NBER WORKING PAPER SERIES

DYNAMIC IDENTIFICATION IN VARS

Paul Beaudry
Fabrice Collard
Patrick Fève
Alain Guay
Franck Portier

Working Paper 32598
http://www.nber.org/papers/w32598

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 2024

We thank participants at conferences and seminars for helpful comments and discussions. We are particularly thankful to Christian Gourieroux for very insightful remarks and discussions. Collard and Fève acknowledge funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program, grant ANR-17-EURE-0010. Collard and Fève also acknowledge funding from the French National Research Agency (ANR) under the project ANR-23-CE26-0011-01. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

At least one co-author has disclosed additional relationships of potential relevance for this research. Further information is available online at http://www.nber.org/papers/w32598

Dynamic Identification in VARs
Paul Beaudry, Fabrice Collard, Patrick Fève, Alain Guay, and Franck Portier
NBER Working Paper No. 32598
June 2024
JEL No. C32,E30,E50

ABSTRACT

Most macroeconomic models view economic outcomes as being generated by a combination of endogenous and exogenous dynamic forces. In particular, the exogenous forces are generally modelled as a set of independent dynamics processes. In this paper we begin by showing that this dual dynamic structure is sufficient to identify the entire set of structural impulse responses inherent to any such model. No extra restrictions are needed. We then use this result to suggest how it can be used to evaluate common SVAR restrictions (impact restrictions, long run restrictions and proxy-VAR).

Paul Beaudry
Vancouver School of Economics,
University of British Columbia
6000 Iona Drive
Vancouver, V6T 1Z1
Canada
and NBER
Paul.Beaudry@ubc.ca

Fabrice Collard
Toulouse School of Economics
fabrice.collard@gmail.com

Patrick Fève
Toulouse School of Economics
21, Allée de Brienne
31000 Toulouse, France
patrick.feve@tse-fr.eu

Alain Guay
Universite du Quebec a Montreal
Dep. Sc. Economique, UQAM
CP 8888 Centre ville, Montreal, Canada
H3C 3P8
guay.alain@uqam.ca

Franck Portier
Department of Economics
University College London
30 Gordon Street
WC1H 0AX London
United Kingdom
and CEPR
f.portier@ucl.ac.uk

Dynamic Identification in VARs

Paul Beaudry*
Vancouver School of Economics and NBER

Fabrice Collard
Toulouse School of Economics (CNRS, UTC) and CEPR

Patrick Fève
Toulouse School of Economics (UTC)

Alain Guay
Université du Québec à Montréal, CIREQ and CMP

Franck Portier
University College London and CEPR

May 25, 2024

Abstract

Most macroeconomic models view economic outcomes as being generated by a combination of endogenous and exogenous dynamic forces. In particular, the exogenous forces are generally modelled as a set of independent dynamics processes. In this paper we begin by showing that this dual dynamic structure is sufficient to identify the entire set of structural impulse responses inherent to any such model. No extra restrictions are needed. We then use this result to suggest how it can be used to evaluate common SVAR restrictions (impact restrictions, long run restrictions and proxy-VAR).
Keywords: Structural Shocks, Dynamic Identification, SVAR, DSGE Model

1 Introduction

Macroeconomists are often interested in knowing how the economy reacts to different types of shocks (see Ramey (2016) and Stock and Watson (2017) for very detailed accounts of the recent macroeconomic literature). There are two main approaches to look at this issue. On the one hand, one can build a fully specified Dynamic Stochastic General Equilibrium (DSGE) model, estimate it using full information methods (see *e.g.* Smets and Wouters (2007), Christiano et al. (2010) and Lindé et al. (2016)), and look at its implied impulse response functions (IRF).

*Beaudry: paul.beaudry@ubc.ca, Collard: fabrice.collard@gmail.com, Fève: patrick.feve@tse-fr.eu, Guay: guay.alain@uqam.ca, Portier: f.portier@ucl.ac.uk. We thank participants at conferences and seminars for helpful comments and discussions. We are particularly thankful to Christian Gourieroux for very insightful remarks and discussions. Collard and Fève acknowledge funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program, grant ANR-17-EURE-0010. Collard and Fève also acknowledge funding from the French National Research Agency (ANR) under the project ANR-23-CE26-0011-01.

These IRF are generally referred to as structural impulse responses. The advantage of such an approach is that the identification of shocks is generally granted (see *e.g.* Canova and Sala (2009), Iskrev (2010), Komunjer and Ng (2011)), given the many restrictions imposed by the (generally small scale) structural model. One caveat though —the flip side of the same coin— is that a DSGE model imposes many restrictions on the data that reflect modelling detail and, consequently, is prone to mis-specification. On the other hand, one can follow the Structural Vector Autoregressive (SVAR) literature (see Section 4 in Stock and Watson (2016) and Kilian and Lütkepohl (2016) for a complete review) and impose a more limited set of identifying restrictions —restrictions more loosely motivated by theory or alternatively motivated by institutions— to derive structural impulse responses using a VAR.[1] SVARs are less prone to mis-specification, but mapping their implications into the language of models and exogenous structural shocks is not uncontroversial.

It is important to note that the SVAR approach is aimed at obtaining the same objects than those obtained using a DSGE, that is, it is aimed at recovering structural impulse responses that can be interpreted as being the outcome of an economy subjected to particular exogenous driving forces. It is this last observation that we want to exploit in this paper. In particular, we will show that when a VAR is viewed as the reduced form of a DSGE model, then one can immediately obtain the full set of structural impulses responses without the need of any additional identifying restrictions. Because of this property, most identifying restrictions used in the SVAR literature can be visually evaluated or formally tested. For instance, this allows for the evaluation of the variety of identifying restrictions imposed in SVARs (Short-run, Long-run, proxy-variables, sign restrictions ...).

Dynamic Identification. Identification is best understood from the simple recognition that IRF implied by DSGE models reflect both propagation mechanisms associated with the functioning of the economy as well as external dynamics associated with the exogenous driving forces. For these exogenous driving forces to have a clear structural interpretation, it is usually assumed that the exogenous processes are linearly independent, both contemporaneously and over time. The fact that the processes for the exogenous driving forces are restricted is key. For example, a common assumption is that the exogenous driving forces are governed by linearly independent AR(1) processes. As we shall show, because DSGE models share this structure, one can recover structural IRF directly from the implied VAR without the need of any of the additional assumption on the loading of the shocks used in the SVAR literature. In other words, the specification of the lag structure and the set of variables of a SVAR is sufficient to identify a set of structural shocks. No additional assumptions are needed to obtain the set of structural IRF. Because it explicitly makes use of restrictions on the dynamic structure of the underlying model, we dub that particular SVAR a "D-SVAR". By *dynamic structure*, we mostly mean the process of the exogenous forcing variables in the model, although the state variables and the lag structure also (obviously) need to be specified. By *identification of structural shocks*,

[1] We use the generic term VAR, which, in our case, also includes Vector Error Correction Models (VECM).

we mean that there generically exists a unique[2] vector of mutually orthogonal shocks in the D-SVAR that satisfies the restrictions imposed by the dynamic structure of the DSGE model. Structural shock identification in VARs usually relies on restrictions imposed on the loading matrix of structural shocks, under the requirement of compatibility with covariance matrix of the unrestricted VAR's innovations. Our approach focuses instead on constraining the dynamic propagation of structural shocks rather than the loading of the shocks. Consequently, identification solely through the covariance matrix is no longer sufficient, we need to fully exploit the autocovariance function at different order of the dynamic representation (that can be a VAR, a VARMA or a State-Space model), or equivalently, the spectral density, to obtain sufficient conditions for local identification. In loose and over-simplified terms, if the economy is moved by exogenous variables that follow linearly independent AR(1) processes with mutually distinct persistence parameters, then the economy follows a D-SVAR and identification of structural shocks is granted. This theoretical result will be shown in Section 3. Importantly all identification results are obtained from the Wold representation of the process, whether this representation obtains from a VAR, VARMA, or a linear state space model. For expositional purposes, we will mainly focus on the case of a VAR model in the main text and, accordingly, will maintain the D-SVAR terminology.

Testing commonly used SVAR restrictions. It is worth mentioning that the structural shocks identified by our D-SVAR approach remain unlabelled (nothing in the identification assigns a name –e.g. technology, monetary, fiscal, etc…– to the identified structural shock). While standard SVAR restrictions (impact, long-run, sign …) can be used to label them *ex-post*, our approach can be used to test these latter restrictions.

To fix ideas, consider the bi-variate environment examined in the seminal paper by Blanchard and Quah (1989). This paper aimed at deriving the impulse responses associated with supply and demand shocks. The identification restriction used to separate the two shocks under consideration imposed that a demand shock has no long-run (permanent) effect on GDP, while a supply shock may. Instead, our approach allows us to first obtain the two unique structural impulse response consistent with a DSGE, and then to examine the extent to which the Blanchard and Quah (1989)'s restrictions are consistent with our D-SVAR.

We will provide three examples drawn from the literature to show how our approach can be used to evaluate SVAR strategies. We first present the Blanchard and Quah (1989) example discussed above. Then we examine the proxy VAR strategy used in Gertler and Karadi (2015) to identify monetary shocks by exploiting a high frequency instrument (see Kuttner (2001), Gürkaynak et al. (2005), Bernanke and Kuttner (2005) and Gürkaynak et al. (2007) for early work on High Frequency Identification of monetary policy shocks). Finally, we assess the validity of the impact restrictions used in Christiano et al. (1999, 2005) to identify monetary policy shocks. We will show that, for two of these examples (Blanchard and Quah (1989) and Christiano et al. (1999)), the identifying restrictions cannot be rejected within our D-SVAR, while

[2]Uniqueness is up to the sign and/or a permutation of the shocks.

they are in the proxy-VAR of Gertler and Karadi (2015).

Related Literature. Our identification result relates to several papers including, among others, McGrattan (2010), Pagan and Robinson (2019), Bai and Wang (2015) and Gourieroux and Jasiak (2022). While several of our theoretical results have precedents in the literature, our main contribution is to show why standard SVAR restrictions can be seen as over-identifying restrictions and therefore can be tested.

McGrattan (2010) derives conditions for identification of an unrestricted state-space representation associated with a specific small-scale Real Business Cycle model.[3] In particular, the paper shows that, when the model includes a permanent technology shock and a stationary labor wedge shock (in the form of a labor income tax), the unrestricted state-space representation is identified. Our paper provides conditions for identification in a broader class of DSGE models, admitting a VAR or a VARMA representation of the solution. Pagan and Robinson (2019) note that SVARs may face difficulty to properly uncover the loading matrix of DSGE models because standard estimation of SVAR models avoids imposing the type of statistical restrictions commonly used in DSGE models —*e.g.* that structural shocks follow mutually orthogonal univariate autoregressive processes. The two authors discuss conditions for local identification in SVARs when the autoregressive matrix is diagonal and shocks are normalized. Our formal analysis shows more generally the conditions on the autoregressive matrix that allows to identify the structural shocks. Bai and Wang (2015) study identification in dynamic factor models similar to our unrestricted state space representation. Their approach, in line with the conventional way of identifying shocks in the VAR literature, imposes restrictions on the loading matrix while leaving unrestricted the autoregressive matrix of factors. In this paper, we take the opposite viewpoint and determine which type of organisation of the autoregressive matrix allows to freely identify the loading matrix in the state-space representation. Finally, Gourieroux and Jasiak (2022) provide conditions for identification in multivariate undetermined convoluted systems when the exogenous shocks (the "*sources*" in their terminology) follow linearly independent autoregressive processes of order one and when there is no intrinsic dynamics of the endogenous variables. They show that when the autoregressive parameters are distinct, the loading matrix (the "*mixing matrix*" in their terminology) is identified. Our paper departs from theirs in at least three dimensions. First, we consider a larger class of dynamic models and makes connections with the DSGE literature. Second, we extend the identification problem to non diagonal autoregressive processes. Third, we determine conditions for partial identification when the practitioner seeks to identify only a subset of structural shocks.

Outline. The paper is structured as follows. Section 2 overviews the main results of the paper. It shows how the D-SVAR representation can be derived, explains heuristically why it is identified and presents an application. Section 3 formally proves local identification. Section 4 discusses estimation and inference in the D-SVAR setup. Section 5 illustrates the use of D-

[3] See also Kascha and Mertens (2009) for simulation experiments in a similar setup.

SVARs to assess SVARs in the context of monetary policy shocks. A last section concludes. All proofs are reported in an appendix.

2 A Primer on D-SVARs

This section shows how to derive our D-SVAR representation, while explaining its relationship with a (linearised) DSGE model. It then explains intuitively why this D-SVAR should be identified, by checking the necessary order condition, leaving proof of identification to the next section.[4] Finally, it presents a simple application to an output growth–unemployment VAR.

2.1 A Basic Setup

Let us assume that the Data Generating Process (DGP hereafter) is an economic model (typically a DSGE model) of the type

$$\begin{aligned} X_t &= M_1 X_{t-1} + M_2 \mathbb{E}_t[X_{t+1}] + M_3 Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{1}$$

where $\mathbb{E}_t[\cdot]$ denotes the expectation operator conditional on period t information set, X_t is a $n_x \times 1$ vector of endogenous variables and Z_t is a $n_z \times 1$ vector of structural shocks. Those shocks are assumed to be autoregressive of order one.[5] The structural innovations ε_t are normally distributed, with zero mean and their covariance matrix is identity. Note that this implies that the loading matrix M_3 encapsulates the size of the shocks. The vector X_t splits between the $(n_y \times 1)$ vector Y_t of observed variables and the $(n_k \times 1)$ vector of (unobserved or observed) endogenous state variables K_t. Note that some substitutions might be needed to obtain a system featuring as many observed variables as shocks ($n_y = n_z$).

Matrices M_1, M_2, M_3 are functions of the vector of the economic model deep parameters, θ, and encapsulate any cross-equation restrictions imposed by the micro-foundations of the DSGE model. Note in particular that those matrices M_i may contain some zero elements. Finally matrix R gathers all the parameters pertaining to the dynamics of the shock processes. In this section, it is assumed that all the variables in X_t are observable, while the shocks Z_t are not. The case of non-observable endogenous state variables is dealt with in Section A of the Online Appendix. The solution of the model admits[6] the following state space representation

$$\begin{aligned} K_{t+1} &= G K_t + F Z_t, \\ Y_t &= \Pi_{yk} K_t + \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{2}$$

[4]Here we refer to the first order condition for identification. However, local identification may still be possible using higher order conditions (see Sargan (1983) and Dovonon and Hall (2018)).

[5]To keep exposition simple at this stage, we present only the case of a model with one lead and one lag and an order one process for shocks. Conceptually, everything extends to higher order models.

[6] This implicitly assumes that the dynamic system admits a saddle path. When the system is locally indeterminate, the Z_t vector can be extended to capture extrinsic uncertainty.

where G, F, Π_{yk} and Π_{yz} are functions of M_1, M_2, M_3 and R, and therefore of θ and R, as represented by the mapping

$$(G, F, \Pi_{yk}, \Pi_{yz}, R) = \Phi(\theta, R).$$

If the System (2) is identified, one can then go one step further and identify the model parameters, provided that the mapping Φ is invertible.

Having set the stage for System (2), we are now in a position to discuss the identification of shocks.

2.2 Heuristic Approach to Identification

For expositional purpose, we consider the case where the vector X_t only consists of observed variables Y_t, which, as we will show, can be directly written as a VAR. The case of latent endogenous state variables is treated in full generality in the next section. We also assume that there are as many such observed variables as shocks ($n = n_y = n_z$). In this case, $\Pi_{yk} = G$ and $\Pi_{yz} = F$, such that the system reduces to

$$\begin{aligned} Y_t &= GY_{t-1} + FZ_t, \\ Z_t &= RZ_{t-1} + \varepsilon_t. \end{aligned} \tag{3}$$

Eliminating Z_t, System (3) can be written as follows, which we dub a D-SVAR:

$$Y_t = \left(G + FRF^{-1}\right) Y_{t-1} - FRF^{-1}GY_{t-2} + F\varepsilon_t. \tag{4}$$

Estimating a VAR(2) on the data, one can obtain the non structural VAR representation:

$$Y_t = \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \nu_t. \tag{5}$$

where ν_t is a vector of canonical innovations with covariance matrix Σ_ν. The representation (5) is referred to as the non structural VAR. Matrices Γ_1, Γ_2 and Σ_ν are functions of G,F and R according to the mapping

$$(\Gamma_1, \Gamma_2, \Sigma_\nu) = \Psi(G, F, R).$$

Note that provided the D-SVAR representation can be recovered –*i.e.* if the mapping Ψ is invertible, one can compute the theoretical impulse responses functions of the structural model, the variance decomposition, conditional correlations …

Identifying the D-SVAR (4) means recovering matrices G, F and R from Γ_1, Γ_2 and Σ_ν. Absent any restrictions, each matrix contains n^2 elements, so that $3n^2$ unknown coefficients need to be recovered. The available information in the non-structural VAR is given by $(\Gamma_1, \Gamma_2, \Sigma_\nu)$. The system of equations that determines the elements of F,G and R is, using (4) and (5):

$$\left\{ \begin{array}{lcl} \Gamma_1 & = & G + FRF^{-1}, \\ \Gamma_2 & = & -FRF^{-1}G, \\ \Sigma_\nu & = & FF'. \end{array} \right. \tag{6}$$

Because Σ_ν is symmetric, this system only provides with $3n^2 - \frac{n(n-1)}{2}$ independent equations for $3n^2$ unknowns. This is the well-known problem of the identification of shocks in SVARs. If one adds some extra identifying restrictions (at least $\frac{n(n-1)}{2}$), then F, G and R can be identified. Of course, this order condition is only necessary, and a rank condition also needs to be satisfied (see Section 3). For now, let us simply count the number of restrictions and check a necessary condition for identification. A restriction commonly imposed in the VAR literature assumes that F is lower triangular, which amounts to restrict the effect of shocks on impact (see Sims (1980)). This puts exactly $\frac{n(n-1)}{2}$ restrictions, so that the VAR is just identified. But, if the loading matrix F is obtained by solving a standard DSGE, F is a complicated function of the matrices M_1, M_2, M_3 and R, and may not necessarily comply with the lower triangular assumption unless some specific assumptions are placed on the timing of agent's decisions (see e.g. Christiano et al. (2005)). Likewise, restricting the long-run, as in Blanchard and Quah (1989), imposes a particular structure on the loading matrix F that not all DSGE share.

