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Marco Battaglini
Thomas R. Palfrey

Working Paper 32473
http://www.nber.org/papers/w32473

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 2024

Support from the National Science Foundation (SES-2343948) is gratefully acknowledged. The paper has benefited from comments and suggestions by seminar audiences at the 2025 Columbia Conference on Economic Theory (CCET), the 2024 SITE Workshop on Dynamic Games, Contracts, and Markets, Brown University, New York University, Northwestern University, NYUAD, Princeton University, Stanford University, the Tippie College of Business, the University of Chicago, Paris School of Economics, University College London, and the University of Wisconsin-Madison. We also wish to thank Sandeep Baliga, Gary Cox, John Ferejohn, Emir Kamenica, Roger Myerson, and Andrea Prat for discussions and comments. Edoardo Bollati and Gabriel Sekeres provided excellent research assistance. We are responsible for any remaining shortcomings. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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May 2024
JEL No. D71, D74, D82

ABSTRACT

Collective action problems arise when a group's common goal can only be achieved if enough members engage in a costly action. We study the equilibrium properties of such problems when decisions are made dynamically over time, delay is costly, and members have heterogeneous and privately known preferences. In these environments, time acts as both a curse and a blessing: individuals have incentives to delay their actions to observe the commitments of others, yet time can also serve as a coordination device. This tension generates an equilibrium dynamic that combines learning about the eventual probability of success with free-riding incentives and coordination challenges. For finite n, dynamic collective decisions are inherently probabilistic: the probability of success is always positive, but there remains a strictly positive probability that the process halts inefficiently—stopping "cold turkey" before success is achieved. As n becomes large, however, the outcome becomes essentially deterministic: success is achieved either almost instantly at minimum cost or not at all, depending on how quickly the threshold number of participants grows with n. We establish this result by uncovering a novel connection between the equilibria of dynamic Bayesian games and the class of optimal static direct mechanisms requiring Honest and Obedient behavior—an insight that may have broader applicability to mechanism design, beyond collective action environments.

Marco Battaglini
Cornell University
Department of Economics
and CEPR
and also NBER
battaglini@cornell.edu

Thomas R. Palfrey
California Institute of Technology
and NBER
trp@hss.caltech.edu

1 Introduction

Collective action problems unfold dynamically, and require time to achieve success. Public protests often start small and, if they do not die out, reach a critical mass only gradually.[1] International agreements are initiated by small group of countries, but then require years to rally further support and collect ratifications from enough participants.[2] It is customary to leave public good and charitable-giving fund drives open for weeks or months. The success of open source projects is determined over time by many dispersed sofware engineers' willingness to contribute their efforts and expertise to develop the software. In these problems a common goal is achieved if and only if individual participation is sufficiently high: time allows participants to better coordinate, getting a better sense of whether the goal is achievable, which avoids wasting resources if a cause does not have enough support from early contributors.

In these situations, time is both a curse and a blessing. It is a curse because it creates incentives for individuals to delay their participation, waiting to see what others do; and this moral hazard problem is not just with respect to other players, but also against an individual's future selves. It is however also a blessing because it enables coordination and information transmission.

Some progress has been made in the study of dynamic moral hazard problems with free riding, identifying some inefficiencies that arise, and even characterizing conditions under which efficient allocations are possible in special cases (Schelling [1960], Fershtman and Nitzan [1991], Admati and Perry [1991], Marx and Matthews [2000], Gale [1995, 2001], Matthews [2013], Lockwood and Thomas (2002), Battaglini et al. [2014] among others). A key takeaway from those studies is that the cost of moral hazard is not so much that projects are not completed or that there are excess contributions, but that they are completed with inefficient delays. Under special conditions, for example when the players are very patient and have long horizons, the delay inefficiencies and the costs of overcontribution can disappear entirely.

There is however an important factor affecting the ability of social groups to dynamically achieve common goals that has not been adequately studied, which paints a less optimistic

[1]For example, a classic historical account of the dynamic evolution of the Great English Agricultural Uprising of 1839 is presented by Hobsbaum and Rude [1968]. Lohmann [1994] discusses the dynamic evolution of demonstrations leading to the fall of the Berlin wall in 1989.

[2]The average time of ratification of the 1992 *United Nations Framework Convention on Climate Change* was 810 days. Indeed more than a year passed by the time a quarter of the countries had ratified it, and more than two years before half of the countries had ratified it (Fredrickson and Gaston [2000]).

picture - *private information.* A limitation of the existing studies of dynamic moral hazard is that they focus almost exclusively on environments with complete information, where preferences are common knowledge, so the uncertainty faced by individuals is entirely strategic: How many others will do their part? Will additional participants engage? But in environments with private information about preferences, players may become more (or less) hesitant to volunteer over time because of what they learn about the prospects for success. The relevant question then becomes: Are there a sufficient number of committed citizens for whom it is worthwhile to participate in the collective action? Because of the strategic effects of information transmission and learning, there is an additional curse from time: as time progresses, players are uncertain whether delays are simply due to procrastination (i.e., moral hazard), or because other players lack a willingness to participate due to high private costs (i.e., adverse selection). Players gradually accumulate information on the others' preferences and willingness to contribute. Because delays from procrastination are unavoidable, this process generates a *systematic bias* toward failure. It is indeed possible that any further participation stops "cold turkey" after some histories because uncommitted individuals become too pessimistic about the prospects for eventual success and simply give up. How and to what extent can the passage of time still help solve the collective action problem? These questions have not been addressed and this paper provides some answers.

We study these and related questions in a simple but natural dynamic collective action model with private information about preferences. The collective action problem is modeled as a dynamic participation game with n individuals, or group members. The game takes place over a possibly infinite sequence of discrete periods. In period one, each of the n members independently and simultaneously decides whether or not to join in the collective action. Participation decisions are binary and the associated sunk cost of participation for member i, c_i, is private information and is borne immediately. If the threshold number of required participants for success, $m \geq 1$, is met in period one, the game ends; each non-participant member receives the benefit v and each participant receives $v - c_i$. If the threshold is not met, the game continues to the second period, and all non-participants again must decide whether or not to join the action. The game continues indefinitely like this, with discounting, until the threshold is met at which point the game ends and each member receives the success payoff of v, discounted by the number of periods it took to reach the threshold; in addition each participant loses $-c_i$, discounted by the period when they participated. If the threshold is never met, then the game continues forever, with final payoffs equal to 0 for each non-participant, and $-c_i$ for each participant, discounted by the period when they participated.

The special case of $m = 1$, which we will call the *Dynamic Volunteer Dilemma*, is of independent interest[3] and can be interpreted as a classic war of attrition. The prize is achieving the public good without paying for it (i.e. v); at any time, a player can terminate the game by participating and secure a lower payoff $v - c_i$. When $m > 1$, however, the game is fundamentally different from a war of attrition. If a player quits, the payoff now is endogenous, because it depends on the *externality* generated by the *other* players completing the game. The key difference is that there is no exogenous flow cost to be paid for each period in which a player stays in the game, and it would not be natural to assume any: the cost of procrastination is already captured by the delay in receiving the public good.[4]

Two basic lessons emerge from the analysis. The first is that when $m > 1$ collective decisions are probabilistic in all equilibria: the process starts for sure in the sense that some group members volunteer with positive probability in early periods; the final outcome, however, depends on the trajectory of participation decisions, which in turn depends on the exact realization of types. With strictly positive probability the required threshold m is reached and the public good is obtained. On the other hand, the process will also get "stuck" with strictly positive probability, where additional participation ceases forever, resulting in failure, even if there is a sufficiently large number of players whose cost of contributing is strictly smaller than the private benefit of success. This fundamentally differs from the case with $m = 1$, in which if there is at least one player i with type $c_i < v$, the collective goal is achieved for sure in finite time.

The second lesson is that outcomes become essentially deterministic with large populations. This however does not imply that success is guaranteed (or even possible in some equilibrium) nor that other sources of inefficiencies disappear. We indeed show that the group's success depends on the speed with which the threshold fraction of players required for success $\alpha_n = m_n/n$ converges to zero as $n \to \infty$: if α_n converges faster than the *cube root of* $1/n$, then there is a sequence of equilibria converging to the efficient collective decision with no delay; if instead it converges slower than the cube root of $1/n$, then the expected utility in the game converges to zero for all types of all players in *all* Perfect Bayesian Equi-

[3]The classic volunteer's dilemma was introduced by Diekmann [1985] and refers to a static simultaneous-move with complete information about preferences in which a group can achieve a collective goal if at least one member volunteers to pay a fixed cost c. It has been adopted as a paradigm of cooperation in many disciplines including economics (Bergstrom [2018], Battaglini and Palfrey [2025]), biology (Patel et al. [2018]), neuroscience (Park et al. 2019), engineering (Roberto et al. 2011) and other fields.

[4]While this is true when $m = 1$ as well, in that case it does not have qualitative implications since the game ends as soon as one player contributes. As we will explain, however, it is essential to the strategic analysis of the game when $m > 1$.

libria. We call this phenomenon the *Curse of Large Numbers* for collective action because the requirement for efficiency is very strong: even if an arbitrarily small share ε of population is required to contribute, the project is doomed for failure for large n.[5]

The result that success is achieved if and only if α_n decreases faster than $n^{-1/3}$ offers two important qualitative insights: one regarding the limits of dynamic collective action, and the other concerning a deep but yet unexplored connection between dynamic games with incomplete information and optimal mechanisms without transfers.

Regarding the first point, a key result in Bliss and Nalebuff (1984) is that inefficiencies in the Dynamic Volunteer Dilemma disappear in the limit as $n \to \infty$. But that result is based on the restrictive assumption that $m_n = 1$ for all n. Our results significantly strengthen these findings by showing that full efficiency in the dynamic collective action game can still be achieved in the limit, even when m_n increases without bound, as long as the number of required contributors, m_n, grows slower than $n^{2/3}$. Therefore, if the prevailing lesson in the literature was that large numbers solve the free rider problem in the Dynamic Volunteer Dilemma, our result significantly qualifies this conclusion: while our condition is substantially less restrictive than previous ones, it is still very demanding. The $n^{2/3}$ rule implies that efficiency fails for sure if success always requires even an arbitrarily small positive fraction of the population in the limit. These results show that collective action is difficult, but by no means impossible, even when players are motivated exclusively by self-interest. Indeed, they offer new insights into questions about the driving forces behind the success or failure of collective action in a wide range of applications, some of which we mentioned above, from public protests to international environmental agreements to open source projects. We return to a deeper discussion of these applications, in light of the results, to Section 5.3.

As for the connection with mechanism design, we show that the condition for asymptotic efficiency in dynamic collective action problems is exactly the same as the condition for efficiency in the optimal Honest and Obedient (static) direct mechanism without transfers. This finding is important not only because it shows that the expected outcome generated by spontaneous contributions over time cannot be improved upon in limit economies through more sophisticated forms of communication or coordination, but also because it highlights a deep connection between dynamic games with incomplete information and optimal Honest and Obedient (HO) mechanisms, which we are able to exploit in the impossibility result

[5] To prove the limit efficiency result, we rely on the characterization of symmetric PBE, thus making the result stronger since we show there is no loss in assuming symmetric equilibria. The finding that the expected utility in the game converges to zero when α_n converges slower than the cube root of $1/n$, however, is not restricted to symmetric equilibria, and actually holds for all equilibria thus giving us a fully general impossibility result for dynamic collective action games.

described above — a point we now explain.

In seminal contributions, Ausubel and Deneckere [1989a, 1989b] characterized the equilibria of a dynamic two-person bargaining game with one-sided private information in terms of the equilibria of a suitable static incentive-compatible (IC) and individually rational (IR) mechanism. This characterization allowed them to study the welfare properties of the equilibrium payoffs of the dynamic game using the static IC and IR mechanism.

As we will explain in greater detail, the equivalence highlighted by Ausubel and Deneckere does not hold in games with multilateral private information—such as the one we study—in which the equilibria cannot be characterized in terms of simple IC and IR constraints of a direct mechanism. In our work, we instead generalize the approach by demonstrating that for any perfect Bayesian equilibrium (PBE) of the dynamic contribution game, we can construct a corresponding static *Honest and Obedient (HO) mechanism* that achieves exactly the same expected payoffs for each player type.[6] Given this, the payoffs attainable in any PBE can be bounded above by the payoff attainable in the best HO mechanism. This argument allows us to apply Theorem 4 in Battaglini and Palfrey [2024], which shows that if α_n converges to zero slower than the cube root of $1/n$, then the payoff in the best Honest and Obedient mechanism converges to zero, thereby proving the impossibility part of the result.

The connection between the PBE of dynamic games with multilateral private information and static HO mechanisms that we exploit in our analysis is, to our knowledge, new and potentially applicable for characterizing properties of equilibria in dynamic games with incomplete information in other environments as well.

The connection between HO mechanisms and the PBE of the dynamic game, however, does not allow us to prove that if α_n converges to zero faster than the cube root of $1/n$, an efficient limit equilibrium exists, since the set of payoffs obtainable in HO mechanisms is larger than the set of payoffs supported in PBE. We therefore prove the possibility result directly using the characterization of the game. The reason the cube root of $1/n$ plays a key role for efficiency stems from fundamental features of the strategic interaction and will be intuitively explained in Section 5.

The reminder of the paper is organized as follows. In the next subsection we discuss the related literature. We presents the model in Section 2. In Section 3, we study the special case of $m_n = 1$, a case of independent interest that we refer to as the *Dynamic Volunteer Dilemma.* Building on this analysis, in Section 4, we study the *Dynamic Collective Action Problem* in which $m_n > 1$, showing that it is qualitatively and substantively different than

[6] HO mechanisms are a more demanding class of mechanisms, first introduced by Myerson (1982).

the Dynamic Volunteer Dilemma. Section 5 is dedicated to the study of the properties of equilibria in large economies as $n \to \infty$. In Section 6 we present extensions and variations of the model.

1.1 Related literature

Our paper is most closely related to three lines of research. The first line, briefly mentioned above, studies dynamic contributions to public goods. This research has focused on settings with perfect information in which there is no uncertainty regarding the environment (say, for example, regarding the players' evaluations of the public good or their cost of contributing). The key issue in these works is the moral hazard problem faced by participants who would like the public good, but prefer for others to contribute and thus may postpone their contributions.[7] Our paper extends this analysis by considering environments where players' costs of contribution are private and heterogeneous. The key new feature of this environment is that, as time unfolds, players learn about the distribution of types and re-evaluate whether it is optimal to contribute.

The second line of research to which our work is connected is the war of attrition. Bliss and Nalebuff [1984] present a continuous time model in which a public good is achieved if at least one player volunteers for it. Players have private and heterogeneous costs of volunteering and may choose to wait hoping that other players will do it for them. In this problem, there is the usual moral hazard problem with public goods, but in addition there is uncertainty regarding the conditions under which other players will be willing to contribute. As time progresses, players update their beliefs about the cost of contributing of the remaining players. Our work extends this volunteer's dilemma framework by addressing the collective action problem with *multiple volunteers* ($m_n > 1$), and even allowing m_n to grow with n without bounds. These differences are essential to model collective action in realistic environments, since it seems natural to require multiple contributors for success in common projects with even moderately sized groups. As highlighted above, the analysis

[7] Seminal contributions are Admati and Perry [1991] and Marx and Matthews [2000]. The first paper characterizes the unique equilibrium in a game in which two players alternate contributing until the sum of contributions pass a threshold. They show that equilibrium generally implies delay and characterized conditions for efficiency, showing that they are demanding. Marx and Matthews [2000] extend the analysis to environments in which players can contribute simultaneously in each period. They show that, although equilibrium completion may be delayed and non-contribution equilibria may also exist, there are conditions under which perfect Bayesian equilibria eventually complete the project and become nearly efficient. These conditions include environments with a positive payoff jump at the completion threshold, as in our model, and the linear-benefit case under their stated condition. Battaglini et al. [2014] show that efficient outcomes are attainable even in environments with continuous, non-linear utilities and no threshold.

of the general collective action problem with $m_n > 1$ is qualitatively different from the volunteers dilemma. While the war of attrition has been used as leading framework in numerous important economic problems (see Alesina and Drazen [1991] for a prominent example), applications restrict the analysis to setting in which only one player needs to concede to terminate the game, as in the volunteers dilemma.[8]

A key feature of the war of attrition, and in our dynamic colletive action problems as well, is that players learn over time the distribution of other players' types. Recent works have studied this dynamic signaling problem in contribution games that are related but different from ours. Mutluer [2024] studies a Dynamic Volunteer Dilemma with $m = 1$ under the assumption that players decide sequentially, one at a time. In this model, with an exogenous order of play, players cannot self-select as in our model. However, observing previous decisions is informative in that game because players have common values and receive correlated signals. The author shows that, in this environment, there is a unique finite population size above which increasing the group size is associated with a reduction in the probability of providing the public good. Deb et al. [2024] study a model of reward-based crowdfunding, in which contributions to a business project can be made by randomly arriving private investors who make one-shot decisions in the period of arrival, and by a long-lived donor who has an interest in the success of the project. They study the extent to which the donor can alleviate the coordination problem of the private investors by signaling his valuation of the project with donations.

The third related line of research studies the optimal design of static public good mechanisms (d'Aspremont et al. [1990], Mailath and Postlewaite [1990], Hellwig [2003], Battaglini and Palfrey [2024], among others). As in our work, this literature focuses on environments with private information; but, in addition to restricting attention to static settings, it has a normative flavor, allowing for potentially complex communication mechanisms that require commitment power. Our analysis is positive and explicitly dynamic: we are not interested in characterizing the optimal mechanism, but in studying collective action in a natural dynamic environment in which society cannot commit to complex cooperation mechanisms. As discussed above, however, there is a deep connection between our work and this literature,

[8]The only other papers we are aware of that consider extensions of the war of attrition with m multiple volunteers are Haigh and Cannings [1989] and Bulow and Klemperer [1999], but these works study different economic environments, restrict attention to fixed m, and lead to very different results. The first restricts the analysis to environments with complete information. Bulow and Klemperer [1999] study an all pay auction with $m+n$ players and m prizes in which players pay a strictly positive exogenous cost κ per period to stay in the game. In the all pay auction the payoff of quitting is exogenous even with $m > 1$ since the auction allocates private goods among the other players. In this setting, they show there is a unique equilibrium in which the allocation is ex post efficient and the ex ante inefficiency converges to zero as $n \to \infty$.

especially insofar as it studies HO mechanisms (Battaglini and Palfrey [2024]).

Two other recent contributions are related to our work. Madsen and Schmaya [2025] study optimal mechanisms for public good provision in dynamic environments and characterize the optimal mechanism for public goods requiring recurrent non-monetary contributions for maintenance. Besides taking a normative approach, a key difference between Madsen and Schmaya's environment and ours is that they focus on settings with excludable public goods, allowing the mechanism to restrict access in order to elicit preferences. Jehiel and Leduc [2025] present a dynamic model of collective action in which a principal can intervene to punish volunteers. In the model, both the agents and the principal receive opportunities at random times to volunteer or punish a volunteer, respectively. The main result is that deterrence of collective action is possible only if the principal can intervene sufficiently quickly after an agent's arrival.

2 The Dynamic Collective Action Model

A *Dynamic Collective Action Problem* is a public good game played over a possibly infinite sequence of periods, $t = 1, 2, ..., \infty$. There is a group with n members, and each member i has a privately known participation cost c_i that is an independent draw from a commonly known cost distribution $F(c)$ with a continuous density function, $f(c)$ that is strictly positive on the interval $[0, 1]$. In each period before the game ends, each member simultaneously and independently decides whether to participate or not, a binary choice. A decision by any member i to contribute in period t is irreversible and incurs the cost c_i in the period at which i contributes. If at least m members of the group have chosen to participate up to and including in period t the game ends, and we say the group *succeeds in period* t. If fewer than m members have contributed by period t, the game continues to period $t + 1$. The required share of participation for success is denoted $\alpha_n = m/n$. Participation decisions are publicly observed.

Group success yields a common benefit of $v \in (0, 1)$ to all group members. Payoffs are discounted, and the discount factor is $\delta \in [0, 1)$. Hence, if the group succeeds in period t the payoff to inactive members is $v\delta^{t-1}$ and the payoff to each member who contributed in period $\tau \leq t$ is $v\delta^{t-1} - c_i\delta^{\tau-1}$. If the game continues indefinitely and success is never achieved, then all members who never chose to participate receive a payoff of 0 and each member who participated in period τ receives a payoff $-c_i\delta^{\tau-1}$.[9]

[9]The standard collective action problem is a one shot game and corresponds to an extreme case in our model where $\delta = 0$.

We study the set of Perfect Bayesian Equilibria of this dynamic game. A strategy is a function that assigns, for each (public) history of play at period t and for each type, c, a (possibly mixed) current action to either participate or not.[10] A history at t, h_t, has two components. The first component is the sequence of participation decisions by all members in the previous periods, $t = 1, ..., t-1$. The sequence of the sets of members choosing to participate in each period before t is $I_t = (I^1, ..., I^{t-1})$, where I^τ for $\tau \leq t-1$ is the set of players who volunteer at τ. We define $\kappa_t = (\kappa^1, ..., \kappa^{t-1})$ as the sequence of the numbers of members choosing to participate in each period before t, where $\kappa^\tau = |I^\tau|$ for $\tau \leq t-1$ is the number of players who volunteer at τ. The second component of the public history is a public signal that is observed at the beginning of each period t, θ^t, which is the outcome of a single independent draw from the uniform distribution on $[0,1]$.[11] Thus, a history at period t is denoted by $h_t = (I_t, \theta_t)$, where $\theta_t = (\theta^1, ..., \theta^{t-1})$. Given a history h_t we denote by k_t the minimum number of remaining members at period t who must contribute in order for the group to achieve success. That is, $k_t = m - \sum_{\tau=1}^{t-1} \kappa^\tau$.

In Sections 2 and 3, we characterize the set of symmetric Perfect Bayesian Equilibria (PBE) in the game. Focusing on symmetric equilibria is both natural in this setting and standard in the literature (e.g., Bliss and Nalebuff [1984], Bulow and Klemperer [1999]). However, in our analysis, this focus is also without loss of generality and strengthens our results. Indeed, one of our key results is that limit efficiency can be achieved in equilibrium when α_n converges to zero faster than the cube root of $1/n$ as $n \to \infty$. The focus on symmetric PBE allows us to conclude that efficiency is achievable with symmetric equilibria. Another key result is that if α_n converges to zero slower than the cube root of $1/n$, then in all sequences of PBE we have $\lim_{n\to\infty} EU_n(c) = 0$. Notably, for this impossibility result, we do not need the restriction to symmetric equilibria.

As we will prove, the equilibrium strategies of the game are characterized as a sequence of history-dependent cutpoints, $\{c(h_t)\}_{t=1}^{\infty}$, whereby, in period t, following history h_t, any player with a type $c \leq c(h_t)$ who has not yet participated chooses to participate. Hence, in an equilibrium, the game ends in period 1 if there are at least m members with $c \in [0, c(h_1)]$, where $h_1 = (\varnothing, \theta_1)$; the game ends in period 2 if, for some $j < m$, there are exactly j members with $c \in [0, c(h_1)]$, and at least $m - j$ members with $c \in [c(h_1), c(h_2)]$, where $h_2 = (j, \theta_2)$; and so forth. In a symmetric equilibrium, moreover, players are treated anonymously by the other players: two histories h_t and h'_t, that differ only in the identity of the volunteers (so

[10] Because participation decisions are irreversible, any individual who chooses to participate in some period τ is inactive in all future periods $t > \tau$.

[11] The public signal does not affect the characterization of equilibrium, but simplifies the existence proof.

$\kappa_t = \kappa'_t$) are associated with the same cutpoint for the active players: $c(h_t) = c(h'_t)$.

In an equilibrium, as the game progresses each remaining member's belief about the distribution of the other remaining members' types are updated simply by increasing the lower bound of the distribution of types, which we denote by $l_{h_t} = c(h_{t-1})$, with $l_{h_1} = 0$. An equilibrium consists of a history-dependent cutoff strategy, $c(h_t)$, and conditional beliefs about the distribution of remaining members, derived by Bayes rule:

$$\widetilde{F}(c; l_{h_t}) = \max \left\{ 0, \frac{F(c) - F(l_{h_t})}{1 - F(l_{h_t})} \right\}. \tag{1}$$

Associated with an equilibrium are two value functions. For any history h_t, $Q(h_t)$ is the continuation value for a member who has previously participated (i.e., any member with $c \leq l_{h_t}$); and $V(c, h_t)$ is the continuation value for a member with cost c who has not yet contributed (i.e., any member with $c > l_{h_t}$).

To solve this class of games, a key observation is that, for any given cutoff, strategy each public history of play results in a new continuation game $\Gamma(h_t)$ defined by the lower bound on the cost distribution, l_{h_t}, and the minimum number of contributors that are still needed for success, k_t. This is just another collective action problem with a new group size, $n' = n - m + k_t$, a new threshold, $m' = k_t$, and a new distribution F' which is F truncated below at l_{h_t}. Hence, a solution to the original collective action involves solving for all games in this general class of collective action problems.

To characterize the equilibria of this more general class of games, we initially begin by solving the case where the continuation game is a *Dynamic Volunteer Dilemma*, i.e., the special case of $k = 1$, and any lower bound on the type distribution, l, and proceed inductively. That is, given the solutions for $k = 1$ for all $l \in [0, 1]$, and assuming we have a characterization for each $k' = 2, ..., k - 1$ for all $l \in [0, 1]$ we characterize the PBE for k.

3 The Dynamic Volunteer Dilemma

3.1 Equilibrium

In this case, we solve the continuation game for a group of n members following a history h_t, at which point exactly $m - 1$ members have already contributed, so $k_t = 1$ and there are $n-m+1$ remaining uncommitted members, and the lower bound on the distribution of types is l_{h_t}. We call this continuation game, where the group is missing exactly one contributor to succeed, the *Dynamic Volunteers Dilemma*. We start with a preliminary result. A PBE is in *cutoff strategies* if there is a cutoff $c(h_t)$ such that any type $c \leq c(h_t)$ find it optimal to contribute, and any type $c > c(h_t)$ find it strictly optimal to wait. It is straightforward to show:

Lemma 1. *All PBE of a continuation game starting from any history h_t in which only one contributor is missing for success are in cutoff strategies.*

Proof: See the appendix. ■

We next show that the symmetric equilibrium of the continuation game, i.e., the equilibrium cutoff strategy function, $c(\cdot)$, is *uniquely determined.*

Suppose that at some history h_t we have reached a point at which the lower bound of the support is $l_{h_t} = l$ and exactly $m-1$ members have already contributed, so $k_t = 1$. Denote by $V^-(c, h_t)$ the expected value for a type c who does not contribute in the current period, and $V^+(c, h_t)$ the expected value for a type c who chooses to contribute in the current period. Since $k_t = 1$, success is automatically achieved if the member contributes, so $V^+(c, h_t) = v - c$. The expression for $V^-(c, h_t)$ is slightly more complicated and depends on the continuation value if some other member contributes, which in turn depends on the current cutpoint, $c(h_t)$ (to be solved for) and the continuation value of the game if no other member contributes in the current period, $V(c, h_{t+1})$, in which case the lower bound of the distribution of types will change in the next period to $l_{h_{t+1}} = c(h_t)$:

$$V^-(c, h_t) = v\left[1 - \left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l\right)}\right)^{n-m}\right] + \delta\left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l\right)}\right)^{n-m} V(c, h_{t+1}) \qquad (2)$$

In equilibrium, type $c(h_t)$ must be indifferent between volunteering and abstaining. Thus, we must have

$$V^+(c(h_t), h_t) = V^-(c(h_t), h_t), \qquad (3)$$

and both values must be equal to $v - c(h_t)$. To understand the implications of (3), the following observation is useful. Suppose that the lower bound on types, l, is below v, but that $c(h_t) = l$, so that no type is expected to volunteer. Then a type $c' \in (l, v)$ expects a payoff of zero. This leads to a contradiction, since such a type can secure a payoff of $v - c' > 0$ by volunteering. Therefore, in any equilibrium with $l < v$, the cutoff must satisfy $c(h_t) > l$. Indeed, we have:

Lemma 2. *Consider an history h_t with lower bound on types $l < v$ and $k = 1$ missing volunters. Then $c(h_t) > l$. Furthermore, $\lim_{t\to\infty} c(h_t) = v$.*

Proof: See the appendix ■.

