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Zhengyang Jiang
Arvind Krishnamurthy
Hanno Lustig
Jialu Sun

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ABSTRACT

We characterize the cyclicality and predictability of exchange rates without committing to a specific model of preferences, endowment or menu of traded assets. When investors can trade home and foreign currency risk-free bonds without frictions, the exchange rate conditionally appreciates in states of the world that are worse for home investors than foreign investors. We show that its unconditional counterpart can only be overturned if the deviations from U.I.P. are large and exchange rates are highly predictable, conditions that do not hold in the data. Thus, without frictions in bond markets, it is impossible to match the empirical exchange rate cyclicality (the Backus-Smith puzzle) and the deviations from U.I.P. (the Fama puzzle) as well as the lack of predictability (the Meese-Rogoff puzzle). To relax this trade-off, we need small Euler equation wedges consistent with a home currency bias, home bond convenience yields, or financial repression, but we do not need market segmentation.

Zhengyang Jiang
Northwestern University
Kellogg School of Management
and NBER
zhengyang.jiang@kellogg.northwestern.edu

Arvind Krishnamurthy
Stanford University
and NBER
akris@stanford.edu

Hanno Lustig
Stanford University
Stanford Graduate School of Business
Department of Economics
and NBER
hlustig@stanford.edu

Jialu Sun
Northwestern University
Kellogg School of Management
jialu.sun@kellogg.northwestern.edu

1 Introduction

We consider the following five key facts about exchange rates. First, real exchange rates are only weakly positively correlated with relative aggregate consumption growth [Kollmann, 1991, Backus and Smith, 1993]. The home currency tends to depreciate when the home investors experience adverse macroeconomic shocks and thus have high marginal utility growth. In other words, exchange rates are weakly pro-cyclical. Second, exchange rates seem disconnected from the other macro variables that should determine them [Obstfeld and Rogoff, 2000]. Third, as documented by Tryon [1979], Hansen and Hodrick [1980] and Fama [1984], interest rate differences do not predict changes in exchange rates with the right sign to enforce the uncovered interest rate parity (U.I.P.). Instead, currency returns are predictable, but exchange rates themselves are not. In order to explain the negative slope coefficients, risk premia have to be extremely volatile [Fama, 1984]. Fourth, other macro variables also fail to predict exchange rates, as shown by Meese and Rogoff [1983]. It is hard to beat a random walk when predicting exchange rates, especially out of sample and at frequencies lower than daily or weekly [see, e.g. Rossi, 2013, for a survey of the literature]. Order flow seems to predict exchange rates only at higher frequencies [Evans and Lyons, 2002b]. Finally, exchange rates are much less volatile than the growth rate of the marginal utility of investors backed out from asset market data [Brandt, Cochrane, and Santa-Clara, 2006b].

In a large class of international real business cycle models, as long as investors can trade home and foreign risk-free bonds frictionlessly, bond investors' Euler equations impose strong restrictions on the relationship between exchange rates and marginal utilities. Let $m_{t,t+1}$ and $m^*_{t,t+1}$ denote the home and foreign SDF in log, let r_t and r^*_t denote the home and foreign risk-free rates in log, and let s_t denote the log spot exchange rate in units of foreign currency per dollar. A higher s_t means a stronger home currency. Then, the four bond Euler equations are given by:

$$\begin{aligned}
1 &= \mathbb{E}_t\left[\exp(m_{t,t+1} + r_t)\right], \\
1 &= \mathbb{E}_t\left[\exp(m_{t,t+1} - \Delta s_{t+1} + r^*_t)\right], \\
1 &= \mathbb{E}_t\left[\exp(m^*_{t,t+1} + r^*_t)\right], \\
1 &= \mathbb{E}_t\left[\exp(m^*_{t,t+1} + \Delta s_{t+1} + r_t)\right].
\end{aligned}$$

The first two equations are the Euler equations for the home investor investing in domestic and foreign currency risk-free bonds. The second set of two Euler equations pertains to the foreign investor.

We first consider these equations' implications for the conditional exchange rate cyclicality, which is defined as the covariance between the exchange rate movement Δs_{t+1} and the SDF differential $m_{t,t+1} - m^*_{t,t+1}$. Building on the work by Lustig and Verdelhan [2019], we obtain a stark result: the conditional covariance is always positive, which means that the exchange rate has to be conditionally counter-cyclical to enforce these bond Euler equations.

In complete markets, the reason for this is well understood. The exchange rate of the home currency has to appreciate in the bad state to keep the Arrow-Debreu prices at home and abroad aligned state-by-state.[1] When the stand-in investor has power utility over consumption, this induces a perfectly negative correlation between the exchange rate and the aggregate consumption growth differential. This result carries over to the incomplete market setting. As long as investors can trade home and foreign risk-free bonds frictionlessly, an average version of this prediction survives [Lustig and Verdelhan, 2019].

It is worth building intuition for why bond trading alone is sufficient to generate this result. Consider any bilateral currency pair — say the Euro and the dollar. A U.S. investor holding Euro bonds and a Eurozone investor holding dollar bonds are both taking on exchange rate risk. When we sum their expected excess returns on these positions, the interest rate differential and the expected change in the exchange rate cancel out exactly — they appear with opposite signs for the U.S. and the Eurozone investors. What remains is the conditional variance of the Euro/dollar exchange rate, which has to equal the aggregate risk premium that FX investors in both countries earn on their long-short positions. The larger the variance, the larger this aggregate premium must be, and this premium can only be zero if there is no exchange rate risk at all.

