tends to make individuals want to reduce their accumulation. This mitigates the initial desire to increase e_t, and thus in equilibrium the increase e_t will ultimately be smaller than in the autarky case. That is, substitutability tends to be a *dampening* force.

If instead a higher level of $\bar{e}_t$ favors a higher level of e_t, then we have strategic complementarity. As is well known (see Cooper and John [1988]), multiple equilibria may exist when strategic complementarity is strong enough that e_t reacts positively and more than one-for-one to an increase in $\bar{e}_t$. While such multiple equilibria are intriguing, our focus in this paper will be on exploring the case where complementarities, if they exist, are always weak, as stated formally in the following assumption.

Assumption A3. (Weak Complementarities) *Given the current levels of the state variable X_t and the entire sequence of future private and aggregate actions $\{e_\tau, \bar{e}_\tau\}_{\tau>t}$, an agent's best response of e_t to $\bar{e}_t$ at t always satisfies $de_t/d\bar{e}_t < 1$.*

With complementarities, an initial desire by all agents to accumulate more raises $\bar{e}_t$, which causes agents to want to increase e_t by even more, further increasing $\bar{e}_t$, etc. Assumption A3 means that an individual never responds more than one-to-one to an increase in aggregate action $\bar{e}_t$. Under Assumption A3, it can be verified that in equilibrium the resulting increase in e_t will ultimately be larger than in the autarky case, so that complementarity tends to be an *amplifying* force.[2] It is also worth emphasizing that Assumption A3 does not necessarily

[2] If complementarities are instead strong, then it is actually possible for the equilibrium level of e_t to paradoxically *fall* when all agents initially want to increase their accumulation. Note that such equilibria would however not be stable under a tâtonnement process.

rule out all types of equilibrium multiplicity that may occur. In particular, as we shall see below, it does not generally rule out the possibility of dynamic indeterminacy.

Our primary goal will then be to understand, in the context of the above environment, when and how, as the type and degree of strategic interaction varies (within the bounds established by Assumption A3), non-standard dynamics will emerge. The emergence of non-standard dynamics can potentially come in a variety of forms depending on the particulars of the model and the nature of the strategic interaction. We broadly consider two different (but not mutually exclusive) types: the transition from monotonic dynamics to oscillatory ones, and the transition from saddle-path stability of the SS to some other configuration. The question of when and how these transitions come about can be answered by determining how the eigenvalues of the linearized system evolve as the nature of the interactions changes. Many of our results below characterizing emergent properties accordingly rely on analyzing the eigenvalues of the SS.

For cases where the system transitions from saddle-path stability to local instability, we also wish to characterize as much as possible what happens immediately after the transition. This requires analyzing the non-linear components of the system. Since non-linear dynamics are notoriously difficult to pin down, we are generally much more limited in what can be said. Nonetheless, we are able to provide some intuitive and illuminating results for certain special cases.

As in Cooper and John [1988], strategic interactions in this model always fundamentally involve aggregate actions affecting the optimal choice of private actions. In the static environment of Cooper and John [1988], this interaction necessarily took a static form: an agent's current private marginal value (MV) of their action was affected only by the current aggregate action. In our dynamic model, this "static interaction" is also present, and corresponds to the case $V_{e\bar{e}}(e, X, \bar{e}) \neq 0$. In addition, however, there is also a second kind of interaction in our model, which occurs when the current aggregate action $\bar{e}$ affects the current MV of the agent's stock variable (i.e., where $V_{X\bar{e}}(e, X, \bar{e}) \neq 0$), and this in turn affects the optimal accumulation of that stock as controlled by the choice of e in some prior period. We refer to this second type of interaction as "dynamic interaction".

2 Static Interaction

While in general a given dynamic macro model will simultaneously feature both static and dynamic interaction, in order to better understand their effects, we start by focusing in this section on the case of static interaction only.[3] Dynamic interaction will be discussed in

[3] The example economy presented in Section 1.3 corresponds to this case.

Sections 3 and 5 below. Focusing on static interactions also allows for a closer comparison to the static environment of Cooper and John [1988]. Accordingly, for the remainder of Section 2 we shut down dynamic interaction by assuming that $V_{X\bar{e}}(e, X, \bar{e}) = 0$ for all $e, X, \bar{e}$.

Before discussing our analytical results for this case in detail, we first present here a brief graphical overview of the essential nature of our results in order to better help the reader make sense of them. In particular, in Figure 1 we plot the deterministic evolution of X starting from a situation where $X_0 \neq X_{SS_0}$ under several different scenarios. Absent of interactions, by Assumption A1 the steady state is saddle-path stable and convergence is monotone, which is illustrated by the dashed gray curves in each of the panels of Figure 1. Each of these panels then contrasts this no-interaction path with one that would occur *with* interactions for a particular scenario.

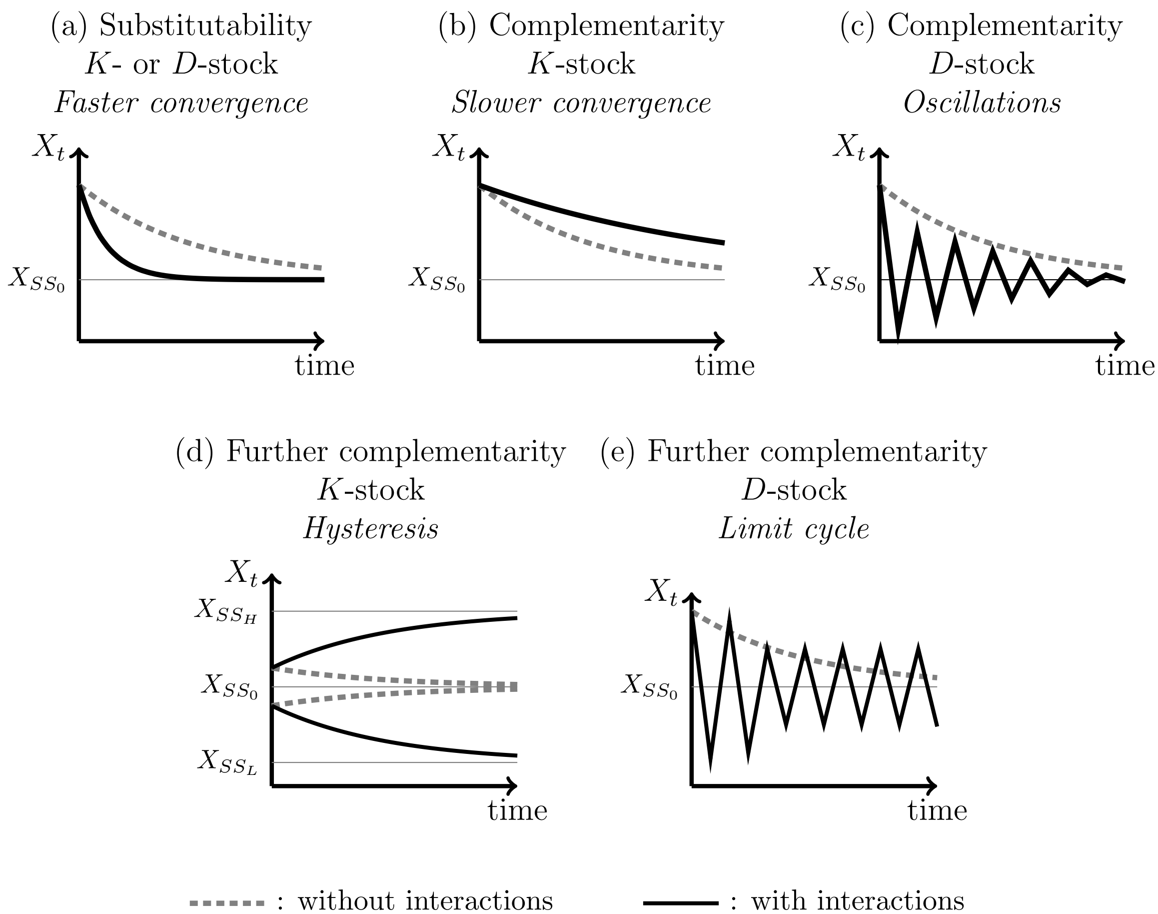
Our first result is that increasing substitutability always makes the economy "more stable", in the sense that convergence to the steady state X_{SS_0} is faster. This is illustrated in panel (a) of Figure 1, where the solid black curve (representing the model with substitutability) converges to the SS faster than the broken gray curve (representing the model with no interactions). Our second result is that, in the K-stock case, increasing complementarity always makes the economy "less stable", in the sense that convergence is slower (panel (b) of Figure 1), while in the D-stock case, given enough complementarity the dynamics will be oscillatory (panel (c) of Figure 1). Our third result is that, if one continues to increase complementarities, the steady state always X_{SS_0} loses stability, and it does so while the complementarities are still weak (in the sense of Assumption A3). What emerges at this point depends on the the nature of the stock. With a K-stock, two locally saddle-path stable steady states emerge (X_{SS_L} and X_{SS_H}), and the economy features hysteresis (panel (d) of Figure 1). With a D-stock, a saddle-path stable limit-cycle emerges (panel (e) of Figure 1). Our final result, which is illustrated only by its absence in Figure 1, is that static interactions never lead to indeterminacy.

We now turn to proving these results and providing some interpretation. Note that the agent's FOCs (4)-(5) can be combined to obtain the Euler equation

$$V_e\left(e_t, X_t, \bar{e}_t\right) = \beta\left[\left(1-\delta_X\right) V_e\left(e_{t+1}, X_{t+1}, \bar{e}_{t+1}\right) - V_X\left(e_{t+1}, X_{t+1}, \bar{e}_{t+1}\right)\right] . \tag{6}$$

This Euler equation, along with the accumulation equation $X_{t+1} = (1-\delta_X)X_t + e_t$, the TVC $\lim_{T\to\infty} \beta^T V_e(e_T, X_T, \bar{e}_T)X_T = 0$, and the symmetric equilibrium condition $\bar{e}_t = e_t$ together characterize the evolution of the equilibrium system. Under Assumptions A1 and A2, the autarkic system has a unique SS, SS_0, that is saddle-path stable, and where convergence along the saddle path occurs monotonically and non-degenerately (in particular, the eigenvalue

Figure 1: Increasing Interactions in the Case of Static Interactions Only



associated with the saddle path is strictly between 0 and 1).

2.1 Linearized Dynamics

Let partial derivatives of V with no arguments denote the corresponding derivatives evaluated at the no-interaction SS, SS_0 (e.g., $V_{ee} \equiv V_{ee}(e^s, X^s, e^s)$, etc.). The main form of our analysis will be to explore the consequences of varying the sign and degree of interactions for the local dynamics of the system around its SS. In order to isolate the direct effect of such a change on the dynamics, we suppose that we can parameterize the model in such a way that varying this parameter alters $V_{e\bar{e}}$—which controls the sign and magnitude of the direct interactions at SS_0—without affecting any of the other first- or second-order derivatives of V at SS_0. Note that this in turn implies that SS_0 will continue to be a SS even when $V_{e\bar{e}} \neq 0$. Without loss of generality, we may take $V_{e\bar{e}}$ itself to be the parameter that we vary. For convenience and without loss of generality, let us also normalize $e^s = X^s = 0$, so that we may henceforth interpret e and X as being expressed in deviations from the autarkic SS.

Taking a first-order approximation to (6) around SS_0 and then using $X_{t+1} = (1-\delta_X)X_t + e_t$, we may obtain the linearized private "decision rule"

$$e_t = \frac{\beta\chi}{\xi} e_{t+1} - \frac{\zeta}{\xi} X_t - \frac{V_{e\bar{e}}}{\xi} \bar{e}_t + \frac{\beta (1-\delta_X) V_{e\bar{e}}}{\xi} \bar{e}_{t+1} \, , \tag{7}$$

where we have defined

$$\begin{aligned} \chi &\equiv (1-\delta_X) V_{ee} - V_{eX} \, , \qquad \xi \equiv V_{ee} - \beta (1-\delta_X) V_{eX} + \beta V_{XX} \, , \\ \zeta &\equiv (1-\delta_X) \xi - \chi = \left[1 - \beta (1-\delta_X)^2\right] V_{eX} + \beta (1-\delta_X) V_{XX} \, . \end{aligned} \tag{8}$$

It can be verified that, under Assumptions A1-A2, we have $\xi < \chi < 0$,[4], which in turn implies that the coefficient on e_{t+1} in (7) is strictly between 0 and 1.

Next, note that Assumption A2*(ii)* requires that $\zeta \neq 0$. Recall our distinguishing characteristic that, conditional on e_{t+1} and the sequence of aggregate actions, X is a D-stock if e_t is affected negatively by X_t, and X is a K-stock if e_t is affected positively by X_t. Thus, since $\xi < 0$, it follows from (7) that X is a D-stock if $\zeta < 0$, and a K-stock if $\zeta > 0$. Note that this composite parameter ζ gives the combined net effect of X_t on the optimal choice of e_t coming through three different channels. First, assuming $V_{eX} \neq 0$, an increase in X_t directly affects the MV—and therefore optimal choice—of e_t. The remaining two channels relate to the fact that, since X is partially durable (assuming $\delta_X < 1$), all else equal more X_t translates into more X_{t+1}. Specifically, the second channel is that the additional X_{t+1} affects the MV of

[4]See Lemma A.1 in Appendix A.

e_{t+1} (again, assuming $V_{eX} \neq 0$), and this in turn affects the optimal *timing* of accumulation. For example, if $V_{eX} > 0$, then all else equal the increase in X_{t+1} makes accumulation at $t+1$ less costly relative to t, giving agents an incentive to delay some accumulation from t to $t+1$. Finally, the third channel relates to the concavity of V in X: when X_{t+1} increases, its MV falls, and this in turn reduces the desire to accumulate at t.

Next, Assumption A3 requires that the coefficient on $\bar{e}_t$ in (7) be strictly less than one. Thus, the upper bound for values of $V_{e\bar{e}}$ that satisfy Assumption A3 is

$$\bar{V}_{e\bar{e}} \equiv -\xi \,.$$

With this definition, one may verify that the linearized equilibrium system can be written

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{\beta}{\xi}\left[\zeta + (1-\delta_X)\left(\bar{V}_{e\bar{e}} - V_{e\bar{e}}\right)\right] \end{pmatrix}}_{M_1} \begin{pmatrix} X_{t+1} \\ e_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1-\delta_X & 1 \\ \frac{\zeta}{\xi} & -\frac{\bar{V}_{e\bar{e}} - V_{e\bar{e}}}{\xi} \end{pmatrix}}_{M_0} \begin{pmatrix} X_t \\ e_t \end{pmatrix} . \tag{9}$$

The qualitative dynamics of (9) are determined by the generalized eigenvalues λ that solve $M_0 v = \lambda M_1 v$, where v is an associated eigenvector.[5] Let λ_0 always denote the smallest generalized eigenvalue in absolute value, let λ_1 be the other eigenvalue if it exists, and set $\lambda_1 = \pm\infty$ if there is only one eigenvalue. Letting v_j denote an eigenvector associated with λ_j, the standard Blanchard and Kahn [1980] conditions then characterize the dynamics of the linearized model as follows:

(i) Local (saddle-path) stability: If $|\lambda_0| < 1 < |\lambda_1|$, then there is a unique equilibrium path satisfying the TVC, and given X_t, the unique equilibrium value of e_t is the one that projects the system onto the stable eigenspace $\{av_0 : a \in \mathbb{R}\}$.

(ii) Dynamic indeterminacy: If $|\lambda_1| < 1$, then there is a continuum of equilibrium paths satisfying the TVC.

(iii) Local instability: If $|\lambda_0| > 1$, then all equilibrium paths diverge from SS_0.

Whenever there is a unique solution, its qualitative dynamics are determined by the smaller eigenvalue λ_0, and in particular X will evolve as $X_{t+1} = \lambda_0 X_t$. If $\lambda_0 > 0$ then this system will exhibit monotonic dynamics, while if $\lambda_0 < 0$ it will exhibit oscillations. More specifically, we refer to the oscillations in this case as being "jagged", in the sense that the variables will tend to flip back and forth over their SS values each period, so that plot of their time

[5] If M_1 is invertible, these eigenvalues are exactly the two eigenvalues of the matrix $M \equiv M_1^{-1} M_0$. On the other hand, if the second diagonal element of M_1 is zero (which is possible in principle), then M_1 is singular and there is only one generalized eigenvalue. For practical purposes, the "missing" eigenvalue in that case may be treated as infinite. See Klein [2000].

paths will exhibit (possibly damped or exploding) zigzag patterns. We contrast this type of oscillations with "smooth" oscillations, which occur in systems governed by a pair of complex conjugate eigenvalues, and which are characterized by variables tending to spend multiple periods on one side of the SS before transitioning to the other side. While smooth oscillations cannot possibly play a role in a determinate equilibrium system that features only one state variable—as this one does—they could potentially play a role if additional state variables were added to the model. We return to this issue below in Section 4.

Note that Assumptions A1-A2 imply that, for the autarkic case ($V_{e\bar{e}} = 0$), we have $0 < \lambda_0 < 1 < \lambda_1 < \infty$.[6]

We now present our first key proposition, which characterizes which qualitative forms the dynamics can and cannot take depending on the type and strength of the direct interactions and the type of stock X is.

Proposition 1. *Suppose Assumptions A1-A2 hold. Then indeterminacy never emerges, regardless of the value of $V_{e\bar{e}}$. If in addition Assumption A3 holds, then:*

(i) As $V_{e\bar{e}}$ decreases from 0 to $-\infty$, SS_0 remains saddle-path stable, with convergence to SS_0 occurring monotonically.

(ii) As $V_{e\bar{e}}$ increases from 0 towards $\bar{V}_{e\bar{e}}$:

  (a) Local instability always emerges at some point, and once it does it remains permanently thereafter.

  (b) Oscillations emerge if and only if $\zeta < 0$ (D-stock case), in which case they emerge while SS_0 is still saddle-path stable, and they are always jagged.

Forward-looking behavior in principle opens the door to the possibility of dynamic indeterminacy. However, Proposition 1 establishes that indeterminacy never actually emerges here, even if we were to allow complementarities to become so strong that Assumption A3 is violated. We discuss the reasons for this below in Section 3.