Our D-SVAR approach does not hinge on restricting the loading matrix F but rather relies on assumptions placed on the dynamic structure of the shocks only —*i.e.,* on the autoregressive matrix R. To fix ideas, let us assume that the shocks in Z_t are mutually orthogonal at all leads and lags – *i.e.* that R is a diagonal matrix with distinct diagonal elements. In other words, let us assume that all the shocks in the DSGE model follow linearly independent AR(1) processes. While this assumption may appear very restrictive at first sight, it is shared by a vast majority of the DSGE literature. In that case, the necessary order condition is satisfied: R only consists of n non zero diagonal elements, so that we have $2n^2+n$ unknowns to determined. Note however that, as we will show in the next section, counting restrictions does not guarantee identification. For example, a diagonal autoregressive matrix with identical elements (*i.e.* $R = \rho I_n$) cannot be identified as, in that case, F drops from the first two equations of System (6). We will follow another strategy to prove formally identification in the next section

2.3 Application to a Bivariate VAR

Here we apply our dynamic identification to a bivariate VAR featuring the growth rate of output per capita, Δy_t, and a measure of the unemployment rate gap, u_t, computed as the gap between the actual and non-cyclical rate of unemployment. Blanchard and Quah (1989) (BQ hereafter) used this type of VAR to uncover the permanent, ε_P, and transitory, ε_T, component of output by imposing that the latter has no long-run effect on the level of output.

Motivated by Blanchard and Quah (1989), we estimate a VAR for the 1960Q1-2007Q4 period and recover the impulse responses consistent with the D-SVAR representation.[7] The two resulting impulse responses are displayed in Figure 1. Most importantly, we have not imposed the restrictions used by Blanchard and Quah (1989). Table 1 reports the associated forecast error variance decomposition at various horizons.

We uncover an interesting and familiar pattern. There is a shock, ε_2, that increases output and decreases unemployment on impact. Then the response of output is hump-shaped and goes back to almost zero in the long-run. This shock explains more than 80% of unemployment

[7]We use 2 lags of data as selected by BIC and we use an Asymptotic Least Squares estimation method, as discussed in Corollary 1, using the unrestricted VAR as auxiliary model.

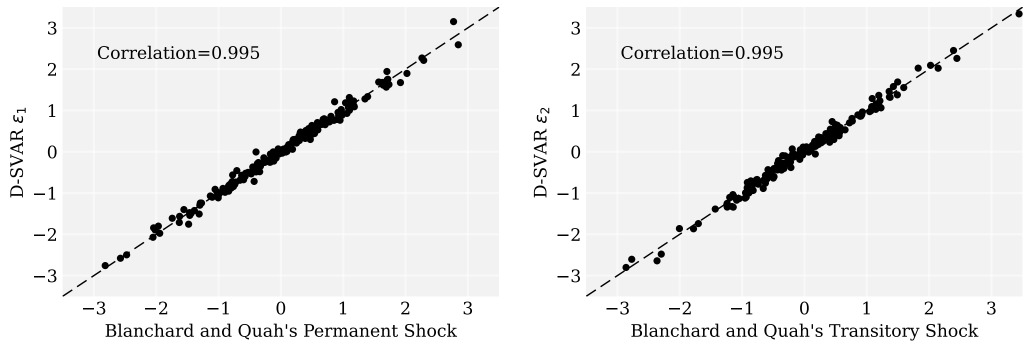
Sample is 1960Q1-2007Q4. y is the real GDP, u is the unemployment rate gap. Estimation is done with (Δy,u) using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

Figure 1: Comparing the D-SVAR with Blanchard and Quah (1989) Identification

Table 1: Forecast Error Variance Decomposition, $(\Delta y, u)$, D-SVAR and Blanchard and Quah (1989)

	Output				Unemployment gap			
Horizon	ε_1	ε_2	ε_P	ε_T	ε_1	ε_2	ε_P	ε_T
1	24.7	75.3	34.0	66.0	15.9	84.1	9.2	90.8
4	18.4	81.6	27.0	73.0	4.7	95.3	1.3	98.7
8	22.3	77.7	31.3	68.7	2.6	97.4	1.2	98.8
20	42.1	57.9	49.1	50.9	2.0	98.0	1.7	98.3
∞	99.8	0.2	100.0	0.0	2.0	98.0	1.7	98.3

Sample is 1960Q1-2007Q4. Estimation is done with (Δy,u) using two lags, where y is the real GDP and u is the unemployment rate gap. ε_1 and ε_2 correspond to the D-SVAR, ε_P and ε_T to Blanchard and Quah (1989).



Sample is 1960Q1-2007Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap.

Figure 2: Correlation between D-SVAR and BQ shocks

volatility at any horizon, about 75% of the volatility of output on impact, but about 0% in the long-run. The other shock, ε_1, exerts a permanent effect on output and little effect in unemployment. The two shocks look pretty much like the permanent and temporary shocks of BQ. This is confirmed by the dash lines on Figure 1, which correspond to the BQ identification. Responses are indeed very similar. Figure 2 shows scatter plots of the BQ's permanent and transitory shocks against ε_1 and ε_2: the correlation between the D-SVAR and BQ shocks is almost perfect in both cases.

As we will make it explicit in Section 4, the estimation of the D-SVAR allows to test the BQ identification restriction. If one restricts the data to be generated by a model in which the two latent shocks are linearly independent AR(1) processes, then, as can be seen on Figure 1, one cannot reject at 0% that only one shock has a permanent effect on output. Figure 1 seems to indicate that the impact response of the unemployment gap differs across the two identifications. However, a formal test of the null hypothesis of equality between the two IRF does not reject the null hypothesis, with a p-value of 40%.

Our dynamic identification hence recovers dynamics that are extremely similar to Blanchard and Quah (1989), whose identifying restrictions can be tested (and not rejected) under the D-SVAR.[8]

In the two next sections, we formally prove the results we have been using informally in this preview section.

3 D-SVAR Identification

This section formally proves local identification of the D-SVAR. We start by fixing some notations that will be used throughout the proof, we then analyze identification relying on covariance matrix restrictions only and finally add dynamic restrictions.

[8] As illustrated in the Online Appendix D, an advantage of our approach though is that the identification of the shocks does not require the estimation of the spectral density of, at least, one variable at frequency 0 —an object which is usually hard to estimate and at best very imprecise (see, *e.g.* Fernald (2007)).

3.1 Setup

Consider an economy whose DGP is described by the following state-space representation (we abstract from constant vectors without any loss of generality)

$$\underbrace{K_{t+1}}_{n_k\times 1} = \underbrace{G}_{n_k\times n_k}\underbrace{K_t}_{n_k\times 1} + \underbrace{F}_{n_k\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{7}$$

$$\underbrace{Y_t}_{n_y\times 1)} = \underbrace{\Pi_{yk}}_{n_y\times n_k}\underbrace{K_t}_{n_k\times 1} + \underbrace{\Pi_{yz}}_{n_y\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{8}$$

$$\underbrace{Z_t}_{n_z\times 1} = \underbrace{R}_{n_z\times n_z}\underbrace{Z_{t-1}}_{n_z\times 1} + \underbrace{\varepsilon_t}_{n_z\times 1}, \tag{9}$$

where the $(n_y \times 1)$ vector Y_t gathers all observed variables, $(n_k \times 1)$ vector K_{t+1} collects all of possibly unobserved endogenous state variables, whose values are predetermined at time t, Z_t represents the $(n_z \times 1)$ vector of unobserved exogenous variables and ε_t is the $(n_z \times 1)$ vector of structural innovations to Z_t. In particular, ε_t satisfies $\mathbb{E}_{t-1}\varepsilon_t = 0$, where $\mathbb{E}_{t-1}$ denotes the expectation operator conditional on the information set of histories until period $t-1$, *i.e.* all past realizations and histories of $\{K_t, Y_t, Z_t\}$.[9] This framework is general enough to represent the (log-)linear solution of most (dynamic general) equilibrium model, including among others DSGE models. This solution usually depends on a limited number of parameters, that we denote by θ gathering all "*deep*" structural parameters (representing preferences, technology, institutions, policies ...), together with the stochastic process of the exogenous forcing variables present in the structural model. In this case, the matrices in the state-space representation (7)–(9) will be functions of θ. In this paper, we do not consider the identification of θ, but instead the identification of the state-space parameters that freely enter in the matrices of the state-space representation. We denote this vector of state-space parameters ψ. It must be clear to the reader that System (7)–(9) imposes less restrictions than the (possibly underlying) DSGE model. The only restrictions that we will explore apply to the matrix R and the covariance matrix of the structural innovations ε_t in Equation (9). Finally, the shocks ε_t are zero mean weak white noise processes with covariance matrix $\mathbb{E}(\varepsilon_t\varepsilon_t') = \Sigma_\varepsilon$. It is customary in the literature to identify ψ by imposing a set of restrictions on the way the structural shocks load into the system (Short-run restriction à la Sims (1980), Long-run restrictions à la Blanchard and Quah (1989), sign restrictions à la Uhlig (2005), ...). In this case, only the loading matrix and information on the covariance matrix of non-structural innovations are needed to achieve identification. In this paper, we follow another route, and do not restrict the loading but rather place restrictions on the dynamic propagation of shocks. The sole knowledge of the covariance matrix is not sufficient to achieve identification. The econometrician needs to make use of the whole autocovariance function, or equivalently the spectral density, of the process.

[9] Shall some elements K_t be observed, those elements should be reassigned to vector Y_t and the matrices Π_{yk} and Π_{yz} should be adjusted accordingly.

System (7)-(9) can be rewritten in the more compact form

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{10}$$

$$Y_t = \Pi S_t, \tag{11}$$

with

$$\underbrace{S_t}_{n_s\times 1} = \begin{bmatrix} K_t \\ Z_t \end{bmatrix}, \qquad \underbrace{A}_{n_s\times n_s} = \begin{bmatrix} G & F \\ 0_{n_z\times n_k} & R \end{bmatrix}, \qquad \underbrace{B}_{n_s\times n_z} = \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} \quad \text{and} \quad \underbrace{\Pi}_{n_y\times n_s} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix},$$

where $n_s = n_k + n_z$.[10] System (10)-(11) can be expressed as the Fernández-Villaverde et al. (2007) ABCD representation

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{12}$$

$$Y_{t+1} = CS_t + D\varepsilon_{t+1}, \tag{13}$$

where $C = \Pi A$ and $D = \Pi B$. Note that, in the sequel, we consider the case where the number of observables is equal to the number of shocks. Some assumptions need to be placed on the ABCD representation (10)-(11).

Assumption 1 *For any $z \in \mathbb{C}$, $det(I - Az) = 0$ implies $|z| > 1$.*

Assumption 1 restricts the class of matrices A to those with eigenvalues lying inside the unit circle. Under Assumption 1 and using (10)-(11) and/or (12)-(13), the process $\{Y_t\}$ admits the following infinite Vector Moving Average (VMA) representation:

$$Y_t = \Pi(I - AL)^{-1}B\varepsilon_t = \left[C(I - AL)^{-1}BL + D\right]\varepsilon_t = H(L,\psi)\varepsilon_t$$

where $H(z,\psi) = \sum_{j=0}^{\infty} h(j;\psi)z^j$ is the transfer function and L denotes the lag operator. For every $\psi \in \Psi$, $\mathbb{E}(Y_t) = 0$ and

$$\mathbb{E}(Y_t Y_s') \equiv \Gamma(s-t;\psi) = \sum_{j=0}^{\infty} h(j;\psi)\Sigma_\varepsilon h(j+s-t;\psi)',$$

for all $t, s \geq 1$, where $\psi = (vec(A)', vec(B)', vec(C)', vec(D)', vech(\Sigma_\varepsilon)')'$ is the vector collecting all the parameters of the state-space representation (7)–(9).

For any weakly stationary process $\{Y_t\}$ implied by Assumption 1 and under the assumption that the shocks ϵ_t are Gaussian, the unconditional mean and auto-covariance function completely characterize the properties of the process. Let us therefore define the auto-covariance generating function as:

$$\Omega(z,\psi) = \sum_{j=-\infty}^{\infty} \Gamma(j;\psi)z^j,$$

for any $z \in \mathbb{C}$. Evaluating $\Omega(z,\psi)$ at $z = \exp(i\omega)$ for any $\omega \in [-\pi,\pi]$ and rescaling it by $(2\pi)^{-1}$ yields the spectral density of the observable $\{Y_t\}$ which is always positive semi-definite. To simplify, we hereafter refer to $\Omega(z,\psi)$ as the spectral density as well.

[10] In the specific case where there are no endogenous state variables K_t, $n_k = 0$, $n_s = n_z$ and the matrices are defined accordingly. For example, $A = R$.

As will be clear later, it will prove useful to defined observational equivalence for this class of multivariate covariance stationary process. We closely follow Komunjer and Ng (2011) and define it with respect to the entire auto-covariance function of the observable (or the spectral density).

Definition 1 *Two sets of state-space parameters ψ and $\widetilde{\psi}$ are observationally equivalent if $\Omega(z;\psi) = \Omega(z;\widetilde{\psi})$, for all $z \in \mathbb{C}$ or, equivalently, $\Gamma(j;\psi) = \Gamma(j;\widetilde{\psi})$ at all $j \geq 0$.*

In other words, two stationary state-space models are observationally equivalent is they share the same auto-covariance (spectral) properties. This then allows us to define local identification.

Definition 2 *The state-space representation (10)-(11) (or equivalently (12)-(13)) is locally identifiable from the spectral density of Y_t (or equivalently from the auto-covariances of Y_t) at $\psi \in \Psi$ if there exists an open neighborhood of ψ such that for every $\tilde{\psi}$ in this neighbourhood, ψ and $\widetilde{\psi}$ are observationally equivalent if and only if $\tilde{\psi} = \psi$.*

In state space system, the spectral density function can be simply obtained from the transfer function and the covariance matrix of the shocks, Σ_ε, as

$$\Omega(z, \psi) = H(z;\psi)\Sigma_\varepsilon H(z^{-1};\psi)',$$

where, in our ABCD representation of the dynamics,

$$H(z;\psi) = \Pi(I - Az)^{-1}B \equiv C(I - Az)^{-1}B + D,$$

with $z = \exp(i\omega)$ for any $\omega \in [-\pi, \pi]$. As explained in Komunjer and Ng (2011), this representation of the spectral density makes it clear that equivalent spectral density function can obtain because *(i)* for given Σ_ε, two distinct vectors of state space parameters, ψ and $\widetilde{\psi}$, yield the same transfer function ($H(z;\psi) = H(z;\tilde{\psi})$) or *(ii)* many pairs of $H(z;\psi)$ and Σ_ε give rise to the same spectral density.

