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ABSTRACT

Why did the real interest rate decline and the equity premium increase over the last 30 years? This paper assesses the role of uncertainty and credit market frictions. We quantify a model with heterogeneous households using data on asset prices and macro aggregates, as well as on households' debt and equity positions. We find that compensation for both uncertainty and frictions is reflected in asset prices. Moreover, a secular increase in frictions is important to understand jointly the decline in real rate and the relative scarcity of debt. Modeling uncertainty as ambiguity allows for tractable characterization of asset premia and precautionary savings effects in steady state.

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1 Introduction

The last three decades have seen a decline in the real interest rate – all the way into negative territory – accompanied by high stock returns and an increase in the equity premium. Economic theory offers two ways to think about medium run changes in average returns. One view, traditionally popular in financial economics, holds that asset prices reflect compensation for bearing aggregate uncertainty. According to this view, higher uncertainty is a candidate explanation for the observed changes in returns: when uncertainty increases, debtholders who want precautionary savings require less compensation (a lower interest rate). At the same time, shareholders require more compensation (a higher premium).

A second view, prominent in macroeconomics and corporate finance, holds that asset prices reflect compensation for financial frictions, in particular leverage costs in credit markets. Under this view an increase in the leverage cost provides an alternative explanation for the stylized facts on asset returns. Indeed, when leverage becomes more costly, borrowers require more compensation (a lower interest rate). At the same time, equity positions become more costly to fund, so shareholders require more compensation (a higher premium). While large literatures discuss how premia in financial markets depend on aggregate uncertainty and leverage costs, there is no consensus on which view is more relevant and how change in compensation for frictions or uncertainty accounts for recent history.

This paper compares the role of uncertainty and frictions for recent price trends. While compensation for both uncertainty and frictions is reflected in asset prices, we find that frictions have recently become more important. We are led to this conclusion by an additional stylized fact we document: debt has become scarce relative to equity. In particular, the sharp decline in market leverage of nonfinancial businesses since the 1980s was not offset by an increase in government or household debt, so the equity share in households' aggregate portfolio is now much higher. We therefore need to explain a *joint* decline in interest rates and the relative quantity of debt. An increase in the cost of leverage does exactly that: it shifts borrowers' supply of bonds, and lower supply leads to lower quantity and a higher price (or a lower interest rate). An increase in uncertainty, in contrast, implies a shift in debtholders' demand that should have gone along with relatively more debt.

We study the joint determination of asset prices and asset positions in a quantitative borrower-saver model with aggregate uncertainty and costly borrowing. Our evidence on asset positions comes from the Survey of Consumer Finances (SCF) and the Financial Accounts of the US (FA). We show in particular that indirect borrowing through business leverage is an important way for a small share of rich shareholders to borrow from other households who hold mostly debt. This result motivates a model with two agent types:

shareholders who borrow from debtholders. We further show that the relative importance of such borrowing has declined recently: since the peak of the debt boom of the 1980s the effective leverage of shareholders has fallen by about 20%. Over the same period, the share of debt in aggregate household financial assets declined by about a quarter.

Our quantitative exercise infers the relative contribution of uncertainty and frictions to recent history by fitting our model to statistics on asset prices, inequality in asset positions and income, as well as macro aggregates for 1989 and 2016, two SCF years that are representative of market conditions in the 1980s versus today. In addition to a higher cost of borrowing in 2016, we find lower dividend uncertainty perceived by shareholders as well as higher labor income uncertainty perceived by debtholders. These shifts in uncertainty help explain why the decline in the interest rate was relatively larger than the increase in the equity premium. Put together, changes in uncertainty *and* frictions thus imply that the equity premium has morphed from what was mostly an uncertainty premium towards a frictional premium. We use our model to explore how this move to a more frictional economy affects welfare inequality as well as the potential for insurance through fiscal policy.

Our model is an endowment economy with two agent types who differ in ambiguity aversion. In equilibrium, shareholder/borrower agents hold a tree with uncertain dividends. They supply debt to debtholder/saver agents whose uncertain labor income induces a greater desire for precautionary savings. Debt is issued until gains from ambiguity sharing are equal to the marginal cost of borrowing, a tax wedge in the credit market that is increasing in borrowers' leverage. Asset prices then reflect compensation for *both* aggregate uncertainty and financial frictions. An advantage of modeling aggregate uncertainty as ambiguity is that both precautionary savings effects and asset premia can be analyzed in a deterministic steady state of the model.[1] We exploit this feature to derive transparent formulas for average returns and debt as functions of preference parameters governing perception of ambiguity and the borrowing cost. We then infer those parameters from statistics for our two benchmark years.

Our quantitative approach considers claims on business capital in the US economy that equal equity plus "debt", identified in the data with all fixed income instruments. While we allow for government debt, the majority of debt is private and represents indirect borrowing by equity holders via leverage of firms that are provided to bondholders via intermediaries. We trace out those connections using SCF and FA data. Dividing households based on the equity share in their portfolios, we contrast 20% of rich effective borrowers who hold essentially all capital in the US economy and 80% poor and middle class effective savers who

[1] We assume agents have multiple priors utility with sets of priors that differ in means. As a result, ambiguity, unlike risk, has first order effects on utility. See Ilut and Schneider (2022) for a comparison of different models of aversion to uncertainty and their use in macro and finance applications.

hold mostly debt. The main new stylized fact we develop here is that market leverage – defined as the value of fixed income securities relative to the value of equity – has declined from 29% in 1989 to 23% in 2016. In terms of changes in asset prices, we document a large fall in the real rate, using the Cleveland Fed measure, from 3.5% to -0.5%, and an increase in the equity premium from 6% in 1989 to 7.4% in 2016.

We model the cost of borrowing as a tax schedule in order to infer it from price and quantity data. This approach allows us capture general equilibrium effects of two forces that work like a tax on credit. The first is tighter regulation of leverage in the financial system. Our theory speaks to borrowing and lending between groups of households. Intermediation is part of the technology through which such borrowing and lending occurs. Regulatory reform that discourages leverage is thus captured in our model by a higher borrowing cost per unit of leverage. The second force is a change in the composition of firm assets, away from physical capital towards intangibles and the present value of rents, as reflected in growing market-to-book ratios.[2] In our model, this effect also maps into a higher cost of borrowing.

The key changes in frictions and uncertainty we infer in our quantitative exercise are consistent with direct evidence on recent trends. There is evidence for both forces captured by a higher cost of borrowing, stricter leverage regulation implemented after the 1990s and 2010s financial crises (see Duffie (2018) for a discussion), as well as an increase in the intangible asset share (eg. Crouzet et al. (2022)). Second, uncertainty about profits has been trending down. To show this, we construct a measure of ambiguity about capital income from the dispersion of profit growth forecasts in the Survey of Professional Forecasts. Dispersion in experts' forecasts has been used as an index of ambiguity perceived by economic agents: when experts disagree, agents naturally perceive more uncertainty (see Ilut and Schneider (2014), Ilut and Saijo (2021) or Masolo and Monti (2021)). Our measure of capital income ambiguity has fallen by more than one half since the late 1980s. Third, our model implies a relative increase in insecurity on the part of households who do not participate in equity markets and receive most income from wages. This finding is plausibly connected to divergence of incomes between top income earners and the rest of the population (eg. Acemoglu and Autor (2011), or recently Guvenen et al. (2022) and Heathcote et al. (2023)).

We perform counterfactuals to show that changes in both uncertainty and frictions are essential for understanding asset prices and leverage. We first consider a "pure uncertainty story": we re-estimate the model to infer perceived ambiguity, holding the cost of borrowing constant. We find that such a model can capture both a decline in the real rate and an

[2]High market-to-book (MTB) is a classic predictor of low market leverage in the cross section of firms: firms with high MTB have either few tangible assets or low current cash flow relative to future growth opportunities, which makes it difficult to borrow against either collateral assets or cash flow. See Frank and Goyal (2024) for a recent survey of work that relates market leverage and MTB in the cross section.

increase in the equity premium. It does so by increasing perceived uncertainty not only about bondholders' labor income but also about shareholders' dividend income, the opposite of our baseline result. At the same time, it badly fails to capture the decline in leverage. Further counterfactuals explore the role of borrowing cost without the baseline changes in uncertainty. The main takeaway here is that, while an increase in borrowing cost on its own can qualitatively push the interest rate, the equity premium and leverage in the right direction, its effect on the riskfree rate is quantitatively much smaller than that on the equity premium. Without accompanying changes in uncertainty, the moments cannot be matched jointly. A second key conclusion from our counterfactuals is that incorporating quantity information is important since a focus on asset prices alone can result in misleading results.[3]

Methodologically, we propose a *first order* perturbation approach to characterize equilibrium and solve systems of expectational difference equations with *permanent* differences in beliefs. The approach helps solve models with ambiguity aversion since agents behave *as if* they evaluate the future under a worst case belief. It is of independent methodological interest, as it can also handle other models with heterogeneous beliefs. The challenge with permanent differences in beliefs is that agents disagree on the steady state: it is therefore not possible to up front solve for a state where the system is at rest and everybody knows this, and then perturb around that state. Instead, steady state represents a situation where the system is at rest even though agents believe they are on different transition paths. We show how to solve jointly for steady state and linear dynamics.

What are the welfare implications of recent changes in uncertainty and frictions? We show that welfare inequality between concentrated shareholders and other households has increased substantially. Shareholders enjoy higher welfare from both negative real rates on their levered positions and, even more importantly, from lower capital income uncertainty. At the same time, debtholders' welfare is lower due to higher perceived ambiguity about labor income. A natural question is then whether fiscal policy can help better allocate ambiguity. We study tax-transfer policies that provide partial insurance to one agent type funded by lump-sum taxes on the other agent type with a balanced budget. In particular, with a "welfare state" policy, shareholders pay a transfer to bondholders when bondholders' labor income is low and the reverse occurs when bondholders' labor income is high. With a "bailout" policy, transfers are instead contingent on dividends earned by shareholders.

We show that the recent increase in frictions in private debt markets has reduced the scope for government insurance. At parameter values calibrated to 1989, we provide examples of

[3]We find relatively small effects from two other trends in quantity data. The increase in the ratio of government debt to GDP substituted for the decline in private debt and thus relieved downward pressure on risk-free rates. The decline of the labor share increased the value of capital to output and contributed to wealth inequality.

both "welfare state" and "bailout" policies that make both agents better off. It is plausible that the agent who is insured always benefits. When marginal borrowing costs are relatively low, however, general equilibrium effects spread those benefits to the other agent as well. For example, while a welfare state policy lowers demand for private bonds and increases shareholders' funding cost, shareholders delever in equilibrium and are overall better off. At parameter values calibrated to 2016, however, the same policy makes shareholders worse off. When marginal borrowing costs are high, a drop in demand leads to a large increase in interest rates that lowers shareholder consumption.

Our paper contributes to a large literature on uncertainty sharing with bonds. Early work building on Imrohoroğlu (1989), Huggett (1993) and Aiyagari (1994) emphasized labor income uncertainty but abstracted from aggregate risk. Our interest in aggregate uncertainty connects to borrower-saver models with aggregate risk. The classic two-agent DSGE models of Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), Iacoviello (2005) are early examples of linearized models where premia come entirely from frictions, typically in the form of collateral constraints. Nonlinear models where premia also compensate for aggregate risk include the theoretical studies of Brunnermeier and Sannikov (2014), Caballero and Farhi (2018), Caballero and Simsek (2020) and Coimbra and Rey (2023), as well as the recent quantitative studies of Kekre and Lenel (2022), Barro et al. (2022) and Fernández-Villaverde et al. (2023). Our paper differs from this literature in three ways. First, technically, we propose a linear approach with ambiguity where we can still have uncertainty compensation and frictions jointly, allowing for simple characterizations and comparison of economic mechanisms. Second, we are interested in secular changes, rather than business cycles or monetary policy shocks. Third, we distinguish uncertainty in capital and labor income, as opposed to concern driven by a single risk factor, which is also important for fitting the data.

We share an interest in secular changes with recent work analyzing trends in asset prices, for example Farhi and Gourio (2018) and Eggertsson et al. (2019). The main new fact we add is the behavior of leverage, which plays a central role in our identification of economic forces. We also relate to representative agent quantitative DSGE models with ambiguity, eg. Ilut and Schneider (2014), Bianchi et al. (2018), Krivenko (2023). Here we study an incomplete markets model with ambiguity and emphasize heterogeneity. Technically, we share with that work the insight that linear methods can be used to study uncertainty when modeled as ambiguity. In contrast to such representative-agent models, our environment features effective disagreement even in steady state, leading us to propose a novel solution method that jointly solves for steady-state and linear dynamics.

Finally, the dual model interpretation of a higher effective borrowing cost as tighter leverage regulation as well as lower share of tangible assets connects our estimated higher degree of financial friction to two other trends emphasized in the literature. One is stricter regulatory for intermediaries proposed in the literature as accounting for the emergence of risk-free positive excess returns in post Global Financial Crises markets (see for example Boyarchenko et al. (2018), Avdjiev et al. (2019) and Du et al. (2023)). The second trend is a significant fall in the asset share of tangibles over the last 30 years. In particular, here see Corrado et al. (2009) and Corrado and Hulten (2010) for measurement and recent trends on intangibles, Crouzet and Eberly (2019) and Crouzet et al. (2022) for a comparison of the role of intangibles and rents and Falato et al. (2022) for a study of trends in tangibility and debt in firm level data.

2 A model of ambiguity sharing

In this section, we present our model of ambiguity sharing between a borrower and a saver household. To exposit the main mechanisms and the solution technique in a transparent way, we leave out a number of features that are important for our quantitative analysis below, in particular government debt, long term debt and persistent shocks.

2.1 Setup

Agents and assets. Time is discrete and indexed by t. A unit mass of infinitely-lived agents inelastically supply one unit of labor and derive utility from the consumption of a numeraire good. Individual agents belong to one of two types. Type A agents receive a stream of labor income w_t^A and can save in one period risk-free bonds that trade at a price q_t. Type B agents receive a stream of labor income w_t^B. In addition to trading bonds, they have access to a tree that pays a dividend stream d_t, and trades at price p_t. For now we allow general stochastic processes for the exogenous streams of labor income and dividends.

Financial frictions and asset tangibility. Type B agents can borrow by issuing bonds collateralized by trees. We capture financial frictions in credit markets by introducing a *borrowing cost*: we impose a proportional tax on the borrower that is rebated lump sum to the lender. The tax rate moreover increases with leverage, defined as the ratio of debt to the value of trees. Formally, a type B agent who owns θ_t shares of the tree and issues b_t units of bonds to achieve leverage $\ell_t = q_t b_t / p_t \theta_t$ has to pay $k(\ell_t)\, q_t b_t$. We assume that the function $k(\ell)$ is strictly increasing and convex.

The borrowing cost function captures key features of more complicated contracting arrangements dealing with information or agency problems. First, the marginal borrowing cost increases with the debt amount. Second, this increase in marginal cost is weaker if debt is better collateralized. As curvature of the convex cost function increases, we approximate models with a collateral constraint, as in Kiyotaki and Moore (1997). Such models typically feature binding constraints and hence fixed leverage in equilibrium. In contrast, in our model the cost schedule allows borrowers to optimally choose leverage. At the same time, no resources are actually lost in the economy, as is the case with a collateral constraint.

The borrowing cost function further serves to describe the role of asset tangibility for borrowing. As an example, consider a quadratic specification

$$k(\ell) = .5\hat{\kappa}(\ell/\phi)^2 \tag{1}$$

Here the parameter ϕ describes how good trees are as collateral, and the parameter $\hat{\kappa}$ scales the cost of leverage after correcting for tangibility. We think of the former as reflecting the characteristics of capital: it is lower when capital consists of ideas that are hard to seize or value after default. In contrast, the latter reflects market features such as how easy it is to enforce contracts in court. The equation clarifies that both forces affect the economy in the same way: in our quantitative analysis below we therefore define a single constant $\kappa = \hat{\kappa}/\phi^2$ that we allow to change over time and estimate from data on leverage and prices.

Budget constraints. The date t budget constraint for an agent of type $i = A, B$ is

$$c_t^i = w_t^i + T_t^i + q_t b_t^i \left(1 - k\left(\ell_t^i\right)\right) - b_{t-1}^i + \left(p_t + d_t\right)\theta_{t-1}^i - p_t\theta_t^i \tag{2}$$

Expenditures on consumption must be equal to available resources, collected on the right hand side. Resources consist of the realized endowment w_t^i of goods, a lump sum transfer T_t^i as well as the change in resources from trading in bond and tree markets. For bond markets, we have the value of bonds sold or bought less net debt b_{t-1}^i that matures in the current period. For borrowers with $\ell_t^i > 0$ we subtract the borrowing cost. For tree owners, we further add net wealth accumulation from participating in the tree market, given by the cum-dividend value of shares brought into the period less the value of shares bought.

Preferences. Agents perceive uncertainty as risk and ambiguity (or Knightian uncertainty). Aversion to ambiguity is described by recursive multiple priors preferences.[4] In particular, the utility a type i agent receives from a consumption plan c_t^i is defined recursively by

$$U_t^i = \log c_t^i + \beta \min_{P^i \in \mathcal{P}_t^i} \mathbb{E}^i \left[U_{t+1}^i\right] \tag{3}$$

[4]See Epstein and Schneider (2003) for axiomatic foundations and Ilut and Schneider (2022) for a discussion of alternative models as well as applications in macroeconomics and finance.

Here $\mathcal{P}_t^i$ is a stochastic process of sets of one-step-ahead conditional probabilities, the (conditional) beliefs of the agent. After any history, the agent evaluates expected continuation utility using the worst case conditional probability from the set that describes beliefs given that history. If the set is a singleton, the recursion (3) defines standard expected utility.

