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ABSTRACT

We develop a dynamic model of firm investment under uncertainty that captures firms' risk attitude using quantile preferences. The firm maximizes its present value, defined as current profits and investment plus the discounted value of the τ-quantile of its value next period. In our framework, $\tau \in (0, 1)$ parametrizes the firm's attitude toward downside risk. The model implies that the firm's investment policy equates the marginal cost of capital with the τ-quantile of the discounted present value of future marginal profits — investment depends directly on the firm's risk attitude. We further integrate our model into a "q-theory" of investment. Numerical solutions show how heterogeneity across τ-quantiles impacts the value of the firm and investment decisions. Empirical estimations of the quantile investment model show that the strength of the relation between investment and Tobin's q increases as downside risk aversion decreases. Estimates of firms' risk attitude reveal evidence of high levels of downside risk aversion.

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1 Introduction

Understanding corporate investment is paramount to informing economic policy. To this end, dynamic investment models with emphases on uncertainty and nonconvexities have become the workhorse of the literature and motivated a large number of empirical studies. The vast majority of these models assume that the firm is risk neutral when making investment decisions. In the real world, however, corporate managers exhibit attitudes toward risk that are not aligned with risk neutrality.[1] Ample existing evidence suggests that firms present downside risk aversion, but the literature lacks a tractable model of investment decisions that includes this finding.[2]

This paper develops a novel dynamic model of firm investment under uncertainty in which downside risk aversion is incorporated using quantile preferences. In this model, the firm maximizes the τ-quantile of future cash flows and the parameter $\tau \in (0,1)$ captures the firm's attitude toward downside risk. Our framework builds naturally on models of investment by, among others, Hayashi (1982), Abel and Eberly (1994), Dixit and Pindyck (1994), Cooper and Haltiwanger (2006), Bloom et al. (2007), Bloom (2009), and Asker et al. (2014). In the quantile model, each period profit, net of costs and investments, is exactly as in the standard framework. The difference consists in how uncertainty is evaluated: instead of using expectations, uncertainty is resolved using a quantile certainty equivalent. That is, the firm maximizes the current period net profit plus the τ-quantile of the uncertain discounted stream of future net profits. Intuitively, the τ-quantile of a random variable is the value that is exceeded with probability $1-\tau$, for $\tau \in (0,1)$. As such, in this model, a firm considering how much to invest in a risky project chooses the policy ensuring the highest τ-quantile. This justifies the assertion that quantile-maximizers are concerned with downside risk (possibility of losses below some critical level).

The quantile model is tractable and easy to interpret. We build on general results for dynamic quantile preferences in de Castro and Galvao (2019) to establish the properties of the firm investment model. In particular, the problem is dynamically consistent, yields a value function via a fixed point argument, and this value function is concave and differentiable. Additionally, we derive the corresponding Euler equation. We are able to maintain the standard assumption that firms maximize the present value of profits, regardless of the time at which profits are obtained. Notice that in the standard expected utility model, this time-indifference assumption implies risk neutrality, that is, a linear utility function of profits. Indeed, any concave function would distort how profits accrued in different times are evaluated. The quantile model, in contrast, can separately incorporate the present value framework, and the downside

[1]For discussions on firms deviations from risk neutrality, see, e.g., Smith and Stulz (1985), Graham et al. (2013), Brenner (2015), Rabin and Bazerman (2019), Kerr et al. (2019), and Lovallo et al. (2020). Koudstaal et al. (2016) provide a review of the experimental literature on risk aversion.

[2]See, e.g., Mao (1970), Veld and Veld-Merkoulova (2008), List and Haigh (2010), and Stulz (2022). Risk aversion has been considered in models of firm decision making by applying the standard expected utility framework, or variations of discounted cash flow methods. See Sections 2.3 and 2.4 for a more detailed review of the methods usually developed.

risk aversion. Hence, an important new feature of the dynamic quantile firm investment model is that, since the risk attitude is captured by the parameter τ, it allows for empirical estimation of the firm's risk attitude, which is valuable to policy makers, and could be confronted with the common theoretical notion that firms are risk neutral.

Our investment model yields three main results. First, the Euler equation shows that the firm's investment policy equates a measure of the marginal cost of capital accumulation (which includes the direct cost of new capital as well as the marginal adjustment cost) with the τ-quantile of the marginal gains of newly added capital (which is measured through the derivative of the value function). In the literature, this is conventionally termed "marginal q" or "Tobin's q," after Tobin (1969). In other words, the measure of the marginal cost of capital accumulation is equal to the τ-quantile of the marginal gains of more capital investment. This new result is intuitive and shows that investment decisions are a function of the risk attitude.

Second, the firm's investment policy implies that the marginal value of investment today is equal to the τ-quantile of the discounted present value of marginal profits in the future. This result reconciles the intuitive notion that the purchase price of capital is represented as a discounted present value, while also capturing its dependence with respect to risk attitude through a simple parameter (τ). This finding has important practical implications for the valuation of corporate investment: the marginal value of investment is an explicit function of the firm's risk attitude and varies accordingly. This contrasts with usual methods adopted by practitioners, that capture risk through risk-adjusted discount rates, where firm risk aversion is not a primitive concept.

Third, we specialize the model for a Cobb-Douglas production function and quadratic cost function to establish a *quantile q-theory of investment*. In short, we extend the standard q-theory of investment developed by Hayashi (1982) considering uncertainty under quantile preferences. The standard q-theory predicts that in the absence of fixed investment costs and no financial market frictions, the firm optimally chooses investment to equate the marginal value of capital with the marginal cost of capital (including adjustment costs). With a homogeneous production technology, the marginal value of capital ("marginal q") equals the average value of capital ("average q"). We show that under quantile preferences, the τ-quantile of the firm's marginal value is linear in investment. Moreover, under the linear-quadratic formulation of capital accumulation — constant returns of scale and quadratic cost function — the value function is proportional to the stock of capital and the investment rate is independent of the current level of the capital stock, rendering the average and marginal value of capital to be equal. The model shows that investment is equal to the τ-quantile of average q, and that investment depends directly on the firm's attitude toward risk. In this framework, a firm's investment decisions are independent of its financial condition, but related to the quantile of its investment opportunities.

We investigate the properties of our model using numerical solutions and empirical data.

First, we compute the value functions, policy functions, and optimal investment functions for different values of the parameters in the model. We use a linear profit function, fix the discount rate, curvature of the profit function, and cost parameters, and vary the risk attitude and depreciation. Our computations reveal strong heterogeneity in these functions. In short, our results show that the value function increases with risk attitude τ for any combination of depreciation and cost parameters. Optimal capital and investment decrease with τ when depreciation is large enough, while with small depreciation optimal capital and investment are the same across risk attitudes. Our numerical exercises demonstrate that the quantile model of investment is flexible enough to produce different valuations across firms and yet only small differences in their investment rates.

Next, we take the quantile investment model to the data. Using COMPUSTAT data from 1970 to 2021 and quantile regressions methods, we estimate linear models of average q on investment. The regression equation is derived from the structural model after showing that the marginal q is equal to the average q, and consequently the conditional quantile of the average q is a linear function of investment. Our results show that the sensitivity of Tobin's q to investment varies with risk attitude. For high quantiles (low downside risk aversion), there is a strong relation between Tobin's q and investment; that is, investment opportunities and investment spending are strongly related. For low quantiles (high downside risk aversion), that relation is weak. Our empirical results are new and point to strong heterogeneity in the investment–q relation across risk attitudes.

Finally, we identify and estimate firms' risk attitude. To do so, we adopt a parametric Asymmetric Laplace assumption on the underlying distribution characterizing the residual function in the conditional quantile model (see, e.g., Bera et al., 2016). Under this assumption, the asymmetry, or skewness, in the distribution of the residuals, that is, the distribution of investment opportunities net of investment spending, allows us to identify τ. Intuitively, when observed investment spending is small, relative to investment opportunities, the distribution of the residuals is skewed to the right, and hence the risk attitude is small (small quantile). On the other hand, a larger quantile (low risk aversion) would be elicited when observed investment spending is large, relative to investment opportunities, such that the distribution is skewed to the left. The Laplace condition induces additional moment restrictions that are used for estimation.

Our risk attitude estimates are around 0.39–0.45, depending on model specification. Since risk neutrality would correspond to a quantile of 0.5, and statistical tests show that the estimates are different from this value, the results reveal evidence of downside corporate risk aversion. This finding is both surprising and important given the wide use of the risk neutrality assumption for firms. Moreover, we reestimate the risk attitude for subsamples of the data, where we consider different size and age of firms. The results show that smaller and younger firms are slightly more risk taker, but for all cases the firms continue to be risk averse.

The remaining of the paper is organized as follows. Section 2 presents a simple version of the dynamic model for investment using quantile preferences. Section 3 describes a more general model with convex adjustment costs and derives its properties. Section 4 illustrates the model using numerical properties. Empirical application results are presented in Section 5. Section 6 concludes. We relegate all proofs to the Appendix.

2 A simple model of investment under uncertainty

Most of the literature on risk management for firms (reviews in Section 2.4 below) does not focus on downside risk. However, there is significant evidence that downside risk is important. Indeed, in an influential study, Mao (1970, p. 354) reports, among many other similar quotes, the following comment by a business executive: "I never worry about the project return going above the target return. Risk is what might happen when the return is going to be less." The famous investor Warren Buffet has made similar statements.[3] In an experimental study, Veld and Veld-Merkoulova (2008) report that the most common risk measure used by investors was downside risk. List and Haigh (2010, p. 974) report that "the downside investment state influences the investment decision, whereas the upside investment state is ignored." Stulz (2022, p. 32) argues that corporate risk management should focus on "costly lower-tail outcomes."

The purpose of this paper is to develop a model of firm investment taking downside risk aversion as its central piece. This section introduces the core concepts of our theory within a simplified framework. This enables a clearer understanding of the main ideas, without the complexities of the full model developed in Section 3.

2.1 The firm investment problem

The basic model is a partial equilibrium model cast in discrete time, with an infinite horizon. The firm aims to maximize the value of the plant by choosing the variable factors of production that are rented for the production period, such as capital.

Let A denote the productivity and K the capital. We will ignore labor decisions. Given A and K, the firm profit is given by $\Pi(A, K) = AK^{\alpha}$, where $\alpha > 0$. In other words, the production function is a standard Cobb-Douglas with parameter α capturing the returns to scale. Let p denote the purchasing price of new capital, which will be assumed constant. Therefore, the only source of variation in this economy is the productivity shock A, which is a function of $z \in \mathcal{Z} \subset \mathbb{R}$. For simplicity and without loss of generality, we may assume $A(z) = z$. The capital

[3]For instance, Buffett (2001): "We regard volatility as a measure of risk to be nuts. And the reason it's used is because the people that are teaching want to talk about risk. And the truth is, they don't know how to measure it in business. I mean, that would be part of our course on how to value a business. It would also be, how risky is the business? And we think about that in terms of every business we buy. And risk with us relates to – Well, it relates to several possibilities. One is the risk of permanent capital loss. And then the other risk is just an inadequate return on the kind of capital we put in. It does not relate to volatility at all."

accumulation equation is

$$K' = (1-\delta)K + I, \tag{1}$$

where I denotes the investment, such that investment at the current time becomes productive next period, and $\delta \in [0,1]$ represents the rate of depreciation of the capital stock. Moreover, unprimed variables are current values and primed variables refer to next period values.

Managers choose the capital stock each period to maximize the value of the firm, which is the present value of the stream of future cash flows to (or from) shareholders. These cash flows, which we denote by Π, can be expressed as:

$$\Pi(A,K) - pI = AK^{\alpha} - pI = zK^{\alpha} - p\left[K' - (1-\delta)K\right].$$

The next step in describing the model requires the introduction of quantiles. For this, it is useful to recall its definition. For a given random variable Y and $\tau \in (0,1)$,[4] we define its τ-quantile as

$$Q_{\tau}[Y] \equiv \inf\{y \in \mathbb{R} : \Pr[Y \leqslant y] \geqslant \tau\}. \tag{2}$$

A simpler definition of quantiles uses the cumulative distribution function (cdf) of the random variable Y. Let $F_Y : \mathbb{R} \to [0,1]$ be the continuous and strictly increasing cdf of Y. If F_Y has an inverse, then the quantile is simply the inverse of the cdf, that is, $Q_{\tau}[Y] \equiv F_Y^{-1}(\tau)$, for $\tau \in (0,1)$. These definitions can be easily extended to the conditional case: if $F_{Y|X=x}(\cdot)$ is the conditional cdf of Y given $X = x$, $Q_{\tau}[Y|X = x] \equiv F^{-1}_{Y|X=x}(\tau)$. In general, we will write only $Q_{\tau}[Y|X]$ for the corresponding conditional τ-quantile function of the random variable, as usual.

Firms maximize current profits plus the discounted value of the τ-*quantile* of their future value, that is,

$$V(K,z) = \max_{K' \in \Gamma(K)} \left\{ zK^{\alpha} - p\left[K' - (1-\delta)K\right] + \beta Q_{\tau}\left[V(K',z')\Big|z\right]\right\}, \tag{3}$$

where $\beta \in (0,1)$ is the discount factor. The firm solving the optimization problem in (3) is concerned with downside risk. There are at least two reasons rationalizing this objective function. First, as discussed previously, downside risk is an important concept in risk management documented in the literature. Second, this model allows for firms to have different attitudes toward risk, while still focusing in the present value of its cash flow.

We show below that the Euler equation for this problem is[5]

$$Q_{\tau}\left[z'\alpha(K')^{\alpha-1}|z\right] = \frac{p}{\beta} - p(1-\delta). \tag{4}$$

[4] In this paper we will not consider the cases in which $\tau \in \{0,1\}$. de Castro and Galvao (2019) state and prove a number of results about quantiles, some of which are well-known. For instance, their Lemma A.2 shows that if $g : \mathbb{R} \to \mathbb{R}$ is weakly increasing and left-continuous, then $g(Q_{\tau}[Y]) = Q_{\tau}[g(Y)]$.

[5] The results stated in this section are formally established in Appendix A.2.

For comparison, we also recall the Euler equation from the standard expected utility (EU) model of investment. For this model, similar calculations yield the following Euler equation:

$$\mathrm{E}\left[z'\alpha(K')^{\alpha-1}|z\right] = \frac{p}{\beta} - p(1-\delta). \tag{5}$$

An examination of equations (4) and (5) show that they have the same basic format, with the main difference being the substitution of the expectation by the quantile to assess uncertainty. Moreover, in the EU in (5), investment is a function of three parameters: (β, α, δ), while in the quantile model (4), the investment is a function of four parameters: $(\beta, \alpha, \delta, \tau)$.