Part *(i)* of Proposition 1 establishes that, when individual actions are strategic substitutes, this favors continued stability of the system. Roughly speaking, instability arises when, in equilibrium, agents' actions e react strongly to changes in the state variable X, so that a deviation of the initial state vector X_0 from the SS causes such a large response in e_0 that X_1 is even further from the SS than X_0, and this in turn causes e_1 and thus X_2 to be even more extreme, and so on. In the present environment, Assumption A1 implies that, in the autarkic case, agents do not react strongly enough to create instability. In the presence of strategic substitutability ($V_{e\bar{e}} < 0$), meanwhile, agents' equilibrium actions actually respond

[6] See the proof of Lemma A.1.

even less strongly than in the autarkic case: an initial impetus by all agents to, say, increase their actions raises $\bar{e}$, which, as can be seen in (7), has a mitigating effect on individual actions. Thus, in equilibrium, e actually reacts less strongly to deviations of X from SS than in the autarkic case. Since, by assumption, the sensitivity of e to the current state is too weak to generate instability in the autarkic case, it cannot possibly generate instability in the substitutability case when it is even weaker. Note that strategic substitutability is typical of Walrasian settings, where the price mechanism mediates all strategic interactions, which may explain why dynamic Walrasian environments are typically stable.

Part *(ii)(a)* of Proposition 1 establishes that not only *can* instability emerge via weak (in the sense of Assumption A3) strategic complementarity, this is in fact guaranteed to happen. Intuitively, unlike substitutability which, in equilibrium, weakens the sensitivity of e to the state X, weak complementarity ($V_{e\bar{e}} \in (0, \bar{V}_{e\bar{e}})$) strengthens it by creating a positive feedback mechanism: an initial desire by all agents to increase their actions raises $\bar{e}$, which favors even higher action by individuals, further raising $\bar{e}$, etc. As long as the complementarity is sufficiently strong, e will in equilibrium react strongly enough to deviations in the state vector that the SS will become locally unstable.

Part *(ii)(b)* of Proposition 1 establishes that, in the D-stock (i.e., $\zeta < 0$) case only, as the degree of complementarity increases, (saddle-path-stable) jagged oscillations will necessarily emerge, and this occurs before the system becomes unstable. Indeed, if the equilibrium response de_t/dX_t is sufficiently negative, having $X_t > X^s$ will cause e_t to be so low that we actually end up with $X_{t+1} < X^s$. And since de_t/dX_t is sufficiently negative precisely when complementarity is strong enough (but still weak), that part of the proposition should not be surprising.

2.2 Bifurcations of the Non-Linear System

Proposition 1 establishes that complementarity will eventually cause SS_0 to lose local stability. In this section, we address the question of what the dynamics of the system look like following such a loss of stability.

In the theory of dynamical systems, a change in the local stability properties when a parameter varies—such as the one described in Proposition 1 for the case where $V_{e\bar{e}}$ increases—is referred to as a bifurcation. For a system with the structure we consider here, there are essentially two possible types of bifurcations that can occur as $V_{e\bar{e}}$ changes. The first, known as a flip bifurcation, occurs as an eigenvalue passes through -1. The second, known as a Hopf bifurcation (more accurately referred to as a Neimark-Sacker bifurcation in discrete-time systems), occurs when a pair of complex conjugate eigenvalues crosses the unit circle together. The last, which we refer to simply as a $+1$ bifurcation, occurs as an eigenvalue passes through $+1$.[7] We have already established that a bifurcation will necessarily occur as $V_{e\bar{e}}$ increases toward $\bar{V}_{e\bar{e}}$, and in particular this bifurcation involves the system transitioning from (saddle-path) stability to local instability (Proposition 1*(ii)(a)*). Let $\widetilde{V}_{e\bar{e}} \in (0, \bar{V}_{e\bar{e}})$ denote the value of $V_{e\bar{e}}$ at which the first such bifurcation occurs. We have the following proposition.

Proposition 2. *Suppose Assumptions A1-A2 hold. Then the bifurcation that occurs as $V_{e\bar{e}}$ increases through $\widetilde{V}_{e\bar{e}}$ is a flip bifurcation if X is a D-stock, and a $+1$ bifurcation if X is a K-stock.*

When a system loses local stability, the dynamic paths generated by the *linearized* version of the model generically explode. However, this is not necessarily the case for the original system, which may in general feature non-linearities that prevent such explosive behavior. Thus, to understand what the dynamics of our model look like following a loss of stability, we will need to move beyond simply analyzing the linearized model. However, the non-linear dynamics of our model are ultimately driven by the derivatives of V of third-order and higher. Such high-order derivatives do not generally lend themselves to clear economic interpretations, and therefore general results of the type we have presented so far would involve conditions on the parameters that are very complicated and yield little in the way of interesting or useful economic intuition. Thus, rather than consider the general case, we highlight here one particular special case that we believe does yield some useful intuition. In particular, let us suppose that V has the form

$$V\left(e, X, \bar{e}\right) = \frac{1}{2} V_{ee}\, e^2 + V_{eX}\, eX + \frac{1}{2} V_{XX}\, X^2 + V_{e\bar{e}}\, eF\left(\bar{e}\right) \ ,$$

where F is a potentially non-linear function of $\bar{e}$. With this special form for V, it can be verified that the exact decision rule for e_t for the non-linear case (i.e., the analog to the linearized decision rule (7)) can be written

$$e_t = \frac{\beta\chi}{\xi} e_{t+1} - \frac{\zeta}{\xi} X_t - \frac{V_{e\bar{e}}}{\xi} F\left(\bar{e}_t\right) + \frac{\beta\left(1-\delta_X\right) V_{e\bar{e}}}{\xi} F\left(\bar{e}_{t+1}\right) \ , \tag{10}$$

Thus, we see that the only potential source of non-linearity is in the type and strength of the strategic interactions as controlled by the function F. In order to allow for a cleaner characterization of the model dynamics, we will impose several restrictions on F, combined

[7] $+1$ bifurcations can be sub-categorized (e.g., as a fold, transcritical, pitchfork, cusp, etc.) according to the non-linear structure of the system. Since we have not yet specified a non-linear structure for our model, when a $+1$ bifurcation occurs in our model we cannot at this point say which type it will be.

together here as Assumption A4.

Assumption A4. *The function F is three times continuously differentiable and satisfies the following:*

(i) $F(e^s) = 0$ *and* $F'(e^s) = 1$*;*

(ii) $0 < F'(\bar{e}) \leq 1$ *for all* $\bar{e}$*;*

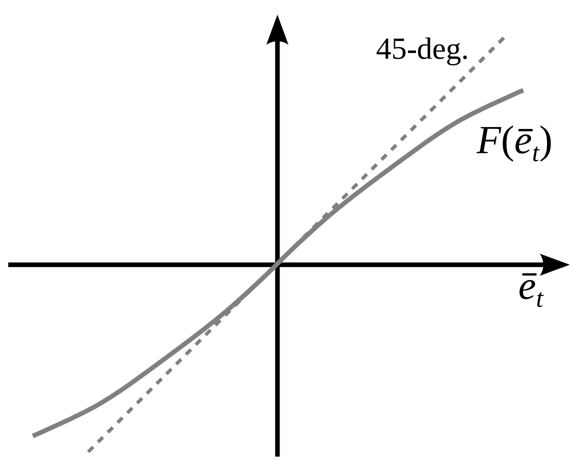
(iii) $\lim_{\bar{e}\downarrow \underline{e}} F'(\bar{e}) = \lim_{\bar{e}\to\infty} F'(\bar{e}) = 0$*;*[8]

(iv) $F''(\bar{e}) > 0$ *for* $\bar{e} < e^s$*, and* $F''(\bar{e}) < 0$ *for* $\bar{e} > e^s$*.*

Under Assumption A4*(i)*, SS_0 continues to be an equilibrium SS for all values of $V_{e\bar{e}}$, and to a first-order approximation around SS_0 the exact decision rule coincides with the linearized one. This in turn implies that the parameter $V_{e\bar{e}}$ continues to determine both the type and strength of the interactions at SS_0, just as in the linearized case.

The remaining parts of Assumption A4 together imply that F is a strictly increasing (possibly unbounded) sigmoid function whose slope attains a unique maximum of 1 at $e = e^s = 0$.[9] As a result, as $\bar{e}$ moves away in either direction from its SS_0 level, the strength of the strategic interactions—that is, $|\partial e_t/\partial \bar{e}_t|$—declines. An example of a function with this shape is given in Figure 2.

Figure 2: Example F Function



Thus, in summary, we see that the sign of $V_{e\bar{e}}$ globally controls the type of interaction (i.e., whether there is substitutability or complementarity), the magnitude of $V_{e\bar{e}}$ controls the maximum strength of the interactions (which occurs at SS_0), and the nature of the

[8]Recall that $\underline{e}$ is the minimum possible level of e.

[9]Recall that we have normalized SS_0 to $e^s = X^s = 0$.

non-linearities in F controls how quickly that strength dissipates as $\bar{e}$ moves away from SS_0. We then have the following proposition.

Proposition 3. *Suppose Assumptions A1-A4 hold. Then:*

(i) *SS_0 is the unique SS for all $V_{e\bar{e}} \leq 0$.*

(ii) *If X is a D-stock, then a flip bifurcation occurs as $V_{e\bar{e}}$ increases through $\widetilde{V}_{e\bar{e}}$, so that a 2-period cycle emerges, while the SS remains unique. There is a 1-dimensional invariant manifold that passes through SS_0 and both points on the 2-cycle, and the 2-cycle attracts all nearby trajectories contained in that manifold, while trajectories not on that manifold diverge from it.*

(iii) *If X is a K-stock, then a pitchfork bifurcation occurs as $V_{e\bar{e}}$ increases through $\widetilde{V}_{e\bar{e}}$, with two additional SSs—SS_H and SS_L—appearing, where SS_H features $e > 0$, and SS_L features $e < 0$. There is a 1-dimensional invariant manifold that passes through all three SSs, and the two new SSs each attract all nearby trajectories contained in that manifold, while trajectories not on that manifold diverge from it.*

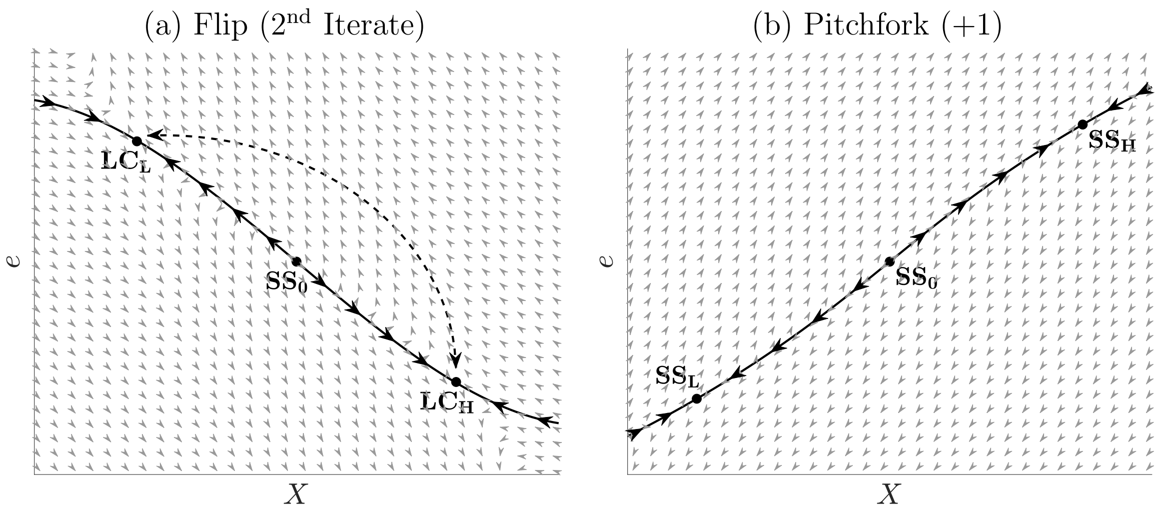
The 1-dimensional invariant manifolds described in parts *(ii)* and *(iii)* of Proposition 3, which we will refer to as "saddle manifolds", are directly analogous to the saddle paths encountered in many standard macro models: trajectories beginning on them converge to some bounded (though not necessarily constant) state, while all other trajectories diverge (and ultimately violate the TVC). Thus, only trajectories lying on the saddle manifold are equilibrium paths, and for a given value of the pre-determined variable X_t, there is a unique e_t that places the system on the saddle manifold.

Figure 3 illustrates, using phase diagrams[10] in (X, e)-space, the dynamics following the bifurcations outlined in Proposition 3. Panel (a) shows the flip bifurcation (D-stock) case. Since this case features jagged oscillations that are difficult to make sense of in a phase diagram, for this case we show the dynamics of the 2nd iterate of the system (i.e., the system that gives the position of the original system two periods later, rather than one). The two points on the 2-cycle from Proposition 3*(ii)* (which are SSs of the 2nd iterate) are indicated by LC_H and LC_L. As indicated by the dashed curve with arrows at either end, starting from either of these points the original system jumps back and forth each period, exhibiting jagged oscillations. The light gray arrows indicate the local direction of motion (for the 2nd-iterate system), while the saddle manifold is indicated as the thick solid curve, which passes through LC_H, LC_L, and SS_0. Trajectories beginning on the saddle manifold (except for those beginning exactly at SS_0) converge to one of the 2-cycle points LC_H or LC_L. In

[10] In discrete time, pahse diagrams are only suggestive of the dynamics.

terms of the original system, this convergence corresponds to jagged oscillations in X and e, each jumping back and forth across its SS_0 value each period, with the system converging eventually to the 2-cycle. As indicated by the light gray arrows, all trajectories beginning anywhere other than the saddle manifold diverge (and end up violating the TVC). Thus, for a given X_t, the unique value of e_t satisfying the TVC is the one that places the system on the saddle manifold.

Figure 3: Non-Linear Dynamics After the Bifurcation



Panel (b) illustrates the phase diagram for the pitchfork bifurcation (K-stock) case. In this case, two new SSs, SS_L and SS_H, emerge. The saddle manifold is again indicated by the thick solid curve, which passes through all three SSs. Trajectories beginning on the saddle manifold (except for those beginning exactly at SS_0) converge to either SS_L or SS_H, with all other trajectories diverging. Thus, again, given X_t, the unique solution for e_t is the one that places the system on the saddle manifold. Note that we may describe SS_L and SS_H as each being saddle-path stable, with the respective saddle path given locally by the saddle manifold. Multiple saddle-path-stable SSs can, in some environments, imply indeterminacy if the saddle paths of the SSs overlap, i.e., if, for some values of the pre-determined variables, there is more than one saddle path that the jump variables can place the system onto. Since, in the present case, the saddle paths for SS_L and SS_H are given by separate sections of the saddle manifold, we do not have any indeterminacy here: for a given X, there is always a unique e that places the system on the saddle manifold. Which SS the system eventually converges to is then determined entirely by which side of SS_0 the initial level of X is: if $X_0 < 0$ then the system converges to SS_L, while if $X_0 > 0$ the system converges to SS_H. Thus, we have hysteresis in this case: small variations in the initial conditions of the system can impact its long-run state.

3 Indeterminacy and Dynamic Interaction

Proposition 1 established that indeterminacy never arises for any value of $V_{e\bar{e}}$, regardless of whether it's positive or negative, and even regardless of whether it satisfies Assumption A3. To understand why, consider a simple linear univariate system made up of a single jump variable x that must satisfy a forward-looking Euler equation of the form $x_t = \eta x_{t+1}$ with $|\eta| \neq 1$, and with a TVC of the form $\lim_{t\to\infty} x_t = 0$. Indeterminacy will arise in this environment if every sequence satisfying the Euler equation necessarily also satisfies the TVC. This is the case if and only if $|x_t|$ is always a declining sequence, i.e., if $|\eta| > 1$. Since η can be interpreted as a parameter governing how strongly $|x_t|$ responds to anticipated changes in $|x_{t+1}|$, we can interpret indeterminacy as arising when this reaction is more than one-for-one.

This basic intuition carries over to our more complicated environment: while local instability can be thought of as a situation where the current equilibrium action e_t is highly sensitive to the current state, indeterminacy arises when e_t is highly sensitive to the anticipated *future* equilibrium action e_{t+1}.

With this in mind, consider how e_t responds in equilibrium when e_{t+1} is anticipated to increase. In the autarky ($V_{e\bar{e}} = 0$) case, Assumption A1 implies that agents will want to increase e_t, though less than one-for-one (as noted earlier, the coefficient on future private accumulation e_{t+1} in the decision rule (7) is strictly between 0 and 1). In order for indeterminacy to appear once we introduce static interactions ($V_{e\bar{e}} \neq 0$), it would have be the case that, relative to the autarkic case, these interactions increase the responsiveness of e_t to e_{t+1} by enough that e_t ends up responding more than one-for-one. However, it turns out that adding static interactions produces two effects that always work in opposite directions on this responsiveness. In the complementarities case ($V_{e\bar{e}} > 0$), these are as follows. First, a higher anticipated $\bar{e}_{t+1}$ makes accumulation at $t+1$ relatively cheap (since $V_{e,t+1}$ will increase), which favors delaying accumulation to the future, so that agents have less desire to increase e_t in response to an anticipated increase in e_{t+1} than in the autarkic case. This effect, which tends to make e_t *less* responsive to e_{t+1}, is captured by the last term in the private decision rule (7). Second, for a given initial desire to increase e_t in response to an anticipated increase in equilibrium e_{t+1} (i.e., the net result of the autarkic effect and the first effect), complementarity produces the usual amplification force (see Section 1.4), which favors e_t being *more* responsive to e_{t+1}. It turns out that these two effects largely cancel each other out, and as a result the equilibrium sensitivity of e_t to e_{t+1} will never become high enough in the static-interaction case to create indeterminacy, no matter how large $V_{e\bar{e}}$ becomes.[11]

A natural question is whether there are any circumstances under which indeterminacy *can* arise in this model. Given the above discussion, one possibility would be to seek an additional mechanism that works to increase agents' initial desire to increase e_t in response to an anticipated increase in e_{t+1} (rather than decrease it, as the first effect above does). As we now show, dynamic interaction—that is, a situation where a change in the current aggregate action $\bar{e}$ affects the current private MV of a stock variable—which we have so far shut down, provides such a mechanism.

In particular, let us now allow $V_{X\bar{e}}(e, X, \bar{e}) \neq 0$, i.e., dynamic interaction. To make the results as clear as possible, we also shut down static interaction by assuming that $V_{e\bar{e}}(e, X, \bar{e}) = 0$ for all $e, X, \bar{e}$. Taking a first-order approximation to (6) around SS_0 and then using $X_{t+1} = (1 - \delta_X)X_t + e_t$, we may obtain the private decision rule for this case as

$$e_t = \frac{\beta\chi}{\xi} e_{t+1} - \frac{\beta V_{X\bar{e}}}{\xi} \bar{e}_{t+1} - \frac{\zeta}{\xi} X_t \,, \tag{11}$$

where χ, ξ and ζ are as defined in Section 2.1.[12] One may verify that the linearized equilibrium system satisfies

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{\beta}{\xi}\left(\chi - V_{X\bar{e}}\right) \end{pmatrix}}_{M_1} \begin{pmatrix} X_{t+1} \\ e_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 - \delta_X & 1 \\ \frac{\zeta}{\xi} & 1 \end{pmatrix}}_{M_0} \begin{pmatrix} X_t \\ e_t \end{pmatrix} . \tag{12}$$

The qualitative dynamics of (12) are again determined by the generalized eigenvalues λ that solve $M_0 v = \lambda M_1 v$, where v is an associated eigenvector. We have the following proposition:

Proposition 4. *Suppose Assumptions A1 and A2 hold. Then as $V_{X\bar{e}}$ increases or decreases from zero, at some point dynamic indeterminacy emerges, and once it does it remains thereafter as $V_{X\bar{e}}$ continues to increase or decrease all the way to $\pm\infty$.*

Thus, unlike in the static interaction case where dynamic indeterminacy was not possible, we see that sufficiently strong dynamic interaction (either positive or negative) always produces indeterminacy. Recall from the discussion above that we may characterize indeterminacy as arising in cases where the current equilibrium action e_t is highly sensitive to

[11] The intuition for the substitutability ($V_{e\bar{e}} < 0$) case is essentially the same, just with the directions of the first and second effects each reversed.