By Lemma 2, we know that $c(h_{t+1}) > c(h_t)$, so type $c(h_t)$ contributes for sure in period $t+1$ if the game arrives to this stage. Hence, we must have: $V(c(h_t), h_{t+1}) = v - c(h_t)$. We conclude that the indifference condition characterizing the equilibrium cutpoint is:

$$v - c(h_t) = v\left[1 - \left(\frac{1-F\left(c(h_t)\right)}{1-F\left(l\right)}\right)^{n-m}\right] + \delta\left(\frac{1-F\left(c(h_t)\right)}{1-F\left(l\right)}\right)^{n-m}\left(v - c(h_t)\right) \tag{4}$$

Condition (4) leads immediately to a complete characterization of the unique sequence of equilibrium cutpoints:

Proposition 1. *Starting from any history h_t with lower bound on types $l < v$ and $k = 1$ missing volunteers, the equilibrium cutoff thresholds $c(h_\tau)$ are uniquely characterized for all $\tau \geq t$ by the difference equation:*

$$c(h_\tau) = \left[\frac{(1-\delta)\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}{1-\delta\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}\right]v \tag{5}$$

with initial condition $c(h_{t-1}) = l$.

Proof: Condition (5) follows directly from rewriting (4). ■

Condition (5) is illustrated in the top panel of Figure 1. The bottom panel of Figure 1 illustrates the dynamics of the cutoff thresholds over time.

The left hand side of (5) is the opportunity cost of contributing in the current period for the cutoff type, $c(h_\tau)$; the right hand side is the discounted net expected benefit of waiting and contributing next period, for the cutoff type. The right hand side is a function of $c(h_\tau)$ itself, since it depends on the strategy followed by the other players, which is itself determined by the cutoff $c(h_\tau)$. The equilibrium cutoff is a fixed-point of (5). As illustrated in the top panel of Figure 1, the right hand side of (5) is strictly decreasing, greater than c at $c = c(h_{\tau-1})$, and less than c at $c = 1$: so there is a unique interior fixed-point where the curve intersects the diagonal: $c(h_\tau) \in (c(h_{\tau-1}), 1)$. Difference equation (5) can therefore be used to iteratively construct all the equilibrium cutpoints for all h_t and the associated value functions, and thus fully characterize the unique symmetric PBE for the case of $k = 1$.

The characterization of the equilibrium for the Dynamic Volunteers Dilemma has two immediate implications. First, even when only one contributor is needed, the equilibrium is inefficient since the probability of immediate realization of the public good, $\Phi_t = 1 - [1 - F(c_1)]^n$, is strictly less than 1. Second, if there is at least one player with cost lower than v, then the project would be eventually realized.

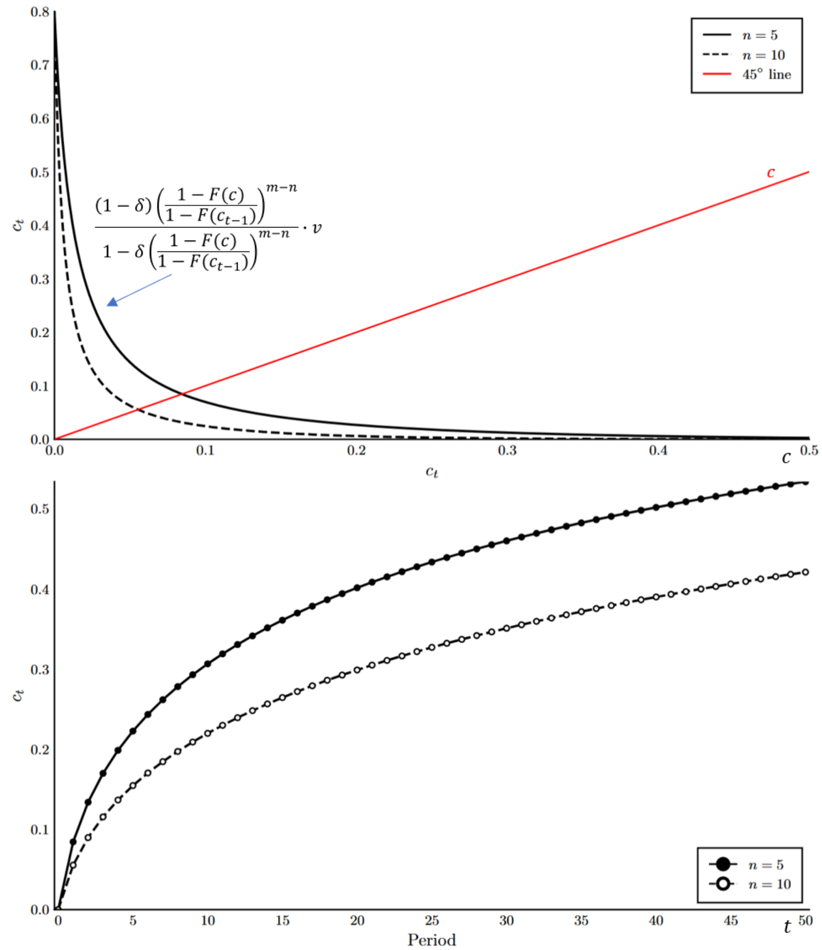

Figure 1: The equilibrium for $m = 1$ when F is uniform, $\delta = 0.95$, $v = 0.8$ and $c_{t-1} = 0$. The solid and dashed curves in the top panel are, respectively, the right hand side of (5) when $n = 5$ and $n = 10$.

3.2 Value functions

The unique characterization of equilibrium cutpoints in the difference equation (5) implies two relevant value functions: (1) the equilibrium continuation value for an agent who has committed before t, given that the lower-bound of types is l, which we denote as $Q^1(l)$; and (2) the equilibrium continuation value of a player of type c when the lower-bound on types is l who is still uncommitted, that we denote as $V^1(c, l)$. These will be useful for the full characterization when $k > 1$. Note that $Q^1(l)$ and $V^1(c, l)$ depend only on the payoff-relevant component of the history, namely the number of missing volunteers (i.e., 1 in this case), the lower bound of types l and, in the case of $V^1(c, l)$, the agent's type. The reason is that, once the game reaches a history h_t in which only one volunteer is missing, the equilibrium is unique and it is independent of the payoff irrelevant component of h_t.

Consider first the value in period t of a member who has already committed in some previous period before t, when the lower-bound on types is $l = c(h_{t-1})$ and the group is still missing *exactly* $k = 1$ contributors for success. Note there are $n - m + 1$ other uncommitted players. The value $Q^1(l)$ for such a committed player (net of the sunk cost of contributing) when the lower-bound on the types is l can be written as:

$$Q^1(l) = v\left(1 - B(0, n-m+1, \widetilde{F}(c(h_t); l))\right) + \delta Q^1(c(h_t))\, B(0, n-m+1, \widetilde{F}(c(h_t); l)) \quad (6)$$

where $B(0, n-m+1, x)$ is the binomial probability of 0 successes out of $n - m + 1$ trials when the probability of success equals x.[12] Using the notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ we can use recursion to solve for $Q^1(l)$ and write (6) as:

$$Q^1(l) = \sum\nolimits_{\tau=1}^{\infty} \delta^{\tau-1} \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1 - F(c_j)}{1 - F(c_{j-1})}\right)^{n-m+1}\right] \left[1 - \left(\frac{1 - F(c_{\tau+1})}{1 - F(c_\tau)}\right)^{n-m+1}\right] v \qquad (7)$$

where, by convention, $c_0 \equiv c_1 = l$, so $\frac{1-F(c_1)}{1-F(c_0)} = 1$. Note that (7) is defined only as a function of the primitives and the current and future cutpoints $(c_\tau)_{\tau=1}^{\infty}$, defined by (5).

We can also define the value of being uncommitted for a type c at $k = 1$ when the lower-bound on types is l and the group is missing $k = 1$ contributors for success as follows. Note that in this case there are $n - m$ other uncommitted players. If $c \in (l, c(h_t)]$, we have $V^1(c, l) = v - c$ since the game ends when they contribute. Hence, $V^1(c(h_t), l) = v - c(h_t)$. If $c > c(h_t)$, we can define $V^1(c, l)$ when $c > c(h_t)$ as follows:

$$V^1(c, l) = \left[1 - \left(\frac{1 - F(c(h_t))}{1 - F(l)}\right)^{n-m}\right] v + \delta \left(\frac{1 - F(c(h_t))}{1 - F(l)}\right)^{n-m} V^1(c, c(h_t)) \qquad (8)$$

Using the same notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ as in the derivation of $Q^1(l)$, define $T(c)$ as the minimal τ such that $c_\tau \geq c$. Solving recursively, as with $Q^1(l)$, gives:

$$\begin{aligned} V^1(c, l) \;&=\; \sum\nolimits_{\tau=1}^{T(c)-1} \delta^{\tau-1} \cdot \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1 - F(c_j)}{1 - F(c_{j-1})}\right)^{n-m}\right] \left[1 - \left(\frac{1 - F(c_{\tau+1})}{1 - F(c_\tau)}\right)^{n-m}\right] v \\ &\quad + \delta^{T(c)-1} \cdot \left[\prod\nolimits_{j=1}^{T(c)} \left(\frac{1 - F(c_j)}{1 - F(c_{j-1})}\right)^{n-m}\right] \cdot (v - c) \qquad (9) \end{aligned}$$

Note again that $V^1(c, l)$ is fully determined by the cutpoints $(c_\tau)_{\tau=1}^{\infty}$.

[12] Here and henceforth we denote binomial probabilities by $B(i, N, p) \equiv \binom{N}{i} p^i (1-p)^{N-i}$.

3.3 The effects of n and δ on success and welfare

Since the distortion depends on a delay in realization, it is natural to ask whether the distortion may be mitigated by an increase in group size or a decrease in the delay costs. Appendix B contains a detailed analysis of the effect of n and Δ on equilibrium behavior when $m = 1$. With respect to n, we prove that an increase in n always leads to a uniform reduction in the equilibrium cutoff points, implying that players are individually more reluctant to contribute. This free-rider effect, however, is mitigated by the addition of more potential volunteers. A simple revealed preference argument allows us to prove that this second effect always dominates: no type c is worse off after an increase in n, and some types c are strictly better off; and the probability of success increases. The comparative statics with respect to δ is less clear-cut. An increase in δ, corresponding to a decrease in delay costs, also leads to a uniform reduction in the equilibrium cutoff points, since more patient players are more willing to delay contributions. The number of players, however, remains constant, so an increase in δ always reduces the probability of success in each period. However, since the costs of delays are lower when δ is higher, an increase in δ has a welfare effect that cannot be unambiguously signed. In Section 5, we characterize the condition under which the expected utility converges to the efficient level, which includes $m_n = 1$ as a special case. For any given n, however, full efficiency is unattainable even in the limit as δ converges to one.

4 The General Dynamic Collective Action Problem

If $k > 1$ contributors are required, the analysis is more complicated and we modify the notation accordingly. Consider an history h_t at which there are exactly k missing contributors. Let $Q(l, h_t)$ denote the expected utility of a committed player (net of the sunk contributing cost) at history h_t when the lower bound on types is l; and define $V(c, l; h_t)$ to be the expected utility of an active (uncommitted) player of type c when the lower bound is l at history h_t. These functions may now depend on the payoff-irrelevant component of h_t, since when $k > 1$ there may be multiple equilibria. In that case, h_t may be used to select the relevant continuation values.

As in the previous section, we say a PBE is in cutoff strategies if, for every history h_t with lower bound l, there is a critical cost $c(h_t)$ with the property that every type $c < c(h_t)$ finds it strictly optimal to contribute at h_t, every type $c > c(h_t)$ finds it strictly optimal to wait at h_t, and a player with type $c(h_t)$ is indifferent between contributing and waiting at h_t. We have:

Lemma 3. *All PBE of a continuation game starting from any history h_t in which k contributors are missing for success are in cutoff strategies.*

Proof: See the appendix. ■

The previous section uniquely characterized the equilibrium for $k = 1$, denoted here by $Q^1(l)$, $V^1(c,l)$, and $c^1(l)$ for all histories h_t with one missing volinteer and lowerbound l. As we said, the dependence on the history h_t that led to the subgame with $k = 1$ can be omitted, since the continuation equilibrium is unique and depends only on l. We now proceed by induction on k. Assume that the functions $Q(l, h'_t)$, $V(c, l; h'_t)$, and $c(h'_t)$ are fully defined for all l and all histories h'_t following h_t with fewer missing volunteers than h_t. In the next subsection, we first show that this information allows us to characterize the cutpoints $c(h_t)$ for all l and all histories h_t with k missing volunteers. Then, given these cutpoints, the value functions $Q(l, h_t)$ and $V(c, l; h_t)$ can be derived. This procedure characterizes all continuation games after all histories, for every $k \leq m$ and every $l \in [0, 1]$. In Section 4.2, we complete the finite-n equilibrium analysis by studying when these equilibria lead to success.

In what follows, we study the PBE starting from a general history h_t at which k volunteers are still missing. To simplify notation, we omit this history whenever this creates no confusion, and write $Q^k(l)$, $V^k(c,l)$, and $c^k(l)$ for $Q(l, h_t)$, $V(c, l; h_t)$, and $c(h_t)$, respectively. Similarly, for continuation values at $t+1$, if j players volunteer at t, we write $Q^{k-j}(l)$, $V^{k-j}(c)$, and $c^{k-j}(l)$ for the corresponding continuation objects that follow history h_t when at t there are j volunteers, leaving their dependence on h_t implicit. Finally, for a given history h_t, and whenever this creates no confusion, define the sequence $\{c^k_s\}_{s\geq 0}$ recursively by $c^k_s = c^k(c^k_{s-1})$, with initial condition $c^k_0 = l$, when no player contributes in the periods from $t - j$ to $t - 1$. This is the sequence of cutpoints following l at h_t when there are no volunteers.

4.1 Characterization and existence

Consider first the value of an uncommitted player of type c who contributes at an history h_t with k missing volunteers and lowerbound on types l. Note that when we are missing $k > 1$ contributors, for an uncommitted player, there are $n - 1 - m + k$ other uncommitted players in the game. Hence, the value is:

$$\left[V^k\right]^+(c,l) = v \sum_{j=k-1}^{n-1-m+k} B\left(j, n-1-m+k, \widetilde{F}(c^k(l);l)\right) \tag{10}$$
$$+\delta \sum_{j=0}^{k-2} B\left(j, n-1-m+k, \widetilde{F}(c^k(l);l)\right) Q^{k-j-1}(c^k(l)) - c.$$

The first term on the right hand side of (10) is the probability the group reaches success at t times the prize v; the second term collects the probabilities that an insufficient number $j < k$ of players volunteers times the discounted expected continuation value for a contributor, $\delta Q^{k-j-1}(c^k(l))$; the third term is the cost of contributing.

The function $Q^{k-j-1}\left(c^k(l)\right)$ used in (10) does not depend on the type of the agent, but it depends on $c^k(l)$ because $c^k(l)$ becomes the minimal type at $t+1$. One can simplify $\left[V^k\right]^+(c,l)$ by adding and subtracting v times the probability the game does not end at t to obtain:

$$\left[V^k\right]^+(c,l) = v - c - \delta \sum\nolimits_{j=0}^{k-2} \left[\begin{array}{c} \left(\frac{v}{\delta} - Q^{k-j-1}(c^k(l))\right) \\ \cdot\, B\left(j, n-1-m+k, \widetilde{F}(c^k(l);l)\right) \end{array} \right] \tag{11}$$

Consider next the expected continuation payoff of an uncommitted agent of type c who does not contribute at history h_t. Similarly as in (11), we can derive it as:

$$\left[V^k\right]^-(c,l) = v - \delta \sum\nolimits_{j=0}^{k-1} \left[\begin{array}{c} \left(\frac{v}{\delta} - V^{k-j}(c, c^k(l))\right) \\ \cdot\, B\left(j, n-1-m+k, \widetilde{F}(c^k(l);l)\right) \end{array} \right] \tag{12}$$

Note that $[V^k]^+(c,l)$ and $[V^k]^-(c,l)$ differ in subtle ways, beyond the fact that in the first case the player pays the cost c. After contributing at t, in (11) the player's continuation value with j other volunteers, that is, the third term, depends on $Q^{k-j-1}(c^k(l))$: the expected continuation value for a committed player when the lower bound is $c^k(l)$ and the remaining number of missing volunteers is $k-j-1$. In (12), instead, the continuation value with j other volunteers depends on $V^{k-j}(c, c^k(l))$: the expected value of an uncommitted player with the same lower bound but with $k-j$ missing volunteers. Thus, although the realization j is the same in the two cases, the number of missing volunteers differs depending on whether the player volunteers. As a result, the player expects the other players to behave differently.

There are now two possibilities. The first case arises if the equilibrium cutoff is a corner solution in which $c^k(l) = l$ and all types $c \geq l$ choose not to contribute. This is possible only if $\left[V^k\right]^+(l,l) \leq \left[V^k\right]^-(l,l)$, which (as it can be easily verified from (11) and (12)) is equivalent to $\delta Q^{k-1}(l) \leq l$. When this condition is satisfied, even a player of type l (the lowest possible cost at h_t^k) is unwilling to commit if no other player with a higher cost will ever contribute.[13] Intuitively, if this player contributes alone, then the discounted benefit is $\delta Q^{k-1}(l)$ and the cost is l. On the other hand, if this condition is not satisfied, then any player of type close to l is willing to contribute even if s/he expects no participation from

[13]Strictly speaking, we have a corner solution if $\left[V^k\right]^+(l,l) < \left[V^k\right]^-(l,l)$. The properties of a PBE when type l is indifferent between contributing or not are the same as when the inequality is strict.

the other players at t, in the hope that other players will continue contributing in history h_{t+1}^{k-1} in which the missing contributors are $k-1$.

The other possibility is that we have an interior solution with $c^k(l) > l$. In this case the threshold $c^k(l)$ is given by an indifference equation similar to the indifference condition for the volunteer's dilemma in equation (4):

$$\left[V_t^k\right]^+ (c^k(l), l) = \left[V^k\right]^- (c^k(l), l) \tag{13}$$

The right and left hand side of (13) are defined as functions of exclusively the cutpoints $c^k(l)$ for histories in which k contributors are missing. These cutpoints can be found solving (13) for any h_t^k. After some algebra, these conditions can be written as:

$$c^k(l) = \delta \sum\nolimits_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}\left(c^k(l)\right) - V^{k-j}(c^k(l), c^k(l))\right) \\ \cdot B\left(j, n-1-m+k, \widetilde{F}(c^k(l); l)\right) \end{array} \right]. \tag{14}$$

where, by convention, we define $Q^0(\cdot) = v/\delta$. The system of equations defined in (14) characterizes $c^k(l)$ in the same way as condition (5) defined the cutpoints in the case with $k = 1$.[14] It also has a similar interpretation. The left hand side is the cost of contributing by the cutpoint type $c^k(l)$; the right hand side is the net expected discounted utility of contributing: the difference between the utility after contributing minus the continuation in the absence of a contribution. In order to prove existence of a PBE, in Theorem 2 below we prove that a fixed-point of (14) exists for any h_t, a step that will be discussed below.

Once we have the cutpoints for all continuation games when k contributors are missing, we can close the circle and define an equilibrium in all continuation games up to $k = m_n$. Given the cutpoints $c^k(l)$ defined above we can indeed define the expected continuation payoffs at history h_t^k to all the players as a function of l. The payoff to a player who has contributed in previous periods does not depend on the player's cost, c, and is given by:[15]

$$Q^k(l) = v - \delta \cdot \sum\nolimits_{j=0}^{k-1} \left[\left(\frac{v}{\delta} - Q^{k-j}(c^k(l))\right) B\left(j, n-m+k, \widetilde{F}(c^k(l); l)\right)\right]. \tag{15}$$

Using (11),(12) and (15), we can finally obtain $V^k(c, l)$, which is equal to $\left[V^k\right]^+(c, l)$ for $c \leq c^k(l)$, and equal to $\left[V^k\right]^-(c, l)$ otherwise. With this, we have all the ingredients to complete the induction argument. Using $Q^j(l)$ and $V^j(c, l)$ for $j \leq k$ we can now obtain $c^{k+1}(l)$, and then $Q^j(l)$ and $V^j(c, l)$ for $j \leq k+1$. We can therefore obtain $c^j(l)$, $Q^j(l)$, and $V^j(c, l)$ for $j \leq m_n$, which fully characterizes the equilibrium.

[14] One can verify that, for $k = 1$, equation (14) reduces to equation (5).

[15] To find (15) we first write $Q^k(l)$ in terms of expected payoffs then, as we did for (11) and (12), we rewrite it by adding and subtracting v times the probability the game does not ends at t.

We now have a full characterization of the equilibria.

Theorem 1. *Every symmetric PBE is characterized by a monotonically non-decreasing sequence of cutpoints $c_t = c(h_t) < v$ for histories h_t with $k \leq m$ missing volunteers and $t = 0, ..., \infty$, where $c(h_t)$ is inductively defined by (11), (12), (14) and (15) as described above. Furthermore if $c_t = c_{t-1}$ for some t, then $c_\tau = c_{t-1}$ for all $\tau > t$.*

Proof. At each generic history, h_t with lower bound on types l, we use the following shorthand to denote the evolution of cutpoints. For state k we write $c_t^k := c^k(l)$ where $l = c_{t-1}^k$ and $c_{t-1}^k := c(h_{t-1}^j)$ for some history h_{t-1}^j with $j \geq k$ missing volunteers. The game is initialized at the history, h_1 with m missing volunteers and lowerbound $l = 0$. Given the analysis above, we only need to prove that $c_t^k < v$. To see this, note that the right hand side of (14) is always strictly less than v since

$$\delta\left(Q^{k-j-1}\left(c^k(l)\right) - \left[V^{k-j}\right]^+\left(c^k(l), c^k(l)\right)\right) \leq \delta\left(Q^{k-j-1}\left(c^k(l)\right)\right) < v.$$

for all $j = \{0, ...k-1\}$, and $\delta\left(Q^0\left(c^k(l)\right)\right) = v$. So we have that $c^k(l) < v$ for any $l < v$. It follows that $c_t = c^k(c_{t-1}) < v$. ■

We complete the analysis proving the existence of a PBE.

Theorem 2. *A symmetric PBE exists.*

Proof: See the discussion below. The details are provided in the appendix. ■

To understand this result, consider (14). As discussed above, all continuation functions defining the right hand side are defined by the induction step. It is however the case that the value of the right hand side of (14) does not only depend on the cutpoint at t, i.e. $c^k(l)$. The reason is that at $t+1$ the lower bound has moved to $c_1^k = c^k(l)$, so the other players use the strategy $c_2^k = c^k(c^k(l)) = c^k(c_1^k)$. If at $t+1$ we have an interior solution then, from equation (10), $\left[V^k\right]^+(c_1^k, c_1^k)$ can be written as:

$$\left[V^k\right]^+(c_1^k, c_1^k) = v \cdot \sum_{j=k-1}^{n-1-m+k} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k\right)$$
$$+\delta \cdot \sum_{j=0}^{k-2} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k)\right) Q^{k-j-1}(c_2^k) - c,$$

so c_1^k depends on c_2^k. Analogously, c_2^k is itself a function of c_3^k, and so on. This implies that (14) defines c_1^k as a function of all equilibrium cutoffs that follows it along the "worst history" in which there are no additional contributors. When condition (14) is required for all histories h_t, it defines the equilibrium cutoffs $\left(c_j^k\right)_{j=1}^\infty$ as a fixed-point of a correspondence that maps the sequence of cutoffs to itself. To prove existence, we proceed as follows. We first define an auxiliary truncated game in which the players give up and stop contributing if there are T attempts at which no player contributes (for some finite $T > 0$). We then show that, in this game, equilibrium cutpoints $\left(c_j^{T,k}\right)_{j=1}^{T}$ exist and are defined as fixed points of a condition similar to (14). Finally, we prove that as $T \to \infty$, these cutpoints converge to a limit $\left(c_j^k\right)_{j=1}^{\infty}$ that is an equilibrium of the original game. A key step to prove that the truncated game has a fixed-point is to show that the set of continuation values, and thus the right hand side of (14), is a non-empty, convex-, closed- valued and upper-hemicontinuous correspondence in $c^k(l)$. We can prove this with an inductive argument over k. To this goal, note that Section 3.1 established, by construction, the existence of a unique PBE for any lower bound on types l when only one contributor is needed. Moreover, we showed that the associated value functions $Q^1(l)$ and $V^1(c, l)$ are continuous both in l and c for any $c \geq l$. For the induction hypothesis, assume that, for all $j = 1, ..., k-1$, the set of continuation value functions $Q^{k-j-1}(l)$ and $V^{k-j}(c; l)$ corresponding to a PBE in the continuation game is non-empty, convex-, closed- valued and upper-hemicontinuous in l. Using this property, we can prove that the the set of continuation value functions at k, $Q^k(l)$ and $V^k(c; l)$ are non empty, convex valued and upper-hemicontinuous in l.

4.2 Participation and comparative statics

In the previous section we mentioned that it is possible the equilibrium cutoff gets stuck, implying contributions stop after some history reached with strictly positive probability in equilibrium (and this possibility has to be contemplated in the characterization). We however did not prove such an event occurs in equilibrium. Indeed, the following result shows that, while *the probability of success is always strictly positive* in every equilibrium, *the probability of prematurely getting stuck is also strictly positive* in equilibrium:

Theorem 3. *For any $n > 2$, any $1 < m < n$, and any $\delta \in (0, 1)$, in every symmetric PBE:*

1. *The probability of success is strictly positive.*

2. *There is a strictly positive probability of reaching an effectively terminal history h_t with lower bound on types l with $c(h_t) = l < v$ for all $\tau \geq t$.*

Proof: See the appendix. ■

To understand the intuition behind Theorem 3, first observe that point 1 was already established for $m = 1$ in Section 3, where we showed that the first-period cutpoint $c^1(0)$ is strictly positive. Theorem 3 generalizes this result using an inductive argument. For point

2, suppose instead that the probability of success is 1. Then we must have $c_\tau^m \to c_\infty^m = v$, where c_τ^m denotes the sequence of cutpoints after τ periods without a volunteer; otherwise, there would be a positive-probability history leading to failure. But then, for large values of τ, a player must assign most of the probability to other players' types in $(v, 1]$, that is, to types who never volunteer. It follows that players with costs below, but sufficiently close to v will also prefer to abstain: they will not expect any other player to contribute in subsequent periods with significant probability, and thus they will fear wasting their effort. This yields a contradiction. We therefore conclude that $c_\tau^m \to c_\infty^m < v$, implying a strictly positive probability of failure.