In general, the state-space parameter ψ is not identifiable from the second order moments of the observable variables. As an illustration, let us consider the following two state-space representations (K_t is observable):

$$\mathcal{S} = \begin{cases} K_{t+1} &= GK_t + FZ_t, \\ Y_t &= \Pi_{yk}K_t + \Pi_{yz}Z_t, \\ Z_t &= RZ_{t-1} + \varepsilon_t, \end{cases} \qquad \widetilde{\mathcal{S}} = \begin{cases} K_{t+1} &= GK_t + \widetilde{F}\widetilde{Z}_t, \\ Y_t &= \Pi_{yk}K_t + \widetilde{\Pi}_{yz}Z_t, \\ \widetilde{Z}_t &= \widetilde{R}\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t, \end{cases}$$

where $\widetilde{Z}_t = U^{-1}Z_t$, $\widetilde{F} = FU$, $\widetilde{\Pi}_{yz} = \Pi_{yz}U$, $\widetilde{R} = U^{-1}RU$ and $\widetilde{\Sigma}_\varepsilon = U^{-1}\Sigma_\varepsilon U^{-1\prime}$ for some full rank matrix U. The two representations $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are observationally equivalent with respect to the spectral function since $\Omega(z,\psi) = \Omega(z,\widetilde{\psi})$ for all $z \in \mathbb{C}$ where the vectors ψ and $\tilde{\psi}$ gather, respectively, the elements of the vectorization of matrices defining, respectively, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

Following Komunjer and Ng (2011) (see Proposition 1-S), the following property obtains in the case of the $ABCD$ representation

Property: *Two distinct vectors of state-space parameters $\psi, \widetilde{\psi} \in \Psi$ are observationally equivalent respective to the transfer function and the spectral density if and only if there exists a full rank $n_s \times n_s$ matrix T and a full rank $n_z \times n_z$ matrix U such that $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TBU$, $\widetilde{C} = CT^{-1}$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$.*

This property obtains as follows. First, the equalities $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, $\widetilde{C} = CT^{-1}$ are necessary and sufficient for the equivalence of the transfer function $H(z;\tilde{\psi}) = H(z;\psi)$. Sufficiency follows directly from the observation of the transfer function in the ABCD representation

$$\begin{aligned} H(z;\tilde{\psi}) &= \widetilde{C}(I - \tilde{A}z)^{-1}\widetilde{B} + D \\ &= CT^{-1}(I - TAzT^{-1})^{-1}TB + D \\ &= CT^{-1}T(I - Az)^{-1}T^{-1}TB + D = H(z;\psi). \end{aligned}$$

The necessary condition follows directly from a well known result in control theory under the condition of minimality of the state-space representation (See Theorem 3.10 in Antsaklis and Michel (1997) and Chapter 8 in Gouriéroux and Monfort (1995)). A system is minimal if and only if it is controllable and observable (see Appendix A for formal definitions), implying that, among all systems leading to the same output spectral density, it is driven by the minimal number of state variables. This equivalence class is function of a non-singular transformation that corresponds to a rotation of the state, i. e., $T^{-1}S_t$.

The equivalence of the spectral density is obtained for a full rank matrix U such that:

$$H(z;\psi)\Sigma_\varepsilon H(z^{-1};\psi)' = H(z;\psi)UU^{-1}\Sigma_\varepsilon U^{-1\prime}U'H(z^{-1};\psi)'.$$

Fixing $\widetilde{B} = BU$, $\widetilde{D} = DU$ and $\Sigma_{\tilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$, this yields $H(z;\tilde{\psi}) = H(z;\psi)U = DU + C\left[I - Az\right]^{-1}BU$. Observational equivalence follows immediately.

We therefore established that identification of ψ or a subset of ψ cannot obtain without placing additional restrictions on the covariance matrix, Σ_ε. This is what we do in the next section.

3.2 Covariance Matrix Restrictions

This section provides a key proposition on local identification when restrictions are placed on the covariance matrix only. In particular, we consider the case where the structural innovations are mutually orthogonal and their covariance matrix is normalised to the identity matrix such that $\mathbb{E}(\varepsilon_t\varepsilon_t') = I_{n_z}$ – a common identifying assumption in the SVAR literature (see Ramey (2016)). In this case, matrices F and Π_{yz} in (7)–(9) encapsulate any scale effect from the shocks —*i.e.* contains information about the volatility of the shocks. A direct implication of this assumption is that the only admissible matrix U which allows for $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime} = I_{n_z}$ is an orthonormal matrix—*i.e.* $UU' = I_{n_z}$ (see Corollary 1 of Kocięcki and Kolasa (2018)).

We further make the following assumption, that adapts Assumption 1 to our initial state space representation (7)–(9).

Assumption 1′ *In presence of endogenous state vaiables ($n_k > 0$), for any $z \in \mathbb{C}$, $det(I - Az) = 0$ implies $|z| > 1$ and matrices G and R have no eigenvalues in common.*

When $n_k > 0$, since matrix A is block triangular (see System (10)–(11)), we have $det(I - Az) = det(I - Gz)det(I - Rz)$. Matrices G and R have all their eigenvalues lying within the unit circle and do not have any common eigenvalue. This assumption is necessary to disentangle the dynamics of the endogenous variables K_t from those of the exogenous process Z_t.

Assumption 2 *i)* $n_k \leq n_z = n_y$*, ii) when* $n_k = 0$, $\Pi = \Pi_{yz}$ *is of full rank i.e.* $rank(\Pi_{yz}) = n_s$*, iii) when* $n_k > 0$*, matrices* G*,* F*,* Π_{yk} *and* $\left(F', \Pi_{yz}'\right)'$ *are of full rank and linearly independent of* $\left(G', \Pi_{yk}'\right)'$*.*

Without any endogenous state variables, Π_{yz} is of rank n_z, the system is then defined only by equations (8)–(9) without $\Pi_{yk}K_t$ on the right-hand side and $det(I - Az) = det(I - Rz)$. When $n_k > 0$, Assumption 2 guarantees that no endogenous state variables are redundant. The next proposition follows.

Proposition 1 *Under Assumption* $1'$ *and Assumption 2, the state-state (7)-(9) representation is minimal.*

Proof : See Appendix B.

Assumption 2 ensures that the dynamics of Z_t can be recovered from the observables Y_t. To see this, consider the transfer function of System (7)–(9):

$$H(z; \psi) = \left[\Pi_{yk}(I - Gz)^{-1}Fz + \Pi_{yz}\right](I - Rz)^{-1}.$$

Under iii), the $rank(H(z; \psi)) = n_z$ for every $|z| > 1$, then $H(z; \psi)$ is invertible. When $n_y = 0$, $H(z; \psi) = \Pi_{yz}(I - Rz)^{-1}$ and the rank is also equal to n_z for every $|z| > 1$. In the following, we consider the case where $n_k > 0$.

Endowed with Assumption $1'$ and Assumption 2, the next proposition shows that the rotation matrix T is block upper triangular.

Proposition 2 *Under Assumption* $1'$ *and Assumption 2 we have*

- *When the endogenous state variables* K_t *are unobserved, the full rank matrix* T *has the form*
$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{14}$$
with T_{11} *a full rank* $(n_k \times n_k)$ *matrix.*

- *When the endogenous state variables* K_t *are observed, the full rank matrix* T *has necessarily the following form:*
$$T = \begin{bmatrix} I_{n_k} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{15}$$

In both cases, V *is an orthonormal* $(n_z \times n_z)$ *matrix such that* $VV' = I_{n_z}$ *and* $V = U' = U^{-1}$ *defined above.*

Proof : See Appendix C.

Proposition 1 implies that, as long as endogenous state variable K_t is observed, matrices G and Π_{yk} can be identified using the observed spectral density function.[11] However, it does not allow for proper identification of the loading matrix F relying on the properties of Z_t. Indeed, a direct implication of the proposition is that there exists at least one equivalent exogenous process to (9):

$$Z_t = VRV'Z_{t-1} + \varepsilon_t. \tag{16}$$

Observational equivalence in terms of transfer function (and spectral density) holds if and only if sub-matrix V in matrix T defined in (15) is orthonormal ($VV' = I_{n_z}$). In that case, pre-multiplying (16) by V' we get

$$\widetilde{Z}_t = R\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t,$$

where $\widetilde{Z}_t = V'Z_t$, $\widetilde{\varepsilon}_t = V'\varepsilon_t$ and $\mathbb{E}(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t') = I_{n_z}$. In other words, the sole knowledge of the spectral properties of Z_t is not sufficient to identify F. Further (dynamic) restrictions need to be placed on the autoregressive matrix R.

3.3 Covariance Matrix and Dynamic Restrictions

This section focuses on the local identification of the exogenous process Z_t relying on second order moments information (spectral density of observables Y_t). Our strategy is to show that the only admissible permutation matrix in equation (16) is $V = I_{n_z}$ (up to changes of sign and/or permutation of the identity matrix) and hence that there indeed exists only one Z_t process that is compatible with the spectral properties of observables Y_t. We examine the following four cases (commonly encountered in the DSGE literature):

1. R is a diagonal matrix;
2. R is a lower (identically upper) triangular matrix;
3. R is a symmetric matrix;
4. R is a block diagonal matrix with blocks corresponding to cases 1 and 2;

and prove the identification problem in each case.

3.3.1 R is a Diagonal Matrix

The case of a diagonal matrix is of particular interest. It implies, together with the restriction on the covariance matrix, that all processes in the Z_t vector are mutually orthogonal at any leads and lags. While this assumption may sound very restrictive, it actually corresponds to the common practice in the DSGE literature, and hence echoes economic theory. The next proposition derives the sufficient condition for local identification.

[11] The observed spectral density (or equivalently the auto-covariance) function can be obtained by the estimation of a VAR, a VARMA or by the ML estimation of a state space representation.

Proposition 3 *If R is a diagonal matrix with distinct diagonal elements ($r_{i,i} \neq r_{j,j}$, $\forall i \neq j$) then the state-space model (10)-(11) is locally identifiable.*

Proof : See Appendix D.

In other words, the loading matrix F and the autoregressive matrix R are locally identifiable if all the autoregressive parameters of the n_z linearly independent forcing variables are all different. Henceforth, if one has in mind a "standard" DSGE model featuring mutually orthogonal shocks at any leads and lags, dynamic identification easily obtains. From an intuitive point of view, identification obtains because, given an economic structure, differences in the persistence of shocks implies that the impulse response functions to each shock all bring different information regarding the dynamics. To see this more concretely, it may prove useful to consider the following example.

Example: Identification of Demand/Supply Shocks in a New-Keynesian Model. Consider the following textbook 3-equation New Keynesian (NK) model (see Galí (2015)) featuring two structural shocks

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - E_t[\pi_{t+1}]) + z_{1,t}, \\ \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa y_t + z_{2,t}, \\ i_t &= \phi_\pi \pi_t, \end{aligned}$$

where y_t, π_t and i_t denote respectively aggregate output, the rate of inflation and the nominal interest rate and $\mathbb{E}_t[\cdot]$ denotes the conditional expectation operator. Parameter $\beta \in (0, 1)$ is the discount factor, $\kappa \geq 0$ denotes the slope of the Phillips curve and ϕ_π is the degree of aggressiveness of monetary policy to inflation. The random shock $z_{1,t}$ can be interpreted as a demand shock shifting the IS curve, whereas $z_{2,t}$ is a cost-push shock shifting the Phillips curve. For expositional purposes, we assume that the demand shock, $z_{1,t}$ is serially uncorrelated (as a benchmark case), while $z_{2,t}$ exhibits serial correlation. Assuming the Taylor principle holds ($\phi_\pi > 1$), the solution takes the form

$$Y_t = \Pi_{yz} Z_t,$$

where Π_{yz} is a 2×2 matrix with elements $\pi_{ij} \in \mathbb{R}$ that depends on the structural parameters and the persistence of the cost-push shock (ρ). The vector $Y_t = (y_t, \pi_t)'$ collects the two endogenous variables and $Z_t = (z_{1,t}, z_{2,t})'$ is the vector of the two structural shocks which is assumed to follow an autoregressive process of the form

$$Z_t = RZ_{t-1} + \varepsilon_t \quad \text{with} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix} \quad \text{and} \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

where ε_t is a zero mean weak noise and where we impose the normalization $E(\varepsilon_t \varepsilon_t') = I_2$.

Our problem is then to identify the free elements of matrix Π_{yz} without imposing the cross-equation restrictions created by the NK model together with the autoregressive parameter ρ of the supply shock. We then seek to identify the vector of five parameters $\psi =$

$\{\rho, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}\}$ from the auto-covariance function of y_t and π_t. Each element of the vector Y_t can be expressed as a linear combination of the innovations of the shocks as

$$y_t = \pi_{11}\varepsilon_{1,t} + \pi_{12}\sum_{i=0}^{\infty}\rho^i\varepsilon_{2,t-i}$$
$$\pi_t = \pi_{21}\varepsilon_{1,t} + \pi_{22}\sum_{i=0}^{\infty}\rho^i\varepsilon_{2,t-i}$$

from which the auto-covariances of y_t and π_t can easily be obtained. For instance the variance and auto-covariances of output express as (similarly for inflation)

$$\gamma_y(0) = \pi_{11}^2 + \frac{\pi_{12}^2}{1-\rho^2},$$
$$\gamma_y(h) = \rho^h\frac{\pi_{12}^2}{1-\rho^2} \quad \text{for} \quad h > 0.$$

Note that computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for any $h > 0$ allows to immediately identify ρ. Given ρ, the knowledge of any $\gamma_y(h)$ for $h > 0$ is sufficient to identify π_{12} (up to its sign). Then, π_{11} (up to its sign) straightforwardly obtains from $\gamma_y(0)$. Using the same approach with the auto-covariance function of inflation identifies π_{21} and π_{22}. It is worth noting that when $\rho = 0$, so the two shocks $z_{1,t}$ and $z_{2,t}$ display the same dynamic properties and the parameters π_{ij} are not identified. Indeed, in this case, the model reduces to

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

which is not identifiable from the covariance matrix of Y_t (3 moments to identify 4 parameters). This example therefore illustrates in what sense the dynamic structure of exogenous forcing variables is key to identify the state–space representation.

3.3.2 R is a Lower Triangular Matrix

We now extend the autoregressive matrix R to the case of lower triangular shape, hence relaxing its diagonal feature. Let us define the matrix

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_{z1}} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{n_{z2}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{n_{zL}} \end{bmatrix},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$, and n_{zl} be positive integers for $l = 1, \ldots, L$ such that $n_{z1}+\ldots+n_{zL} = n_z$ and $n_{zl} = m(\lambda_l)$ denotes the multiplicity of eigenvalue λ_l.

Proposition 4 *If R is a lower triangular matrix with the same main diagonal as Λ and all elements on the first sub-diagonal are different from zero, i.e. $r_{i+1,i} \neq 0$ for $i = 1, \ldots, n_z - 1$, the state-space model is locally identifiable.*

Proof : See Appendix E.

Proposition 3 indicates that, contrary to the strict diagonal case, the lower triangular case does not require all diagonal elements to be different to permit identification. Should there be some identical elements on the diagonal, they should appear in consecutive order —*i.e.* all gathered in one of the matrices $\lambda_\ell I_{n_{z\ell}}$. The proposition also implies that while not all elements below the diagonal have to be non-zero, those lying on the first sub-diagonal have to be non zero. Again, intuitively, the condition for proper local identification is that all shocks should lead to distinguishable dynamics (as captured for example by IRF), hence some elements have to be distinct to guarantee that different shocks generate different dynamics.

3.3.3 R is a Symmetric Matrix

For simplicity, we restrict the presentation the case with two shocks. The matrix R is symmetric with the same value on the diagonal, and the same value on the anti-diagonal. While such a configuration may appear as a curiosity at first sight, it is actually of very practical interest in the international macroeconomic literature since the seminal work of Backus et al. (1992) [12]. For this specification, the autoregressive matrix takes the following form:

$$R = \begin{bmatrix} \rho & \tau \\ \tau & \rho \end{bmatrix} \text{ with } \tau \neq 0. \tag{17}$$

The previous non-zero restriction on τ is critical. When $\tau = 0$, matrix (17) reduces to a diagonal matrix with identical elements and therefore does not satisfy Proposition 2. Provided $\tau \neq 0$, the following proposition holds.

Proposition 5 *For the 2×2 symmetric matrix R (eq. 17), the state-space model is locally identifiable.*

In particular, and contrary to the diagonal case, we show in the Appendix F that the diagonal elements must necessarily be identical for local identification of the state-space model.[13]

3.3.4 Partial Identification.

Finally, Proposition 6 establishes that the model is partially identifiable. In other words, even though some shocks may not be identified, it is still possible to identify a subset (Online Appendix B provides a simple illustrative example).

Proposition 6 *In the case of a block diagonal organisation of the R matrix with one block corresponding to one of the preceding cases, the state–space model is partially locally identifiable for this block.*

Proof : See Appendix G.

[12]This dynamic structure of autoregressive matrix has widely been used, among others, by Backus et al. (1994), Baxter and Crucini (1995), Heathcote and Perri (2002) and Kehoe and Perri (2002).

[13]Appendix G offers an illustration of the symmetric autoregressive matrix, R, with an application to the international transmission of shocks between the US and the Euro area.

4 Estimation and Inference

This section discusses estimation and inference in the D-SVAR approach and shows how this approach can be used to evaluate DSGE and SVAR models.

4.1 Estimation

To simplify the exposition, let us consider the simplified state space representation[14]

$$\begin{aligned} Y_t &= \Pi_{yz} Z_t && (18) \\ Z_t &= R Z_{t-1} + \varepsilon_t \quad , && (19) \end{aligned}$$

where $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{nz}$. In the sequel, we will restrict ourselves to cases where the parameter vector $\psi = (vec(\Pi_{yz})', vec(R)')'$ is indeed locally identified, such that the conditions for the validity of Propositions 2-4 are satisfied. Vector ψ can thus be estimated either by Maximum Likelihood (ML) from (18)-(19) or equivalently by a two step Asymptotic Least Square (ALS) approach (See Corollary 1 below). Let us denote $\widehat{\psi}_T$ the ML estimator of ψ for a sample size T. Absent any unobserved variables, the D-SVAR representation rewrites as a VAR(1) model:

$$Y_t = \left(\Pi_{yz} R \Pi_{yz}^{-1} \right) Y_{t-1} + \Pi_{yz} \varepsilon_t \tag{20}$$

where the Π_{yz} is identified using the dynamic structure of unobserved structural shocks. So the loading matrix Π_{yz} is obtained without any restriction. Consider now the reduced-form VAR(1) representation :

$$Y_t = \Gamma Y_{t-1} + u_t \tag{21}$$

with $E(u_t) = 0$ and $E(u_t u_t') = \Sigma_u$. This implies that $u_t = \Pi_{yz} \varepsilon_t$.