To gain further intuition, we study the effects of the parameters on optimal capital, and hence investment, for the dynamic quantile model. Rearranging the Euler equation in (4), we obtain

$$K' = \left(\frac{\alpha\beta}{p[1-\beta(1-\delta)]} Q_\tau\left[z'|z\right]\right)^{\frac{1}{1-\alpha}}. \tag{6}$$

Equation (6) shows that the optimal capital is a function of the four parameters. Notice that, the smaller α, the smaller is the capital (and thus investment). Intuitively, the more concave the production function, the less the production function shifts in response to the shock, z. In economic terms, if the firm has sharply decreasing returns, then technology shocks have only a modest effect on the marginal profit of capital, so the firm responds less strongly to these shocks. When considering fixed decreasing returns of scale $\alpha < 1$, the optimal capital is decreasing on the depreciation δ, and price p; and it is increasing on the productivity/demand shock z, on the discount factor β, and on the quantile τ. In particular, as τ increases (downside risk aversion decreases), the conditional quantile increases, and the firm responds more strongly to shocks because it knows that the marginal product of capital is likely to last. Moreover, if the conditional quantile, $Q_\tau[\cdot|\cdot]$, equals a constant, the firm never alters its optimal capital stock, as it understands that any productivity increase will be temporary.

One could also assume that the shock z follows a specific Markov process to obtain a closed form solution to $Q_\tau\left[z'|z\right]$, and consequently capital and investment. For instance, as it is common in the literature, one could consider a first-order autoregressive process, AR(1), in logs:

$$\log(z') = \rho\log(z) + \varepsilon'.$$

Here, ρ is an autoregressive coefficient, and ε' is the error term in this AR(1) process. If ε is assumed to have a normal distribution with zero mean and variance σ_ε^2, then, z' given z is lognormally distributed. From the expression of the τ-quantile of a lognormal distribution, one obtains:

$$Q_\tau\left[z'|z\right] = \exp\left(\rho\log(z) + \sqrt{2\sigma_z^2}\,\mathrm{erf}^{-1}(2\tau-1)\right),$$

where $\text{erf}(z) \equiv (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$. We do not follow this path in this paper and leave it for future research.

Notice also that, in equation (6), the next period's optimal capital stock, K', is set as a function of all parameters and the shock z. In this simplified model, the optimal choice of K' only depends on the shock, z, and does not depend on K. For further illustration, we consider the following two special cases:

Case 1: $\delta = 0$. In this case, there is no depreciation and the investment is $I = K' - K$, the budget set is $\Gamma(K) = [K, M]$, and the capital becomes

$$K' = \left(\frac{\alpha\beta}{p(1-\beta)} Q_\tau \left[z' | z \right] \right)^{\frac{1}{1-\alpha}}.$$

Thus, given the budget set, the optimal policy is

$$K' = \max \left\{ K, \left(\frac{\alpha\beta}{p(1-\beta)} Q_\tau \left[z' | z \right] \right)^{\frac{1}{1-\alpha}} \right\}.$$

This equation shows that capital next period, and hence investment, is a function of the three remaining parameters in the model – discount factor (β), returns to scale (α), and quantile (τ). It is also increasing on τ, which means that capital and investment are increasing with respect to the downside risk aversion parameter τ.

Case 2: $\delta = 1$. When depreciation is equal to one, $I = K'$, the budget set is $\Gamma(K) = [0, M]$, and

$$K' = \left(\frac{\alpha\beta}{p} Q_\tau \left[z' | z \right] \right)^{\frac{1}{1-\alpha}}.$$

Since $K' = I$ in this case, the firm's problem in equation (3) can be simplified to:

$$\max_{I \in \Gamma(K)} \left\{ -pI + \beta Q_\tau \left[z' | z \right] I^\alpha \right\}.$$

The first order condition leads to:

$$\alpha\beta Q_\tau \left[z' | z \right] I^{\alpha - 1} = p \Rightarrow I = \left(\frac{\alpha\beta}{p} Q_\tau \left[z' | z \right] \right)^{\frac{1}{1-\alpha}},$$

which is the same solution obtained in (6) above.

In addition to the formalization of the discussion on optimal investment and capital, Appendix A.2 provides an explicit expression for the value of the firm — the value function —, which can be written as follows:

$$V(K,z) = zK^{\alpha} + p(1-\delta)K + S(z), \tag{7}$$

where $S(z)$ is an infinite sum depending on quantiles of the shocks; see details in the appendix. This closed form solution of the value function shows that, in this simple case of no adjustment cost, the value of the firm is separable on capital and shocks.

2.2 Risk neutrality versus downside risk aversion

The main difference between the quantile model described above and the standard model, based on the expected value, is the different treatment of risk and the risk attitude. While the standard model incorporates the usual notion of risk neutrality by the firm, the quantile model allows to study downside risk aversion through the quantile parameter $\tau \in (0,1)$. Before explaining in more detail why τ captures downside risk aversion, it is useful to discuss with a couple of simple examples how the two models lead to different choices for risky prospects.

Consider an example where a firm needs to decide between two investment opportunities, π_1 and π_2. Let us say that project π_1 pays \$100 with certainty, while project π_2 pays $\$\frac{101}{\alpha}$ with probability $\alpha > 0$ and nothing with probability $1-\alpha$. If the firm is risk neutral, it strictly prefers project π_2, no matter how small α is. In contrast, any τ-quantile maximizer will prefer project π_1, provided that $\tau < 1-\alpha$. This choice ($\pi_1 \succ \pi_2$) seems a more reasonable choice in this context.

Of course one can create other examples in which quantiles seem to indicate the wrong answer. For instance, let π_3 be the prospect of receiving zero for sure and π_4, of losing \$1 with probability 50% and winning \$1,000,000 with probability 50%. Any τ-quantile maximizer with $\tau < 0.5$ will give up the opportunity of receiving that large prize for the fear of losing just \$1. Definitively, this does not seem reasonable. The point, however, is not to develop a theory that can work for any conceivable pair of alternative, but for those that are more likely to be faced by real firms. This last example, for instance, consists of asking whether or not the firm should buy, for the price of \$1, a lottery ticket that pays \$1,000,001 with probability 50%. Do real firms face the opportunity to buy such generous lottery tickets? Once we restrict our attention to choices that actual firms make, downside risk aversion seems more realistic than risk neutrality. However, these comments lead to the natural question on why quantile preferences capture downside risk aversion. We discuss this next.

2.3 Risk attitude in the quantile model: downside risk aversion

Manski (1988) was the first to study quantile preferences and observed that the risk attitude for these preferences is determined by the quantile τ that is maximized.[6] In this subsection,

[6] Static quantile preferences were axiomatized in the context of risk by Chambers (2009) and in the Sav-

We discuss the risk attitude for the static quantile model in two steps. First, we show that the concavity of the utility function has no relation with risk aversion, as it has for the expected utility model. Second, we show that the quantile τ captures downside risk aversion.

To see that the curvature of a utility (profit) function does not relate to the risk attitude for quantile preferences, it is sufficient to recall that quantiles are invariant with respect to monotone transformations, that is, quantiles commute with increasing and continuous functions.[7] Indeed, if we assume that $U : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous, we can use this property to obtain:

$$Q_\tau[U(X)] \geqslant Q_\tau[U(Y)] \Leftrightarrow U(Q_\tau[X]) \geqslant U(Q_\tau[Y]) \Leftrightarrow Q_\tau[X] \geqslant Q_\tau[Y].$$

Hence, the utility (profit) function is not relevant for ranking of alternatives by a quantile preference and, hence, has no relation with the risk attitude in quantile preference.

In order to relate τ to downside risk aversion, we consider Rostek (2010, p. 354)'s definition of *downside risk* or the equivalent definition of *quantile-preserving spreads* introduced by Mendelson (1987).

Definition 2.1 (More downside risk/quantile-preserving spread)**.** *The random variable* Y involves (or has) more downside risk than *or, alternatively, is a* quantile preserving spread *of* X *if there exists* $q \in \mathbb{R}$ *and* $\tau \in (0,1)$ *such that:* (i) $Q_\tau[X] = Q_\tau[Y] = q$*;* (ii) $F_X(x) \leqslant F_Y(x)$ *for any* $x \leqslant q$ *and* (iii) $F_X(x) \geqslant F_Y(x)$ *for any* $x \geqslant q$*.*

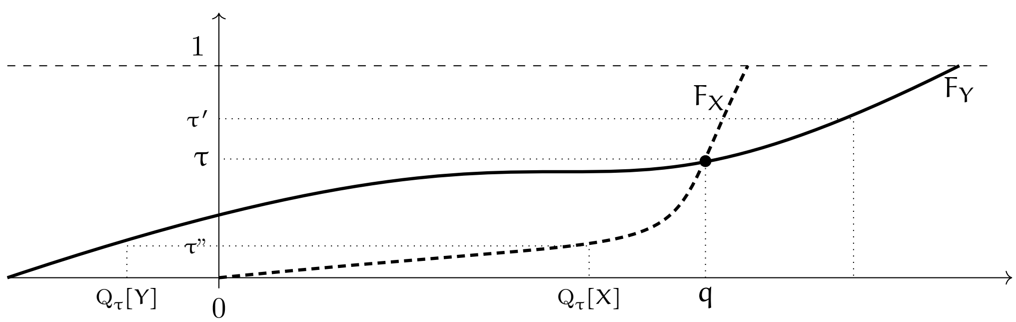


Figure 1: Y is a τ-quantile preserving spread of X

Figure 1 illustrates the concept in Definition 2.1. In this picture, we can see that the c.d.f. of X, F_X, crosses the c.d.f. of Y, F_Y, from below exactly at $q = Q_\tau[X] = Q_\tau[Y]$. Thus, Y is more likely to lead to worse outcomes below q, and to better outcomes above q, that is, Y is more widespread than X. This justifies the notion that Y is a quantile-preserving of X (since the τ-quantile is preserved). Also, Y involves more downside risk than X because it leads to worse outcomes with higher probability.

age setup by Rostek (2010). de Castro and Galvao (2022) axiomatize both the static and dynamic quantile preferences.

[7]See footnote 4 for a formal statement and reference for this property.

It is useful to see that a decision maker (DM) who maximizes the τ'-quantile for a $\tau' > \tau$ prefers the riskier Y to X. On the other hand, a DM that maximizes a τ''-quantile for $\tau'' < \tau$, prefers the safer (less spread) alternative X to Y. That is, a DM maximizing a lower quantile prefers the less risky option; a DM maximizing a higher quantile prefers a riskier option. This clarifies the sense in which τ captures the risk attitude of the DM. This was already observed by Manski (1988), but was summarized by Rostek (2010) as follows:

Lemma 2.2. *If Y has higher downside risk than X, than a quantile maximizer that has sufficiently high downside risk aversion, that is, a sufficiently low τ, prefers X to Y.*

In other words, a firm that is a τ-quantile maximizer always prefers the alternative with lower downside risk, provided that the quantile $\tau \in (0, 1)$ is low enough. This result formalizes how quantile preferences capture downside risk aversion.

It should be noted, however, that there are alternative approaches to the study of downside risk and downside risk aversion that are not based on quantile preferences. Nawrocki (1999) describes a history of the efforts in this field that goes back at least to Markowitz (1959), who defined below-mean and below-target semivariances as the most convenient ways to evaluate risk in a portfolio. He only adopted variance for this purpose out of technical convenience, thus creating his famous mean-variance framework. Bawa (1975) argues that third order stochastic dominance is optimal for comparing uncertain prospects with equal means. He also relates this criterion to concave utility functions whose third derivative is strictly positive. Fishburn (1977) extends the concept of below-target semivariance to arbitrary exponents α (instead of just $\alpha = 2$ as in the case of semivariance) and show that the adoption of his mean-risk analysis is compatible with expected utility maximization, for utility functions u that are linear above a target t and satisfy $u(x) = x - k(t - x)^{\alpha}$ for values of x below t. Notice that $u'''(x) > 0$ if $\alpha \geqslant 3$ and $x < t$. Following this approach, Menezes et al. (1980) considers three definitions of downside risk and show that they are equivalent. A large literature derives from these papers; see, for instance, Sortino and van der Meer (1991), Grootveld and Hallerbach (1999), Estrada (2006), and Crainich and Eeckhoudt (2008), among others. Another branch of the literature departs from expected utility models, such as Gul (1991)'s disappointment aversion, which was used by Ang et al. (2006) to estimate that stock returns reflect a downside risk premium of approximately 6% per year.

It is certainly interesting to compare these alternative definitions of downside risk and downside risk aversion, but this exercise is beyond the scope of this paper, which focus exclusively on the development of a model of investment based on quantile preferences. Thus, when we refer to downside risk aversion, we are referring exclusively to the notion introduced by Rostek (2010), as mentioned above.

2.4 Standard approaches to capturing risk aversion by firms

There is substantial evidence that firms do not follow risk neutral behavior. See, e.g., Swalm (1966), Graham et al. (2013), Brenner (2015), Koudstaal et al. (2016), Rabin and Bazerman (2019), Kerr et al. (2019), Lovallo et al. (2020), among many others. Smith and Stulz (1985, p. 403) point out three explanations for hedging that could be perceived as deviations from risk neutrality: "(1) taxes, (2) costs of financial distress, and (3) managerial risk aversion."

The literature has considered alternative approaches to incorporate risk aversion in models of firms' choice. We will briefly review some of these alternative approaches here. Froot et al. (1993) develop a model in which risk averse behavior by firms is justified out of financial frictions. They argue (p. 1633) that "there are additional (deadweight) costs to the firm of external finance," which with some standard assumption, make the firm maximize the expectation of a concave function (see their eq. 16), just as in the standard expected utility framework with risk aversion.

Another common approach is based on production-based asset pricing models in a general equilibrium framework, with consumers and firms. See, among others, Detemple (1986), Jermann (1998), Brock (2001), Tallarini (2000), Tuzel (2010), Croce (2014), and Branger et al. (2016). In these models, consumers are risk averse and the corresponding Euler equation lead to stochastic discount factors (SDF). These SDF can be used by firms to price risky assets. Note, however, that in these models, the risk aversion comes from the consumer side, not from the firms, and consequently, there is no primitive risk parameter associated with the firm's preference. Moreover, in this approach, all assets are discounted by the same SDF. An alternative is to consider specific discount rates, as those provided by the Capital Asset Pricing Model (CAPM).