Since no player with a cost $c > v$ would find it optimal to contribute, the highest possible probability of success achievable is $\overline{p}_n = \sum_{j=m}^n B\left(j, n, F(v)\right)$.We use this benchmark to evaluate the performance of an equilibrium in a dynamic collective action game. We define a group to be *constrained-successful* if $\lim_{t\to\infty} c_t^k = v$, so that conditional on at least m players having $c < v$, success is achieved with probability one.[16] The following corollary follows immediately from Theorem 3:

Corollary 1. *If $m > 1$ then the group is not constrained-successful, and the probability of success is strictly less than $\overline{p}_n$ in all symmetric equilibria: types with sufficiently low c contribute in early periods, but there is a strictly positive probability of reaching an effectively terminal history h_t^k with $k < m$ and $l_{h_t^k} < v$.*

The main implication of Corollary 1 is that the dynamic process of contributing is probabilistic: the process starts for sure, in the sense that the initial cutpoint is strictly greater than 0, so players initially contribute with strictly positive probability; the final outcome, however, depends on the dynamics of participation which in turn depends on the realization of types. With strictly positive probability the required threshold m is reached and the public good is obtained; but with strictly positive probability the process gets "stuck". This finding shows that the general Dynamic Collective Action Problem with $m > 1$ is fundamentally different than the Dynamic Volunteer Dilemma with $m = 1$ (in discrete or continuous time as in Bliss and Nalebuff [1984]). It also illustrates a sharp contrast with the multi-person war of attrition studied by Bulow and Klemperer [1999]. In all these cases, the probability the efficient allocation is reached equals 1.

The dynamics of beliefs on the equilibrium path is characterized by Theorem 1, Theorem

[16]Being constrained-successful does not imply that the equilibrium is efficient. An equilibrium is efficient if the sum of the costs is lower than nv. When m is finite or when we have a threshold m_n that depends on n but such that $m_n/n \to \alpha < 1$, this condition is always satisfied for n sufficiently large. But these efficient equilibria are unachievable in a honest and obedient mechanism with no transfers.

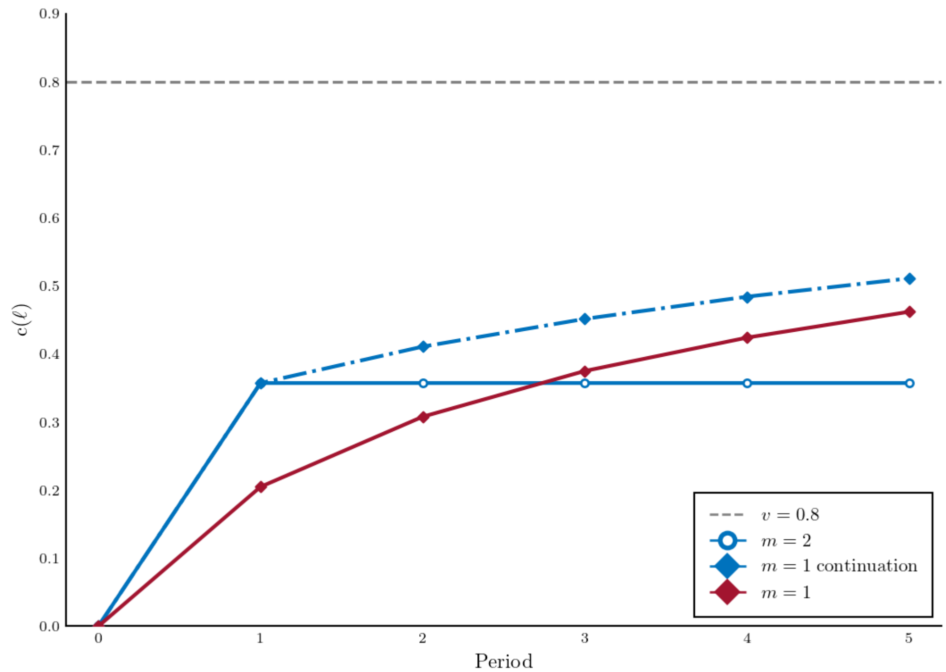

Figure 2: Equilibrium dynamics of cutpoints for $m = 1$ (red) and $m = 2$ (blue).
$n = 3$, $v = 0.8$, $\delta = 0.8$, F Uniform.

3 and equation (1). At any history h_t, all types $c < c(h_t)$ volunteer, so the posterior at h_{t+1} is given by the truncation of the prior: $\widetilde{F}(c; c(h_t))$. Since the cutoffs are monotonically non-decreasing over time, the players who are still active assign progressively greater probability to types above v, i.e., types that will never volunteer. Eventually, one of two events occurs: either the game stops because success is achieved, which happens when the threshold rises above the m_nth lowest realized cost; or beliefs become so pessimistic that the process reaches a history at which the cutoff remains forever constant at some $c < v$. In that case the collective goal is obviously never achieved.

Figure 2 displays the equilibrium dynamics for a very simple example ($n = 3$, $v = 0.8$, $\delta = 0.8$, F Uniform) to illustrate some of the key differences in the evolution of equilibrium cutpoints between $m = 1$ and $m > 1$. The horizontal axis is the period number, with cutpoints on the vertical axis.

The solid red upward sloping curve marks the sequence of cutpoints for each period for the Dynamic Volunteer Dilemma ($m = 1$), along the path where there has not yet been a contribution. The initial cutpoint is approximately 0.2 and increases each period after that, until a contribution is made.

The solid and dashed blue curves mark the history-dependent sequence of cutpoints for $m = 2$. The key difference to note is that the initial cutpoint of 0.34 is much higher (by more than 60%) than the initial cutpoint for $m = 1$. If there are zero contributions in the first period, the game is effectively over, since the equilibrium cutpoints never increase again, as indicated by the solid blue line. The higher period 1 cutpoint is needed to support an equilibrium for two reasons, first to reduce the probability of immediate and complete failure (which doesn't happen if $m = 1$), and second to significantly increase the probability of success. These two reasons are complementary, since the second is needed to induce more members to be willing to take the risk of making an initial contribution. The dashed blue line shows the evolution of cutpoints following a single contribution in period 1, and corresponds to the the $m = 1$ equilibrium with an initial lower bound is 0.34. Another reason that the initial cutpoint for $m = 1$ is lower than the initial cutpoint for $m = 2$ is that the dynamic free riding problem is lessened with $m > 1$, due to the threat of immediate failure. One can interpret this as a *bandwagon effect*, since an early contribution increases the likelihood of future contributions.

Equation (14) and, more generally, the characterization of Theorem 1 allow us to compute the equilibria and evaluate the ex ante expected utility to the players as we vary the parameters. The equilibrium ex ante expected value to a member of the group of size, n, for given parameters δ, v, m, F is defined as:

$$E_n(\delta, v, m, F) = \int_0^1 V^m(c, 0|\delta, v, n)dF(c).$$

Figures 3 and 4 illustrate how E_n varies with n, fixing $m = 2$ and F Uniform in the best PBE, that is the PBE that maximizes the agents expected utility. Group size is on the horizontal axis with E_n on the vertical axis. Figure 3 illustrates how E_n varies with v with a given discount factor $\delta = 0.8$. Figure 4 illustrates how E_n varies with the discount factor δ for $v = 0.65$.

An increase in n leads to a higher (and earlier) probability of success and hence a higher ex ante per capita value of the game. Moreover, in the examples of Figures 3 and 4, E_n converges rapidly to the fully efficient ex ante value for any v and δ, so in the limit, $E_\infty = v$. We will characterize the exact conditions for this property in the next section, which is independent of v or δ, but instead exclusively depends on the speed with which the share of required volunteers m/n converges to zero.

The comparative statics of changing v are perhaps unsurprising: an increase in v is associated with higher utility. The comparative statics of changing the discount factor are more complicated than the effect of changing v. Increasing the discount factor leads to greater cost efficiency, but exacerbates the adverse incentive to delay contributions. So, an

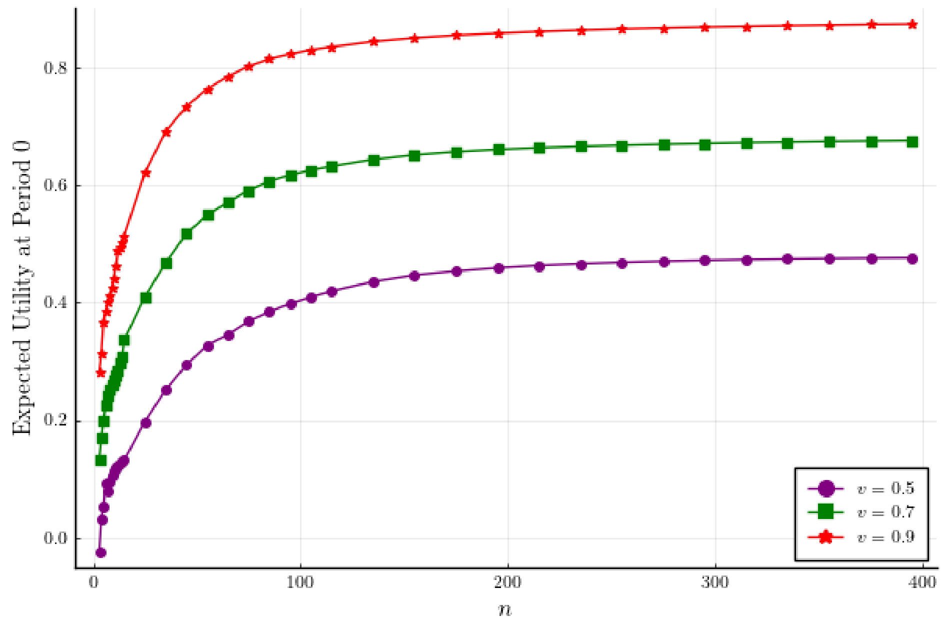

Figure 3: $E_n(\delta, v, m, F)$ for $v = 0.5$, 0.7, 0.9; $m = 2$, $n \leq 400$, $\delta = 0.8$, F Uniform.

increase in patience, δ, results in more delay, a lower probability of success, and lower ex ante group value. The examples of Figure 4 suggest that for any positive discount factor $\delta = 0.2, 0.8$ and 0.95 and any $n = 1, ..., 400$, the players would be better off in a static game where they could commit to immediate termination after a single period (corresponding to $\delta = 0$). While this is true in the example, it is not generally the case that the delay costs of dynamic free riding outweigh the coordination benefits of information transmission.[17] If m_n grows at the same rate as n (as opposed to this example, where $m_n = 2$ for all n), then for sufficiently large n the probability of success in the static game is exactly 0, while the probability of success in the dynamic game is strictly positive. We elaborate on this in Section 6.4.

5 Large groups

We next turn to an analysis of the properties of the PBE and welfare as $n \to \infty$. In the continuous time model of the Dynamic Volunteer Dilemma with $m_n = 1$ by Bliss and Nalebuff [1984], the equilibrium is asymptotically efficient as $n \to \infty$. Large numbers, therefore, eliminate the free rider problem in a collective action.[18] In this section we study

[17]For the special case of $m = 1$ more can be said. See Battaglini and Palfrey (2024b) for an analysis of the effects of changing n and δ.

[18]The same is true in the all pay auction model by Bulow and Klemperer [1999] in which $m > 1$, but fixed. The existence of an asymptotically efficient equilibrium is also the typical result in the literature with

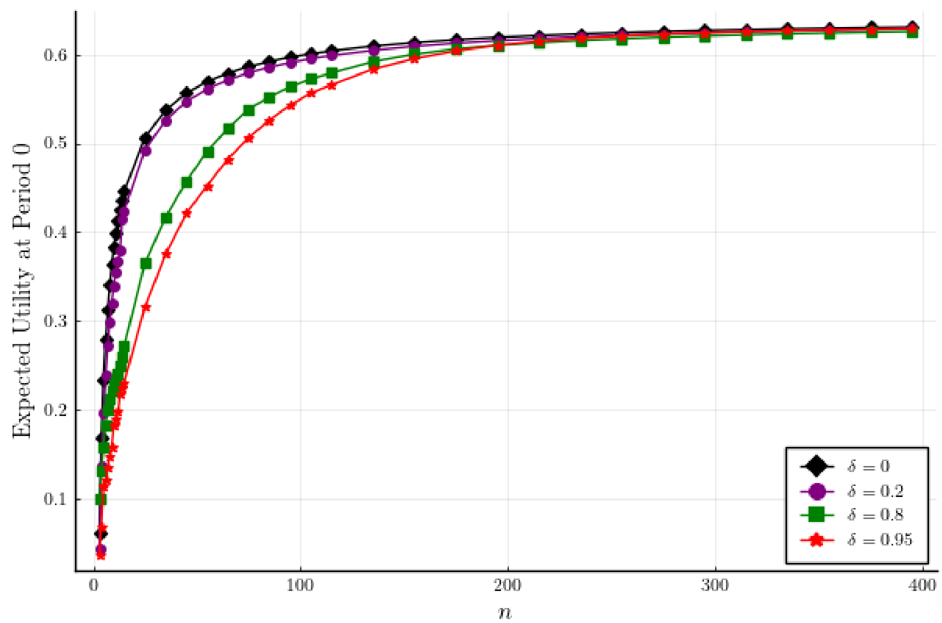

Figure 4: $E_n(\delta, v, m, F)$ for $v = 0.65$, $m = 2$, $n \leq 400$, $\delta = 0, 0.2, 0.8, 0.95$, F Uniform.

the welfare properties of our more general environment in which m_n contributions are needed. In collective action problems with large groups is natural to assume that m_n is larger than one, and indeed that it grows with n. In these cases, we will show that the existence of an efficient equilibrium critically depends on the relative speed with which m_n grows with n.

As previously defined, the share of required contributors is $\alpha_n = m_n/n$. For any sequence $\{\alpha'_n\}_{n=1}^{\infty}$, we say that α_n converges to zero slower (resp., faster or at the same rate) than α'_n if $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = \infty$ (resp., $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = 0$, or $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = l$ for some finite l). For future reference, for two sequences a_n, b_n with $a_n \to 0$, $b_n \to 0$, we write $a_n \succ b_n$ if $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, and $a_n \prec b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$; and for two sequences a_n, b_n with $a_n \to \infty$, $b_n \to \infty$, we write $a_n \succ b_n$ if $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, and $a_n \prec b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

In the following, we maintain the assumption that $\lim_{n\to\infty} \alpha_n < F(v)$. Under this assumption, the probability that the share of players with costs below v exceeds α_n converges to one as $n \to \infty$. This is the most interesting case. If, instead, $\lim_{n\to\infty} \alpha_n > F(v)$ then the probability of failure converges to one, since the share of players with costs below v is, with probability approaching one, smaller than α_n (and the boundary case in which $\lim_{n\to\infty} \alpha_n = F(v)$ is non-generic).

In the first best, the expected per capita utility, as $n \to \infty$, is

$$W_n^* = v - E\left[\sum_{j=1}^{m_n} c^{[j]}(n)/n\right],$$

perfect information (see Marx and Matthews [2000] and Battaglini et al. [2014] for instance).

where $c^{[j]}(n)$ is the jth lowest cost with n samples from $F(c)$. Note that when $\alpha_n \to 0$, then $W_n^* \to v$ as $n \to \infty$; when $\alpha_n \to \alpha < F(v)$, then we still have $W_n^* \to W^* \geq v - F^{-1}(\alpha) > 0$ as $n \to \infty$. It is also immediate to see that if $m_n > 1$, then efficiency is not guaranteed by large numbers, since equilibria with arbitrarily low probability of success are possible for any n if δ is sufficiently small, even if m_n is constant.

As anticipated in the introduction, the key threshold for efficiency is the speed at which α_n converges to zero: if $\alpha_n \prec n^{-1/3}$, then we there is a sequence of equilibria that achieves asymptotic efficiency; if $\alpha_n \succ n^{-1/3}$, then the expected utility converges to zero in all equilibria. Before we formally prove the results, it is useful to discuss intuitively the reason the cube root of $1/n$ plays such a key role in the characterization and why this threshold stems from fundamental features of the strategic interaction.

Suppose here for simplicity that types are uniformly distributed in $[0,1]$, and consider a sequence of equilibria converging to a limit in which success is achieved with no delay. A necessary condition for this to happen is that for large n the first period cutoff for participation, $c_{1,n}^*$, is sufficiently high that the at least $\alpha_n n$ members have a cost less than or equal to $c_{1,n}^*$ with probability close to 1. This, in turn, requires that $c_{1,n}^*$ converges to zero slower or at the same rate as α_n as $n \to \infty$ (hence, $c_{1,n}^* \gtrsim \alpha_n$), otherwise, with probability close to 1, the share of volunteers would fall short of the requirement. The cutoff type $c_{1,n}^*$, moreover, must be on the order of the expected benefit of contributing, i.e. the probability for an individual player to affect the decision in the first period, which can be shown to be proportional to $B(\alpha_n n, n, c_{1,n}^*)$, the binomial probability of $\alpha_n n$ contribution out of n trials when the probability of an individual contribution is $c_{1,n}^*$.[19] If $c_{1,n}^*$ does not converge to zero at the same rate as $B(\alpha_n n, n, c_{1,n}^*)$, then an agent with cost $c_{1,n}^*$ would not be indifferent, contradicting the assumption that $c_{1,n}^*$ is a valid cutoff value. Putting these observations together yields:

$$\alpha_n \lesssim c_{1,n}^* \simeq B(\alpha_n n, n, c_{1,n}^*) \lesssim B(\alpha_n n, n, \alpha_n),$$

where the last step uses the well-known (and intuitive) fact that $B(\alpha_n n, n, c_{1,n}^*)$ is maximized at $c_{1,n}^* = \alpha_n$. That is, the probability of obtaining exactly $\alpha_n n$ successes in n trials is highest when the success probability is α_n. By the Central Limit Theorem, the binomial distribution $B(\cdot, n, \alpha_n)$ is approximately normal with mean $\alpha_n n$ and variance $\alpha_n(1-\alpha_n)n$. It follows that the probability of observing exactly $\alpha_n n$ realizations is of order $1/\sqrt{\alpha_n n}$. We therefore

[19]Indeed, a contribution may be beneficial even if the agent is not pivotal at $t = 1$, since it changes the state at which the game is played in the following periods. A key step in our analysis is to show that, in any sequence of equilibria converging to a limit efficient equilibrium, these expected benefits have only a second order effect, and can thus be ignored in this discussion.

conclude that we need:

$$\alpha_n \precsim \frac{1}{\sqrt{\alpha_n n}}.$$

But this condition cannot hold if $\alpha_n/(1/n)^{1/3} \to \infty$, since in that case α_n converges to zero more slowly than $1/\sqrt{\alpha_n n}$ as $n \to \infty$.

Note that this argument only establishes this rate of convergence as a necessary condition for instant success: not that it is a necessary and sufficient condition for efficiency. To prove the result, in Theorem 4 of Section 5.1 we show that there is guaranteed to exist a sequence of PBEs for which the probability of being pivotal converges to zero at the rate of $B(\alpha_n n, n, \alpha_n)$ whenever $\alpha_n \prec n^{-1/3}$, which in turn implies instant success with probability 1 in the limit. In Theorem 5 of Section 5.2, we prove that the expected utility converges to zero in all equilibria if $\alpha_n \succ n^{-1/3}$.

5.1 A possibility result

The following theorem establishes an asymptotic efficiency result for large populations for the case in which α_n converges to zero sufficiently fast, showing that there is always a sequence of equilibria such that the benchmark W_n^* is asymptotically achieved as $n \to \infty$. To prove this result we rely on the characterization of symmetric equilibria of the previous section. We can, therefore, not only prove that asymptotic efficiency is achievable in equilibrium, but also that it can be achieved with symmetric equilibria.

We start from a preliminary lemma that is useful to prove the result, but has independent interest. Define $c_1^n(0)$ to be the initial cutpoint in a symmetric equilibrium at the start of the game with n members, where $t = 1$, $k = m$ and $l = 0$. The lemma shows that, if α_n converges to zero faster than the cube root of $1/n$, then there exists a sequence of symmetric equilibria in which the share of players who volunteer in the first period, $F(c_1^n(0))$, converges to zero no faster than α_n. Specifically, we have the following result.

Lemma 4. *If α_n converges to zero faster than the cube root of $1/n$, then there exists a sequence of symmetric equilibria with period $t = 1$ cutpoints $c_1^n(0)$ such that $F(c_1^n(0))/\alpha_n > L$ for some $L > 1$.*

Proof. See appendix. ■

Before we discuss the intuition behind this result, it is useful to see its implications. Lemma 4, directly implies that:

$$\lim_{n\to\infty} F(c_1^n(0))/\alpha_n > 1. \tag{16}$$

Using this property and Chebyshev's inequality we conclude that as $n \to \infty$, the share of players willing to contribute is larger than α_n with probability converging to 1 (and it is comprised only of types with cost arbitrarily close to zero). It follows that the collective action succeeds with probability converging to 1. We have:

Theorem 4. *If α_n converges to zero faster than the cube root of $1/n$, then for all $\delta \in (0,1)$ there is a sequence of symmetric equilibria in which $\lim_{n\to\infty} EU_n(c) = v$ for all $c \in (0,1)$.*

Proof. See appendix. ■

The logic of Lemma 4 can be explained using Panel A of Figure 5. The solid curve represents the right-hand side of (14) at $t = 1$, when $k = m$ and $l = 0$, which we denote here by $Z_n(c)$. For the purpose of this discussion, we assume without loss of generality that it is a continuous function of c.[20] Condition (14) admits a fixed point larger than $F^{-1}(\alpha_n)$ if the solid curve, evaluated at $F^{-1}(\alpha_n)$, is above the 45° line. In this case, it admits an intersection to the right of $F^{-1}(\alpha_n)$, since the curve converges to zero as $c \to 1$.[21] In Lemma 4, we prove $Z_n(c)$ is bounded below by:

$$z_n(c) = \xi \cdot B(m_n - 1, n-1, F(c)),$$

where ξ is a positive constant. In the top panel of Figure 5, $z_n(c)$ is illustrated by the dashed curve. Hence, a sufficient condition for (16) is that $z_n(c)$, evaluated at $F^{-1}(\alpha_n)$, is above the 45° line. In this case, the dashed curve intersects the 45° line to the right of $F^{-1}(\alpha_n)$, since this curve converges to zero as $c \to 1$. A fortiori, so does the solid curve, which implies the existence of a fixed point $c_1^n(0) > F^{-1}(\alpha_n)$. The key condition is therefore that $z_n^{m_n}(F^{-1}(\alpha_n)) > F^{-1}(\alpha_n)$, or:

$$\frac{\xi B\left(m_n - 1, n-1, \alpha_n\right)}{F^{-1}\left(\alpha_n\right)} > 1 \tag{17}$$

To see that this is the case, note that using Stirling's approximation, we have:

$$\frac{\xi B\left(m_n - 1, n-1, \alpha_n\right)}{F^{-1}\left(\alpha_n\right)} \simeq \xi f(0) \cdot \sqrt{\frac{1}{2\pi \cdot \alpha_n^3 \left(1-\alpha_n\right) n}}$$

[20]In general, $Z_n(c)$ is a correspondence in c, since there may be multiple equilibria and, therefore, multiple continuation value functions for each c. The set of continuation values defines a nonempty, convex-valued, upper-hemicontinuous correspondence in c. These properties are sufficient for the argument outlined here to go through.

[21]Note that, in Figure 5, $Z_n(c)$ has a positive intersection at $c = 0$. This reflects the fact that the expected benefit of contributing for a type $c = 0$ is positive even if no other player contributes in the current period. Recall that, in $Z_n(c)$, c is the cutoff adopted by the other players. In this case, when $m_n > 1$, although success would be impossible in the current period, a contribution may move the game to a state in which success is more likely in the future.

When $\alpha_n \to 0$ faster than the cube root of $\frac{1}{n}$, the denominator of the right hand side converges to zero, so (17) is guaranteed for sufficiently large n.

5.2 An impossibility result

We have proven above that if α_n converges to zero faster than the cube root of n, then the equilibrium payoff in the most efficient PBE converges to the efficient allocation: immediate success with probability 1. The next result shows that the cube root of n is the critical dividing threshold between complete efficiency and complete failure in large groups: if α_n converges to zero slower than the cube root of n, then in all equilibria—both symmetric and asymmetric—the expected utility of every type of every player converges to zero. In an asymmetric equilibrium, players may adopt different cutpoints, $(c_1^* (h_t), ..., c_n^* (h_t))$, where player $i's$ cutpoint $c_n^* (h_t)$ may depend on the identity of the past contributors, not just their numbers.

The key step in proving this result consists in establishing a relationship between a PBE of our dynamic game and the associated class of static Honest and Obedient mechanisms. A static *Honest and Obedient Mechanism* for our game, henceforth an *HO-mechanism*, is a direct mechanism $\mu : [0, 1]^n \to \Delta(2^I)$, that maps each profile of types, c, to a probability distribution over subgroups of agents who volunteer, and satisfies two properties: honesty and obedience. Honesty means that players find it optimal to report their type truthfully to the mechanism. Obedience means that, if a player is selected by the mechanism, then he or she finds it optimal to volunteer; if the player is not selected, then abstaining is optimal.[22] We have:

Lemma 5. *For any PBE, possibly asymmetric, there is a payoff equivalent (static) Honest and Obedient (HO) direct mechanism that achieves an expected utility for each type that is equal to that type's expected payoff in the PBE.*

Lemma 5 is useful to prove our result because it implies the supremum of the payoffs achievable in a PBE can be bounded above by the maximal payoff achievable in the best static Honest and Obedient mechanism. As proven in Battaglini and Palfrey (2024), the per capita payoff in the best Honest and Obedient mechanism of the static game converges to

[22] HO-mechanisms were introduced by Myerson [1982] as a way to characterize correlated equilibria in static Bayesian games. They differ conceptually from direct incentive-compatible and individually rational mechanisms (hence IC- and IR-mechanisms) because, in HO-mechanisms, the choice of the allocation is decentralized to the players, whereas in IC and IR mechanisms it is centralized by the mechanism. The obedience constraint requires that the action recommended by the mechanism be optimal for the players after the see the recommendation. We stay that a HO-mechanism is payoff equivalent to a PBE of the dynamic game if it generates the same expected utility for all types.