Our D-SVAR representation imposes as many or more restrictions on the dynamic structure of the data Y_t than the unrestricted VAR model. This offers an opportunity to use the information contained in the parameters of the unrestricted estimated VAR model (21) to estimate ψ in the D-SVAR (18)-(19). Let us define $\eta = (vec(\Gamma)', vec(\Sigma_u)')'$ and the binding function $\widetilde{\eta}(\psi)$. We derive a version of the Corollary in Gouriéroux and Monfort (1995) (Chap. 10, Section 10.4.2, Corollary 10.2) adapted to our D-SVAR.

Corollary 1 *Let $\widehat{\eta}_T$ be a consistent and asymptotically normal estimator of η from (21) and let $\eta(\psi)$ be the binding function. The ALS estimator $\widehat{\psi}_T$ obtained by solving*

$$\min_{\psi} \left[\widehat{\eta}_T - \widetilde{\eta}(\psi) \right]' S_T \left[\widehat{\eta}_T - \widetilde{\eta}(\psi) \right] \quad , \tag{22}$$

where S_T is an estimator of the inverse of the asymptotic covariance matrix of $\widehat{\eta}_T$, is a consistent estimator of ψ and is asymptotically equivalent to the ML estimator $\widehat{\psi}_T$ of ψ obtained from (18)-(19).

[14] This discussion can be extended to a VAR(p) or VARMA(p,q) representation.

Intuitively, Corollary 1 states that estimating the parameter vector ψ in one direct step (ML approach) or relying on a two step procedure using η as an auxiliary parameter (ALS approach) yields asymptotic equivalent estimator of ψ. The two-step estimation uses the constraints $\Gamma = \left(\Pi_{yz} R \Pi_{yz}^{-1}\right)$ and $\Sigma_u = \Pi_{yz}\Pi'_{yz}$ to uncover the elements of the R and Π_{yz} matrices. This corollary therefore illustrates the strong connection between our approach and standard VAR modelling.

Consider now the DSGE model that underlies System (18) and (19). This DSGE model imposes cross-equation restrictions on the elements of the $\Pi_{yz}(\theta)$ matrix together with those contained in the $R(\theta)$ matrix. Let us define the binding function $\widetilde{\psi}(\theta)$ that expresses the vector of state-space parameters, ψ, as a function of θ. Under the conditions provided in Komunjer and Ng (2011), vector of parameters θ is also identifiable and can thus be estimated by ML, a usual practice in the applied macroeconomic literature. Let us denote by $\widehat{\theta}_T$ the ML estimator of θ. Because our D-SVAR imposes less restrictions on the state-space representation (18) and (19) than the DSGE model and provided dim $\theta <$ dim ψ, vector ψ can be used as an auxiliary parameter to estimate θ. The following corollary, again adapted from Gouriéroux and Monfort (1995), holds.

Corollary 2 *Let $\widehat{\psi}_T$ be the ML estimator of ψ from the unconstrained state-space version of the representation (18) and (19) and $\widetilde{\psi}(\theta)$ the binding function. The estimator $\widetilde{\theta}_T$ obtained by solving*

$$\min_{\theta} \left[\widehat{\psi}_T - \widetilde{\psi}(\theta)\right]' S_T \left[\widehat{\psi}_T - \widetilde{\psi}(\theta)\right] \quad ,$$

where S_T converges to the inverse of the asymptotic covariance matrix of $\widehat{\psi}_T$ which is given by the information matrix $\mathcal{I}(\psi)$ of the log-likelihood function, is asymptotically equivalent to the ML estimator $\widehat{\theta}_T$ of θ obtained from the constrained state-space version of the representation (18) and (19)

Estimating θ is one step or by a two step procedure using ψ as an auxiliary parameter yields asymptotically equivalent estimator of θ. Our unrestricted state-space representation features less restrictions than the DSGE model and thus contains potentially useful information about the relevance of the structural restrictions imposed by the DSGE model. This corollary illustrates the tight relationship between the DSGE model and the D-SVAR.

4.2 Inference

The D-SVAR offers the possibility to conduct statistical inference both on DSGE and SVAR models. Let us first consider the DSGE and our D-SVAR, and let us remind the reader that the D-SVAR imposes no cross-equation restrictions during estimation. Using ML estimates of the two models, it is then possible to test the relevance of the cross-equation restrictions imposed by the DSGE model, and therefore guide modelling. Let us consider the null hypothesis that the DSGE model can mimic the unconstrained state-space representation (18) and (19),

$H_0 : \psi = \widetilde{\psi}(\theta)$. A Wald-type statistic to test for this null hypothesis is then given by:

$$W_T = T(\hat{\psi}_T - \widetilde{\psi}(\hat{\theta}_T))'\mathcal{I}(\hat{\psi}_T)(\hat{\psi}_T - \widetilde{\psi}(\hat{\theta}_T)) \tag{23}$$

which is asymptotically distributed as χ^2 with $p-q$ degrees of freedom under the null hypothesis, where $p = \dim \psi$ and the assumption that $rank\left(\frac{\partial\psi(\theta)}{\partial\theta'}\right) = \dim \theta = q$, where $\mathcal{I}(\hat{\psi}_T)$ is an estimator of the information matrix evaluated at the unconstrained estimator $\hat{\psi}_T$ (see Gouriéroux and Monfort (1995), Section 17.4.1).[15] A score test and a Likelihood Ratio test can also be constructed (see Gouriéroux and Monfort (1995), Section 17.4.2 and Section 17.4.3).

As a particular but interesting case, a test for the equality of the loading matrix F (restricted (DSGE) and unrestricted (D-SVAR)) can also be performed. The associated statistic is given by

$$W_T^F = T\left(vec(\hat{\Pi}_{yz,T}) - vec(\tilde{\Pi}_{yz}(\hat{\theta}_T)\right)'\mathcal{I}^{11}(\hat{\psi}_T)\left(vec(\hat{\Pi}_{yz,T}) - vec(\tilde{\Pi}_{yz}(\hat{\theta}_T))\right)$$

where $vec(\Pi_{yz})$ is a p_1-vector. $\mathcal{I}^{11}(\hat{\psi}_T)$ is an estimator of the corresponding block to Π_{yz} of the information matrix and $\tilde{\Pi}_{yz}(\theta)$ is the binding function linking θ to the loading matrix Π_{yz}. Under the assumption that $rank\left(\frac{\partial vec(\tilde{\Pi}_{yz}(\theta))}{\partial\theta'}\right) = q_1$ with $q_1 < p_1$ this statistics is asymptotically distributed as a χ^2 with $p_1 - q_1$ degrees of freedom under the null hypothesis. Since, the loading matrix Π_{yz} collects the impact response of each variable to each shock, this test immediately assesses the relevance of the structural restrictions. Inspecting point by point the impact responses and/or the overall dynamic responses is also straightforward.

Let us now consider the D-SVAR and VAR models. First and foremost, a specification test (J-test) can be performed in the case of over-identification, *i.e.* when the dimension of the vector η is greater than the dimension of ψ, by multiplying the objective function (22) by the number of observations. This allows to assess whether the dynamic restrictions imposed by the D-SVAR model are satisfied and hence evaluate the reliability of our D-SVAR approach regarding an unconstrained VAR model.

The D-SVAR also offers the opportunity to assess the relevance of various identification schemes used in SVAR modelling. In particular, it allows to test general null hypotheses on the loading matrix Π_{yz}. For example, one may be interested in the relevance of the timing imposed by short-run restrictions. In this case, the null hypothesis writes $H_0 : Hvec(\Pi_{yz}) = 0$, where the selection matrix H is such that the elements above the diagonal of Π_{yz} are all equals to zero. Likewise, matrix H can be adapted to test for long-run restrictions à la Blanchard and Quah (1989). A Wald statistic can then be computed with a consistent estimator of the appropriate variance covariance matrix. Likewise, one may be interested in testing for the dynamic response to a particular shock, as identified using two competing identification schemes. This can be achieved with a Wald statistic and using the appropriate asymptotic distribution or by bootstrapping techniques as proposed by Inoue and Kilian (2016).[16]

[15]Under possible misspecification, the estimator of the information matrix $\mathcal{I}(\hat{\psi}_T)$ can be replaced by an estimator of the inverse of the sandwich formulae $\mathcal{J}(\hat{\psi}_T)^{-1}\mathcal{I}(\hat{\psi}_T)\mathcal{J}(\hat{\psi}_T)^{-1}$ where $\mathcal{I}(\hat{\psi}_T)$ is an estimator of the variance covariance matrix of the score and $\mathcal{J}(\hat{\psi}_T)$ an estimator of minus the second derivative of the log-likelihood function.

[16]In particular, in the case when the number of structural impulse responses exceeds the number of the VAR

5 Testing for SVARs Restrictions: Two Examples With Monetary Policy Shocks

In this section we revisit two seminal papers that both proposed to identify monetary policy shocks through the lens of the D-SVAR. The first one, Gertler and Karadi (2015), relies on an external instrument to identify the shock — the so-called proxy VAR approach. The second, Christiano et al. (1999), identifies a monetary policy shock by imposing zero restrictions on its impact effect on key economic variables.

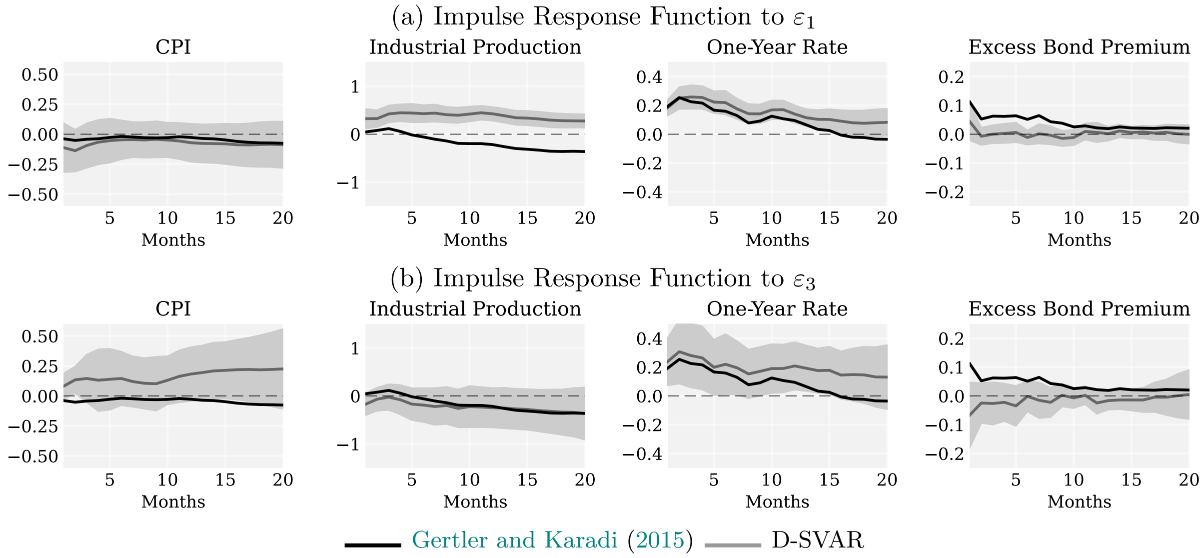
5.1 Revisiting Gertler and Karadi (2015) Proxy VAR

In this section, we revisit Gertler and Karadi (2015), who identified a monetary policy shock relying on a proxy-VAR approach with an external instrument (see Beaudry and Saito (1998), Stock (2008), Stock and Watson (2012) and Mertens and Ravn (2013) among others). Such an approach avoids imposing timing restrictions on both the behavior and the impact of the policy rate. The instrumental variable needs to satisfy two assumptions to identify a given structural shock: *i)* the instrument must be *relevant*, *i.e.* the contemporaneous correlation between the structural shock and the external instrument must be non-zero; *ii)* the instrument must be *exogenous*, *i.e.* the instrument must be uncorrelated with the other structural shocks. According to Gertler and Karadi (2015), an advantage of proxy VARs is that it does not impose a special organization (and thus restrictions) of the loading matrix F. This is also the case in our D-SVAR, and it is therefore interesting to compare the two approaches.

We first replicate Gertler and Karadi (2015) and estimate a VAR featuring the log consumer price index, the log industrial production, the one year government bond rate, and Gilchrist and Zakrajšek (2012) excess bond premium. The data are evaluated at the monthly frequency for the period running from July 1979 to June 2012. Following Gertler and Karadi (2015), the unrestricted VAR includes 12 lags. We then recover the impulse response function of these variables to a monetary policy shock identified relying on the proxy VAR approach where, like Gertler and Karadi (2015), the external instrument is the surprise in the three month ahead futures rate. The identified contractionary monetary policy shock shifts the one-year rate upward, decreases economic activity after one year, raises the excess bond premium persistently, which signals the presence of financial frictions, and leads to a very small negative response of the CPI, without exhibiting any price puzzle (see black lines in Figure 3).

Then we proceed to estimating our D-SVAR by an ALS approach, using the unrestricted VAR as auxiliary model. As in the Blanchard and Quah (1989) example of Section 2.3, the autoregressive matrix R is assumed to be diagonal. We recover four unlabelled structural shocks that we arbitrarily index $\varepsilon_1, \ldots, \varepsilon_4$. Is one of these shocks the Gertler and Karadi (2015) monetary policy shock? We actually find that ε_1 and ε_3 are two shocks for which Gertler and Karadi (2015) instrument would be valid, as the p-values for non-zero correlation between

parameters, the asymptotic distribution of a Wald statistic is degenerated but the bootstrap test can still be implemented following the transformation of the statistic.



The black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR with diagonal R matrix. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

Figure 3: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks ε_1 and ε_3

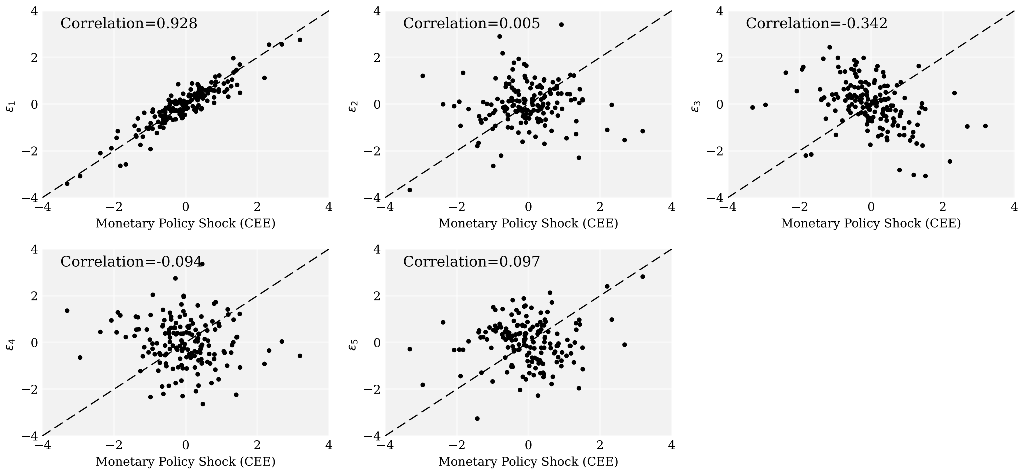
these two shocks and the instrument are less than 5%, whereas they are above 5% for ε_2 and ε_4. Through the lens of model with four AR(1) shocks, this shows that in Gertler and Karadi (2015), the instrument identifies a shock that is a combination of ε_1 and ε_3, and not a single exogenous monetary policy shocks. Figure 3 plots the responses of the four variables to the two shocks ε_1 and ε_3, together with the responses to the Gertler and Karadi (2015) shock.

The shocks ε_1 gives a very similar response of the one-year government bond rate as compared to Gertler and Karadi (2015). Although the response is more persistent, the response of the one-year government bond rate is also pretty similar. Inspection of the IRF for the other variables reveals that none of these two shocks gives similar responses for the four variables. The credit channel narrative of Gertler and Karadi (2015) requires an increase in the excess bond premium, which is obtained with ε_1, but comes with a persistent increase in industrial production. For the shock ε_3, there is indeed an increase in the one-year rate, a decrease in industrial production but an increase in inflation and a decrease in the excess bond premium, which does not square with the credit channel narrative of Gertler and Karadi (2015).

5.2 Revisiting Christiano et al. (1999) Impact Restrictions SVAR

This section revisits Christiano et al. (1999) who identifies a monetary policy shock by means of impact restrictions.[17] We ask whether the obtained macroeconomic dynamics following a monetary policy shock can be uncovered with a D-SVAR. The D-SVAR makes no assumptions on the impact responses, but only assumes that underlying shocks follow mutually orthogonal

[17] See also Christiano et al. (2005).