[12] Note that Assumption A3 is trivially always satisfied in this case, as, for given values of X_t, e_{t+1}, and $\bar{e}_{t+1}$, the optimal private choice of e_t does not depend on $\bar{e}_t$.

changes in the anticipated future equilibrium action e_{t+1}. In this environment, indirect interaction provides a mechanism to increase this sensitivity without bound. For example, from the perspective of an agent at date t, if $V_{X\bar{e}} > 0$ and the anticipated aggregate $\bar{e}_{t+1}$ increases, this raises the future MV of X_{t+1}. This increases agents' desire to accumulate at t, i.e., to increase e_t. The larger $V_{X\bar{e}}$ is, the more sensitive e_t is to $\bar{e}_{t+1}$ (see (11)), and thus in equilibrium e_t can be made arbitrarily sensitive to e_{t+1} by sufficiently increasing $V_{X\bar{e}}$. If this sensitivity is high enough, the system will feature indeterminacy, which is the content of Proposition 4. An analogous argument applies when $V_{X\bar{e}}$ is sufficiently negative.

While indeterminacy always eventually emerges for $|V_{X\bar{e}}|$ large enough, other qualitative behaviors are possible for intermediate values of $|V_{X\bar{e}}|$, as outlined in the following proposition.

Proposition 5. *Suppose Assumptions A1 and A2 hold.*

(i) As $V_{X\bar{e}}$ decreases from zero, the system exhibits saddle-path stable monotonic dynamics up until the point that indeterminacy first emerges.

(ii) If $\zeta \leq -\delta_X^2\xi/(1+\delta_X)$, then as $V_{X\bar{e}}$ increases from zero the system exhibits saddle-path stable monotonic dynamics up until the point that indeterminacy first emerges.

(iii) If $\zeta > -\delta_X^2\xi/(1+\delta_X)$, then as $V_{X\bar{e}}$ increases from zero, the system transitions at some point from saddle-path stability (with monotonic dynamics) to local instability via a $+1$ bifurcation, before eventually transitioning to indeterminacy.

Note that, since $\xi < 0$, the condition under which local instability emerges for some range of $V_{X\bar{e}}$ necessarily requires $\zeta > 0$, i.e., for X to be a K-stock. While the mechanism is more subtle, when local instability emerges here it does so for the same fundamental reason as in previous sections; namely, because equilibrium e_t is highly sensitive to the current state X_t. In this case, one can show that, in response to changes in the conditions it faces at date t, individual agents have a smoothing motive that causes them to initially respond by optimally adjusting e_t and e_{t+1} in the same direction. Thus, when $\zeta > 0$, a higher level of X_t causes agents to initially respond by increasing both e_t and e_{t+1}. In equilibrium, there must be a corresponding increase in $\bar{e}_{t+1}$, and from (11) we see that if $V_{X\bar{e}} > 0$ this causes a further rise in e_t. Because of the smoothing motive, e_{t+1} will also again rise, and in equilibrium this is met by a further increase in $\bar{e}_{t+1}$, causing a further increase in e_t, and so on. In this way, the initial effect of the increase in X_t on e_t is amplified in equilibrium. If ζ is positive and large enough, then Proposition 5 states that there is a range of positive values of $V_{X\bar{e}}$ where this amplification mechanism is strong enough to create local instability, but where $V_{X\bar{e}}$ is not so large that indeterminacy arises.

4 Determinate Smooth Oscillations

Returning to the direct-interaction case, in Proposition 1*(ii)(b)*, we established that if oscillations emerge, then they are jagged, in the sense that the model variables will flip back and forth across their SS levels each period. Similarly, Proposition 3*(ii)* established (for the non-linear model) that, once the system loses local stability, such jagged oscillations are associated with a 2-period limit cycle.

While understanding the conditions under which jagged oscillations—which are associated with negative eigenvalues—may arise in our simple environment is helpful for thinking about what kinds of mechanisms may produce endogenous cyclical fluctuations in a fully micro-founded model, we do not view jagged oscillations themselves as being particularly empirically relevant. After all, real-world data series rarely exhibit the kind of sawtooth patterns illustrated in panels (c) and (e) of Figure 1, which are characteristic of jagged oscillations. Rather, if endogenous oscillations are in fact an important driver of business cycle fluctuations, then they are likely to be smooth, not jagged. As noted in Section 2.1, however, smooth oscillations, which are associated with complex eigenvalues, cannot occur in a model that features only a single state variable: the evolution of a determinate model can be reduced to a system in the state variables only, and univariate systems cannot feature complex eigenvalues.

With this in mind, suppose we augment the model we set up in Section 1 with a second stock variable, Z, which evolves according to

$$Z_{t+1} = (1 - \delta_Z)\, Z_t + e_t \,, \tag{13}$$

with $\delta_Z \in (0, 1]$. The agent's objective function is now

$$\sum_{t=0}^{\infty} \beta^t V\left(e_t, X_t, Z_t, \bar{e}_t\right) \,, \tag{14}$$

so that V now depends on both X and Z. We assume that the period payoff function V is strictly concave in its first three arguments. All other elements of the model are unchanged.[13]

If $\beta > 0$ then the resulting dynamic system will be 4-dimensional.[14] As a result, conditions that analytically characterize the dynamics for this case are quite complicated and difficult to interpret. Nonetheless, we can gain some useful insight by considering the myopic case

[13]Note that we update Assumptions A2 and A3 in a natural way to reflect the presence of the additional state variable. See Appendix A.3.1.

[14]The two state variables X and Z, plus two jump variables, i.e., the Lagrange multipliers associated with the accumulation equations (2) and (13).

where $\beta = 0$. It can be verified in that case that Assumption A3 is now equivalent to the condition $V_{e\bar{e}} < \bar{V}_{e\bar{e}} \equiv -V_{ee}$.[15] Without loss of generality, let us assume that $V_{eX} \leq V_{eZ}$ (if not, then we may simply reverse the roles of X and Z in what follows). We have the following proposition.

Proposition 6. *Suppose Assumptions A1-A3 hold, and that $\beta = 0$. Then:*

(i) Smooth oscillations will emerge for some range of $V_{e\bar{e}}$ if and only if $V_{eX} < 0 < V_{eZ}$ and $\delta_Z > \delta_X$.

(ii) As $V_{e\bar{e}}$ decreases from 0 towards $-\infty$, the system always remains locally stable.

(iii) As $V_{e\bar{e}}$ increases from 0 towards $\bar{V}_{e\bar{e}}$:

- *(a) Local instability always emerges at some point, and once it does it remains permanently thereafter.*
- *(b) If the conditions in part (i) hold, then the loss of stability will occur while the system is experiencing smooth oscillations (i.e., via a Hopf bifurcation) if and only if in addition V_{eZ} is neither too small nor too large relative to $|V_{eX}|$.*[16]

There are a few things worth emphasizing about Proposition 6. First, the fact that we have added a second stock variable and focused on the myopic case has not changed the conclusions we obtained in Section 2 that substitutability does not produce instability, while sufficient weak complementarity always produces instability.

Second, adding another stock variable can indeed lead to smooth oscillations, but a necessary condition for this to happen is that one of the stocks is a D-stock (X in this case) and the other a K-stock (Z). Roughly speaking, the D-stock causes a higher level of past accumulation to favor a lower level of current accumulation, which is precisely the kind of reversal that characterizes oscillations. The K-stock, meanwhile, can be thought of as imparting "momentum" to the system that prevents higher past accumulation from *immediately* leading to lower current accumulation, which would be the case for jagged oscillations. Instead, it takes multiple periods for any reversal favored by the D-stock to actually occur, so that we end up with smooth oscillations.

Finally, neither stock variable strongly dominates the other in driving the dynamics of the system, part (iii)(b) of the proposition states that when the system loses stability it will do so via a Hopf bifurcation. One can show that if we were to introduce non-linearities into the model via an F function of the form used in Section 2.2, upon losing local stability a

[15] See Appendix A.3.1.

[16] Specifically, if and only if $(\frac{2-\delta_Z}{2-\delta_X})^2 |V_{eX}| < V_{eZ} < (\frac{\delta_Z}{\delta_X})^2 |V_{eX}|$.

limit cycle would emerge that would attract all nearby trajectories, with such trajectories exhibiting smooth oscillations.[17]

Note that while the results in Proposition 6 were obtained for the $\beta = 0$ case, they apply more generally. For example, in the next section we provide a fully micro-founded example with $\beta > 0$ that reflects the basic lessons of Proposition 6.

5 Beyond the Tractable Case: Numerical Examples

The key lessons we take away from the above analysis are that *(i)* substitutability generally favors stability, whereas instability will generally arise only if complementarity is strong enough; *(ii)* jagged oscillations only tend to emerge with enough complementarity when a D-stock strongly dominates the dynamics, in which case instability emerges in the form of a 2-period limit cycle; *(iii)* when a K-stock strongly dominates the dynamics, instability emerges in the form of hysteresis; *(iv)* when a D-stock and a K-stock both exert an important degree of influence over the dynamics, smooth oscillations may emerge; and *(v)* indeterminacy arises when e_t is highly sensitive to the anticipated future equilibrium action e_{t+1}.

These results were obtained in an abstract environment using a particular analytical approach (i.e., directly varying $V_{e\bar{e}}$ or $V_{X\bar{e}}$ without allowing anything else to change). It is natural to ask whether we may expect similar results would hold in actual economic models of the kind typically encountered in macro. In this section, we perform some numerical analysis of such a model, with an eye to illustrating that the above lessons are not simply artifacts of the particular form of the analysis done in the previous sections.

The model we use will feature at the same time a K-stock and a D-stock, and two types of externalities, in production and in utility. We suppose that household preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \left\{ \log\left(\psi d_t + c_t - (1-\psi)\, h_t^{\mu} \bar{c}_t^{1-\mu} + \tilde{c}\right) - \frac{\omega_0}{1+\omega} \ell_t^{1+\omega} \right\},$$

where c is consumption, d is a stock of durable goods, h a stock of habit, and $\psi \in [0,1]$ is a parameter that controls the importance of the durable goods (an D-stock) relative to the habit (a K-stock). We introduce a sort of "Keeping up with the Joneses" externality when the parameter $\mu \in [0,1]$ is not equal to one: the strength of habits is larger when aggregate consumption $\bar{c}$ is high. Meanwhile, $1/\omega$ controls the elasticity of labor supply, while $\tilde{c}$ and ω_0 are parameters that will control, respectively, the SS curvature of the utility function and

[17] A formal proof of this result, which we have omitted for considerations of length, can be provided upon request.

the SS labor supply. The stocks accumulate according to

$$d_{t+1} = (1 - \delta_d)d_t + \gamma_d c_t \,,$$
$$h_{t+1} = (1 - \delta_h)h_t + \gamma_h c_t \,.$$

Production of each firm is given by $y_t = \chi_t \ell_t^{\alpha}$, but we now assume that the production externality takes the more general form

$$\chi_t = \chi \bar{\ell}_t^{\,\rho(\bar{\ell}_t)\alpha},$$

where we define the function

$$\rho\left(\bar{\ell}\right) = \nu \exp\left\{-\frac{1}{2}\left(\frac{\bar{\ell} - \ell_{ss}}{\sigma}\right)^2\right\}.$$

Here, ℓ_{ss} denotes the SS level of labor. Note that, in the limiting case $\sigma \to \infty$, $\rho(\bar{\ell})$ is constant, in which case the form of the externality collapses back to the simple version seen in Section 1.3. For $\sigma < \infty$, meanwhile, $\rho(\ell)$ declines monotonically as ℓ moves away from the SS in either direction, and as ρ falls so will the strength of the productive externality. This echoes one of the key features of the F function described in Assumption A4.

Much like $V_{e\bar{e}}$ in the theoretical analysis above, in our numerical analysis the quantity ν parametrizes the strength of the productive externalities at the SS, and will be the main parameter that we vary in our experiments. For the remaining parameters, we use three different configurations, chosen to illustrate the different kinds of behavior that can emerge as ν increases from zero. The parameters in Table 1 are taken to be the same across all three configurations. Table 2 then shows the remaining parameters that vary across the two configurations, with the exception of ω_0, which we set in order to normalize the SS level of ℓ to one in all cases.

Table 1: Common Parameters Values

β	α	χ	ω	$\tilde{c}$	δ_d	σ
0.98	0.7	1	0.5	1	0.025	4

Notes: ω_0 is set so that steady state level of hours is one.

The equilibrium version of the model can be reduced to a dynamic system in four variables: two state variables (d and h), and two jump variables (the Lagrange multipliers

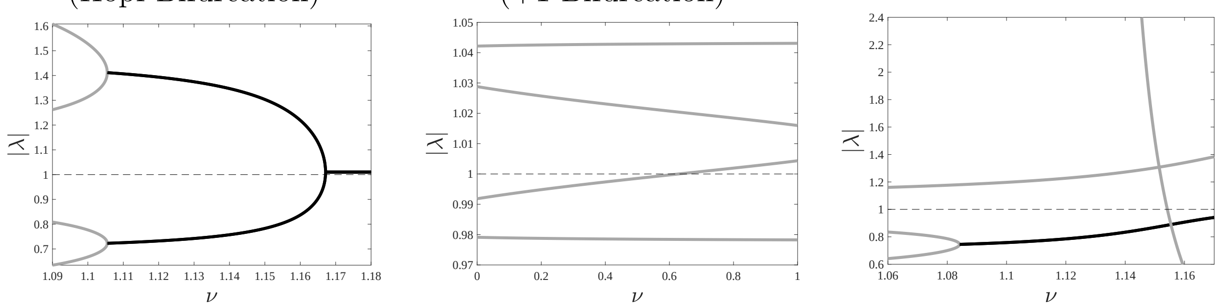
Table 2: Specific Parameters Values

	ψ	γ_d	γ_h	δ_h	μ
Case I: Local instability (Hopf)	0.5	0.4	0.4	0.5	1
Case II: Local instability (+1)	0.1	0.05	0.08	0.05	1
Case III: Indeterminacy	0.5	0.4	0.4	0.5	0.5

attached to the two accumulation equations). The linearized system therefore has four eigenvalues. In the benchmark case in which $\nu = 0$ (no productive externalities) and $\mu = 1$ (no externalitis in preferences), we always have saddle-path stable monotonic dynamics, and in particular all four eigenvalues are real and positive, with two stable and two unstable. In Figure 4, we plot the moduli of the four eigenvalues over a range of $\nu > 0$ that contains the first bifurcation of the system. Gray portions of curves correspond to real eigenvalues, black portions to conjugate pairs of complex eigenvalues.

Figure 4: Eigenvalue Moduli as a Function of ν

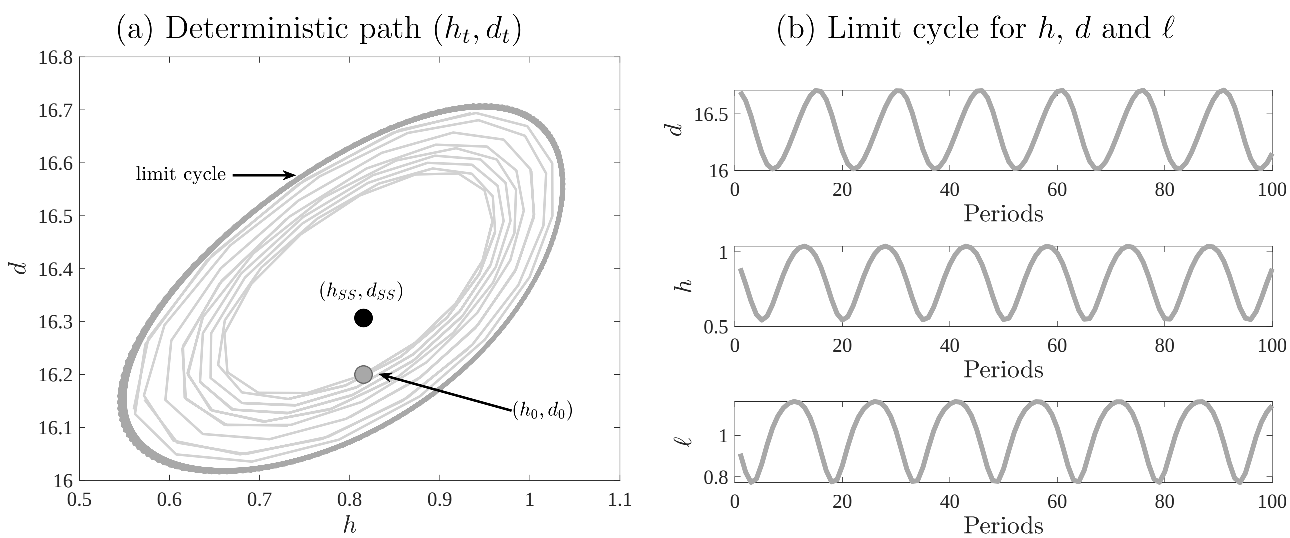
(a) Case I: Local Instability (Hopf Bifurcation) (b) Case II: Local Instability (+1 Bifurcation) (c) Case III: Indeterminacy



Notes: The four curves show the moduli of the four eigenvalues for the respective parameterization over a range of ν containing the first bifurcation. Gray portions of curves correspond to real eigenvalues, black portions to complex eigenvalues.