Practitioners have been considering different approaches for dealing with risk. Damodaran (2007, Ch. 5) discusses four ways in which risk adjustment are made in practice. The first two approaches are based upon discounted cash flow valuation, where one values an asset by discounting the expected cash flows on it at a discount rate. The risk adjustment here can take the form of a higher discount rate or as a reduction in expected cash flows for risky assets, with the adjustment based upon some measure of asset risk.[8] The third approach is to do a post-valuation adjustment to the value obtained for an asset, with no consideration given for risk, with the adjustment taking the form of a discount for potential downside risk or a premium for upside risk. In the final approach, risk is adjusted by observing how much the market discounts the value of assets of similar risk. These approaches seem to offer an ad-hoc treatment of risk attitude.

Although a comparison of these models could be an interesting exercise, this paper focus on studying and deriving the properties of an investment model based on quantile preferences.

[8] See Welch (2011) for a treatment of the Capital Asset Pricing Model together with the net present value.

3 Investment under uncertainty

Since the traditional investment theory considers adjustment cost (see, e.g., Adda and Cooper (2003, Chapter 8) and Miao (2013, Chapter 8)), in this section, we generalize the model presented above and introduce a quantile theory of firm investment with convex adjustment costs. We establish a general dynamic optimization problem for investing, as well as a new formula for the price of investment as a function of the τ-quantile. We also extend the standard q-theory of investment developed by Hayashi (1982) considering uncertainty under quantile preferences.

3.1 The dynamic quantile model

As before, K represents the stock of capital used by the firm; L, the variable factors, such as labor; and A, the shock to revenue and/or productivity. Demand for the variable inputs L is optimally determined given its factor prices w, and the shock and state variable, represented by (K, A). The result of the firm's optimization leaves a profit function, denoted by $\Pi(A, K)$. This can be modeled using a profit function $\Pi(A, K)$ given by

$$\Pi(A, K) = \max_{L} R(A, K, L) - wL,$$

where $R(A, K, L)$ denotes revenues given the capital input K, and labor input L, with price w, which may also be stochastic and vary with time. However, we are not interested in L and w, and the analysis below will not depend on them, such that capital is the only production factor.

Recall that p denotes the purchasing price of new capital. Now, we allow p to vary. Thus, the source of variation in this economy is (A, p), which we assume are functions of the shocks $z \in \mathcal{Z} \subset \mathbb{R}$. Further, firms cannot costless and instantaneously adjust capital. They face building lags, delivery lags, and costs of installation or disinstallation. We consider costs of adjustment given by $C(K', A, K)$, where K' stands for the next period choice of capital, while K denotes the current level of capital. The capital accumulation equation is

$$K' = (1 - \delta)K + I,$$

where $\delta \in [0, 1]$ is the depreciation.

Therefore, the problem can be written in the following general format. The firm maximizes its current value, which can be represented in a recursive format as the current profits net of costs and investment expenditures plus the discounted value of the τ-quantile of the value of the firm next period. In this model, the uncertainty with respect to the future realizations of z are evaluated by a quantile — instead of the traditional expectation —, which captures the firm's attitude toward risk. Formally, $K \in \mathcal{X}$ is the state variable, and $K' \in \mathcal{K}$ is the decision variable, where $\mathcal{K} = \mathcal{X} = [0, M]$, for $0 < M < \infty$ is the state space.[9] Therefore, the intertemporal firm choices can be represented by the maximization of a value function $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ that satisfies the recursive equation:

$$V(K, z) = \max_{K' \in \Gamma(K)} \left\{ \Pi(A(z), K) - C(K', A(z), K) - p(z) \left[K' - (1 - \delta)K \right] + \beta Q_\tau \left[V(K', z') \middle| z \right] \right\}, \tag{8}$$

where the budget set $\Gamma : \mathcal{K} \to \mathcal{K}$ is given by $\Gamma(K) = [(1 - \delta)K, M]$ such that M is just an upper bound for the capital. Effectively, this condition restricts the budget set to be bounded, and the capital to be an interior point. The discounting factor is $\beta \in (0, 1)$, and the quantile $\tau \in (0, 1)$ is the *downside risk aversion parameter* of the firm. For simplicity, there are no financial frictions, as for example borrowing constraints, in this framework. Hence, the choice of investment, and thus future capital, is unconstrained by current profits or retained earnings.

The Bellman equation in (8) is simple to interpret. The value of the firm is represented by two terms. The first term corresponds to the current time t (the first period's net profit). For later reference, it is convenient to summarize the current period t 'utility' by the function $u : \mathcal{K} \times \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ defined by:

$$u(K, K', z) \equiv \Pi(A(z), K) - C(K', A(z), K) - p(z) \left[K' - (1 - \delta)K \right]. \tag{9}$$

Then, $u(K, K', z)$ constitutes the current period flow to shareholders: profits less adjustment costs less investment. The second term in (8), $\beta Q_\tau \left[V(K', z') \middle| z \right]$, is called the continuation value, that is, the quantile of next period's firm value, given that optimal policies will be chosen in the future. This term measures the discounted value of next period, which is affected by uncertainty, evaluated through a quantile.

In this model there are two potential costs of obtaining new capital. The first is the direct purchase price, denoted by p. Second, there are potential adjustment costs given by the function $C(K', A, K)$. These costs are assumed to be internal and might include: installation and disinstallation costs, disruption of productive activities in the plant, retention of workers, reconfiguration aspects of the production process, etc. The function $C(K', A(z), K)$ is able to allow for components of both convex and non-convex costs, as well as different transactions costs. Finally, note that the quantile model in equation (8) is similar to the usual dynamic programming problem, in which the expectation operator $E[\cdot]$ is in place of quantile $Q_\tau[\cdot]$.

3.2 Properties of the investment model

Now we build on the general results for dynamic quantile preferences in de Castro and Galvao (2019), to establish the properties of the new investment model considered in this paper. The intertemporal firm choices is represented by the value function $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ defined by (8),

[9]Notice that choosing investment is equivalent to choose capital next period.

reproduced here by convenience:

$$V(K,z) = \max_{K'\in\Gamma(K)} \left\{\Pi(A,K) - C(K',A,K) - p\left[K' - (1-\delta)K\right] + \beta Q_\tau \left[V(K',z')\middle|z\right]\right\},$$

where $\Gamma(K) = [(1-\delta)K, M]$.

Throughout the paper, we maintain the following assumption:

Assumption 1. *The following hold:*

(i) The function $u : \mathcal{K}\times\mathcal{K}\times\mathcal{Z} \to \mathbb{R}$ *defined by* (9) *is continuously differentiable, strictly concave in* (K,K')*, and strictly increasing in* z*.*

(ii) $z \in \mathcal{Z} \subset \mathbb{R}$*, where* $\mathcal{Z}$ *is an interval,* z *follows a Markov process with pdf* $f : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}_+$*, which is continuous, symmetric,* $f(z,w) > 0$*,* $\forall(z,w) \in \mathcal{Z}\times\mathcal{Z}$ *and for any weakly increasing function* $h : \mathcal{Z} \to \mathbb{R}$ *and* $z, z' \in \mathcal{Z}$ *such that* $\mathrm{E}\left[h(w)\middle|z\right] \leqslant \mathrm{E}\left[h(w)\middle|z'\right]$*.*

Assumption 1 is very mild and standard in the investment literature. Part (i) imposes standard assumptions of continuity and differentiability on the cash flow, and requires a standard convex condition on the cost function. This requirement can be imposed separately on the functions Π and C, but it is simpler to state it to u. Part (ii) is a simple condition on the innovations of the model to follow a Markov process. Now we can state results for the value function of the dynamic firm investment model.

Theorem 3.1. *Let Assumption 1 hold. Then, problem* (8) *is dynamically consistent and there exists a unique continuous and bounded function* $V : \mathcal{K}\times\mathcal{Z} \to \mathbb{R}$ *satisfying it. This function is differentiable and strictly concave for any interior point* $K \in \mathcal{K}$*, strictly increasing in* z *and satisfies*

$$\frac{\partial V}{\partial K}(K,z) = \partial_2\Pi(A(z),K) - \partial_3 C(K',A(z),K) + (1-\delta)p(z), \tag{10}$$

for interior K *and* K'*, where* K' *is optimal given* (K,z)*, and* $\partial_2\Pi$ *and* $\partial_3 C$ *denote respectively the derivative of* Π *and* C*, with respect to their second and third arguments.*

Theorem 3.1 establishes that the model is dynamically consistent, a property that follows from the recursive structure of the problem. Moreover, (8) has a unique fixed point, $V(K,z)$, that is strictly increasing in both variables, strictly concave in K, and differentiable with respect to K. This uniqueness comes from a standard contraction argument, which is also useful for the numerical computation of the model. Indeed, we will use a value function iteration to numerically compute and illustrate the model in Section 5 below.

The next step is derivation of the Euler equation, that is, the equilibrium condition. To do so, we strengthen our assumptions as follows:

Assumption 2. *For* u *given by* (9)*,* $z \mapsto A(z)$ *and* $z \mapsto \frac{\partial u}{\partial K}(K,K',z)$ *are strictly increasing.*

The next result derives the quantile Euler equation for the firm investment model.

Theorem 3.2. *Let Assumptions 1 and 2 hold. Then, the following Euler Equation holds for an optimal path* $\{K_t\}_{t=1}^{\infty}$ *that is interior:*

$$\partial_1 C(K', A, K) + p = \beta Q_\tau \left[\partial_2 \Pi(A', K') - \partial_3 C(K'', A', K') + (1-\delta)p' \middle| z \right], \tag{11}$$

where $A' = A(z')$, $p' = p(z')$, *and* z' *denotes the next period shock.*

Equation (11) in Theorem 3.2 is the Euler equation characterizing the equilibrium condition. The interpretation of this Euler equation is similar to that of the standard expectation case (see, e.g., Adda and Cooper, 2003) where the marginal cost equalized to the marginal benefit. The left hand side is a measure of the marginal cost of capital accumulation, which includes the direct cost of new capital ($p(z)$) as well as the marginal adjustment cost ($C_{K'}(\cdot)$). The right hand side of the expression measures the marginal gains of more capital through the derivative of the value function, which in the literature, is conventionally termed "marginal q" or Tobin's q and denoted by q, after Tobin (1969). The uncertainty in the model is solved by the quantile, that is, the right hand side is measured by the τ-quantile utility at time $t+1$, which viewed as of time t, and is equal to $\beta Q_\tau \left[\frac{\partial V}{\partial K}(K, z) \middle| z_t \right]$.

It is also common to interpret Euler equations — as (11) — using small perturbations. Consider increasing the current investment by a small amount. The cost of investing is measured on the left side of (11), which has the two components, the direct cost of the capital and the marginal adjustment cost, as mentioned previously. The gain of investing comes in the next period, measured on the right side. The additional capital increases profits. Moreover, as the manager "returns" to the optimal path following this deviation, the undepreciated capital is valued at the future market price p' and adjustment costs are reduced. It is important to notice that the uncertainty here is resolved through the τ-quantile, which is a measure of the risk attitude of the firm.

As mentioned above, equation (11) is closely related to the corresponding one from the expected utility model, where uncertainty is resolved by taking expectation instead of quantile. For completeness, we present the Euler equation for the expected utility, which is given by

$$\partial_1 C(K', A, K) + p = \beta E \left[\partial_2 \Pi(A', K') - \partial_3 C(K'', A', K') + (1-\delta)p' \middle| z \right]. \tag{12}$$

Detailed discussion for derivation of (12) can be found, for example, in Adda and Cooper (2003). The main difference, however, is that in the quantile Euler equation (11) the τ parameter captures the risk attitude of the firm, and hence the model is able to account for this risk attitude while maintaining a linear profit function capturing the intertemporal substitution. Notably, in the expected utility case of equation (12) the firm is risk neutral and there is no separation between risk and intertemporal substitution.

In turn, we discuss two properties of the dynamic quantile investment model. First, we derive the marginal value of investment – the purchase price of capital – which is commonly used as a proxy for the price of the firm. Second, we investigate the quantile investment model in the context of the Tobin's q-theory.

3.3 Marginal value of investment

In this section we discuss the marginal value of investment under the dynamic quantile model. Recall that p_t is the price of capital at time t, which must be equal to the marginal value of investment. We present the marginal value of the investment in a recursive form. For this, we change the notation and use subscript t to identify the time period. To simplify the presentation, we concentrate on the case without adjustment cost, i.e., $C(K_{t+1}, A_t, K_t) = 0$.

It is interesting to notice that equation (11) is closely related to the corresponding one from the expected utility (EU), where uncertainty is resolved by taking expectation in (8). For completeness, we first present the Euler equation in (12) for EU, under no cost, which is

$$p_t = \beta \mathrm{E}\left[\partial_2 \Pi(A_{t+1}, K_{t+1}) + (1-\delta) p_{t+1} \middle| z_t\right]. \tag{13}$$

It is common in the literature to use the EU model to interpret the Euler equation in (13) via recursive substitution. Iterating it forward, the price of capital at time t, p_t, is given by:

$$p_t = \mathrm{E}\left[\sum_{s=0}^{\infty} (1-\delta)^s \beta^{s+1} \partial_2 \Pi(A_{t+s+1}, K_{t+s+1})\right], \tag{14}$$

Accordingly, the firm's investment policy equates the purchase price of capital today with the *expected value* of the discounted present value of marginal profits in the future.

Now, we move our attention to the quantile model. The quantile Euler equation derived in equation (11) of Theorem 3.2 specializes as following

$$p_t = \beta Q_\tau\left[\partial_2 \Pi(A_{t+1}, K_{t+1}) + (1-\delta) p_{t+1} \middle| z_t\right]. \tag{15}$$

Notice that the main difference between (15) and (13) is that in the former case risk neutrality for the firm is imposed *a priori*, and the equilibrium solution is not able to account for firm's risk attitude.