ε_1 to ε_5 are the structural shocks as obtained with the D-SVAR. Sample is 1965Q1-2007Q4.

Figure 4: Correlation between Christiano et al. (1999) Monetary Shock and D-SVAR Shocks

AR(1) processes. If Christiano et al. (1999) approach identifies a monetary shock, then we should recover it using our D-SVAR approach —under the assumption that this shock is an independent AR(1) shock.

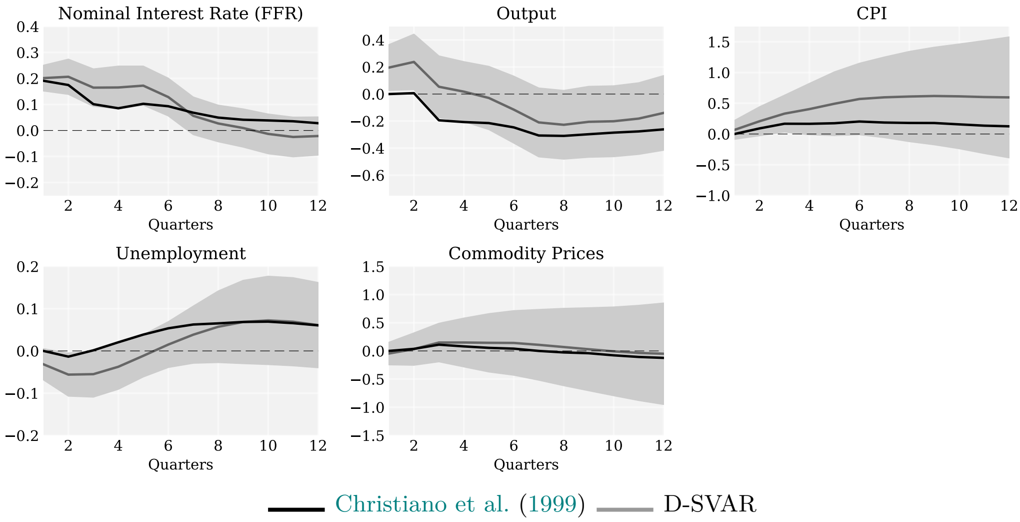
We estimate a VAR featuring real GDP, the unemployment rate, CPI inflation, commodity price inflation and the federal funds rate, in that order, for the period 1965Q1-2007Q4.[18] The VAR includes four lags. We then recover the impulse response function of these variables to a monetary policy shock identified by impact restrictions as in Christiano et al. (1999): the monetary policy shock corresponds to the shock that shifts the federal funds rate while leaving the other variables unaffected on impact. As well known, the identified contractionary monetary policy shock decreases output with a lag, and increases unemployment after a few quarters. Prices increase for about two years, which is known as the price puzzle, and fall below their initial level after this initial phase.

Then we proceed to estimating our D-SVAR by an ALS approach, using the unrestricted VAR as auxiliary model. The J-test indicates that the restrictions imposed by the D-SVAR are not rejected by the data. We recover five unlabelled structural shocks, $\varepsilon_1, \ldots, \varepsilon_5$. Is the Christiano et al. (1999) monetary shock one of these shocks? As illustrated in Figure 4, ε_1 is the only shock that exhibits strong positive correlation with Christiano et al. (1999) monetary policy shock shock. Moreover, one cannot reject that this correlation be 1 (p-value=0.986), while one cannot reject that all other shocks are uncorrelated with Christiano et al. (1999) monetary policy shock (p-value<0.035).[19]

Figure 5 plots the responses to ε_1 (grey lines), together with the responses to the Christiano et al. (1999) monetary policy shock. The figure indicates that the dynamics following a Christiano et al. (1999) monetary policy shock and those following ε_1 are very similar. Just

[18]We end the sample period in 2007Q4 to avoid dealing with the zero lower bound, which would require an explicit non-linear modelling of the dynamics of the nominal interest rate to account for the presence of an occasionally binding constraint (see *e.g.* Mavroeidis (2021)).

[19]See Table 4 in Online Appendix F.



On the five panel, the black line is the response to a monetary policy shock, as identified following Christiano et al. (1999). The grey line is the response to shock ε_1 in the D-SVAR. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 5: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_1

like for Christiano et al. (1999) monetary policy shock, a tightening of monetary policy leads to a prolonged recession with output (resp. unemployment) reaching it trough after about seven (resp. nine) quarters, and eventually reverting back in the longer run. Accordingly, both output and unemployment exhibit persistent hump shaped dynamics to the shock, just like in the aftermaths of a Christiano et al. (1999) monetary policy shock. Just like in Christiano et al. (1999), prices exhibit a persistent price puzzle, although it is more severe. Such a "price puzzle" is reminiscent of the results in Beaudry et al. (2020), and can be rationalized in a model with a flat Phillips curve and a cost channel.

All in all, the response of the economy is very much in line under the two identification schemes, and hence so are the forecast error variance decomposition (see Table 5 in Appendix F).[20]

Under the D-SVAR, we can go a step further and test if ε_1 satisfies the zero impact restriction imposed by Christiano et al. (1999). Table 2 reports the p-value associated to the zero impact effect of ε_1 on, respectively, output, the CPI, unemployment and the commodity price. The results do not reject the restriction imposed by Christiano et al. (1999) as none of the p-values lies below 25%, well above the 5% standard level. Furthermore, we do not reject (p-value is 32%) that the responses of output and unemployment are jointly zero. This exercise shows that the D-SVAR recovers a shock that is quite similar to the Christiano et al. (1999) monetary policy one, although there is a more pronounced "price puzzle". It also shows that the Christiano et al. (1999) zero restrictions, which have been criticised as often not compatible with a DSGE model (*e.g.* Carlstrom et al. (2009)), are not rejected by our D-SVAR.

[20]Figures 10–13 in Appendix F report the IRF to ε_2, ε_3, ε_4 and ε_5. Inspection of the figure indicate that none of these shocks is a good candidate to a Christiano et al. (1999) monetary policy shock.

Table 2: Test of Zero Impact Restriction (p-values)

	Output	CPI	Unemp.	Com. Price
ε_1	0.2470	0.6587	0.3890	0.8169

ε_1 is the D-SVAR structural shock that is the closest to the Christiano et al. (1999) monetary policy shock. Sample is 1965Q1-2007Q4.

6 Conclusion

In this paper, we have shown that one can identify structural shocks in a SVAR under the identifying assumption that the economy shares the dynamic structure of the vast majority of DSGE models. To put it loosely, if the economy is moved by exogenous variables that follow mutually orthogonal AR(1) processes (or more general specification of the autoregressive matrix), then a D-SVAR will allow for the identification of structural shocks, without the need for zero-impact, long-run or sign restrictions nor instruments. We have given a formal proof for identification and have shown how to conduct estimation and inference with D-SVAR.

We have then applied our methodology to the evaluation of two approaches in the literature used to identify monetary shocks, namely the proxy-var strategy ofGertler and Karadi (2015) and the impact zero restriction strategy of Christiano et al. (1999). Our approach provides support for the Christiano et al. (1999) results but suggests that the proxy-var strategy used by Gertler and Karadi (2015) is likely confounding the effects of two distinct structural shocks.

References

Panos J. Antsaklis and Anthony N. Michel (1997): *Linear Systems*. McGraw-Hill College.

David K Backus, Patrick J Kehoe, and Finn E Kydland (1992): "International Real Business Cycles," *Journal of Political Economy*, 100(4), 745–775.

——— (1994): "Dynamics of the Trade Balance and the Terms of Trade: The J-Curve?," *American Economic Review*, 84(1), 84–103.

Jushan Bai and Peng Wang (2015): "Identification and Bayesian Estimation of Dynamic Factor Models," *Journal of Business & Economic Statistics*, 33(2), 221–240.

Marianne Baxter and Mario J Crucini (1995): "Business Cycles and the Asset Structure of Foreign Trade," *International Economic Review*, 36(4), 821–854.

Paul Beaudry, Chenyu Hou, and Franck Portier (2020): "Monetary Policy when the Phillips Curve is Locally Quite Flat," CEPR Discussion Papers 15184, forthcoming American Economic Journal: Macroeconomics.

Paul Beaudry and Makoto Saito (1998): "Estimating the Effects of Monetary Shocks: An Evaluation of Different Approaches," *Journal of Monetary Economics*, 42(2), 241–260.

Ben S. Bernanke and Kenneth N. Kuttner (2005): "What Explains the Stock Market's Reaction to Federal Reserve Policy?," *Journal of Finance*, 60(3), 1221–1257.

Olivier Jean Blanchard and Danny Quah (1989): "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review*, 79(4), 655–673.

Fabio Canova and Luca Sala (2009): "Back to Square One: Identification Issues in DSGE Models," *Journal of Monetary Economics*, 56(4), 431–449.

Charles T. Carlstrom, Timothy S. Fuerst, and Matthias Paustian (2009): "Monetary policy shocks, Choleski identification, and DNK models," *Journal of Monetary Economics*, 56(7), 1014–1021.

Lawrence J. Christiano, Martin Eichenbaum, and Charles L. Evans (1999): "Monetary Policy Shocks: What Have we Learned and to What End?," in *Handbook of Macroeconomics*, vol. 1, chap. 2, pp. 65–148. Elsevier.

——— (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113(1), 1–45.

Lawrence J. Christiano, Mathias Trabandt, and Karl Walentin (2010): "DSGE Models for Monetary Policy Analysis," in *Handbook of Monetary Economics*, ed. by B. M. Friedman, and M. Woodford, vol. 3 of *Handbook of Monetary Economics*, chap. 7, pp. 285–367. Elsevier.

Prosper Dovonon and Alastair R. Hall (2018): "The Asymptotic Properties of GMM and Indirect Inference under Second-order Identification," *Journal of Econometrics*, 205(1), 76–111.

John G. Fernald (2007): "Trend Breaks, Long-run Restrictions, and Contractionary Technology Improvements," *Journal of Monetary Economics*, 54(8), 2467–2485.

Jesús Fernández-Villaverde, Juan F. Rubio-Ramírez, Thomas J. Sargent, and Mark W. Watson (2007): "ABCs (and Ds) of Understanding VARs," *American Economic Review*, 97(3), 1021–1026.

Jordi Galí (2015): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, no. 10495 in Economics Books. Princeton University Press, second edn.

F. R. Gantmacher (1959): *The Theory of Matrices. Vols. 1, 2*, Translated by K. A. Hirsch. Chelsea Publishing Co., New York.

Mark Gertler and Peter Karadi (2015): "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics*, 7(1), 44–76.

Simon Gilchrist and Egon Zakrajšek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102(4), 1692–1720.

Gene H. Golub and Charles F. Van Loan (2013): *Matrix Computations*. The Johns Hopkins University Press, fourth edn.

Christian Gourieroux and Joann Jasiak (2022): "Dynamic Deconvolution and Identification of Independent Autoregressive Sources," *Journal of Time Series Analysis*.

Christian Gouriéroux and Alain Monfort (1995): *Statistics and Econometric Models*, Cambridge Books. Cambridge University Press.

Refet S Gürkaynak, Brian Sack, and Eric Swanson (2005): "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements," *International Journal of Central Banking*, 1(1).

Refet S Gürkaynak, Brian P Sack, and Eric T Swanson (2007): "Market-Based Measures of Monetary Policy Expectations," *Journal of Business & Economic Statistics*, 25(2), 201–212.

Jonathan Heathcote and Fabrizio Perri (2002): "Financial Autarky and International Business Cycles," *Journal of Monetary Economics*, 49(3), 601–627.

Roger A. Horn and Charles R. Johnson (2013): *Matrix Analysis*. Cambridge University Press, Cambridge; New York, 2nd edn.

Atsushi Inoue and Lutz Kilian (2016): "Joint Confidence Sets for Structural Impulse Responses," *Journal of Econometrics*, 192(2), 421–432.

Nikolay Iskrev (2010): "Local Identification in DSGE Models," *Journal of Monetary Economics*, 57(2), 189–202.

Christian Kascha and Karel Mertens (2009): "Business Cycle Analysis and VARMA Models," *Journal of Economic Dynamics and Control*, 33(2), 267–282.

Patrick J. Kehoe and Fabrizio Perri (2002): "International Business Cycles with Endogenous Incomplete Markets," *Econometrica*, 70(3), 907–928.

Lutz Kilian and Helmut Lütkepohl (2016): *Structural Vector Autoregressive Analysis*. Cambridge University Press.

Andrzej Kocięcki and Marcin Kolasa (2018): "Global Identification of Linearized DSGE Models," *Quantitative Economics*, 9(3), 1243–1263.

Ivana Komunjer and Serena Ng (2011): "Dynamic Identification of Dynamic Stochastic General Equilibrium Models," *Econometrica*, 79(6), 1995–2032.

Kenneth N Kuttner (2001): "Monetary Policy Surprises and Interest Rates: Evidence from the Fed Funds Futures Market," *Journal of monetary economics*, 47(3), 523–544.

J. Lindé, F. Smets, and R. Wouters (2016): "Challenges for Central Banks' Macro Models," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2 of *Handbook of Macroeconomics*, chap. 0, pp. 2185–2262. Elsevier.

Sophocles Mavroeidis (2021): "Identification at the Zero Lower Bound," *Econometrica*, 89(6), 2855–2885.

Ellen R. McGrattan (2010): "Measurement with Minimal Theory," *Quarterly Review*, pp. 2–13.

Karel Mertens and Morten O. Ravn (2013): "The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States," *American Economic Review*, 103(4), 1212–47.

Adrian Pagan and Tim Robinson (2019): "Implications of Partial Information for Econometric Modeling of Macroeconomic Systems," CAMA Working Papers 2019-41, Centre for Applied Macroeconomic Analysis, Crawford School of Public Policy, The Australian National University.

V.A. Ramey (2016): "Macroeconomic Shocks and Their Propagation," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2 of *Handbook of Macroeconomics*, chap. 0, pp. 71–162. Elsevier.

J D Sargan (1983): "Identification and Lack of Identification," *Econometrica*, 51(6), 1605–1633.

Christopher A Sims (1980): "Macroeconomics and Reality," *Econometrica*, 48(1), 1–48.

Frank Smets and Rafael Wouters (2007): "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 97(3), 586–606.

James Stock (2008): "What's New in Econometrics: Time Series," Lecture 7 short course lectures, NBER Summer Institute.

J.H. Stock and M.W. Watson (2016): "Dynamic Factor Models, Factor-Augmented Vector Autoregressions, and Structural Vector Autoregressions in Macroeconomics," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2 of *Handbook of Macroeconomics*, chap. 0, pp. 415–525. Elsevier.

James H. Stock and Mark Watson (2012): "Disentangling the Channels of the 2007-09 Recession," *Brookings Papers on Economic Activity*, 43(1 (Spring)), 81–156.

James H. Stock and Mark W. Watson (2017): "Twenty Years of Time Series Econometrics in Ten Pictures," *Journal of Economic Perspectives*, 31(2), 59–86.

Harald Uhlig (2005): "What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics*, 52(2).

Appendix

A Preliminaries: Controllability, Observability and Minimality

In the literature on linear systems, invariant finite-dimensional dynamical systems are described by the following state-space system:

$$S_{t+1} = AS_t + B\epsilon_t \tag{24}$$

$$Y_t = CS_t + D\epsilon_t \tag{25}$$

where the $n_s \times 1$ vector S_t denotes the state variables, the $n_y \times 1$ vector Y_t denotes the outputs, and the $n_\epsilon \times 1$ vector ϵ_t contains the inputs (see, for example, Antsaklis and Michel (1997) and Gouriéroux and Monfort (1995)). Note that this ABCD state-space representation differs from that of Fernández-Villaverde et al. (2007), but it allows us to define the concepts of controllability, observability, and minimality in accordance with the standard linear system literature.

The system (or the pair (A, B)) is considered controllable when any state S_t can be driven to the initial state in a finite number of steps for a given input sequence ϵ_t. To illustrate the concept of controllability, the state variables at time $t = k$ can be expressed as:

$$S_k = A^k S_0 + A^{k-1}B\epsilon_0 + A^{k-2}\epsilon_1 + \ldots + AB\epsilon_{k-2} + B\epsilon_{k-1}$$

where S_0 is the vector of initial states. The state S_k is said to be reachable at time k if, for some finite $t = 0 \leq k$, there exists an input sequence $\epsilon_t, t \in [0, k]$, that transfers the state from the origin at $t = 0$ to S_k at time $t = k$. This implies that the system is reachable if and only if the rank of the matrix $\mathbb{C} = \left[B, AB, A^2B, \ldots, A^{k-1}B\right]$, which contains the matrices related to the input sequence in the above system, is of full rank. Moreover, if the system is reachable, then it is also controllable.

According to the ABCD system representation (24) and (25), a formal definition of controllability is given by:

Definition 3 Controllability *: A system is controllable if and only if the controllability matrix* $\mathbb{C} = [B, AB, A^2B, \ldots, A^{n_s-1}B] \in \mathbb{R}^{n_s \times n_\epsilon n_s}$ *has full row rank i.e., rank*$(\mathbb{C}) = n_s$.