The first parameter configuration (Case I, panel (a) of Figure 4) is one in which the two stocks d and h have equal weight in the utility function: $\psi = 0.5$, and $\gamma_d = \gamma_h = 0.4$. For ν below approximately 1.105, the four eigenvalues are real and positive, with two stable and two unstable. As ν increases through that point, two pairs of complex eigenvalues emerge, one pair stable and the other unstable. Thus, the system continues to be saddle-path stable, but now exhibits smooth oscillations rather than monotonic dynamics. Finally, as ν increases through approximately 1.167, the stable pair of complex eigenvalues exits the unit circle, resulting in a Hopf bifurcation. This bifurcation was not possible in our analytic example because the dynamics was of order two. Therefore, as the two eigenvalues had to be conjugate if complex, it was not possible for one eigenvalue to stay larger than one in modulus, for the other one to cross 1 from above in modulus and for the two to be complex. In the present model where the dynamics is of order four, we can have a determinate equilibrium (two eigenvalues are larger than one in modulus) and at the same time the two eigenvalues smaller than one in modulus that first become complex and conjugate and then leave the unit disk, hence a Hopf bifurcation. Starting away from the steady state, the economy will jump onto a saddle path that converges to a smooth limit cycle. The economics is very similar to the one of a 2-cycle with a D-stock, but the cycle is here smoother as we have two state variables. Such a deterministic simulation is shown on Figure 5, where we have solved the model using the perturbation algorithm of Galizia [2021] as implemented by Duineveld [2021]. This illustrate the convergence to a smooth limit cycle. In Beaudry, Galizia, and Portier [2020], we have have estimated a specific model of unemployment risk and incomplete markets whose solution features limit cycle forces.

Figure 5: Convergence to a Smooth Limit Cycle in Case I



Notes: This Figure shows the deterministic path when the economy starts with h at the steady state and d below the steady state (the grey dot in panel (a)). The black dot on panel (a) represents the steady state. Panel (b) displays time path of d, h and ℓ along the limit cycle.

Compared with Case I, the second parameter configuration (Case II) puts significantly more weight on h than d in the utility function: $\psi = 0.1$, and $\gamma_h > \gamma_d$. In addition, h also now depreciates more slowly than in Case I. The implications of this case for the eigenvalues are shown in panel (b) of Figure 4. For ν close to zero we again have four real positive eigenvalues, with two stable and two unstable. As ν increases, one of the stable eigenvalues gradually increases, eventually passing through 1 at around $\nu = 0.62$, resulting in a +1 bifurcation. The initial steady state SS_0 becomes unstable and two new saddle-path stable SS emerge.

Comparing the results for Cases I and II, recall that as ν increases this tends to reduce the degree of substitutability/increase the degree of complementarity between the aggregate and private actions $\bar{c}$ and c. In Case I, where the effects of d (an X-stock) and h (a Z-stock) are balanced, we obtain smooth oscillations and a Hopf bifurcation as ν increases. In contrast, in Case II, which is strongly dominated by h (the K-stock), the first bifurcation as ν increases is a +1 bifurcation.

Consider now Case III in which we also introduce the externality in the utility function ($\mu < 1$), so that the future marginal value of the stock h_{t+1}, that depends on current action c_t, is directly impacted by *future* aggregate action $\bar{c}_{t+1}$, which corresponds to the mechanism highlighted in the "dynamic interactions" analytical case. In this case, as shown on panel (c) of Figure 4, the eigenvalues are again initially real and positive for $\nu = 0$, with two stable and one unstable. As ν increases, as in Case I at some point the two stable eigenvalues become complex, so that smooth oscillations emerge. Unlike in Case I, however, as ν increases beyond this point, the first bifurcation involves one of the (real) unstable eigenvalues decreasing through one in absolute value at around $\nu = 1.155$, so that, for the reasons discussed in Section 3, indeterminacy emerges.

Conclusion

In this paper, we extend the static framework of Cooper and John [1988] to a dynamic environment that embeds two fundamental economic forces that both arise in virtually all modern macro models: the accumulation of stocks, and strategic interactions between agents. The basic form of our analysis is as follows. We suppose that, in the absence of interactions, dynamics exhibit monotonic convergence to a unique saddle-path stable SS, which we refer to as standard dynamics. We then ask when and how non-standard dynamics may appear as the type and strength of strategic interactions increases.

The main take-aways of our analysis are as follows. First, non-standard dynamics can be interpreted as arising from a certain type of sensitivity of the system to economic conditions. Second, local instability tends to emerge when the system is sensitive to the current state, whereas indeterminacy tends to emerge when the economy is sensitive to the anticipated future state. Third, the kind of sensitivity that leads to non-standard dynamics is largely associated with strategic complementarity in accumulation decisions, since this complementarity exerts an amplifying force in equilibrium on agents' initial desire to accumulate. On the other hand, substitutability does not generally produce non-standard dynamics since it tends to exert a dampening force rather than an amplifying one. Fourth, cyclicality only tends to emerge when there is a stock variable that exerts a negative influence on agents' desire to accumulate more of it. Fifth, the emergence of instability is typically associated with the emergence of a limit cycle when the system exhibited cyclical dynamics just prior to the loss of stability, and with the emergence of multiple steady states when the system exhibited monotonic dynamics. Finally, indeterminacy tends to appear only in the case of dynamic interactions, i.e., where an anticipated change in the future aggregate level of accumulation causes agents to anticipate a change in the future value of a stock variable, and this in turn affects their current desire to accumulate that stock.

We believe this framework is useful for understanding many situations covered in the dynamic macroeconomic literature. Although many of the models we briefly discuss below do not fit perfectly into our simplified framework, most of their dynamic properties can be meaningfully interpreted through our lens.

To begin, we showed that substitutability tends to weaken dynamic forces and thereby favors quick monotonic convergence. As conventional competitive models most often generate substitutability through price incentives, such frameworks are likely to lack strong internal propagation forces. This is consistent with the finding of Cogley and Nason [1995] for the first generation of Real Business Cycle models. In the same vein, granular fluctuations in competitive models (Gabaix [2011], Carvalho and Grassi [2019]) are likely to be dampened by competitive forces and reallocation dynamics. For example, if there is a positive idiosyncratic productivity shock to a very large firm, input prices should increase as this big firm produces more, and other firms should then scale down their production. Although this effect is at play for example in Carvalho and Grassi [2019], it is second-order because of decreasing returns and the Pareto firm-size distribution.

Turning to models with complementarity, a classic example is Benhabib and Farmer [1994], in which an externality in production creates complementarity. As the variable accumulated is capital, the complementarity is dynamic, and as we have shown this is a case in which the equilibrium is prone to indetermincay. When the externality is large enough, the return to capital becomes constant (the endogenous growth case in Benhabib and Farmer [1994]) and there is also multiplicity of steady states (more precisely of Balanced Growth Paths), just as in our K-stock configuration. Another example of a K-stock configuration is Fajgelbaum, Schaal, and Taschereau-Dumouchel [2017]. The complementarity comes from an information externality. The stock that accumulates is the precision of belief about the return of investment. The more information has been accumulated because of past high investment, the higher is precision and the higher is the current incentive to invest, hence the multiplicity of steady state and hysteresis obtained by Fajgelbaum, Schaal, and Taschereau-Dumouchel [2017], as predicted in our simple framework. Fernández-Villaverde, Mandelman, Yu, and Zanetti [2019] is an example of complementarity generated by search. Firms that produce intermediate and final goods must build long-lasting trading relationships to produce a final good by exerting costly effort. The matching function among firms is supermodular, which overcomes the congestion effects of many conventional search environments and generates search complementarities. That model is a case where static complementarity are strong enough to violate our Assumption A3 of weak complementarities. In Fernández-Villaverde, Mandelman, Yu, and Zanetti [2019], the indeterminacy then generated is associated with a selection mechanism to generate endogenous switches between two regimes in the economy.

The role of financial frictions in shaping the dynamics of an economy can also be understood in our framework. In the financial accelerator model of Bernanke, Gertler, and Gilchrist [1999], the net worth of potential borrowers lowers the external finance premium, and therefore makes investment and wealth accumulation easier. This is a K-stock type model, so we should expect it to generate more persistence, but not financial cycles in the sense of endogenous (dampened or not) boom-bust dynamics. This is indeed what the model generates. In contrast, Gorton and Ordoñez [2019] study a model of financial frictions with periods in which lenders do not carefully screen investment project qualities. There is a credit boom, but also a depreciation of information in credit markets over time, in which case lenders increasingly have incentives to acquire information and restrict credit. There are complementarities between the volume of credit and the incentives for information acquisition. The model fits in the D-stock case: the more loans have been distributed in the past, the poorer information (that encodes past history of loans quantity), the lesser the incentive to distribute further loans. Therefore, not surprisingly, the model generates endogenous self-sustained boon-bust cycles. The D-stock configuration is also the one of Beaudry, Galizia, and Portier [2020]. The complementarity works as follows. Households spend to accumulate a durable good and face uninsurable unemployment risk. When aggregate spending is high, firms post many vacancies to meet demand, unemployment risk is low so that each household faces low unemployment risk, does little precautionary savings and hence spends more. As the marginal utility of the stock of the durable good is decreasing, the higher the stock the lower new expenditures. Beaudry, Galizia, and Portier [2020] show that an estimated version of the model displays smooth self-sustained oscillations, as our theory would predict.

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Appendix

A Proofs

A.1 Section 2.1 Proofs

A.1.1 Preliminaries

Lemma A.1. *Suppose Assumptions A1 and A2 hold. Then $\xi < \chi < 0$ and $\chi/\xi < 1/(1+\beta\delta_X)$.*

Proof. To see that $\xi < 0$, suppose to the contrary that $\xi \geq 0$. Since $V_{ee}, V_{XX} < 0$ by the assumed strict concavity of V (see Section 1.1), looking at the definition of ξ in (8) this would necessarily require $V_{eX} < 0$. Thus,

$$\begin{pmatrix} 1 & -\sqrt{\beta} \end{pmatrix} \begin{pmatrix} V_{ee} & V_{eX} \\ V_{eX} & V_{XX} \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{\beta} \end{pmatrix} = \xi - \left[2 - \sqrt{\beta}\,(1-\delta_X)\right] \sqrt{\beta} V_{eX} > \xi \geq 0\,.$$

But this contradicts concavity of V. Thus, we cannot have $\xi \geq 0$, which confirms $\xi < 0$.

Next, for the autarkic case ($V_{e\bar{e}} = 0$), the linearized decision rule (7) becomes

$$e_t = \frac{\beta\chi}{\xi} e_{t+1} - \frac{\zeta}{\xi} X_t\,.$$

If $\chi = 0$, then $\zeta = (1-\delta_X)\xi$, and thus this decision rule becomes simply $e_t = -(1-\delta_X)X_t$, and therefore the X accumulation equation (2) implies $X_{t+1} = 0$, so that the lone eigenvalue of the system is zero. This contradicts Assumption A2*(i)*, and therefore we cannot have $\chi = 0$. The autarkic linearized system can therefore be written

$$\begin{pmatrix} X_{t+1} \\ e_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\delta_X & 1 \\ \frac{\zeta}{\beta\chi} & \frac{\xi}{\beta\chi} \end{pmatrix} \begin{pmatrix} X_t \\ e_t \end{pmatrix}.$$

The trace and determinant of the transition matrix in this expression are given respectively by $T = 1 - \delta_X + \xi/(\beta\chi)$ and $D = 1/\beta$. Assumptions A1-A2 together require the eigenvalues to be real with one strictly between 0 and 1, and the other greater than 1 in absolute value. Since the product of the eigenvalues is $D > 0$ and one of them is strictly between 0 and 1, the unstable eigenvalue must also be positive. That is, the two eigenvalues λ_0, λ_1 must satisfy $0 < \lambda_0 < 1 < \lambda_1$. This is the case only if $T > D + 1$, i.e., if $\xi/\chi > 1 + \beta\delta_X > 1$. Thus, ξ and χ must be of the same sign, with $|\xi| > |\chi|$, and since $\xi < 0$ it therefore follows that $\xi < \chi < 0$, and therefore $\chi/\xi < 1/(1+\beta\delta_X)$, which completes the proof. ■

Define $\alpha_e \equiv \beta\chi/\xi$, $\alpha_X \equiv -\zeta/\xi = \alpha_e - (1-\delta_X)$, and $\rho \equiv -V_{e\bar{e}}/\xi \equiv V_{e\bar{e}}/\bar{V}_{e\bar{e}}$. Under Assumptions A1-A2, Lemma A.1 implies that (i) $0 < \alpha_e\beta/(1+\beta\delta_X) < 1$, (ii) α_X is non-zero and of the same sign as ζ, and (iii) ρ is strictly increasing in and of the same sign as $V_{e\bar{e}}$, with $\rho = 0$ corresponding to autarky. We will henceforth assume these properties. Assumption A3, meanwhile, is equivalent to the condition $\rho < 1$. Given the tight relationship between ρ and $V_{e\bar{e}}$, we henceforth take ρ as the parameter controlling the type and strength of interactions, with $\rho < 0$ indicating substitutability, and $\rho \in (0,1)$ indicating weak complementarity.

With these definitions, the matrices M_0 and M_1 in (9) may be written

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \beta\widetilde{D}_\rho \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 1-\delta_X & 1 \\ -\alpha_X & 1-\rho \end{pmatrix},$$

where

$$\widetilde{D}_\rho \equiv |M_0| = \alpha_X + (1-\delta_X)(1-\rho) = \alpha_e - (1-\delta_X)\rho\,.$$

The generalized eigenvalues are then solutions to $|M_0 - \lambda M_1| = 0$, or, equivalently, $P_\rho(\lambda) = 0$, where

$$P_\rho(\lambda) \equiv \beta\widetilde{D}_\rho\lambda^2 - \left[1-\rho+\beta(1-\delta_X)\widetilde{D}_\rho\right]\lambda + \widetilde{D}_\rho\,.$$

If $\widetilde{D}_\rho \neq 0$, then P_ρ is quadratic and therefore there are two generalized eigenvalues. This is equivalent to the case where M_1 is invertible so that the generalized eigenvalues λ_0, λ_1 are exactly the eigenvalues of $M \equiv M_1^{-1}M_0$. In this case, it can be verified that

$$\lambda_0 + \lambda_1 = T_\rho \equiv 1 - \delta_X + \frac{1-\rho}{\beta\widetilde{D}_\rho}\,, \tag{A.1}$$

$$\lambda_0\lambda_1 = D \equiv \frac{1}{\beta}\,, \tag{A.2}$$

where T_ρ and D are the trace and determinant, respectively, of M. Note that (A.2) implies for this case that if λ_0, λ_1 are real then they are of the same sign, and since $|\lambda_0| \leq |\lambda_1|$ by definition, we either have $0 < \lambda_0 < \lambda_1$ or $\lambda_1 < \lambda_0 < 0$. Note also that, by definition, λ_j satisfies $P_\rho(\lambda_j) = 0$. From the definition of P_ρ, we therefore have

$$\begin{aligned}\frac{\partial P_\rho(\lambda_j)}{\partial\rho} &= \lambda_j + \left[\beta\lambda_j^2 - \beta(1-\delta_X)\lambda_j + 1\right]\frac{\partial\widetilde{D}_\rho}{\partial\rho} \\ &= \frac{\alpha_X\lambda_j}{\widetilde{D}_\rho}\,,\end{aligned}$$

where the second line uses $P_\rho(\lambda_j) = 0$ (to replace the quadratic term in λ_j), the definition of $\widetilde{D}_\rho$, the fact that $\partial\widetilde{D}_\rho/\partial\rho = -(1-\delta_X)$, and then simplifies. If λ_0, λ_1 are real and distinct, then we can implicitly differentiate the expression $P_\rho(\lambda_j) = 0$ with respect to ρ to obtain

$$\frac{d\lambda_j}{d\rho} = -\frac{\partial P_\rho(\lambda_j)/\partial\rho}{P'_\rho(\lambda_j)} = -\frac{\alpha_X\lambda_j}{\widetilde{D}_\rho P'_\rho(\lambda_j)}. \tag{A.3}$$

Lemma A.2. *Suppose Assumptions A1 and A2 hold. Define ρ_0 as the value of ρ, if it exists, such that $\widetilde{D}_\rho = 0$. Then ρ_0 exists if and only if $\delta_X < 1$. If ρ_0 does exist, then $\rho_0 > 0$, and further $\rho_0 < 1$ if $\zeta < 0$, while $\rho_0 > 1$ if $\zeta > 0$. Further, there is exactly one generalized eigenvalue when $\rho = \rho_0$, which is given by $\lambda_0 = 0$.*

Proof. Since $\alpha_e > 0$ and $\widetilde{D}_\rho = \alpha_e - (1-\delta_X)\rho$, clearly there exists a ρ such that $\widetilde{D}_\rho = 0$ if and only if $\delta_X < 1$. Assuming this is the case, we have $\rho_0 = \alpha_e/(1-\delta_X) > 0$. Since $\alpha_e = \alpha_X + 1 - \delta_X$, we may alternatively write $\rho_0 = 1 + \alpha_X/(1-\delta_X)$. If $\zeta < 0$ then $\alpha_X < 0$, in which case $\rho_0 < 1$. If $\zeta > 0$ then $\alpha_X > 0$, in which case $\rho_0 > 1$.

Next, for the case $\rho = \rho_0$, substituting $\widetilde{D}_{\rho_0} = 0$ into the expression for P_{ρ_0} above, when $\widetilde{D}_\rho = 0$ a generalized eigenvalue λ must satisfy $(1-\rho_0)\lambda = 0$. Since $\rho_0 \neq 1$, there is exactly one such value of λ, which is $\lambda_0 = 0$. ∎

A.1.2 Proof of Proposition 1

To see the claim that indeterminacy never arises, note that, from above, if $\widetilde{D}_\rho \neq 0$, then $\lambda_0\lambda_1 = 1/\beta > 1$, and thus we must have $|\lambda_1| > 1$. Thus, there is at least one unstable eigenvalue in this case, and therefore there cannot be indeterminacy. If instead $\widetilde{D}_\rho = 0$, then from Lemma A.2 there is only one generalized eigenvalue ($\lambda_0 = 0$), and therefore there cannot be two stable eigenvalues, so that there cannot be indeterminacy in this case either.

Next, we have the following lemma.