An alternative representation for the well-known EU result uses the same recursive substitution of the Euler equation for the dynamic quantile model. To see this, write the Euler equation in (13) at time $t+1$ as following

$$p_{t+1} = \beta Q_\tau\left[\partial_2 \Pi(A_{t+2}, K_{t+2}) + (1-\delta) p_{t+2} \middle| z_{t+1}\right], \tag{16}$$

by substituting the future price of capital in (16) into (13) we obtain

$$\begin{aligned} p_t &= \beta Q_\tau\left[\partial_2\Pi(A_{t+1}, K_{t+1}) + (1-\delta)\beta Q_\tau\left[\partial_2\Pi(A_{t+2}, K_{t+2}) + (1-\delta)p_{t+2}\middle|z_{t+1}\right]\middle|z_t\right], \\ &= Q_\tau\left[Q_\tau\left[\beta\partial_2\Pi(A_{t+1}, K_{t+1}) + (1-\delta)\beta^2\partial_2\Pi(A_{t+2}, K_{t+2}) + (1-\delta)^2\beta^2 p_{t+2}\middle|z_{t+1}\right]\middle|z_t\right], \end{aligned}$$

where we can move the term $\partial_2\Pi(A_{t+1}, K_{t+1})$ into $Q_\tau[\cdot\,|z_{t+1}]$ because they are constant given the conditioning information. By iterating forward and recursively repeating this procedure, we have

$$\begin{aligned} p_t = Q_\tau\big[Q_\tau\big[\cdots\big[Q_\tau\big[&\beta\partial_2\Pi(A_{t+1}, K_{t+1}) + (1-\delta)\beta^2\partial_2\Pi(A_{t+2}, K_{t+2}) + (1-\delta)^2\beta^3\partial_2\Pi(A_{t+3}, K_{t+3}) + \cdots \\ &+ \cdots + (1-\delta)^{T-1}\beta^T\partial_2\Pi(A_{t+T}, K_{t+T}) \mid z_{t+T-1}\big]\big|z_{t+T-2}\big]\big|\cdots\big|z_{t+1}\big]\big|z_t\big]. \end{aligned} \tag{17}$$

It is convenient to use the notation introduced by de Castro and Galvao (2019), $Q_\tau^T[\cdot]$, to denote T applications of the quantile operator, conditioned on the information sets z_{t+T-1}, z_{t+T-2}, …, z_{t+1}, z_t, consecutively, as above. With this notation, (17) can be written as

$$p_t = Q_\tau^T\left[\sum_{s=0}^{T-1}(1-\delta)^s\beta^{s+1}\partial_2\Pi(A_{t+s+1}, K_{t+s+1}) + (1-\delta)^{T-1}\beta^T\partial_2\Pi(A_{t+T}, K_{t+T})\right]. \tag{18}$$

Notice that as $T \to \infty$, $(1-\delta)^{T-1}\beta^T\partial_2\Pi(A_{t+T}, K_{t+T}) \to 0$ if $\partial_2\Pi(\cdot)$ is bounded. In fact, one can prove that the right side of (18) also converges to a function that is denoted using the notation $Q_\tau^\infty[\cdot]$, that is,

$$\begin{aligned} p_t &= Q_\tau^\infty\left[\sum_{s=0}^{\infty}(1-\delta)^s\beta^{s+1}\partial_2\Pi(A_{t+s+1}, K_{t+s+1})\right] \\ &\equiv \lim_{T\to\infty} Q_\tau^T\left[\sum_{s=0}^{T-1}(1-\delta)^s\beta^{s+1}\partial_2\Pi(A_{t+s+1}, K_{t+s+1}) + (1-\delta)^{T-1}\beta^T\partial_2\Pi(A_{t+T}, K_{t+T})\right]. \end{aligned}$$

Therefore, we can write the purchase price of capital today in (11) as

$$p_t = Q_\tau^\infty\left[\sum_{s=0}^{\infty}(1-\delta)^s\beta^{s+1}\partial_2\Pi(A_{t+s+1}, K_{t+s+1})\right]. \tag{19}$$

Equation (19) has an interesting interpretation. It implies that the firm's investment policy equates the purchase price of capital today with the τ-*quantile* of the discounted present value of marginal profits in the future. Notice that the resolution of uncertainty in the present value is given by the τ-quantile. Therefore, the model shows that the investment depends directly on the firm's attitude toward the risk. Moreover, we note that in stating (19), one is assuming that the firm will be optimally resetting its capital stock in the future so that (13) holds in all subsequent periods.

An interesting specialization of (19) is for the case that shocks z_t are independent, such that the successive conditional quantiles in (17) and (18) become simple unconditional quantiles and can be applied directly to each term $\partial_2\Pi(A_t, K_t)$. Therefore, the right hand side of equation (19) simplifies to $\sum_{s=0}^{\infty}(1-\delta)^s\beta^{s+1}Q_\tau\left[\partial_2\Pi(A_{t+s+1}, K_{t+s+1}))\right]$, and we obtain

$$p_t = \sum_{s=0}^{\infty}(1-\delta)^s\beta^{s+1}Q_\tau\left[\partial_2\Pi(A_{t+s+1}, K_{t+s+1}))\right]. \tag{20}$$

This equation is the exact analogue of the expectation case,

$$p_t = \sum_{s=0}^{\infty}(1-\delta)^s\beta^{s+1}\mathrm{E}\left[\partial_2\Pi(A_{t+s+1}, K_{t+s+1})\right],$$

which one can obtain from the linearity of expectations and equation (14).

The result in this section has important practical implications for computing the value of investment. Intuitively risk averse investors will assign lower values to investments, or assets, that have more risk associated with them than to otherwise similar investments that are less risky. This intuitive result is confirmed in equations (19) and (20) for the quantile model. For a smaller quantile, that is, for a firm with more downside risk aversion, the price of investment is smaller, everything else constant.

Moreover, another important feature of this model is that the value of investment depends explicitly on the risk attitude parameter. As discussed in more detail in Subsection 2.4 above, usual methods for dealing with risk involves computing a value that is risk adjusted. However, these methods do not have a structural parameter capturing the risk attitude nor have a clear connection with standard EU preferences. In contrast, our treatment is directly connected to quantile preferences and the value of any investment depends explicitly on the downside risk attitude parameter τ through the quantile of the present value of its cash flows, corresponding quantile maximization theory, as equations (19) and (20) clearly reveal.

3.4 A quantile q-theory of investment

We now discuss the Tobin's q-theory in the context of the quantile investment model. Since Hayashi (1982), it has been standard in the literature to use linear regressions to learn how much the marginal gains of newly added capital, the Tobin'q, is able to explain aggregate investment. Later we investigate this relation empirically for the quantile model.

As usual in the literature, we consider a particular case where Π is proportional to K, C is quadratic, and p is constant. That is, assume the following:

Assumption 3. *The following hold:*

(i) $\Pi(A, K) = AK^\alpha$, *with* $\alpha \in (0, 1)$*;*

(ii) $C(K', A, K) = \frac{\gamma}{2}\left(\frac{K'-(1-\delta)K}{K}\right)^2 K$, *with* $\gamma > 0$*;*

(iii) p *is constant and strictly positive;*

(iv) $A(z) = z$, *and Assumption 1(ii) holds.*

(v) There exists $0 < \underline{q} < \overline{q} < \infty$, *such that* $Q_\tau[z'|z] \in [\underline{q}, \overline{q}], \forall z \in \mathcal{Z} \subset \mathbb{R}_{++}$.

Assumption 3(i) assumes a constant returns of scale production function with curvature parameter α, and part (ii) considers a quadratic adjustment cost function with parameter γ for the level of adjustment cost. Moreover, we focus on constant prices, in part (iii). These conditions are standard in the literature. Assumption 3(iv) and (v) impose some mild conditions on the Markov process and behavior of its quantiles. Since the shocks are now identified with the productivity shock A, they need to be strictly positive. However, their distribution can be unbounded, like a log-normal. Only the τ-quantile needs to be bounded for all $z \in \mathcal{Z}$ by Assumption 3(v). Assumption 3 implies Assumptions 1 and 2, as the proof of the next result (in the appendix) shows. Under Assumption 3 we can write the value function in equation (8) as:

$$V(K, z) = \max_{K' \in \Gamma(K)} \left\{ AK^\alpha - \frac{\gamma}{2}\left(\frac{K' - (1-\delta)K}{K}\right)^2 K - p\left[K' - (1-\delta)K\right] + \beta Q_\tau\left[V(K', z')\Big|z\right] \right\}. \quad (21)$$

Let i denote the ratio of the investment with respect to K, that is, $i \equiv \frac{I}{K}$. The first order condition for the choice of the investment level implies that the investment rate can the written as a function of the τ-quantile of the Tobin's q. The following proposition summaries the result.

Proposition 3.3. *Under Assumption 3, Theorems 3.1 and 3.1 hold and the Euler equation is given by:*

$$i = \frac{1}{\gamma}\left(\beta Q_\tau\left[\partial_K V(K', z')|z\right] - p\right), \quad (22)$$

where $\partial_K V$ *is the partial derivative of the value function with respect to capital, given by* (10).

In the literature, the term $\partial_K V$ that appears in (22) is conventionally termed "marginal q" or Tobin's q and denoted by q. The Euler equation (22) is similar to that in the standard expected utility model, with the difference that it contains the τ-quantile value of the "marginal q," instead of its usual expected value.

Equation (22) establishes that the investment rate is linearly related to the difference between the τ-quantile of the future marginal value of new capital and the current price of capital. The future marginal value of new capital is uncertain, and the quantile is the operator solving the uncertainty.

It is possible to provide a more explicit form of the Euler equation than (22). From (11), the Euler equation can be written as:

$$\beta Q_\tau\left[z'\alpha(K')^{\alpha-1}+\gamma\frac{K''}{K'}\left(\frac{K''}{K'}-(1-\delta)\right)-\frac{\gamma}{2}\left(\frac{K''}{K'}-(1-\delta)\right)^2+p(1-\delta)\Big|z\right]=p+\gamma\left(\frac{K'}{K}-(1-\delta)\right). \tag{23}$$

The Euler equation in (23) has a conditional quantile function form that can be used to estimate parameters of interest. Indeed, it has been standard in the empirical literature to use the Euler equation in investment models for estimation. It forms the basis for a wide range of empirical exercises since it allows the researcher to substitute the average value of q (observable from the stock market) for marginal q (unobservable).

We show now that the *marginal* value of capital can be replaced by the *average* value of capital if $\alpha = 1$ in Assumption 3. With this modification, Assumption 3 no longer implies Assumption 1, since u is now just concave, not strictly concave. However, we are still able to establish the existence of a value function, as the next result states. This framework sets the basis for the quantile q-theory empirical approach that we discuss in the next section. Since p is constant by Assumption 3, we can identify z with A. Therefore, for simplicity, we substitute z by A in the expressions below.

Proposition 3.4. *Let Assumption 3 hold, but with* $\alpha = 1$. *Then, there exist a unique* $V : \mathcal{K}\times\mathcal{Z}\to\mathbb{R}$ *functional equation* (8) *has a solution of the form*

$$V(K, A) = \varphi(A)K, \tag{24}$$

where φ *is an increasing function satisfying the implicit relation*

$$\begin{aligned}\varphi(A) \;=\;& A-\frac{1}{2\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right)-p\right]^2-\frac{p}{\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right)-p\right]\\ &+\beta\varphi\left(Q_\tau\left[A'|A\right]\right)\left[\frac{1}{\gamma}\left(\beta\varphi\left(Q_\tau\left[A'|A\right]\right)-p\right)+1-\delta\right].\end{aligned} \tag{25}$$

Moreover, the rate of investment, $i = I/K$ *does not depend on the current capital level* K.

The result in Proposition 3.4 shows that, under the linear-quadratic formulation of the capital accumulation – constant returns of scale and quadratic cost function – the value function is proportional to the stock of capital and the investment rate is independent of the current level of the capital stock. These results are similar to the expected utility case. In this case, from Proposition 3.4 we obtain

$$Q_\tau\left[\partial_K V(K', A')|z\right] = Q_\tau\left[\varphi(A)\right],$$

and importantly, from equation (24), the marginal and average q are the same, that is

$$\partial_K V(K,A) = V(K,A)/K = \varphi(A).$$

Hence, equation (22) simplifies to

$$\frac{1}{\gamma}\left(\beta Q_\tau\left[\varphi(A')|z\right] - p\right) = \frac{1}{\gamma}\left(\beta Q_\tau\left[\frac{V(K,A)}{K}\Big|z\right] - p\right) = i. \tag{26}$$

This equation relates investment to the average Tobin's q. In particular, the τ-quantile of the average Tobin's q, $V(K,A)/K$, is a linear function of investment. Hence, this result has an important practical implication, since standard quantile regression econometric methods can be used to estimate the strength of this relation.

4 Numerical computation

It is straightforward to employ numerical solutions to characterize investment decisions in the quantile model. One can use the value function iteration method to compute the optimal capital, investment, and the value of the firm. The algorithm for computation is very similar to the case of expectation, as discussed in de Castro et al. (2023a). The main difference is that we compute a conditional quantile instead of a conditional average.

Recall that the intertemporal firm choice is represented by the value function in equation (8). We reproduce it here for easy of exposition.

$$V(K,z) = \max_{K'\in\Gamma(K)}\left\{\Pi(A,K) - C(K',A,K) - p\left[K' - (1-\delta)K\right] + \beta Q_\tau\left[V(K',z')\Big|z\right]\right\}.$$

We numerically compute the optimal value function and the associated policy function K'. From the restriction $I = K' - (1-\delta)K$, we are able to calculate the optimal investment decision.

To implement the procedure, we specify the model as in Assumption 3, such that $\Pi(A,K) = AK^\alpha$ and $C(K',A,K) = \frac{\gamma}{2}\left(\frac{K'-(1-\delta)K}{K}\right)^2 K$. This exercise assumes a quadratic cost of adjustment specification, and we focus on computing the optimal investment and value function (value of the firm). We investigate the effects of varying some parameters on these functions. In particular, we set the following parameters at levels found in previous studies (see, e.g., Cooper and Ejarque, 2003): discount factor $\beta = 0.95$; curvature of the profit function $\alpha = 0.7$; adjustment cost $\gamma = 0.25$; and price $p = 1$. We then vary the level of depreciation δ, and risk attitude (quantiles τ).

To complete the model, we represent the functions through matrices or arrays and initialize the variables of the following algorithm:

1. Consider a discretization for capital and investment as: $\mathcal{K} = \{k^1, k^2, ..., k^p\}$ and $\mathcal{I} =$

$\{i^1, i^2, ..., i^q\}$ from zero to three. We set different values for $p = q = 500$.

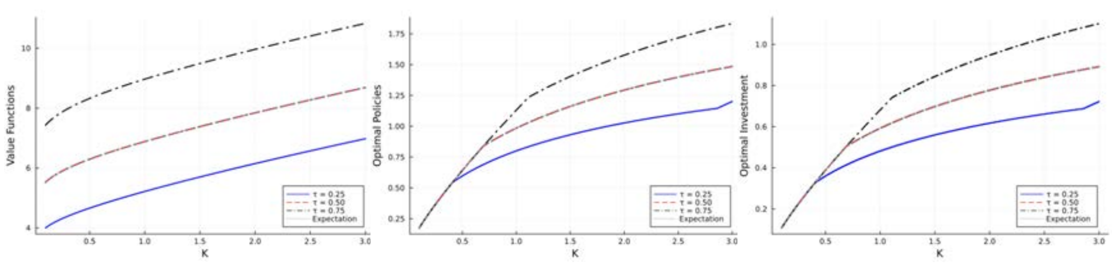
2. Consider five possible states, and the following values for shocks: $\mathcal{Z} = \{0.80, 0.90, 1, 1.1, 1.2\}$.