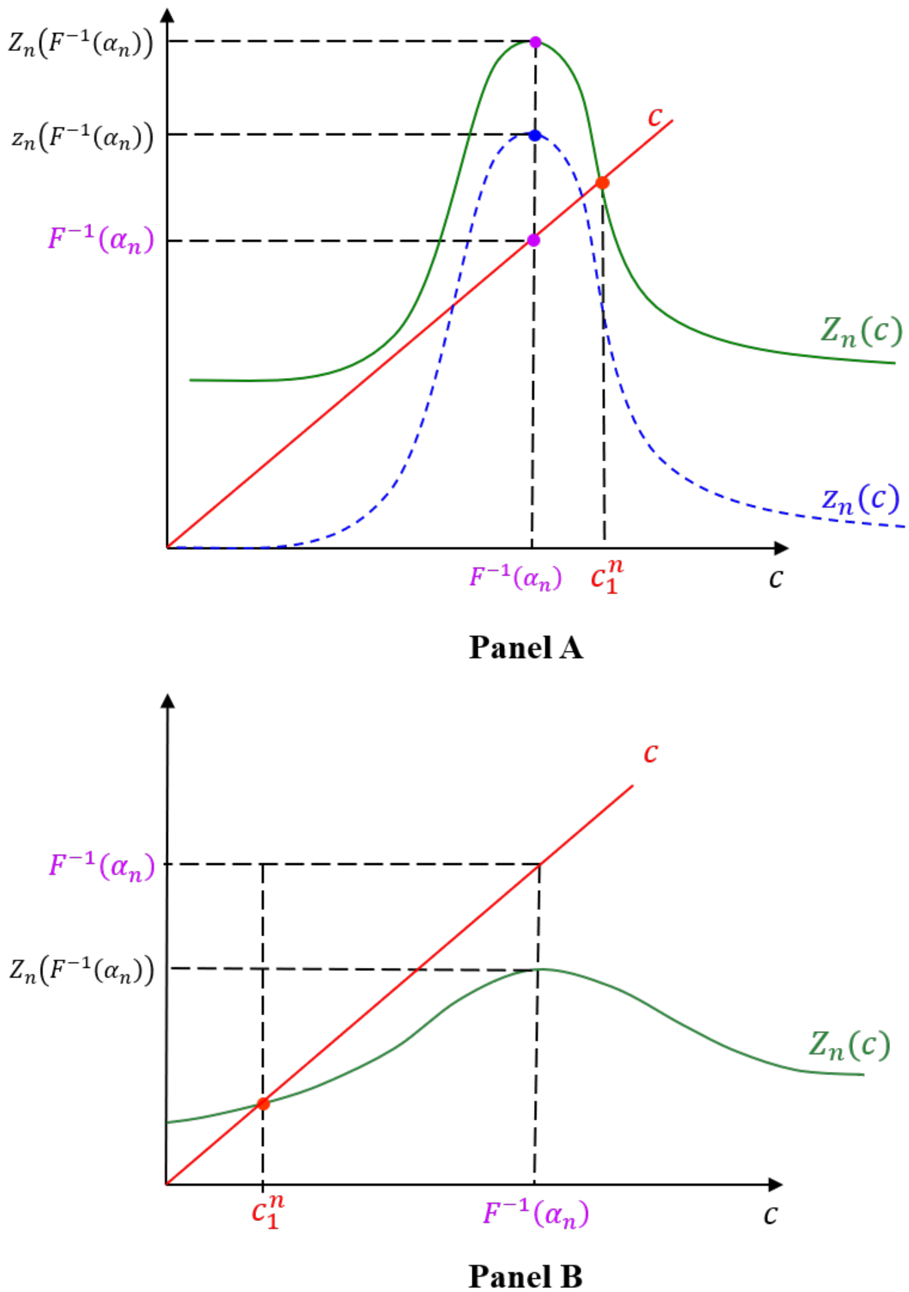

Figure 5: Panel A illustrates the condition characterizing the cutpoint at $t = 1$ when $m_n \prec n^{2/3}$. Panel B illustrates the condition when $m_n \succ n^{2/3}$.

zero as $n \to \infty$ if $\alpha_n/\sqrt[3]{1/n} \to \infty$. Since the payoff of a player in any PBE is non-negative, this implies that the per capita payoff in a PBE converges to zero as well if $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$. This gives us:

Theorem 5. *If α_n converges to zero slower than the cube root of $1/n$, then, for all $\delta \in (0, 1)$, in every sequence of PBE, both symmetric and asymmetric, we have $\lim_{n\to\infty} EU_n(c) = 0$ for all $c \in (0, 1)$.*

Proof. See appendix. ■

The idea behind the proof of Lemma 5 and Theorem 5 may be understood through the lens of the revelation principle, but it is important to highlight its differences. In the revelation principle it is shown that for any mechanism, there is a direct mechanism with the same payoffs for the players: the game forms associated to the mechanisms differ in terms of the players' action space; but they share the same outcome space and utility functions. Here, the dynamic game has a very different outcome space and thus different preferences over it than the corresponding static game: in the latter, the outcome is just a vector of activated players (and the associated success/failure of the common project); in the dynamic game, the outcome is a distribution over time of activated players evaluated over time. Lemma 5, moreover, does not show a 2-way relationship: it is indeed not true that for any HO-mechanism we have a payoff equivalent game. The one-way relationship in Lemma 5, however, is sufficient for our result, because it establishes a tight upper bound on payoffs as $n \to \infty$: since payoffs are weakly positive, the fact that admit an upper bound that converges to zero as $n \to \infty$ characterizes them in the limit.

As observed in the introduction, Ausubel and Deneckere [1989a, 1989b] previously characterized the equilibria of a two-person, dynamic bargaining game with one-sided private information in terms of the allocation achievable through a direct IC, IR mechanism, under the assumption that agents are sufficiently patient. This important result, however, does not extend to more general games with multilateral private information. The key distinction is that with one-sided private information, only the agent possesses private information. When the agent receives a recommendation from the mechanism, the agent's information set remains unchanged. In contrast, in a model with multilateral incomplete information, the agent must update beliefs about the types of other agents after any recommendation, as the recommendation depends on the information collected by the mechanism. This belief updating is precisely what is incorporated into the Obedience constraint. The Obedience constraint is key for the result, since it is the reason why the expected utility converges to zero in the best direct HO-mechanism when $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$. Efficient outcomes are indeed possible in the best IC and IR (but not Obedient) mechanism even when $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$.

To intuitively see why any PBE of the dynamic game defines an equivalent static HO mechanism, consider here, for the sake of the argument, an equilibrium in which the cutoffs $(c_\tau(\mathbf{c}))_{\tau=1}^{\infty}$ depend only on the realized types $\mathbf{c}$ (so there is no other public signal observed by the players).[23] Note that, because we are allowing for asymmetric equilibria, here $c_\tau(\mathbf{c}) = \{c_{\tau,1}(\mathbf{c}), ..., c_{\tau,n}(\mathbf{c})\}$, where $c_{\tau,i}(\mathbf{c})$ is player i's cutpoint at τ when the type profile is $\mathbf{c}$. Even if we fix the PBE, this sequence is stochastic since it depends on the realized profile of types $\mathbf{c}$. But, for a given PBE and a given $\mathbf{c}$, it is deterministic.[24] Given a realized profile of individual costs, $\mathbf{c}$, these cutoffs define $S(\mathbf{c})$, i.e. the first period in which there are m volunteers (which may be never); $T_i(\mathbf{c})$, the first period in which player i volunteers (i.e. the first t in which $c_{t,i}(\mathbf{c}) \geq c_i$); $I_t(\mathbf{c})$, the set of volunteers up to and including period t; and $k_t(\mathbf{c})$ is the number of missing volunteers for success at the end of t.

To construct a payoff equivalent HO-mechanism associated to a PBE, consider the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals i with a type below $c_{1,i}(\mathbf{c})$ are asked to volunteer (i.e., the set $I_1(\mathbf{c})$). If there are at least m such individuals, i.e., $k_1(\mathbf{c}) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c}) = 1$ and $\mu^{DYN}_{I_1(\mathbf{c})}(\mathbf{c}) = 1$ (we denote by $\mu^{DYN}_g(\mathbf{c})$ the probability a group g is asked to volunteer when the profile is $\mathbf{c}$). If $k_1(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > 1$, then with probability $1-\delta$ the algorithm also stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c}) > 1$ and $\mu^{DYN}_{I_1(\mathbf{c})}(\mathbf{c}) = 1 - \delta$. With probability δ, instead, the algorithm proceeds to Step 2. In Step 2, all individuals i with a type in the interval $(c_{1,i}(\mathbf{c}), c_{2,i}(\mathbf{c})]$ are also asked to volunteer and the process continues. In general, at any Step t at which the algorithm has not yet stopped, individuals i with a type in the interval $(c_{t-1,i}(\mathbf{c}), c_{t,i}(\mathbf{c})]$ are asked to volunteer. If there are at least $k_{t-1}(\mathbf{c})$ such individuals, i.e., $k_t(\mathbf{c}) = 0$ and $S(\mathbf{c}) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c})$ without proceeding to step $t+1$. If $k_t(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > t$, then with probability $1-\delta$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability δ the algorithm proceeds to step $t+1$.

[23] Given a type profile $\mathbf{c}$, and ignoring here for simplicity the public signals, the equilibrium cutoff strategy $c(h_t)$ defines a unique history $h_t(\mathbf{c})$, describing the sets of volunteers in all periods $\tau \leq t$. The cutpoint at t can therefore be written as $c_t(\mathbf{c}) = c\,(h_t(\mathbf{c}))$.

[24] In the presence of public signals, the sequence may also depend on the realization of the signals, which determines the continuation equilibrium that is chosen. See the proof of Lemma 5 in the appendix for details.

The algorithm described above defines the following static mechanism:

$$\mu_g^{DYN}(\mathbf{c}) = \begin{cases} \sum_{\{\tau | I_\tau(\mathbf{c})=g\}} (1-\delta)\,\delta^{\tau-1} & |g| < m \\ \delta^{S(\mathbf{c})-1} & \text{if } |g| \geq m \text{ and } g = I_{S(\mathbf{c})}(\mathbf{c}) \\ 0 & \text{else} \end{cases} \tag{18}$$

which mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1-\delta$ after any step at which the threshold m has not yet been achieved.

The proof is completed by showing that $\mu_g^{DYN}(\mathbf{c})$ is an Honest and Obedient mechanism. This fact is intuitive. By construction, the static mechanism asks a player to contribute if and only if the player is in an event that mimics a history in which the player finds it optimal to contribute in the dynamic game. The event "mimics" such a history in the sense that conditioning on such an event, the player has a the same posterior on the other players types as after such an history. It follows therefore that if the static mechanism is not honest and obedient, then we would have a deviation in the PBE, a contradiction.

The arguments behind Theorems 4 and 5 use very different proof strategies, yet the two results complement each other. Theorem 4 shows that efficiency is possible when $m_n \prec n^{2/3}$, while Theorem 5 shows that failure is almost surely guaranteed when $m_n \succ n^{2/3}$. Together, they provide an almost complete characterization of behavior in large economies.[25] There is a deeper geometric intuition for why this happens. As discussed above, the efficiency result is proved by showing that, when $m_n \prec n^{2/3}$, the right-hand side of (14) at $t = 1$, when $k = m$ and $l = 0$, $Z_n(c)$, lies above the 45° line at α_n. This guarantees the existence of an equilibrium cutoff point c_1^n such that $F(c_1^n) > \alpha_n$, as illustrated in Panel A of Figure 5. By contrast, when $m_n \succ n^{2/3}$, the reverse occurs: as illustrated in Panel B of Figure 5, $Z_n(c)$ remains below the 45° line at α_n for large n and admits only fixed points to the left of α_n. These fixed points are incompatible with success with significant probability in large economies. The two cases require different arguments because, in the first case, it is enough to establish the existence of an equilibrium with the required property. In the second case, the property must be established for all equilibria.

5.3 Discussion

The characterizations in Theorems 4 and 5 convey what may seem a pessimistic lesson about the effectiveness of collective action with purely instrumental players (that is, players who care only about outcomes): ceteris paribus, if even an arbitrarily small share of players is required to contribute, the project may fail in the theoretical limit as the population grows

[25] We qualify this statement with "almost" because, when $m_n \prec n^{2/3}$, inefficient equilibria may also exist.

without bound; success occurs only in the demanding case in which the required threshold diverges at a rate of less than $n^{2/3}$ as $n \to \infty$.

Theorems 3 and 4, however, do not imply that collective action should generally be expected to fail and provide new insights about which environments should be expected to be more (or less) conducive to success. For *finite populations*, the characterization in Theorem 3 establishes that the group succeeds with a strictly positive probability, regardless of how large m_n is. Thus, whether this probability is high or low *empirically* would depend on the structural variables of the model: n, m_n, v, and the cost distribution F, even when n is large. For relatively major change requiring participation thresholds above $n^{2/3}$, there could still be a significant probability of success if the distribution of costs is relatively low compared with v (in a first order stochastic dominance sense).

Simple variants of the model within the same framework can also help explaining significant rates of success even in the most challenging scenarios. For instance, some individuals may enjoy a non-instrumental benefit $b > 0$ from volunteering, i.e., a positive utility derived directly from the act of participation. This is equivalent to assuming that the cost of volunteering is distributed according to F with support $[\underline{c}, \overline{c}]$, where $\underline{c}$ is negative. In this case, a fraction $F(0)$ of the population have a dominant strategy to participate immediately regardless of others' behavior. The analysis with this minor variant can be executed in the same way as above, and leads to a more optimistic result about limiting behavior. In fact, collective action is guaranteed as long as $\lim_{n\to\infty} \alpha_n < F(0)$, a condition that is compatible even with a constant $\alpha_n = \alpha > 0$.[26]

The extent to which we should expect the conditions for successful dynamic collective action, therefore, depends on the specific features of the environment. We discuss several prominent applications below.

Public protests A natural interpretation of our model, mentioned in the introduction, is that it describes behavior in public protests. Commentators typically perceive these are problems in which achieving a sufficiently large threshold of participation is key for success. For example, commenting the Iran uprising of 2025, the Iranian scholar Abbas Milani noted that "if you are in numbers strong enough, you can force the regime to back down".[27] Based on an analysis of a large dataset on nonviolent mass campaigns, Chenoweth and Stephan (2011) conjecture that nonviolent campaigns aimed at regime change that actively mobilize

[26]Evidence that protesters may receive social-image benefits from participation, and that these benefits may depend on population size, is presented by Cantoni et al. (2019).

[27]PBS NewsHour, interview with Abbas Milani, January 16, 2025, 21:18. https://www.youtube.com/watch?v=C3sYBAL34Ss

at least about 3.5 percent of the population tend to achieve their main objective. They refer to this empirical regularity as the "3.5 percent rule."[28]

Importantly for our theory, moreover, there is ample evidence that public protests have an important dynamic component, as agents look at past behavior to assess the likelihood that the protest will succeed. For instance, empirical studies have demonstrated that participation fosters further participation and generates dynamic bandwagon effects.[29] Studying the Hong Kong anti-authoritarian student movement, Cantoni et al. (2019) show that, consistently to our model, participants update their beliefs about actual protest turnout in response to information about others' planned turnout, with evidence of strategic substitutability: higher expected turnout reduces an individual's own incentive to participate.

Using the $n^{2/3}$ rule as a rough approximation suggests that collective action should have a reasonable chance to succeed when the participation threshold is below about 150,000 people in a country such as Italy, with a population of roughly 60 million; or below about one-half million people in the United States, with a population of roughly 340 million. For smaller collectives, such as Ithaca, NY, with a population of roughly 32,000, the corresponding threshold is below about 1,000 participants.[30] Of course one would expect thresholds to vary a lot, depending on the significance of the policy change that a protest movement is seeking. These thresholds are unlikely to be sufficient for drastic policy changes, such as regime change, but they may be relevant in many other contexts. For instance, in 2012, United Airlines revised its "PetSafe" policy, which had restricted dogs from flying in the main cabin on the basis of breed, in direct response to a customer petition organized through Change.org. Similarly, in 2007, the UK government reconsidered road-pricing and car-tracking proposals after a large citizen petition.[31] In the context of public protests, the requirement of a threshold of no more than $n^{2/3}$ is further relaxed by the fact that there is evidence that protesters may receive a non-instrumental benefit $b > 0$ from protesting, thus the effective threshold should be seen as $\alpha_n - F(0)$, as discussed above.[32]

[28]Such a sharp threshold should be interpreted with caution, the "3.5% rule" is however noteworthy because it has attracted considerable media attention and has been used as a motivating benchmark for civil action. See, for instance, Leingang (2025) and Pod Save America (2025).

[29]See, for example, Finkel (1985) and Gonzalez-Rostani and Nonnemacher (2025). Further anecdotal evidence abounds. See for instance Hobsbawm and Rude(1969) for a description of the 1830s "swing riots" by agricultural workers in the United Kingdom, and Lohmann (1994) for an analysis of the demonstrations leading to the fall of the Berlin Wall in 1989.

[30]The asymptotic results in Theorem 4 and 5 identify a rate of convergence, not a sharp numerical threshold. These calculations use $n^{2/3}$ with an implicit constant of proportionality equal to one: for example, $60,000,000^{2/3} \approx 153,000$. They should therefore be read only as illustrative orders of magnitude, not as calibrated predictions.

[31]For the first example, see Karp [2012]. For the second example, see BBC News [2007].

[32]For instance, Cantoni et al. (2019) provide suggestive evidence that participation in the Hong Kong

International Environmental Agreements Participation in environmental agreements can be seen as contributions toward a public good performed at the national level.[33] Joining and complying with International Environmental Agreements (IEAs) entails costs, such as the adoption of costly technologies or regulations that constrain economic activity; the benefits of these actions can be fully realized only if participation exceeds certain thresholds. A prominent example is the *Kyoto Protocol*, which explicitly required ratification by at least 55 countries and by countries accounting for at least 55 percent of 1990 Annex I CO_2 emissions. The Protocol entered into force only in 2005, after Russia ratified it. The United States never ratified the agreement sharply limiting its effectiveness. For these reasons, the protocol is widely considered a failure. Consistently with our model, these agreements also typically unfold over time. For instance, the average time to ratification of the *United Nations Framework Convention on Climate Change* was 810 days, and more than one year elapsed before a quarter of the countries ratified it (Fredriksson and Gaston [2000]). Clearly, for problems of this type the appropriate assumption is that the number of participants is small, which implies our theory predicts success is possible, but not guaranteed. The characterization in Theorem 3 can still be used to generate sharp predictions about behavior and outcomes, similar to the predictions of Figures 3 and 4.

Open source software, vaccination drives and other applications A variety of other important applications of collective action can be understood in light of our model. For instance, in open source development, software engineers independently contribute to the construction of code that is enjoyed by the entire community as a public good. Contributing entails a significant private sunk cost, as it requires time and effort to learn the existing code base and to develop it further. As in the case of public protests, the success of projects depends on the participation of a sufficiently large number of developers. These features make open source development closely related to the free-riding problem studied in this paper. Software development unfolds over time, and early contributions affect both the perceived viability and the eventual usefulness of the project, thereby shaping subsequent participation. At the same time, contributors may obtain immediate private benefits from their actions, since they can directly use the code incorporating their additions. Empirical evidence consistent with this interpretation is provided by Hann et al. (2004). Lerner and Tirole (2002) also emphasize that contributions may generate additional private returns by

protests was partly motivated by political identity and by the desire to signal that identity to one's social network.

[33] See, for example, Carraro and Siniscalco [1993], Dixit and Olson (2000), and Battaglini and Harstad [2016], among others.

signaling competence, thereby increasing the value of contributors' human capital. All these factors contribute to increase $F(0)$ and thus the likelihood of success, as discussed in the previous point.

A similar discussion applies to vaccination drives. Individuals receive a collective benefit if the immunization threshold for society is reached, so that the disease can no longer spread. At the same time, they incur a private cost of vaccination, which may be positive or negative, since immunization can be partially effective even if the societal threshold is not attained. Diseases differ widely in the size of the immunization threshold required for herd immunity. In addition, individuals face heterogeneous net private costs that depend on a variety of unobserved factors, such as age, overall health, and other personal characteristics. For older individuals, the private benefits of vaccination may exceed the costs, while for younger individuals the reverse may be true. With respect to the analysis presented above, in these types of problems, it is plausible to assume that the individual benefit from immunization exceed the cost if the share of immunized players is sufficiently small: this suggests that equilibria with very small levels of immunizations may not exist.

6 Extensions

6.1 Heterogeneous values

Heterogeneous private values for the public good, in addition to private volunteering costs, can be easily incorporated into the model and do not change the analysis in any substantial way. Suppose that, instead of having only privately known costs, the private information for each member is a pair, (c_i, v_i), with $c_i \in [0,1]$ and $v_i \in (0,\infty)$. Let G be a joint distribution on $[0,1] \times (0,\infty)$ with full support and continuous strictly positive and bounded density function g, and assume each member's private information is an independent draw from G. All that matters in terms of a member's optimal decision at any point in the game is the ratio of their cost to value, $b_i = \frac{c_i}{v_i}$. The joint distribution G implies a distribution on b_i, call it $\hat{G}$, from which the cost/value ratio is an independent draw for each member.

This model inherits all the same properties as the simpler model with only private costs, with the exception that $\hat{G}$ has full support on $[0,\infty)$, whereas F only had full support over $[0,1]$ and v was a fixed positive constant less than 1. But this difference is only cosmetic since all voters with $b_i > 1$ have a dominant strategy to never volunteer, exactly like all members with very high cost $c_i > v$. Members with extremely high v_i behave the same as voters with low cost in the original model. Dynamic cutpoint strategies would be in terms of $b_i = \frac{c_i}{v_i}$, and belief updating would be in terms of $\hat{G}$. Hence, all of the characterization results continue to hold.

6.2 On the time horizon and the effectiveness of imposed deadlines

Theorem 6 showed that, while the ex ante probability if success is strictly positive for any n, the limit probability of success always converges to zero unless $\alpha_n \to 0$ sufficiently fast. This suggests the question of whether there are simple modifications in the strategic interaction of the players that may avert this "curse of large numbers" for the collective action problem. We leave the general problem of studying the optimal dynamic mechanism for future research, but focus here on the discussion of simple rules that the group could adopt hoping to improve the performance: *deadlines*, where the group commits to terminate the volunteering game if the goal is not reached by some specified finite period T (if such a commitment power is granted to the group). While terminating the game at T may be suboptimal once period T arrives, commitment to such a rule may be beneficial if it stimulates more volunteering in periods $\tau = 1, ...T$. Intuitively, in the Dynamic Collective Action Problem players have a free rider problem not only with respect to other participants, but to the future selves of all participants (including themselves); imposing a terminal period for contributions could, in principle, limit this problem.

It is not difficult to show that in some cases with finite n, deadlines can improve expected welfare. This follows from the earlier examples in Figure 4, where welfare in the static collective action game is strictly greater than welfare in the unconstrained Dynamic Collective Action Problem. In other words, in those examples, imposing a deadline of $T = 1$ improves welfare.

However, the following result shows that in large enough groups, setting a deadline of any kind has the especially undesirable side effect by generating a unique equilibrium in which participation is exactly zero, so the group does not even try to achieve the goal.

Proposition 2. *Assume $m_n = \alpha n$ for some $\alpha < 1$. For any finite deadline T, participation is exactly zero if n is sufficiently large.*

Proof: See the appendix. ■

Proposition 2 implies that, for a sufficiently high n a deadline of T periods is strictly suboptimal; it generates a payoff of exactly zero, while by Theorem 3 we know that in the game with no deadline the probability of success is always strictly positive, thus the expected payoff is strictly positive.

To see the intuition of this result, consider here the special case in which $T = 1$. In this case the equilibrium cutpoint is determined by the equation:[34]

$$c_n^{1,1} = vB\left(m_n - 1, n - 1, F\left(c_n^{1,1}\right)\right) \tag{19}$$

The left hand side is the cost of contributing for the marginal type; the right hand side is the expected benefit, that is v times the probability of being pivotal. An equilibrium cutoff is a fixed point of this equation. The right hand side of (19) is a function of $c_n^{1,1}$ with derivative equal to zero at $c_n^{1,1} = 0$, increasing up to a maximum at $c_n^* > 0$, and then declining to zero as $c_n^{1,1}$ approaches 1. If v is sufficiently large, when n is small, the right hand side crosses the 45^{o} line twice, from below and from above: so equilibria with positive participation exist. But as n increases, the right hand side of (19) shifts down, since the probability of $m_n - 1$ players with type below $c_n^{1,1}$ is reduced for any $c_n^{1,1} > 0$. For a sufficiently large but still finite n, the right hand side is uniformly strictly lower than $c_n^{1,1}$, except at $c_n^{1,1} = 0$, implying that the only equilibrium cutpoint is $c_n^{1,1} = 0$. In Proposition 2 we indeed show that when the number of missing volunteers m_n grows at the speed of n (as when $m_n = \alpha n$ for some $\alpha < 1$) and $T = 1$ (or there is only one period left before termination), then there exists a threshold $n^{(1)}$ such that for $n > n^{(1)}$ there is no equilibrium in which players with $c > 0$ contribute. Proposition 2 extends this result to the case in which the deadline is in T periods with an induction argument.

A notable implication of this result is that when m_n grows at the speed of n, the static one-shot game leads to *zero* probability of success for large enough n. Therefore, in such environments, the dynamic game leads to better outcomes in terms of welfare than the static game: i.e., the benefits of information transmission and coordination from the equilibrium dynamics outweighs the delay costs of dynamic free riding. This contrasts with the example in Section 4.2 of the paper, where welfare is higher in the static game than the dynamic game when m_n is a fixed constant that does not increase with n. An interesting conjecture is that dynamics produces welfare gains (losses) relative to the static game when the free riding problem is more (less) severe, where more severe corresponds to environments where m_n grows faster than speed of $n^{2/3}$.

6.3 Coordination Mechanisms

Besides self-imposed commitments to deadlines, another mechanism that may improve welfare is mediated communication. The Dynamic Collective Action Problem studied in the

[34]We denote $c_n^{\tau,T}$ to be the cutoff at period τ in a model with a deadline with T periods.

previous sections allows for some coordination through the observation of past actions, but it does not allow agents to use communication to correlate their actions. Communication may, in principle, improve outcomes, since it can allow players to avoid allocations that are clearly dominated for everyone, and perhaps do even better. For instance, if more than m_n players are willing to contribute, the mechanism can select exactly m_n of them and ask only those players to volunteer, thereby avoiding unnecessary waste. If fewer than m_n players are willing to contribute, the mechanism can ask them to wait until more players become available. All players agree that, if at least m_n of them have a cost below 1, which is the value of the collective goal, then the project should be carried out. Yet we have seen that contributions may stop in all PBEs even in this event. Can the players design a mechanism that avoids these "sudden stops"?

A simple communication mechanism that allows for explicit coordination is the *Volunteer-Based Organization* mechanism introduced by Battaglini and Palfrey (2024) for static environments, but that can be extended in a natural way to our dynamic environment.

The static VBO works as follows. Players report to the mechanism whether they are willing to volunteer if asked. If at least m_n players are willing, the mechanism randomly selects exactly m_n of them and asks only them to volunteer. If fewer than m_n players are willing to volunteer, none of the willing players are asked to volunteer, so there is no waste. It is proved in Battaglini and Palfrey [2024] that this mechanism has a unique symmetric Bayesian equilibrium and that it performs strictly better than the static contribution game for all parameter configurations. Figure 6 illustrates that it is not only better than the static contribution game, but can improve on the best PBE of the dynamic contribution game. Thus, in this example, not only does the group do strictly better by adopting a deadline with $T = 1$ than in the dynamic contribution game; the VBO also performs strictly better than the game with a deadline.

The static VBO described above relies both on communication and on the ability to impose a deadline on the game, with the threat of terminating it if there are insufficient volunteers at $t = 1$. This implies that the mechanism, by its very design, stops at $T = 1$ even if the number of players with cost below 1 is at least m_n.[35] Just as for deadlines, moreover, it is also generally dynamically inconsistent, since after $t = 1$ the players may find it optimal to continue contributing. The following mechanism, which we call the *Dynamic Volunteer*

[35] As shown by Battaglini and Palfrey [2024], the equilibrium of the VBO is characterized by a threshold $c_n^{VBO} < 1$ such that players contribute if and only if their cost is less than or equal to this threshold. It follows that the probability of failure is strictly positive even when the costs of more than m_n players are below 1.

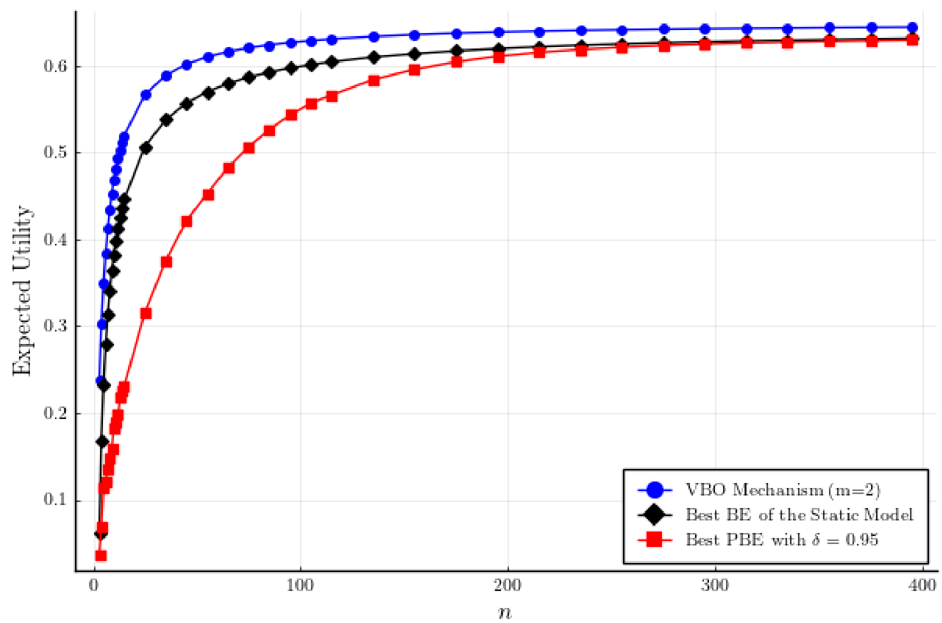

Figure 6: Comparison of welfare in the best PBE of the dynamic collective action game with welfare under the simple VBO and in the best equilibrium with a deadline $T = 1$. The parameters are: $m = 2$, $\delta = 0.95$ and $v = 0.65$.