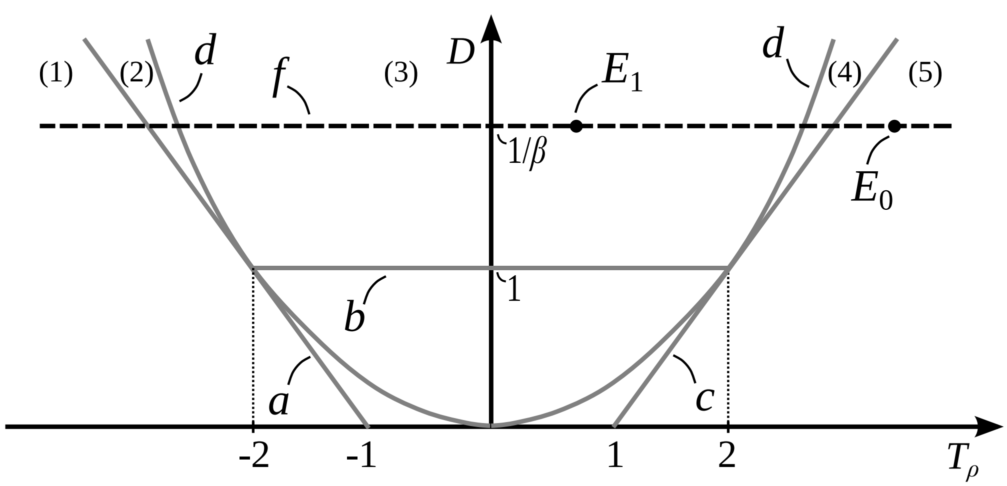
Lemma A.3. *Suppose Assumptions A1-A3 hold, and that $\zeta > 0$. Then $\widetilde{D}_\rho > 0$ for all ρ, and thus there are always two generalized eigenvalues, which are the eigenvalues of M. Further, there exist $\widetilde{\rho}, \rho' \in (0,1)$ with $\widetilde{\rho} < \rho'$ such that (i) the eigenvalues are real if and only if $\rho \leq \rho'$; (ii) $\lambda_0 < 1$ for $\rho = \widetilde{\rho}$ and $\lambda_0 > 1$ for $\rho \in (\widetilde{\rho}, \rho')$; and (iii) λ_0, λ_1 are continuous in ρ on $\rho < \rho'$, with λ_0 strictly increasing and λ_1 strictly decreasing.*

Proof. Note first that Assumption A3 implies $\rho < 1$. Further, since $\zeta > 0$, we have $\alpha_X > 0$. From the definition of $\widetilde{D}_\rho$, we then see that $\widetilde{D}_\rho \geq \alpha_X > 0$ for all $\rho < 1$, which in turn confirms that the generalized eigenvalues λ_0, λ_1 are simply the eigenvalues of M.

Next, recall that the trace and determinant of M are given in (A.1)-(A.2). Clearly $D = 1/\beta > 0$ regardless of ρ. From (A.1), since $\widetilde{D}_\rho > 0$ under Assumption A3, we see that T_ρ is a continuous function of ρ on $\rho < 1$, with $T_\rho > 0$, and $\partial T_\rho/\partial\rho = -\alpha_X/(\beta\widetilde{D}_\rho^2) < 0$. Further, $T_1 = 1 - \delta_X$.

These basic properties are illustrated in Figure A.1. (T_ρ, D) must lie somewhere on the horizontal dashed line f in the figure corresponding to $D = 1/\beta$. Line f is made up of five sections which are bounded by intersections with line a ($D = -T_\rho - 1$), line c ($D = T_\rho - 1$), and the parabola d ($D = T_\rho^2/4$). From left to right, it can be verified that these five sections correspond to the eigenvalue configurations (1) $\lambda_1 < -1 < \lambda_0 < 0$ (left of a); (2) $\lambda_1 < \lambda_0 < -1$ (between a and d); (3) λ_0, λ_1 complex with $|\lambda_j|= 1/\sqrt{\beta}$ (between the two arms of d); (4) $1 < \lambda_0 < \lambda_1$ (between d and c); and (5) $0 < \lambda_0 < 1 < \lambda_1$. The $\rho = 0$ case must be in section (5), which is indicated as point E_0. Since $T_1^2 = (1-\delta_X)^2/4 < 1 < D$, the $\rho = 1$ case must be in section (3), which is indicated as point E_1.

Figure A.1: (T_ρ, D)-space for $\beta > 0$ Case



Notes: Line a is $D = -T_\rho - 1$. Line b is $D = 1$. Line c is $D = T_\rho + 1$. Curve d is the parabola $D = T_\rho^2/4$.

Since T_ρ is continuous and strictly decreasing in ρ on $\rho < 1$, as ρ decreases from 0 (T_ρ, D) moves rightward along line f, and therefore remains in section (5) for all $\rho < 0$. Thus, the eigenvalues are real with $0 < \lambda_0 < 1 < \lambda_1$ for all $\rho \leq 0$.

As ρ increases from 0 to 1, meanwhile, the system moves leftward along line f from E_0 to E_1, and therefore the system first crosses into section (4) ($1 < \lambda_0 < \lambda_1$), and then finally into section (3) (λ_0, λ_1 complex) where it remains as ρ increases the rest of the way to 1. Thus, there clearly exist $\widetilde{\rho}, \rho' \in (0, 1)$ with $\widetilde{\rho} < \rho'$ satisfying parts *(i)* and *(ii)* of the lemma.

To see part *(iii)*, suppose $\rho < \rho'$, so that the eigenvalues are real with $0 < \lambda_0 < \lambda_1$. Since $\widetilde{D}_\rho > 0$, T_ρ is continuous in ρ, and since λ_0, λ_1 are continuous in T_ρ, it follows that λ_0, λ_1 are continuous in ρ. Next, since $\widetilde{D}_\rho > 0$, $P_\rho(\lambda)$ is a convex parabola, and therefore we must have $P_\rho'(\lambda_0) < 0$ and $P_\rho'(\lambda_1) > 0$. Thus, since $\alpha_X, \lambda_0, \lambda_1, \widetilde{D}_\rho > 0$, from (A.3) it follows that $d\lambda_0/d\rho > 0$ and $d\lambda_1/d\rho < 0$, which completes the proof. ■

Lemma A.4. *Suppose Assumptions A1-A3 hold, and that $\zeta < 0$. Then ρ_0 as defined in Lemma A.2 exists and satisfies $\rho_0 \in (0,1)$, and there exist $\widetilde{\rho}, \rho' \in (0,1)$ with $\rho_0 < \widetilde{\rho} < \rho'$ such that (i) the generalized eigenvalues are real if and only if $\rho \leq \rho'$; (ii) $\lambda_0 \in (0,1)$ for $\rho < \rho_0$; (iii) $\lambda_0 \in (-1,0)$ for $\rho \in (\rho_0, \widetilde{\rho})$; (iv) $\lambda_0 < -1$ for $\rho \in (\widetilde{\rho}, \rho')$; (v) λ_0 is continuous and strictly decreasing in ρ on $\rho < \rho'$; and (vi) λ_1 is continuous and strictly increasing in ρ everywhere except at $\rho = \rho_0$, with $\lim_{\rho \uparrow \rho_0} \lambda_1 = \infty$ and $\lim_{\rho \downarrow \rho_0} \lambda_1 = -\infty$.*

Proof. Note first that $\zeta < 0$ implies $\alpha_X < 0$, and since $\alpha_e > 0$ and $\alpha_X = \alpha_e - (1 - \delta_X)$, this in turn requires $\delta_X < 1$, which we henceforth assume. From Lemma A.2, this immediately implies that ρ_0 exists and satisfies $\rho_0 \in (0,1)$.

Next, for $\rho \neq \rho_0$, we have $\partial T_\rho/\partial\rho = -\alpha_X/(\beta\widetilde{D}_\rho^2) > 0$, so that T_ρ is continuous and strictly increasing in ρ, except possibly at the point $\rho = \rho_0 > 0$. Note that $T_{-\infty} \equiv \lim_{\rho \to -\infty} T_\rho = 1 - \delta_X + 1/[\beta(1 - \delta_X)]$. It can be verified that $T_{-\infty} > D + 1$. Thus, as ρ decreases from 0, starting from point E_0 in Figure A.1, (T_ρ, D) moves leftward along line f, approaching the point $(T_{-\infty}, D)$ which is also to the right of line (c). Thus, the system remains in section (5)—and therefore the eigenvalues are real with $0 < \lambda_0 < 1 < \lambda_1$—for all $\rho \leq 0$.

Consider now what happens as ρ increases from 0. Note that $\partial\widetilde{D}_\rho/\partial\rho = -(1 - \delta_X) < 0$. Thus, $\widetilde{D}_\rho$ is strictly decreasing in ρ, with $\widetilde{D}_\rho > 0$ for $\rho < \rho_0$ and $\widetilde{D}_\rho < 0$ for $\rho > \rho_0$. Since $\rho_0 < 1$, we therefore see that $\lim_{\rho \uparrow \rho_0} T_\rho = \infty$ and $\lim_{\rho \downarrow \rho_0} T_\rho = -\infty$. Looking at Figure A.1, this corresponds to the system initially moving rightward along line f, with $T_\rho \to \infty$ as $\rho \uparrow \rho_0$. As ρ increases through ρ_0, T_ρ then "jumps" to $-\infty$ and increases continuously from then on, eventually reaching point E_1 for $\rho = 1$. Ignoring for the moment exactly what happens at $\rho = \rho_0$, we see that, as ρ increases from 0 to 1, the system is initially in section (5) ($0 < \lambda_0 < 1 < \lambda_1$), then jumps to section (1) ($\lambda_1 < -1 < \lambda_0 < 0$) as ρ increases through ρ_0, before eventually transitioning into section (2) ($\lambda_1 < \lambda_0 < -1$), and finally section (3) (λ_0, λ_1 complex). At $\rho = \rho_0$, meanwhile, we have only one generalized eigenvalue, $\lambda_0 = 0$, which is clearly real. Thus, there clearly exists $\widetilde{\rho}, \rho' \in (0,1)$ with $\widetilde{\rho} < \rho'$ satisfying parts *(i)*-*(iv)* of the lemma.

To see parts *(v)* and *(vi)*, suppose first $\rho < \rho_0$, so that the eigenvalues are real with $0 < \lambda_0 < \lambda_1$. Since $\widetilde{D}_\rho > 0$ on $\rho < \rho_0$, T_ρ is continuous in ρ on that range, and since λ_0, λ_1

are continuous in T_ρ, it follows that λ_0, λ_1 are continuous in ρ on $\rho < \rho_0$. Further, since $\widetilde{D}_\rho > 0$, P_ρ is convex, and therefore $P'_\rho(\lambda_0) < 0$ and $P'_\rho(\lambda_1) > 0$. Since $\alpha_X < 0$, from (A.3) it therefore follows that $d\lambda_0/d\rho < 0$ and $d\lambda_1/d\rho > 0$, which confirms that, for $\rho < \rho_0$, λ_0, λ_1 are continuous in ρ with λ_0 decreasing and λ_1 increasing.

Next, suppose $\rho \in (\rho_0, \rho')$, so that the eigenvalues are real with $\lambda_1 < \lambda_0 < 0$. Since $\widetilde{D}_\rho < 0$ on this range of ρ, T_ρ is continuous in ρ, and therefore similar to above we may conclude that λ_0, λ_1 are continuous in ρ on (ρ_0, ρ'). Further, since $\widetilde{D}_\rho < 0$, P_ρ is concave, and since $\lambda_1 < \lambda_0$, we have $P'_\rho(\lambda_0) < 0$ and $P'_\rho(\lambda_1) > 0$. Since $\alpha_X, \lambda_0, \lambda_1, \widetilde{D}_\rho < 0$, from (A.3) it therefore follows that $d\lambda_0/d\rho < 0$ and $d\lambda_1/d\rho > 0$, which confirms that, for ρ on (ρ_0, ρ'), λ_0, λ_1 are continuous in ρ with λ_0 decreasing and λ_1 increasing.

We have therefore verified parts (v) and (vi) for all values of ρ except at $\rho = \rho_0$. Recall from above that, for ρ near (but not equal to) ρ_0, we have $T_\rho > 0$ if $\rho < \rho_0$ and $T_\rho < 0$ if $\rho > \rho_0$, with $\lim_{\rho\to\rho_0}|T_\rho| = \infty$. Recall also that, for $\rho \neq \rho_0$, we may write $\lambda_0, \lambda_1 = (T_\rho \pm \sqrt{T_\rho^2 - 4D})/2$, with λ_0 the smaller root for $\rho < \rho_0$ and the larger root for $\rho \in (\rho_0, \rho')$. Thus, for ρ near ρ_0, $\rho \neq \rho_0$, we may write

$$\lambda_0 = \frac{1 - \sqrt{1 - 4DT_\rho^{-2}}}{2T_\rho^{-1}}\,.$$

Taking the limit as $\rho \to \rho_0$ of this expression is equivalent to taking the limit as $T_\rho \to \pm\infty$. Applying l'Hopital's rule, it can therefore be verified that $\lim_{\rho\to\rho_0} \lambda_0 = 0$. Recalling that, when $\rho = \rho_0$, we have the single generalized eigenvalue $\lambda_0 = 0$, we conclude that λ_0 is continuous in ρ at ρ_0, which confirms the rest of part (v) of the lemma. Further, since $\lambda_0 + \lambda_1 = T_\rho$ for $\rho \neq \rho_0$, and $\lim_{\rho\to\rho_0} \lambda_0 = 0$, it follows that $\lim_{\rho\uparrow\rho_0} \lambda_1 = \lim_{\rho\uparrow\rho_0} T_\rho = \infty$ and $\lim_{\rho\downarrow\rho_0} \lambda_1 = \lim_{\rho\downarrow\rho_0} T_\rho = -\infty$, which confirms the rest of part (vi). ■

The elements of Proposition 1 all follow immediately from the properties outlined in Lemmas A.3 and A.4. ■

A.1.3 Proof of Proposition 2

The proposition follows immediately from Lemmas A.3 and A.4. ■

A.1.4 Proof of Proposition 3

Let $\widetilde{\rho} \equiv \widetilde{V}_{e\bar{e}}/\bar{V}_{e\bar{e}} \in (0,1)$ be the value of ρ at which the system loses local stability.

Lemma A.5. *Suppose Assumptions A1, A2 and A4 hold. Then:*

(i) If $\zeta < 0$, then SS_0 is the unique SS for all $\rho < 1$.

(ii) If $\zeta > 0$, then SS_0 is the unique SS for all $\rho \le \widetilde{\rho}$.

(iii) If $\zeta > 0$ and $\rho \in (\widetilde{\rho}, 1)$, then there are either one or two additional SSs, the first (which necessarily exists) with $e > 0$, and the second (which may or may not exist) with $e < 0$. Both additional SSs necessarily exist as long as $\rho > \widetilde{\rho}$ is sufficiently small.

Proof. In SS, we have $X = e/\delta_X$. Substituting this along with the equilibrium condition $\bar{e} = e$ into the Euler equation (6), we may obtain that a SS level of e is a solution to the equation $G^*_\rho(e) = 0$, where $G_\rho(e) \equiv \rho^* e - \rho F(e)$, and

$$\rho^* \equiv \left(1 + \frac{\beta\alpha_X}{\kappa}\right)^{-1} = \frac{(1+\beta\delta_X)\,\kappa}{[1-\beta\,(1-\delta_X)]\,\beta\delta_X}\,, \tag{A.4}$$

$$\kappa \equiv \frac{\beta}{1+\beta\delta_X} - \alpha_e\,. \tag{A.5}$$

As noted in the proof of Proposition A.1, Assumptions A1-A2 imply that $\xi/\chi > 1 + \beta\delta_X$, which it can be verified is equivalent to $\kappa > 0$. This in turn implies that $\rho^* > 0$, with $\rho^* < 1$ if and only if $\alpha_X > 0$ (i.e., $\zeta > 0$). It is also straightforward to verify that if $\zeta > 0$ then $\rho^* = \widetilde{\rho}$.

Note that $Q_\rho(0) = 0$ for all ρ, and therefore, as expected, SS_0 is always a SS. Also, $Q'_\rho(e) = \rho^* - \rho F'(e)$. Since $F'(e) > 0$ (from Assumption A4*(ii)*), it follows that $Q'_\rho(e) \ge 0$ if and only if $\rho \le \rho^*/F'(e)$. Since $F'(e)$ achieves a unique maximum of 1 at $e = 0$, $Q'_\rho(e)$ achieves a unique minimum of $\rho^* - \rho$ at $e = 0$.

If $\rho < \rho^*$, then $Q'_\rho(e) > 0$ for all e, while $Q'_{\rho^*}(0) = 0$ and $Q'_{\rho^*}(e) > 0$ for all $e \neq 0$. Thus, if $\rho \le \rho^*$ then Q_ρ is strictly increasing, in which case there is a unique solution to $Q_\rho(e) = 0$, and thus a unique SS (which must be SS_0). We therefore see that the SS is unique if $\rho \le \rho^*$. Since $\rho^* \ge 1$ if $\zeta < 0$, while $\rho^* = \widetilde{\rho}$ if $\zeta > 0$, parts *(i)* and *(ii)* of the lemma follow immediately.

To see part *(iii)*, suppose henceforth that $\zeta > 0$ and $\rho \in (\widetilde{\rho}, 1) = (\rho^*, 1)$. Since ρ^* is strictly positive, so is ρ, so that Assumption A4*(iv)* implies that $Q''_\rho = -\rho F''(e) < 0$ for $e > 0$, and therefore Q_ρ is strictly convex on $e \in [0, \infty)$. Since a convex function can have at most two roots, we see that there can be at most two SS levels of e in $[0, \infty)$, one of which we know is $e = 0$. Thus, there can be at most one strictly positive SS level of e. By a similar argument, Assumption A4*(iv)* implies that Q_ρ is strictly concave on $e \in [\underline{e}, 0]$, so that it can have at most two roots in $e \in [\underline{e}, 0]$, one of which is $e = 0$, and therefore there can be at most one strictly negative SS level of e.

Next, since $Q'_\rho(0) = \rho^* - \rho < 0$, we have $Q_\rho(e) < 0$ for $e > 0$ sufficiently small.

Further, under Assumption A4*(iii)*, we have $\lim_{e\to\infty} Q'_\rho(e) = \rho^* > 0$, which implies that $\lim_{e\to\infty} Q_\rho(e) = \infty$, and therefore $Q_\rho(e) > 0$ for $e > 0$ sufficiently large. By the intermediate value theorem, it therefore follows that there exists an $e > 0$ such that $Q_\rho(e) = 0$, and thus a strictly positive SS necessarily exists.

Thus, we have verified most of part *(iii)* of the lemma. It remains only to confirm that the strictly negative SS also necessarily exists as long as $\rho > \widetilde{\rho}$ is sufficiently small. To see this, note first that, since $Q'_\rho(0) < 0$, we must have $Q_\rho(e) > 0$ for $e < 0$ sufficiently large. Recall also that $Q_{\rho^*}(e)$ is a strictly increasing function, and therefore $Q_{\rho^*}(e) < 0$ for $e < 0$, and in particular $Q_{\rho^*}(\underline{e}) < 0$. Since $F(\underline{e}) < 0$ by the properties of F from Assumption A4, $Q_\rho(\underline{e})$ is a continuous and strictly increasing function of ρ. Thus, as long as $\rho > \rho^*$ is not too large, we will have $Q_\rho(e_0) \leq 0$. Assuming this is the case, combined with the fact that $Q_\rho(e) > 0$ for $e < 0$ sufficiently large, we may again apply the intermediate value theorem to conclude that $Q_\rho(e) = 0$ for some $e \in [\underline{e}, 0)$, so that a strictly negative SS exists as long as $\rho > \rho^*$ is sufficiently small, which completes the proof. ■

Part *(i)* of Proposition 3 follows immediately from Lemma A.5, as do the claims about the SSs in parts *(ii)*-*(iii)*. We turn now to verifying the claims regarding the dynamics in parts *(ii)*-*(iii)*. It will be convenient to first state a number of somewhat involved lemmas that apply to more generic dynamic systems.