Ambiguity aversion is reflected in non-singleton belief sets: agents do not think in terms of probabilities, and resolve the ambiguity due to multiple beliefs by taking the worst case. The size of $\mathcal{P}_t^i$ describes agent $i's$ lack of confidence in probability assessments: a larger set corresponds to more ambiguity. A larger belief set typically means a lower certainty equivalent for continuation utility that lowers overall utility and induces more cautious behavior. In general, the evolution of the set and its correlation with data describe how perceptions of uncertainty compare to data.

We note two other properties of multiple priors utility that distinguish it from expected utility with risk aversion. First, belief sets can contain probabilities that differ in mean, which leads to first order effects of uncertainty on utility. Second, it is straightforward to allow for ambiguity aversion only towards specific *sources* of uncertainty. In our model, for example, there are three exogenous shocks: the labor income processes w_{t+1}^A and w_{t+1}^B, as well as the tree dividend d_{t+1}. We can thus make an agent's beliefs disagree only about one of the shocks. The interpretation is that the agent particularly worries about that one shock, but is relatively confident about the other shocks. Belief sets may differ across agents' to capture different worries. We provide examples in the next section.

Equilibrium. A competitive equilibrium consists of stochastic processes for consumption, asset holdings as well as tree and bond prices such that agents maximize utility (3) subject to the budget constraints (2) and markets clear. In particular, the resource constraint, or goods market clearing condition, is

$$c_t^A + c_t^B = w_t^A + \overline{w}^B + d_t$$

Market clearing for trees means simply that type B agents own the tree. Finally, bonds are in zero net supply, so bond market clearing is

$$b_t^A + b_t^B = 0.$$

2.2 Characterizing equilibrium

In any equilibrium, individual agent optimization not only implies choice of a consumption plan and asset holdings, but also an optimal worst case belief that supports choice. As a result, choice is observationally equivalent to a model with biased beliefs: agents choose optimal consumption and portfolios at their worst case beliefs. We can exploit this observational equivalence to characterize equilibrium: asset prices satisfy Euler equations under worst case expectations. At this stage, we do not need to take a stand on how the worst case belief is selected: we only postulate that it can be written as a function of past exogenous shocks. We then return to characterization of worst case beliefs for specific preferences below.[5]

Optimality conditions. In any equilibrium with nonzero bond positions, type B, the sole owner of collateral, must be the borrower and type A must be the lender. For type A, we thus expect a standard Euler equation equating the bond price to the marginal rate of substitution of current for future consumption. For type B, the borrowing cost puts a wedge between the marginal rate of substitution and the bond price. Combining first order conditions for bonds, we have

$$q_t = \beta E_t^A \left(\frac{c_t^A}{c_{t+1}^A} \right) = \beta E_t^B \left(\frac{c_t^B}{c_{t+1}^B} \right) + q_t \left[k\left(\ell_t\right) + \ell_t k'\left(\ell_t\right) \right] \tag{4}$$

The last term in type B's first order condition is positive when borrowing costs are present: it reflects the extra marginal borrowing cost of the last dollar of debt. Financial frictions make bonds issued by type B more expensive: relative to the benchmark of the MRS, they increase the price at which bonds are sold and lower the interest rate at which type B borrows. Comparing formulas, for both equalities to hold, we need agent A to be more pessimistic about consumption *under the worst case belief* than agent B.

Intuitively, for an equilibrium with bond trading to exist, agent A's stronger desire for precautionary savings drives up the price at which agent A is willing to hold bonds. Due to this stronger relative demand for safety, agent A becomes the lender in equilibrium, that is, $b_t^A > 0$. The more optimistic agent B is willing to issue debt at the high price q_t agent A is willing to pay. In fact, in the absence of borrowing costs, agent B would want to sell bonds without bound: a high q_t implies a low interest rate that makes borrowing cheap for agent B. The borrowing cost is thus important for equilibrium to exist: when it is present, agent B's marginal cost of borrowing increases with debt until equilibrium is achieved at an interior solution for leverage where both agents are happy and the bond market clears.

Consider now type B agents' choice of trees. Their first order condition implies

$$p_t = \beta E_t^B \left[\frac{c_t^B}{c_{t+1}^B} \left(p_{t+1} + d_{t+1} \right) \right] + \ell_t^2 k'\left(\ell_t\right) p_t \tag{5}$$

[5] We emphasize that worst case beliefs should be interpreted as a tool to describe precautionary behavior under ambiguity, as opposed to a biased distribution of future outcomes. In contrast to an exogenous belief, the endogenous worst case belief changes when there is a change in the environment that alters what the worst case is, and this matters for welfare.

Ambiguity about dividends matters for pricing through the worst-case belief E_t^B. At the same time, the financial friction matters by raising the tree value due to the collateral benefit, given by the last component of equation (5). The collateral benefit is increasing in leverage: at higher value for leverage, collateral is more scarce so the price of trees increases, and the return on trees is lower.

Recursive equilibrium. We summarize equilibrium dynamics by a collection of time invariant functions that depend on a vector of state variables. As usual, those functions describe not only the evolution of endogenous variables, but also agents' expectations about endogenous variables. Importantly, in a model with ambiguity, the latter includes also agents' worst case expectations supporting choice. We make the standard assumption that, while agents disagree about exogenous primitives, they all use the same functions to reason about future endogenous variables. In other words, disagreement about endogenous variables can only come from disagreement about primitives given knowledge of the structure of the economy.

Let x_t denote a vector of state variables that is a sufficient statistic for forecasting future exogenous shocks under both the true data generating process *and every agent's worst case belief.* The only endogenous state variable we consider is type B debt $b_t = -b_t^B$. Since trees can only be held by type B agents, debt is sufficient to describe the wealth distribution. A recursive equilibrium then consists of functions for consumptions $c^i(x, b)$, the debt allocation $b'(x, b)$ as well as prices $q(x, b)$ and $p(x, b)$. The equilibrium functions satisfy the Euler equations (4) and (5) and the budget constraints (2). Since debt is the only connection between the two agent problems, a solution to Euler equations and budget constraints also satisfies the resource constraint by Walras' law.

2.3 Dirac beliefs

We now introduce a set of assumptions that jointly restrict the true data generating process and agents' belief sets. All probabilities are Dirac measures that put probability one on a single realization. Under the true DGP, there are therefore no shocks: all three sources of endowment are constant at, say, $w_t^A = \overline{w}^A$, $w_t^B = \overline{w}^B$ and $d_t = \overline{d}$. Belief sets also contain only Dirac measures. We focus on two concerns with ambiguity: type A agents worry about their own labor income, whereas type B agents worry about dividend income. Otherwise, agents' beliefs coincide with Dirac measures on the true endowments, as they would under the standard assumption of rational expectations. We view this as a minimal way to introduce precautionary savings effects on interest rates as well as uncertainty premia on equity.

Consider first type A agents' worry about labor income. We assume that type A agents entertain an interval of one-period-ahead endowment realizations for labor income

$$\log w_{t+1}^A \in [\log \overline{w}^A - a^w, \log \overline{w}^A + a^w] \tag{6}$$

where $a^w > 0$. Low future labor income w_{t+1}^A has a direct negative impact on future available resources for consumption and asset trading in agent A's budget constraint (2). A natural conjecture is therefore that along the equilibrium consumption path this primitive set of beliefs leads agent A to maximize utility defined in equation (3) under the worst-case belief

$$E_t^A \log w_{t+1}^A = \log \overline{w}^A - a^w. \tag{7}$$

Agent B perceives ambiguity about dividends. In particular, at date t she entertains an interval of one-period ahead dividend realizations

$$\log d_{t+1} \in [\log \overline{d} - a^d, \log \overline{d} + a^d], \tag{8}$$

where $a^d \geq 0$. Echoing the argument for agent A above, lower future dividend income d_{t+1} lowers future available resources for agent B in the budget constraint (2). The natural conjecture is that agent B's worst case belief is always

$$E_t^B \log d_{t+1} = \log \overline{d} - a^d \tag{9}$$

We verify the conjectures for the worst cases below. For now, we note the role of ambiguity parameters in creating gains from trade, or "sharing of ambiguity", as well as ambiguity premia. On the one hand, the parameter a^w makes type A agents behave *as if* labor income falls, whereas type B believes it is constant. This will incentivize type B to borrow from type A who would like precautionary savings. On the other hand, the parameter a^d makes type B agents behave *as if* dividends are too low, and hence value trees at a price below their present value as observed by an outsider such as an econometrician. The econometrician thus records a premium: the realized return is larger than the riskfree interest rate.

Recursive equilibrium. With the parsimonious belief specification used here, we need to keep track of three state variables. In addition to debt, the exogenous state x includes the exogenous endowments w_t^A and d_t about which agents perceive ambiguity. This may be initially surprising, since all components of the endowment are constant under the true data-generating process. However, those state variables are important for describing the evolution of agents' worst case beliefs: since agents entertain worst case beliefs about exogenous primitives that are different from the true constants, agents typically also fear non-constant transition paths of endogenous variables.

Consider a steady state of the economy, that is, an equilibrium such that allocations and prices are constant. In DSGE models with rational expectations, steady state equilibria are found by deriving constant solutions to the functional equation characterizing equilibrium. Such constant solutions typically exist when the exogenous variables are constant. While we also assume constant exogenous variables here, the steady state cannot be found without studying also the dynamics, that is, without characterizing how endogenous variables vary with the state of the economy.

To illustrate, consider the first order conditions for bonds (4). Under rational expectations, the steady state expectation of future consumption equals current consumption, and hence cancels out of the Euler equations, leaving both MRSs equal to β. In contrast, with ambiguity the expectation of future worst case consumption need not be equal current consumption. In fact, for both equalities to hold, we need differences in expected consumption growth across agents. Since agents use the consumption function (in state variables) to forecast future consumption, knowledge of that function is required to discuss bond pricing.

Rather than work with a general form for consumption and other equilibrium functions, we directly move now to log-linear approximations to the equilibrium functions around the steady state. This approach leads to intuitive formulas, and also mirrors our general solution procedure below. We thus represent the logarithm of an endogenous variable by an affine function with intercept equal to the steady state value and elasticities that measure the response to the state variables. For example, we write consumption of type A agents as,

$$\widehat{c}_t^A = \log c_t^A - \log \bar{c}^A = \varepsilon_{wA}^{cA} \widehat{w}_t^A + \varepsilon_d^{cA} \widehat{d}_t + \varepsilon_b^{cA} \widehat{b}_t \tag{10}$$

where hats denote log-deviations from the steady state.

2.4 Steady state asset pricing

Precautionary savings and the risk free rate. With this notation in place, consider bond pricing in the Dirac economy. Ambiguity about income encourages precautionary savings by both agents. Mechanically, agent A, who perceives ambiguity about labor income, acts *as if* consumption will drop from the steady state value $\bar{c}^A$ to a lower value

$$\tilde{c}^A = c^A \left(\overline{w}^A \exp\left(-a^w\right), \overline{d}, \overline{b} \right) \tag{11}$$

In steady state, both the level of consumption and the worst case level next period are constant. Every period, agent A thus behaves *as if* a surprisingly high realization of labor income has just occurred, and consumption needs to be smoothed optimally. This behavior captures precautionary behavior when fearing a worst case labor income realization.

Since agent A perceives ambiguity only about his own labor income w^A and not about other variables, a log-linear approximation to the function $c^A(.)$ as in (10) delivers an approximation to expected consumption $\tilde{c}^A$:

$$\tilde{c}^A \approx \bar{c}^A \left(\overline{w}^A, \overline{d}, \bar{b}\right) \exp\left(-\varepsilon^{cA}_{wA} a^w\right) \tag{12}$$

Agent A expects consumption to fall in proportion to ε^{cA}_{wA}, the elasticity of his *equilibrium* consumption function to the ambiguous labor income. The latter is expected to be lower by a^w and thus future consumption is lower, in logs, by $\varepsilon^{cA}_{wA} a^w$.

At the same time, agent B only perceives ambiguity about dividend income. She thus acts *as if* expecting her consumption to drop from the steady state $\bar{c}^B$ to the lower value

$$\tilde{c}^B = c^B \left(\overline{w}^A, \overline{d} \exp\left(-a^d\right), \bar{b}\right) \approx \bar{c}^B \left(\overline{w}^A, \overline{d}, \bar{b}\right) \exp\left(-\varepsilon^{cB}_{d} a^d\right). \tag{13}$$

where we have now approximated $c^B(.)$. Mirroring the worst-case beliefs of agent A in equations (11) and (12), agent B acts *as if* she is surprised by high dividends and expects her consumption to fall by $\varepsilon^{cB}_{d} a^d$, where ε^{cB}_{d} is the equilibrium elasticity response of her equilibrium consumption function to the ambiguous dividend.

Precautionary savings motives are thus relevant for both agents, with uncertainty working *as if* there is pessimism about future consumption. This pessimism induces an incentive to save to smooth the consumption profile. In equilibrium, the agent with a stronger precautionary motive becomes the saver and the other agent is then willing to provide safe assets. We can therefore have nonzero bond trading even though agents discount the future at the same rate. The coefficients of the loglinear approximation must satisfy

$$\varepsilon^{cA}_{wA} a^w > \varepsilon^{cB}_{d} a^d. \tag{14}$$

Substituting the formula for agent A's future consumption (12) into the Euler equation (4), the steady state bond price is $\bar{q} = \beta \exp\left(\varepsilon^{cA}_{wA} a^w\right)$.

We obtain the steady state net interest rate

$$r^f = \bar{q}^{-1} - 1 \approx -\log\beta - \varepsilon^{cA}_{wA} a^w. \tag{15}$$

More ambiguity, in the form of a larger a^w, lowers the interest rate, as agent A is willing to pay an even larger bond price to achieve more stability in his consumption profile. While such behavior is similar to what would happen with a higher discount factor, we emphasize that it is *endogenous*, in the sense that the equilibrium response ε_{wA}^{cA} matters for the size of the effect. With sufficiently large ambiguity the interest rate can turn negative in steady state even if $\beta < 1$.[6]

Equilibrium leverage. Comparing Euler equations for types A and B clarifies how the equilibrium volume of credit depends on heterogeneity among the agent types. To see this, observe first that the bond price and interest rate must also satisfy the type B Euler equation (4). Substituting our approximation for agent B's future consumption, we obtain

$$r^f \approx -\log\beta - \varepsilon_d^{cB} a^d - \left(k(\bar{\ell}) + \bar{\ell}k'(\bar{\ell})\right). \tag{16}$$

As a borrower, agent B needs to be compensated in equilibrium for the higher cost of borrowing, given by $k(\bar{\ell}) + \bar{\ell}k'(\bar{\ell})$, by paying a lower interest rate on debt. Since this cost increases with leverage, a higher equilibrium $\bar{\ell}$ lowers r^f.

Combining the two equations for agents' intertemporal tradeoffs, we substitute r^f from equation (15) into (16) to express the equilibrium marginal cost of borrowing as

$$k(\bar{\ell}) + \bar{\ell}k'(\bar{\ell}) = \varepsilon_{wA}^{cA} a^w - \varepsilon_d^{cB} a^d. \tag{17}$$

The right hand side can be interpreted as the equilibrium marginal gain from uncertainty sharing: it is the difference between the worst-case belief over consumption of agent A and that of agent B. Through equilibrium elasticities ε_{wA}^{cA} and ε_d^{cB}, this gain from trade is endogenous to the environment. If condition (14) holds, it is positive, so agent B is willing to supply safe assets to agent A, leveraging up to the equilibrium marginal borrowing cost.

Premia on trees. We now turn to the equilibrium tree price. Loglinearizing type B's Euler equation for tree holdings (5), we obtain

$$r^{tree} = \bar{d}/\bar{p} \approx -\log\beta + (\varepsilon_d^P - \varepsilon_d^{cB})a^d - \bar{\ell}^2 k'(\bar{\ell}), \tag{18}$$

where ε_d^p is the elasticity if the equilibrium tree price with respect to dividends. Agent B values the tree under a worst case belief of low dividends. As for bond pricing by agent A above, ambiguity enters via its effect on worst case consumption. Since the tree is a long-lived asset, however, ambiguity also enters via the worst case price next period.

The tree return (18) thus consists of three components. Two are familiar from other

[6]In addition, curvature in utility matters. A lower intertemporal elasticity of substitution, equal here with log utility to one, means that the agent is less willing to accept an intertemporally volatile consumption profile and thus is willing to accept an even lower interest rate to smooth his consumption.

linearized asset pricing models with financial frictions. The first term is standard discounting: higher β increases the present discounted value of expected dividends and thus the tree price, so the return on the tree is lower. The last term in equation (18) captures the collateral value of the tree. Borrower types B benefit from the tree not only because of its payoff, but also because it lowers the tax rate on debt. In other words, the tree earns a convenience yield – other things equal, borrowers are willing to hold it even if provides a lower return than what the MRS implies. Moreover, the convenience yield is increasing in leverage $\bar{\ell}$.

The middle term is a unique feature of our model: an uncertainty premium in a linear asset pricing equation. Its size is driven by competing wealth and substitution effects of uncertainty. Indeed, more ambiguity over dividends (higher a^d) works like pessimism about future dividends. On the one hand, more ambiguity increases pessimism about the future equilibrium price, because of the elasticity $\varepsilon_d^p > 0$. This pessimism lowers the current price and increases the return r^{tree} required as compensation for holding the tree in equilibrium. On the other hand, consumption is also exposed to dividends via $\varepsilon_d^{cB} > 0$. More ambiguity a^d thus raises future marginal utility used to value the tree's future payoff, raising $\bar{p}$ and lowering r^{tree}. The net pricing effect of ambiguity is thus proportional to $(\varepsilon_d^p - \varepsilon_d^{cB})$.