Based Organization (DVBO), generalizes the VBO to environments in which players cannot impose a deadline. The DVBO starts in any period t with the pool of members who have already committed by period $t-1$, denoted A_{t-1}. We call this pool the "available players": those who have declared that they are available to volunteer if they are eventually selected. The initial condition is $A_{-1} = \varnothing$ at $t = 0$. The mechanism is defined as follows:

- At the beginning of period t, players are asked to declare whether they are "available". Let R_t denote the set of players who declare themselves available at t. The set of available players is updated according to $A_t = A_{t-1} \cup R_t$.

- If $|A_t| \geq m_n$, then exactly m_n volunteers are selected from A_t with uniform probability. If they all volunteer (and possibly other players out of equilibrium) volunteer, the project is realized at t. If at t some of the agents who were asked do not obey and the the number of volunteers is not sufficient, then the project is not realized at t, and at $t+1$ the same agents who were asked at t but have not volunteered are asked again to volunteer.

- If $|A_t| < m_n$, then no agent is asked to volunteer and the project is not realized at t. The mechanism continues to period $t+1$ with state variable A_t.

The next result shows that the DVBO guarantees that in all PBE whenever the number of agents with cost below v is at least m_n, then the probability of contributions and therefore of success is always strictly positive. We have:

Proposition 3. *If the realized cost profile is such that the number of players with cost strictly lower than v is greater than or equal to m_n, then contributions cannot stop "cold turkey" under the DVBO: in all PBE, the probability of success is strictly positive in all periods until the project is realized.*

Proof. See the appendix. ■

Despite these results, there are two important limitations of the DVBO described above. First, the fact that the mechanism guarantees contributions with strictly positive probability whenever at least m_n agents' types are below 1 does not imply that agents' expected utility is substantially improved relative to the original game, since success may still take a long time to occur. Indeed, we conjecture that, under the DVBO as well, expected utility converges to zero as $n \to \infty$ if m_n/n converges to zero more slowly than $n^{-1/3}$. A full analysis of the DVBO, and more generally of the optimal design of contracts in this dynamic context, would however require a more extensive characterization of equilibrium behavior, which is beyond the scope of the present paper. We leave such an analysis for future research.

Second, and perhaps more importantly for the results of this paper, mediated mechanisms require agents to agree on an organization, that is, on a mechanism. While the dynamic collective action game arises naturally in many environments without prior agreement, the DVBO requires both a communication technology and prior agreement on a specific mechanism, which may be infeasible in many settings, especially when the collective action problem arises spontaneously, as in the case of public protests. This is perhaps the most significant obstacle for the agents in the quest for collective action, the fact that they often need to rely on "spontaneous" forms of coordination such the (unmediated) dynamic collective action game, which is the main focus of the paper.

6.4 Other extensions

We consider four additional extensions that further explore the robustness of the model and the possibility of alternative dynamic mechanisms to improve group success. We discuss them briefly here and offer a full analysis Appendix C.

First we extend the basic model to environments where there is aggregate uncertainty about the distribution of types. Second, we show how the analysis can be extended to incorporate nonstationary environments, in which the cost of inaction for a group increases over time, as it might occur for example in environmental problems. Third, we explore the case of uncertain thresholds, so success is probabilistic and agents update their beliefs about the probability distribution of possible thresholds for success over time, as they simultaneously update their prior belief about the distribution of costs. In all three of these extensions to more complex environments, the characterization of equilibrium and limit results generalize.

Fourth, we consider "high-value" environments, where $v > 1$. In these environments, it is common knowledge that it is individually rational for an agent to contribute if they are pivotal. Hence, it is possible that there are interior equilibria of the dynamic game where the equilibrium cutpoints strictly increase over time and success is eventually attained. However, we are able to show for all symmetric equilibria that v must be strictly bounded away from 1 in order for that to happen. That is, there is a critical value, $v^* > 1$, which can vary with n, m and δ, such that the main finding of Theorem 3 continues to hold if $v < v^*$, while for $v \geq v^*$ all equilibria are interior with eventual success guaranteed.

7 Conclusions

Collective action problems arise when a group's collective goal can only be achieved if at least some fraction of its members engage in a costly action to help the group succeed. We study the equilibrium properties of collective action problems when decisions are taken dynamically over a potentially long horizon, delay is costly, and members have heterogeneous and privately known preferences.

We present two categories of characterization results. The first half of the paper characterizes the properties of dynamic equilibrium, as well as providing efficiency results, for any fixed group size and any fixed participation threshold for group success. The simplest such case is known as the volunteers dilemma, or bystander intervention problem, where exactly one member must undertake the action - to rescue a drowning swimmer, or call 911 to report an accident or ongoing violent crime. This also happens to be the only case that had been analyzed as a dynamic stochastic game with incomplete information (Bliss and Nalebuff, 1984). Our first finding establishes that the Dynamic Volunteers Dilemma is a very special case: except in the limit with arbitrarily large groups, the equilibrium properties of the dynamic version of this game do not extend to the more general, and arguably more realistic, case where group success requires the action of more than one member.

In the volunteers dilemma case where only one member is needed, the group always succeeds as long as at least one member of the group has an action cost that is less than the benefit of success. But if group success requires the coordinated action of multiple members, this is no longer true. In this more complicated environment, a member who contemplates taking an action early on faces a real risk that their action will be useless because there will never be a sufficient number of members who decide to activate at later dates. In fact, we show that in such environments, while there will always be a strictly positive probability of success, there is also always a strictly positive probability that the dynamic process of accumulating more activists and getting closer to the goal can fizzle out. In that case, all members who have activated lose out and nobody benefits. The possibility of such failure always exists unless it is common knowledge that every group member would be willing to activate if pivotal. Hence, there are two sources of inefficiency: delay (time is costly); and the probability that the goal is never achieved but many members suffer their action cost.

Our second category of results explores the efficiency properties of dynamic equilibrium in large groups. For these results, we allow both the group size as well as the required threshold to grow without bound and obtain a characterization of efficiency in the limit, which depends on the relative rate at which the threshold grows relative to the group size.

If the fraction of members required for the threshold converges to zero very fast as group size increases –at a rate *faster than the cube root of the inverse of the group size*– then there is always a limiting equilibrium where the goal is instantly achieved with probability 1. There is no delay and full efficiency is achieved. The volunteers dilemma is a special case of this.

On the other hand, if the fraction of members required to meet the threshold converges to zero more slowly than the cube root of the inverse group size, then players' expected utility converges to zero *in every limiting equilibrium*. A special case of this arises if a constant fraction of group size is required. Thus, in both cases, delay costs disappear, as does the deadweight loss incurred when some group members' action costs are wasted. The only efficiency issue in the limit is the probability of group success, which is either 0 or 1.

In collective action problems, bandwagon effects are key, and their role in the equilibrium dynamics warrant further study. We do not have detailed results in the form of the finer properties of equilibrium in our model, but this is a useful direction to pursue. If a protest movement or petition drive catches on quickly and exhibit heavy participation from the outset, this early activity snowballs and encourage others to join in because the prospects of success are higher. On the flip side, if there are only a handful of visible activists in the early stages, then other who were sitting on the sidelines might decide to just forget about it and the movement would fizzle out.

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Appendix A: Proofs

Proof of Lemma 1: We proceed in two steps.

Step 1. We first prove that the expected continuation value of a player who does not volunteer is convex in c and admits right and left derivatives, respectively denoted $\partial^r \left[V^1\right]^- (c, h_t)/\partial c$ and $\partial^l \left[V^1\right]^- (c, h_t)/\partial c$, with $\partial^d \left[V^1\right]^- (c, h_t)/\partial c > -\delta$ for $d = r, l$. Consider any history h_t^1 in which only one volunteer is missing for success, and denote h_{t+j}^1 any history following h_t^1 in which there is no volunteer from t to $t + j$. Let $\beta(h_t^1)$ be the probability that there is at least 1 volunteer at history h_t^1. Define $W^{1,\lambda}(c, h_t^1)$ to be the expected value to a player of type c at h_t^1 who does not volunteer at t, but instead volunteers after $\lambda \in [1, \infty)$ periods (if there is not a volunteer before):

$$\begin{aligned} W^{1,\lambda}(c, h_t^1) &= \sum\nolimits_{\tau=0}^{\lambda-1} \delta^\tau \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \beta(h_{t+j-1}^1)\right]\right] \beta(h_{t+\tau}^1) v \\ &\quad + \delta^\lambda \cdot \prod\nolimits_{j=0}^{\lambda} \left[1 - \beta(h_{t+j-1}^1)\right] \cdot (v - c) \end{aligned} \tag{20}$$

where we define by convention $\beta(h_{t-1}^1) = 0$. Note that all these expressions are linear in c and $\partial W^{1,\lambda}(c, h_t^1)/\partial c > -\delta > -1$. The value for a player who does not volunteer at h_t^1 is: $\left[V^1\right]^- (c, h_t^1) = \max_\lambda W^{1,\lambda}(c, h_t^1)$, which is convex in c and so admits right and left derivatives with $\partial^d \left[V^1\right]^- (c, h_t^1)/\partial c > -\delta$ for $d = r, l$.

Step 2. Note that at h_t^1, the expected utility of a type c who volunteers at t is $\left[V^1\right]^+ (c, h_t^1) = v - c$. Suppose now a type $c(h_t^1)$ is indifferent between volunteering or not, so:

$$v - c(h_t^1) = \left[V^1\right]^- (c(h_t^1), h_t^1) \tag{21}$$

Consider a type $c' > c(h_t^1)$ with $\Delta c = c' - c(h_t^1)$. We have:

$$\begin{aligned} v - c' &= v - c(h_t^1) - \Delta c = \left[V^1\right]^- (c(h_t^1), h_t^1) - \Delta c \\ &< \left[V^1\right]^- (c(h_t^1), h_t^1) - \delta \Delta c < \left[V^1\right]^- (c', h_t^1) \end{aligned}$$

where in the second we use (21), and in the last inequality we use the convexity of $\left[V^1\right]^- (c', h_t^1)$. The proof that $c' < c(h_t^1)$ implies $v - c > \left[V^1\right]^- (c', h_t^1)$ is analogous. ■

Proof of Lemma 2: Note that $c(h_t) \in [l_{h_t}, v)$ and suppose to the contrary that $c(h_t) = l_{h_t}$ in the PBE, i.e., no member will activate in period t. This implies $V^-(c, h_t) > V^+(c, h_t)$ for all $c(h_t) \in (l_{h_t}, v)$. If no member activates in period t, then it must be that no member activates in period $t + 1$ as well: if a member with cost $c \in [l_{h_t}, v)$ were using a strategy to activate in period $t + 1$ and the member knows that no member is activating in period t, then their payoff in the continuation game is $V^-(c, h_t) = \delta(v - c) < v - c = V^+(c, h_t)$, a contradiction.

It follows that if no player volunteers at t, $V^{-}(c, h_t) = 0 < v - c = V^{+}(c, h_t)$, implying again a contradiction. We conclude that at t the probability a player volunteers is strictly positive. To prove that $\lim_{t\to\infty} c(h_t) = v$, suppose to the contrary that $\lim_{t\to\infty} c(h_t) = \overline{c} < v$. Then $\lim_{t\to\infty} V^{-}(\overline{c}, h_t) = 0 < V^{+}(\overline{c}, h_t) = v - \overline{c} > 0$, a contradiction. ■

Proof of Lemma 3: We proceed in three steps.

Step 1. Lemma 1 already established that there is a unique PBE when $k = 1$, which is in cutoff strategies. In that equilibrium, for any history h_t^1 at which $k = 1$, the value for an active player who volunteers is $[V^1]^+(c, h_t^1) = v - c$, which is linear in c with $\partial [V^1]^+(c, h_t^1)/\partial c = -1$. The proof of Lemma 1 also established that $[V^1]^-(c, h_t^1)$ is piecewise linear, convex and hence admits right and left derivatives $\partial^r [V^1]^-(c, h_t^1)/\partial c$ and $\partial^l [V^1]^-(c, h_t^1)/\partial c$, both bounded below by $-\delta > -1$.

Step 2. At any history h_t^k at which $k > 1$, the continuation value of a player who has *previously* committed (i.e. $Q^k(l_{h_t^k})$) does not depend on his/her own type c; it depends only of the behavior of the other players. It follows that the value of an active player at h_t^k, with cost c, who volunteers at h_t^k, i.e. $\left[V^k\right]^+(c, h_t^k)$, is linear in c with $\partial \left[V^k\right]^+(c, h_t^k)/\partial c = -1$. We now proceed by induction. Assume that, for all $\kappa = 1, ..., k-1$, $[V^\kappa]^-(c, h_t^\kappa)$ is convex in c, with $\partial^d [V^\kappa]^-(c, h_t^\kappa)/\partial c > -\delta$ for $d = l, r$. (Step 1 established this property for $\kappa = 1$.) We need to prove that the same is true for $\left[V^k\right]^-(c, h_t^k)$.

To see this, first observe that $[V^\kappa](c, h_t^\kappa) = \max\left\{[V^\kappa]^+(c, h_t^\kappa), [V^\kappa]^-(c, h_t^\kappa)\right\}$ for any $\kappa < k$. Since $[V^\kappa]^+(c, h_t^\kappa)$ is linear in c and $[V^\kappa]^-(c, h_t^\kappa)$ is convex in c , then $[V^\kappa](c, h_t^\kappa)$ is convex in c for all $\kappa < k$. Define $\vartheta(h_t^k)$ to be the probability of having at least one volunteer at h_t^k. Define $\Phi^k(c, h_t^k)$ to be the expected utility of a type c player at history h_t^k conditioning on at least one volunteer at history h_t^k. Because of the properties of $[V^\kappa](c, h_t^\kappa)$ for $\kappa < k$ proven above, $\Phi^k(c, h_t^k)$ is convex in c and $\partial^d \Phi^k(c, h_t^k)/\partial c \geq -\delta$ for any $d = l, r$. Finally, define $W^{k,\lambda}(c, h_t^k)$ to be the value of a player of type c at h_t^k who does not volunteer at t, but volunteers instead after at most $\lambda \in [1, \infty)$ periods in which there is no other volunteer, if there is no volunteer before. We can write:

$$
\begin{aligned}
W^{k,\lambda}(c, h_t^k) \quad = \quad & \sum\nolimits_{\tau=0}^{\lambda-1} \delta^\tau \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \vartheta(h_{t+j-1}^k)\right]\right] \cdot \vartheta(h_{t+\tau}^k) \cdot \Phi^k(c, h_{t+\tau}^k) \\
& + \delta^{\tau\lambda} \cdot \left[\prod\nolimits_{j=0}^{\lambda} \left[1 - \vartheta(h_{t+j-1}^k)\right]\right)\right] \cdot \left[V^k\right]^+(c, h_{t+\lambda}^k) \qquad (22)
\end{aligned}
$$

where $\vartheta(h_{-1}^k) = 0$ by convention. Since $\Phi^k(c, h_{t+\tau}^k)$ and $\left[V^k\right]^+(c, h_t^k)$ are convex $\forall \tau = 1, ..., \lambda - 1$, $W^{k,\lambda}(c, h_t^1)$ is convex. And by the same argument as before, this also implies that $\partial^d W^{k,\lambda}(c, h_t^k)/\partial c > -\delta$ for any $d = l, r$. Finally, note that $\left[V^k\right]^-(c, h_t^k) = \max_\lambda W^{k,\lambda}(c, h_t^k)$. It follows that $\left[V^k\right]^-(c, h_t^k)$ is convex in c and $\partial^d \left[V^k\right]^-(c, h_t^k)/\partial c > -\delta$ for $d = l, r$.

Step 3. Assume now a type $c(h_t^k)$ is indifferent between volunteering or not at history

h_t^k. That is: $\left[V^k\right]^+(c(h_t^k), h_t^k) = \left[V^k\right]^-(c(h_t^k), h_t^k)$. Consider a type $c' > c(h_t^k)$ with $\Delta c = c' - c(h_t^k)$. We have:

$$\begin{aligned}
\left[V^k\right]^+(c', h_t^k) &= \left[V^k\right]^+(c(h_t^k), h_t^k) - \Delta c = \left[V^k\right]^-(c(h_t^k), h_t^k) - \Delta c \\
&< \left[V^k\right]^-(c(h_t^k), h_t^k) - \delta\Delta c < \left[V^k\right]^-(c', h_t^k)
\end{aligned}$$

where in the first equality we use the linearity of $\left[V^k\right]^+(c', h_t^k)$, in the second equality we use the indifference condition, in the third (inequality) we use the induction hypothesis, and finally in the last inequality we use the convexity of $\left[V^k\right]^-(c', h_t^k)$. The proof that $c' < c(h_t^k)$ implies $\left[V^k\right]^+(c', h_t^k) > \left[V^k\right]^-(c', h_t^k)$ is analogous. ■

Proof of Theorem 2: Consider a modified version of the game, which we call the T-truncated game. In this version, if only one volunteer is missing, the game is unchanged. If more than one volunteer is missing and T periods pass with no player volunteering, the game stops and all players receive payoff 0. The original game corresponds to the case in which players are always given an opportunity to volunteer until success is reached. We first prove the existence of a PBE for the T-truncated game. We then show that the limit of equilibria of the T-truncated game as $T \to \infty$ are equilibria of the original game. We proceed in three steps.

Step 1. Assume the number of missing volunteers is k and $T-1$ periods have passed with no volunteer, so that only one chance to volunteer is left. In this subgame an equilibrium cutpoint is characterized by:

$$\Omega_1(l) = \left\{ c \geq l \;\middle|\; \begin{array}{c} c = \delta \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q_T^{k-j-1}(c) - V_T^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \\ \text{or } c = l \text{ if } \delta Q_T^{k-1}(l) \leq l, \\ \text{for some } \{Q_T^{k-j}(c), V_T^{k-j}(c,c)\}_{j=1}^{k-1} \in \{\mathcal{V}_T^{T,k-j}(c)\}_{j=1}^{k-1}, \\ \text{and } Q_T^0(c) = v,\, V_T^k(c,c) = 0 \end{array} \right\} \tag{23}$$

The set of all possible future equilibrium value functions with $k-j$ missing volunteers in the definition of $\Omega_1(l)$, $\{\mathcal{V}_T^{T,k-j}(c)\}_{j=1}^{k-1}$, is defined inductively starting from $k-j = 1$ for all $j \in 1, 2, ..., k-1$, i.e. when at least one volunteer activates in stage t; and $\{Q_T^{k-j}(c), V_T^{k-j}(c,c)\}_{j=1}^{k-1}$ is a selection from $\{\mathcal{V}_T^{T,k-j}(c)\}_{j=1}^{k-1}$: i.e., $\{Q_T^{k-j}(c), V_T^{k-j}(c,c)\}_{j=1}^{k-1}$ is the collection of future equilibrium value functions associated with one specific PBE. Note that we have proven in Section 3 that $Q^1(c)$ and $V^1(c,c)$ are uniquely defined continuous function of c, so a fortiori $\mathcal{V}_T^{T,1}(c)$ is a non empty, compact- and convex-valued, upper-hemicontinuous correspondence in c. We now assume as induction hypothesis that $\mathcal{V}_T^{T,k-j}(c)$ has these properties for all $j \in [1, k-1]$, and we prove it is true for $\mathcal{V}_T^{T,k}(c)$ as well.

If $\delta Q_T^{k-1}(l) \leq l$, then $l \in \Omega_1(l)$, hence $\Omega_1(l)$ is non-empty. Assume $\delta Q_T^{k-1}(l) > l$. Fix l and define the correspondence $\Phi_1(\cdot; l)$ on the compact interval of feasible cutoffs $[l, 1]$ by the right-hand side of the fixed-point equation in the first line in (23), allowing all admissible selections from the continuation-value correspondences. By the induction hypothesis, the continuation-value correspondences are non-empty, compact- and convex-valued, and upper-hemicontinuous. Since the binomial weights are continuous in (c, l), $\Phi_1(\cdot; l)$ is non-empty, compact- and convex-valued, and upper-hemicontinuous. Moreover, Φ_1 maps the compact feasible cutoff interval into itself. Kakutani's fixed-point theorem therefore implies that Φ_1 has a fixed point. Hence, $\Omega_1(l)$ is non-empty.

For any possible equilibrium cutoff, $c_1 \in \Omega_1(l)$, and associated set of future equilibrium value functions $\{Q_T^{k-j-1}(c), V_T^{k-j}(c, c)\}_{j=0}^{k-1}$, one obtains the corresponding value functions $Q_1^k(l), \left[V_1^k\right]^+(c, l)$, $\left[V_1^k\right]^-(c, l)$ and $V_1^k(c, l) = \max\{\left[V_1^k\right]^+(c, l), \left[V_1^k\right]^-(c, l)\}$. These functions directly depend on c_1. Let $\mathcal{E}_1^k(l)$ be the set of possible equilibrium values for an uncommitted player of type l when the lower bound on types is l and k volunteers are missing; and denote the convex hull of $\mathcal{E}_1^k(l)$ by $\Delta\mathcal{E}_1^k(l)$. Note that the set $\Delta\mathcal{E}_1^k(l)$ corresponds to the equilibrium values if the expected continuation values are generated by PBEs when at least one volunteer activates in period t, and the value is 0 if there is no contribution. Values in the interior of the convex hull correspond to situations in which the public randomization device is used to mix between equilibria in the continuation game, for example mixing between the equilibria generating $V_1^k(l, l)$ and $\widetilde{V}_1^k(l, l)$, which are equilibrium continuation values constructed as described above. The set of values $\Delta\mathcal{E}_1^k(l)$ is non-empty, compact- and convex- valued and upper-hemicontinuous. To verify upper hemicontinuity, consider any sequence $l_\iota \to l$ and any convergent sequence of cutoffs $c_{1,\iota}^k \in \Omega_1(l_\iota)$ with $c_{1,\iota}^k \to c_1^k$. We first show that $c_1^k \in \Omega_1(l)$. Since $c_{1,\iota}^k \in \Omega_1(l_\iota)$, for each ι either the fixed-point condition in the first line of (23) holds, or the boundary condition $c_{1,\iota}^k = l_\iota$ in the second line holds. Passing to a subsequence if necessary, the same alternative holds for all ι. Suppose first that the fixed-point condition holds along the subsequence. By the induction hypothesis, the continuation-value correspondences $Q^{T,k-j-1}$ and $V^{T,k-j}$ are compact-valued and upper-hemicontinuous. Hence any convergent sequence of selected continuation values has a limit that belongs to the corresponding limiting correspondence. Moreover, the binomial probability term is continuous in (c, l), since F is continuous and $1 - F(l) > 0$. Therefore, passing to the limit in the fixed-point equation shows that c_1^k satisfies the first line of (23) at l. Suppose instead that the boundary condition holds along the subsequence. Then $c_{1,\iota}^k = l_\iota$ for all ι, and therefore $c_1^k = l$. By upper hemicontinuity of $Q^{T,k-1}$, the associated limiting continuation value belongs to $Q^{T,k-1}(l)$. Hence the boundary condition in the second line of (23) is also preserved in the limit. Thus, in either case, $c_1^k \in \Omega_1(l)$. This proves that the graph of Ω_1 is closed.

Now consider any sequence $l_\iota \to l$ and any convergent sequence of equilibrium values $w_\iota \in \mathcal{E}_1^k(l_\iota)$, $w_\iota \to w$. Each w_ι is generated by some cutoff $c_{1,\iota}^k \in \Omega_1(l_\iota)$ and by selected continuation values satisfying (23). By compactness, pass to a subsequence along which these cutoffs and selected continuation values converge. The closed-graph argument above implies that the limiting cutoff belongs to $\Omega_1(l)$, and upper hemicontinuity of the continuation-value correspondences implies that the limiting continuation values are admissible. Therefore the limiting value w is generated by an equilibrium continuation at l, so $w \in \mathcal{E}_1^k(l)$. Thus $\mathcal{E}_1^k$ is upper-hemicontinuous. Since $\Delta\mathcal{E}_1^k(l) = co\mathcal{E}_1^k(l)$, and since the convex hull of a compact-valued upper-hemicontinuous correspondence is upper-hemicontinuous, $\Delta\mathcal{E}_1^k$ is also upper-hemicontinuous.

Step 2. Assume we have a non-empty, compact- and convex-valued, and upper-hemicontinuous set of continuation values $\Delta\mathcal{E}_{\tau-1}^k(l)$ for the game with $\tau - 1 \geq 1$ attempts and k missing volunteers. We can now characterize the equilibrium cutpoints when $T-\tau$ period have passed as:

$$\Omega_\tau(l) = \left\{ c \geq l \;\middle|\; \begin{array}{c} c = \delta \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q_T^{k-j-1}(c) - V_T^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \\ \text{or } c = l \text{ if } \delta Q_T^{k-1}(l) \leq l \\ \text{for some } \{Q_T^{k-j}(c), V_T^{k-j}(c,c)\}_{j=1}^{k-1} \in \{\mathcal{V}_T^{T,k-j}(c)\}_{j=1}^{k-1}, \\ \text{and } Q_T^0(c) = v,\, V_T^k(c,c) \in \Delta\mathcal{E}_{\tau-1}^k(c) \end{array} \right\} \tag{24}$$

this correspondence is similar to (23), except that now $V_T^k(c,c)$ is allowed to take values in $\Delta\mathcal{E}_{\tau-1}^k(c)$: the set of expected values when $\tau - 1$ attempts are possible.

The same fixed-point argument used in Step 1 applies here, with $V_T^k(c,c) \in \Delta\mathcal{E}_{\tau-1}^k(c)$ replacing the terminal payoff 0. Since $Q^{T,k-j-1}$, $V^{T,k-j}$, and $\Delta\mathcal{E}_{\tau-1}^k$ are nonempty, compact- and convex-valued, and upper-hemicontinuous, and since the binomial weights are continuous in (c,l), the associated cutoff correspondence is nonempty, compact- and convex-valued, and upper-hemicontinuous on the feasible compact interval of cutoffs. Kakutani's theorem therefore implies that $\Omega_\tau(l)$ is nonempty. As in Step 1.1, it allows to define the set of expected values when τ attempts are possible $\Delta\mathcal{E}_\tau^k(l)$. We can iterate until we arrive to $\Omega_T(l)$, which allows us to define the expected values $Q_T^k(l)$ and $V_T^k(c,l)$. Let $\widehat{\mathcal{V}}_T^{T,k}(c,l)$ denote the set of expected values for a type c across PBEs of the T-game, when the number of missing volunteers is k and the lower bound on types is l. Since players' strategies may depend on the public correlation device, the relevant set of expected values is the convex hull of this set denoted by $\mathcal{V}_T^{T,k}(c,l) = co\widehat{\mathcal{V}}_T^{T,k}(c,l)$.