3. Consider a simple stochastic transition matrix $P[i,j] = p_{ij} = 0.20$ that describes the probability of moving from state j to i, for all i and j. For simplicity, this matrix considers an equal probability, $p_{ij} = 0.2$, for moving from state j to i for all five states considered.

4. Specify relevant parameters as follows: the discount factor $\beta = 0.95$; adjustment cost $\gamma = 0.25$; quantile risk attitudes $\tau = \{0.25, 0.50, 0.75\}$; depreciation $\delta = \{0.50, 0.25, 0.10\}$.

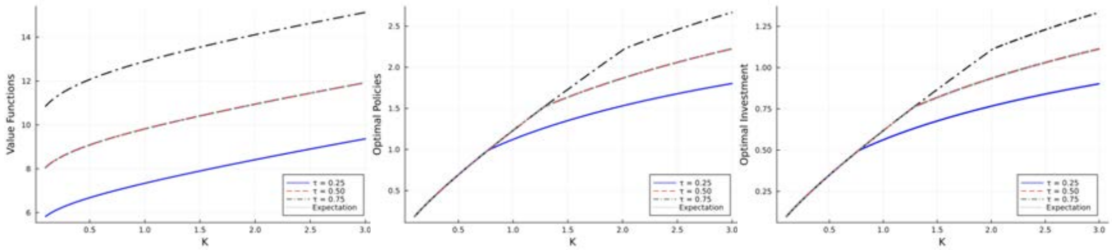
The results for value functions, optimal capital and investment are presented in Figures 2, 3, and 4 for $\delta = 0.50$, $\delta = 0.25$, $\delta = 0.10$, respectively. Solid blue, dashed red, and black dashed-dotted lines represent quantiles 0.25, 0.50, and 0.75, respectively. For comparison, results for the standard expected utility (EU) case are in gray dotted lines. However, the curves corresponding to the EU case cannot be easily distinguished because they coincide with the curves for the median $\tau = 0.5$. This is expected, since the distributions considered in these examples are symmetric and the shocks independent and identically distributed (iid).

Figure 2 shows that the value functions are increasing with K for all quantiles and the EU case. One can also see that, for a fixed value of capital K, value functions increase with the quantiles. Regarding the optimal capital and investment, it can be observed that they have the same shape with different scales. This happens because $I = K' - (1 - \delta)K$. Optimal capital and investment grow at the same rate and are equal across quantiles up to a point, and then start to differ across quantiles. For large values of capital, the larger the quantile, the larger the capital and investment.



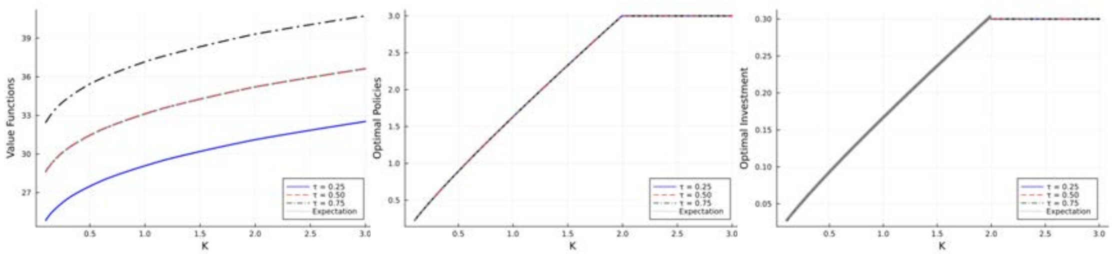
(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 2: Shocks iid, and $\delta = 0.50$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 3: Shocks iid, and $\delta = 0.25$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 4: Shocks iid, and $\delta = 0.10$

Figure 3 considers the case where $\delta = 0.25$. In this case, $I = K' - 0.25K$. Panel (a) of Figure 3 shows the same general features for value functions as in the previous case. But panels (b) and (c) are somewhat different. The slope of capital and investment less steep. As in the previous case, the optimal capital and investment coincide for low values of capital for all quantiles, but differ for larger level of K.

Figure 4 considers the case where $\delta = 0.10$. Panel (a) of Figure 4 shows the same increasing features for value functions are in the previous case. But panels (b) and (c) are different, showing that the optimal capital and investment collapse to the same values across quantiles. This shows that it is possible that firms have different risk attitudes, and thus different values, but still invest equally depending on the depreciation parameter. One can also observe that optimal capital is increasing on K until it reaches the maximum.

We note that these numerical results connect with a recent literature on corporate finance documenting low investment rates. Gutiérrez and Philippon (2018) and Alexander and Eberly (2018) document that recently US investment has been lackluster, despite rising valuations. The two leading explanations for this pattern relate to are intangible capital and rents; see, e.g., Crouzet and Eberly (2019, 2021). Results using the dynamic quantile model are able to provide an alternative explanation for this observation, nonetheless. The numerical solutions show that firm valuations as well as investments vary substantially across quantiles. When depreciation is relatively large, risk averse firms have lower value and investment. In contrast, low risk averse firms have higher valuations and investment. When depreciation is close to zero, there is still large differences among firm's valuation across quantiles, but the investment rates are the same. The quantile model shows that it is possible to observe large differences across firm valuations together with a much smaller gap in their investment rates.

Appendix B contains some additional robustness results for the case of $\gamma = 0$ and dependent shocks. The results are qualitatively similar to those in this section.

5 Empirical investigation

Finally, we assess predictions of the dynamic quantile model discussed in Section 3. Since Hayashi (1982), it has been standard in the literature to use linear regressions to learn how much Tobin'q is able to explain investment. Using the quantile model, we shed new light in this matter by exploring a number of related points. The first is how strong is the relation between Tobin's q and investment along the conditional quantile function of Tobin's q. Second, whether there is any evidence of heterogeneity in this relation across the distribution. Third, and importantly, we estimate the firm's risk attitude, which is captured by the quantile τ.

5.1 Regression analysis

The investment regression is obtained from the structural model after showing that the marginal q is equal to the average q in Proposition 3.4. The theory developed previously can be used as a baseline econometric model for the quantile q theory. We estimate a quantile regression model, where a proxy for investment demand (Tobin's q) is regressed on firm's investment.[10] In particular, from a panel data of firms, the conditional quantile of q in equation (26) can be rewritten as a linear function of investment spending as following

$$Q_\tau \left[q_{it} | i_{it} \right] = \alpha_\tau + \theta_\tau i_{it}, \tag{27}$$

where the subscripts index different firms and time, respectively. The quantity $i_{it} = I_{it}/K_{i,t-1}$ is so that the base capital stock comes from the previous period, with I denoting investment, K capital stock, q the observed Tobin's q. The parameter of interest is θ_τ, which is allowed to depend on quantile τ when estimating the model.

The model in equation (27) has a standard quantile regression (QR) representation, see, e.g., Koenker (2005), where the conditional quantile function of the average Tobin's q is a linear function of investment. We note that the slope coefficient in the regression (27) is not able to separately identify both structural parameters γ and β, hence we simply interpret this slope coefficient as the sensitivity of investment opportunities with respect to investment spending.

A review of the corporate investment literature shows that it is common in empirical work in the area to consider panel data models with firm-fixed effects (see, e.g., Kaplan and Zingales (1997) and Almeida and Campello (2007)). From an estimation point of view, there are

[10] For simplicity, we abstract from potential measurement error problems in Tobin's q. For a discussion on this, see, e.g., Almeida et al. (2010).

distinct advantages in exploiting repeated observations from individuals to identify the model (Blundell et al., 1992). In an investment model setting, exploiting individual firm specific effects allows for unobserved firm heterogeneity, for model consistency in the presence of unobserved idiosyncrasies that may be simultaneously correlated with investment and Tobin's q, and contributes to estimation precision.

Panel data QR is a suitable tool for analyzing the behavior of investment equations since it allows controlling for individual specific intercepts, and most importantly, it allows exploring a range of conditional quantile functions exposing a variety of forms of conditional heterogeneity. The linear fixed effects QR version of the baseline equation described in (27) are given by the following conditional quantile functions

$$Q_\tau[q_{it}|\alpha_i, i_{it}] = \alpha_{i\tau} + \theta_\tau i_{it}. \tag{28}$$

The empirical estimation of panel data fixed effects QR (FE-QR) in (28) is challenging because of the nonlinearity of the problem, and associated incidental parameters problem. We refer the reader to Galvao and Kato (2017) for a brief review of the literature. In this paper, we use the simple methods discussed in Canay (2011). To assess the potential heterogeneity in investment relative to the Tobin's q predicted by the quantile investment model, we estimate the QR model in equation (28) for several different quantiles.

The quantile τ carries important information about the downside risk attitude, and we wish to identify and estimate it from the data. The quantile τ can be recovered from the data, provided the underlying distribution characterizing the residual function in the conditional quantile model follows an Asymmetric Laplace distribution.[11] Let $u_{it} := q_{it} - \alpha_i - \theta i_{it}$. The distribution of the residuals, u_{it}, is the distribution of investment opportunities net of the investment spending. Hence, statistically, the quantile τ captures the asymmetry, or skewness, of the distribution of this residual function. Hence, the risk attitude, τ, is identified from the skewness of this distribution. Intuitively, when observed investment spending is small, relative to investment opportunities, the distribution of the residuals is skewed to the right, and hence the risk aversion is large (small quantile). On the other hand, a larger quantile (low risk aversion) would be elicited when observed investment spending is large, relative to investment opportunities, such that the distribution is skewed to the left.

In terms of practical estimation, the key idea is to consider additional moment conditions resulting from the Asymmetric Laplace Distribution assumption. These additional moments can be characterized as

$$E\left[\frac{1-2\tau}{\tau(1-\tau)} - \frac{(Y - X^\top\beta)}{\sigma}\right] = 0, \quad \text{and} \quad E\left[-\frac{1}{\sigma} + \frac{1}{\sigma^2}\rho_\tau(Y - X^\top\beta)\right] = 0, \tag{29}$$

[11]This approach is fully discussed by Yu and Zhang (2005) and Bera et al. (2016). The Asymmetric Laplace distribution, $\text{ALD}(\mu, \sigma, \tau)$, is skewed to right when $\tau < 1/2$ (positive skewness), and skewed to left when $\tau > 1/2$ (negative skewness).

where $\rho_\tau(u) = u(\tau - I\{u < 0\})$ is the check function (cf. Koenker and Bassett, 1978), $Y = q_{it}$, $X = (1, i_{it})^\top$, and $\beta = (\alpha, \theta)$. Parameters in equations in (29) can be estimated using their sample counterparts. Regarding inference for quantile estimates, we compute standard errors by using subsampling methods – see, e.g., Politis and Romano (1994), and Politis et al. (1999). We use the subset size of $b = 2n/3$, with 199 replications.

For completeness, we also estimate the corresponding model using a conditional average restriction as

$$E[q_{it}|i_{it}] = \mu_i + \phi i_{it}. \tag{30}$$

Notice that both regression models in equations (27) and (30) are derived from their respective, quantile and average, Euler equations.[12]

5.2 Data description

The data are taken from COMPUSTAT and cover 1970 to 2021. The sample consists of manufacturing firms with fixed capital of more than \$ 5 million (with 1976 as the base year for the CPI), and the sample firms have growth of less than 100% in both assets and sales; cf. Almeida and Campello (2007). We compute variables as is usual in the literature. The average q is the firm's market value divided by the book value of the asset. For robustness reasons, we use two different proxies for average q: (1) q_1 as in Almeida et al. (2010); and (2) q_2 as in Daines (2001). We also use two proxies for investment spending: (1) i_1 as capital expenditure divided by past capital stock; and (2) i_2 as growth in capital stock.

Table 1: Descriptive Statistics

Variable	Obs.	Mean	Std. Dev.	Min	Max	Skewness
Investment i_1	44798	0.215	0.155	-0.033	2.310	2.339
Investment i_2	44798	1.037	0.171	0.152	1.997	1.153
Average q_1	44798	0.863	0.502	0.036	22.792	9.444
Average q_2	44798	0.617	0.487	0.001	14.409	8.368

Summary statistics for all investment and q variables are presented in Table 1. These statistics are similar to those reported in the literature (e.g. Almeida and Campello (2007)). We also report the skewness (Fisher's moment coefficient of skewness) of the variables. They show a positive skewness of all variables considered, especially the average q. To save space, we omit further discussion of descriptive statistics.

[12] Notably, when the true regression model is the one featuring Tobin's q as the dependent variable as in (27) and (30), but we estimate the traditional investment regression in which investment is the dependent variable, then the parameter of interest may be inconsistent (see, e.g., Hansen, 2022).

We also provide estimate results for different subsamples of the data. First, we construct a size category, where we a use the percentile of the total assets to obtain firms in percentile 70 and above as Large, and firms in the percentile 30 and below as Small. In addition, we categorize firms by age, where we label firms with over 30 years in the data as Old, and firms with less than 15 years in the data as Young.

5.3 Estimation results

We use a panel data fixed effects quantile regression (FE-QR) methodology for estimating the slopes of interest, as well as the quantile risk attitude. Table 2 collects the main results for the FE-QR using both measures of investment and Tobin's q. The top portion of Table 2 reports results for slope coefficients of the regression of Tobin's q on investment, for given fixed values of τ. The middle part of the table provides estimates of the risk attitude τ, together with the corresponding slope coefficients. Finally, the lower part of the table displays fixed effects ordinary least squares (FE-OLS) estimates. As discussed above, for robustness reasons, we use two measures of Tobin's q (q_1 and q_2) and two measures of investment (i_1 and i_2). The table displays point estimates as well as standard errors, which are inside parenthesis.

Our FE-QR estimates document a positive relation between investment demand and the investment spending across quantiles of the conditional distribution of Tobin's q. The coefficients are statistically different from zero at usual levels of significance. The results document important heterogeneity on the response of Tobin's q to changes in investment spending. Firms in different quantiles of the conditional distribution of investment opportunities (Tobin's q) respond differently to marginal changes in investment spending. In particular, Table 2 shows evidence of increasing coefficients across quantiles. The marginal value of the investment is more responsive to investment spending (large magnitude of the coefficients) for firms at high quantiles. In addition, the FE-OLS estimates in Table 2 show statistically significant positive coefficients for the standard OLS, for all measures of Tobin's q and investment.