The difference between the tree and bond returns gives rise to a premium

$$\bar{pr} \equiv r^{tree} - r^f = \underbrace{k(\bar{\ell}) + \bar{\ell}k'(\bar{\ell}) - \bar{\ell}^2 k'(\bar{\ell})}_{\substack{\text{frictional} \\ \text{premium}}} + \underbrace{\varepsilon_d^P a^d}_{\substack{\text{uncertainty} \\ \text{premium}}} \tag{19}$$

This excess return is the equilibrium financial return to agent B to a strategy of borrowing one unit of the risk free bond and investing it as an additional share in the tree. We decompose the premium in equation (19) as a frictional and uncertainty premium.

Consider the frictional premium. The term $k(\bar{\ell})+\bar{\ell}k'(\bar{\ell})$ is the additional marginal cost of borrowing that such a strategy entails, while the term $\bar{\ell}^2 k'(\bar{\ell})$ is the marginal collateral value of owning a tree share. The frictional premium is the equilibrium compensation arising from the financial friction, as the difference between this collateral cost and benefit. At low leverage $\bar{\ell}$, the frictional premium is positive and increasing with leverage for general functional forms of $k(\ell)$.[7]

The uncertainty premium is the equilibrium compensation of facing ambiguity about dividends when investing in the tree compared to the bond. Agent B's *as if* pessimism about her consumption induced by ambiguity about dividends lowers both r^{tree} and r^f by the same amount $\varepsilon_d^{cB} a^d$. A strategy of borrowing risk free and investing in the tree is thus

[7]Imposing only the properties that $k(\ell), k'(\ell), k''(\ell) > 0$ for $\ell > 0$, a sufficient condition for the frictional premium to be positive and increasing in $\bar{\ell}$ is that $\bar{\ell} < 0.5$, a condition met in our model's empirical evaluation.

not affected by pessimism about consumption. Instead, the *as if* pessimism about the future tree price induced by ambiguity about dividends, i.e. the term $\varepsilon_d^p a^d$, increases r^{tree} compared to r^f and is thus the uncertainty-driven source of the excess return in (19).

Welfare costs of uncertainty. Since ambiguity has first order effects on welfare, welfare costs of uncertainty are present in steady state and easily computed via a first order approximation to the equilibrium value function. Let $\bar{U}^i$ denote agent i's steady state utility. We are interested in computing the constant certainty equivalent level of consumption $\log \bar{c}^{i,det} = (1-\beta)\bar{U}^i$. Agent i is indifferent between receiving $\bar{c}^{i,det}$ for sure forever and the ambiguous equilibrium consumption plan. The difference between actual steady state consumption $\bar{c}^A$ and counterfactual consumption $\bar{c}^{A,det}$ thus tells us how much consumption agent A would be willing to give up in order not to endure any ambiguity.

Consider the welfare of agent A who evaluates the future *as if* he expects the value function to drop from its steady state $\bar{U}^A$ to

$$\tilde{U}^A \approx \bar{U}^A\left(\overline{w}^A, \overline{d}, \bar{b}\right) \exp\left(-\varepsilon_{wA}^{UA} a^w\right) \tag{20}$$

The sign of the endogenously determined elasticity ε_{wA}^{UA} of the function $U^A(.)$ with respect to the labor income innovation w^A is particularly useful to assess the worst-case conjectured initially in section 2.3. In particular, a positive value for ε_{wA}^{UA}, which we do find in all of our numerical exercises below, verifies the worst-case belief of low mean for w_{t+1}^A in equation (7).

From the Bellman equation (3), the steady state value function for type A is then

$$\bar{U}^A = \log \bar{c}^A + \beta \tilde{U}^A = \frac{\log \bar{c}^A}{1 - \beta \exp\left(-\varepsilon_{wA}^{UA} a^w\right)} \tag{21}$$

where the second equality follows by substituting the worst-case expected value $\tilde{U}^A$ from (20). The resulting representation shows how welfare decreases when there is more ambiguity over labor income (i.e. a higher a^w) and when, for a given a^w, the value function endogenously depends more on that uncertain labor income through ε_{wA}^{UA}. Analogous expressions are available for agent B.

2.5 Solution method

The basic idea behind our computational approach is that an approximation based on a first order perturbation remains possible if we solve jointly for the steady state and the transition dynamics, described by an approximating law of motion for the endogenous variables. Suppose we start from a candidate steady state and law of motion. We can compute agents' one-step-ahead worst case forecasts in steady state and thus find a set of steady state actions, using the formulas from Section 2.4 above. Perturbing the system characterizing equilibrium around this new steady state, we obtain a new law of motion. The true steady state and law of motion is a fixed point of this operation. The individual steps of the algorithm can be implemented using standard tools – the main additional task is to loop over candidate law of motions.

We leave the detailed description of our method to the Appendix. In particular, Section A.1 in the Appendix presents the approach in the context of the simple model introduced above. More generally, Section A describes the method for a general class of linear recursive models with long-run heterogeneity in beliefs. There we also note the observational equivalence of the heterogeneous ambiguity model with an expected utility (EU) model, given the worst-case beliefs. Once we focus on ambiguity about conditional means, we develop an approach to solve for this EU model with disagreement using log-linearization. In the heterogeneous ambiguity model we also need to verify the conjectured worst-case belief by analyzing the equilibrium value functions. This step is not required for EU models, which impose dogmatic disagreement, but still can be solved using our solution approach, and thus are of independent interest for that literature. Once the linear policy functions are determined, the equilibrium dynamics are found by simulating under the true DGP.

2.6 Model extensions

We build towards our quantitative evaluation by introducing three new ingredients: a government, trend growth as well as persistent shocks, and long term debt. From now on, we also allow for different (constant) population weights of the two agent types – we normalize total population to one and define n^A as the population share of type A agents.

Government. We introduce a one-period risk-free asset issued by the government. In equilibrium, type A agents can therefore save by holding government debt b_t^g in addition to private debt b_t introduced in Section 2. Government debt is set constant at b^g. Service on government debt is funded by lump sum taxes levied on both agents, with a share s^A paid by type A. We also introduce a proportional income tax applied equally to labor and dividend income that funds government spending. The relatively simple government structure assumed here is meant to capture the steady state effects of taxes on asset returns as well as redistribution due to tax-financed debt service.

The budget constraints for types A and B then become

$$\begin{aligned} n^A c_t^A &= (1-\tau) n^A w_t^A + b_{t-1} - q_t b_t + q_t b_t^k (\ell_t) + (1-s^A) b^g (1-q_t) \qquad (22) \\ (1-n^A) c_t^B &= (1-\tau) \left[(1-n^A) w_t^B + d_t \right] + q_t b_t^p (1 - k(\ell_t)) - b_{t-1}^p - (1-s^A) b^g (1-q_t) \end{aligned}$$

In the first equation, we have already substituted here for the lump sum transfers taken as given by agent A: the financial cost paid by the lender plus the net payment from the interest on government debt. The latter is the interest rate received on the held government debt minus the lump-sum payment borne by agent A to finance a share s^A of that government debt service. Equations (22) further indicate that, compared to the baseline of $b^g = 0$, introducing positive government debt acts as a zero-sum redistribution of lump-sum transfers across the two agents, proportional to the equilibrium risk free and the share $1 - s^A$.

Growth and persistence in exogenous shocks. For our calibration, we allow for trend growth. Moreover, we let agents worry about persistent deviations from trend. As the true DGP for the exogenous income variables, we assume an AR(1) processes with constant drift

$$\begin{aligned} \log w_t^i &= g + \rho \log w_{t-1}^i + (1-\rho) \log \overline{w}^i + \nu_{w,t}^i;\ \text{i = A,B} \\ \log d_t &= g + \rho \log d_{t-1} + (1-\rho) \log \overline{d} + \nu_{d,t} \end{aligned} \tag{23}$$

where g is a constant long run growth rate, $\rho \in [0,1)$, and the innovations $\nu_{w,t}^i$ and $\nu_{d,t}$ have mean zero and are mutually uncorrelated as well as iid over time.

Ambiguity perceived by agents A and B is now captured by sets of Dirac priors for the one-step ahead *innovations*: we have $\nu_{w,t+1}^A \in [-a^w, a^w]$ and $\nu_{d,t+1} \in [-a^d, a^d]$, respectively. At the same time, agents are confident about the new parameters ρ and g. The range of possible one-step-ahead incomes or dividends then follows from (23) by substituting all possible innovations. They reduce to our earlier specification (6)-(8) when there is no growth and shocks are iid (that is, $g = \rho = 0$). Computation of equilibrium thus proceeds as before, except that linearization is around a balanced growth path, and the true steady state values of the income variables take persistence into account.

Persistence matters because it increases the effect of uncertainty about an innovation on utility and actions. The intuition is familiar from models of risk. In our model, we can see it by inspecting the linearization, and in particular, the elasticities of equilibrium objects, such as consumption or the tree price, with respect to the ambiguous innovations. Consider for example the interest rate equation (15). It continues to hold exactly as long as growth g is zero. However, the elasticity ε_{wA}^{cA} of consumption with respect to the ambiguous labor income innovation ν_w^A is larger when labor income w^A is more persistent. As a result, an increase in ambiguity about the innovation lowers the interest rate by more.

Long term bonds. We finally extend our model to incorporate long term riskfree bonds. This step is useful to motivate our calibration to fixed income positions below and to use long term bond prices to discipline agents' perception of the persistence parameter ρ. We introduce zero coupon bonds of arbitrary maturity. We denote the date t price of a bond that pays one unit of consumption at date $t+n$ by $q_t^{(n)}$. As before, type A agents hold bonds issued by type B agents subject to the borrowing cost. Leverage ℓ_t is still the value of outstanding bonds relative to the value of tree holdings - the only additional detail now is that we sum over outstanding bonds of all maturities.

In equilibrium, bond prices satisfy both types' Euler equations, generalizing equation (4) above. We define $q_t^0 = 1$ for all t to write bond prices recursively as

$$q_t^{(j)} = \beta E_t^A \left(\frac{c_t^A}{c_{t+1}^A} q_{t+1}^{(j-1)} \right) = \beta E_t^B \left(\frac{c_t^B}{c_{t+1}^B} q_{t+1}^{(j-1)} \right) + q_t^{(j)} \left[k\left(\ell_t\right) + \ell_t k'\left(\ell_t\right) \right]$$

Equilibrium further requires that bond markets for all maturities clear. In steady state, returns are deterministic so all bonds are perfect substitutes from the perspective of both agents. We can thus still work with one state variable, total private debt b_t, and one market clearing condition.

In steady state, the optimality condition for agent A implies

$$\overline{q}^{(j)} = \beta \exp\left(a^w \varepsilon_{wA}^{cA}\right) \overline{q}^{(j-1)} \exp\left(-\varepsilon_{wA}^{q^{(j-1)}} a^w\right) = \overline{q}^{(1)} \overline{q}^{(j-1)} \exp\left(-\varepsilon_{wA}^{q^{(j-1)}} a^w\right) \tag{24}$$

where $\overline{q}^{(j)}$ is the steady state bond price of maturity j. For short term debt with $j = 1$ we reproduce equation (15): the bond price reflects only discounting and worry about consumption growth one period ahead. For longer term bonds, the payoff – that is, the price next period – is also uncertain. This uncertainty is captured by the third term in equation (24): under the equilibrium worst-case belief, the agent expects the bond's future payoff to be $\overline{q}^{(j-1)} \exp\left(-\varepsilon_{wA}^{q^{(j-1)}} a^w\right)$. Here the elasticity is the equilibrium response of the next period's bond price $q_t^{(j-1)}$ with respect to the ambiguous labor income shock w_t^A.

The steady state pricing equation implies an ambiguity premium on long term bonds of maturity $j > 1$. To see this, define the steady state return from holding a j period bond for one period as $R_j \equiv \log(\overline{q}^{(j-1)}/\overline{q}^{(j)})$. By equation (24), its excess return over a short (one period) bond with return equal to the riskfree rate is

$$R_j - r^f = \log(\overline{q}^{(j-1)}/\overline{q}^{(j)}) - \log(1/\overline{q}^{(1)}) = \varepsilon_{wA}^{q^{(j-1)}} a^w. \tag{25}$$

The sign of the excess return follows from the equilibrium elasticity $\varepsilon_{wA}^{q^{(j-1)}}$. Consider for example a two period bond. A positive income shock increases the desire to save for agent A wanting to smooth consumption, and hence pushes up the one period bond price. We therefore expect that $\varepsilon_{wA}^{q^{(1)}}$ is generally positive and the ambiguity premium is positive.

Equation (24) further describes the entire equilibrium yield curve, using the standard definition for the j-period interest rate $r_t^{(j)} = -\log q_t^{(j)}/j$. There is a close relationship between the sign of uncertainty premia on holding period returns and term spreads. A sequence of positive elasticities $\varepsilon_{wA}^{q^{(j-1)}}$ thus not only generates a sequence of positive premia on long bonds, but also an upward sloping yield curve. Intuitively, an agent who worries about future low income also worries, in equilibrium, about future low short-term bond prices or higher short term interest rates. To compensate for this worry, long term bond prices are lower and long term interest rates higher, as are excess returns on long term bonds.

In equation (25) the excess return is increasing in the ambiguity parameter a^w and the elasticities $\varepsilon_{wA}^{q^{(j-1)}}$. For the latter, the persistence ρ of the labor income shock matters. In particular, holding all else constant, this positive elasticity is larger when the shock is more transitory in nature. Intuitively, in that case, the desire for consumption smoothing in stronger, putting more equilibrium pressure on the shorter-term bond prices to rise. This larger exposure of bond prices to more transitory shocks thus leads to a steeper profile for elasticities $\varepsilon_{wA}^{q^{(j-1)}}$ and thus an upward sloping curve of returns R_j. Conversely, the yield curve would be flatter when the shock persistence ρ is larger.

The pricing equation (24) also clarifies why long term bond prices help identify perceived persistence. Start from the case $j = 1$ and fix a pair of parameters (ρ, a^w). The same equilibrium one period bond price $\overline{q}^{(1)} = \beta \exp\left(\varepsilon_{wA}^{cA} a^w\right)$ can also be generated by (i) higher persistence $\tilde{\rho} > \rho$, which would increase effective caution ε_{wA}^{cA}, and (ii) an appropriately *lower* ambiguity parameter $\tilde{a}^w < a^w$. The yield curve can be used to break that equivalence. Indeed, long prices also depend on ambiguity as well as on ρ through the elasticities $\varepsilon_{wA}^{q^{(j)}}$. They thus provide additional restrictions that help select ρ and a^w. We exploit this feature in our quantitative evaluation below.

3 Quantitative analysis

In this section, we first describe the data we use to calibrate our model and present a number of stylized facts on debt and leverage in US financial markets. In particular, we document a trend of declining leverage that was only temporarily disrupted by the housing boom of the early 2000s. Moreover, we show that equity holdings are highly concentrated and indirect debt via ownership of businesses is an important component of borrowing and lending between households who own claims to businesses and those who do not.

Together, these facts motivate our approach to compare two episodes, anchored by benchmark years 1989 and 2016, and to work with two broad classes of agents corresponding to savers (type A) and borrowers (type B). We then explain our calibration and in particular present estimates for the key parameters describing preferences and financial frictions.

3.1 Data and stylized facts

We draw mainly on two data sources. The 1989 and 2016 waves of the Survey of Consumer Finances (SCF) provide micro data on household balance sheets. SCF balance sheet items consist of financial products purchased by households, many of which are supplied by intermediaries and not by the ultimate borrowers such as firms or the government. We thus use the Financial Accounts of the United States (FAUS) to "see through" the financial system and get at links between ultimate borrowers and lenders that our theory speaks to.

Defining assets. We divide household financial assets into *business equity* and *debt*. Business equity includes (i) equity in publicly traded firms, held directly or through an investment intermediary such as a mutual fund or a defined-contribution pension plan, (ii) stakes in private businesses, including closely held shares, proprietorships or stakes in partnerships and (iii) investment property. We label all other financial assets as debt: this category includes direct and indirect ownership of bonds as well as other fixed income products such as loans, bank deposits and life insurance policies.

Appendix B summarizes three preliminary steps that we have taken in our quantitative strategy, in particular to map financial positions in our model to the data. First, the credit market in our model describes net borrowing by shareholders from debtholders. We view the financial system as a technology to implement such borrowing. We thus consolidate different intermediaries to derive a measure of total debt that reflects net credit between domestic nonfinancial borrowers and lenders and hence corresponds to total debt in our model. Second, we report additional evidence to show that the trend in leverage we emphasize is robust to alternative debt definitions. Finally, we explain how we treat owner-occupied housing and the foreign sector, two data features that our model abstracts from.

Aggregate positions. Figure 1 presents aggregate statistics for key variables in our model. We identify $q_t b_t^p$ with net private debt held by households, or equivalently net debt of businesses. We measure $q_t b_t^g$ as government debt held by households and define the value of the tree p_t as the sum of net private debt and equity held by households. Figure 1 reports these variables as multiples of GDP to take out economic growth. The top left panel shows total asset value and private debt issued against that value. The top right panel has the resulting leverage ratio. The bottom left panel computes a "safe asset share", that is, the ratio of bonds to all assets in nonzero net supply $p_t + q_t b_t^g$, with government debt broken out separately.