Step 3. For any $T > 0$, the procedure described above generates the cutpoints $\left(c_{T,\tau}^k(l)\right)_{\tau=1}^T$

where $c^k_{T,\tau}(l) = c^k_{T-\tau}(l)$ is the cutpoints when there are $T-\tau$ attempts, i.e. under the constraint that the game is terminated if there are $T-\tau$ periods without volunteers. By compactness, for every finite history and every k, there exists a subsequence $T_r \to \infty$ along which the relevant cutoffs and continuation values converge. Let the limiting strategy profile be defined by these limits: $c^k_\tau(l) = \lim_{r\to\infty} c^k_{T_r,\tau}(l)$ for any τ; and the associated expected value functions $Q^{k-j}(l)$ and $V^{k-j}(c,l)$ for $j = 0, ..., k-1$. We claim that this is a PBE. Assume this is not true. Then for some $k \le m$ there is a deviation for a player i that yields $\overline{V}^k_i(c,l)$ such that $\overline{V}^k_i(c,l) - V^k(c,l) > 2\varepsilon$ for some $\varepsilon > 0$. We now make two observations. First, let $\overline{V}^k_{i,T}(c,l)$ be the value of the strategies used in $\overline{V}^k_i(c,l)$ in the T-truncated game. Since utilities are bounded and $\delta \in (0,1)$, the truncated game is continuous at infinity (as defined in Fudenberg and Levine, 1983). We must therefore have $\overline{V}^k_{i,T}(c,l) \ge \overline{V}^k_i(c,l) - \varepsilon/2$ for T sufficiently large. Moreover, by construction, $V^k_T(c,l) \le V^k(c,l) + \varepsilon/2$ for T sufficiently large. It follows that there exists a T^* such that for $T > T^*$:

$$\overline{V}^k_{i,T}(c,l) - V^k_T(c,l) > \varepsilon$$

But this is in contradiction with the fact that $V^k_T(c,l)$ is the equilibrium value function of a PBE in the T-truncated game. ■

Proof of Theorem 3: Part (1): Define c^j_1 to be the cutoff poiny at $t = 1$ with $l = 0$ and $j = 1, ..., m$ missing volunteers. A sufficient condition for the probability of success to be strictly positive is that $c^m_1 > 0$, since the probability of success would then be bounded below by $1 - (1 - F(c^m_1))^n > 0$. First, notice that we already proved that $c^1_1 > 0$. We now proceed by induction on the number of volunteers that are needed. Assume that for all $j = 2, ..., m-1$, $c^j_1 > 0$ in every equilibrium and therefore $Q^j(0) > 0$ for $j = 2, ..., m-1$. Now suppose that there is some equilibrium of the (n, m, δ, v) game in which the probability of success is 0. This implies that $c^m_1 = 0$ and hence $[V^m]^+(0,0) = 0$. But $[V^m]^+(0,0) \ge \delta Q^{m-1}(0)v > 0$, a contradiction. Hence, $c^m_1 > 0$ in every equilibrium, so the probability of success is strictly positive in all equilibria.

Part (2) We proceed in two steps.

Step 1. We first prove that, for any convergent subsequence, $\lim_{\tau\to\infty} c^m_\tau < v$. Suppose by way of contradiction that there exists an equilibrium that contains a sequence $\{c^m_\tau\}^\infty_{\tau=1}$ with $\lim_{\tau\to\infty} c^m_\tau = c^m_\infty = v$. (If not, then the result is proved.) This is the sequence of equilibrium cutpoints along those histories where no player has contributed up to period τ. Along such a sequence, it must be that $c^m_\tau - c^m_{\tau-1} \to 0$. Hence:

$$\widetilde{F}(c^m_\tau; c^m_{\tau-1}) \to \frac{F(c^m_\tau) - F\left(c^m_{\tau-1}\right)}{1 - F(v)} \to 0$$

It follows from equation (14) that:

$$\begin{aligned}\lim_{\tau\to\infty} c_\tau^m &= \delta \lim_{\tau\to\infty} \sum_{j=0}^{m-1} \left[\left(Q^{m-j-1}\left(c_\tau^m\right) - V^{m-j}(c; c_\tau^m)\right) B\left(j, n-1, \widetilde{F}(c_\tau^m; c_{\tau-1}^m)\right)\right] \\ &\to \delta\left(Q^{m-1}\left(v\right) - V^m(c; v)\right) = 0 < v, \text{ a contradiction.}\end{aligned}$$

The last step follows from the fact that if the lower bound on types is v, then the expected probability that an active player contributes is zero, so $Q^{m-1}(v) = V^m(c; v) = 0$.

Step 2. We now prove that contributions stop at some finite time τ^*. By Step 1, for any convergent subsequence, $\lim_{\tau\to\infty} c_\tau^m = c_\infty^m < v$. For large τ, a type $c \in (c_\tau^m, c_\infty^m)$ would strictly prefer not to contribute, since contributing entails a cost $-c$ and induces an arbitrarily small probability of success, namely the probability that a type lies in (c_τ^m, c_∞^m). It follows that there must exist a finite τ^* such that $c_\tau^m = c_{\tau^*}^m$ for all $\tau \geq \tau^*$. ■

Proof of Lemma 4: We proceed in three steps.

Step 1. We first prove that if $m_n \prec n^{2/3}$ then $\frac{F[vB(m_n-1,n-1,\alpha_n)]}{\alpha_n} \to \infty$, where $\alpha_n = m_n/n$. To establish this property, note that we can write:

$$B\left(m_n-1, n-1, \alpha_n\right) = \binom{n-1}{m_n-1} \alpha_n^{m_n-1}\left(1-\alpha_n\right)^{n-m_n} \simeq \frac{1}{\sqrt{2\pi\alpha_n(1-\alpha_n)n}}$$

by Stirling's formula. Furthermore, F is approximately uniform in the neighborhood of 0, and $vB\left(m_n-1, n-1, \alpha_n\right) \to 0$, so:

$$\frac{F\left[vB\left(m_n-1,n-1,\alpha_n\right)\right]}{\alpha_n} \simeq \frac{f(0)\cdot vB\left(m_n-1,n-1,\alpha_n\right)}{\alpha_n} \simeq \frac{vf(0)}{\sqrt{2\pi\left(\frac{m_n}{n^{2/3}}\right)^3\left(1-\frac{m_n}{n}\right)}}$$

which diverges to infinity if $m_n \prec n^{2/3}$. This implies $\frac{F[vB(m_n-1,n-1,L\alpha_n)]}{L\alpha_n} \to \infty \; \forall L > 1$.

Step 2. Recall from Theorem 3 that in every equilibrium the cutpoint at the initial period $t = 1$ is strictly positive, and hence is given by (14) evaluated at $k = m_n$ and $l = 0$:

$$c_1^n(0) = \delta \sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c_1^n(0)\right)\right) \left[\begin{array}{c} Q^{m_n-j-1}\left(c_1^n(0)\right) \\ -\left[V^{m_n-j}\right]^+\left(c_1^n(0), c_1^n(0)\right)\end{array}\right]. \tag{25}$$

The maximal fixed-point consistent with (25) can be bounded below by $\overline{c}_n^{m_n}(0)$ defined as follows:

$$\overline{c}_n^{m_n}(0) = \max_{c\in[0,1]} \left[c | c \leq \delta \sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c\right)\right)\left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+\left(c, c\right)\right]\right] \tag{26}$$

For any arbitrary constant $L > 1$, define $\widehat{c}_n^L$ by $F\left(\widehat{c}_n^L\right) = L\frac{m_n}{n}$ for n large enough so that

$L\frac{m_n}{n} < 1$. That is, given L, $\widehat{c}_n^L$ is a hypothetical cutpoint with the property that the expected fraction of types lower than or equal to $\widehat{c}_n^L$ is greater than the required threshold fraction by a factor of $L > 1$.

We next show that if we choose a sufficiently large (but still finite) value of L then there will exist a critical group size n_L such that for $n > n_L$:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right) \equiv \delta \sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(\widehat{c}_n^L\right)\right)\left[\left(Q^{m_n-j-1}\left(\widehat{c}_n^L\right) - \left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L, \widehat{c}_n^L\right)\right)\right] \quad (27)$$

$$> \varsigma B\left(m_n-1, n-1, F(\widehat{c}_n^L)\right). \quad (28)$$

where ς is a strictly positive number that does not depend on n. Notice that $\Psi_{m_n,n}\left(\widehat{c}_n^L\right)$ is the right hand side of (25) evaluated at $\widehat{c}_n^L$.

From the definition of $\widehat{c}_n^L$ we have:

$$\frac{B\left(m_n-2, n-1, F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1, n-1, F\left(\widehat{c}_n^L\right)\right)} = \frac{m_n}{n-m_n}\frac{1-F(\widehat{c}_n^L)}{F(\widehat{c}_n^L)} = \frac{\frac{n}{L}-m_n}{n-m_n} \to \frac{1}{L}$$

as $n \to \infty$. Similarly, one obtains, for $j = 2, ..., m_n - 1$: $\frac{B\left(m_n-1-j,n-1,F\left(\widehat{c}_n^L\right)\right)}{B(m_n-1,n-1,F(\widehat{c}_n^L))} \to (1/L)^j$ as $n \to \infty$. So, for large values of L, the probability of exactly j volunteers, conditional on having less than or equal to $m_n - 1$ volunteers becomes highly concentrated on $j = m_n - 1$.

Define $\overline{B}_j^L$ as the probability of exactly $j \leq m_n - 1$ volunteers, conditional on having less than or equal to $m_n - 1$ volunteers, when the cutpoint is $\widehat{c}_n^L$: $\overline{B}_j^L = \frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1} B(k,n-1,F(\widehat{c}_n^L))}$. Hence, we have:

$$1 = \sum_{j=0}^{m_n-1} \overline{B}_j^L = \sum_{j=0}^{m_n-1} \frac{B\left(j, n-1, F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1} B\left(k, n-1, F\left(\widehat{c}_n^L\right)\right)} \quad (29)$$

$$= \sum_{j=0}^{m_n-1} \frac{B\left(j, n-1, F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1, n-1, F\left(\widehat{c}_n^L\right)\right)} \cdot \frac{B\left(m_n-1, n-1, F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1} B\left(k, n-1, F\left(\widehat{c}_n^L\right)\right)} \quad (30)$$

$$\to \sum_{j=0}^{m_n-1} \left(\frac{1}{L}\right)^j \overline{B}_{m_n-1}^L = \overline{B}_{m_n-1}^L \frac{(1-(1/L)^{m_n})}{1-1/L}. \quad (31)$$

This implies that there is a n_L sufficiently large such that for $n > n_L$:

$$\overline{B}_{m_n-1}^L > \frac{1}{1+\epsilon}\frac{1-1/L}{1-(1/L)^{m_n}} \quad (32)$$

for all $\epsilon > 0$. That is, $\overline{B}_{m_n-1}^L$ approaches 1 for large L.

Next observe that:

$$\begin{aligned}\Psi_{m_n,n}\left(\widehat{c}_n^L\right) &= \delta\sum_{j=0}^{m_n-1} B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\left[Q^{m_n-j-1}\left(\widehat{c}_n^L\right)-\left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L,\widehat{c}_n^L\right)\right]\\ &\geq \delta\left(1-\delta\right)vB\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)-\delta\sum_{j=0}^{m_n-2} B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)v\end{aligned}$$

since for all c:

$$\delta\left[Q^0\left(c\right)-\left[V^1\right]^+\left(c,c\right)\right]=v-\delta\left[V^1\right]^+\left(c,c\right)\geq\left(1-\delta\right)v,$$
$$\text{and } \delta\left[Q^{m_n-j-1}\left(c\right)-\left[V^{m_n-j}\right]^+\left(c,c\right)\right]\geq -\delta v$$

for $j=0,...m_n-2$. Substituting the inequality (32), it follows that for $n>n_L$:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right)\geq\sum_{j=0}^{m_n-1} vB\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\cdot\left[\begin{array}{c}\left(1-\delta\right)\frac{1}{1+\epsilon}\frac{1-1/L}{1-(1/L)^{m_n}}\\ -\delta\left(1-\frac{1}{1+\epsilon}\frac{1-1/L}{1-(1/L)^{m_n}}\right)\end{array}\right]$$

for all $\epsilon>0$. For any discounting δ, we can choose value of L large enough so that:

$$\left(1-\delta\right)\frac{1-1/L}{1-\left(1/L\right)^{m_n}}>\delta\left(1-\frac{1-1/L}{1-\left(1/L\right)^{m_n}}\right)\Leftrightarrow\frac{1-1/L}{1-\left(1/L\right)^{m_n}}>\delta \tag{33}$$

since the right hand side of the first line in (33) is strictly less than 1 and the left hand side converges to 1 as L increases. It follows that for such values of L we have: for $n>n_L$

$$\begin{aligned}\Psi_{m_n,n}\left(\widehat{c}_n^L\right)\geq\left(1-\delta\right)v\frac{1-1/L}{1-\left(1/L\right)^{m_n}}B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)\\ =\frac{\varsigma}{1-\left(1/L\right)^{m_n}}B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)>\varsigma B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)\end{aligned}$$

for all $n>n_L$, where $\varsigma=\left(1-\delta\right)v(1-1/L)$ is the desired strictly positive constant.

Step 3. From the definition of $\widehat{c}_n^L=F^{-1}(L\frac{m_n}{n})$ and Step 1, we have that for n sufficiently large:

$$\rho_{L,n} < F\left[vB\left(m_n-1,n-1,\rho_{L,n}\right)\right]\Leftrightarrow\widehat{c}_n^L<vB\left(m_n-1,n-1,F(\widehat{c}_n^L)\right) \tag{34}$$
$$\Leftrightarrow F^{-1}\left(L\frac{m_n}{n}\right)=\widehat{c}_n^L\leq\max_{c\in[0,1]}\left[c\left|c\leq\Psi_{m_n,n}\left(c\right)\right.\right]=\overline{c}_n^{m_n}(0) \tag{35}$$

where $\rho_{L,n}=L\alpha_n$, and therefore: $F(\overline{c}^{m_n}(0))>L\frac{m_n}{n}$.

Note now that, as proven in Theorem 2, the set of possible continuation values is a non empty, closed valued and upperhemicontinuous correspondence in c, so $\Psi_{m_n,n}\left(c\right)$ has these properties as well. Let $\underline{\varphi}_{m_n,n}\left(c\right)$ and $\overline{\varphi}_{m_n,n}\left(c\right)$ be the the minimal and maximal values that can be assumed by $\Psi_{m_n,n}(c)$ in equilibrium. Since $\Psi_{m_n,n}(1) = 0 < 1$ and we have just proven above there is a $\overline{c}^{m_n}(0)$ such that $\Psi_{m_n,n}(\overline{c}^{m_n}(0)) \geq \overline{c}^{m_n}(0)$, there must be a $c_1^n(0) \geq F^{-1}(L\frac{m_n}{n})$ such that $\underline{\varphi}_{m_n,n}(c_1^n(0)) \geq c_1^n(0)$ and $\underline{\varphi}_{m_n,n}(c_1^n(0)) \leq c_1^n(0)$. Since, as also proven in Theorem 2, the set of continuation value functions is convex valued in c, we must also have that $c_1^n(0) \in \Psi_{m_n,n}(c_1^n(0))$. We conclude that $c_1^n(0)$ solves (25) and satisfies $F(c_1^n(0)) \geq L\frac{m_n}{n}$. ■

Proof of Theorem 4: Suppose $m_n \to \infty$ as $n \to \infty$ and $m_n \prec n^{2/3}$. We proceed in two steps.[36]

Step 1. We first prove that $\lim_{n\to\infty} \frac{F(c_1^n(0))}{\alpha_n} \to L > 1$, where L is either bounded but strictly larger than 1 or infinite. Since $\frac{F(c_1^n(0))}{\alpha_n} \geq 1$ by Lemma 4, assume by way of contradiction that $\frac{F(c_1^n(0))}{\alpha_n} \to 1$. As in Step 1, a standard approximation gives us: $B(\alpha_n n-1, n-1, F(c_{n,1}^{m_n}(0))) \simeq \sqrt{\frac{1}{2\pi\alpha_n(1-\alpha_n)n}}$. Similarly as in Step 1, by the definition of $c_1^n(0))$ and the fact that $m_n \prec n^{2/3}$, for large n, we have: $1 \geq \frac{f(0)B(m_n-1,n-1,F(c_1^n(0)))}{c_1^n(0)} \simeq \sqrt{\frac{1}{2\pi(\alpha_n)^3(1-\alpha_n)n}} \to \infty$, a contradiction. We must therefore have that in equilibrium: $\frac{F\left(c_1^n(0)\right)}{\alpha_n} \to L > 1$, with L possibly arbitrarily large.

Step 2. We next prove that this implies the probability of success in the first period converges to 1. Define for convenience here, $\zeta_n = \frac{F(c_1^n(0))}{\alpha_n}$. Note that the probability of failure in the first period is equal to the probability that the number of volunteers in period 1, j, is less than or equal to $\alpha_n n$ agents, which using Chebyshev's inequality can be bounded above as follows:

$$\Pr(j \leq \alpha_n n) \leq \Pr\left[\left|\frac{j}{n} - F(c_1^n(0))\right| \geq \alpha_n(\zeta_n - 1)\right] \leq \left(\frac{\sqrt{\zeta_n(1 - F(c_1^n(0)))}}{\sqrt{n\alpha_n}(\zeta_n - 1)}\right)^2 \tag{36}$$

The result follows since $\zeta_n \to L > 1$, so $\left(\frac{\sqrt{\zeta_n(1-F\left(c_1^n(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n-1)}\right)^2 = \lim_{n\to\infty} \frac{1}{m_n}\frac{L}{(L-1)^2} = 0.$ ■

Proof of Lemma 5: For any history h_t with $k_{h_t} = k$ and $l_{h_t} = l$ and public signals $\theta_t = (\theta^1, ..., \theta^t)$, the PBE cutpoint at that history can be written as a function $c(h_t) = \{c_1(h_t), ..., c_n(h_t)\}$, where $c_i(h_t)$ is the cutpoint of player i. For any profile of types $\mathbf{c} = (c_1, ...c_n)$, vector of signals θ_t, a PBE defines a deterministic sequence of

$$c_t(\mathbf{c},\theta_t) = \{c_{t,1}(\mathbf{c},\theta_t), ..., c_{t,n}(\mathbf{c},\theta_t)\},$$

where $c_{t,i}(\mathbf{c},\theta_t)$ is the cutpoint of agent i in period t given the history generated in correspondence to a realized profile $\mathbf{c} = (c_1, ...c_n)$, and a vector of signals θ_t up to t. At $t = 1$,

[36]The proof for the case where $m_n \to m$, a constant, is similar.

$c_1(\mathbf{c},\theta_1) = c(h_1)$, where h_1 is just the realization of the public signal at $t = 1$, i.e. θ^1. Given the realization of types $\mathbf{c}$ and θ_2, $c_1(\mathbf{c},\theta_1)$ defines a unique history $h_2(\mathbf{c},\theta_2)$ in which the set of players who participate at 1 is given by the players with type $c_i \in [0, c_{1,i}(\mathbf{c},\theta_1)]$ and the public signals are θ_2. Suppose we have defined the history up to t. Then the cutpoints at t are given by $c_t(\mathbf{c},\theta_t) = c\left(h_t(\mathbf{c},\theta_t)\right)$.

Define a direct HO mechanism as follows. Denote $k_t(\mathbf{c},\theta_t)$ as the number of missing volunteers at the end of period t after history $h_t(\mathbf{c},\theta_t)$. For any profile $\mathbf{c}$ and corresponding PBE thresholds, $(c_t(\mathbf{c},\theta_t))_{t=0}^{\infty}$, define $T_i(\mathbf{c},\theta)$ for each i as the period at which i would volunteer in the PBE when the profile of types is $\mathbf{c}$, if there is such a period. That is:

$$\begin{aligned} T_i(\mathbf{c},\theta) &= \min\left\{t | c_{t,i}(\mathbf{c},\theta_t) \geq c_i\right\} \text{ if } \exists t \text{ such that } c_{t,i}(\mathbf{c},\theta_t) \geq c_i \\ &= \infty \text{ otherwise} \end{aligned}$$

where $\theta = (\theta^{\tau})_{\tau=1}^{\infty}$; and let $S(\mathbf{c},\theta)$ denote the period at which the game ends with success if there is ever success at $\mathbf{c}$. That is:

$$\begin{aligned} S(\mathbf{c},\theta) &= \min\left\{t | k_t(\mathbf{c},\theta_t) = 0\right\} \text{ if } \exists t \text{ such that } k_t(\mathbf{c},\theta_t) = 0 \\ &= \infty \text{ otherwise} \end{aligned}$$

Denote by $I_t(\mathbf{c},\theta) = \{i | T_i(\mathbf{c},\theta) \leq t\}$ the set of agents who have activated up to and including t. We can now define the activity function μ^{DYN} for a static mechanism as follows, where, for each subset of agents, $g \subseteq I$, $\mu_g^{DYN}(\mathbf{c})$ specifies the probability that only the agents in g are activated when the reported cost profile is $\mathbf{c}$:

$$\mu_g^{DYN}(\mathbf{c}) = \begin{cases} \int_{\theta} \left[\sum_{\{\tau | I_{\tau}(\mathbf{c},\theta) = g\}} (1-\delta)\,\delta^{\tau-1}\right] d\Pi(\theta) & |g| < m \\ \int_{\theta} \left[1_{\left\{\theta | I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta) = g\right\}} \cdot \delta^{S(\mathbf{c},\theta)-1}\right] d\Pi(\theta) & \text{for } |g| \geq m \end{cases} \tag{37}$$

where $\Pi(\theta)$ is the distribution of the public signals; and $1_{\left\{\theta | I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta) = g\right\}}$ is the indicator function equal to 1 when θ is such that $I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta) = g$, and zero otherwise.

The activity function for the static mechanism, $\mu_g^{DYN}(\mathbf{c})$, is constructed from the PBE by the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals i with a type below $c_{1,i}(\mathbf{c},\theta_1) = c_i(h_1(\mathbf{c},\theta_1))$ are asked to volunteer (i.e., the set $I_1(\mathbf{c},\theta)$). If there are at least m such individuals, i.e., $k_1(\mathbf{c},\theta_1) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c},\theta) = 1$ and $\mu_{I_1(\mathbf{c},\theta)}^{DYN}(\mathbf{c}) = 1$. If $k_1(\mathbf{c},\theta_1) > 0$, i.e., $S(\mathbf{c},\theta) > 1$, then with probability $1-\delta$ the algorithm also stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c},\theta) > 1$ and $\mu_{I_1(\mathbf{c},\theta)}^{DYN}(\mathbf{c}) = 1-\delta$, as in (37). With probability δ, instead, the algorithm proceeds to Step 2. In Step 2, a public signal θ^2 is drawn; the vector of cutpoints

$c_2(\mathbf{c},\theta_2) = \{c_{2,1}(\mathbf{c},\theta_2),...,c_{2,n}(\mathbf{c},\theta_2)\}$ is determined; and an individual i with a type in the interval $(c_{1,i}(\mathbf{c},\theta_1), c_{2,i}(\mathbf{c},\theta_2)]$ is asked to volunteer and the process continues. In general, at any step t at which the algorithm has not yet stopped, any individual i with a type in the interval $(c_{t-1,i}(\mathbf{c},\theta_{t-1}), c_{t,i}(\mathbf{c},\theta_t)]$ is asked to volunteer. If there are at least $k_{t-1}(\mathbf{c},\theta_t)$ such individuals, i.e., $k_t(\mathbf{c},\theta_t) = 0$ and $S(\mathbf{c},\theta) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c},\theta)$ without proceeding to Step $t+1$. If $k_t(\mathbf{c},\theta_t) > 0$, i.e., $S(\mathbf{c},\theta_t) > t$, then with probability $1-\delta$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability δ the algorithm proceeds to step $t+1$. In all cases, the probabilities are given at each step by (37). Thus, the static mechanism mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1-\delta$ after any step at which the threshold m has not yet been achieved.

From the above construction of μ_g^{DYN}, we can represent the probability of success and that a player i is asked to volunteer by:

$$P(\mathbf{c}) = \int_\theta \delta^{S(\mathbf{c},\theta)-1} d\Pi(\theta)\,,\ A_i(\mathbf{c}) = \int_{\{\theta | S(\mathbf{c},\theta)\geq T_i(\mathbf{c},\theta)\,\}} \delta^{T_i(\mathbf{c},\theta)-1} d\Pi(\theta)$$

where $P(\mathbf{c})$ is the probability of obtaining the public good at profile $\mathbf{c}$, and $A_i(\mathbf{c})$ is the probability that i is asked to volunteer at $\mathbf{c}$. The expected utility for an individual with type c at profile $\mathbf{c} = (c, \mathbf{c}_{-i})$ is $U_i(\mathbf{c}) = vP(\mathbf{c}) - c_iA_i(\mathbf{c})$. This is exactly equal to the expected utility for an individual with type c_i at profile $\mathbf{c}$ in the corresponding PBE of the dynamic game. We only need to prove that this direct mechanism is Honest and Obedient (Myerson, 1982). We need to show that every type c is weakly better off reporting c and obeying all recommendations, than they would be reporting c and disobeying some recommendations or reporting $c' \neq c$ and then following some optimal strategy in terms of obedience/non-obedience of the subsequent recommendation. Suppose there is a player i of type c who is strictly better off reporting to be a type $c' > c$. The analysis of the case in which i reports be a type $c' < c$ is analogous and omitted. There are two cases, corresponding to the two information sets in which i can find himself/herself: when the recommendation is to volunteer; and when it is to not volunteer.

Case 1. Consider first the case in which the recommendation is to volunteer. We prove here that reporting $c' > c$ and obeying a recommendation to volunteer is not a strictly optimal deviation for an agent i. The proof that reporting $c' > c$ and disobeying a recommendation to volunteer is not a strictly optimal deviation is similar. We first observe that if by reporting c' the recommendation is to volunteer, then the same recommendation must be received by reporting c. Since c' has received a recommendation to volunteer, it must be that $\mathbf{c}_{-i}$ is such that $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) \geq c'$ for some some $t \leq S(\mathbf{c}, \theta)$ and a sequence of cutoffs $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the smallest period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, then $c \in I_{t'}(c', \mathbf{c}_{-i}, \theta)$. Note, moreover, that by definition $c_{t'',i}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and $c_{t',i}(\widetilde{c}, \mathbf{c}_{-i}, \theta_{t'})$ is the same if $\widetilde{c} = c'$ or $\widetilde{c} = c$, so $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'})$ and $I_{t'}(c', \mathbf{c}_{-i}, \theta) = I_{t'}(c, \mathbf{c}_{-i}, \theta)$. Since $t' \leq t$, we have $I_{t'}(c, \mathbf{c}_{-i}, \theta) = I_{t'}(c', \mathbf{c}_{-i}, \theta) \subseteq I_t(c', \mathbf{c}_{-i}, \theta)$: we conclude that a recommendation to volunteer to a type c', implies the same recommendation to a player who reports to be a type c as well. When the recommendation is to volunteer and the player obeys, then reporting $c' > c$ cannot be strictly superior than reporting c.

Case 2. In the case in which the recommendation is to abstain, we have two subcases to consider:

Step 2.1. Consider first the case in which i reports c' and obeys to a recommendation to abstain. In this case, again, $\mathbf{c}_{-i}$ must be such that $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) \leq c'$ for some $t \leq S(\mathbf{c}, \theta)$ and a sequence of cutoffs $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the minimal period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Then $c_{t'',i}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and so by the same argument as in Case 1 in the paper, $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'})$. If $t' > t$, then $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) < c$, so in this event reporting c' induces the same action as reporting c: it cannot generate a strictly superior deviation in this event. If instead, $t' \leq t$, then $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Player i does not know $\mathbf{c}_{-i}$ and t, but s/he knows that conditioning on being asked to volunteer, $\mathbf{c}_{-i}$ is such that it is as if s/he is a period $t \leq S(\mathbf{c}, \theta)$ with public signals θ_t in which $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c$. This implies that i has the same expected values as in the PBE, and weakly prefers to volunteer. So reporting c' and obeying a recommendation to abstain cannot yield a higher expected utility than reporting truthfully and obeying the recommendation of the mechanism.