If a system is state observable, its present state can be determined from the knowledge of the present and future outputs Y_t and inputs ϵ_t. The expression for output Y_k can be rewritten as:

$$Y_k = CA^k S_0 + CA^{k-1}B\epsilon_0 + CA^{k-2}B\epsilon_1 + \ldots + CAB\epsilon_{k-2} + CB\epsilon_{k-1} + D\epsilon_k.$$

Observability refers to the ability to uniquely determine the state of a system from knowledge of current and future outputs and inputs. Specifically, a system is said to be observable if it is possible to determine the state at time $t = 0$ from knowledge of future system outputs Y_k and system inputs ϵ_k for $k \geq 0$. A state is considered unobservable if it remains indistinguishable from the zero state under any possible output observation, mathematically expressed as $CA^k S = 0$ for all $k \geq 0$. Conversely, a system is completely state observable if the only state that satisfies this condition is the zero state, $S = 0$. The degree of observability is determined by the rank of the matrices CA^k for all k, with full rank indicating complete observability.

A formal definition of observability is given by:

Definition 4 Observability: *A system is observable if and only if the observability matrix* $\mathbb{O}$ *has full column rank, i.e.,* $rank(\mathbb{O}) = n_s$*, where*

$$\mathbb{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n_s-1} \end{bmatrix} \in \mathbb{R}^{n_y n_s \times n_s}.$$

Theorem 1 *A state-space representation is minimal if and only if it is both controllable and observable.*

See Antsaklis and Michel (1997), Theorem 3.9, p. 395 for a proof of minimality under controllability and observability For a minimal state-space representation, the state vector is of the smallest possible dimension.

B Proof of Proposition 1

First, consider the case without endogenous state variables. In this scenario, $A = R$ and $\Pi = \Pi_{yz}$, which is of full rank. This implies that S_t follows a VAR(1) process given by $S_t = RS_{t-1} + \epsilon_t$, where ϵ_t has the same dimension as S_t. This directly implies that the system is controllable according to the definition above. Moreover, since $Y_t = \Pi_{yz} S_t$ and Π_{yz} is of full rank, the system is also observable, which implies that the system is minimal.
Second, consider the case where $n_y > 0$. The state variables at time $t = k$ can be expressed as:

$$S_k = A^k S_0 + A^{k-1} B\epsilon_1 + A^{k-2} B\epsilon_2 + \ldots + AB\epsilon_{k-1} + B\epsilon_k$$

where the matrices S, A, and B are defined by the system (10)-(11).

For controllability, it is sufficient to show that a submatrix $\mathbb{C}_k = \left[B, AB, \ldots, A^{k-1}B\right]$ has full rank for any k. Indeed, by the Cayley-Hamilton Theorem, if the submatrix $\mathbb{C}_k$ is of full rank, then the rank of any matrix $\mathbb{C}_n = [B, AB, A^2B, \ldots, A^{n-1}B]$ is equal to the rank of $\mathbb{C}_k$ for $n \geq k$. Since the matrix F is full rank, so is the matrix

$$\mathbb{C}_2 = \left[B, AB\right] = \begin{bmatrix} 0 & F \\ I & R \end{bmatrix}$$

which implies that $\mathbb{C}$ is also full rank. Therefore, the system is controllable.

Now, for observability, the expression for Y_k can be rewritten as:

$$Y_k = \Pi A^k S_0 + \Pi A^{k-1} B\epsilon_1 + \Pi A^{k-2} B\epsilon_2 + \ldots + \Pi AB\epsilon_{k-1} + \Pi B\epsilon_k.$$

according to the system (10)-(11). The corresponding matrix $\mathbb{O}$ is then given by:

$$\mathbb{O} = \begin{bmatrix} \Pi \\ \Pi A \\ \vdots \\ \Pi A^{n_s-1} \end{bmatrix}.$$

Under Assumption 2 iii), the rank of the matrix $(A', \Pi')'$ is full and equal to n_s which is a necessary and sufficient condition for observability (see Corollary 4.7 in Antsaklis and Michel (1997)). Therefore, this system is observable. By controllability and observability, the system is minimal.

C Proof of Proposition 2

Consider a general form for the matrix T such that $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$.

We first show that the relation $\tilde{A}T = TA$ implies that $T_{21} = 0$. Since $\tilde{A}$ is necessarily block upper triangular, $\tilde{R}T_{21} = T_{21}G$ which can be rewritten as:

$$[(I_{n_k} \otimes \tilde{R}) - (G' \otimes I_{n_z})]vec(T_{21}) = 0. \tag{26}$$

Matrix $\tilde{A}$ is *similar* to matrix A which implies that the eigenvalues of $\tilde{A}$ are the same than the eigenvalues of A.[21] Since A is block upper diagonal, for the set of eigenvalues of A denoted $\lambda(A)$, we have $\lambda(A) = \lambda(G) \cup \lambda(R)$. The same property holds for $\tilde{A}$, *i.e.* $\lambda(\tilde{A}) = \lambda(\tilde{G}) \cup \lambda(\tilde{R})$. Since the matrices A and $\tilde{A}$ have the same block upper triangular structure, this implies that the eigenvalues of the matrix $\tilde{R}$ are the same as the eigenvalues of the matrix R. Under Assumption $1''$, R and G share no common eigenvalues, this also holds for $\tilde{R}$ and G. The expression $[(I_{n_k} \otimes \tilde{R}) - (G' \otimes I_{n_z})]$ is then of full rank, Equation (26) holds only for $T_{21} = 0$.[22]

Now, for the block upper triangular matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & T_{22} \end{bmatrix}$, the equation $\widetilde{B} = TBU$ gives for the left lower block $I_{n_z} = T_{22}U$. Since U is orthonormal, this implies $T_{22} \equiv U^{-1} = U' = V$ an orthonormal matrix. The matrix T has the following form: $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & V \end{bmatrix}$. The result for the case where the state variables K_t are observed follows directly.

[21] If X is a square matrix and nonsingular, then A and $B = X^{-1}AX$ are similar and X is called a similarity transformation. If two matrices A and B are similar, they have the same eigenvalues, i.e. $\lambda(A) = \lambda(B)$, and the same number of independent eigenvectors but probably not the same eigenvectors (see Golub and Van Loan (2013) p. 349). Moreover, if X is an orthonormal matrix, i.e. $XX' = I$, A and B real matrices and $A = XBX'$, A is said to be real orthogonally similar to B (see Horn and Johnson (2013), p. 94).

[22] If λ is an eigenvalue of A and μ an eigenvalue of B, $\lambda - \mu$ is an eigenvalue of $(I \otimes A) - (B \otimes I)$ and all eigenvalues of $(I \otimes A) - (B \otimes I)$ is on this form. Thus $(I \otimes A) - (B \otimes I)$ has zero as an eigenvalue if and only if A has an eigenvalue λ and B has an eigenvalue μ such that $\lambda - \mu = 0$.

D Proof of Proposition 3

The only admissible orthonormal matrix V such that all elements (i,i) of the diagonal matrix R are identified is $V = I$ when the diagonal element $r_{i,i} \neq r_{j,j}$ for $\forall i \neq j$. For any other orthonormal matrix V, it is easy to verify that the resulting $\tilde{R} = VRV'$ matrix is not diagonal. Since V is a square matrix and is non singular, V corresponds to a *similarity transformation*, the eigenvalues of matrix $\tilde{R}$ are the same as the eigenvalues of matrix R which implies that the diagonal elements of $\tilde{R}$ are the same as the diagonal elements of R. Moreover, for diagonal matrices $\tilde{R}$ and R, the system of equations $\tilde{R}V - VR = 0$ leads to

$$
\begin{aligned}
(\tilde{r}_{i,i} - r_{i,i})V_{i,i} &= 0 \quad (\text{ for } i = 1, \ldots, n_z), &(27)\\
(\tilde{r}_{i,i} - r_{j,j})V_{i,j} &= 0 \quad (\text{for } i \neq j \text{ and } i,j = 1, \ldots, n_z), &(28)
\end{aligned}
$$

where $r_{i,j}$ and $\tilde{r}_{i,j}$ are respectively the element (i,j) of R and $\tilde{R}$.[23] The first set of equation implies that $\tilde{r}_{i,i} = r_{i,i}$ for diagonal elements of V which are different from zero. Since the diagonal elements of R (and then $\tilde{R}$) are different, the second set of equations implies necessarily that $V_{i,j} = 0$ for all $i \neq j$. Matrix V is necessarily diagonal and the only orthonormal matrix which is diagonal is the identity. The result also holds up to changes of sign and/or permutation of the identity matrix.

Assume now that some elements on the diagonal are the same. Denote the multiplicity of similar diagonal elements $r_{i,i}$ by $m(r_{i,i})$. One can first check that all diagonal elements are the same, in which case any orthonormal matrix V is admissible by equations (27) and (28). Now consider that a subgroup of elements has the same value. The elements in the V matrix corresponding to multiple values are not uniquely defined but only up to the post multiplication by an $m(r_{i,i}) \times m(r_{i,i})$ orthogonal matrix. Without lost of generality, suppose that the diagonal elements with the same value are ordered as the first $m(r_{1,1})$ elements on the diagonal and the other elements on the diagonal $r_{j,j}$ are different from $r_{1,1}$, then define the matrix V such that $V = \begin{bmatrix} V_{m(r_{1,1}) \times m(r_{1,1})} & 0 \\ 0 & I_{(n_z - m(r_{1,1}))} \end{bmatrix}$, where $V_{m(r_{1,1}) \times m(r_{1,1})}$ is an orthonormal matrix. Consequently, there exists an infinity of admissible V matrices such that $V_{m(r_{1,1}) \times m(r_{1,1})}$ is orthonormal. This argument can be generalised to more than one diagonal element with multiplicity. A sufficient condition for local identification is therefore that matrix R be diagonal with distinct diagonal elements $(r_{i,i} \neq r_{j,j},\ \forall i \neq j)$.

E Proof of Proposition 4.

By $\tilde{R} = VRV'$, one show that the only admissible matrix V which satisfies $\tilde{R}V = VR$ for lower triangular matrices $\tilde{R}$ and R is $V = I_{n_z}$. Since V is of full rank and orthonormal and $\tilde{R} = VRV'$, $\tilde{R}$ and R have the same eigenvalues. Moreover, the eigenvalues of a lower triangular matrix are

[23] The system of equations $\tilde{R}V - VR = 0$ is of the well known form $AX - BX = C$ in control theory called the Sylvester equation. For $C = 0$, this corresponds to the homogeneous Sylvester equation (see Gantmacher (1959), chap VIII).

the elements on the diagonal. By $\tilde{R}V = VR$, we have

$$[(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]vec(V) = 0. \tag{29}$$

This implies that $vec(V)$ belong to the null space of $A = [(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]$. Since $\tilde{R}$ and R have n_z common roots, the null space of A has a dimension equal to n_z. The system of equations (29) can be written more explicitly as:

$$\begin{bmatrix} (\tilde{R}-\mu_1 I) & -r_{2,1}I & -r_{3,1}I & \cdots & -r_{n_z,1}I \\ 0 & (\tilde{R}-\mu_2 I) & -r_{3,2}I & \cdots & -r_{n_z,2}I \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & (\tilde{R}-\mu_{n_z-1}I) & -r_{n_z,n_z-1}I \\ 0 & 0 & \cdots & 0 & (\tilde{R}-\mu_{n_z}I) \end{bmatrix} \begin{bmatrix} V_{[.,1]} \\ V_{[.,2]} \\ \vdots \\ V_{[.,n_z-1]} \\ V_{[.,nz]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

with $V_{[.,i]}$ a vector containing the i-th column of matrix V, μ_j for $j = 1, \ldots, n_z$ are the eigenvalues of R and the eigenvalues of $\tilde{R}$ and R are the same which are given by the elements on the diagonal. Under Proposition 3, the eigenvalues μ_j for $j = 1, \ldots, n_z$ are the same as the matrix Λ. Moreover, all sub-matrices $(\tilde{R} - \mu_j I)$ are lower triangular since $\tilde{R}$ is lower triangular and the entire matrix is a block upper triangular matrix.

This system of equations can be solved by forward substitution for the lower triangular block $(\tilde{R} - \mu_{n_z} I)$ to obtain the vector $V_{[.,n_z]}$ and by backward substitution for the upper block diagonal matrices. Thus, the lower triangular form of the system of equations $(\tilde{R} - \mu_{n_z} I)V_{[.,n_z]} = 0$ implies that the elements of $V_{[.,n_z]}$ are equal to zero except the last one V_{n_z,n_z}. Indeed the recursive form of the equations allows to rewrite the system of equations as $\sum_{j=1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 1, \ldots, n_z$, which imply that $V_{j,n_z} = 0$ except for V_{n_z,n_z} where $\hat{r}_{i,j}^{n_z}$ is the element (i, j) of the sub-matrix $(\tilde{R} - \mu_{n_z} I)$ and $\hat{r}_{n_z,n_z}^{n_z} = 0$ since R and $\tilde{R}$ share the same eigenvalues. According to Proposition 3, this holds under the weaker restriction that only the first sub-diagonal elements are different from zero which implies that $\hat{r}_{1,1}^{n_z} V_{1,n_z} = 0$ and $\sum_{j=i-1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 2, \ldots, n_z$.

Now, for the next upper block, we have

$$\begin{bmatrix} (\tilde{R} - \mu_{n_z-1} I) & -r_{nz,nz-1}I \end{bmatrix} \begin{bmatrix} V_{[.,n_z-1]} \\ V_{[.,n_z]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

using the preceding result above for $V_{[.,n_z]}$ yields $V_{j,n_z-1} = 0$ for all $j < n_z - 1$. We can continue to solve the following upper blocks by backward substitution to obtain that all elements of the matrix V above the diagonal are equal to zero: $(\tilde{R} - \mu_j I)V_{[.,j]} = 0$ for $j = n_z$ and $(\tilde{R} - \mu_j I)V_{[.,j]} = \sum_{i=j+1}^{n_z} (r_{i,j} I)V_{[.,i]}$ for $j = 1, \ldots n_z - 1$. The resulting matrix V has all elements above the diagonal equal to zero which implies that the only admissible orthonormal matrix is $V = I$ (up to changes of sign and/or permutation of the identity matrix).

F Proof of Proposition 5.

Consider the following general 2×2 symmetric matrix $R = \begin{bmatrix} \rho_1 & \tau \\ \tau & \rho_2 \end{bmatrix}$ where ρ_1, ρ_2 and τ are real numbers and $\tau \neq 0$. The eigenvalues are the roots of the following characteristic equation:

$$\begin{vmatrix} \rho_1 - \lambda & \tau \\ \tau & \rho_2 - \lambda \end{vmatrix} = (\rho_1 - \lambda)(\rho_2 - \lambda) - \tau^2 = \lambda^2 - \lambda(\rho_1 + \rho_2) + \rho_1\rho_2 - \tau^2.$$

The two roots can be written as $\lambda_{1,2} = \frac{1}{2}\left[\rho_1 + \rho_2 \pm \sqrt{(\rho_1-\rho_2)^2 + 4\tau^2}\right]$. Since $(\rho_1-\rho_2)^2+4\tau^2 > 0$, the two eigenvalues are necessarily real. By the expressions of λ_1 and λ_2, there exists an infinity of values for ρ_1, ρ_2 and τ that gives the same eigenvalues. In other words, for any orthonormal matrix V and $\tilde{R} = VRV'$, the matrices $\tilde{R}$ and R are similar and therefore have the same eigenvalues. The matrix R is then not identifiable.

Now consider the case where the elements on the diagonal have the same value, i.e. $\rho_1 = \rho_2 = \rho$. The two eigenvalues are now given by $= \lambda_{1,2} = \rho \pm \tau$, so that $\lambda_1 + \lambda_2 = 2\rho$ and $\lambda_1 - \lambda_2 = 2\tau$. This implies that there does not exist another 2×2 symmetric matrix $\tilde{R}$ with $\tilde{\rho} \neq \rho$ and/or $\tilde{\tau} \neq \tau$ with $\tilde{\tau} \neq 0$ having the same eigenvalues as matrix R. In this particular case, matrix R is then locally identifiable.

G Proof of Proposition 6.

We can consider cases with a block diagonal matrix R with blocks corresponding to the two preceding cases. For example $R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & R_2 \end{bmatrix}$, where $\tilde{R}$ is a $(n_{z1} \times n_{z1})$ diagonal matrix with different elements on the diagonal and R_2 is any $(n_{z2} \times n_{z2})$ matrix with $z_1 + z_2 = z$. In this case, matrix V has the form $V = \begin{bmatrix} I & 0 \\ 0 & V_{n_{z2}\times n_{z2}} \end{bmatrix}$, where $V_{n_{z2}\times n_{z2}}$ is an orthonormal matrix. The first z_1 exogenous processes are then locally identified. Matrix $\tilde{R}$ could be also lower triangular.