The figure illustrates the key facts that our model needs to capture and how they are part of a broader trend in the US economy. As risky assets increased in value, the quantity of debt did not keep pace. As a result, leverage declined. While there is some cyclicality in leverage with upticks in recessions when the market value of capital falls, the downward

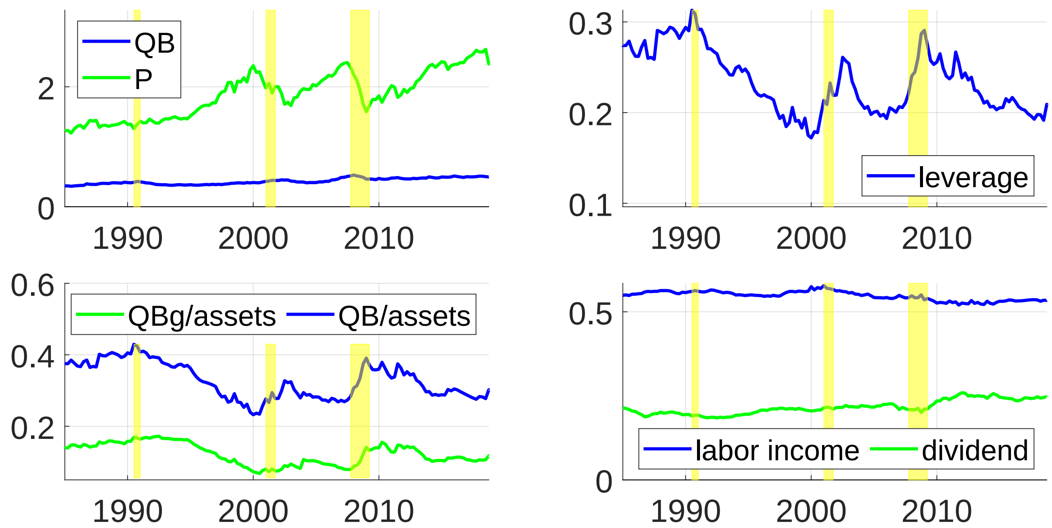

Figure 1: Aggregate statistics that correspond to model inputs. Top left: value of equity plus private debt as share of GDP. Top right: private debt as a ratio of value of equity plus private debt. Bottom left: US public debt and total (public plus private) debt as a share of value of equity plus total debt. Bottom right: labor income and dividends as a share of GDP. Source: FAUS.

trend is very clear. The same force generates a downward trend in the safe asset share. The decline is much more pronounced in the private component of safe assets, although the share of safe government assets also fell. What matters here is that a good chunk of increased US government debt was absorbed by foreigners.

Household heterogeneity. We now turn to household heterogeneity and how we discretize the distribution of agents. In the SCF, we observe household wealth portfolios as well as income. A key difference to the FAUS is that we cannot distinguish between public and private debt.[8] We thus aggregate all bond holdings in the SCF and subtract non-mortgage debt to arrive at an overall bond position. Other relevant positions – equity, mortgages backed by the primary residence and other household debt – can be defined in the same way as in the FAUS. We then define net worth as the sum of equity and bond holdings.

Figure 2 illustrates the key fact that motivates our approach: equity holdings, and hence indirect borrowing via business debt is highly concentrated. The left (right) panel shows statistics for 1989 (2016). In each year, we first sort all SCF households by the ratio of equity to assets. We then group them into bins indicated by the steps in the lines: the leftmost bin contains the large share of households without any equity, and the rightmost bins represent deciles. The equity-asset ratio used to select bins in plotted as a black solid

[8]While it is possible to observe direct holdings of bonds by issuer, say Treasury bonds held directly by a household, we are interested also in indirect holdings of Treasuries via intermediaries, which are large.

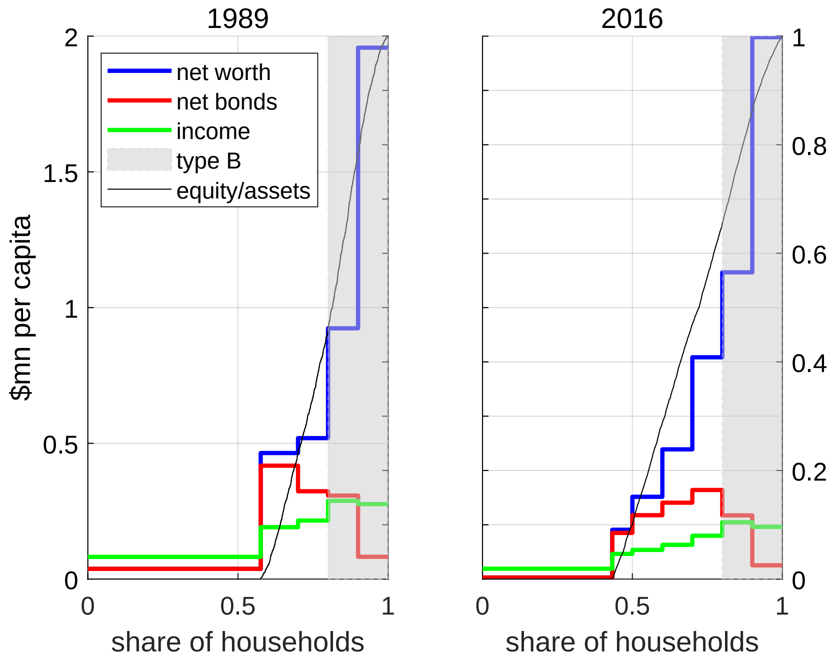

Figure 2: Wealth positions and income for households sorted by equity/assets. Left panel 1989 wave, right panel 2016 wave of SCF. Horizontal axis shows percentile of households in distribution of equity/assets ratio. Solid black line (right axis) shows value of equity/assets ratio. Colored lines show per household net worth, net fixed income position and income as a function of percentile, where distribution has been discretized to a finite number of bins. Shaded area indicates top 20% of households by equity/assets who represent type B agent in the model.

line, with values on the right axis. The step functions represent per capita net worth, net bonds and income, with value in 2016 dollars on the left axis.

The figure reflects the familiar fact that wealth is much more concentrated than income. Moreover, it shows that portfolio composition changes dramatically with wealth. While poor and even relatively wealthy households in the middle of the figure hold mostly bonds, the wealthy households in the right tail are almost entirely invested in equity. Across years, concentration has remained strong even if equity participation has somewhat broadened. We define agent B as the top 20% of households by the equity-asset ratio, represented by the gray shaded area in the figure. The threshold equity-asset ratio separating types is 64% for 2016 and 46% for 1989.

Both panels indicate that our stark two-type distribution does a decent job discretizing the continuous distribution in the data. Our model assumes that only type A holds bonds and only type B holds equity. Mismatch is therefore captured by the area under the red function in the shaded area (net bonds of type B) as well as the area between the blue and red functions outside the shaded area (equity held by type A). In both years, those areas are relatively small relative to outstanding debt and equity, respectively, where the latter is measured by the area between the blue and red lines.

Since we would like to work with correct aggregates but two types, we reassign asset position to create "clean" versions of type A and B. At the same time, we would like to respect the original numbers for per capita income as well as the correlation of income with asset positions in the cross section of households. Our correction envisions that per capita equity of type B as well as population weights are correctly measured, but some type B agents were mis-classified as type A and vice versa. We find the number of mis-classified agents by dividing type A equity holdings by type B per capita equity holdings, and adjust group incomes accordingly. The correction makes type B agents on average about 10% richer, but has a small effect on our main conclusions.

3.2 Quantifying the model

The model period is a year. Our quantitative exercise selects two sets of parameters, one for 1989 and one for 2016, so the steady state of our model matches a set of moments for the respective years, presented in Table 1. We distinguish two types of moments: the top panel groups endogenous variables – the riskfree rate, leverage and the premium on the tree – whereas the middle panel collects moments for exogenous variables that directly identify particular parameters. All quantities are measured as shares of output, which we define as national income less corporate tax less tariffs from the FAUS.

Moments. We construct moments by combining numbers from the SCF and FAUS. To measure the leverage ratio, we start from equity in the SCF that reports a value of 45 trn in 2016 and 23 trn in 1989 (in 2016 dollars). In both years, the value of private businesses amounts to about one half – it stood at \$19trn in 2016 and \$12trn in 1989. We then measure the debt-equity ratio in FAUS as

$$DE = \frac{\text{net private debt of financial \& nonfinancial business}}{\text{combined equity of financial and nonfinancial business}}.$$

Here the numerator incorporates the correction for mortgages described above.[9] The debt-equity ratio was .41 in 1989 and .29 in 2016.

We define private bonds qb^p held by households as equity in SCF multiplied by DE and define the value of the tree p as the sum of equity and qb^p. Implied leverage declined from 29% in 1989 to 23% in 2016, as shown in Table 1. We measure public bonds qb^g held by

[9] Mechanically, the mortgage correction somewhat increases leverage, as it lowers the bond holdings of business. But this does not affect any of our main conclusions, since the mortgage share overall is small.

households as public bonds held by households in FAUS. We thus have total bond holdings of \$19trn in 2016 (of which are \$6trn are public) and \$14trn of bonds in 1989 (of which \$5trn are public). The bond totals inferred in this way are slightly larger than bond totals in the SCF, which were \$16trn in 2016 and \$12trn in 1989. This type of discrepancy is known to exist between SCF and FAUS aggregates because of differences in measurement via survey versus regulatory data.

We construct a measure of labor income for each type of agent from the SCF. It contains wages and two corrections. The first is for owner-occupied housing: we compute housing equity as the difference between the value of the primary residence and the mortgage balance, and convert to flows by applying a 5% discount rate. Results are not sensitive to the exact discount rate since this correction is not large relative to income for most households. Our second correction accounts for labor compensation for private business owners, where we use proprietors' income from the BLS.

Real interest rates are difficult to measure, especially before the introduction of real Treasury bonds in the US. We work with a rate of 3.75% for 1989 and -.5% for 2016. For comparison, during the first half of 1989 when the SCF was administered, the Cleveland Fed's measure of the one year real interest rate increased from 3.2% to 4.5%. During the first half of 2016, that measure hovered between +.2% and -1.2%. With a real interest rate in hand, we can compute the premium on trees by calculating the steady state dividend yield and using the growth rate to derive the tree return. Subtracting the real rate yields a 6% premium in 1989 that increased slightly to 7.4% in 2016.

Table 1: Moments and calibrated parameters

			Period 1 (1989)		Period 2 (2016)	
			Data	Model	Data	Model
Moments	Risk free rate %	r^f	3.7	3.6	-.5	-.5
	Leverage %	$\bar{\ell}$	29	29	23	23
	Premium %	$\bar{pr}$	6	6	7.4	7.4
Direct inputs	Dividend share %	$\tilde{d}$	23		26	
	Labor share A %	$\tilde{w}^A$	56		51	
	Gov't debt %	b^g	35		40	
Calibrated parameters	Discount factor	β		.94		.94
	Leverage cost	κ		.08		.97
	Agent A ambiguity	a^w		.06		.11
	Dividend ambiguity	a^d		.07		.00

Parameters. Among parameters that vary over time, some are *direct inputs*, that is, parameters we can measure independently of the model structure. Direct inputs are collected in the middle panel of Table 1: they consist of the level of government debt (b^g), as well as the ratio of agent A income ($\tilde{w}^A$) and dividends ($\tilde{d}$) to total output.[10] Another set of parameters is set up front to constant values. We set the proportional tax rate τ to a standard value of 25% and let the share s^A of debt service paid by agent A simply equal the labor share $\tilde{w}^A$. In addition, we set the growth rate g equal to 2% per year and fix the persistence parameter ρ at .95. The results that follow are not sensitive to this choice as long as persistence is close to one, a common assumption for aggregate shocks in many macro models. Moreover, we show below that this choice results in a plausible term structure of interest rates.

The remaining parameters are listed in the bottom panel of Table 1. We specify preferences, by choosing a discount factor β that is common to the 1989 and 2016 episodes, as well as type A and B ambiguity parameters a^w and a^d, respectively, that are allowed to differ. We further work with a quadratic borrowing cost function $k(\ell) = 0.5\kappa\ell^2$ for which we choose a time-dependent scale parameter κ. Altogether we thus look for 7 parameters in order to match 6 moments. However, all parameters are constrained to be positive, and the nonnegativity constraint on type B's ambiguity about dividends a^d is binding in 2016. As a result, our model is just identified with 6 parameters adjusting to hit 6 moments.

Parameter values for the two episodes are reported in the second and third column of Table 1. A discount factor of .94 is relatively low for an annual calibration. We emphasize, however, that in our model uncertainty plays an important role in pushing interest rates down. Indeed, consider agents of type A, for whom the standard Euler equation holds: in 1989, they behaved *as if* income was going to drop by 6% over the next year, while in 2016, they behaved *as if* they feared an even larger decline of 11%. For type B agents, the evolution of ambiguity is the opposite: while their 1989 behavior reflected a 7% drop in dividends, this fear was gone in 2016.

The leverage cost parameter κ describes credit market frictions captured by a tax on borrowing. For the 1989 episode, a period of historically high business leverage, the implied average tax rate is 35bp. For the 2016 episode, characterized by substantially lower leverage, the implied tax rate is 2.5%, a large increase. In light of equation (1), we can also interpret this change as a shift in asset tangibility, captured by the parameter ϕ. If we assume the borrowing cost parameter $\hat{\kappa}$ has stayed fixed, the increase in tax rate is equivalent to a 63% drop in the tangible share ϕ.

[10]The labor share of agent A is defined as $\tilde{w}^A \equiv n^A\overline{w}^A/\overline{y}$, and dividend share $\tilde{d} \equiv \overline{d}/\overline{y}$, where total output $\overline{y}$ equals $n^A\overline{w}^A + (1-n^A)\overline{w}^B + \overline{d}$.

3.3 Evidence on trends in frictions and uncertainty

We recover a three-part narrative: (1) a higher leverage cost, (2) lower ambiguity perceived by capital owners over capital income, and (3) higher ambiguity perceived by savers over labor income. We conclude this section by briefly reviewing evidence from outside the model.

Consider the higher estimated leverage parameter κ. By equation (1), it can be interpreted as a combination of a lower tangible share ϕ or a higher scale factor on the borrowing cost $\hat{\kappa}$, that is, costlier leverage holding fixed collateral values. The literature has described trends in credit markets that fit both parameter shifts.

First, there has been a secular change in the composition of firm assets, away from physical capital towards intangibles and the present value of rents. In particular, recent work such Crouzet and Eberly (2019), Crouzet et al. (2022) or Falato et al. (2022) document a fall in the asset share of tangibles of roughly 40% over the last 30 years.[11]

Our inferred higher borrowing cost in 2016 is also consistent with a tougher regulatory regime. Since intermediation is part of the technology through which borrowing and lending between groups of households occurs, regulation that discourages leverage is captured in our model as an increase in this borrowing cost $\hat{\kappa}$. Such regulation has been a major focus of reform, especially after the Global Financial Crisis. Examples are rules based on the Basel III leverage ratio or the U.S. supplementary leverage ratio - see Duffie (2018) for a discussion. At the same time, a higher marginal cost of leverage is consistent with higher frictional premia on returns to other trading strategies, not simply equity. In particular, recent literature has described positive excess returns on otherwise riskfree trading strategies, including the covered-interest parity, bond-CDS, or swaps spreads (see for example Boyarchenko et al. (2018), Avdjiev et al. (2019) and Du et al. (2023)).

Consider next supporting evidence for a secular decline in ambiguity over capital income. We construct a measure of this ambiguity using dispersion of corporate profits forecast in the Survey of Professional Forecasters. Forecast dispersion has been used in various contexts as a proxy for ambiguity (eg. Ilut and Schneider (2014), Ilut and Saijo (2021) or Masolo and Monti (2021)). The idea is that decision makers – here capital owners, that is, ambiguity-averse agent of type B – sample experts' forecasts of capital income before making decisions. Stronger disagreement among experts leads to lower confidence in probability assessments on the part of the capital owner. Formally, in our model type B's beliefs are parameterized by an interval of one-step conditional mean of capital income. If this interval is increasing in, say, the interquartile range (IQR) of expert forecasts of corporate profit growth, then

[11] Intangible capital typically includes components emphasized by the literature on innovation (eg. Corrado et al. (2009), Corrado and Hulten (2010)), as knowledge, organizational and informational capital. This intangible capital measurement usually relies on proxies such as R&D, SG&A or computerized expenditures.

more disagreement among experts indicates more ambiguity perceived by agent B.

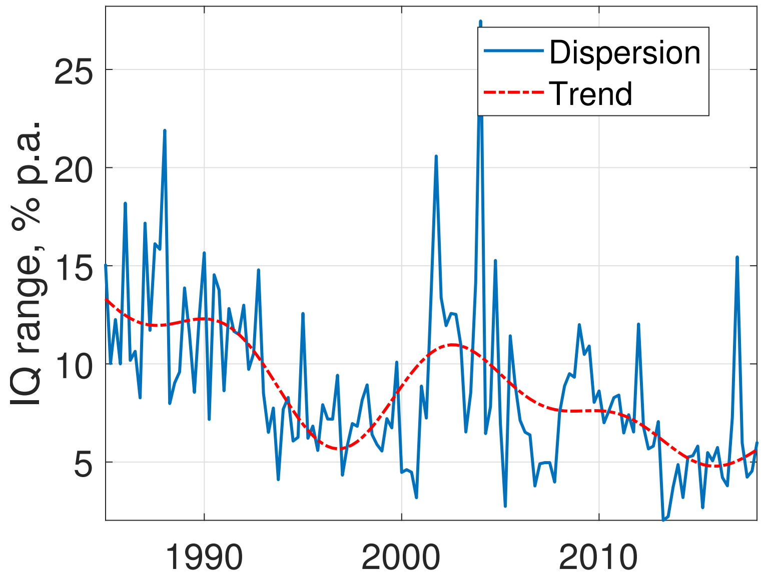

Figure 3: Cross-sectional dispersion of quarterly forecasts for nominal corporate profit, after taxes, from Survey of Professional Forecasters. Units: Annualized percentage points. Dispersion measure is the 75th percentile minus 25th percentile of the forecasts for Q/Q growth. Blue solid line is raw data. Dotted red line is the trend, defined as fluctuations with a frequency lower than 8 years.