Step 2.2. Finally, consider the case in which i reports c' and disobeys a recommendation to abstain. Again, let t' be the minimal period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. If $t' > t$, then $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) < c$ for all $t' \leq t$. So a type c would find it optimal to abstain. If instead, $t' \leq t$, then $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, and a type c would receive the same expected payoff from reporting truthfully and obeying than from reporting c' and disobeying a recommendation to abstain.

Since there is no scenario in which the player finds it strictly optimal to report to be a type $c' > c$, we conclude that the player is never strictly better off by reporting to be $c' > c$, no matter what obedience policy s/he follows afterwards. ■

Proof of Theorem 5: From Lemma 5 we know that the expected utility in the PBE is equal to the expected utility in a specific direct, static mechanism that is honest and obedient. Battaglini and Palfrey (2024) have proven that the expected utility of a player in the best direct, static mechanism that is honest and obedient converges to 0 as $n \to \infty$ when $m_n \succ n^{2/3}$, i.e. when $\alpha_n / \sqrt[3]{1/n} \to \infty$ (See Theorem 4). The same must be true in the PBE. $\blacksquare$

Proof of Proposition 2: We proceed in two steps.

Step 1. Consider period $T > 1$ and assume that the number of missing volunteers is larger or equal than $k_n = \frac{m_n}{T} > 0$; and that the lower bound of types is $l \geq 0$. We now prove that there is $n^{(T)}$ such that for $n > n^{(T)}$, the probability of contributing is zero for all players. It follows that at any history in which $k_n \geq \frac{m_n}{T}$ at stage T, then the continuation values are $Q_T^{k_n}(l) = V_T^{k_n}(c, l) = 0$ for any $c \geq l$ and $l \geq 0$.

At period T, let the equilibrium cutpoint be $c_n^{T,T}$ (we omit here for simplicity the dependence on h_t and l), which must satisfy:

$$c_n^{T,T} = vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(c_n^{T,T}; l)\right) = \Psi\left(c_n^{T,T}\right) \tag{38}$$

where we define the function $\Psi\left(c_n^{T,T}\right)$, and $z_n = (1-\alpha)\, n + k_n$ and $\beta_n = \frac{k_n}{z_n} \geq \frac{\alpha}{T} > 0$. We now prove that for n large enough, it must be $c_n^{T,T} = l$. If $l > 0$ and $c_n^{T,T} > l$, then the right hand side of (38) converges to zero, but the left hand side converges to a strictly positive value, a contradiction. Assume therefore that $l = 0$ and $c_n^{T,T} > 0$. Define $\widehat{\beta}_n$ to be the value such that $\widetilde{F}(\widehat{\beta}_n; l) = \frac{\beta_n}{1 - 1/z_n}$. This is the value that maximizes the right hand side of (38), i.e. $\Psi(\cdot)$. It is straightforward to verify that it must be that $\widehat{\beta}_n > 0$ for any n. Moreover, since $\beta_n \geq \frac{\alpha}{T}$, we have that for n large enough:

$$vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(\widehat{\beta}_n; l)\right) < \widehat{\beta}_n \simeq \beta_n / f(0), \tag{39}$$

given that the first and second terms converge to zero, but $\widehat{\beta}_n$ converges to $\beta_n / f(0)$, which is strictly positive for all n. From the inequality in 39, we have that $\Psi'\left(c_n^{T,T}\right) < 1$, at any fixed point $c_n^{T,T}$ of Ψ. This follows from the fact that the right hand side of (38) has a maximum below the 45^o line, so if there is a strictly positive fixed point, there must be a fixed point at which $\Psi(c)$ intersects the 45^o line from above. But, as we now show, this is impossible. To see this note that:

$$B'\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right) = B\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right) \begin{bmatrix} \frac{\beta_n z_n - 1}{F\left(c_n^{T,T}\right)} \\ -\frac{z_n - \beta_n z_n}{1 - F\left(c_n^{T,T}\right)} \end{bmatrix} f\left(c_n^{T,T}\right) \tag{40}$$

$$\to \frac{f(0)\, c_n^{T,T}}{v} \left[\frac{\beta_n z_n - 1}{f(0)\, c_n^{T}} - (1 - \beta_n)\, z_n\right] \to \frac{f(0)}{v} \beta_n \cdot z_n \geq \frac{f(0)}{v} (\alpha/T) \cdot z_n \to \infty$$

So we have a contradiction, since the right hand side of (39) converges to a bounded value. We conclude that if $k_n \geq \frac{m_n}{T}$, then $c_n^{T,T} = l$ is the unique fixed point of (38) for any $l \geq 0$.

Step 2. Assume as an induction step that for some $t < T$ and all $\tau \geq t+1$ we have: there is a $n^{(\tau)}$ such that for $n > n^{(\tau)}$ we have $V_\tau^{k_n^\tau}(c, l_n) = Q_\tau^{k_n^\tau}(l_n) = 0$ for all $c \geq 0$ when $k_n^\tau \geq \overline{k}_n^\tau = (T - \tau + 1)\frac{m_n}{T}$. This property is true for $t+1 = T$ by Step 1. We prove the result if we prove that:

$$c_n^{t,T} = \delta \sum_{j=0}^{k_n^t - 1} \left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T}) \right) B\left(j, n - 1 - m_n + k_n^t, \widetilde{F}(c_n^{t,T}; l_n)\right) \right]. \tag{41}$$

has a no strictly positive fixed point $c_n^{t,T}$ when $k_n^\tau \geq \overline{k}_n^\tau = (T - \tau + 1)\frac{m_n}{T}$. To this goal define, similarly as in Step 1, $z_n^t = n - m_n + k_n^t$, $\beta_n^t = k_n^t / z_n^t$ and $\widetilde{F}_n^t = \widetilde{F}(c_n^{t,T}; l_n)$.

By the induction step we have

$$c_n^{t,T} = \delta \sum_{j=\frac{m_n}{T}+1}^{k_n^t - 1} \left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T}) \right) B\left(j, z_n^t - 1, \widetilde{F}_n^t\right) \right] \tag{42}$$

The right hand side of (42) can be bounded above by:

$$\begin{aligned} & \delta v \cdot \sum_{j=\frac{m_n}{T}+1}^{k_n^t - 1} \left[B\left(j, z_n^t - 1, \widetilde{F}_n^t\right) \right] \\ \leq \;& \exp\left(-n \left(\frac{\alpha}{T} \log \frac{\alpha/T}{\widetilde{F}_n^t} + (1 - \frac{\alpha}{T}) \log \frac{1 - \alpha/T}{1 - \widetilde{F}_n^t} \right) \right) = D_n(\widetilde{F}_n^t) \end{aligned}$$

where for the inequality we used the Chernoff bound of the upper tail of the Binomial distribution (see, for example, Ash [1990, 4.7.2)]. Without loss of generality we can assume that $\widetilde{F}_n^t < \frac{\alpha}{T}$ for n sufficiently large. Indeed, if this were not the case then we would have some $c^{\alpha/T} > 0$ such that $c_n^{t,T} > c^{\alpha/T}$, but this is impossible in equilibrium since the expected benefit of contributing for a single player converges to zero as $n \to \infty$. Note that for any $\widetilde{F}_n^t > \underline{F}$ for some $\underline{F} > 0$, we have $D_n(\widetilde{F}_n^t) < \underline{F}$, since $D_n(\widetilde{F}_n^t) \to 0$. Moreover a Taylor approximation tells us that for $F < \underline{F}$ with $\underline{F}$ sufficiently small we have: $D_n(\widetilde{F}_n^t) = D_n(0) + D_n'(0)\widetilde{F}_n^t + o(\widetilde{F}_n^t)$, where $o(\widetilde{F}_n^t)/\widetilde{F}_n^t \to 0$ as $\widetilde{F}_n^t \to 0$. But then, if we have a positive fixed $c_n^{t,T}$ point, we have:

$$c_n^{t,T} \leq D_n(0) + D_n'(0) f(0) c_n^{t,T} + o(c_n^{t,T}) = c_n^{t,T} \left[D_n'(0) f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}} \right] < c_n^{t,T}$$

since $\left[D_n'(0) f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}} \right]$ can be chosen to be arbitrarily small. This gives us a contradiction. We can iterate the argument up to the first period. We must therefore have $m_n \geq \overline{k}_n^1 = (T)\frac{m_n}{T} = m_n$, which implies that $c_n^1 = 0$ for $n > n^{(1)}$. ■

Proof of Proposition 3: Let E be the event in which the realized cost profile is such that the number of players with cost less than or equal than v is greater than or equal to m_n. For any history h_t of the DVBO, define $K(h_t)$ to be the number of volunteers still missing for success at h_t, and $S(h_t)$ to be be the probability of success in the continuation game. Assume that there exists some history h_t^* reached with strictly positive probability in some PBE of the DVBO in event E, such that $S(h_t^*) = 0$. We now prove this leads to a contradiction. We proceed in four steps:

Step 1. Suppose $K(h_t^*) = 1$. Each player i such that $c_i < v$ must obtain a payoff at least equal to $v - c_i > 0$, since they can volunteer and thereby secure this payoff. But this is possible only if $S(h_t^*) > 0$, a contradiction. We conclude that if h_t is such that $K(h_t) = 1$, then conditioning on E the game terminates in finite time and $Q(c; h_t) > v - c > 0$ for all $c \in [0, v)$, where $Q(c; h_t)$ is the expected utility at h_t for a type c after volunteering.

Step 2. Next, suppose $K(h_t^*) = 2$ and $P_i(E; h_t^*) > 0$ for some i with $c_i < v$ for some i who is still active, where $P_i(E; h_t^*)$ is i's posterior that the true state is in E conditioning on h_t^*. If $S(h_t^*) = 0$, then $V(c_i; h_t) = 0$, where $V(c_i; h_t)$ is the utility at h_t for an uncommitted type c_i. But this leads immediately to a contradiction. By volunteering at h_t^*, either the game ends immediately because some other player also volunteers i, leading to a positive payoff, or i moves the history to h_{t+1}^* with $K(h_{t+1}^*) = 1$, $Q(c_i; h_{t+1}) > 0$ and $P_i\left(E; h_{t+1}^*\right) = P_i(E; h_t^*) > 0$, implying $V(c_i; h_t) \geq \delta Q(c_i; h_{t+1}) \cdot P_i(E; h_{t+1}) > 0$, a contradiction.

Step 3. Assume now that we have established the following statement: for all histories h_t such that $K(h_t) \in \{1, 2, ..., k-1\}$ and $P_i(E; h_t) > 0$ for some player i with $c_i < v$, we have that $S(h_t) > 0$, $V(c_i; h_t)$ and $Q(c_i; h_t) > 0$. In Steps 1 and 2 we have established this property for $k = 1, 2$. We now prove that if $K(h_t) = k$ and $P_i(E; h_t) > 0$ for some player i with $c_i < v$, we have that $S(h_t) > 0$, $V(c_i; h_t) > 0$ and $Q(c_i; h_t) > 0$. Following the same logic as in Step 2, suppose by contradiction that $K(h_t) = k$ and $P_i(E; h_t) > 0$ but $S(h_t) = 0$, and so $V_i(c_i; h_t) = 0$. But by volunteering such a player i moves the history to h_{t+1} with $K(h_{t+1}) = k - 1$, $Q(c_i; h_{t+1}) > 0$ and $P_i(E; h_{t+1}) > 0$, so we must have $V(c_i; h_t) \geq \delta Q(c_i; h_{t+1}) \cdot P_i(E; h_{t+1}) > 0$, a contradiction. We conclude that we must have $S(h_t) > 0$, $V(c_i; h_t) > 0$ and $Q(c_i; h_t) > 0$.

Step 4. To complete the argument note that if, conditioning on E, an history h_t is reached with positive probability in a PBE, so $\Pr(h_t; E) > 0$, then:

$$P_i(E; h_t) = \frac{\Pr(h_t; E, c_i)\Pr(E, c_i)}{\left[\Pr(h_t; E, c_i)\Pr(E, c_i) + \Pr\left(h_t; \overline{E}, c_i\right)\Pr(\overline{E}, c_i)\right]} > 0$$

for any i with $c_i < v$, where $\overline{E}$ denotes the complement of E. The inequality follows from the fact that $\Pr(E, c_i) > 0$, and the fact that $\Pr(h_t; E) > 0$ implies $\Pr(h_t; E, c_i) > 0$ for any i with $c_i < v$. We conclude that if $K(h_t) = k$ and $\Pr(h_t; E) > 0$, then we have $\Pr_i(E; h_t) > 0$

for some $c_i < v$ and therefore, by Step 3, $S(h_t) > 0$. It follows that there is no history h_t^* reached with strictly positive probability in some PBE of the DVBO in event E, such that $S(h_t^*) = 0$. ■

Appendix B: Formal statements and proofs of results in Section 3.3: The effects of n and δ on success and welfare

To study the overall effect of an increase in n, the following preliminary result is useful. To make the dependence on n explicit, define $c_t(n)$ to be the cut-point with n players at period t (for some given lower bound on types, l, and δ). Define as above $\Phi_t(n)$ as the cumulative probability of success up to and including the current period t, for an active player who chooses not to contribute. We say that an increase in n generates an *improvement in success probability* if $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. This implies that for a higher n the distribution is more skewed toward low values of t, which is good for the players (so an improvement). We have:

Lemma B1. *An increase in n shifts the cut-points downward so that $c_t(n) < c_t(n-1)$ for all t, but it generates an improvement in success probability.*

Proof: We proceed in two steps.

Step 1. We first prove that the cutpoints are declining in n so: $c_t(n) < c_t(n-1)$ for all $t > 0$. To see this note that $(1-\delta)\, x / [1 - \delta \cdot x]$ is a strictly increasing function of x if, as always verified in our environment, $\delta \cdot x < 1$. Since $1 - F(c)/[1-F(l)] < 1$ for $c > l$, it follows that the function:

$$\lambda_n(c; l, \delta) = \frac{(1-\delta)\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1 - \delta \cdot \left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}} \tag{43}$$

is strictly decreasing in n for any c, l, δ. It can also be verified that $\lambda_n(c; l, \delta)$ is strictly decreasing in c for all n, l, δ. From (23) in the paper, it follows that the fixed point $c^1_{n,1}$ is decreasing in n for any l, δ. To see this, note that:

$$0 = \lambda_n(c_1(n); l, \delta) - c_1(n) < \lambda_{n-1}(c_1(n); l, \delta) - c_1(n)$$

where the equality follows by the definition of $c_1(n)$, and the inequality follows by the monotonicity of $\lambda_n(c; l, \delta)$ in n. Since $\lambda_{n-1}(c; l, \delta) - c$ is strictly decreasing in c, it follows that we must have $c_1(n-1) > c_1(n)$, else it would be $\lambda_{n-1}(c_1(n-1); l, \delta) - c_1(n-1) > 0$, a contradiction. Assume the induction hypothesis that $c_j(n)$ is decreasing in n for all $j \leq t$. We prove the same is true for $j = t+1$. To see this note that an increase in n shifts the function $\left(\frac{1-F(c)}{1-F(c_t(n))}\right)^{n-1}$ downward for any c since it increases the exponent while reducing $F(c_t(n))$, thus increasing the denominator. It follows that $\lambda_n(c; c_t(n), \delta)$ shifts downward for any c after an increase in n, implying as above that:

$$0 = \lambda_n(c_{t+1}(n); c_t(n), \delta) - c_{t+1}(n) < \lambda_{n-1}(c_{t+1}(n); c_t(n), \delta) - c_{t+1}(n)$$

thus implying that $c_{t+1}(n-1) > c_{t+1}(n)$. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. The probability of success at or before period t, $\Phi_t(n)$, can be written as:

$$\Phi_t(n) = 1 - (1 - F(c_t(n)))^{n-1} = 1 - \prod_{j=1}^{t} \left(\frac{1 - F(c_j(n))}{1 - F(c_{j-1}(n))} \right)^{n-1}$$

From the cutpoint condition in equation (4) in the paper we have:

$$c_j(n) = \left[\frac{(1-\delta) \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))} \right)^{n-m}}{1 - \delta \cdot \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))} \right)^{n-m}} \right] v$$

Since the right hand side is increasing in $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))} \right)^{n-m}$, it follows that $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))} \right)^{n-1}$ is decreasing in n for any j, since $c_j(n)$ is decreasing in n. We conclude that an increase in n induces an increase in $1 - \prod_{j=1}^{t} \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))} \right)^{n-1}$, and thus in $\Phi_t(n)$. It follows that $\Phi_t(n) \geq \Phi_t(n-1)$ for all t and $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. ■

An increase in the size of the population induces players to be more reluctant to contribute in every period. Lemma B1 however shows that, from the point of view of any player, the increase in n more than compensates for this effect and generates an unambiguous improvement in the timing of the realization of the public good until a player decides to contribute.

The next result shows that this implies an unambiguous improvement in welfare for all players. Indeed, in the paper, where we generalize the analysis to allow for $m_n > 1$, we show that the utility of all players converges to the first best v as $n \to \infty$ for any finite m (and thus for $m = 1$ as a special case as well). Let $EU_n(c)$ be the expected utility of a player of type c with n players.

Theorem B1. *An increase in n induces an increase in welfare for all types, strict for sufficiently high types: i.e., $EU_n(c) \geq EU_{n-1}(c)$ for all $c \in [0,1]$ and $EU_n(c) > EU_{n-1}(c)$ for $c_1(n)$.*

Proof. If $c \leq c_1(n)$, then a type c has a payoff of $v - c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v - c$ and with n players the payoff of a type c is not lower than $v - c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. We now prove the result by induction, using these findings as first step. Assume we have proven that for all types $c \leq c_t(n-1)$ for $t \leq j$ a player of type c with n players has utility $EU_n(c)$, weakly higher than the utility of a type c with $n-1$ players $EU_{n-1}(c)$. We have just proven this result for $j = 1$.

Consider first a type $c \in [c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$, if not empty. When there are $n-1$ players, the payoff of a type c is:

$$EU_{n-1}(c) = v\Phi_j(n-1)\delta^{t-1} + [1 - \Phi_j(n-1)]\,\delta^j\,(v-c)\,. \tag{44}$$

A type c with n players instead receives:

$$EU_n(c) = v\Phi_j(n)\delta^{t-1} + [1 - \Phi_j(n)]\,\delta^j \cdot (v-c) > V_{n-1}(c).$$

where the last inequality follows from the fact that $\Phi_t(n-1)$ strictly first order stochastically dominates $\Phi_t(n)$ and $v > v - c$.

Alternatively, it could be that $[c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$ is empty. In that case, consider $c \in [\min\{c_{j+1}(n), c_{j+1}(n-1)\}, c_{j+1}(n-1)]$, which must be non-empty. In this case the payoff of a type c with $n-1$ players is again (44), which can be rewritten as:

$$EU_{n-1}(c) = v\Phi_j(n-1)\delta^{t-1} + \sum_{t=j}^{\infty} [\Phi_{t+1}(n-1) - \Phi_t(n-1)]\,\delta^j \cdot (v-c)$$

The payoff with n of a player with type c instead is:

$$EU_n(c) = v\Phi_j(n)\delta^{t-1} + \sum_{t=j}^{\infty} [\Phi_{t+1}(n) - \Phi_t(n)]\,\delta^j \cdot V_{n,j}(c)$$

where $V_{n,j}(c)$ is the expected continuation value function for a type c when there are n players and j contributors; and where we have $V_{n,j}(c) \geq (v-c)$ by revealed preferences, since $c > c_{j+1}(n)$. Once again we have that $V_{n-1}(c) < V_n(c)$. We have therefore proven that for all $c \leq c_{j+1}(n)$, we have $V_{n-1}(c) < V_n(c)$. It follows that for all types $c < \lim_j c_{j+1}(n) = v$, we have $EU_{n-1}(c) < EU_n(c)$, which proves the result. ∎

The proof of this result follows from revealed preferences and a simple inductive argument on types. The core of the argument runs as follows. If $c \leq c_1(n)$, then a type c has a payoff of $v-c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v-c$ and with n players the payoff of a type c is not lower than $v-c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. In both cases the utility of a type $c \leq c_1(n-1)$ weakly increases with n. Assume we have proven this property for all types $c \leq c_t(n-1)$. Then this property together with Lemma B1 can be used to prove that a type c in $[c_t(n-1), c_{t+1}(n-1)]$ is strictly better off with n players than with $n-1$: even if type c does not change behavior the other players volunteer more with a higher n by Lemma B1; and if c behaves differently, then by revealed preferences this must induce an even higher utility. We can therefore extend the inductive assumption to types $c \leq c_{t+1}(n-1)$. Since

$c_{t+1}(n-1) > c_t(n-1)$ and indeed we have proven above that $c_t(n-1) \to v$ as $t \to \infty$, this argument allows to show that all types $c \in [0,1]$ obtain a higher expected utility with n than with $n-1$.

Consider now the comparative statics with respect to δ. Similarly as for an increase of n, an increase in δ has an ex ante unambiguous effect on participation, leading to a downward shift in all cutpoints. To make the dependence on δ explicit, define $c_t(\delta)$ and $\Phi_t(\delta)$ similarly as above (for some given lower bound on types, l, and fixed values of n), say that an increase from δ' to δ generates an *improvement in success probability* if $\Phi_t(\delta')$ first order stochastically dominates $\Phi_t(\delta)$. Differently from n, now an increase in δ implies an deterioration in $\Phi_t(\delta)$. This implies that although players are more patient, now success takes more time.

Lemma B2. *An increase in δ shifts the cut-points downward so that $c_t(\delta) < c_t(\delta')$ for all t and $\delta > \delta'$, and a downward shift in $\Phi_t(\delta)$ in the sense first order stochastic dominance, i.e. $\Phi_t(\delta) < \Phi_t(\delta')$ for all t and $\delta > \delta'$.*

Proof: We proceed again in two steps.
Step 1. We first prove that the cutpoints are declining in δ so: $c_t(\delta) < c_t(\delta')$ for all t and $\delta > \delta'$. To see this note that $\frac{(1-\delta)\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1-\delta\cdot\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}$ is a strictly decreasing function of δ. It follows that $\lambda_n(c; l, \delta)$, as defined in (43), is strictly decreasing in δ for any c, l, n. By the same argument as in Lemma B1, it follows from (43) that the fixed point $c_1(\delta)$ is decreasing in δ for any l, n. Assume the induction hypothesis that $c_j(\delta)$ is decreasing in δ for all $j \leq t$. We now prove the same is true for $j = t+1$. To see this note that a decrease in δ certainly increases $\left(\frac{1-F(c)}{1-F(c_t(\delta))}\right)^{n-1}$ for any given c, since it increases $F(c_t(\delta))$. It follows that $\lambda_n(c; c_t(\delta), \delta)$ shifts upward after an decrease in δ, implying that $c_{t+1}(\delta)$ increases. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(\delta)$ first order stochastically dominates $\Phi_t(\delta')$ for $\delta > \delta'$. As in the proof of Lemma B1, we define:

$$\Phi_t(\delta) = 1 - \left(\frac{1 - F(c_t(\delta))}{1 - F(l)}\right)^{n-1}$$

Since $\left(\frac{1-F(c_j(\delta))}{1-F(l)}\right)^{n-1}$ is increasing in δ for all j, then $\Phi_t(\delta') \geq \Phi_t(\delta)$ for all t and $\delta > \delta'$. ■

Because $\Phi_t(\delta)$ deteriorates, the increase in δ has an ambiguous marginal effect on the welfare of an agent that cannot be signed: while success takes more time, the cost of delay has decreased, so these two effects go in opposite directions. The ambiguous overall effect, however, implies that contrary to what we prove happens when $n \to \infty$, for a fixed n, efficiency is unattainable even in the limit as $\delta \to 1$.[37]

Proposition B1: *For all n, v, there exists a $\xi > 0$ such that $\lim_{\delta \to 1} EU_n(c) < v - \xi$ for all $c \in [0, 1]$.*

Proof. For any $\varepsilon > 0$, the probability of the event E in which no player has a type c lower than ε is $[1 - F(\varepsilon)]^n > 0$. In this event, no success can occur until we reach a period t in which the cutpoint is strictly larger than ε. Let t_ε be the minimal t such that $c_{t_\varepsilon} \geq \varepsilon$. For δ high, we can assume without loss of generality that $c_{t_\varepsilon - 1} > \frac{\varepsilon}{2}$. To see this note that if $c_{t_\varepsilon - 1} \leq \frac{\varepsilon}{2}$, then (5) in Proposition 1 in the paper implies that for δ sufficiently close to 1, c_{t_ε} is arbitrarily close to $\frac{\varepsilon}{2}$ as well: so $c_{t_\varepsilon} < \varepsilon$, a contradiction. The expected utility of a player in event E is not larger than $(v - \frac{\varepsilon}{2})$ since in period $t_\varepsilon - 1$ no player can obtain a payoff larger than the lowest type. But then $\delta \to 1$, the expected utility of a player is at most

$$(1 - [1 - F(\varepsilon)]^n)\, v + [1 - F(\varepsilon)]^n\, (v - \varepsilon/2) = v - [1 - F(\varepsilon)]^n\, \frac{\varepsilon}{2} < v.$$

The result is proven if we let $\xi = [1 - F(\varepsilon)]^n\, \frac{\varepsilon}{2}$. ■

[37] The fact that equilibrium converges to its efficient value as $n \to \infty$ when $m_n = 1$ is proven as a special case of the case in which $m_n \geq 1$ and can potentially grow with n. See Theorem 4 in the paper.

Appendix C: Additional Extensions in Section 6.4

Aggregate uncertainty and learning

In the previous analysis we assumed the players' types are i.i.d. In such environments, as time progresses players update their beliefs about the types of the other players because they know that the remaining active players must have a type higher than the previous equilibrium cutoffs. Yet, they do not learn anything new about the original environment, since the distribution of the other players' types is common knowledge. It is natural to consider scenarios in which players are also ex ante uncertain about the environment, for example about the shape of the distribution of types. In these environments, as time progresses, players also learn about the shape of the distribution of types and this learning also depends on equilibrium strategies.

To illustrate how the analysis can be generalized to this more complex case, we characterize here the equilibrium in the volunteer's dilemma with $m_n = 1$ as in Section 3. The analysis can extended in similar ways to the case with $m_n > 1$. We assume for simplicity that there are two states of nature $\vartheta = H, L$; and that the distribution of types is $F_\vartheta(c)$ in state ϑ, with density $f_\vartheta(c)$ and $F_H(c)$ first order stochastically dominating $F_L(c)$. In this environment, at $t = 1$, a player's belief that the state is H at the beginning of the first period is necessarily function of their type $\pi^1(c)$ since for any initial common prior π^0, they would update after observing their type. We assume here that $\pi^1(c)$ is increasing and continuous in c.[38]

Consider a threshold equilibrium with cutoffs $(\dot{c_t})_{t=0}^{\infty}$ and $\dot{c_t} > \dot{c}_{t-1}$ as in Section 3, such that at t all types $c \leq \dot{c_t}$ contribute, and types $c > \dot{c_t}$ wait. Given these cutoffs and belief $\pi^{t-1}(c)$ at $t-1$, the belief at the beginning of period $t > 1$ for a type c is given by:

$$\pi^t(c) = \left[1 + \frac{1-\pi^{t-1}(c)}{\pi^{t-1}(c)} \frac{\left(\frac{1-F_H\left(\dot{c}_{t-1}\right)}{1-F_H\left(\dot{c}_{t-2}\right)}\right)}{\left(\frac{1-F_L\left(\dot{c}_{t-1}\right)}{1-F_L\left(\dot{c}_{t-2}\right)}\right)}\right]^{-1} \tag{45}$$

Note that, for any t, if $\pi^{t-1}(c)$ is continuous and increasing in c, then $\pi^t(c)$ is continuous and increasing in c as well since $\frac{1-\pi^{t-1}(c)}{\pi^{t-1}(c)}$ is continuous and decreasing in c.