Lemma A.6. *Suppose a 2-dimensional dynamic system in (x, y), parameterized by some $\eta \in \mathbb{R}$, satisfies*

$$x_{t+1} = b\left(x_t, y_t; \eta\right), \tag{A.6}$$

$$g\left(y_{t+1}; \eta\right) = h\left(x_t, y_t; \eta\right), \tag{A.7}$$

for some smooth functions $b, h : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, satisfying $b(0,0;\eta) = g(0;\eta) = h(0,0;\eta) = 0$ for all η, so that $SS_0 \equiv (0,0)$ is a SS, with h smoothly invertible in y. Given η, we will say a set $\mu_\eta \subset \mathbb{R}^2$ is invariant (with respect to system (A.6)-(A.7)*) if and only if $(x_t, y_t) \in \mu_\eta$ implies there exists a $(x_{t+1}, y_{t+1}) \in \mu_\eta$ satisfying conditions* (A.6)-(A.7).[18]

[18] Note that, given (x_t, y_t), while there is a unique value of x_{t+1} satisfying condition (A.6), since we do not require $g(y;\eta)$ to be invertible in y there could in general be multiple values of y_{t+1} satisfying condition (A.6), and therefore multiple $(x_{t+1}, y_{t+1}) \in \mu_\eta$ satisfying conditions (A.6)-(A.7).

(i) Given η, suppose $f_\eta, \phi_\eta : \mathbb{R} \to \mathbb{R}$ are a pair of smooth functions satisfying

$$g\left(\phi_\eta\left(f_\eta\left(x\right)\right);\eta\right) = h\left(x, \phi_\eta\left(x\right);\eta\right) \;, \tag{A.8}$$

$$f_\eta\left(x\right) = b\left(x, \phi_\eta\left(x\right);\eta\right) \;, \tag{A.9}$$

for all x. Then the 1-dimensional manifold $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ is invariant, and in particular if $(x_t, y_t) \in \mu_\eta$ then we must have $x_{t+1} = f_\eta(x_t)$.

(ii) Given η, let $\mu_\eta \subset \mathbb{R}^2$ be a 1-dimensional manifold that can be written $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ for some function ϕ_η. If μ_η is invariant then, letting f_η be defined by (A.9), the function ϕ_η satisfies (A.8).

(iii) Given η, let Λ_η denote the set of generalized eigenvalues λ associated with the problem $J_{0,\eta}v = \lambda J_{1,\eta}v$, where

$$J_{0,\eta} \equiv \begin{pmatrix} b_x\left(0,0;\eta\right) & b_y\left(0,0;\eta\right) \\ h_x\left(0,0;\eta\right) & h_y\left(0,0;\eta\right) \end{pmatrix} \,, \qquad J_{1,\eta} \equiv \begin{pmatrix} 1 & 0 \\ 0 & g_y\left(0;\eta\right) \end{pmatrix} \,,$$

are the Jacobians associated with the right- and left-hand sides, respectively, of system (A.6)-(A.7) at SS_0. Suppose f_η, ϕ_η are as in part (i). If in addition $\phi_\eta(0) = 0$, then f'_η is equal to one of the generalized eigenvalues $\lambda \in \Lambda_\eta$, and $v \equiv (1, \phi'_\eta)'$ is an associated eigenvector. Thus, the invariant 1-dimensional manifold $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ is tangent at 0 to the associated eigenspace $\zeta \equiv \{av : a \in \mathbb{R}\}$.

Proof. To see part *(i)*, given a pair f_η, ϕ_η satisfying (A.8)-(A.9), take any $(x_t, y_t) \in \mu_\eta$. By definition of μ_η, we must have $y_t = \phi_\eta(x_t)$. By inspection, it is clear that if $x_{t+1} = f_\eta(x_t)$ and $y_{t+1} = \phi_\eta(x_{t+1})$, then (A.8)-(A.9) together imply (A.6)-(A.7). Further, with $y_{t+1} = \phi_\eta(x_{t+1})$, we have $(x_{t+1}, y_{t+1}) \in \mu_\eta$. Thus, for any $(x_t, y_t) \in \mu_\eta$ we have found a $(x_{t+1}, y_{t+1}) \in \mu_\eta$ satisfying (A.6)-(A.7), and therefore μ_η is invariant.

To see part *(ii)*, suppose $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ is invariant. Fix any x_t, and let $y_t = \phi_\eta(x_t)$, so that $(x_t, y_t) \in \mu_\eta$. By definition of invariance, there exists a (x_{t+1}, y_{t+1}) satisfying (A.6)-(A.7) such that $y_{t+1} = \phi_\eta(x_{t+1})$. Fix such a (x_{t+1}, y_{t+1}). Defining f by (A.9), substituting $y_t = \phi_\eta(x_t)$ into (A.6) implies $x_{t+1} = f_\eta(x_t)$. From (A.7) and the fact that $y_t = \phi_\eta(x_t)$ and $y_{t+1} = \phi_\eta(x_{t+1})$, we then have $h(x_t, \phi_\eta(x_t); \eta) = g(y_{t+1}; \eta) = g(\phi_\eta(x_{t+1}); \eta) = g(\phi_\eta(f_\eta(x_t)); \eta)$. Since x_t was chosen arbitrarily, it follows that (A.8) holds for all x.

Finally, to see *(iii)*, differentiating (A.8) and (A.9) with respect to x, evaluating at $x = 0$, and using the fact that $\phi_\eta(0) = b(0, 0; \eta) = g(0; \eta) = h(0, 0; \eta) = 0$, we may write the resulting equations as $J_{0,\eta}v = f'_\eta J_{1,\eta}v$. Thus, clearly f'_η is a generalized eigenvalue, and v is an associated eigenvector. Since $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ is simply the graph of ϕ_η, it is clearly tangent at 0 to the vector v, and thus to the eigenspace ζ. ■

Lemma A.6 establishes a useful isomorphism between 1-dimensional invariant manifolds of system (A.6)-(A.7) and pairs of functions satisfying (A.8)-(A.9). Further, if we restrict system (A.6)-(A.7) to a 1-dimensional invariant manifold μ_η constructed from a pair of functions f_η, ϕ_η satisfying (A.8)-(A.9), then the resulting dynamics are governed by the univariate process $x_{t+1} = f_\eta(x_t)$ (with $y_t = \phi_\eta(x_t)$). It also implies that if μ_η is an invariant 1-dimensional manifold that passes through 0 and can be written as $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ for some function ϕ_η, then if f is given by (A.9), f'_η is a (generalized) eigenvalue of the linearized system, and μ_η is tangent to the associated eigenspace ζ.

Lemma A.7. *Consider the system described in Lemma A.6, and assume $g_y(0; 0) \neq 0$. Suppose that, for η in a neighborhood of 0, the generalized eigenvalues*[19] *are real and distinct, and that one of these eigenvalues λ_η is such that $|\lambda_\eta|$ increases through 1 as η increases through 0. Then for η in a neighborhood of 0 there exists a pair of functions f_η, ϕ_η that are smooth on a neighborhood of 0 and satisfy (A.8)-(A.9) and $\phi_\eta(0) = 0$, and for which $f'_\eta = \lambda_\eta$. Thus, as η increases through 0, the univariate process $x_{t+1} = f_\eta(x_t)$ representing the restriction of (A.6)-(A.7) to the invariant manifold $\mu_\eta \equiv \{(x, y) : y = \phi_\eta(x)\}$ experiences the same type of bifurcation as (A.6)-(A.7) does.*

Proof. Note first that, since $g_y(0; 0) \neq 0$ and g is smooth in its arguments, it follows that $g_y(0; \eta) \neq 0$ for η in a neighborhood of 0. Thus, $J_{1,\eta}$ is invertible, so that the generalized eigenvalues are simply the eigenvalues of $J_\eta \equiv J_{1,\eta}^{-1} J_{0,\eta}$. Furthermore, applying the implicit function theorem to (A.7), there exists a unique function $\tilde{h}(x, y; \eta)$ that is (a) continuously differentiable in a neighborhood of 0, (b) satisfies $\tilde{h}(0, 0; \eta) = 0$, and (c) for (x_t, y_t) in a neighborhood Ξ of 0, $y_{t+1} = \tilde{h}(x_t, y_t; \eta)$ satisfies (A.7).

While in general there could be multiple values of y_{t+1} satisfying (A.7) for a given (x_t, y_t), for our purposes it will be sufficient to consider only $y_{t+1} = \tilde{h}(x_t, y_t; \eta)$, which we henceforth do. In order to economize on notation, for the purposes of this lemma we may therefore simply assume (A.7) is already in this explicit form, which is equivalent to the case where $g(y; \eta) = y$ (so that $\tilde{h} = h$).

With this in place, consider system (A.6)-(A.7) augmented with the trivial equation $\eta = \eta$:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \eta \end{pmatrix} = \begin{pmatrix} b\,(x_t, y_t; \eta) \\ h\,(x_t, y_t; \eta) \\ \eta \end{pmatrix}. \tag{A.10}$$

[19] See part *(iii)* of Lemma A.6.

By construction, $(0, 0, \eta)$ is a SS of (A.10) for every η, which corresponds to SS_0 of the non-augmented system (A.6)-(A.7). Since we have assumed that $b(0, 0; \eta) = h(0, 0; \eta) = 0$ for all η, it is straightforward to verifty that the Jacobian of (A.10) at the SS $(0, 0, 0)$ (i.e., at the bifurcation value of η) is given by

$$\widetilde{J} \equiv \begin{pmatrix} J_0 & 0 \\ 0 & 1 \end{pmatrix} .$$

The upper-left 2×2 block of $\widetilde{J}$ is just J_η for the $\eta = 0$ (i.e., bifurcation) case, which by definition has one eigenvalue $\lambda_\eta = \pm 1$, and the other eigenvalue $\overline{\lambda}_\eta$ satisfies $|\overline{\lambda}_\eta| \neq 1$. Given the block-diagonal structure of $\widetilde{J}$, its eigenvalues are thus clearly given by λ_η, $\overline{\lambda}_\eta$, and 1. Thus, $\widetilde{J}$ has two eigenvalues equal to 1 in absolute value, plus one that is either strictly greater or strictly less than 1 in absolute value.

Let ζ_η denote the eigenspace of J_η associated with λ_η, and note that since the eigenvalues of J_η are distinct for η near 0, ζ_η is 1-dimensional for η in a neighborhood of 0. Let $\zeta^* \equiv \zeta_0 \times \mathbb{R}$ denote the 2-dimensional eigenspace of $\widetilde{J}$ associated with the two eigenvalues of absolute value 1. By the Center Manifold Theorem for maps (see, e.g., Marsden and McCracken [1976], Section 2, Theorem 2.1), there exists a 2-dimensional manifold $W \subset \mathbb{R}^3$ that is invariant with respect to (A.10) and tangent to ζ^* at the origin. Such a manifold is called a "center manifold". Since W passes through the origin, we can write $W = \{(x, y, \eta) : y = \phi_\eta(x), x \in \Xi_x, \eta \in \Xi_\eta\}$, where $\phi_\cdot(\cdot)$ is some function of x and η satisfying $\phi_0(0) = 0$ and $\Xi_x, \Xi_\eta \subset \mathbb{R}$ are two sufficiently small non-empty neighborhoods of 0.

Note that we can write $W = \cup_{\eta \in \Xi_\eta} W_\eta$, where the $W_\eta \equiv \{(x, y, \eta) : y = \phi_\eta(x), x \in \Xi_x\}$ are disjoint. Further, since η is constant in system (A.10), each W_η is also invariant with respect to (A.10), and thus the set $\mu_\eta \equiv \{(x, y) : (x, y, \eta) \in W_\eta\} = \{(x, y) : y = \phi_\eta(x), x \in \Xi_x\} \subset \mathbb{R}^2$ is a 1-dimensional manifold that is invariant with respect to the non-augmented system (A.6)-(A.7). By Lemma A.6*(ii)*, it therefore follows that, for all η in a neighborhood of 0, ϕ_η satisfies (A.8) with f_η given by (A.9).

Next, note that center manifolds are unique only up to their Taylor series. That is, two center manifolds may differ only by flat functions (functions whose k-th-order derivatives are all zero for all $k \in \mathbb{N}$). For practical purposes, without loss of generality we henceforth restrict attention to the *unique* analytic center manifold (see, e.g., Guckenheimer and Holmes [2002], ch. 3.2). We now verify that this analytic center manifold W is such that $\phi_\eta(0) = 0$ for η near 0.

In particular, since ϕ_η satisfies (A.8) with f_η given in (A.9), substituting the latter equation into the former (where we have again used the simplification that $g(y; \eta) = y$) and then setting $x = 0$, we get

$$\phi_\eta \left(b\left(0, \phi_\eta\left(0\right); \eta\right)\right) = h\left(0, \phi_\eta\left(0\right); \eta\right) . \tag{A.11}$$

Since $b(0,0;\eta) = h(0,0;\eta) = 0$ for all η, clearly $\phi_\eta(0) = 0$ $\forall\eta$ solves (A.11), and further $\phi_\eta(0) = 0$ is trivially analytic in η. Thus, the analytic center manifold W satisfies $\phi_\eta(0) = 0$ $\forall\eta$.

We have therefore verified that for all η in a neighborhood of 0, ϕ_η satisfies (A.8) with f_η given by (A.9), and further $\phi_\eta(0) = 0$ for all such η. The remainder of the lemma therefore follows immediately from application of Lemma A.6. ■

Lemma A.8. *Consider the bifurcation described in Lemma A.7, and fix some $\eta > 0$ sufficiently small so that we may apply the results in that Lemma, and in particular the restriction of (A.6)-(A.7) to $\mu_\eta = \{(x, y) : y = \phi_\eta(x)\}$ evolves according to $x_{t+1} = f_\eta(x_t)$, with $f_\eta(0) = \phi_\eta(0) = 0$ and $|f'_\eta| = |\lambda_\eta| > 1$. Let $\overline{\lambda}_\eta$ denote the eigenvalue of J_η that is not λ_η.*[20] *Suppose $\sigma_\eta \subset \mathbb{R}$ is a locally attractive invariant set of $x_{t+1} = f_\eta(x_t)$, and let $S_\eta \equiv \{(x, y) \in \mu_\eta : x \in \sigma_\eta\} \subset \mu_\eta$ be the corresponding invariant set of the full system (A.6)-(A.7). If $|\overline{\lambda}_\eta| < 1$, then S_η attracts all nearby trajectories, while if $|\overline{\lambda}_\eta| > 1$ then S_η repels all nearby trajectories except for those on μ_η, which it attracts.*

Proof. Fix $\eta > 0$ as in the statement of the lemma. Note that the non-bifurcating eigenvalue $\overline{\lambda}_\eta$ exists and satisfies $|\overline{\lambda}_\eta| \neq 1$ as long as $\eta > 0$ is sufficiently small, which we assume.

Since σ_η attracts all nearby trajectories of $x_{t+1} = f(x_t)$, which in turn represents the restriction of (A.6)-(A.7) to the invariant manifold μ_η, it follows that S_η necessarily attracts all nearby trajectories on μ_η. It therefore remains only to check whether trajectories near S_η but not on μ_η converge to or diverge from S_η.

Let $\widehat{S}_\eta \equiv S_\eta \times \{\eta\}$ be the invariant set of the augmented system (A.10) associated with S_η (for (A.6)-(A.7)) and σ_η (for $x_{t+1} = f_\eta(x_t)$). From the proof of Lemma A.7 it is clear that $W_\eta = \mu_\eta \times \{\eta\}$, and since $S_\eta \subset \mu_\eta$, we have $\widehat{S}_\eta \subset W_\eta \subset W$, i.e., $\widehat{S}_\eta$ is a subset of the center manifold W.

Suppose $|\overline{\lambda}_\eta| < 1$. Then Theorem 5.3 in Kuznetsov [1998] implies that the augmented system (A.10) is locally topologically equivalent to one in which all trajectories not on the center manifold W converge linearly to it over time. Thus, for any trajectory of (A.10) starting at a point (x_0, y_0, η) sufficiently near a point in $\widehat{S}_\eta \subset W$, it follows that the trajectory must converge to $\widehat{S}_\eta$. It then follows immediately that any trajectory of the non-augmented

[20] See proof of Lemma A.7.

system (A.6)-(A.7) starting at a point (x_0, y_0) sufficiently near a point in S_η must converge to S_η, so that S_η attracts all nearby trajectories.

Suppose instead $|\overline{\lambda}_\eta| > 1$. Then Theorem 5.3 in Kuznetsov [1998] implies that the augmented system (A.10) is locally topologically equivalent to one in which all trajectories not on the center manifold W are repelled from it. By a similar argument to the above, this property therefore extends to all trajectories of the non-augmented system (A.6)-(A.7) except for those on μ_η, which confirms that S_η attracts nearby trajectories on μ_η, but repels nearby trajectories that are not on μ_η. ■

Returning to the model that is the subject of Proposition 3, we may write the equilibrium system as

$$X_{t+1} = (1 - \delta_X)\, X_t + e_t\,, \tag{A.12}$$

$$H_\rho\,(e_{t+1}) = -\alpha_X X_t + G_\rho(e_t)\,, \tag{A.13}$$

where $G_\rho(e) \equiv e - \rho F(e)$ and $H_\rho(e) \equiv \alpha_e e - \beta(1-\delta_X)\rho F(e)$. Given our assumptions on F, it can be verified that $G_\rho(0) = H_\rho(0) = 0$, $G'_\rho = 1 - \rho$, $H'_\rho = \widetilde{D}_\rho$, $G''_\rho = H''_\rho = 0$, $G'''_\rho > 0$, and $H'''_\rho > 0$.