Figure 3 shows this quarterly IQR measure for the period 1985-2018. The dotted red line adds a trend component, defined as fluctuations occurring with frequencies lower than 8 years. The stylized fact that emerges is a significant decline in trend dispersion between the 1980s and the 2010s.[12] In particular, the red trend line dropped from a IQR value of 12.2% in 1989 to 4.9% in 2016.

Consider now the last ingredient of the model's recovered narrative, namely the increase in the ambiguity perceived by savers over their labor income. The above approach of using disagreement of experts' opinions to proxy for ambiguity does not readily apply in this case. The main reason is that data on such expert forecasts over labor income is not readily available, in contrast to capital income. Moreover, to be consistent with our model we would ideally like to further distinguish trends in (proxies for) ambiguity over labor income perceived by type A agents, savers, relative to that perceived by type B agents, capital-owners. In the absence of such specific survey evidence, we connect here to a vast empirical literature documenting that during the last thirty years labor income has grown

[12] The forecasts in Figure 3 are over nominal profit growth. A potential concern is that the measure reflects a decline in inflation uncertainty. However, the IQR of inflation forecasts exhibits no downward trend since the 1980s and is moreover an order of magnitude smaller than that in profits. Figure 7 in Appendix shows this fact. We thus interpret the downward trend in Figure 3 as a decline in ambiguity over real profit growth.

disproportionately more for workers at the top of the earnings distributions (eg. Acemoglu and Autor (2011), Guvenen et al. (2022), Heathcote et al. (2023)).

Finally, we note that changing ambiguity perceptions can also reflect policy changes. For example, as we introduce later in Section 5.2, fiscal policy might provide insurance against the ambiguous source of income. For agent A this would mean providing additional income when his labor endowment surprisingly falls, a form of 'welfare state'. Similarly, for agent B, such income would insure against bad times for her capital income, a form of a 'bailout' policy. Through such insurances, for a given primitive amount of ambiguity, policy effectively lowers the effective caution of agents. Therefore, our estimated increased worry of agent A can be further interpreted as agent A perceiving a lower degree of 'welfare state' insurance. In turn, the decreased worry of agent B could arise from her perception of a stronger 'bailout' policy intervention. Such a divergence in agents' policy perceptions appears consistent with the increasing trend towards 'winner-take-all' politics, as argued by Hacker and Pierson (2010).

4 Results: the evolution of uncertainty and frictions

In this section, we study the quantitative relevance of the model mechanisms described in Section 2.1 above. We start with identification: what led the calibration procedure to infer higher frictions and a change in the mix of ambiguity in the recent period? It is helpful to turn to the linearized equilibrium conditions. With quadratic leverage cost, the equilibrium conditions (16), (17) and (19) that jointly determine $(r^f, \bar{\ell}, \bar{pr})$ become

$$r^f = -\log\beta + g - \varepsilon^{cA}_{wA}\bar{a}^w \tag{26}$$

$$1.5\kappa\bar{\ell}^2 = \varepsilon^{cA}_{wA}\bar{a}^w - \varepsilon^{cB}_{d}\bar{a}^d \tag{27}$$

$$\bar{pr} = \kappa(1.5 - \bar{\ell})(\bar{\ell})^2 + \varepsilon^{p}_{d}\bar{a}^d \tag{28}$$

At first sight, it looks like the mapping from moments to parameters is straightforward and can be done analytically. We emphasize, however, that the equilibrium elasticities $(\varepsilon^{cA}_{wA}, \varepsilon^{cB}_{d}, \varepsilon^{p}_{d})$ are determined endogenously and change with parameters. Analytic formulas for given elasticities thus only capture the 'direct' effects of changing parameters. We now provide intuition for identification by performing a series of counterfactual experiments.

4.1 Identification

In Table 2 we report a counterfactuals that illustrate the contribution of various forces to the change in observables between 1989 and 2016. In the first column of results, 'Base 1989', we report the baseline calibrated economy that matches the 1989 data and moments. We then gradually feed in the model the direct inputs $(\tilde{d}, \tilde{w}^A, b^g)$ and then the structural parameters (κ, a^w, a^d) that characterize the calibrated economy of 2016, until the last column, 'Base 2016', which uses the full set of 2016 parameters. We discuss the model implications in terms of the row entries of the 'implied caution' terms that appear in equations (26)-(28), which together with the structural parameters can directly be used to read the three implied moments of interest and assess identification.

Table 2: Counterfactuals: role of different economic forces

			Base 1989	Counterfactuals: from 1989 to 2016			Base 2016
	Change:			$(\tilde{d}, \tilde{w}^A, b^g)$	(κ)	(a^w)	(a^d)
Direct inputs	Dividend share %	$\tilde{d}$	23	26	26	26	26
	Labor A share %	$\tilde{w}^A$	56	51	51	51	51
	Gov't debt %	b^g	35	40	40	40	40
Parameters	Leverage cost	κ	.08	.08	.97	.97	.97
	Agent A ambiguity	a^w	.06	.06	.06	.11	.11
	Dividend ambiguity	a^d	.07	.07	.07	.07	0
Implied caution	Caution over c^A %	$\varepsilon_{wA}^{cA} a^w$	4.4	4.2	4.8	9.2	8.4
	Caution over c^B %	$\varepsilon_{d}^{cB} a^d$	3.3	3.3	3.5	4.1	0
	Uncertainty premium %	$\varepsilon_{d}^{p} \bar{a}^d$	4.6	4.7	5	5.2	0
Implied moments	Risk free rate %	r^f	3.6	3.8	3.2	-1.3	-.5
	Leverage %	$\bar{\ell}$	29	26	9	19	23
	Premium %	$\bar{pr}$	6	5.9	6.8	10.1	7.4

Direct inputs. Consider under column $(\tilde{d}, \tilde{w}^A, b^g)$ the role of letting the direct inputs to change to their 2016 values while keeping other structural parameters at their 1989 values.

We first note that part of these changes in direct inputs is a government debt b^g increase. We find that this change alone has very small equilibrium effects, so the changes in elasticities generated by the counterfactual of this column are almost entirely attributable to changes in $\tilde{d}$ and $\tilde{w}^A$.[13] The reason for the small quantitative effect is transparent in equations (22): government debt acts as a zero-sum redistribution of lump-sum transfers across the two agents, with a magnitude of $(1 - s^A) b^g (1 - q_t)$. Compared to other sources of income, like labor or dividend, these interest rate transfers are effectively small and change little

[13]Table 6 confirms this by reporting a series of counterfactuals changing $\tilde{d}, \tilde{w}^A$ and b^g one a time.

in equilibrium when feeding in the variation in government debt observed in the data. As a result, these transfers affect little the equilibrium elasticities that govern the endogenous determination of the asset market moments of interest.

The quantitatively more relevant change in this counterfactual is that agent A gets less labor income. As labor income becomes less important for agent A's consumption, the elasticity ε_{wA}^{cA} falls compared to the initial 1989 economy. In turn, agent B gets richer overall, with a share of dividends out of her consumption similar to that of 1989, so her elasticity ε_{d}^{cB} also stays similar. Similarly, the exposure ε_{d}^{p} of the tree price to fluctuations in the now larger dividend stock increases.

This counterfactual also illustrates how, while parameters (κ, a^w, a^d) are fixed, the new equilibrium elasticities map into and alter the moments of interest. First, by equation (26), the lower effective worry $\varepsilon_{wA}^{cA} a^w$ of agent A now leads to less precautionary savings and a higher rate r^f. Second, as agent B's effective caution $\varepsilon_{d}^{cB} a^d$ remains similar to 1989, the gains from uncertainty sharing get squeezed, i.e. a decrease in $\varepsilon_{wA}^{cA} a^w - \varepsilon_{d}^{cB} a^d$. By equation (27), this decrease in excess demand for safety leads to a lower equilibrium leverage $\bar{\ell}$. Third, the uncertainty premium increases proportionally with the increase in ε_{d}^{p}, but the overall premium in equation (28) ends up falling. This occurs due to its lower frictional component, as leverage $\bar{\ell}$ in this counterfactual has fallen. Overall, the quantitative effects in this counterfactual are still relatively small - the largest contribution is to leverage, which falls from 29% to 26%, halfway to the value of 23% in 2016.

Higher leverage cost. In column (κ) we condition on the new 2016 values of $\tilde{d}, \tilde{w}^A, b^g$, keep ambiguities fixed at their 1989 values, and, critically, allow the leverage cost κ to change to its higher 2016 calibrated value. The fundamental economic effect of this increase is that the economy is now more constrained in sharing uncertainty between the two types of agents. Through the lenses of equations (26)-(28) a higher κ has direct and indirect effects. Consider the former: holding elasticities constant, in equation (27) a higher marginal cost of borrowing reduces the supply of safe assets, lowering leverage $\bar{\ell}$, while in equation (28) the premium $\bar{pr}$ would increase in proportion to the resulting increase in its frictional component.

As the economy is more constrained, equilibrium dynamics and elasticities cannot be held fixed - they do change. In particular, now both agents are less able to smooth out shocks to their own income. Thus, in equilibrium their consumption is more exposed to those income innovations, including the ambiguous sources of income. For agent A, this means a higher elasticity ε_{wA}^{cA} to labor income shocks, while for agent B a higher elasticity ε_{d}^{cB} to dividend income innovations. At the same time, due to the larger collateral benefit in this constrained economy, the value of the tree reacts more to dividend income shocks, so ε_{d}^{P} also increases.

Put together, column (κ) presents the overall effects of a change in leverage cost in equations (26)-(28). First, given the fixed ambiguity a^w, the saver is now effectively more worried about his future consumption. Thus, while there is no direct effect of κ on r^f in equation (26), as the saver faces no direct friction, his stronger equilibrium precautionary savings does lower the risk-free rate, from 3.8% in the comparison economy of column $(\tilde{d}, \tilde{w}^A, b^g)$ to 3.2%. Second, we find that the saver's caution increases by more than the borrowers' caution, i.e. that $\varepsilon^{cA}_{wA} a^w - \varepsilon^{cB}_{d} a^d$ increases. This increase in gains from trade acts as counterforce to the supply of safe assets being more expensive - nevertheless we find that the latter, direct force dominates. The quantitative effect here is very large: leverage $\bar{\ell}$ collapses from 26% to 9% in this economy. Third, both frictional and uncertainty premia increase - the latter due to the heightened exposure of the tree price to dividends. The two components add up to an implied new premium $\bar{pr}$ of 6.8% compared to 5.9%.

Qualitatively, an increase in leverage cost thus provides an economic mechanism that simultaneously fits the three directions of change from 1989 to the 2016 economy. Quantitatively though it is not sufficient - this mechanism alone predicts too large of a drop in equilibrium leverage, without a sufficient increase in premium and particularly without a sufficient drop in the interest rate.

More agent A ambiguity over labor income. The main dimensions of the poor quantitative fit of allowing only a change in κ, namely the very low leverage and too high risk-free rate, point out to the need of the model to recover a stronger demand for safety coming from the saver. In our model this takes the form of a higher a^w - we present its main implications by going from column (κ) to column (a^w) and now letting a^w take on its 2016 value.

Holding constant elasticities, more ambiguity about labor income leads the saver to act under more effective caution $\varepsilon^{cA}_{wA} a^w$ over his future consumption. This higher demand for safety has two immediate effects: (i) lowers the risk free rate in equation (26), and (ii) increases leverage in equation (27), by raising the gains from trade.

A larger demand for safety interacts with the financial friction to have indirect effects on the model dynamics. The reason is that the direct effect of a^w puts upward pressure on steady-state leverage. That force make the economy in steady state effectively more constrained at the margin, since the marginal borrowing cost raises with leverage. In our model, as the steady-state and dynamics are jointly determined, the equilibrium elasticities are therefore also affected by a^w, as shown in column (a^w). The direction is similar to the intuition of a larger κ: an effectively more constrained economy, now caused by more demand for safety, raises the equilibrium exposures ε^{cA}_{wA}, ε^{cB}_{d} and ε^{p}_{d}, compared to the relevant counterfactual economy of column (κ).[14].

Overall, the major mechanism behind the model's estimated increase in ambiguity over

[14]Note that column (a^w) implies an elasticity ε^{cA}_{wA} of 0.87, higher than 0.84 implied by column (κ).

labor income is to help it quantitatively generate a large drop in the real rate r^f, which falls from 3.2% in column (κ) to -1.5% in column (a^w). The heightened demand for safety also counterbalances some of the downward pressure on equilibrium leverage generated by the increase in the borrowing cost κ: due to the increase in a^w, leverage $\bar{\ell}$ now increases substantially, from 9% to 19%. For the premium, its uncertainty component increases endogenously through ε_d^p, while the significantly higher equilibrium leverage $\bar{\ell}$ raises the frictional component in equation (28). The overall effect in column (a^w) is a very significant increase in the total premium $\bar{pr}$, from 6.8% to 10.1%.

Less agent B ambiguity over dividend income. Premium rose in 2016 compared to 1989, but only mildly, to 7.4%, instead of the 10.1% predicted in column (a^w). Put differently, the strong demand for safety and high leverage cost that help the model generate a significantly lower riskfree rate and mildly lower leverage in 2016 also produce a large frictional premium. Without a reduction in the uncertainty premium, which we can capture through a lower a^d, the predicted overall premium would thus be excessively large compared to the data.

The quantitative tension between the three asset market moments allows our model, which has been constructed to capture both uncertainty and friction forces, to recover their relative contribution. In particular, column 'Base 2016' presents results for our baseline 2016 parameterization, where ambiguity a^d takes on its calibrated value as the only parameter change compared to column (a^w). We first note here that the conceptual constraint of $a^d \geq 0$, i.e. that ambiguity aversion can decline until it collapses to expected utility where $a^d = 0$, binds in our calibration. This result also underscores that our approach, of calibrating 7 parameters for 6 moments, is identified.

The mechanics of the model now are even simpler when a^d drops to 0. Perceiving full confidence, agent B now requires no uncertainty premium to hold the tree compared to the riskless bond and lacks a precautionary savings desire in smoothing her consumption. As a result of the latter full confidence over future consumption, the marginal gain from uncertainty sharing $\varepsilon_{wA}^{cA} a^w - \varepsilon_d^{cB} a^d$ rises significantly. Intuitively, for the same bond price, agent B is now more willing to provide safe assets to agent A and thus leverages up until the marginal borrowing cost equals the marginal gain. In column 'Base 2016', compared to column (a^w), the direct effect of setting $a^d = 0$ is thus a decrease in premium in equation (28) and an increase in leverage in equation (27). The higher willingness of agent B to provide safe assets for agent A to smooth intertemporally also matters for equilibrium dynamics. This shows in the lower equilibrium exposure of agent A's consumption to his own income shocks, ε_{wA}^{cA}. In steady-state, agent A thus worries less about his future consumption, raising the equilibrium interest rate in (26) compared to column (a^w). Put together, column 'Base 2016' replicates the targeted empirical moments for the 2016 episode.

Re-calibrating restricted models. So far we have inspected the model's account of the data by gradually changing parameters in Table 2 from the 1989 calibration to the one in 2016. We now ask whether any individual force is essential for explaining the data. For example, could changes in uncertainty or frictions alone do a good (if not perfect) job to fit the data? Formally, we proceed as follows. For the 1989 economy, we keep all parameters at their benchmark values. For the 2016 economy, we allow the direct inputs $(\tilde{d}, \tilde{w}^A, b^g)$ to change to their 2016 values, like in our benchmark approach. But in fitting the 2016 moments of interest $(r^f, \bar{\ell}, \bar{pr})$ we keep one of the three structural parameters (κ, a^w, a^d) fixed at its 1989 value, while calibrating the other two remaining parameters.[15] These counterfactuals complement our earlier analysis and recover different narratives by allowing only two out of the three economic forces that we study to vary.

Table 3: Re-calibrating restricted models

			Base 2016	$\kappa_{2016} = \kappa_{1989}$ new a^w, a^d	$\bar{a}^w_{2016} = a^w_{1989}$ new κ, a^d	$a^d_{2016} = a^d_{1989}$ new κ, a^w
Parameters	Leverage cost	κ	.97	.08	2	.57
	Agent A ambiguity	a^w	.11	.13	.06	.10
	Dividend ambiguity	a^d	0	.10	.06	.07
Implied moments	Risk free rate (%)	r^f	-.5	-.51	3.1	-.37
	Leverage (%)	$\bar{\ell}$	23	54	7	.22
	Premium (%)	$\bar{pr}$	7.4	7.4	6.8	9.1

We perform three such counterfactuals, reported in Table 3. For comparison, the first column 'Baseline 2016' reports values for our benchmark 2016 economy. When the leverage cost is restricted to take on its 1989 value (in the second column $(\kappa_{2016} = \kappa_{1989})$), ambiguity a^w over labor income is higher than the baseline 2016 value. Interestingly, in stark contrast to the baseline model, ambiguity a^d over dividend income is now not only larger than the 2016 baseline, but even higher than in 1989. Intuitively, without allowing for a change in the borrowing cost, the larger a^w is needed to account for the fall in the real rate, while the larger a^d for the increase in premium. The critical miss of this restricted version of the model is in the predicted quantity of safe assets: due to the high excess demand for safety needed to account for the low rate, leverage would more than double in 2016 at 54% compared to the data. While this "pure uncertainty story" ends up fitting almost perfectly the risk-free rate

[15] We search over the two remaining parameters of this counterfactual economy to minimize the equally weighted sum of absolute deviations of the model-implied $(r^f, \bar{\ell}, \bar{pr})$ from their 2016 empirical values.

and premium, this mis-specified model experiment showcases the importance of leverage as an over-identifying restriction. In our baseline quantitative approach leverage significantly alters inference not only in recovering the need for a higher estimated friction κ, but even over how various income sources of uncertainty a^w, a^d have changed.