Additional Online Material

A Presence of Unobserved Endogenous State Variables

We investigate the the case of a vector X_t consisting of both observed, Y_t, and unobserved state variables, K_t (*i.e.* $X_t = (K_t, Y_t)'$). This implies, in particular, that Π_{yk} is not necessarily equal to the identity matrix, and, more importantly, that $\Pi_{yz} \neq 0$. The standard RBC model studied in McGrattan (2010) is a typical example of such a situation as the capital stock may not be directly observed (or at least not without measurement errors) by the econometrician. In that case, the model takes the form of System (2).

As for the previous case, we impose that Y_t has the same dimension as Z_t ($n_y = n_z$). We consider the case where Z_t consists of at least two elements and where there are not more state variables than elements in Z_t. For simplicity, we take K_t to be of the same order as Z_t and impose that G is of full rank. Moreover, we also assume that F, Π_{yk} and $\left(F', \Pi'_{yz}\right)'$ are of full rank. Note that this does not preclude Π_{yz} from being less than full rank and even possibly zero.

Making use of these assumptions, the dynamics of Y_t can be expressed as

$$Y_t = C_1 Y_{t-1} + C_2 Z_t + C_3 Z_{t-1} \tag{30}$$

where $C_1 = \Pi_{yk} G \Pi_{yk}^{-1}$, $C_2 = \Pi_{yz}$ and $C_3 = \Pi_{yk}(F - G\Pi_{yk}^{-1}\Pi_{yz})$. The issue is then whether, when R is diagonal (or lower triangular), C_1, C_2, C_3 and R can be identified.[24]

Making use of the Z_t process in (30), it is easy to show that the vector of observed variables Y_t follows a VARMA(2,1) process of the form

$$Y_t = \left(DRD^{-1} + C_1\right) Y_{t-1} - DRD^{-1} C_1 Y_{t-2} + C_2 \varepsilon_t + (D - DRD^{-1} C_2)\varepsilon_{t-1}.$$

where $D \equiv C_3 + C_2 R$. The matrix D be invertible under the assumption of linear independence of C_3 and C_2.

As in the previous case, counting the number of coefficients to uncover and the number of moments the VARMA(2,1) structure offers, we recover that imposing a lower triangular structure on R provides us with the right number of restrictions. Note that this does not generically guarantees identification, as the system that needs to be solved features a quadratic term implying that a pair of solutions generally arise. However, as long as R is sparser than

[24] Note that while G, F and Π_{yk} cannot (in general) be identified separately. This is however not an issue as far as the identification of the structural impulse responses is concerned as all that is needed is the identification of R and Cs. Indeed, we will identify $\Pi_{yk} G \Pi_{yk}^{-1}$, $\Pi_{yk} F$, Π_{yz} and R.

a triangular matrix, the system features more equations than unknowns. The order condition is then clearly satisfied (in fact, it is over-identified). As before, checking the rank condition is non trivial, and we follow another strategy in the paper to formally prove identification.

B Partial Identification: An Example

Let us consider the textbook 3-equation NK model developed in Example 1 in the main text and let us introduce a monetary policy shock, labeled $z_{3,t}$, that exogenously shifts the interest rate set by the monetary authority —*i.e.* $i_t = \alpha_\pi \pi_t + z_{3,t}$. To ease exposition, we assume that the two shocks $z_{1,t}$ and $z_{2,t}$ are not serially correlated, while third one $z_{3,t}$ is. As will be clear later, the two serially uncorrelated shocks cannot be separately identified, whereas the third one one can be identified as long as it is serially correlated. Just as in Example 1, assuming the Taylor principle holds, the model can be solved forward and yields the state-space representation

$$Y_t = \Pi_{yz} Z_t,$$

where Π_{yz} is the (3×3) matrix

$$\Pi_{yz} = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{33} & \pi_{32} & \pi_{33} \end{bmatrix}$$

The vector $Y_t = (y_t, \pi_t, i_t)'$ collects the three endogenous variables and $Z_t = (z_{1,t}, z_{2,t}, z_{3,t})'$ is the vector of the three structural shocks. Given our assumptions on the dynamics of the shocks, Z_t evolves as

$$Z_t = RZ_{t-1} + \varepsilon_t \quad \text{where} \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \quad \text{and} \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}.$$

ε_t is a zero mean weak white noise and $E(\varepsilon_t \varepsilon_t') = I_3$.

Similarly to Example 1, the moving average representation of output (likewise for the inflation rate and the nominal interest rate) writes

$$y_t = \pi_{11}\varepsilon_{1,t} + \pi_{12}\varepsilon_{2,t} + \pi_{13} \sum_{i=0}^{\infty} \rho^i \varepsilon_{3,t-i},$$

from which we get the output auto-covariance functions as

$$\gamma_y(0) = \pi_{11}^2 + \pi_{12}^2 + \frac{\pi_{13}^2}{1-\rho^2} \quad \text{when} \quad h = 0,$$
$$\gamma_y(h) = \frac{\pi_{13}^2 \rho^h}{1-\rho^2} \quad \text{when} \quad h > 0.$$

The persistence parameter ρ can be directly identified by computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for $h > 0$, which is free from any parameter π_{13}. Given ρ, the direct observation of $\gamma_y(h)$, for any $h > 0$, allows to recover parameter π_{13} (up to a sign term). In other words, the effect of a monetary policy shock on output is identified. Applying the same procedure on inflation and the nominal interest allows for the identification of π_{23} and π_{33} (up to a sign term). It is therefore possible to identify the monetary transmission mechanism for all variables in the D-SVAR model.

Note that, on the contrary, the knowledge of $\gamma_y(0)$ only helps identify the sum $\pi_{11}^2 + \pi_{12}^2$, not π_{11} and π_{12} separately, In other words, the effect of two remaining shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ cannot be separately identified. The reason for this result is that the dynamics of these two shocks do not bring any information to disentangle them. This result obviously extends to the inflation rate and the nominal interest rate. This example simply illustrates that one shock can be identified in so far as it displays a dynamic structure that differs from the other shocks in the economy.

C Data Appendix

C.1 Data from Section 2.3

- **Real GDP** is measured as the ratio of Gross Domestic Product in value (Table 1.1.5 from BEA) divided by the GDP price index (Table 1.1.4 from BEA), and is expressed in per capita term by dividing by the Civilian non-institutional population from 16 years of age and older residing in the 50 states and the District of Columbia (`CNP16OV` in the Federal Reserve DataBase (FRED, `http://fred.stlouisfed.org`)).

- The **unemployment gap** is measured as the difference between the average unemployment rate over a quarter (`UNRATE` from FRED) and the natural rate of unemployment — *i.e.* the rate of unemployment arising from all sources except fluctuations in aggregate demand (`NROU` from FRED).

C.2 Data from Section 5.1

The data from Section 5.1 are borrowed from Gertler and Karadi (2015) and are downloadable from `http://doi.org/10.3886/E114082V1`

- The **price level** is measured by the Consumer Price Index for all urban consumers (`CPIAUCSL` from FRED, All Items in U.S. City Average)

- **Economic activity** is measured by the industrial production index (`INDPRO` from FRED)

- The **nominal interest rate** is measured by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity (`GS1` fom FRED)

- The **credit spread** is measured by the Gilchrist and Zakrajšek (2012) excess bond premium.

- The **external instrument** corresponds to the three month ahead monthly fed funds futures (`FF4` from Gertler and Karadi (2015))

The price level and the economic activity index are both expressed in logs prior to estimation.

C.3 Data from Section 5.2

- The **nominal interest rate** is measured by the Effective Federal Funds Rate (`DFF` from FRED).
- **Output** is measured as the Real Gross Domestic Product expressed in Billions of Chained 2012 Dollars (`GDPC1` from FRED, Quarterly, Seasonally Adjusted Annual Rate).
- The **price level** is measured by the Consumer Price Index for all items for the United States (`CPALTT01USM661S` from FRED, Index 2015=100, Quarterly, Seasonally Adjusted).
- The **unemployment rate** corresponds to the quarterly average of the monthly unemployment rate in the US (`UNRATE` from FRED, Percent, Quarterly, Seasonally Adjusted)
- The **commodity price** is the Producer Price Index of all commodities (`PPIACO` from FRED, Index 1982=100, Quarterly).

Output, the Price level and the commodity price are all transformed by applying the log function prior to estimation.

D The Bivariate D-SVAR of Section 2.3

As is well-know (see, *e.g.* Fernald (2007)), Blanchard and Quah (1989) identification is sensitive to the long-run properties of the variables since it requires the estimation of the spectral density of, at least, one variable at frequency 0 —an object which is usually hard to estimate and at best very imprecise. This makes this approach quite non robust in case of trend breaks. The Great Financial Crisis (GFC) of 2008 presents the econometrician with such a challenge, as illustrated in Figure 6.

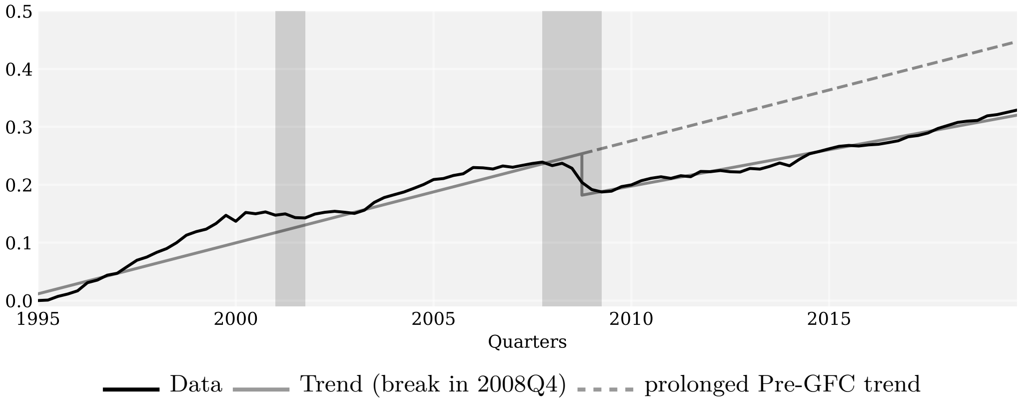
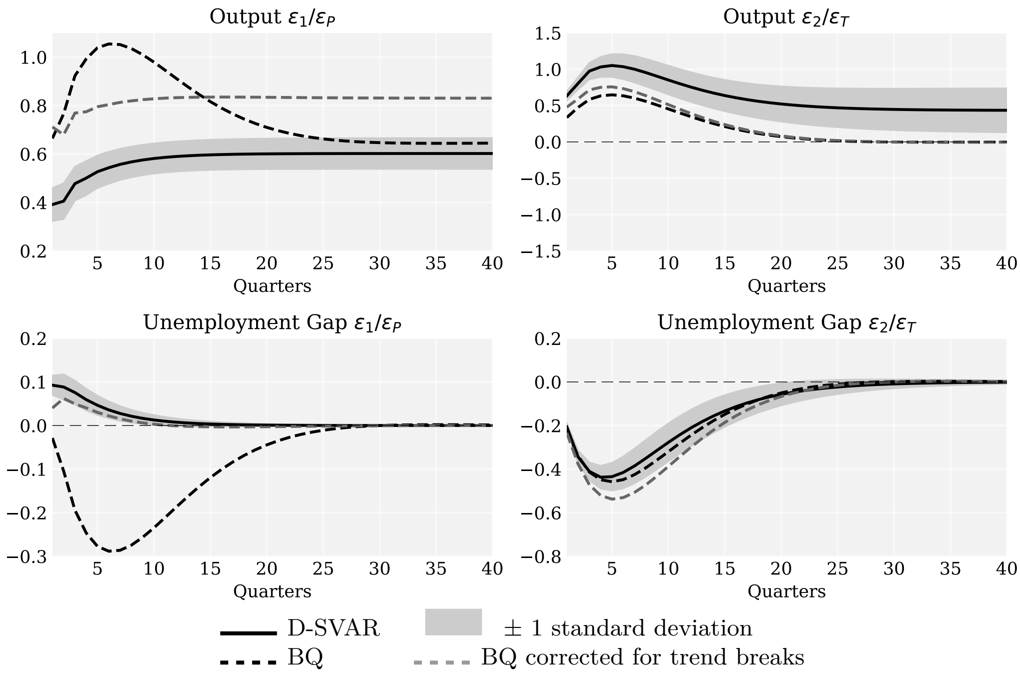


Figure 6: Output per Capita (in logs)



Sample is 1960Q1-20197Q4. y is the real GDP, u is the unemployment rate gap. Estimation is done with (Δy,u) using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

Figure 7: Impulse Response Functions: 1960Q1-2019Q4

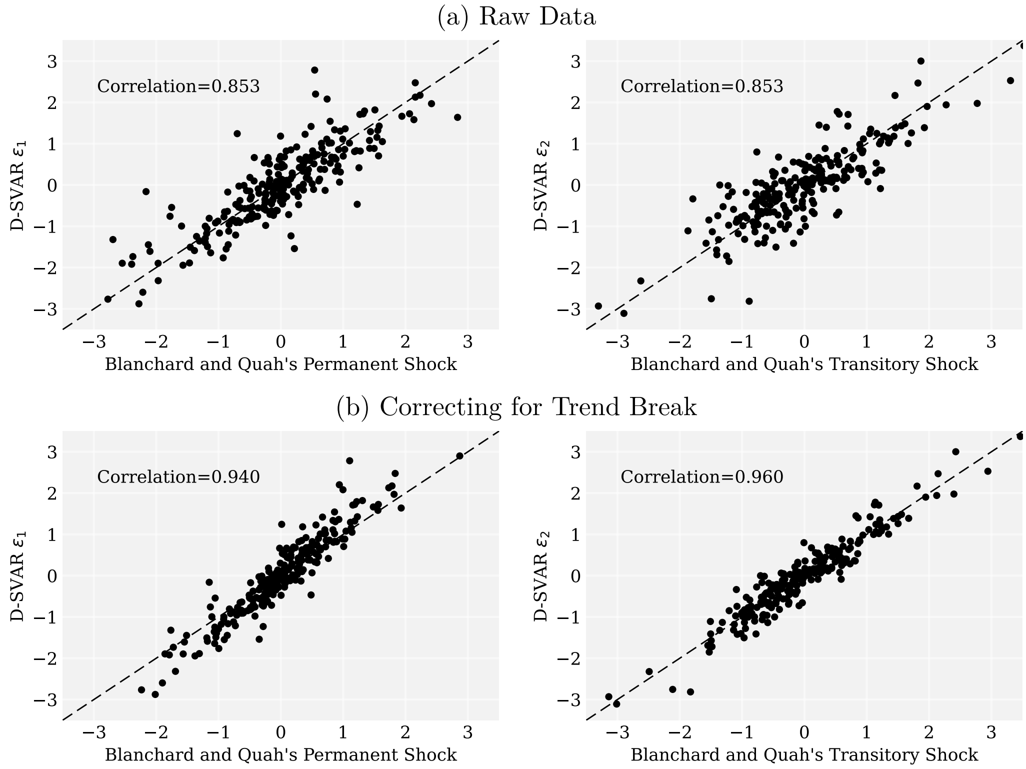
As illustrated in Figure 7 and Table 3, when the sample period is extended up to 2019Q4, the dynamic implications of the two shocks ε_1 and ε_2 essentially remain unaffected both for output and the unemployment gap. ε_1 can still be interpreted as the “permanent” shock, ε_2 as

Table 3: Forecast Error Variance Decomposition, Extended sample 1960Q1–2019Q4

	Output				Unemployment gap			
Horizon	ε_1	ε_2	ε_P	ε_T	ε_1	ε_2	ε_P	ε_T
1	27.7	72.3	79.7	20.3	16.8	83.2	1.6	98.4
4	20.6	79.4	72.6	27.4	4.7	95.3	17.0	83.0
8	21.9	78.1	73.7	26.3	2.6	97.4	25.0	75.0
20	32.6	67.4	81.5	18.5	2.0	98.0	28.5	71.5
∞	65.8	34.2	100.0	0.0	2.0	98.0	28.6	71.4

Sample is 1960Q1-2019Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap. ε_1 and ε_2 correspond to the D-SVAR, ε_P and ε_T to Blanchard and Quah (1989).