The middle part of Table 2 provides results for estimates of the risk attitude τ, along with the corresponding regression slope coefficients for the estimated quantile. Corresponding standard errors are inside parenthesis. Recall the intuition for estimation of the quantile risk attitude. A low estimated quantile (a larger risk aversion) indicates positive skewness of the distribution of investment opportunities net of investment spending, which is associated with a larger mass of the distribution on its left hand side. The results in Table 2 show quantile estimates $\widehat{\tau} = 0.41$ for the different measures of q and investment. Standard errors of the risk attitude estimates are small, varying from 0.003 to 0.010. Since for the dynamic quantile model risk neutrality would correspond to a quantile of 0.5, a simple t-test for the null hypothesis that the firm is risk neutral ($H_0 : \tau = 0.5$) is strongly rejected at any usual level of significance. Hence, these results document empirical evidence of a low estimate of aggregate corporate downside risk aversion.

These results are surprising from the point of view of the finance literature at large, which has relied on models of risk neutral firms in motivating empirical studies. Moreover, these results connect with the existing experimental evidence on eliciting quantile risk attitude of individual decision makers. For instance, de Castro et al. (2022) find that the average of the distribution of the estimated quantile is 0.42, and a majority of subjects are risk averse.

Table 2: Slope coefficients of regression results for FE-QR and FE-OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates.

FE-QR		i_1				i_2	
τ	q_1		q_2		q_1		q_2
0.1	0.137		0.183		0.172		0.206
	(0.017)		(0.014)		(0.014)		(0.012)
0.2	0.172		0.205		0.199		0.221
	(0.009)		(0.009)		(0.007)		(0.007)
0.3	0.195		0.224		0.206		0.223
	(0.007)		(0.007)		(0.006)		(0.006)
0.4	0.215		0.245		0.212		0.225
	(0.007)		(0.007)		(0.006)		(0.005)
0.5	0.243		0.279		0.221		0.238
	(0.007)		(0.007)		(0.006)		(0.006)
0.6	0.279		0.319		0.237		0.262
	(0.007)		(0.007)		(0.006)		(0.006)
0.7	0.312		0.348		0.245		0.278
	(0.008)		(0.008)		(0.007)		(0.007)
0.8	0.381		0.410		0.268		0.311
	(0.013)		(0.011)		(0.009)		(0.009)
0.9	0.568		0.624		0.363		0.438
	(0.027)		(0.028)		(0.015)		(0.016)
$\widehat{\tau} = 0.41$	0.218	$\widehat{\tau} = 0.41$	0.249	$\widehat{\tau} = 0.41$	0.212	$\widehat{\tau} = 0.41$	0.225
(0.006)	(0.007)	(0.010)	(0.007)	(0.003)	(0.006)	(0.005)	(0.005)
FE-OLS	0.388		0.434		0.357		0.395
	(0.016)		(0.015)		(0.013)		(0.013)

Notes: Standard errors inside parenthesis.

Overall, these results show evidence of a statistically positive relation between Tobin's q and investment. Moreover, as we increase quantiles from values close to 0 (corresponding to high risk aversion) to values close to 1 (risk-loving attitude), the estimated (positive) effects increase. Therefore, our results show that there is heterogeneity in this relation across risk attitudes, with low estimates for more downside risk aversion. Finally, and importantly, the results indicate that firms are somewhat risk averse.

Appendix C collects additional results when estimating the model using standard quantile regression without controlling for firm specific fixed effects. We provide results for the entire sample, and the size and age subsamples. The results are qualitative similar, with the difference that risk attitude estimates are smaller.

5.4 Subsample variation

Now we reestimate the model using different subsamples of the data. We estimate the models using panel data fixed effects quantile regression (FE-QR) estimator. We use the same proxies for Tobin's q (q_1 and q_2) and investment (i_1 and i_2) as above. Here we present results for the case of size Small/Large firms and age Young/Old firms. As in the previous cases, we present results for the slope and risk attitude estimates.

Table 3: Slope coefficients of regression results for FE-QR and FE-OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Small firms.

FE-QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.120		0.131		0.232		0.222
	(0.017)		(0.015)		(0.014)		(0.016)
0.2	0.138		0.160		0.242		0.228
	(0.013)		(0.011)		(0.014)		(0.011)
0.3	0.169		0.174		0.244		0.228
	(0.011)		(0.011)		(0.011)		(0.009)
0.4	0.191		0.200		0.248		0.243
	(0.010)		(0.010)		(0.008)		(0.008)
0.5	0.206		0.218		0.267		0.272
	(0.009)		(0.009)		(0.007)		(0.008)
0.6	0.228		0.246		0.284		0.295
	(0.008)		(0.008)		(0.007)		(0.007)
0.7	0.229		0.245		0.263		0.282
	(0.010)		(0.009)		(0.008)		(0.008)
0.8	0.279		0.304		0.273		0.304
	(0.020)		(0.021)		(0.013)		(0.014)
0.9	0.383		0.406		0.313		0.364
	(0.028)		(0.028)		(0.020)		(0.021)
$\hat{\tau} = 0.45$	0.197	$\hat{\tau} = 0.45$	0.211	$\hat{\tau} = 0.45$	0.250	$\hat{\tau} = 0.44$	0.247
(0.011)	(0.009)	(0.008)	(0.009)	(0.006)	(0.008)	(0.008)	(0.008)
FE-OLS	0.251		0.274		0.309		0.335
	(0.020)		(0.019)		(0.017)		(0.017)

Notes: Standard errors inside parenthesis.

Tables 3 and 4 collect results for the FE-QR case when using Small and Large firms, respectively. The first result we notice is that, as in the previous cases, in both tables the slope coefficients of the regression of investment opportunities (Tobin's q) on investment spendings vary across quantiles. More risk aversion (smaller quantile) is associated with smaller responses.

Second, the coefficients for low quantiles for Small firms in Table 3 are larger in magnitude than those for the Large firms in Table 4. However, for top quantiles we observe the reverse, that is, coefficients for Large firms are larger. These results are robust to the different measures of Tobin's q and investment.

Regarding the results for the estimated risk aversion for the FE-QR model in Tables 3 and 4, they are similar to those for all firms in Table 2. Nevertheless, they suggest that small firms may be less risk averse than large firms. For Small firms the risk attitude estimates vary between 0.44–0.45, while for Large firms it varies between 0.41–0.42. Overall, in both Small and Large firms cases, the estimates are still statistically smaller than the median, providing evidence of corporate risk aversion.

Table 4: Slope coefficients of regression results for FE-QR and FE-OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Large firms.

FE-QR		i_1				i_2	
τ	q_1		q_2		q_1		q_2
0.1	0.166		0.253		0.206		0.223
	(0.050)		(0.040)		(0.018)		(0.023)
0.2	0.186		0.256		0.180		0.209
	(0.023)		(0.021)		(0.015)		(0.014)
0.3	0.227		0.284		0.183		0.203
	(0.017)		(0.017)		(0.011)		(0.010)
0.4	0.238		0.320		0.186		0.212
	(0.016)		(0.016)		(0.011)		(0.010)
0.5	0.289		0.365		0.190		0.223
	(0.016)		(0.016)		(0.011)		(0.011)
0.6	0.344		0.432		0.185		0.227
	(0.015)		(0.018)		(0.011)		(0.012)
0.7	0.394		0.491		0.197		0.248
	(0.018)		(0.018)		(0.013)		(0.014)
0.8	0.485		0.554		0.242		0.290
	(0.022)		(0.023)		(0.018)		(0.019)
0.9	0.814		0.892		0.385		0.483
	(0.068)		(0.071)		(0.031)		(0.034)
$\hat{\tau}$ = 0.42	0.244	$\hat{\tau}$ = 0.41	0.323	$\hat{\tau}$ = 0.42	0.190	$\hat{\tau}$ = 0.42	0.208
(0.009)	(0.016)	(0.010)	(0.016)	(0.011)	(0.011)	(0.009)	(0.010)
FE-OLS	0.514		0.598		0.339		0.386
	(0.037)		(0.038)		(0.025)		(0.025)

Notes: Standard errors inside parenthesis.

Now, we reestimate the model using age of the firms, that is Young and Old firms. The corresponding results for the FE-QR estimates are presented in Tables 5 and 6, respectively. The results are very similar to those discussed previously, and describe the same pattern. FE-

QR coefficients show that the slope coefficient of the regression of Tobin's q on investment is increasing across quantiles. Moreover, risk attitude estimates show evidence of risk aversion for both Young and Old firms, with Young firms appearing to be less risk averse than Old firms.

Table 5: Slope coefficients of regression results for FE-QR and FE-OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Young firms.

FE-QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.058		0.097		0.151		0.191
	(0.017)		(0.019)		(0.017)		(0.014)
0.2	0.102		0.113		0.179		0.201
	(0.011)		(0.011)		(0.009)		(0.009)
0.3	0.126		0.145		0.182		0.199
	(0.009)		(0.009)		(0.008)		(0.008)
0.4	0.139		0.163		0.190		0.211
	(0.009)		(0.008)		(0.007)		(0.007)
0.5	0.161		0.175		0.202		0.224
	(0.009)		(0.008)		(0.007)		(0.007)
0.6	0.192		0.207		0.218		0.243
	(0.007)		(0.008)		(0.007)		(0.008)
0.7	0.193		0.211		0.208		0.240
	(0.009)		(0.009)		(0.008)		(0.008)
0.8	0.233		0.250		0.214		0.248
	(0.017)		(0.017)		(0.012)		(0.012)
0.9	0.340		0.395		0.268		0.343
	(0.027)		(0.027)		(0.021)		(0.022)
$\widehat{\tau} = 0.44$	0.151	$\widehat{\tau} = 0.45$	0.167	$\widehat{\tau} = 0.44$	0.191	$\widehat{\tau} = 0.44$	0.215
(0.007)	(0.009)	(0.008)	(0.008)	(0.008)	(0.007)	(0.007)	(0.007)
FE-OLS	0.218		0.248		0.268		0.318
	(0.021)		(0.021)		(0.018)		(0.017)

Notes: Standard errors inside parenthesis.

Table 6: Slope coefficients of regression results for FE-QR and FE-OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Old firms.

FE-QR		i_1				i_2	
τ	q_1		q_2		q_1		q_2
0.1	0.229		0.324		0.199		0.221
	(0.040)		(0.032)		(0.022)		(0.024)
0.2	0.260		0.296		0.198		0.214
	(0.021)		(0.017)		(0.016)		(0.016)
0.3	0.268		0.337		0.200		0.213
	(0.016)		(0.016)		(0.013)		(0.013)
0.4	0.289		0.355		0.201		0.215
	(0.015)		(0.016)		(0.013)		(0.012)
0.5	0.319		0.377		0.211		0.220
	(0.017)		(0.016)		(0.014)		(0.012)
0.6	0.359		0.432		0.216		0.231
	(0.017)		(0.019)		(0.014)		(0.014)
0.7	0.415		0.470		0.239		0.275
	(0.020)		(0.020)		(0.017)		(0.017)
0.8	0.512		0.553		0.299		0.337
	(0.025)		(0.027)		(0.019)		(0.018)
0.9	0.816		0.864		0.424		0.505
	(0.060)		(0.050)		(0.028)		(0.029)
$\widehat{\tau} = 0.39$	0.285	$\widehat{\tau} = 0.38$	0.353	$\widehat{\tau} = 0.39$	0.204	$\widehat{\tau} = 0.39$	0.214
(0.009)	(0.015)	(0.009)	(0.016)	(0.009)	(0.013)	(0.008)	(0.012)
FE-OLS	0.610		0.681		0.475		0.474
	(0.038)		(0.036)		(0.030)		(0.028)

Notes: Standard errors inside parenthesis.

6 Conclusion

This paper develops a novel dynamic model for investment under uncertainty using quantile preferences. The uncertainty in the dynamic model is resolved trough the quantile, and the parameter τ reflects the firm's risk attitude, which captures downside risk. Our framework allows for modeling *both* the firm's attitude toward risk as well as the decision on intertemporal substitution. In our analysis, we first establish the properties of the model. In particular, we show that the dynamic problem yields a value function (via a fixed point argument) that is concave and differentiable. We further derive the corresponding Euler equation, and establish a new expression for the marginal value of investment. We also establish a quantile q-theory of investment.

In this framework, numerical solutions document evidence of heterogeneity in the value of the firm and investment decisions across quantiles. Empirical estimation of the quantile investment model using quantile regression shows that the strength of the relation between Tobin's q and investment increases as risk aversion decreases. The results document evidence of downside risk aversion attitude for firms.

Many issues remain to be investigated. Extensions to of the dynamic quantile model to include several important features, as for example, costly external finance, cash, risk-free debt, cash and debt, and risky debt, are of interest. Inclusion of potential financial frictions is also an important avenue for future research.

Appendix

A Proofs

In this appendix, we first establish the results in Section 3, and then, results in Section 2.