The last two columns allow the leverage cost to vary, but restrict the change in one source of ambiguity. In particular, ambiguity a^w in column ($a^w_{2016} = a^w_{1989}$) is fixed to take on its 1989 value. There the new calibrated friction κ is larger than the 2016 baseline, while ambiguity a^d is close to its 1989 value. This economy thus resembles the counterfactual in column (κ) of Table 3, with the worst overall quantitative fit out of all three counterfactuals considered in Table 2. In particular, here the most informative moments for the need of allowing a change in a^w is that this mis-specified version significantly misses by predicting a leverage that is too low and a risk-free rate that is too high compared to the data.

The last column restricts ambiguity a^d and recovers a similar intuition to the counterfactual (a^w) of Table 2: through a cost κ and ambiguity a^w that are both higher than in 1989, the model can come close in accounting for the fall in the risk-free rate and the mild fall in leverage. But without a change in dividend ambiguity a^d, the predicted premium is significantly larger than in the data. Like in our discussion around Table 2, for this restricted model version, premium appears as the most informative over-identifying moment restriction that points in our quantitative model to a decrease in ambiguity a^d.

4.2 Perceived persistence and the yield curve

Our benchmark calibration in Table 1 is conditional on a persistence value of $\rho = 0.95$. Since shocks are constant under the true DGP, this parameter is best thought of as *perceived* persistence: it determines agents' view of how quickly the economy converges to a worst case steady state. It would therefore be desirable to discipline the parameter with data that directly reflect agents' expectations, and not necessarily observed shocks. We turn to the yield curve for such data. Intuitively, in a model with log utility, the real yield curve reflects expected growth of agents whose Euler equation holds, here agent A.

We focus on the 2016 period, when US Treasury inflation-protected securities (TIPS) were traded, so good data on the real yield curve is available. We use real zero coupon bond rates from the Federal Reserve Board who in turn follow the methodology of Gürkaynak et al. (2010) to fit a real yield curve at the daily frequency, starting at a maturity of two years.[16] The dark diamond line in Figure 4 is the real yield curve, computed as the sample mean for each maturity over 2016. The slope is mildly positive, with a spread of 0.95% on

[16] The data are available available since 1999 and continuously updated online at https:/www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html

the ten year bond compared to the two year bond. This value is typical of the last few decades: average spread over the entire available sample period 1999-2022 is 1%.

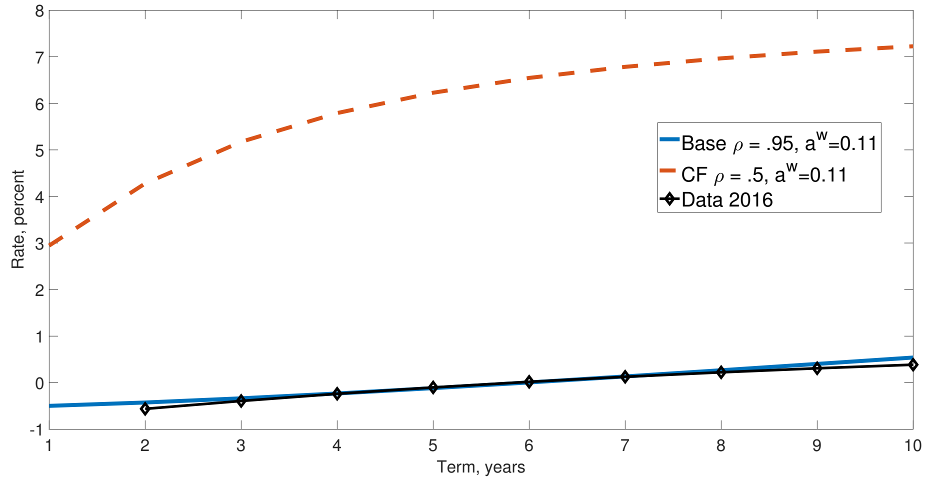

Figure 4: Slope of the yield curve. Dark diamond line plots the data in 2016, using the Gürkaynak et al. (2010) methodology, on zero-coupon real yields across different maturities, starting with 2 years. The blue solid line plots the yield curve implied by our 2016 calibration. The magenta dashed line plots the implied yield curve for a counterfactual that keeps $a^w = .11$ but sets $\rho = .5$.

The blue solid line in Figure 4 is the model-implied yield curve for our 2016 benchmark calibration. Formally, we plot the returns R_j introduced in equation (25), as a function of maturity j, increasing from 1 to 10 years. Qualitatively, the slope is positive, confirming the intuition of section 2.6 that bond price elasticities $\varepsilon_{wA}^{q(j-1)}$ in equation (25) are positive and increasing in horizon j. Intuitively, agent A, being cautious about future low income is also worried, in pricing longer term bonds in equilibrium, about future low short-term bond prices. Steady-state longer-term bond prices thus get pulled down by those expectations, tilting the yield curve upward. Quantitatively, we find that the excess returns $R_{10/1}$, $R_{10/2}$ of a ten year bond compared to the one and two period bond are 1.03% and .97%, respectively. The latter matches the empirical counterpart very well.

To understand identification, the dashed line in Figure 4 plots the yield curve when ambiguity remains fixed at its benchmark value $a^w = .11$ and only persistence is lower, at $\rho = 0.5$. This counterfactual exhibits (i) a higher level, starting from a higher one period rate r^f of 3%, and (ii) steeper slope, with spread 2.9% between the 10-and 2-year return. Effect (i) comes from the lower equilibrium exposure ε_{wA}^{cA}, which falls from .79 in the baseline to .47. With ambiguity a^w fixed, effective caution $\varepsilon_{wA}^{cA} a^w$ and the one-period bond price decline. Effect (ii) comes from the response of longer term bond prices $\varepsilon_{wA}^{q(j-1)}$. As discussed in section 2.6, these elasticities increase when the shock is more transitory, as agents desire for consumption smoothing puts higher pressure on bond prices in that case.

5 Unequal uncertainty sharing and policy implications

In this section we study the implications of our model for inequality and policy.

5.1 Inequality

Table 4 presents agents' consumption and welfare for various model versions. In particular, the first and fifth columns describe the benchmark calibration for 1989 and 2016, respectively. Columns 2-4 correspond to the counterfactual economies from Table 2 where we move only some parameters to 2016 values while leaving others at 1989 values. Moving from left to right, the second column changes only direct inputs (labor income, dividends, and government debt), the third adds the borrowing cost and the fourth adds ambiguity about labor income. For all cases, we consider consumption inequality by comparing agents' steady state consumption levels $\bar{c}^A$ and $\bar{c}^B$; welfare inequality follows from agents' certainty equivalent levels of consumption defined in Section 2.4.

Table 4: Consumption and welfare inequality under different models

		Base 1989	Counterfactuals: from 1989 to 2016			Base 2016
			$(\tilde{d}, \tilde{w}^A, b^g)$	(κ)	(a^w)	
Consumption A	$\bar{c}^A$	.54	.50	.49	.46	.48
Consumption B	$\bar{c}^B$	1.59	1.74	1.79	1.89	1.83
Welfare A	$\bar{c}^{A,det}$	.35	.33	.31	.19	.21
Welfare B	$\bar{c}^{B,det}$	1.15	1.25	1.30	1.37	1.83

Inequality across types has increased between 1989 and 2016: agent A is worse off whereas agent B is better off. While the effect on consumption is modest, there is a large effect on welfare for both agents. For type A, comparing columns 1 and 5 of Table 4, the 2016 values for consumption $\bar{c}^A$ and the welfare measure $\bar{c}^{A,det}$ are 11% and 40% lower than their original 1989 values, respectively. Agent B, in contrast, is better off in 2016, with consumption $\bar{c}^A$ and welfare measure $\bar{c}^{B,det}$ rising by 15% and 64%, respectively. Through the counterfactuals discussed above, we can dissect the economic forces driving this increase in inequality.

The modest increase in consumption inequality mostly reflects the exogenous increase in income inequality across types, with agent A becoming poorer and agent B richer. The second column in Table 4 isolates the effect of the direct inputs and already accounts for the bulk of the change in consumption inequality: $\bar{c}^A$ falls by 7% while $\bar{c}^B$ increases by 10%. As effective caution of both agents is not affected much in this counterfactual, the increase in welfare inequality follows the same patterns.

The main reason for the large increase in welfare inequality is the inferred ambiguity change. We first note that a higher leverage cost, added in column 3, by itself contributes little to inequality: while a lower interest rate is good for type B, it is a sufficiently small share of overall income (or expense for type A). Instead the welfare decline for type A is seen in column 4 which adds the inferred increase in labor income ambiguity, while the welfare increase for type B is seen in column 5 which adds the inferred decline in dividend ambiguity.

5.2 Redistributive fiscal policy

We conclude by studying some policy implications. In particular, we introduce simple redistributive fiscal policies that partly insure against ambiguous sources of incomes.

The policies. A lump-sum fiscal transfer to agent A shifts his actual labor income stream from his endowment w_t^A in equation (23) to a policy-induced income stream $w_t^{A,pol}$:

$$\log w_t^{A,pol} = \rho \log w_{t-1}^{A,pol} + (1-\rho)\log \overline{w}^A + g + (1-\alpha^A)\nu_{w,t}^A$$

When the policy parameter $\alpha^A \in (0,1)$, agent $A's$ actual income has the same persistence as the endowment but an exposure to the ambiguous labor income innovation $\nu_{w,t}^A$ that is still positive but effectively reduced from 1 to $(1-\alpha^A)$. Similarly, with a parameter $\alpha^B \in (0,1)$, policy reduces the exposure of agent B through an induced d_t^{pol} stream

$$\log d_t^{pol} = \rho_d \log d_{t-1}^{pol} + (1-\rho_d)\log \overline{d} + g + (1-\alpha^B)\nu_{d,t}.$$

A simple redistributive policy finances these transfers with a balanced budget. Thus, the resulting budget constraint for type A agent from equations (22) is now

$$n^A c_t^A = (1-\tau)n^A w_t^{A,pol} + b_{t-1}^p - q_t b_t^p + q_t b_t^p k(\ell_t) + (1-s^A)b^g(1-q_t) - \chi_t^{A,B}$$

where $\chi_t^{A,B} = (1-\tau)\left(d_t^{pol} - d_t\right)$ is the transfer to agents B financed by a lump-sum tax on type A. In turn, the budget constraint for type B in equation (22) is now

$$\begin{aligned}(1-n^A)c_t^B = {} & (1-\tau)(1-n^A)(w_t^B + \theta_{i,t-1}d_t^{pol}) + p_t(\theta_{i,t-1} - \theta_{i,t}) + q_t b_t^p(1-k(\ell_t)) - b_{t-1}^p - \\ & -(1-s^A)b^g(1-q_t) - \chi_t^{B,A}\end{aligned}$$

with the corresponding lump-sum transfer $\chi_t^{B,A} = (1-\tau)n^A\left(w_t^{A,pol} - w_t^A\right)$.

According to these policies, when agent $A's$ ambiguous labor income endowment w_t^A suffers a negative shock, i.e. $\nu_{w,t}^A < 0$, the government taxes the capital income owner B to provide additional income to agent A and sustain $w_t^{A,pol}$ (a simple form of 'welfare state'). In turn, in bad times for capital income, i.e. $\nu_{d,t}^A < 0$, a lump-sum tax on agent A partially covers for agent $B's$ income loss to sustain d_t^{pol} (a 'bailout' policy). Conversely, part of the positive innovation to agent $A's$ income gets transferred to agent B, and vice-versa.

These policies only get to be used out-of-steady state, when shocks actually hit. However, since these policies act to reduce agents' exposure to those (ambiguous) shocks they lower agents' effective worries even in steady-state. These insurance transfers therefore have steady-state effects on asset prices, quantities and welfare.

'Bailout' policy. A policy that sets $\alpha^B > 0$ lowers the effective exposure of agent B to her dividend income stream. Table 5 reports results for an illustrative value of $\alpha^B = .2$. In 2016 this policy has trivially no effects (last column) compared to the baseline, since agent B perceives no dividend uncertainty. When the agent does perceive such ambiguity, like in 1989, policy works by decreasing her effective caution about future consumption, continuation utility and tree price. As a result, agent B's welfare measure $\bar{c}^{B,det}$ increases, she is more willing to supply safe assets to the saver, so that equilibrium leverage increases, and the uncertainty premium on the tree, $\varepsilon_d^p a^d$, falls. Partly insuring agent B also has general equilibrium (GE) effects on agent A. The resulting increase in the supply of safe assets allows agent A to better smooth consumption out of his own labor income. His effective caution over consumption ε_{wA}^{cA} thus slightly falls, raising the interest rate. The lower worry, together with the higher leverage and interest rate raise agent A's consumption and welfare measure $\bar{c}^{A,det}$. Put together, in 1989 both agents would gain from this dividend 'bailout' policy.

'Welfare state' policy. Consider the role of activating insurance for agent A, to an illustrative value of $\alpha^A = .2$ (first and second results columns of Table 5). In both 1989 and 2016, this generates a lower effective exposure of agent A's consumption to labor income, i.e. ε_{wA}^{cA}, which for the same a^w lowers the effective caution over future consumption. Agent A precautionary savings motives loosen, leading to a higher equilibrium real rate and a lower leverage.

There are two important differences in this policy effects between 1989 and 2016. The first is that in the former, less constrained economy, the steady-state effects appear stronger in the quantity of safe assets produced in equilibrium. In the recent, more constrained economy, the lower precautionary savings instead moves more equilibrium prices. In particular, the real rate would increase significantly, to a robust positive value of 1.3%, while the premium would fall, reflecting entirely a lower frictional component, to 6%.

The second difference is for welfare. In 1989, both agents stand to gain from this

Table 5: Redistributive fiscal policy

		Insuring ambiguity for A		Insuring ambiguity for B	
		1989	2016	1989	2016
Caution over c^A	$\varepsilon_{wA}^{cA} a^w$	3.7	6.7	4.3	8.4
Caution over c^B	$\varepsilon_{d}^{cB} a^d$	3.1	0	2.9	0
Uncertainty premium	$\varepsilon_{d}^{p} a^d$	4.7	0	3.8	0
Risk free rate	r^f	4.3	1.3	3.7	-.5
Leverage	$\bar{\ell}$	22	21	33	23
Premium	$\bar{pr}$	5.7	6	5.2	7.4
Consumption A	$\bar{c}^A$	.54	.49	.55	.48
Consumption B	$\bar{c}^B$	1.59	1.77	1.56	1.83
Welfare A	$\bar{c}^{A,det}$	.38	.25	.36	.21
Welfare B	$\bar{c}^{B,det}$	1.16	1.77	1.19	1.83

policy. For agent A, the reason is immediate: as insurance lowers his effective exposure of consumption and continuation utility to the ambiguous labor innovation, his steady-state welfare increases, summarized by the higher $\bar{c}^{A,det}$. For agent B the reason is GE effects. In particular, agent B is impacted along two dimensions by the fact that the economy with counterfactual insurance is effectively less constrained. On the one hand, the significantly lower leverage and slightly higher interest rate end up slightly lowering the steady-state interest rate payments of the levered agent B. On the other hand, agent B can now also engage in more consumption smoothing. In particular, her own consumption is also less exposed in equilibrium to her own ambiguous capital income source, so that her effective caution $\varepsilon_{d}^{cB} a^d$ drops (from 3.3 to 3.1). Both of these GE effects serve to increase welfare, as now even $\bar{c}^{B,det}$ increases, even if slightly from the baseline case.

In the recent 2016 economy agent A continues to experience a welfare gain from the insurance transfers, but agent B loses from it. The reason for the latter is that the interest rate turns from its baseline negative value to a positive value, while leverage does not change much. Thus, now agent B instead of receiving payments on the debt held ends up paying interest on it. As a result, her consumption c^B is lower now. Importantly, any potential GE-driven insurance benefits of the policy against capital income are absent in this economy estimated to lack capital income uncertainty. Therefore, her welfare $\bar{c}^{B,det}$ decreases compared to the baseline. Overall, the two agents would thus agree in supporting such a proposed 'welfare state' insurance policy in 1989 but disagree over it in the recent economy.

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Appendix

A Solution method

In this appendix, we describe a general solution method for recursive models with heterogeneous agents who choose actions under different beliefs, which are either exogenously given or endogenous determined because agents are ambiguity averse. Our exposition proceeds in three steps. In Section A.1, we explain the approach using the simplified model Section 2 as a concrete example. In Section A.2, we introduce a general recursive model with heterogeneous ambiguity averse agents. We establish observational equivalence of such models with models with expected utility (EU) who agree to disagree and discuss how to derive beliefs that support choice. Section A.3 introduces a class of such models with agents who disagree about means in the long run. Section A.4 describes our general solution procedure for such models.

A.1 Solving the simple model of Section 2

Here we discuss our solution method in the context of the simple model of ambiguity sharing presented in Section 2. Equilibrium is summarized by agents' budget constraints and their first order conditions, where only the latter contain expectations of future variables. To set up the solution approach, we first rewrite Euler equations by defining new variables that describe agent's reasoning about future prospects under the worst case belief.

Consider first the Euler equations for bonds. We rewrite (4) as

$$q_t = \beta E_t \left(\frac{c_t^A}{\widetilde{c}_{t+1}^A} \right) = \beta E_t \left(\frac{c_t^B}{\widetilde{c}_{t+1}^B} \right) + q_t \left(k \left(\ell_t \right) + \ell_t k' \left(\ell_t \right) \right)$$

where the new variables $\widetilde{c}_{t+1}^A$ and $\widetilde{c}_{t+1}^B$ capture the perspective of agent A and agent B, respectively, about how their own future consumption will evolve. Since agents share structural knowledge of the economy, their view of future consumption can differ only because of disagreement about future state variables: agents must use the correct policy functions to forecast endogenous variables.