Applying the concepts of Lemmas A.6-A.8 to system (A.12)-(A.13), with ρ as the bifurcation parameter rather than η and $\widetilde{\rho}$ as the bifurcation value for this parameter rather than 0, we see that 1-dimensional invariant[21] manifolds μ_ρ that pass through 0 are isomorphic to pairs of functions f_ρ, ϕ_ρ satisfying

$$H_\rho\,(\phi_\rho\,(f_\rho\,(X))) = -\alpha_X X + G_\rho\,(\phi_\rho\,(X))\ , \tag{A.14}$$

$$f_\rho\,(X) = (1 - \delta_X)\, X + \phi_\rho\,(X)\ , \tag{A.15}$$

with $f_\rho(0) = \phi_\rho(0) = 0$, and the system restricted to μ_ρ evolves according to

$$X_{t+1} = f_\rho\,(X_t)\ . \tag{A.16}$$

Further, since $\widetilde{D}_\rho$ is bounded away from 0 for ρ near $\widetilde{\rho}$, $J_{1,\rho} = M_1$ is invertible, and therefore the generalized eigenvalues from part *(iii)* of Lemma A.6 are simply the eigenvalues of $M \equiv M_1^{-1} M_0$. If λ_ρ is such an eigenvalue, then since $v_\rho \equiv (1, \phi'_\rho)'$ is a corresponding eigenvector

[21] Henceforth, unless explicitly stated otherwise, if we refer to a set as "invariant", we mean invariant with respect to system (A.12)-(A.13).

it is straightforward to verify that

$$\phi'_\rho = \lambda_\rho - (1 - \delta_X) \; . \tag{A.17}$$

Henceforth, let λ_ρ denote the eigenvalue of M that increases through 1 in absolute value as part of a flip or $+1$ bifurcation. Thus, for ρ near $\widetilde{\rho}$ we have λ_ρ near ± 1. Further, the other eigenvalue, $\overline{\lambda}_\rho$, is necessarily real and unstable for ρ in a neighborhood of $\widetilde{\rho}$. Assume henceforth that ρ is always in a sufficiently small neighborhood of $\widetilde{\rho}$ that the eigenvalues are real and distinct with λ_ρ of the same sign as $\lambda_{\widetilde{\rho}}$. Note that ϕ'_ρ as given in (A.17) is continuous and differentiable in ρ near $\widetilde{\rho}$, and therefore the eigenspace $\zeta_\rho \equiv \{av_\rho : a \in \mathbb{R}\}$ varies smoothly with ρ as well.

The following lemma establishes all remaining elements of Proposition 3*(ii)*.

Lemma A.9. *Suppose Assumptions A1-A4 hold, and that $\zeta < 0$. Let f_ρ, ϕ_ρ be as in* (A.14)-(A.15) *with $f_\rho(0) = \phi_\rho(0) = 0$, and $\mu_\rho = \{(X, e) : e = \phi_\rho(X)\}$ be the corresponding 1-dimensional invariant manifold. Then as ρ increases through $\widetilde{\rho}$, the univariate system* (A.16) *experiences a flip bifurcation, and a locally attractive 2-cycle emerges. Further, letting $\sigma_\rho \subset \mathbb{R}$ denote this 2-cycle, and $S_\rho \subset \mathbb{R}^2$ the corresponding 2-cycle of* (A.12)-(A.13)*, S_ρ attracts nearby trajectories on μ_ρ, while trajectories not on μ_ρ diverge from it.*

Proof. Since $\zeta < 0$, Proposition 2 implies that λ_ρ decreases through -1 as ρ increases through $\widetilde{\rho}$. Since $f'_\rho = \lambda_\rho$ by Lemma A.7, and f'_ρ is the lone eigenvalue of the univariate system (A.16), this immediately implies that system (A.16) experiences a flip bifurcation as ρ increases through $\widetilde{\rho}$, and in particular $f'_{\widetilde{\rho}} = -1$, and $\partial f'_\rho / \partial \rho < 0$ for ρ in a neighborhood of $\widetilde{\rho}$.

Next, the fact that 2-cycles generically exist on exactly one side of a flip bifurcation or the other (i.e., either on the stable or unstable side) is well known (see, e.g., Kuznetsov [1998], ch. 4.4). A flip bifurcation is said to be supercritical if it emerges on the unstable side and is locally attractive. From Wikan [2013] (Theorem 1.5.1), the flip bifurcation in (A.16) at $\rho = \widetilde{\rho}$ will be supercritical if

$$b \equiv \frac{1}{2} f''^2_{\widetilde{\rho}} + \frac{1}{3} f'''_{\widetilde{\rho}} > 0 \, . \tag{A.18}$$

We turn now to verifying that indeed $b > 0$.

Note first that, since $\lambda_{\widetilde{\rho}} = -1$, we must have $D = -T_{\widetilde{\rho}} - 1$, where T_ρ and D are, respectively, the trace and determinant of M (see Section A.1.1). From (A.1)-(A.2), this implies that

$$\widetilde{\rho} = 1 + \frac{1 + \beta\,(2 - \delta_X)\,\alpha_X}{(2 - \delta_X)\,[1 + \beta\,(1 - \delta_X)]} \, .$$

Note also that, from (A.17), we have $\phi_{\widetilde{\rho}} = -(2 - \delta_X)$

For the case $\rho = \widetilde{\rho}$, differentiating (A.14)-(A.15) twice with respect to X, evaluating at $X = 0$, using the above values of $f_{\widetilde{\rho}}$, $\phi_{\widetilde{\rho}}$ and $\widetilde{\rho}$, and the fact that $G''_{\rho} = H''_{\rho} = 0$, we may solve to obtain that $f''_{\widetilde{\rho}} = \phi''_{\widetilde{\rho}} = 0$.

Next, differentiating (A.14)-(A.15) for the $\rho = \widetilde{\rho}$ case three times with respect to X, evaluating at $X = 0$, using the above values of $f_{\widetilde{\rho}}$, $\phi_{\widetilde{\rho}}$ and $\widetilde{\rho}$, the fact that $G''_{\widetilde{\rho}} = H''_{\widetilde{\rho}} = f''_{\widetilde{\rho}} = \phi''_{\widetilde{\rho}} = 0$, and then substituting in the values of G'''_{ρ} and H'''_{ρ} obtained earlier, we may then solve to obtain

$$f'''_{\widetilde{\rho}} = \phi'''_{\widetilde{\rho}} = \frac{\left[1 + \beta\left(1 - \delta_X\right)\right]^2 \left(2 - \delta_X\right)^4}{\left(1 - \beta\right)\alpha_X} \widetilde{\rho} F''' \,.$$

Since $\widetilde{\rho} > 0$, $F''' < 0$ and $\alpha_X < 0$ (since $\zeta < 0$), it follows that $f'''_{\widetilde{\rho}} > 0$. Thus, b in (A.18) is positive, and therefore the flip bifurcation is supercritical. Thus, the emerging 2-cycle σ_ρ of (A.16) is locally attractive. Further, since $|\overline{\lambda}_\rho| > 1$, Lemma A.8 implies that the corresponding 2-cycle of (A.12)-(A.13), S_ρ, attracts nearby trajectories on μ_ρ, but repels all those not on μ_ρ, which completes the proof. ■

The following lemma establishes all remaining elements of Proposition 3*(iii)*, which completes the proof of the proposition.

Lemma A.10. *Suppose Assumptions A1-A4 hold, and that $\zeta > 0$. Let f_ρ, ϕ_ρ be as in (A.14)-(A.15) with $f_\rho(0) = \phi_\rho(0) = 0$, and $\mu_\rho = \{(X, e) : e = \phi_\rho(X)\}$ be the corresponding 1-dimensional invariant manifold. Then as ρ increases through $\widetilde{\rho}$, the univariate system (A.16) experiences the $+1$ bifurcation known as a pitchfork bifurcation, characterized by the emergence of two additional stable SSs. Further, letting $\sigma_\rho \subset \mathbb{R}$ denote the set containing exactly one of these emerging SSs, and $S_\rho \subset \mathbb{R}^2$ the corresponding SS of (A.12)-(A.13), S_ρ attracts nearby trajectories on μ_ρ, while trajectories not on μ_ρ diverge from it.*

Proof. Since $\zeta > 0$, Proposition 2 implies that λ_ρ increases through $+1$ as ρ increases through $\widetilde{\rho}$. Since $f'_\rho = \lambda_\rho$ by Lemma A.7, and f'_ρ is the lone eigenvalue of the univariate system (A.16), this immediately implies that system (A.16) experiences a $+1$ bifurcation as ρ increases through $\widetilde{\rho}$, and in particular $f'_{\widetilde{\rho}} = 1$, and $\partial f'_\rho / \partial \rho > 0$ for ρ in a neighborhood of $\widetilde{\rho}$.

Recall from the proof of Lemma A.5 that we have $\widetilde{\rho} = \rho^*$ in this case, where ρ^* is given in (A.4). Further, from (A.17) we have $\phi_{\widetilde{\rho}} = \delta_X$.

Similar to the proof of Lemma A.9, for the case of $\rho = \widetilde{\rho}$, we can differentiate (A.14)-(A.15) twice with respect to X, evaluate at $X = 0$, substitute in the properties $G''_{\widetilde{\rho}} = H''_{\widetilde{\rho}} = 0$, and then use the fact that $f_{\widetilde{\rho}} = 1$ and $\phi_{\widetilde{\rho}} = \delta_X$ to conclude that $f''_{\widetilde{\rho}} = \phi''_{\widetilde{\rho}} = 0$.

Similarly, for the case $\rho = \widetilde{\rho}$, differentiating (A.14)-(A.15) three times with respect to X, evaluating at $X = 0$, using the above values of $f_{\widetilde{\rho}}$, $\phi_{\widetilde{\rho}}$ and $\widetilde{\rho}$, the fact that $G''_{\widetilde{\rho}} = H''_{\widetilde{\rho}} = f''_{\widetilde{\rho}} = \phi''_{\widetilde{\rho}} = 0$, and then substituting in the values of $G'''_{\widetilde{\rho}}$ and $H'''_{\widetilde{\rho}}$ obtained earlier, we may solve to obtain

$$f'''_{\widetilde{\rho}} = \phi'''_{\widetilde{\rho}} = \frac{\left[1 - \beta\left(1 - \delta_X\right)\right]^2 \delta_X^4}{\left(1 - \beta\right)\alpha_X} \widetilde{\rho} F''' \,.$$

Since $\widetilde{\rho} > 0$, $F''' < 0$ and $\alpha_X > 0$ (since $\zeta > 0$), it follows that $f'''_{\widetilde{\rho}} < 0$.

The above implies that the following properties hold: $f_{\widetilde{\rho}}(0) = 0$, $f'_{\widetilde{\rho}}(0) = 1$, $\partial f_\rho(0)/\partial\rho|_{\rho=\widetilde{\rho}} = 0$, $f''_{\widetilde{\rho}}(0) = 0$, $\partial f'_\rho(0)/\partial\rho|_{\rho=\widetilde{\rho}} = \partial\lambda_\rho/\partial\rho|_{\rho=\widetilde{\rho}} > 0$, $f'''_{\widetilde{\rho}}(0) < 0$. Thus, system (A.16) experiences a pitchfork bifurcation at $\rho = \widetilde{\rho}$ (see Wiggins [2003], ch. 21.1C). Since a pitchfork bifurcation has three SSs on one side of the bifurcation and one SS on the other, and since we have already established that, for $\rho < \widetilde{\rho}$, $X = 0$ is the unique SS, two additional SSs must emerge as ρ increases through $\widetilde{\rho}$. Further, since a pitchfork bifurcation is such that the two emerging SSs are stable if and only if the other SS is unstable, and since $X = 0$ is unstable for $\rho > \widetilde{\rho}$, it follows that the emerging SSs are in fact stable.

Let σ_ρ be the set containing exactly one of the emerging SSs, and S_ρ the corresponding SS of (A.12)-(A.13). Since $|\overline{\lambda}_\rho| > 1$, Lemma A.8 then implies that S_ρ attracts nearby trajectories on μ_ρ, but repels all those not on μ_ρ, which completes the proof. ■

A.2 Section 3 Proofs

A.2.1 Preliminaries

Define $\alpha_e \in (0, 1)$ and α_X as in Lemma A.1, and let $\omega \equiv -\beta V_{X\bar{e}}/\xi$, and recall that Lemma A.1 implies $0 < \alpha_e\beta/(1 + \beta\delta_X) < 1$. Under Assumptions A1-A2, Lemma A.1 also implies $\xi < 0$, and therefore ω is strictly increasing in and of the same sign as $V_{X\bar{e}}$, with $\omega = 0$ corresponding to autarky. We will henceforth assume these properties.

With these definitions, the matrices M_0 and M_1 in (12) may be written

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_e + \omega \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 1 - \delta_X & 1 \\ -\alpha_X & 1 \end{pmatrix}.$$

Note that $|M_0| = \alpha_e/\beta > 0$, and therefore M_0 is always invertible. Thus, the generalized eigenvalues are given by $\lambda_j = 1/\gamma_j$, $j = 0, 1$, where γ_0, γ_1 are the eigenvalues of the matrix $N \equiv M_0^{-1}M_1$, where we define these eigenvalues such that $\gamma_0 \geq \gamma_1$ whenever they are real (so that $\lambda_0 \leq \lambda_1$), and if $\gamma_j = 0$ we set $\lambda_j = \pm\infty$.

It can be verified that

$$\gamma_0 + \gamma_1 = T_\omega \equiv \beta \left[\frac{1}{\alpha_e} + (1 - \delta_X) \left(1 + \frac{\omega}{\alpha_e} \right) \right] , \tag{A.19}$$

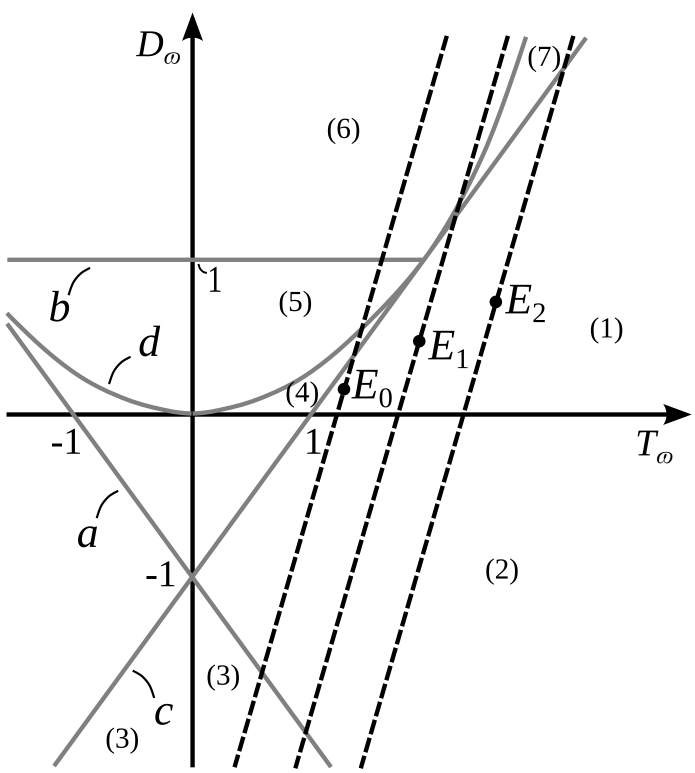
$$\gamma_0 \gamma_1 = D_\omega \equiv \beta \left(1 + \frac{\omega}{\alpha_e} \right) , \tag{A.20}$$

where T_ω and D_ω are the trace and determinant, respectively, of N. Note that T_ω and D_ω are each continuous and strictly increasing in ω and can take on any values in $\mathbb{R}$, and further that we can write

$$D_\omega = \frac{1}{1 - \delta_X} T_\omega - \frac{\beta}{(1 - \delta_X)\, \alpha_e} . \tag{A.21}$$

Figure A.2 illustrates the possible configurations of the eigenvalues γ_0, γ_1. The figure separates the (T_ω, D_ω)-plane into various regions using a number of lines and curves. Line a is $D_\omega = -T_\omega - 1$. Line b is $D_\omega = 1$. Line c is $D_\omega = T_\omega + 1$. Curve d is the parabola $D_\omega = T_\omega^2/4$. The relevant regions for our purposes are as follows:

Figure A.2: (T_ω, D_ω)-space



Notes: Line a is $D_\omega = -T_\omega - 1$. Line b is $D_\omega = 1$. Line c is $D_\omega = T_\omega + 1$. Curve d is the parabola $D = T_\omega^2/4$. E_0, E_1, and E_2 correspond to possible values of (T_0, D_0). The dashed lines passing through these points correspond to line (A.21).

(1) Above the horizontal axis and to the right of line c: γ_0, γ_1 real with $0 < \gamma_1 < 1 < \gamma_0$.

(2) To the right of lines a and c and below the horizontal axis: γ_0, γ_1 real with $\gamma_1 \in (-1, 0)$, $\gamma_0 > 1$.

(3) To the right of line c and to the left of line a: γ_0, γ_1 real with $\gamma_1 < -1$, $\gamma_0 > 1$.

(4) Above the horizontal axis, to the left of line c, and to the right of parabola d: γ_0, γ_1 real with $0 < \gamma_1 < \gamma_0 < 1$.

(5) To the right of the vertical axis, above parabola d, and below line b: γ_0, γ_1 complex with $|\gamma_j| < 1$ and $\mathrm{Re}(\gamma_j) > 0$.

(6) To the right of the vertical axis, above parabola d, and above line b: γ_0, γ_1 complex with $|\gamma_j| > 1$ and $\mathrm{Re}(\gamma_j) > 0$.

(7) To the left of line c, to the right of parabola d, and above line b: γ_0, γ_1 real with $1 < \gamma_1 < \gamma_0$.

Note that regions (1) and (2) correspond to saddle-path stability with monotonic dynamics,[22] regions (3), (6) and (7) to indeterminacy, and region (4) to local instability with monotonic dynamics, and region (5) to local instability with smooth oscillations.

Assumptions A1-A2 require that the point (T_0, D_0) be in region (1) or (2), and since $D_0 = \beta > 0$, it must in fact be in region (1). Examples of such points are given by E_0, E_1, E_2 in Figure A.2. Equation (A.21) is thus a line with slope $1/(1 - \delta_X) > 1$ passing through (T_0, D_0). Examples of such lines corresponding to E_0, E_1, E_2 are shown as the dashed lines in the figure. An increase in ω then corresponds to northeast movement along this dashed line.

A.2.2 Proof of Proposition 4

Note that line c in Figure A.2 has slope 1. Since (T_0, D_0) is below line c, and (A.21) has slope $1/(1 - \delta_X) > 1$, line (A.21) crosses line c only once, and at a point northeast of (T_0, D_0), i.e., for a value $\omega > 0$. Conversely, line a is clearly crossed by line (A.21) only once and at a value southwest of (T_0, D_0), i.e., for a value $\omega < 0$.

Thus, as ω decreases from 0 to $-\infty$, (T_ω, D_ω) is always to the right of line c. As long as (T_ω, D_ω) also remains to the right of line a, the system will be in region (1) or (2), in which case we have saddle-path stability. At the value of $\omega < 0$ where line (A.21) crosses line a, the system moves into region (3)—a region corresponding to indeterminacy—and remains in it thereafter as ω continues to decrease to $-\infty$.

[22] Even though region (2) involves a negative eigenvalue $\lambda_1 = 1/\gamma_1 < -1$, this eigenvalues is unstable, and therefore convergence along the saddle path is governed by the other eigenvalue $\lambda_0 = 1/\gamma_0 \in (0, 1)$, which imparts monotonic dynamics.