A.1 Proofs of the results in Section 3

Proof of Theorem 3.1. Assumptions 1 and 2 of de Castro and Galvao (2019) are implied by Assumption 1 in the main text, with $\mathcal{X} = \mathcal{K} = [0, M]$. Dynamic consistency follows from their Theorem 3.9. Their Theorem 3.11 guarantees that problem (8) has a unique fixed point, and Theorem 3.12 implies that the value function is differentiable, with

$$\frac{\partial V}{\partial K}(K, z) = \frac{\partial u}{\partial K}(K, K', z) = \partial_2 \Pi(A(z), K) - \partial_3 C(K', A(z), K) + (1 - \delta)p(z),$$

if K' is optimal and interior given (K, z). This concludes the proof. $\square$

Proof of Theorem 3.2. Since $z \mapsto A(z)$ and $z \mapsto \frac{\partial u}{\partial K}(K, K', z)$ are strictly increasing by Assumption 2, the conditions of Theorem 3.18 of de Castro and Galvao (2019) are satisfied. From that result, the Euler Equation (11) holds for an optimal path $\{K_t\}_{t=1}^{\infty}$ that is interior. $\square$

Proof of Proposition 3.3. We first show that Assumption 3 implies Assumptions 1 and 2. Notice that Assumption 1(ii) is explicitly required in Assumption 3(iv). To see Assumption 1(i), notice that from equation (9) and Assumption 3, we have

$$\begin{aligned} u(K, K', z) &= A(z)K^{\alpha} - \frac{\gamma}{2}\frac{\left[K' - (1 - \delta)K\right]^2}{K} - p\left[K' - (1 - \delta)K\right] \\ &= zK^{\alpha} - \gamma\frac{(K')^2}{2K} + \gamma(1 - \delta)K' - \frac{\gamma(1 - \delta)^2}{2}K - p\left[K' - (1 - \delta)K\right]. \end{aligned}$$

It is clear that u is C^1 (continuous first derivatives) and strictly increasing in Z. Let us verify that it is strictly concave in (K, K'). For this, observe that

$$\begin{aligned}
\frac{\partial u}{\partial K}(K, K', z) &= A(z)\alpha K^{\alpha-1} + \gamma\frac{(K')^2}{2K^2} - \frac{\gamma(1-\delta)^2}{2} + p(1-\delta); \\
\frac{\partial u}{\partial K'}(K, K', z) &= -\gamma\frac{K'}{K} + \gamma(1-\delta) - p; \\
\frac{\partial^2 u}{\partial K^2}(K, K', z) &= A(z)\alpha(\alpha-1)K^{\alpha-2} - \gamma\frac{(K')^2}{K^3}; \\
\frac{\partial^2 u}{\partial (K')^2}(K, K', z) &= -\frac{\gamma}{K}; \\
\frac{\partial^2 u}{\partial K' \partial K}(K, K', z) &= \gamma\frac{K'}{K^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{\partial^2 u}{\partial K^2} & \frac{\partial^2 u}{\partial K \partial K'} \\ \frac{\partial^2 u}{\partial K \partial K'} & \frac{\partial^2 u}{\partial (K')^2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= x_1^2\left[A(z)\alpha(\alpha-1)K^{\alpha-2} - \frac{\gamma(K')^2}{K^3}\right] + 2x_1x_2\gamma\frac{K'}{K^2} - x_2^2\frac{\gamma}{K} \\
&= -x_1^2 A(z)\alpha(1-\alpha)K^{\alpha-2} - \left(x_1\frac{K'}{K} - x_2\right)^2\frac{\gamma}{K}. \qquad (31)
\end{aligned}$$

Notice that both terms in (31) are nonpositive. Thus, the sum is strictly negative unless both of them are zero. The first term is zero only if $x_1 = 0$. In this case, the second term is zero only if $x_2 = 0$. In other words, this sum is strictly negative unless $(x_1, x_2) = (0, 0)$. This shows that the Hessian matrix is negative definite and, therefore, u is strictly concave in (K, K').

Notice that $\frac{\partial u}{\partial K}$ is strictly increasing in z. Since $A(z) = z$, by Assumption 3(iv), Assumption 2 is also satisfied. Therefore, Theorems 3.1 and 3.2 hold.

Using equation (10), the Euler equation (11) can be written as

$$\partial_1 C(K', A, K) + p = \beta Q_\tau\left[\partial_K V(K', z')|z\right].$$

From Assumption 3, we obtain:

$$\partial_1 C(K', A, K) = \gamma\frac{K' - (1-\delta)K}{K} = \gamma\frac{I}{K} = \gamma i.$$

Substituting this into the previous equation, we obtain (22). □

Proof of Proposition 3.4. Under the modification of Assumption 3 ($\alpha = 1$), u is no longer strictly concave. However, (31) is still nonnegative, which shows that the Hessian is negative semidefinite and u is concave in (K, K'). From de Castro et al. (2023b), the conclusions of Theorems 3.1 and 3.2 still hold, but with weak concavity, instead of strict concavity. That is, there exists a unique continuous and bounded function $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ satisfying (8) and this function is differentiable and weakly concave for any interior point $K \in \mathcal{K}$, strictly increasing in $A = z$ and satisfies (10) for interior points. Now we need to show that V is of the form stated in (24). For this, let $\mathcal{F}$ denote the set of functions from $\mathcal{K} \times \mathbb{R}$ into $\mathbb{R}$ of the form $v(K, A) = h(A)K$, with continuous and bounded h. We want to show that the fixed point $V \in \mathcal{F}$. Since $\mathcal{F}$ is a closed subset of the Banach space of continuous and bounded functions, it is sufficient to show that the operator $\mathbb{M}^\tau$ defined by

$$\mathbb{M}^\tau v(K, A) = \max_{K'} \left\{ AK - \frac{\gamma}{2} \left(\frac{K' - (1-\delta)K}{K} \right)^2 K - p(K' - (1-\delta)K) + \beta Q_\tau \left[v(K', A') \middle| A \right] \right\}$$

takes functions from $\mathcal{F}$ into $\mathcal{F}$. Let $v(K, A) = h(A)K$. We have

$$\begin{aligned} \mathbb{M}^\tau v(K, A) &= \max_{K'} \left\{ AK - \frac{\gamma}{2} \left(\frac{K' - (1-\delta)K}{K} \right)^2 K - p(K' - (1-\delta)K) + \beta Q_\tau \left[h(A')K' \middle| A \right] \right\} \\ &= \max_{K'} \left\{ AK - \frac{\gamma}{2} \left(\frac{K' - (1-\delta)K}{K} \right)^2 K - p(K' - (1-\delta)K) + \beta K' Q_\tau \left[h(A') \middle| A \right] \right\}. \end{aligned}$$

The first order condition over K' is

$$\gamma \left(\frac{K' - (1-\delta)K}{K} \right) + p = \beta Q_\tau \left[h(A') \middle| A \right].$$

In particular, this implies that investment satisfies

$$I = K' - (1-\delta)K = \frac{K}{\gamma} \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right). \tag{32}$$

Hence, once we conclude the proof that $v(K, A) = \varphi(A)K$, (32) will establish that the investment rate

$$i \equiv \frac{I}{K} = \frac{1}{\gamma} \left(\beta Q_\tau \left[\varphi(A') \middle| A \right] - p \right)$$

does not depend on the current capital level K. Continuing our computation substituting the first order condition, we obtain

$$\begin{aligned} \mathbb{M}^\tau v(K, A) \quad &= \quad AK - \frac{1}{2\gamma} K \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right)^2 - \frac{p}{\gamma} K \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right) \\ &\quad + \beta Q_\tau \left[h(A') \middle| A \right] K \left[\frac{1}{\gamma} \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right) + 1 - \delta \right] \\ &= \quad \left\{ A - \frac{1}{2\gamma} \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right)^2 - \frac{p}{\gamma} \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right) \right. \\ &\quad \left. + \beta Q_\tau \left[h(A') \middle| A \right] \left[\frac{1}{\gamma} \left(\beta Q_\tau \left[h(A') \middle| A \right] - p \right) + 1 - \delta \right] \right\} K. \end{aligned} \tag{33}$$

This proves that $\mathbb{M}^\tau v(K, A) = \widetilde{h}(A)K$ for some $\widetilde{h}$, that is, $\mathbb{M}^\tau(\mathcal{F}) \subset \mathcal{F}$. Thus, the fixed point is $V(K, A) = \varphi(A)K$. Since we already know that V is strictly increasing in A, then φ is strictly increasing. This implies that $Q_\tau[\varphi(A')|A] = \varphi\left(Q_\tau[A'|A]\right)$. From (33), we conclude that φ satisfies (25). □

A.2 Proofs of the results in Section 2

This section establishes the claims stated in Section 2.1. In particular, we derive the Euler equation in (4), the optimal capital in (6), and the value function in (7). We notice that the setting discussed there corresponds to the one using Assumption 3, but with $\gamma = 0$. The formal result is as follows:

Proposition A.1. *Let Assumption 3 hold, but with $\gamma = 0$. Then, there exist a unique $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ functional equation (8) has a solution of the form*

$$V(K, z) = zK^\alpha + p(1 - \delta)K + h(z), \tag{34}$$

where h is a continuous, bounded, weakly increasing and nonnegative function. Moreover, the following Euler equation holds:

$$Q_\tau\left[z'\alpha(K')^{\alpha-1}|z\right] = \frac{p}{\beta} - p(1 - \delta), \tag{35}$$

and the optimal K' does not depend on K and it is given by:

$$K' = \left(\frac{\alpha\beta}{p[1 - \beta(1 - \delta)]} Q_\tau\left[z'\middle|z\right]\right)^{\frac{1}{1-\alpha}}. \tag{36}$$

Proof. The argument is similar to the proof of Proposition 3.4. First observe that with $\gamma = 0$, (31) is nonpositive for all $(x_1, x_2) \in \mathbb{R}^2$ and strictly negative if $x_1 \neq 0$. Thus, u is strictly concave in K but only weakly concave in K'. Although Assumption 1 no longer holds, we can use de Castro et al. (2023b) to conclude that there exists a unique continuous and bounded function $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ satisfying (8) and this function is differentiable and weakly concave for any interior point $K \in \mathcal{K}$, strictly increasing in $A = z$ and satisfies (10) for interior points. Moreover, the Euler equation (11) holds, and we can easily see that assumes the form stated in (35) if $\gamma = 0$.

Now, let $\mathcal{F}$ be the set of functions from $\mathcal{K}\times\mathcal{Z}$ to $\mathbb{R}$ of the form $v(K, z) = zK^\alpha + p(1-\delta)K + h(z)$, where h is a continuous, bounded, weakly increasing and nonnegative function. Define the operator $\mathbb{M}^\tau$ on $\mathcal{F}$ as follows:

$$\begin{aligned}
\mathbb{M}^\tau(v)(K, z) &\equiv \max_{K' \in \Gamma(K)} \left\{zK^\alpha - p[K' - (1 - \delta)K] + \beta Q_\tau\left[v(K', z')\middle|z\right]\right\} \\
&= \max_{K' \in \Gamma(K)} \left\{zK^\alpha + p(1 - \delta)K - pK' + \beta Q_\tau\left[(z')(K')^\alpha + p(1 - \delta)K' + h(z')\middle|z\right]\right\}.
\end{aligned}$$

Since $(z')(K')^\alpha$ and $h(z')$ are comonotonic, the quantile of their sum is the sum of their quantiles. If we denote $Q_\tau\left[z'\middle|z\right]$ by $q_\tau(z)$, the above can be simplified to:

$$\mathbb{M}^\tau(v)(K,z) = \max_{K'\in\Gamma(K)} \left\{zK^\alpha + p(1-\delta)K - K'p[1-\beta(1-\delta)] + \beta(K')^\alpha q_\tau(z) + \beta h\left(q_\tau(z)\right)\right\}.$$

The first order condition for the optimal K' gives (36), that is,

$$-p[1-\beta(1-\delta)] + \alpha\beta(K')^{\alpha-1}q_\tau(z) = 0 \Leftrightarrow K' = \left(\frac{\alpha\beta q_\tau(z)}{p[1-\beta(1-\delta)]}\right)^{\frac{1}{1-\alpha}},$$

since $p[1-\beta(1-\delta)] > 0$. Notice that K' is a function of z but not of K. Therefore,

$$\mathbb{M}^\tau(v)(K,z) = zK^\alpha + p(1-\delta)K + \frac{\alpha^{\frac{1}{1-\alpha}}\left[\beta q_\tau(z)\right]^{\frac{2-\alpha}{1-\alpha}}}{\left(p[1-\beta(1-\delta)]\right)^{\frac{1}{1-\alpha}}} + \beta h(q_\tau(z)). \tag{37}$$

Notice that the two last terms above are functions of only z, not K. This shows that $\mathbb{M}^\tau(v) \in \mathcal{F}$ if $v \in \mathcal{F}$. Since $\mathcal{F}$ is a closed set, the fixed point V belongs to it, that is, (34) holds. □

We remark that in the setting in Proposition A.1 we can in fact find an explicit formula for the value function. For this, we need to introduce some notation. First, let us define the constant λ by

$$\lambda \equiv \left(\frac{\alpha\beta}{p[1-\beta(1-\delta)]}\right)^{\frac{1}{1-\alpha}}. \tag{38}$$

Define recursively:

$$r_{\tau,0}(z) \equiv \beta\lambda\left(Q_\tau\left[z'\middle|z\right]\right)^{\frac{2-\alpha}{1-\alpha}}; \text{ and } \quad r_{\tau,n}(z) \equiv r_{\tau,n-1}\left(Q_\tau\left[z'\middle|z\right]\right), \text{ for } n \geqslant 1.$$

Define

$$S(z) \equiv \sum_{n=0}^{\infty} \beta^n r_{\tau,n}(z). \tag{39}$$

We have the following result:

Proposition A.2. *Let Assumption 3 hold, but with* $\gamma = 0$. *Then, the unique continuous and bounded function* $V : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}$ *that satisfies* (8) *is given by:*

$$V(K,z) = zK^\alpha + p(1-\delta)K + S(z). \tag{40}$$

Proof. Let V be defined by (40). Notice that it has the form defined by (34). From (37),

$$\mathbb{M}^{\tau}(V)(K,z) = zK^{\alpha} + p(1-\delta)K + r_{\tau,0}(z) + \beta S\left(Q_{\tau}\left[z'|z\right]\right),$$

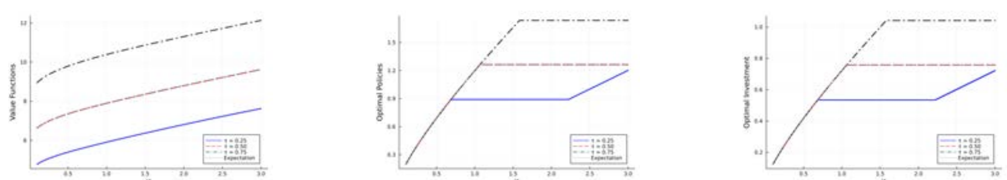
where we have used the notation introduced above. Observe that

$$\begin{aligned} r_{\tau,0}(z) + \beta S\left(Q_{\tau}\left[z'|z\right]\right) &= r_{\tau,0}(z) + \sum_{n=0}^{\infty} \beta^{n+1} r_{\tau,n}\left(Q_{\tau}\left[z'|z\right]\right) \\ &= r_{\tau,0}(z) + \sum_{n=0}^{\infty} \beta^{n+1} r_{\tau,n+1}(z) = S(z), \end{aligned}$$

that is, $\mathbb{M}^{\tau}(V)(K,z) = V(K,z)$. This concludes the demonstration. □

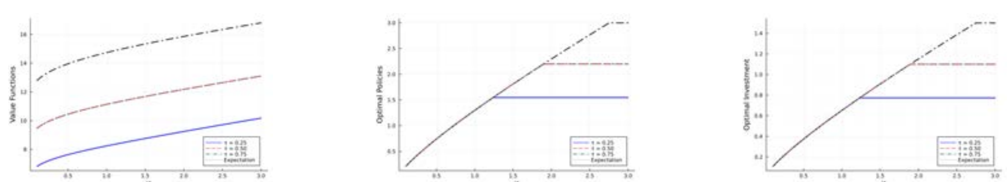
B Additional numerical results

This section reports additional results for Section 4. First, we include no adjustment costs with $\gamma = 0$. We redo numerical computations for $\delta \in \{0.50, 0.25, 0.10\}$. Figures 5, 6, and 7 collect the results, which are qualitatively similar to the previous case. The larger the quantile, the larger the value function and investment. While the depreciation parameter δ does not significantly affect the shape of the value function, it is important for the optimal investment choice.