In a first order approximation, policy functions are parametrized by elasticities that load on the state, as in equation (10) above. Consider the evolution of agents' consumption one period ahead under worst case beliefs. We write consumption as a function of the three state variables

$$\log \widetilde{c}_t^A = \log \bar{c}^A + \widetilde{\varepsilon}_b^{cA}(\log b_{t-1} - \log \bar{b}) + \widetilde{\varepsilon}_{wA}^{cA}(\log w_t^A - \log \overline{w}^A - a^w) + \widetilde{\varepsilon}_d^{cA}(\log d_t - \log \overline{d}) \quad (29)$$

$$\log \widetilde{c}_t^B = \log \bar{c}^B + \widetilde{\varepsilon}_b^{cB}(\log b_{t-1} - \log \bar{b}) + \widetilde{\varepsilon}_{wA}^{cB}(\log w_t^A - \log \overline{w}^A) + \widetilde{\varepsilon}_d^{cB}(\log d_t - \log \overline{d} - a^d) \quad (30)$$

Here we have uses that both agents' worst case beliefs are pessimistic about future the respective consumption: in agent A's policy function, expected future income contains the negative component $-\widetilde{\varepsilon}_{wA}^{cA} a^w$, whereas in agent B's policy function, expected future dividends contain the negative component $\widetilde{\varepsilon}_d^{cB}$.

Finally, the optimality for tree holdings is given by equation (5), which we can write as

$$p_t = \beta E_t \left[\frac{c_t^B}{\widetilde{c}_{t+1}^B} (\widetilde{p}_{t+1} + \widetilde{d}_{t+1}) \right] + \ell_t^2 k' (\ell_t) p_t$$

where the forward looking variable $\widetilde{p}_{t+1}$ captures the perspective of agent B about how the future endogenous price will evolve. We write the price function as

$$\log \widetilde{p}_t = \log \bar{p} + \widetilde{\varepsilon}_b^p(\log b_{t-1} - \log \bar{b}) + \widetilde{\varepsilon}_{wA}^p(\log w_t^A - \log \overline{w}^A) + \widetilde{\varepsilon}_d^p(\log d_t - \log \overline{d} - a^d)$$

Like in the construction of $\widetilde{c}_t^B$, this function captures the pessimistic belief of agent B over dividends, mapping here into the perceived equilibrium price by $\widetilde{\varepsilon}_d^p$.

Putting things together, the two budget constraints and the optimality conditions for bonds and tree holdings constitute a system of difference equations parametrized by the perceived elasticities underlying the conjectured functions $\widetilde{c}_t^A, \widetilde{c}_t^B, \widetilde{p}_t$. For *given* such perceived elasticities, the solution of this system is a set of steady state values $(\bar{c}^A, \bar{c}^B, \bar{b}, \bar{q}, \bar{p})$ and equilibrium elasticities for the functions governing allocations c_t^i, b_t and prices q_t, p_t. The former set can be computed using steady state closed form formulas analyzed in Section 2.4. The latter elasticities can be found by linearizing the system around those steady state values using standard techniques for solving forward-looking linear models (eg. as in Sims (2002)).

Critically, because we are interested in characterizing the solution when agents have structural knowledge of the economy, we impose model-consistent beliefs over the equilibrium functions that solve the linearized system. Thus, we look for the *fixed point* between perceived and actual elasticities, so that the conjectured functions $\widetilde{c}_t^A, \widetilde{c}_t^B, \widetilde{p}_t$ for the endogenous variables need to be consistent with the respective elasticities of the actual laws of motion of those endogenous variables. For example, comparing equilibrium $\log c_t^A$

$$\log c_t^A = \log \bar{c}^A + \varepsilon_b^{cA}(\log b_{t-1} - \log \bar{b}) + \varepsilon_{wA}^{cA}(\log w_t^A - \log \overline{w}^A) + \varepsilon_d^{cA}(\log d_t - \log \overline{d})$$

to the perceived law of motion $\log \widetilde{c}_t^A$ in equation (29), model-consistency in beliefs amounts to finding elasticities $\varepsilon_b^{cA} = \widetilde{\varepsilon}_b^{cA}$, $\varepsilon_{yA}^{cA} = \widetilde{\varepsilon}_{yA}^{cA}$ and $\varepsilon_d^{cA} = \widetilde{\varepsilon}_d^{cA}$. At the fixed point for all the perceived elasticities each agent's expectation of next period's endogenous variables is thus consistent with the linearization of the overall system around the ergodic steady state.

The equilibrium computed above takes as given the two agents' conjectured worst-case beliefs over exogenous variables (see equations (7) and (9)). The final step of our solution method is to verify this conjecture, using the first-order expansion of the value functions U_t^i defined in equation (3). For example, for agent A, this amounts to checking in the expansion

$$\log U_t^A = \log \overline{U}^A + \varepsilon_b^{UA}(\log b_{t-1} - \log \bar{b}) + \varepsilon_{wA}^{UA}(\log w_t^A - \log \overline{w}^A) + \varepsilon_d^{UA}(\log d_t - \log \overline{d}).$$

whether the elasticity $\varepsilon_{wA}^{UA} > 0$. This sign check verifies the conjectured worst-case belief for agent A of a low conditional mean of his labor income. Similarly, for agent B, her conjectured worst-case belief of a low conditional mean for dividends is verified by finding that her continuation utility is monotonic with respect to dividends, i.e that $\varepsilon_d^{UB} > 0$ in a similar expansion. As anticipated earlier in Section 2.3, these sign conditions are easily met, as these ambiguous sources of income affect directly the available resources for the two respective agents.

A.2 General recursive model

We now introduce a general class of recursive models with heterogeneity in ambiguity. Let Z_t denote a $k \times 1$ vector of exogenous state variables, X_t an $m \times 1$ vector of endogenous state variables and Y_t a $n \times 1$ vector of other endogenous variables.

There are I agents, each agent i with a vector of actions, denoted by D^i. These actions are part of X or Y and include the consumption choice C^i. Let $B^i(X, Y, Z)$ denote agent i's constraint set.

Given the evolution of Z, the evolution of the endogenous state variables is given by

$$X' = T(X, Y, Z)$$

An example of such law of motion is the capital accumulation equation, where X will include capital. An important component is the law of motion of the vector of ambiguities, $a^i \in X$, given by some transition

$$a^i = A^i(X, Z)$$

As detailed below, the vector of ambiguities controls the size of the belief sets for each agent, and in turn for each shock. In particular, each agent is characterized by belief sets such that one-step ahead conditional mean of innovation to the jth shock Z_j belongs to the interval $\left[-a_j^i, a_j^i\right]$. In principle, the ambiguity process can be a function of the endogenous state-variables X, as it would happen for example in models of learning.

Finally, the model economy consists of additional restrictions that determine the other endogenous variables, Y, through some relationship

$$\Gamma\left(X, Y, Z\right) = 0$$

An example would be market clearing conditions in the goods or asset markets.

The recursive equilibrium consists of functions for the endogenous variables $X'\left(X, Z\right)$, $Y\left(X, Z\right)$, value functions $V^i\left(X.Z\right)$ that satisfy three conditions.

First, agents optimize. In particular, we describe agents' preferences by the recursive multiple priors according to which

$$V^i\left(X, Z\right) = \max_{D^i \in B^i(Y,X,Z)} \left\{ u^i(C^i) + \beta \min_{|\mu_j^i| \leq A_j^i(X,Z)} E^{\mu}\left[V^i(X', Z')\right] \right\} \tag{31}$$

$$s.t.\ X' = T(X, Y, Z) \tag{32}$$

Agents' valuation of state variables X and Z is given by $V^i\left(X, Z\right)$, which is obtained by maximizing per period utility $u^i(C^i)$ and the continuation value over the available actions D^i from the budget set $B^i\left(Y, X, Z\right)$. A key element of the setup is that each agent has a set of beliefs over the one-step ahead conditional mean, denoted by μ_j^i, of the innovation to each shock Z_j. This set is given by the restriction that $|\mu_j^i| \leq A_j^i\left(X, Z\right)$, where the ambiguity vector A^i is potentially specific to each agent.

In the recursive representation in (31), agents react to the ambiguity (Knightian uncertainty) about the conditional probability distributions of the shocks by taking optimal actions *as if* the true data generating is characterized by each shock Z_j having the probability distribution with the lowest mean μ_j^i out of the set of beliefs. The fact that agents have different sets thus gives rise to heterogeneity in beliefs. Finally, in order the forecast the next period state variables, agents use the law of motion in (32).

Second, the endogenous variables are determined according to the restrictions summarized by $\Gamma\left(Y, X, Z\right) = 0$.

Third, the dynamics of this equilibrium need to be characterized under some law of motion for the exogenous variables Z. In the two steps above we have found the decision rules and laws of motion that arise from agents' aversion to uncertainty, as expressed by the optimization of the recursive multiple priors preference. Given these functions X' and Y, the realized equilibrium outcomes are obtained feeding in the true exogenous Markov state process for Z.

A.3 Recursive equilibrium with general beliefs

The general system consists of three types of equations. There are n equations that determine the other endogenous variables as an implicit function of the state variables:

$$f\left(X_t, X_{t-1}, Y_t, Z_t\right) = 0 \qquad (n \text{ equations})$$

There are m equations that capture intertemporal behavior by agents. In a model with heterogeneous beliefs, we need to distinguish between what belief is used for form expectations. We say there m_i equations involving beliefs of agent i, where $\sum_i m_i = m$.

$$E_t^i \left[g^i\left(X_t, X_{t-1}, Y_t, Y_{t+1}, Z_t\right)\right] = 0 \qquad (m_i \text{ equations})$$

Notice that the expectation operator uses the conditional distribution that emerges as the worst-case distribution from the minimization in the utility represented in (31). Once the worst-case belief is determined, or guessed, the model becomes observationally equivalent with an expected utility model with heterogeneity in beliefs.

There are k equations that describe the law of motion of the exogenous variables. We introduce the risk directly on innovations to the log state variables:

$$\log Z_t = (I - P) \log \bar{Z} + P \log Z_{t-1} + \varepsilon_t \tag{33}$$

We assume that $I - P$ is invertible so the mean of a stationary solution to this difference equation is $\log \bar{Z}$. We assume that agents may disagree about exogenous variables, but they share structural knowledge of the economy, that is, they agree on the mapping from exogenous to endogenous variables.

For a general definition of equilibrium, suppose that each agent has in mind a probability distribution over Z_t, possibly different from (33). A recursive equilibrium is given by functions $X'(X, Z)$ and $Y(X, Z)$ such that the equations above are satisfied for all (X, Z). The resulting system of functional equations is

$$\begin{aligned} f\left(X'(X, Z), X, Y(X, Z), Z\right) &= 0, \\ E^i\left[g^i\left(X'(X, Z), X, Y(X, Z), Y\left(X'(X, Z), Z'\right), Z\right) | Z\right] &= 0, \end{aligned}$$

where in the second equation Z' is uncertain and expectations in equation i are computed using agent i's conditional distribution over it. These are $m + n$ functional equations in

$m + n$ functions.

A.3.1 Recursive equilibrium: zero risk steady state

If agents' beliefs differ in conditional means, then they will disagree in steady state. Denote steady state value by bars. Suppose also there are no shocks, but agents differ in conditional beliefs at the steady state $E^i\left[Z'|\bar{Z}\right]$. The steady state is then characterized by

$$
\begin{aligned}
f\left(\bar{X}, \bar{X}, \bar{Y}, \bar{Z}\right) &= 0, \\
g^i\left(\bar{X}, \bar{X}, \bar{Y}, Y\left(\bar{X}, E^i\left[Z^i|\bar{Z}\right]\right), \bar{Z}\right) &= 0,
\end{aligned}
$$

In general, the beliefs $E^i\left[Z'|\bar{Z}\right]$ matter both directly, in the second equation, and through their effects on future endogenous variables.

A.4 Recursive equilibrium: log-linear approximation

Disagreement at the steady state affects the applicability of perturbation methods. Indeed, if future endogenous variables Y enter intertemporal conditions, we cannot determine the steady state separately from the policy function $Y(X, Z)$. This problem is relevant for example in models with an intertemporal Euler equation of a long lived agent, where Y typically includes consumption, and it is not possible to find consumption in closed form. It does not occur for example in models with 2 period lived agents, which exhibit a closed form solution for consumption as a function of X and Z.

A.4.1 Loglinear approximation

We look for approximate solutions that are loglinear expansions around the steady state. Denote log deviation by lower case letters with hats, that is, $X = \bar{X}e^{\hat{x}}$ and so on. We conjecture solutions

$$
\begin{aligned}
X_t &= \bar{X}\exp\left(\varepsilon_{xx}\hat{x} + \varepsilon_{xz}\hat{z}\right) \\
Y_t &= \bar{Y}\exp\left(\varepsilon_{yx}\hat{x} + \varepsilon_{yz}\hat{z}\right)
\end{aligned}
$$

Here $\hat{x}$ corresponds to the lagged deviation of X from steady state and $\hat{z}$ corresponds to the current deviation of Z from steady state. The undetermined coefficients are therefore $m+n$ constants, and $(m+n)(m+k)$ elasticities.

If all we want is loglinear approximate solutions, we do not have to model explicitly all features of beliefs: all that matters is differences in conditional means. We assume

$$E_t^i \log Z_{t+1} = (I - P) \log \bar{Z} + P \log Z_t + A^i \log Z_t$$

This assumes that conditional means can be written as a linear function of exogenous state variables. The matrix A^i determines the belief adjustment relative to the true law of motion (33). In the exogenous ambiguity case, an ambiguity state variable would be part of Z_t and we can then use different A^i matrices to pick different worst cases.

In a rational expectations setup, we can solve the constants separately from the elasticities. This is not possible here since agents disagree in steady state about what happens to the state variables next period. However, we can still loglinearize to obtain as many equations as coefficients we need to determine.

Loglinearizing the first set of equations, we have (writing the jth partial derivative as f_j and dropping arguments):

$$0 = f\left(\bar{X}, \bar{X}, \bar{Y}, \bar{Z}\right) + \bar{X} f_1 \left(\varepsilon_{xx}\hat{x} + \varepsilon_{xz}\hat{z}\right) + \bar{X} f_2 \hat{x} + \bar{Y} f_3 \left(\varepsilon_{yx}\hat{x} + \varepsilon_{yz}\hat{z}\right) + \bar{Z} f_4 \hat{z}.$$

This delivers n equations for constants, and $n \times (m + k)$ on elasticities in the usual way.

Loglinearizing the second set of equations, for each i, we have

$$\begin{aligned} 0 = {} & g^i\left(\bar{X}, \bar{X}, \bar{Y}, \bar{Y} \exp(\varepsilon_{yz} A^i \log \bar{Z}), \bar{Z}\right) + \bar{X} g_1 \left(\varepsilon_{xx}\hat{x} + \varepsilon_{xz}\hat{z}\right) + \bar{X} g_2 \hat{x} \\ & + \bar{Y} g_3 \left(\varepsilon_{yx}\hat{x} + \varepsilon_{yz}\hat{z}\right) + \bar{Y} g_4 \left[\varepsilon_{yx}(\varepsilon_{xx}\hat{x} + \varepsilon_{xz}\hat{z}) + \varepsilon_{yz}\left(P + A^i\right)\hat{z}\right] + \bar{Z} g_5 \hat{z} \end{aligned}$$

This delivers m equations from the constant terms, and $m \times (m + k)$ equations from matching coefficients on the state variables. Note that all derivatives f_j and g_j also depend on the constants.

A.4.2 Proposed solution strategy: implementation

To solve jointly for the constants and the elasticities, we propose a solution strategy that uses standard linear forward-looking rational expectations solution methods. The strategy consists of the following steps:

1. Guess the elasticities of the other endogenous variables,Y, with respect to the endogenous and exogenous state variables, ε_{yx} and ε_{yz}, respectively. This step is useful so that the agents have a guess on the law of motion of future endogenous variables, which they need to forecast.

2. Solve for candidate steady state from constant terms

$$
\begin{aligned}
f\left(\bar{X}, \bar{X}, \bar{Y}, \bar{Z}\right) &= 0 \\
g^i(\bar{X}, \bar{X}, \bar{Y}, \bar{Y}\exp(\varepsilon_{yz}A^i \log \bar{Z}), \bar{Z}) &= 0
\end{aligned}
$$

   Given the guessed elasticities, this is a nonlinear equation system of dimension $m+n$.

3. Compute all derivatives at steady state and find elasticities

   This becomes finding a linear system of dimension$(m+n)\,(m+k)$.

   To solve for this step we can appeal to standard linear rational expectations model solution methods. To do that, we propose adding a vector of forward-looking endogenous variables $\tilde{Y}^i_{t+1}$, with a size equal to the number of Y variables that enter as leads in the expectation equations $g^i(.)$.

   In particular, we modify the model to read

$$
\begin{aligned}
0 &= f\left(X_t, X_{t-1}, Y_t, Z_t\right) && (34) \\
0 &= E_t\left[g^i(X_t, X_{t-1}, Y_t, \tilde{Y}^i_{t+1}, \bar{Z})\right] && (35) \\
\log Z_t &= (I-P)\log \bar{Z} + P \log Z_{t-1} + \varepsilon_t && (36)
\end{aligned}
$$

   where the new defined variable follows

$$
\tilde{Y}^i_t = \bar{Y}\exp(\varepsilon_{yx}\left(\log X_{t-1} - \log \bar{X}\right) + \varepsilon_{yz}\left(\log Z_t - \log \bar{Z} + A^i \log Z_{t-1}\right)) \qquad (37)
$$

   The variable $\tilde{Y}^i_t$ thus controls for the forward looking variable Y_t. but from the perspective of each agent i. The heterogeneity of beliefs shows up in introducing agent-specific $\tilde{Y}^i_t$ variables. The law of motion for this variable follows from the guess in step 1: it responds to the endogenous and exogenous state variables according to elasticities ε_{yx} and ε_{yz}. Importantly, the agent specific matrix A^i, which determines the belief adjustment about the exogenous state Z, relative to the true law of motion, affects the belief about the endogenous variable through the elasticity ε_{yz}.