Next, consider what happens as ω increases from 0 to ∞. There are three possibilities for what happens, depending on where line (A.21) crosses line c, which is in turn determined by the location of the initial point (T_0, D_0). In the first case, an example of which is given by E_0 in the figure, T_0 is sufficiently small that line c is crossed at a value $T_\omega < 2$, in which case as ω increases through this point the system initially crosses into region (4) (local instability, monotonic dynamics). As ω continues to increase, it must subsequently cross into region (5) (local instability, smooth oscillations), then region (6) (indeterminacy, complex eigenvalues). Since d is a parabola its slope becomes ever steeper as T_ω increases, and thus as ω continues to increase eventually the system must cross d again into region (7) (indeterminacy with real, positive eigenvalues). It then remains in that region as ω continues to increase all the way to ∞.

Thus, we see that regardless of whether ω (equiv., $V_{X\bar{e}}$) increases or decreases from 0, eventually indeterminacy emerges (for the first time in region (3) for the $\omega < 0$ case, and in region (6) for the $\omega > 0$ case), and as ω continues to increase or decrease all the way to $\pm\infty$ that indeterminacy remains thereafter (always in region (3) for the $\omega < 0$ case, eventually in region (7) for the $\omega > 0$ case). ■

A.2.3 Proof of Proposition 5

From the discussion in the proof of Propositon 4, we see that as ω decreases from 0, the system is in either region (1) or (2), which involve saddle-path-stable monotonic dynamics, up until the point that indeterminacy first emerges when line a is crossed, which confirms part *(i)* of the proposition.

Next, as ω increases from 0, the system exhibits saddle-path stable monotonic dynamics up until the point that indeterminacy emerges if and only if the system crosses line c from region (1) into region (7) (e.g., if (T_0, D_0) is a point such as E_1 or E_2). This is in turn the case if and only if $T_\omega \geq 2$ when $D_\omega = 1$. Using (A.21), this is the case if and only if $\alpha_X \leq \delta_X^2/(1+\delta_X)$, which, it can be verified, is equivalent to the condition given in part *(ii)* of the proposition, which confirms that part.

Finally, if $\alpha_X > \delta_X^2/(1+\delta_X)$, then as ω increases from 0 the system crosses line c from region (1) into region (4), which involves γ_0 decreasing through 1, which is a $+1$ bifurcation. Since region (4) is associated with local instability, part *(iii)* of the proposition follows, which completes the proof. ■

A.3 Section 4 Proofs

A.3.1 Preliminaries

In what follows, we only consider the $\beta = 0$ case. When including the second state variable Z, we update Assumptions A2 and A3 as follows.

Assumption A2. (Non-Degeneracy) *In the absence of any interactions,*

(i) the transition matrix governing the evolution of the linearized equilibrium system is diagonalizable, with at least one non-zero eigenvalue; and

(ii) both X_t and Z_t locally affect the equilibrium choice of e_t.

Assumption A3. (Weak Complementarities) *Given the current levels of the state variables X_t and Z_t, and the entire sequence of future private and aggregate actions $\{e_\tau, \bar{e}_\tau\}_{\tau>t}$, an agent's best response of e_t to $\bar{e}_t$ at t always satisfies $de_t/d\bar{e}_t < 1$.*

It can be verified that the FOC for the agent's optimal choice of e_t in the $\beta = 0$ case is simply $V_e(e_t, X_t, Z_t, \bar{e}_t) = 0$. Totally differentiating this FOC, and letting $Y \equiv (e, X, Z)$, we may obtain

$$de = -\frac{V_{eX}(Y, \bar{e})}{V_{ee}(Y, \bar{e})}dX - \frac{V_{eZ}(Y, \bar{e})}{V_{ee}(Y, \bar{e})}dZ - \frac{V_{e\bar{e}}(Y, \bar{e})}{V_{ee}(Y, \bar{e})}d\bar{e}\,. \tag{A.22}$$

Clearly, Assumption A2*(ii)* is equivalent to the condition $V_{eX}, V_{eZ} \neq 0$, while Assumption A3 is equivalent to the condition $V_{e\bar{e}}(Y, \bar{e}) < \bar{V}_{e\bar{e}}(Y, \bar{e}) \equiv -V_{ee}(Y, \bar{e})$ for all $Y, \bar{e}$.

Next, let $\alpha_X \equiv V_{eX}/V_{ee}$ and $\alpha_Z \equiv -V_{eZ}/V_{ee}$, and note that our assumptions imply $\alpha_X, \alpha_Z \neq 0$. Define also $\rho \equiv V_{e\bar{e}}/\bar{V}_{e\bar{e}}$. As in Section A.1, we henceforth take ρ as the parameter that locally controls the type and strength of interactions, with $\rho = 0$ corresponding to autarky, $\rho < 0$ to substitutability, and $\rho \in (0, 1)$ to weak complementarity. Let $Q \equiv (-\infty, 1)$ denote the allowable range for ρ.

With this notation, the evolution of the linearized system can be written

$$\begin{pmatrix} X_{t+1} \\ Z_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 - \delta_X - \frac{\alpha_X}{1-\rho} & \frac{\alpha_Z}{1-\rho} \\ -\frac{\alpha_X}{1-\rho} & 1 - \delta_Z + \frac{\alpha_Z}{1-\rho} \end{pmatrix}}_{M} \begin{pmatrix} X_t \\ Z_t \end{pmatrix}. \tag{A.23}$$

The two eigenvalues of M are given by

$$\lambda, \overline{\lambda} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - D}\,, \tag{A.24}$$

where

$$T = T_\rho \equiv 2 - \delta_X - \delta_Z + \frac{\alpha_Z - \alpha_X}{1-\rho}\,, \tag{A.25}$$

$$D = D_\rho \equiv (1-\delta_X)\,(1-\delta_Z) + \frac{(1-\delta_X)\,\alpha_Z - (1-\delta_Z)\,\alpha_X}{1-\rho}\,, \tag{A.26}$$

are the trace and determinant of M, respectively. Note that T_ρ and D_ρ are both continuous in ρ on Q.

Lemma A.11. *Suppose Assumption A2 holds. Then at least one of $\partial T_\rho/\partial\rho$ and $\partial D_\rho/\partial\rho$ is non-zero on $\rho \in Q$.*

Proof. Suppose $\partial T_\rho/\partial\rho = \partial D_\rho/\partial\rho = 0$ for some ρ. From (A.25), $\partial T_\rho/\partial\rho = 0$ implies $\alpha_X = \alpha_Z$. Since $\alpha_X, \alpha_Z \neq 0$, from (A.26) we see that $\partial D_\rho/\partial\rho = 0$ implies we must have $\delta_X = \delta_Z = \delta$. It can be verified in this case that M has a repeated eigenvalue (specifically, $\lambda, \overline{\lambda} = 1 - \delta$) with associated one-dimensional eigenspace $\{(y,y) : y \in \mathbb{R}\}$. Thus, M is defective, a property ruled out by Assumption A2((i)). We therefore conclude that we cannot have $\partial T_\rho/\partial\rho = \partial D_\rho/\partial\rho = 0$ under Assumption A2, which confirms the lemma. ∎

Note that the eigenvalues $\lambda, \overline{\lambda}$ are both stable if and only if $(T,D) \in S$, where

$$S \equiv \{(T,D) : -1 < D < 1\,,\, -D-1 \leq T < D+1\}\,. \tag{A.27}$$

It can be verified that S is a triangle in (T,D)-space, and is therefore convex and bounded. The eigenvalues both have non-negative real parts if and only if $(T,D) \in \mathbb{R}^2_+$. The eigenvalues are real if and only if

$$D \leq T^2/4\,. \tag{A.28}$$

Finally, note that both eigenvalues are zero if and only if $(T,D) = (0,0)$.

Next, note that Assumptions A1 and A2 together require that $(T_0, D_0) \in S \cap \mathbb{R}^2_+ \setminus \{0,0\}$, and further that $(T,D) = (T_0,D_0)$ satisfies (A.28). Note also that

$$T_{-\infty} \equiv \lim_{\rho\to-\infty} T = 2 - \delta_X - \delta_Z$$

$$D_{-\infty} \equiv \lim_{\rho\to-\infty} D = (1-\delta_X)\,(1-\delta_Z)\,.$$

It can be verified that $(T_{-\infty}, D_{-\infty}) \in S \cap \mathbb{R}^2_+$ and satisfies (A.28). Further, Lemma A.11 implies that

$$\lim_{\rho\uparrow 1} \|(T_\rho, D_\rho)\| = \infty\,.$$

Combining (A.25) and (A.26) to eliminate ρ, we obtain the following relationships between T and D:[23]

$$\begin{cases} D = \Delta(T) & \text{, if } \alpha_X \neq \alpha_Z \\ T = \tau(D) & \text{, if } (1-\delta_Z)\,\alpha_X \neq (1-\delta_X)\,\alpha_Z \end{cases}, \tag{A.29}$$

where

$$\Delta(T) \equiv \frac{(1-\delta_Z)\,\alpha_X - (1-\delta_X)\,\alpha_Z}{\alpha_X - \alpha_Z} T - \frac{(1-\delta_Z)^2\,\alpha_X - (1-\delta_X)^2\,\alpha_Z}{\alpha_X - \alpha_Z}, \tag{A.30}$$

$$\tau(D) \equiv \frac{\alpha_X - \alpha_Z}{(1-\delta_Z)\,\alpha_X - (1-\delta_X)\,\alpha_Z} D + \frac{(1-\delta_Z)^2\,\alpha_X - (1-\delta_X)^2\,\alpha_Z}{(1-\delta_Z)\,\alpha_X - (1-\delta_X)\,\alpha_Z}. \tag{A.31}$$

Together, the above results establish that the set $\{(T_\rho, D_\rho) : \rho \in Q \cup \{-\infty\}\}$ maps out a ray in (T, D)-space emanating from the point $(T_{-\infty}, D_{-\infty})$, and passing through the point (T_0, D_0). Further, as ρ increases in Q, (T_ρ, D_ρ) moves continuously along this ray in the direction away from $(T_{-\infty}, D_{-\infty})$ and without bound.

A.3.2 Proof of Proposition 6 *(i)*

To see part *(i)* of Proposition 6, note that the eigenvalues are complex if and only if (A.28) fails to hold, which is equivalent to the condition

$$(\delta_X - \delta_Z)^2 (1-\rho)^2 + 2(\delta_X - \delta_Z)(\alpha_X + \alpha_Z)(1-\rho) + (\alpha_X - \alpha_Z)^2 < 0. \tag{A.32}$$

If $\delta_X = \delta_Z$, this condition clearly cannot hold for any value of ρ. Thus, a necessary condition for complex eigenvalues to emerge is $\delta_X \neq \delta_Z$. Assuming henceforth that this is the case, the left-hand side of condition (A.32) is a convex quadratic polynomial in $1-\rho$. Thus, there exists values of ρ such that (A.32) holds if and only if the discriminant of that polynomial is strictly positive. It can be verified that this discriminant is equal to

$$16\,(\delta_X - \delta_Z)^2\,\alpha_X \alpha_Z,$$

which is strictly positive if and only if $\alpha_X \alpha_Z > 0$, i.e., $V_{eX} V_{eZ} < 0$. Thus, V_{eX} and V_{eZ} must be of opposite signs, and since we have assumed $V_{eX} \leq V_{eZ}$, this in turn is the case if and only if $V_{eX} < 0 < V_{eZ}$, which is one of the necessary conditions for complex eigenvalues

[23] Note that the case where both $\alpha_X = \alpha_Z$ and $(1-\delta_Z)\,\alpha_X = (1-\delta_X)\,\alpha_Z$ corresponds to the case where $\partial T_\rho / \partial \rho = \partial D_\rho / \partial \rho = 0$, which is ruled out by Lemma A.11. We therefore ignore this case.

stated in the proposition. Assuming henceforth that this condition holds, note that this implies that $\alpha_X, \alpha_Z > 0$.

Next, to see that the condition $\delta_Z > \delta_X$ is necessary for there to exist values of $\rho < 1$ that produce complex eigenvalues, suppose to the contrary that $\delta_Z \leq \delta_X$. Note that we have already ruled out the possibility of complex eigenvalues for the case $\delta_Z = \delta_X$. Thus, suppose $\delta_Z < \delta_X$. Then condition (A.32) holds if and only if

$$1 + \frac{\left(\sqrt{\alpha_X} - \sqrt{\alpha_Z}\right)^2}{\delta_X - \delta_Z} < \rho < 1 + \frac{\left(\sqrt{\alpha_X} + \sqrt{\alpha_Z}\right)^2}{\delta_X - \delta_Z}.$$

But this requires $\rho > 1$. Thus, we cannot have complex eigenvalues for $\rho < 1$ if $\delta_Z \leq \delta_X$, so that $\delta_Z > \delta_X$ is indeed a necessary condition.

Finally, to see that $V_{eX} < 0 < V_{eZ}$ and $\delta_Z > \delta_X$ are together sufficient for there to exist values of $\rho < 1$ producing complex eigenvalues, suppose these conditions hold. Then condition (A.32) holds if and only if

$$1 - \frac{\left(\sqrt{\alpha_X} + \sqrt{\alpha_Z}\right)^2}{\delta_Z - \delta_X} < \rho < 1 - \frac{\left(\sqrt{\alpha_X} - \sqrt{\alpha_Z}\right)^2}{\delta_Z - \delta_X}, \tag{A.33}$$

which clearly identifies a non-empty range of $\rho < 1$. Thus, $V_{eX} < 0 < V_{eZ}$ and $\delta_Z > \delta_X$ are sufficient for there to exist values of $\rho < 1$ featuring complex eigenvalues, which completes the proof of part *(i)* of Proposition 6. ■

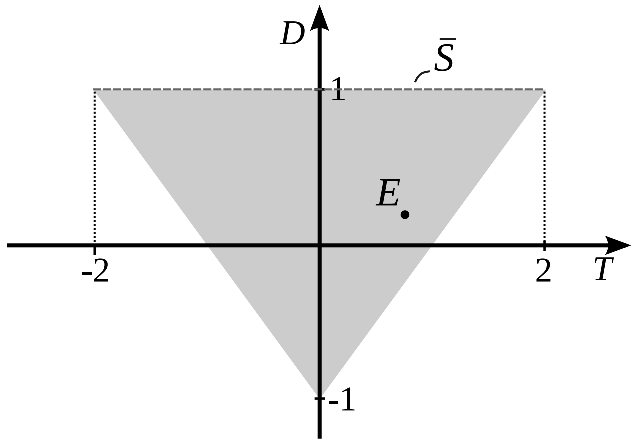
A.3.3 Proof of Proposition 6 *(ii)*

We need to show that $\tau^* \equiv \{(T_\rho, D_\rho) : \rho \leq 0\} \subset S$, where S was defined in (A.27). Based on our earlier discussion, we know that τ^* is a half-open segment of line (A.29), with the closed endpoint given by $(T_0, D_0) \in S$, and the open endpoint by $(T_{-\infty}, D_{-\infty}) \in S$. Since S is a convex set, and both ends of the line segment τ^* are contained in S, it therefore follows that $\tau^* \subset S$, which completes the proof. ■

A.3.4 Proof of Proposition 6 *(iii)*

To see part *(iii)(a)*, recall that the stability triangle S is a triangle in (T, D)-space (see Figure A.3), and starting from the point $(T_0, D_0) \in S$ (indicated as E in the figure), as ρ increases from 0 towards 1, (T_ρ, D_ρ) moves continuously and unboundedly away from (T_0, D_0) along a ray. It therefore follows that (T_ρ, D_ρ) must eventually leave S as ρ increases (i.e., the system becomes unstable), and once it leaves S, it can never return (i.e., it remains unstable thereafter). This confirms part *(iii)(a)*.

Figure A.3: Stability Triangle (S)



Notes: Gray triangle is S. The autarky ($\rho = 0$) case corresponds to the point E, which is inside S and in the upper-right quadrant (i.e., in $\mathbb{R}^2_+$).

To see part *(iii)(b)*, assume the conditions in part *(i)* hold, so that $\alpha_X, \alpha_Z > 0$ and $\delta_Z > \delta_X$. By part *(iii)(a)*, we know that the system loses stability as ρ increases from 0 to 1, and this happens as (T_ρ, D_ρ) crosses out of triangle S. If the eigenvalues are complex at this point then both must have the same modulus, namely, 1, and since D is the product of the eigenvalues, we must have $D_\rho = 1$ at this point. From Figure A.3, we see that this corresponds to crossing the upper boundary of triangle S, which we denote $\bar{S}$ (indicated by the dashed line in the figure). Further, it can be verified (see (A.28)) that the only points on $\bar{S}$ that are *not* associated with complex eigenvalues are the endpoints $(-2, 1)$ and $(2, 1)$.

Thus, we may conclude that loss of stability will occur while the system is experiencing smooth oscillations if and only if (T_ρ, D_ρ) crosses the interior of $\bar{S}$. Recalling the definition of τ in (A.31), this is in turn clearly the case if and only if (I) $dD_\rho/d\rho > 0$ (i.e., (T_ρ, D_ρ) moves in an upward direction as ρ increases), (II) $\tau(1) > -2$ (i.e., (T_ρ, D_ρ) passes to the right of the left endpoint of $\bar{S}$), and (III) $\tau(1) < 2$ (i.e., (T_ρ, D_ρ) passes to the left of the right endpoint of $\bar{S}$).

Since $\delta_X < \delta_Z \leq 1$, from (A.26) we see that condition (I) is equivalent to $\alpha_Z > \frac{1-\delta_Z}{1-\delta_X}\alpha_X$, while from (A.31) condition (b) is equivalent to $\frac{(2-\delta_X)^2\alpha_Z-(2-\delta_Z)^2\alpha_X}{(1-\delta_X)\alpha_Z-(1-\delta_Z)\alpha_X} > 0$. Given condition (I), condition (II) holds if and only if $\alpha_Z > (\frac{2-\delta_Z}{2-\delta_X})^2\alpha_X$. Since $\delta_Z > \delta_X$, we have $(\frac{2-\delta_Z}{2-\delta_X})^2 > \frac{1-\delta_Z}{1-\delta_X}$, and therefore $\alpha_Z > (\frac{2-\delta_Z}{2-\delta_X})^2\alpha_X$ in fact implies both of conditions (I) and (II). Next, condition (III) is equivalent to $\frac{\delta_X^2\alpha_Z-\delta_Z^2\alpha_X}{(1-\delta_X)\alpha_Z-(1-\delta_Z)\alpha_X} < 0$. If condition (I) holds, then this is equivalent to $\alpha_Z < (\frac{\delta_Z}{\delta_X})^2\alpha_X$. Thus, if $\alpha_Z > (\frac{2-\delta_Z}{2-\delta_X})^2\alpha_X$ and $\alpha_Z < (\frac{\delta_Z}{\delta_X})^2\alpha_X$ then (I), (II) and (III) all hold. Substituting the definitions of α_X and α_Z into these conditions, part *(iii)* of the proposition follows immediately. ■