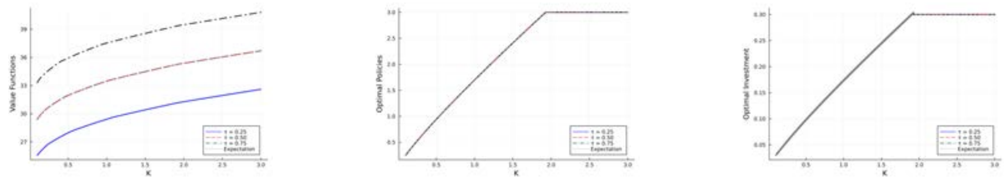
(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 5: Shocks iid, $\gamma = 0$ and $\delta = 0.50$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

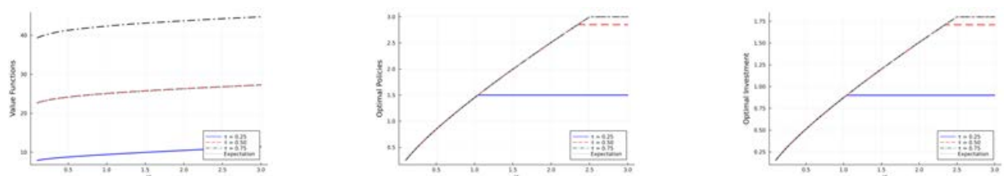
Figure 6: Shocks iid, $\gamma = 0$ and $\delta = 0.25$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

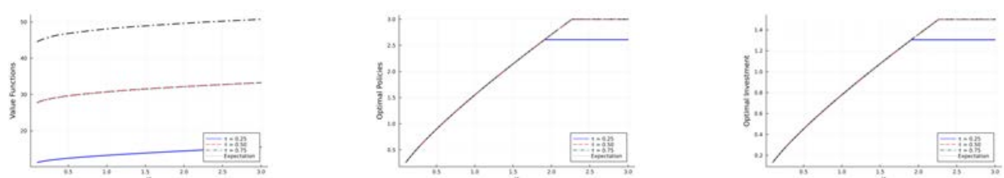
Figure 7: Shocks iid, $\gamma = 0$ and $\delta = 0.10$

Now, we allow for dependent shocks. We create states vector and transition matrix for the discrete Markov process approximation of AR(1) process $z' = \rho * z + e$, $e \sim N(\mu, \sigma)$, by Tauchen's method. We set $\mu = 0.75$, $\sigma = 0.15$, $\rho = 0.5$, and the number of states in discretization of z to be five. We use the following combination of adjustment cost and depreciation parameters: $\gamma = \{0, 0.25\}$ for the adjustment cost function, and $\delta \in \{0.5, 0.25\}$ for the depreciation.



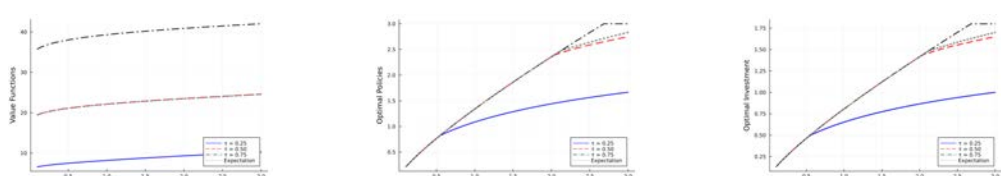
(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 8: Dependent shocks, $\gamma = 0$ and $\delta = 0.5$



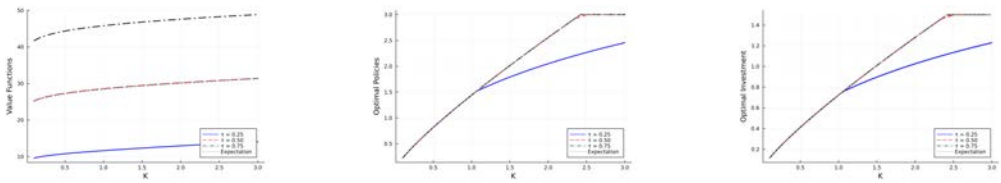
(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 9: Dependent shocks, $\gamma = 0$ and $\delta = 0.25$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 10: Dependent shocks, $\gamma = 0.25$ and $\delta = 0.5$



(a) Value Function (b) Optimal Capital (c) Optimal Investment

Figure 11: Dependent shocks, $\gamma = 0.25$ and $\delta = 0.25$

First, Figures 8 and 9 collect the results fixing zero cost adjustment ($\gamma = 0$) and varying $\delta = 0.5$ and $\delta = 0.25$, respectively. The first observation is that these two results are qualitatively similar to those in the main text using iid shocks. Hence, the shapes of value and policy functions are robust to the presence of dependent shocks. Figure 8 documents a linear increase in optimal capital and investment along K before they flatten out and become constant. There is also heterogeneity in all functions across quantiles τ. Moreover, Figure 9 shows that when the depreciation is small, both optimal capital and investment coincide across quantiles.

Second, Figures 10 and 11 collect results for $\gamma = 0.25$, and $\delta = 0.5$ and $\delta = 0.25$, respectively. As in the previous case, the results are similar to those in the main text. However, in this case the optimal capital and investment are always increasing for $\delta = 0.5$.

C Additional empirical results

In this appendix we present additional results when estimating the model using the standard quantile regression (QR). We present results for the entire sample, as well as using the subsample for size and age of firms.

First, we estimate the model for all firms. The results for the QR estimates are in Tables 7. The overall results are qualitative the same as those discussed in the main text in Table 2. QR coefficients show that the slope coefficient of the regression of Tobin's q on investment is increasing across quantiles. Moreover, for this case, the risk attitude estimate is smaller than those for the model controlling for fixed effects. For the QR we otain estimates varying from 0.22 to 0.27.

Table 7: Slope coefficients of regression results for QR and OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates.

QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.166		0.174		0.228		0.240
	(0.009)		(0.010)		(0.008)		(0.008)
0.2	0.185		0.206		0.216		0.233
	(0.009)		(0.008)		(0.007)		(0.007)
0.3	0.212		0.226		0.211		0.239
	(0.008)		(0.008)		(0.007)		(0.007)
0.4	0.232		0.250		0.208		0.248
	(0.009)		(0.009)		(0.007)		(0.007)
0.5	0.264		0.279		0.213		0.258
	(0.010)		(0.010)		(0.008)		(0.008)
0.6	0.296		0.329		0.224		0.288
	(0.012)		(0.012)		(0.009)		(0.010)
0.7	0.355		0.399		0.241		0.325
	(0.016)		(0.014)		(0.012)		(0.011)
0.8	0.459		0.527		0.287		0.396
	(0.022)		(0.022)		(0.015)		(0.016)
0.9	0.780		0.885		0.464		0.623
	(0.042)		(0.044)		(0.024)		(0.025)
$\widehat{\tau} = 0.27$	0.204	$\widehat{\tau} = 0.25$	0.222	$\widehat{\tau} = 0.26$	0.210	$\widehat{\tau} = 0.24$	0.238
(0.006)	(0.008)	(0.005)	(0.008)	(0.005)	(0.007)	(0.005)	(0.007)
OLS	0.436		0.480		0.395		0.463
	(0.015)		(0.015)		(0.014)		(0.013)

Notes: Standard errors inside parenthesis.

Now we move our attention to the subsample with size of firms. Tables 8 and 9 collect results for the QR case when using Small and Large firms, respectively. The overall results for the QR case are qualitatively similar to the previous ones in Table 7. Regarding risk attitude estimates, for Small firms risk attitude estimates vary between 0.22–0.27, while for Large firms it varies between 0.21–0.23. Thus, Larger firms are slightly more risk averse than Small firms.

Table 8: Slope coefficients of regression results for QR and OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Small firms.

QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.251		0.242		0.287		0.269
	(0.015)		(0.016)		(0.017)		(0.016)
0.2	0.248		0.261		0.262		0.284
	(0.015)		(0.016)		(0.017)		(0.014)
0.3	0.259		0.272		0.251		0.273
	(0.016)		(0.015)		(0.016)		(0.014)
0.4	0.290		0.292		0.246		0.282
	(0.018)		(0.016)		(0.015)		(0.015)
0.5	0.302		0.316		0.232		0.279
	(0.018)		(0.018)		(0.015)		(0.016)
0.6	0.328		0.349		0.232		0.279
	(0.021)		(0.021)		(0.017)		(0.017)
0.7	0.365		0.396		0.222		0.302
	(0.025)		(0.025)		(0.019)		(0.019)
0.8	0.428		0.463		0.239		0.339
	(0.030)		(0.028)		(0.025)		(0.025)
0.9	0.590		0.692		0.303		0.466
	(0.058)		(0.046)		(0.035)		(0.036)
$\widehat{\tau} = 0.27$	0.249	$\widehat{\tau} = 0.23$	0.280	$\widehat{\tau} = 0.26$	0.253	$\widehat{\tau} = 0.22$	0.285
(0.008)	(0.015)	(0.008)	(0.015)	(0.007)	(0.016)	(0.009)	(0.014)
OLS	0.438		0.475		0.350		0.438
	(0.025)		(0.025)		(0.024)		(0.023)

Notes: Standard errors inside parenthesis.

Table 9: Slope coefficients of regression results for QR and OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Large firms.

QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.155		0.159		0.146		0.157
	(0.019)		(0.019)		(0.013)		(0.013)
0.2	0.212		0.185		0.156		0.171
	(0.016)		(0.016)		(0.012)		(0.012)
0.3	0.240		0.214		0.165		0.190
	(0.019)		(0.018)		(0.012)		(0.012)
0.4	0.282		0.240		0.165		0.206
	(0.018)		(0.021)		(0.013)		(0.013)
0.5	0.292		0.311		0.171		0.228
	(0.020)		(0.023)		(0.014)		(0.015)
0.6	0.345		0.371		0.183		0.274
	(0.026)		(0.024)		(0.018)		(0.019)
0.7	0.426		0.521		0.226		0.341
	(0.036)		(0.038)		(0.023)		(0.023)
0.8	0.622		0.761		0.317		0.489
	(0.051)		(0.053)		(0.036)		(0.041)
0.9	1.225		1.315		0.578		0.798
	(0.138)		(0.115)		(0.045)		(0.059)
$\widehat{\tau} = 0.21$	0.215	$\widehat{\tau} = 0.23$	0.199	$\widehat{\tau} = 0.21$	0.154	$\widehat{\tau} = 0.23$	0.174
(0.009)	(0.016)	(0.008)	(0.017)	(0.009)	(0.012)	(0.008)	(0.012)
OLS	0.538		0.587		0.355		0.436
	(0.033)		(0.033)		(0.026)		(0.026)

Notes: Standard errors inside parenthesis.

Finally, we estimate the models discussed in the main text for Young and Old firms. Tables 10 and 11 provide the respective QR estimation results. The results are similar to the previous cases.

Table 10: Slope coefficients of regression results for QR and OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Young firms.

QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.137		0.124		0.207		0.223
	(0.014)		(0.014)		(0.013)		(0.012)
0.2	0.142		0.147		0.196		0.205
	(0.013)		(0.014)		(0.013)		(0.012)
0.3	0.175		0.174		0.201		0.212
	(0.013)		(0.012)		(0.012)		(0.012)
0.4	0.185		0.180		0.189		0.226
	(0.014)		(0.013)		(0.012)		(0.012)
0.5	0.214		0.211		0.198		0.232
	(0.015)		(0.016)		(0.013)		(0.013)
0.6	0.234		0.256		0.210		0.251
	(0.017)		(0.018)		(0.015)		(0.015)
0.7	0.261		0.285		0.204		0.266
	(0.020)		(0.021)		(0.018)		(0.017)
0.8	0.312		0.368		0.244		0.330
	(0.030)		(0.032)		(0.023)		(0.025)
0.9	0.511		0.600		0.368		0.511
	(0.059)		(0.053)		(0.033)		(0.033)
$\widehat{\tau} = 0.28$	0.169	$\widehat{\tau} = 0.26$	0.174	$\widehat{\tau} = 0.28$	0.197	$\widehat{\tau} = 0.25$	0.212
(0.007)	(0.013)	(0.008)	(0.012)	(0.008)	(0.012)	(0.008)	(0.012)
OLS	0.296		0.322		0.317		0.383
	(0.021)		(0.020)		(0.019)		(0.019)

Notes: Standard errors inside parenthesis.

Table 11: Slope coefficients of regression results for QR and OLS of Tobin's q (q_1 and q_2) on investment (i_1 and i_2), and risk attitude estimates. Old firms.

QR	i_1				i_2		
τ	q_1		q_2		q_1		q_2
0.1	0.226		0.263		0.217		0.259
	(0.018)		(0.018)		(0.012)		(0.013)
0.2	0.256		0.280		0.211		0.245
	(0.017)		(0.017)		(0.013)		(0.012)
0.3	0.292		0.312		0.216		0.244
	(0.017)		(0.016)		(0.013)		(0.013)
0.4	0.322		0.329		0.220		0.250
	(0.018)		(0.017)		(0.014)		(0.013)
0.5	0.365		0.358		0.215		0.261
	(0.021)		(0.020)		(0.015)		(0.015)
0.6	0.417		0.428		0.226		0.300
	(0.025)		(0.025)		(0.018)		(0.018)
0.7	0.495		0.543		0.256		0.358
	(0.033)		(0.033)		(0.025)		(0.022)
0.8	0.636		0.769		0.303		0.475
	(0.055)		(0.059)		(0.030)		(0.034)
0.9	1.136		1.340		0.569		0.767
	(0.105)		(0.111)		(0.046)		(0.050)
$\widehat{\tau} = 0.24$	0.272	$\widehat{\tau} = 0.24$	0.299	$\widehat{\tau} = 0.23$	0.213	$\widehat{\tau} = 0.23$	0.250
(0.009)	(0.016)	(0.009)	(0.016)	(0.009)	(0.013)	(0.008)	(0.012)
OLS	0.654		0.716		0.513		0.560
	(0.037)		(0.035)		(0.031)		(0.030)

Notes: Standard errors inside parenthesis.

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