4. Finally, for the solution of the economy represented by the equations in (34)-(37) to be the equilibrium in which we have imposed structural knowledge of the economy, we need to check the consistency of these beliefs. This amounts to check if the guess of the elasticities ε_{yx} and ε_{yz} is correct. If the guess is correct, then agents are forming model-consistent expectations of the future endogenous variables.

Having reached consistency of beliefs about the structure of the economy, the dynamics and the steady state of the rational expectations model described by the equilibrium conditions in (34)-(37) produces the dynamics and the zero-risk steady state of our economy with heterogeneous beliefs. Indeed, this system contains: (i) the original static equations $f(.)$ in (34), (ii) the forward looking equations $E_t g^i(.)$ in (35), where expectations about future endogenous variables are agent specific. These functions contain the agent specific views about future variables, reflected in the guessed laws of motion (37); (iii) dynamics are determined by the true law of motion for the exogenous state, as in (36). At the fixed point for the elasticities $(\varepsilon_{yx}, \varepsilon_{yz})$ agent i's expectation of next period's endogenous variables, as described by equation (37), is consistent with the linearization of the overall system at the zero risk steady state.

B Details on financial positions

In this Appendix, we add detail and additional motivation for our quantitative strategy. We first explain how we map financial positions in the data to our model. We then provide additional evidence on debt positions to show that the trend in leverage we emphasize is robust to alternative definitions. Finally, we describe how we handle housing and the foreign sector, two features we do not explicitly model.

Defining assets and sectors. We divide household financial assets into *business equity* and *debt*. Here business equity includes (i) equity in publicly traded firms, held either directly or through an investment intermediary such as a mutual fund or a defined-contribution pension plan, (ii) stakes in private businesses, including closely held shares, proprietorships or stakes in partnerships and (iii) investment property. We label all other financial assets as debt: this category includes direct and indirect ownership of bonds as well as other fixed income products such as loans, bank deposits and life insurance policies.

The FAUS allow us to distinguish private and public debt. We define the public sector as the Federal government, municipal (state and local) governments, the Federal Reserve System, and retirement funds for municipal and Federal government workers. We count government-sponsored enterprises (GSEs) as well as mortgage pools insured by GSEs, as part of the (private) financial system. Public debt consists of all fixed income instruments issued by the public sector. Private bonds comprise all other fixed income instruments.

Nonfinancial sectors in the FAUS are households, government, business and the rest of the world. Foreign liability positions in equity and bonds represent foreign securities held by domestic sectors, including the US financial system. Foreign asset positions represent US-issued securities held by foreigners. For the facts documented in this section, we adopt this convention as well. We count US subsidiaries of foreign-owned banks as part of the rest of the world sector.

The FAUS contain detailed information on the US financial sector. For our purposes, it is helpful to distinguish two types of intermediaries. *Investment intermediaries* comprise mutual funds, money market funds and defined contribution pension funds. They issue shares that amount to stakes in a portfolio of assets. We consolidate investment intermediaries with their shareholders by attributing portfolios pro rata. For example, we multiply households' position in mutual fund shares by mutual funds' portfolio weight on equity, and add the resulting position to households' equity position.

The remaining intermediaries are *fixed income transformers*, most prominently banks, broker-dealers, finance companies and insurers. They hold instruments such as loans and bonds and issue instruments such as deposits or repos that have different maturity or liquidity properties. They also issue equity. The main position here is the market value of public equity for financial corporations such as bank holding companies. For firms like broker-dealers that are not corporations, FAUS reports book value.

We define *total debt* as all fixed income instruments on the liability side of nonfinancial sectors plus net debt (liability minus asset positions) of fixed income transformers. Importantly, this approach does not double count positions when intermediaries simply transform fixed income instruments. It makes sense in our context since there is no explicit financial system in our theory. We define *total equity* as the sum of all outstanding equity, including both nonfinancial business and fixed income transformers. Both types of business equity represent claims to risky cash flows of businesses.

The debt share in US financial markets. Figure 5 and 6 present the share of debt in financial assets in US markets. The left panel of Figure 5 summarizes the overall scale of debt as well as its intermediation between 1985 and 2019. The blue lines show fixed income positions of the nonfinancial sector relative to the sum of total debt and equity. The solid line is total debt holdings, or the sum of fixed income positions on the asset of the balance sheet, and the dotted line is total supply or the sum of liability side positions. The red lines are holdings (solid) and supply (dotted) of fixed income transformers.

An important takeaway from the figure is that both nonfinancial sectors and fixed income transformers have very small net positions relative to their gross positions. We can therefore think of fixed income transformers as simply altering the properties of a certain set of debt instruments that they pass on from nonfinancial borrowers to nonfinancial lenders. The scale of debt markets, in the sense of trade between ultimate borrowers and lenders, is therefore given by the debt of nonfinancial sectors. Our measure of total debt as defined above captures this feature.

Figure 6 provides detail on nonfinancial borrowers and lenders. The two panels show total holdings and supply of debt relative to the sum total debt and equity. The left panel shows contributions by gross borrower sectors, distinguishing business, government, the rest of the world and households, with mortgages secured by the primary residence broken out separately. The right panel shows contributions by same sector, but to gross lending. Differences between areas in the two panels indicate net positions of nonfinancial sectors: it is easy to see that throughout households and foreigners are net lender, whereas nonfinancial business and government are net borrowers.

The figure establishes a number of stylized facts. First, according to pretty much any definition *the share of debt in total financial assets has declined since the late 1980s*. According to the left panel, the main driver is lower business borrowing that was partially, but not entirely, offset by more government borrowing. An increase in household mortgage borrowing during the housing boom years temporarily slowed this decline, but was overall not particularly large relatively to the broad trend. Finally, international investment in bonds has increased since the financial crisis, but its contribution is quantitatively small.

Another key fact is that the *lower relative supply of debt is reflected in lower debt holdings of US households*. According to the right panel, the rest of the world in contrast increased holdings of US debt, albeit not enough to offset the drop in domestic holdings. Two other positions that have received attention recently do not contribute much to lower holdings at the aggregate level relative to total assets. On the one hand, government holdings through retirement funds and the central bank, are about the same size as in the late 1980s. On the other hand, the share of nonfinancial business holdings of debt, including in particular cash holdings by corporations, has barely moved.

Housing and mortgages. Our theory does not speak to changes in the financing of owner-occupied housing. We follow some of the household finance literature in modeling owner-occupied housing as well as mortgages as fully illiquid: positions enter the model as fixed streams of flow payments rather than adjustable asset positions. Our quantitative analysis thus takes into account payments made by households to service mortgages and flows received due to those payments. This approach requires not only removing borrower household mortgage positions, but also adjusting lenders' positions.

Our working assumption is that all nonfinancial sectors hold fixed income instruments created from mortgages backed by the primary residence in the same proportions as other private bonds. In terms of Figure 6, we want to remove household mortgages from the total in the left panel and reduce positions in the right panel by the same amount. We look up, for every sector, the private component of debt holdings and then reduce it by the share in private debt of mortgages backed by the primary residence. As an example, that share was about 25% in both 1989 and 2016.

Sectoral net positions. Since we look at one bond market, our theory only speaks to net positions. The right panel of Figure 5 illustrates net position of the net lender sectors, households and the rest of the world. Household positions are in blue, with a solid line for private debt holdings and a dotted line for public debt. Rest of the world positions are in magenta, again solid for private and dotted for public. Here private bond positions have been adjusted for mortgages, as discussed above. For the rest of the world sector, we further compute a net equity position, shown in cyan.

Several interesting messages emerge. First, the net household position shows a strong decline, by 10pp or 40 percent, with about half of the effect due to lower government debt holdings. Since mortgages have been removed and other household debt has been netted out, the remaining decline is a drop in business debt holdings. Second, while foreigners' net position is sizeable, it is almost exclusively a position in government bonds. Finally, foreigners net equity position is close to zero throughout, and especially in 1989 and 2016.

These facts motivate our strategy for dealing with foreign positions. The key point is that foreigners do not play an important role transforming risk, for example holding debt and issuing equity. Instead they hold government debt, and their equity positions are a wash. Since government debt is exogenous in our model, we can adjust it by the amount held by foreigners and thus study the effect of government debt available to the domestic economy. For equity, we simply remove the small net foreign position and work only with equity that is domestically issued and held.

Table 6: Counterfactuals: changing one parameter at a time

			Base 1989	Counterfactuals: from 1989 to 2016					Base 2016
	Change one force at a time			$\tilde{d}$	$\tilde{w}^A$	b^g	κ	a^w	a^d
Direct inputs	Dividend share (%)	$\tilde{d}$	23	26	26	26	26	26	26
	Labor A share (%)	$\tilde{w}^A$	56	56	51	51	51	51	51
	Gov't debt (%)	b^g	35	35	35	40	40	40	40
Parameters	Leverage cost	κ	.08	.08	.08	.08	.97	.97	.97
	Agent A ambiguity	a^w	.06	.06	.06	.06	.06	.11	.11
	Dividend ambiguity	a^d	.07	.07	.07	.07	.07	.07	0
Implied moments	Bond rate (%)	r^f	3.6	3.7	3.8	3.8	3.2	-1.3	-.5
	Leverage (%)	$\bar{\ell}$	29	25	26	26	9	19	23
	Premium (%)	$\bar{p}r$	6	5.9	5.9	5.9	6.8	10.1	7.4
Implied caution	Caution over c^A (%)	$\varepsilon^{cA}_{wA} a^w$	4.4	4.3	4.2	4.2	4.8	9.2	8.4
	Caution over c^B (%)	$\varepsilon^{cB}_{d} a^d$	3.3	3.4	3.3	3.3	3.5	4.1	0
	Uncertainty premium	$\varepsilon^{p}_{d} a^d$	4.6	4.7	4.7	4.7	5	5.2	0
Implied inequality	Consumption A	$\bar{c}^A$	.54	.52	.50	.50	.49	.46	.48
	Consumption B	$\bar{c}^B$	1.59	1.67	1.74	1.74	1.79	1.89	1.83
	Welfare A	$\bar{c}^{A,det}$	.35	.34	.33	.33	.31	.19	.21
	Welfare B	$\bar{c}^{B,det}$	1.15	1.19	1.25	1.25	1.30	1.37	1.83

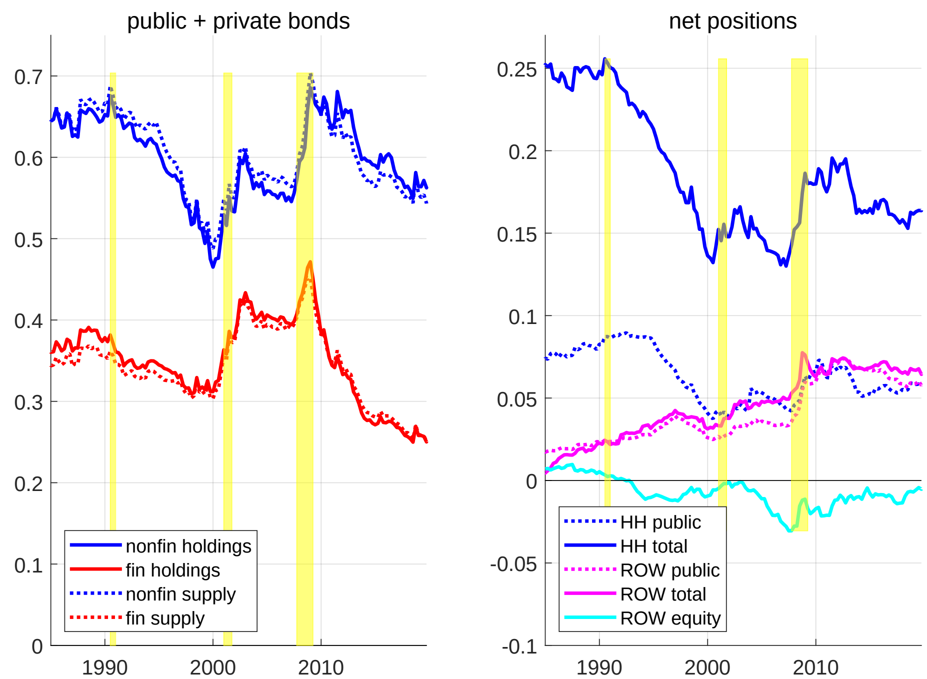

Figure 5: Left panel: gross fixed income positions by type of sector as a share of GDP. Blue lines are aggregate positions of nonfinancial sectors (households, businesses, government, rest of the world), red lines are for financial sectors. Holdings (solid lines) sum positions on asset side of sector balance sheet, supply (dotted lines) sum liability side. Right panel: net asset positions (assets minus liabilities) by rest of the world and household sector as a share of GDP; solid blue and pink lines are net private fixed income holdings. Dotted blue and pink lines are US public debt holdings; cyan line is ROW net equity position. Source: Financial Accounts of the United States.

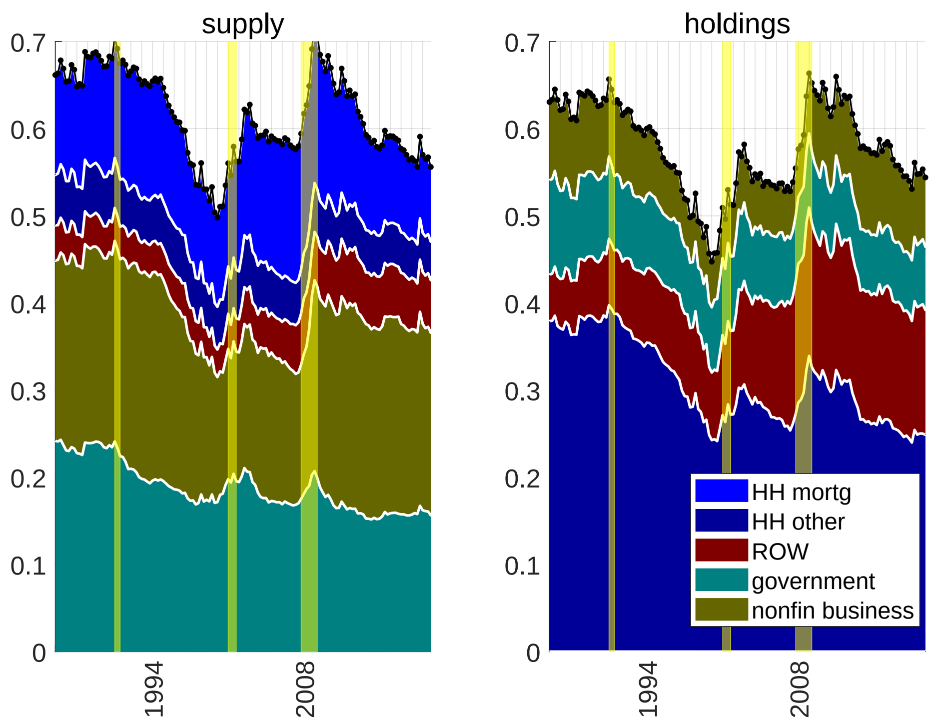

Figure 6: Decomposition of aggregate holdings (left panel) and supply (right panel) of fixed income instruments, as a share of GDP. Colors indicate sectors. Left panel breaks out residential mortgages supplied by households with a separate color (lighter blue). Source: Financial Accounts of the US

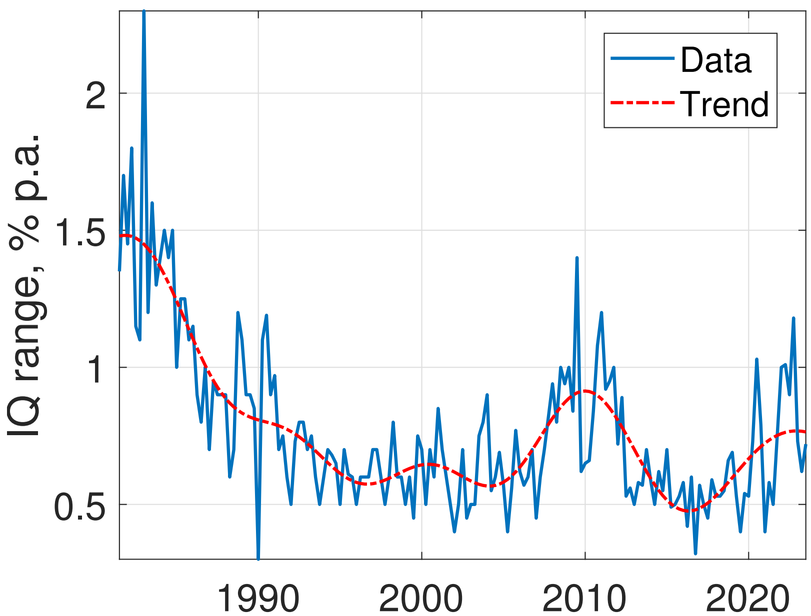

Figure 7: Cross-sectional dispersion of quarterly forecasts for CPI inflation rate. Units: Annualized percentage points. Dispersion measure is the 75th percentile minus 25th percentile of the forecasts for CPI inflation. Blue solid line is raw data. Dotted red line is the trend, defined as fluctuations with a frequency lower than 8 years.