Given the posterior at t, $\pi^t(c)$, and the equilibrium lower bound $\dot{c}_{t-1}$, an argument analogous to the argument leading to (3) in Section 3, gives us the cutoff at t as the fixed-

[38] For an event $E = [c-\varepsilon, c+\varepsilon]$, we have $Pr(H;E) = \frac{\pi^0 \Delta F_H(E)}{\pi^0 \Delta F_H(E)+(1-\pi^0)\Delta F_L(E)}$, where we define $\Delta F_\vartheta(E) = [F_\vartheta(c+\varepsilon) - F_\vartheta(c-\varepsilon)]$. Taking the limit as $\varepsilon \to 0$, we have that $Pr(H;c) = \frac{\pi^0 f_H(c)}{\pi^0 f_H(c)+(1-\pi^0)f_L(c)}$ is increasing if $f_H(c)/f_L(c)$ is increasing in c.

point of:

$$v - c_t^{\cdot} = \sum\nolimits_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left\{ v \left[1 - \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} \right] + \delta \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} (v - c_t^{\cdot}) \right\} \tag{46}$$

note that now $c_t^{\cdot}$ does not only affect $\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)}$ in the right hand side of (46); but also the prior probabilities $\pi_\theta^t(c_t^{\cdot})$ at t with which the shape of the distribution is evaluated: this because all players of different types have different posteriors at each history of the game, since they start from heterogeneous priors $\pi^1(c)$. After some algebra, we have:

$$c_t^{\cdot} = \frac{(1-\delta) \cdot \sum_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}}{1 - \delta \sum_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}} v \equiv G(c_t^{\cdot}) \tag{47}$$

Condition (47) alone is no longer sufficient to characterize the equilibrium. The equilibrium now is determined by the system of difference equations (45) and (47): $\pi^1(c)$ and the initial lower bound $c_0^{\cdot}$ determine $c_1^{\cdot}$; $\pi^1(c)$, $c_0^{\cdot}$ and $c_1^{\cdot}$ determine $\pi^2(c)$; $c_1^{\cdot}$, $c_2^{\cdot}$ and $\pi^2(c)$ determine $c_3^{\cdot}$; and so on so forth for any $t > 0$.

Condition (47) can be used to show that a cutoff equilibrium has similar properties to the equilibria as in Section 3. To see this, consider the right hand side of (47), $G(c_t^{\cdot})$. This function is continuous in $c_t^{\cdot}$ and has the properties that $G(c_{t-1}^{\cdot}) = v$ and $G(1) = 0$; hence it has a fixed point $c_t^{\cdot} > c_{t-1}^{\cdot}$ for any $c_{t-1}^{\cdot}$ and $t-1$. We can moreover verify that $c_t^{\cdot} > c_{t-1}^{\cdot}$ and $c_t^{\cdot} \to v$. For this, assume by way of contradiction that $\lim_{t\to\infty} c_t^{\cdot} = c_\infty^{\cdot} < v$. Then we would still have $\lim_{t\to\infty} \pi^t(c_t^{\cdot}) = \pi^t(c_\infty^{\cdot})$ and $\sum_{\theta=H,L} \pi_\theta^t(c_\infty^{\cdot}) = 1$. It follows that:

$$\lim_{t\to\infty} c_t^{\cdot} = \frac{(1-\delta) \cdot \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}}{1 - \delta \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}} v = v,$$

a contradiction.

When the players learn about the distribution of types, however, an additional complication may arise regarding whether an equilibrium is necessarily in cutoff strategies. To see this point, consider (46) which characterizes the indifferent type $c_t^{\cdot}$. For a type $c < c_t^{\cdot}$, the left hand side is $v - c$, so it decreases linearly with a slope of -1. The right hand side now is:

$$\sum\nolimits_{\theta=H,L} \pi_\theta^t(c) \left\{ v \left[1 - \left(\frac{1-F_\theta\left(\dot{c}_t\right)}{1-F_\theta\left(\dot{c}_{t-1}\right)} \right)^{n-1} \right] + \delta \left(\frac{1-F_\theta\left(\dot{c}_t\right)}{1-F_\theta\left(\dot{c}_{t-1}\right)} \right)^{n-1} (v-c) \right\}$$

where we note that c enters only in the posterior $\pi_\theta^t(c)$ and in the last term $v-c$, since $\dot{c}_t$ is the strategy used by the other players. When $f_H(c)/f_L(c)$ (and therefore a fortiori $\pi^1(c)$ and $\pi^t(c)$) does not increase too sharply in c, then this term certainly declines in c at a rate slower than -1, so types $c > \dot{c}_t$ find it optimal to abstain and types $c < \dot{c}_t$ to contribute (just as in the case with no aggregate learning). But when $f_H(c)/f_L(c)$ can change sharply (as for example when the support of costs changes with the state, so that the posterior is discontinuous in c), then the equilibrium may not be in cutoff strategies. It is interesting that even in the simplest case with $m=1$ we might have these complications.

Non-stationary environments

There are applications for collective action problems in which it seems natural to assume that the value of a group's success changes over time. In environmental problems, for example, the consequences of not solving the collective problem (i.e. failing to succeed in collective action) become more severe over time. In this section we show how non-stationarities can be easily incorporated in the analysis and lead to new insights.

Assume that if the group does not obtain the common goal at $t-1$, then at the beginning of t each player suffers a loss of η^t for $\eta > 1$. The loss from not obtaining the common goal (say closing the ozone hole) grows exponentially over time.[39] The equilibrium condition now becomes:

$$v - \ddot{c}_t - \eta^t = v \left[1 - \left(\frac{1-F\left(\ddot{c}_t\right)}{1-F\left(\ddot{c}_{t-1}\right)} \right)^{n-1} \right] + \delta \left(\frac{1-F\left(\ddot{c}_t\right)}{1-F\left(\ddot{c}_{t-1}\right)} \right)^{n-1} \left(v - \ddot{c}_t - \eta^{t+1}\right) - \eta^t$$

At time t, the cost η^t is sunk, and thus irrelevant for the decision. But the cost at $t+1$ still matters, since if the group is successful at t, it can avoid the cost η^{t+1}. As in the previous sections, the left hand side is the utility from contributing, the right hand side is the utility for not contributing. In both cases, a players suffers a cost η^t, which then can be simplified. In the utility for not contributing we now have η^{t+1} times the probability that none of the other players contributes. The loss η^{t+1} does not affect the decision at $t+1$, when it is a

[39]There are of course other ways to introduce non-stationary elements in the model (for example we could have assumed that the distribution of c or v changes over time). We chose to model non-stationarity as above because it seems it better captures the phenomenon described in the example of environmental protection described above.

suck cost, but it affects the decision at t. This condition can be rewritten as:

$$c_t^{\cdot\cdot} = \left[\frac{\left(1-\delta(1-\frac{\eta^{t+1}}{v})\right)\left(\frac{1-F(c_t^{\cdot\cdot})}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}}{1-\delta\left(\frac{1-F(c_t^{\cdot\cdot})}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}} \right] v \tag{48}$$

For a given $c_{t-1}^{\cdot\cdot}$ and $c_t^{\cdot\cdot}$, the right hand side increases in η and t. The monotonic worsening of the environment reduces the utility of not contributing and leads to a lower utility of not contributing. It can be shown that the cutpoints $c_t^{\cdot\cdot}$ are strictly increasing in $\mathbf{t}$ and eventually success is achieved for all realizations of types.

Unknown thresholds

In the preceding analysis we have assumed the agents play a simple threshold public good game in which the public good is obtained by the group if and only if there are at least m_n contributors. It is easy to consider extensions in which success is determined under alternative, less stylized rules. A natural case to consider is when the realization of the public good depends on the number of contributors in a probabilistic way. As an illustration, consider here the case in which at t the public good is realized with probability $\xi(k_t)$ if contributors up to t are k_t, where $\xi(\cdot)$ is a non-decreasing function with $\xi(k_t) = 0$ if $k_t < \lceil \underline{\theta} m_n \rceil$, $\xi(k_t) > 0$ for $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$, and $\xi(k_t) = 1$ if $k_t \geq \lceil \overline{\theta} m_n \rceil$ for $\underline{\theta} \in (0,1)$ and $\overline{\theta}_n \in \left(1, \frac{n}{m_n}\right)$ (here $\lceil x \rceil$ is the smallest integer that is not smaller than x). This implies that the players are uncertain about the minimal number of volunteers that are required for success which may take values from $\lceil \underline{\theta} m_n \rceil$ to $\lceil \overline{\theta}_n m_n \rceil$.

It is not difficult to see that the characterization of the equilibrium, the existence proof and the limit results of Section 4 all generalize to this more complex environment. The equilibrium conditions are qualitatively similar to equations (10), (12), and (14) in the paper, although now they are complicated by the fact that for any $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$ success is probabilistic. Consider now the limit results presented in Section 5. A simple argument shows that, in this case too, a sufficient condition for the existence of a limit efficient equilibrium is that $\alpha_n = m_n/n$ converges to zero sufficiently fast, faster than the cube root of $1/n$; the fact that the probability of success is lower than one for $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$ is completely irrelevant. To see this, let $\xi_* = \min_{k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)} \xi(k_t)$, which is strictly positive by assumption. Consider a modified game in which the value of the public good is $v' = v\xi_* > 0$, and the probability of success is 1 for $k_t \geq \lceil \overline{\theta}_n m_n \rceil$. This gives a game similar to the games studied in the previous sections, in which an asymptotically efficient equilibrium exists if $\alpha_n / \sqrt[3]{1/n} \to 0$. But the incentives to contribute in the original game with probabilistic success are even higher than in this modified game, since in the modified game we assume the value of the public good v' and the probability of success are always lower than in the original game: thus a limit efficient equilibrium must exists in the game with probabilistic success. To show the impossibility of an equilibrium with positive payoff in the limit when $\alpha_n/\sqrt[3]{1/n} \to \infty$, the analysis is similar. We can now define a modified game in which the threshold for success is at $m_n = \lceil \underline{\theta} m_n \rceil$ and $\xi(k_t) = 1$ if $k_t \geq \lceil \underline{\theta} m_n \rceil$. The preceding analysis shows that all PBE generate zero utility in the limit in this game; thus it can be shown that the same must be true in the game with probabilistic public good, since in this game the expected utility of contributing is always strictly lower.

High-value environments ($v \geq 1$)

Up to this point we assumed $v < 1$. If $v \geq 1$, then it is common knowledge that all players would willingly participate if they are pivotal. This has a number of implications, which we explain here. First, there exist asymmetric equilibria that achieve success instantly: in equilibrium, at $t = 1$ a subset of exactly m members participates, regardless of their private cost, and the remaining $n - m$ members free ride. Of course, this requires some coordination device among the players. If such coordination devices are not readily available, then we are back to characterizing the symmetric PBE of the game. In this high value case, results are not entirely negative. For symmetric PBE, we have the following results. We extend Theorem 3 to the high-value case as follows:

Theorem A1. *If $v \geq 1$ then for all $n > 2$, for all $1 < m < n$, and for all $\delta \in (0, 1)$, in every symmetric equilibrium:*

1. *The probability of success is strictly positive.*
2. *There exists a minimum value threshold, $1 < v^*(n, m, \delta) < \infty$, such that all equilibria are interior iff $v > v^*(n, m, \delta)$, and success is achieved with probability 1.*
3. *For all $v \in [1, v^*(n, m, \delta)]$, there is at least one equilibrium in which there is a strictly positive probability of reaching an effectively terminal history h_t^k with $c^k(l_{h_\tau^k}) = l_{h_t^k} < v$* for *all $\tau \geq t$.*

Proof: To simplify notation, we suppress the dependence of the lower bound of the posterior beliefs, l, on h_t^k. For any lower bound, l, define $\underline{c}^k(l)$ as the minimal x such that:

$$v - x - \delta \cdot \sum\nolimits_{j=0}^{k-2} \left(\frac{v}{\delta} - Q^{k-j-1}(x) \right) B\left(j, n-1-m+k, \widetilde{F}(x; l)\right) \tag{49}$$
$$\geq v - \delta \cdot \sum\nolimits_{j=0}^{k-1} \left(\frac{v}{\delta} - V^{k-j}(x, x) \right) B\left(j, n-1-m+k, \widetilde{F}(x; l)\right).$$

The left hand side is the utility of a cutpoint type x who volunteers when the cutpoint used by the other agents is x. The right hand side is the utility of a type x who does not volunteer, when the others are using cutpoint x. Note that the left hand side may be strictly lower than the right hand side for any $x \in [l, v]$: in this case all types c strictly prefer not to volunteer and $c^k(l) = l$, in which case the cutpoint is not defined by an equality as in (49). It follows that there is no type $x < v$ that is willing to volunteer; when $\underline{c}^k(l) > l$, then any type $x \leq \underline{c}^k(l)$ is willing to volunteer. When $\underline{c}^k(l) = l$, then type l is willing to volunteer only (49) holds with equality.

We can write (49) as:

$$\underline{c}^k(l) = \min_{c \geq [l,1]} \left\{ c | c \leq \delta \sum\nolimits_{j=0}^{k-1} \left[\left(Q^{k-j-1}(c) - V^{k-j}(c,c) \right) B\left(j, n-1-m+k, \widetilde{F}(c;l) \right) \right] \right\} \tag{50}$$

where $Q^0(c) = v/\delta$. We now prove that there is a $v^*(n, m, \delta)$ such that for $v > v^*(n, m, \delta)$, $c^k(l) > l$ for any $k \leq m$ and $l < \min\{v, 1\}$. We proceed in four steps.

Step 1. We have already proven in Lemma 2 that for any $l < v$, we have $\underline{c}^1(l) > l$. Moreover it is easy to verify that there must be a $v^*(n, 1, \delta)$ such that for $v \geq v^*(n, 1, \delta)$ we have $\delta Q^1(l) - l > 0$ for any $l \in [0, \min\{v, 1\}]$: since as it can be verified using (4) in the paper, $c_t^1(l)$ is strictly increasing in v for all t; and $Q^1(l)$ is increasing in both v and $c_t^1(l)$ for all t.

Step 2. For the induction hypothesis, assume that for any $l \in [0, \min\{v, 1\}]$ and for all $j = 1, ..., m-1$ there is a $v^*(n, j, \delta) > 0$ such that for $v \geq v^*(n, j, \delta)$, we have: $c^j(l) > l$ and $\delta Q^j(l) - l > 0$. We prove that there is a $v^*(n, m, \delta)$ such that for $v \geq v^*(n, m, \delta)$, we have: $c^j(l) > l$ for all $l < v$ and $\delta Q^j(l) - l > 0$ for any $j \leq m$ and $l \in [0, 1]$. There are two sub-cases to consider.

Step 2.1. Suppose by contradiction that for any v even arbitrarily large, $c^k(l) = l$ and we have that $\left[V^k\right]^-(c^k(l), l) > \left[V^k\right]^+(l, l)$ for some $l \leq \min\{1, v\}$. Hence, at l we therefore have a strict corner solution when there are k missing volunteers. In this case, the value function for a type l must be $V^k(l, l) = 0$, since the project will never be realized: all players expect that no other player of type $c \geq l$ is willing to contribute. Suppose that $v \geq v^*(n, k-1, \delta)$, as defined by the induction step. If a player of type l volunteers, s/he obtains: $\left[V^k\right]^+(l, l) \geq \delta \cdot Q^{k-1}(l) - l > 0$, where the last inequality follows from the induction step: we thus have a contradiction.

Step 2.2. From the previous step, we conclude that, if Part 1 the theorem is not true, then for $v > v^*(n, k-1, \delta)$, if $c^k(l) = l$ then $V^k(c^k(l), l) = \left[V^k\right]^+(l, l)$. From $c^k(l) = l$, we have:

$$\begin{aligned} B\left(j, n-1-m+k, \widetilde{F}(c;l)\right) &= 0 \text{ for } j > 0 \\ B\left(0, n-1-m+k, \widetilde{F}(c;l)\right) &= 1 \end{aligned} \tag{51}$$

since $V^k(c^k(l), l) = \left[V^k\right]^+(l, l)$. Hence $\underline{c}^k(l)$ satisfies (50) at equality, which implies that (50) can be written as: $l = \underline{c}^k(l) = \delta \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right)$. Note that when $\underline{c}^k(l) = l$, there are no other volunteers, so: $\left[V^k\right]^+(l,l) = \delta \cdot Q^{k-1}(l) - l$. But then we have:

$$\begin{aligned} Q^{k-1}(l) - \left[V^k\right]^+(l,l) &= Q^{k-1}(l) - \left(\delta \cdot Q^{k-1}(l) - l\right) \\ &= l + (1-\delta)Q^{k-1}(l) \end{aligned}$$

This implies that we have $\delta \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right) > l$ if $\delta l + \delta(1-\delta)Q^{k-1}(l) > l$: or equivalently $\delta Q^{k-1}(l) > l$, an inequality that is always true if $v > v^*(n, k-1)$. But then we have $l = \underline{c}^k(l) = \delta \cdot \left(l + (1-\delta)Q^{k-1}(l)\right) > l$, a contradiction. We conclude that for for any $k \leq m$ and $l \leq \min\{1, v\}$, $c^k(l) > l$ if $v > v^*(n, k, \delta)$.

Step 3. Finally, we conclude the inductive argument by proving that there is a $v^*(n,k,\delta) \geq v^*(n, k-1, \delta)$ such that for $v > v^*(n,k,\delta)$, then $\delta Q^k(l) > l$ for any $l \in [l_0, \min\{1, v\}]$. It is sufficient to prove $\delta Q^k(l) > 1$, for v sufficiently high. Assume not. Then it must be that $c^k(l)$ converges to l as v increases, since if it converges to a constant $\widetilde{c} > l$. We must therefore have that $B\left(j, n-1-m+k, \frac{F(\widetilde{c})-F(l)}{1-F(l)}\right) > 0$ for all $j \geq k$ and $Q^k(l) \to \infty$ as $v \to \infty$, since it is strictly increasing in v (and diverging at infinity as v increases, given the lower bounds on the probabilities of $j \geq k$ volunteers): a contradiction. But if $c^k(l) \to l$, then we have: $\left[V^k\right]^+(l,l) \to \delta Q^{k-1}(l) - l$. Since, by the previous step, the equilibrium is interior in stage k for $v > v^*(n, k-1, \delta)$, we have: $c^k(l) = \delta \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right)$. Note moreover that for $v > v^*(n, k-1, \delta)$, we have $\delta Q^{k-1}(l) > l$. It follows that as v increases, we have:

$$c^k(l) \to \delta \cdot \left(l + (1-\delta)Q^{k-1}(l)\right) > \delta \cdot \left(l + (1-\delta)\frac{l}{\delta}\right) = l$$

where the last inequality follows from $v > v^*(n, k-1, \delta)$. We thus have a contradiction. We conclude that there is a $v^*(n,k)$ such that for $v > v^*(n,k,\delta)$, $\delta Q^k(l) > l$. ■

The proof of part (1) is the same as in Theorem 3. Part (2) is proved by induction. When $m = 1$, all equilibria are interior by Lemma 2 for all v, (2) therefore holds for $m = 1$. The properties of the interior equilibrium when $k = 1$ moreover guarantee that we have $\delta Q^1(l) - l > 0$ for any $l \in [0, \min\{v, 1\}]$. For the induction hypothesis, assume that for all $j = 1, ..., k-1$ there exists a $v^*_{k-1}(n, j, \delta) < \infty$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^j(l) > l$ and $\delta Q^j(l) - l > 0$, if and only if $v > v^*_{k-1}(n, j, \delta)$. The next step of the proof is to show that the induction hypothesis implies the existence of a $v^*_k(n,k,\delta) > 1$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^k(l) > l$ and $\delta Q^k(l) - l > 0$ if and only if $v > v^*_k(n,k,\delta)$. This requires some care since $Q^k(l)$ is typically a complicated function of l for $k > 1$. This argument allows us to conclude that for any $k \leq m$, a player with type close to l finds it optimal to contribute, even if s/he expects all other active players not to contribute.

Part (3) of Theorem A1 can be seen as the residual case and follows as a corollary to Part (2). If $v \leq v^*(n, m, \delta)$, then there is a history with strictly positive probability at which if a player expects no other player to contribute, then s/he does not find it optimal to contribute as well: thus we have an equilibrium in which players stop contributing. This in itself, however, does not imply that there is no interior equilibrium in which success is eventually achieved since the payers' decisions to contribute may be strategic complements, implying that we could have additional equilibria in which participation is stimulated by the expectation that other players contribute with strictly positive probability. As shown in Theorem 3, however, this is impossible if $v < 1$.

Theorems 3 and 3′ highlight the fact that, except when v is very high, there is uncertainty regarding whether the group can get stuck at an effectively terminal history where no player is willing to contribute anymore, or, alternatively, the cutpoints continually increase in all periods. However it leaves open two issues. First, it is possible that for all $k < m$ we have $c_t^k > c_{t-1}^k$, but $c_t^k \to c_\infty^k < 1$. In this case, the equilibrium is interior but the project may still remain unrealized even if all types are below v. The second question concerns the size of $v^*(n, m, \delta)$. Should we expect $v^*(n, m, \delta)$ to be close to 1, at least when players are patient (i.e., $\delta \to 1$)? The following proposition addresses the first issue.

Proposition A1. *If $v > v^*(n, m, \delta)$, then the group is constrained-successful, i.e., $\overline{p}_n = 1 - [1 - F(v)]^n = 1$ in all symmetric equilibria. If $v \in [1, v^*(n, m, \delta)]$, then the group is constrained successful in any interior equilibrium.*

Proof: We show here that if $v > v^*(n, m, \delta)$, then the group achieves the objective if there are at least m players with type lower than v. Note that for $v > v^*(n, m, \delta)$, in the history h_t^k with k volunteers needed and a lower bound of types at $l_{h_t^k}$, then in period $t + 1$, there will either be $j < k$ volunteers needed, with a lower bound of $c^k(l_{h_t^k})$; or there will still be k volunteers needed with a higher lower bound of types $c^k(l) > l$.

Now suppose, by way of contradiction, that there is some k for which $c_t^k \to c_\infty^k < v$ for some initial $l_{h_t^k}$, following a sequence of many periods where there are no additional volunteers beyond $m - k$. Note that as $c_t^k \to c_\infty^k$, we have:

$$\widetilde{F}(c_t^k; c_{t-1}^k) \to \frac{\lim_{t \to \infty} \left(F(c_t^k) - F(c_{t-1}^k)\right)}{1 - F(c_\infty^k)} = 0$$

Since (13) in the paper must hold, we therefore have:

$$c_\infty^k = \min_{c \in \left[c_\infty^k, 1\right]} \left\{ c \geq \delta \cdot \left(Q^{k-1}(c) - \left[V^k\right]^+ (c, c) \right) \right\},$$

But then the same argument as Step 2.2. above proves that for $v > v^*(n,m)$ we must have:

$$\min_{c\in\left[c^k_\infty,1\right]}\left\{c \geq \delta\cdot\left[\begin{array}{c} Q^{k-1}(c) \\ -\left[V^k\right]^+(c,c)\end{array}\right]\right\} > \delta\cdot\left(c^k_\infty + (1-\delta)\frac{c^k_\infty}{\delta}\right) > c^k_\infty,$$

a contradiction. We conclude that for all k, $c^k_t \to c^k_\infty = v$. ■

The next result addresses the second issue, the size of $v^*(n,m,\delta)$. We have:

Proposition A2. *For any $n > 2$, for any $1 < m < n$, and for all $\delta \in (0,1)$: $v^*(n,m,\delta) \geq 2$ for any n, m and δ.*

Proof: Call E^{2+} the event comprising histories h_t in which at least two volunteers are missing and they both have a cost $c_i \in (v/2,1)$. Clearly this event has positive probability for any $v \in (1,2)$. We now prove that for any $v \in (1,2)$, there is an equilibrium in which contributions are zero in E^{2+}, no matter what the level of δ is, thus even in the limit as $\delta \to 1$. Consider an history h^2_t with the properties as above. Assume the lower-bound on types is $l > v/2$. An active player i who expects no other active player to contribute obtains from contributing at most a payoff

$$\begin{aligned}\delta Q^1(c^2(l)) - c_i &\leq \delta\left[v - c^2(l)\right] - c_i \leq \delta\left[v - l\right] - l \\ &\leq \left[\delta - \frac{1}{2}(1+\delta)\right] v < 0\end{aligned} \tag{52}$$

where the first inequality follows from the fact that $Q^1(c^2(l))$ must be smaller than the utility of the lowest remaining type, i.e. $c^2(l)$, so it must be $Q^1(c^2(l)) \leq v - c^2(l) \leq v - l$; the second inequality follows from the fact that $c^2(l) \geq l$ and $c_i \geq l$. We conclude that no active player finds it optimal to contribute if s/he does not expect some other player to contribute with positive probability. ■

This result tells us that even if $v \in [1,2]$, where it is common knowledge that all players desire the public good even if their participation is required to get it, the dynamic process of participation is probabilistic, and a strictly positive probability of failure is unavoidable. This is true even if the group is large, the threshold is small, and even if the players are arbitrarily patient. The inability of a group to reach success does not depend on the discount factor δ, so it remains true even in the limit as $\delta \to 1$.

For large groups $v^*(n,m,\delta)$ grows without bound. Specifically, we have:

Proposition A3. *If $m_n > 1$, then $\lim_{\delta\to 0} v^*(n,m_n,\delta) = \infty$*

Proof: We prove here that for any $n > m_n$ and for any $\varepsilon > 0$, there exists a $\delta_{n,\varepsilon} > 0$ such that for $\delta < \delta_{n,\varepsilon}$ the project is realized in an equilibrium with probability less than ε. This implies that for any $\widehat{v}$, there is a $\delta_{n,\widehat{v}}$ such that for $\delta < \delta_{n,\widehat{v}}$ we have $v^*\left(n,m_n,\delta\right) > \widehat{v}$.

For any F and for any $\varepsilon > 0$, there is a $c_\varepsilon > 0$ such that with probability $1-\varepsilon$, at least $n - m_n + 2$ players have cost strictly larger than c_ε (so that there are no more than $m_n - 2$ players with cost lower than c_ε). Consider a continuation game with $k \leq m_n$ and $l \geq c_\varepsilon$, where k is the number of missing volunteers, and l is the lower bound on types. For these continuation games, consider the path of future play along which no uncommitted member volunteers. To see that there is a $\delta_{n,\varepsilon}$ such that for $\delta < \delta_{n,\varepsilon}$ this is an equilibrium, note that with these strategies the expected utility of a player who does not volunteer is zero; the expected benefit of volunteering player is not higher than $D_{n,\varepsilon,\delta} \equiv -c_\varepsilon + \delta(v - c_\varepsilon)$: success can occur no sooner than a period after the deviator volunteers, and the expected payoff the period after a unilateral deviation cannot be larger than the utility of the lowest type, i.e. $v - c_\varepsilon$. For $\delta < \frac{v-c_\varepsilon}{c_\varepsilon} = \delta_{n,\varepsilon}$, we have that $D_{n,\varepsilon,\delta} < 0$, so the equilibrium strategies are optimal. Given these equilibrium strategies, assign to any other k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$, some corresponding equilibrium strategy for the continuation game. In the equilibrium of the overall game it must be that if $\delta < \delta_{n,\varepsilon}$ then with probability $1-\varepsilon$ there are not enough members with cost $c < c_\varepsilon$ to complete the project, which then must fail: indeed, with probability at least $1-\varepsilon$ either we gradually converges to a continuation game corresponding to k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$ in which no player is willing to contribute; or we reach a state k, l with $k \leq m_n$ and $l \geq c_\varepsilon$, in which case again no player finds it optimal to contribute by construction. ■

It is important to note that Proposition A3 does not preclude the possibility that even if δ is small, there can be efficient or approximately efficient equilibria in the limit. In particular, it is easy to see that Theorem 4 in the paper holds for all $v > 0$, including for the high value case.