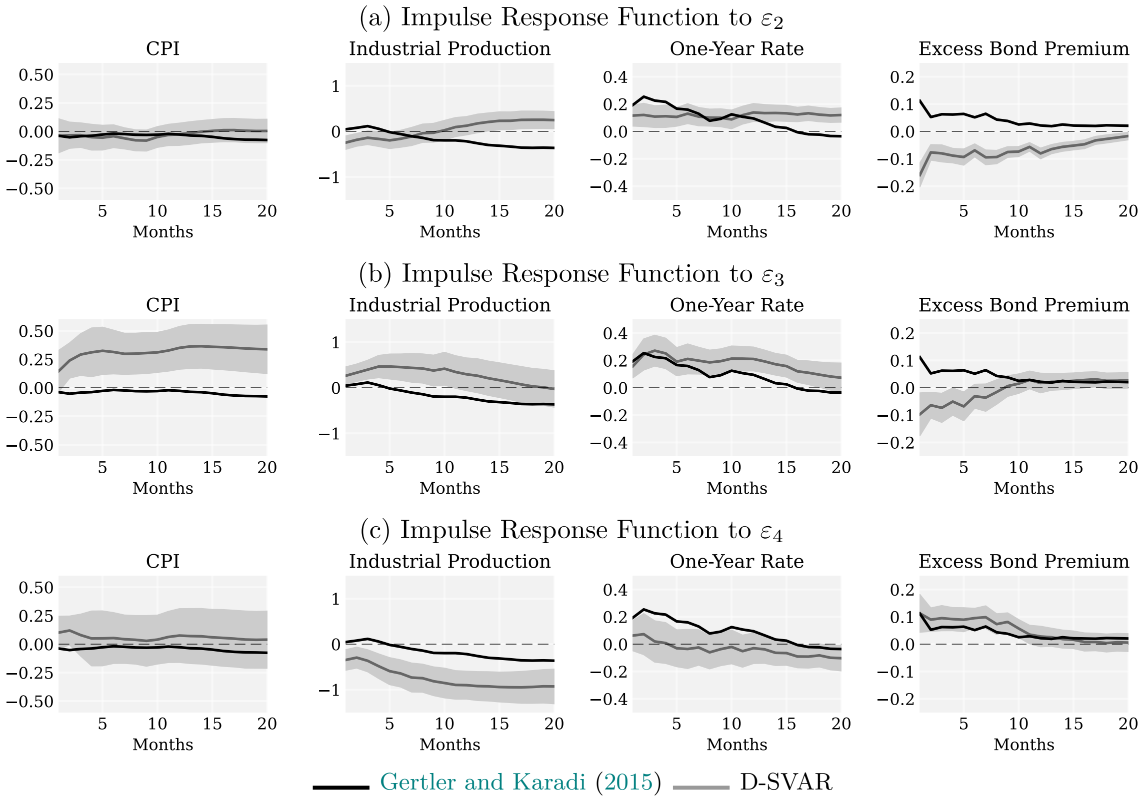
the "transitory" shock, and the shape of the response of both output and the unemployment gap to both shocks and the associated forecast error variance decomposition are essentially unaffected by the shift to the extended sample. Things are different in so far as the BQ decomposition is concerned. The black dash line in Figure 7 reports the dynamics of output and the unemployment gap to both the permanent and transitory shocks, as recovered by BQ's identification. As evident by comparing Figures 1 and 7 the response of both variables to the permanent shock are largely affected by the extension of the sample. More strikingly, this extension leads to a reversal in the respective contribution of the shocks to output dynamics: the permanent shock now becomes the main driver of output dynamics (see Panel (b) of Table 1). As soon as output dynamics is corrected for the trend break, the responses to a permanent shock resembles those to ε_1 in our VAR. Figure 8 shows that, correcting for trend breaks leads to an increase in the correlation between BQ shocks and the shocks as identified by the D-SVAR (from 0.85 to 0.95). The gains from the dynamic identification are then clear: by not directly relying on long-run restrictions, the D-SVAR is much less sensitive to breaks in variables featuring a trend.



Sample is 1960Q1-2019Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap.

Figure 8: Correlation between D-SVAR and BQ shocks: 1960Q1-2019Q4

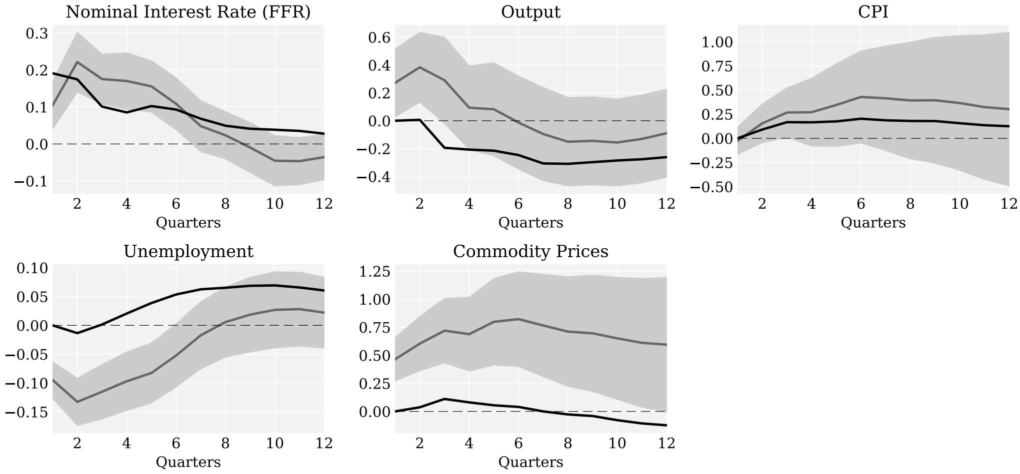
E Additional Material for Section 5.1



On the four panels, the black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR. Shaded area represent ± 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

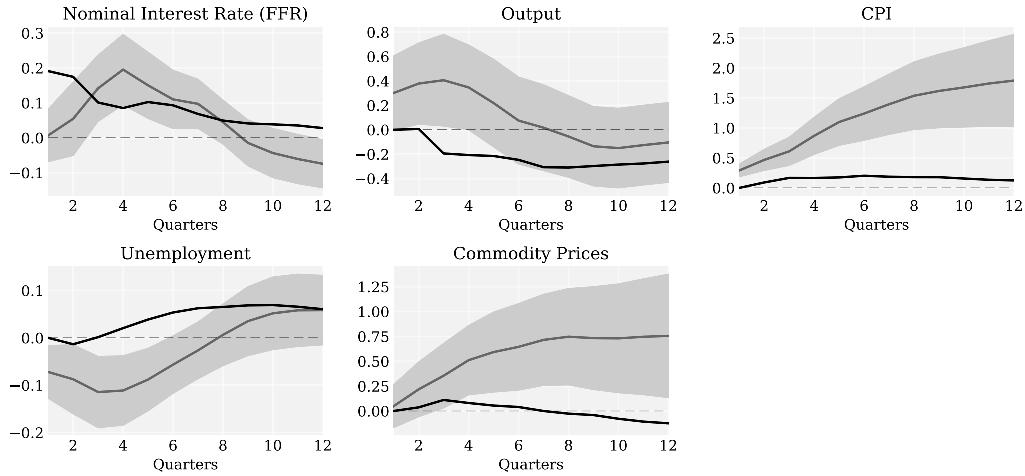
Figure 9: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks with Lower Triangular R Matrix (ε_2–ε_4)

F Additional Material for Section 5.2



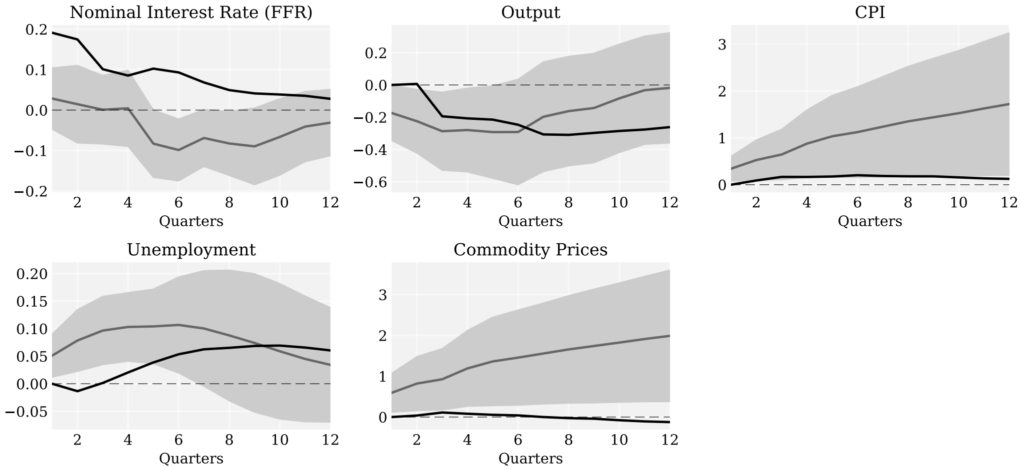
Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 10: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_2



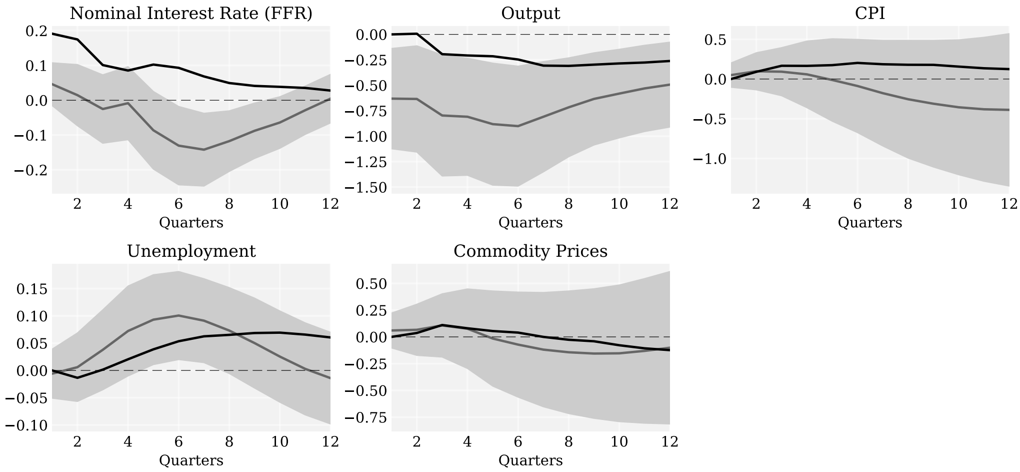
Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 11: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_3



Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 12: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_4



Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 13: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_5

Table 4: Correlation between CEE's MP Shock and DSVAR Shocks

	ε_1	ε_2	ε_3	ε_4	ε_5
Corr	0.928	-0.005	-0.342	0.094	0.097
p(0)	0.000	0.986	0.353	0.695	0.652
p(1)	0.665	0.013	0.015	0.035	0.011

Note: Corr reports the correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. p(0) (resp. p(1)) reports the p-value associated to the Wald test of a unit (resp. zero) correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. This p-value is obtained by means of 1,000 bootstrap, so as to accommodate for the fact that both shocks are generated by an estimated VAR model and are hence contaminated by the uncertainty surrounding the estimated coefficients of the VAR.

Table 5: Forecast Error Variance Decomposition of (in %)

	Christiano et al.(1999)	D-SVAR				
Horizon	ε_R	ε_1	ε_2	ε_3	ε_4	ε_5
Nominal Interest Rate						
1	66.79	74.37	20.16	0.09	1.50	3.89
4	26.03	42.62	37.11	19.00	0.32	0.94
8	17.74	34.46	29.15	19.75	5.36	11.28
20	13.39	31.30	27.11	21.18	7.42	12.99
Output						
1	0.00	6.08	11.72	14.39	4.77	63.04
4	2.47	3.01	9.65	15.92	7.38	64.04
8	5.47	3.22	5.47	8.92	7.38	75.01
20	5.95	3.48	4.12	6.23	4.81	81.36
Price Index (CPI)						
1	0.00	2.16	0.30	40.50	55.86	1.17
4	1.80	9.12	4.80	40.62	44.78	0.68
8	1.04	8.83	4.36	46.40	39.72	0.70
20	0.23	5.44	1.66	46.05	45.27	1.58
Unemployment						
1	0.00	5.61	50.38	29.17	14.64	0.20
4	0.46	6.51	37.46	29.12	21.81	5.11
8	5.70	5.93	25.66	21.69	29.76	16.97
20	6.47	13.22	20.87	22.49	28.28	15.14
Commodity Price						
1	0.00	0.37	37.40	0.42	61.19	0.62
4	0.37	0.89	29.19	8.12	61.34	0.47
8	0.13	0.56	21.06	12.05	65.98	0.35
20	0.49	0.14	9.86	12.86	76.95	0.19

In this table we compare the variance decomposition as obtained by Christiano et al. (1999) for their monetary policy shocks and for the five shocks of the D-SVAR. Sample is 1965Q1-2007Q4.

G A Two-Country VAR

In this section, we consider the modelling of a 2-country VAR featuring the log-difference of US GDP and residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP for the 1995Q1-2019Q4 period as reported by OECD (https://stats.oecd.org/, VPVOBARSA, US Dollars, volume estimates at fixed PPP, seasonally adjusted). In this example, we illustrate how the D-SVAR allows to recover a shock structure à' la Backus et al. (1992) involving dynamic symmetric spillovers. More precisely the shock process is assumed to take the form

$$Z_t \equiv \begin{pmatrix} z_t^{us} \\ z_t^{can} \end{pmatrix} = \begin{bmatrix} \rho & \nu \\ \nu & \rho \end{bmatrix} Z_{t-1} + \varepsilon_t \text{ with } \varepsilon_t \rightsquigarrow N(0_2, I_2),$$

where ν captures the dynamic spillovers. Both the AIC, BIC and Hannan-Quinn information criteria led us to select a VAR(2) specification. The D-SVAR identification then leads to a value of $\rho = 0.43$ and $\nu = 0.20$. The loading matrix Π_{yz} then takes the form

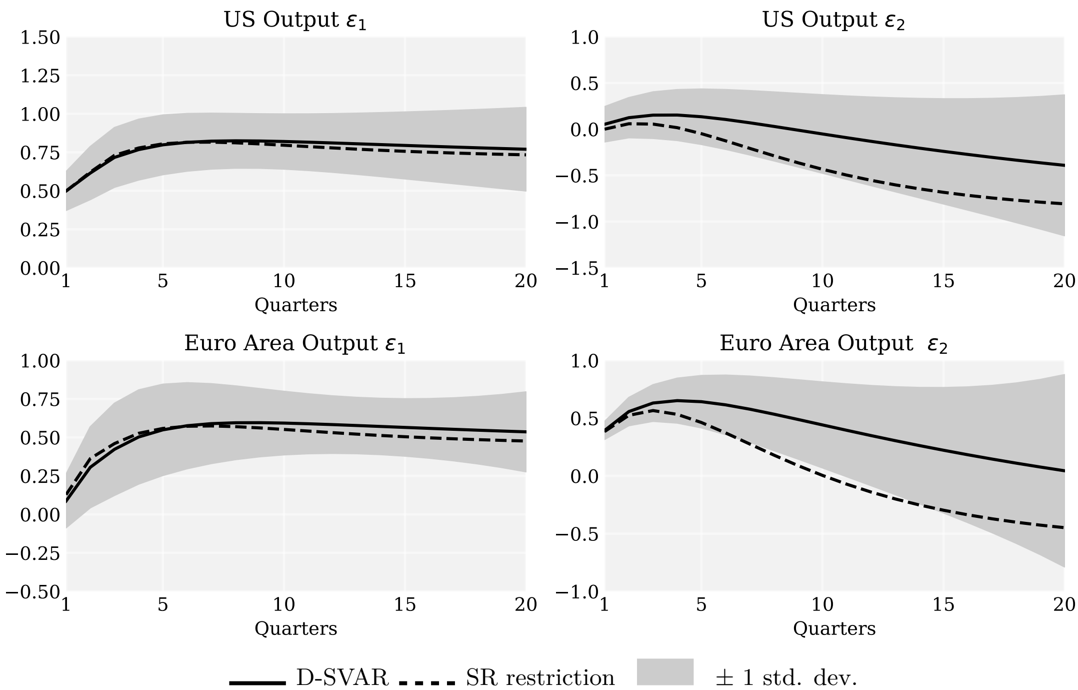
$$\Pi_{yz} = \begin{bmatrix} 0.498 & 0.053 \\ -0.229 & 0.360 \end{bmatrix}$$

The forecast error variance decomposition of levels is reported in Table 6.. While the US shock explain essentially all of US GDP, it accounts for less than 5%the Euro volatility in the very short-run and 56% at the 20 quarters horizon. In other words, in the very short-run, the US and Euro Area economies are essentially insulated from each other, while the shocks are transmitted in the medium run. Figure 14 reports the IRF of US and Euro GDP to both shocks. These IRF confirm and illustrate the broad picture conveyed by forecast error volatility decomposition: US and Euro GDP only responds to their respective shocks on impact, and are essentially insulated from exogenous developments in the other economy in the very short-run. This is actually slightly contrasts with an identification of the US shock as the only shock that affects US GDP on impact. In that latter case, the US shock is transmitted faster to the Euro Area: the US shock accounts for about 10% of the Euro output volatility on impact and about 40% after 1 year (30% in our case). In the longer run, the Cholesky decomposition indicates that while the US economy is essentially not affected by Euro shocks, US shocks account for 60% of GDP volatility in the Euro Area.

Table 6: Forecast Error Decomposition

	US GDP		Euro Area GDP	
Horizon	ε_t^{us}	ε_t^{euro}	ε_t^{us}	ε_t^{euro}
D-SVAR				
1	98.9	1.1	4.6	95.4
4	96.4	3.6	29.3	70.7
8	97.8	2.2	40.9	59.1
20	93.6	6.4	60.8	39.2
Short-Run Restriction				
1	100.0	0.0	10.0	90.0
4	99.6	0.4	38.3	61.7
8	96.7	3.3	56.5	43.5
20	68.7	31.3	67.0	33.0

The variance decomposition is obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.



These IRFs are obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.

Figure 14: Impulse Response Functions: US vs Euro Area