As a result, FX investors in the Euro/dollar market on average have to perceive investments in the other country's currency as risky for them. And for the foreign currency to be risky, it must tend to depreciate in bad times for the average investor — which is precisely the statement that exchange rates are counter-cyclical.

Having established that exchange rates must be *conditionally* counter-cyclical — in any given period, the home currency must be expected to appreciate in bad states for the home investor — we ask whether this conditional force could be offset *unconditionally* to generate pro-cyclical exchange rates (stylized fact 1). In other words, if we evaluate the

[1] More precisely, when domestic investors have a higher marginal willingness than foreign investors to pay for consumption in some state tomorrow, i.e. to save into that state, then the state-contingent interest rate is correspondingly lower at home, and the real exchange rate of the home currency has to appreciate in that state in order to rule out arbitrage from borrowing domestically and investing abroad in that state of the world.

covariance between the exchange rate movement and the SDF differential in a regression like most empirical settings, could the unconditional covariance be negative even though the conditional covariance is positive?

We find that the answer is yes, but only if interest rates predict exchange rate movements strongly enough — and with the right sign — periods of expected appreciation could systematically coincide with periods when home investors are expected to be well off and interest rates are high, generating a negative unconditional covariance large enough to overcome the positive conditional one. The data rules this out. The Fama [1984] puzzle tells us that higher interest rates do predict further currency appreciation rather than depreciation — the right sign to offset conditional counter-cyclicality (stylized fact 3) — but the Meese and Rogoff [1983] result tells us this predictive relationship is extremely weak, not strong enough to overturn the first result. When exchange rate predictability is near zero (stylized fact 4), the offsetting force is negligible — no matter what the model.

We thus end up with an impossibility result: when investors can trade domestic and foreign currency bonds, we cannot generate pro-cyclical exchange rates (stylized fact 1) while matching the observed U.I.P. deviations (stylized fact 3) and the lack of exchange rate predictability (stylized fact 4).[2] The literature has studied these moments of exchange rates largely in isolation. However, they are all tied together through the bond investor Euler equations, implying a close relationship between exchange rate cyclicality and predictability. We therefore obtain an unconditional impossibility results that extends the conditional result in Lustig and Verdelhan [2019].

To be clear, currency risk premium shocks that drive U.I.P. deviations do not resolve this impossibility result. In the literature, both risk premium shocks and cross-currency bond Euler equation wedges are often referred to collectively as U.I.P. or financial shocks [see, e.g., Farhi and Werning, 2014, Itskhoki and Mukhin, 2021]. We show that as long as all of the cross-currency bond Euler equations hold without wedges, the Backus-Smith puzzle reappears. Financial shocks that only drive risk premium variation cannot overturn this result, unless the Euler equations are violated.

Turning to a resolution, we find that market segmentation, i.e., completely denying investors access to currency markets, is not required. We analytically characterize the wedges in cross-currency bond Euler equations needed to simultaneously address all of

[2]Recently, some economists have reported more success using non-standard predictors such as a measure of the net foreign asset position [Gourinchas and Rey, 2007], as well as CIP deviations [Jiang, Krishnamurthy, and Lustig, 2021] and global risk measures (e.g., VIX) as proxies for global safe asset demand when forecasting dollar exchange rates, especially since the GFC [Lilley, Maggiori, Neiman, and Schreger, 2022]. Researchers have also found some evidence that survey expectations can predict exchange rates at longer horizons [Kremens, Martin, and Varela, 2023].

these facts. Investors need to act as if they face a tax on foreign currency bond investments or as if they have a preference for domestic currency bond investments. Surprisingly, even a constant wedge may suffice to generate pro-cyclical exchange rates — that is, our resolution does not rely on these Euler equation wedges driving time variations in the exchange rate dynamics. This is because the wedge's level directly enters the investors' risk-return trade-off, adjusting the investors' required risk premia on foreign currency bonds and thus the exchange rate cyclicality. Quantitatively, a 'tax' of around 40 bps. per annum, roughly one half of the variance of exchange rates, may suffice to close the gap between the model and the data.

In closely related work, Chernov, Haddad, and Itskhoki [2023b] develop a framework that maps the space of tradable assets to restrictions on the exchange rate moments. They rule out the asset market structure with four bond Euler equations, and conclude that the exchange rate moments require the asset markets to be segmented or intermediated, and that, in this environment, the local financial markets are uninformative about the exchange rate. They focus only on the conditional Backus-Smith puzzle, not the unconditional version, the focus of our paper. In comparison, we rule in the asset market structure with four bond Euler equations, but we find that wedges in these Euler equations are required to match the moments of exchange rates. Small and constant wedges suffice. We ultimately interpret these wedges as home bias, domestic convenience yields, or intermediation frictions. In doing so, we show how financial markets are informative about exchange rates by disciplining the properties of these Euler equation wedges.

We offer four interpretations of these wedges. First, investors act as if they face large (perceived or actual) transaction costs associated with buying bonds denominated in a foreign currency, consistent with a large body of empirical evidence [Lewis, 1995a]. Recently, Maggiori, Neiman, and Schreger [2020] report strong evidence of a home currency bias in international mutual fund holdings of corporate bonds.[3]

Second, home investors derive large convenience yields on their home bonds, which can arise for example because home bonds function as private liquidity [Krishnamurthy and Vissing-Jorgensen, 2012]. Diamond and Van Tassel [2021] show that government bonds around the world carry convenience yields.

Third, home investors are forced to hold home bonds, e.g., due to financial repression. Governments routinely adopt measures to allow themselves to borrow at below-market rates. This is usually referred to as financial repression [see Reinhart, Kirkegaard, and

[3]The only exception is the dollar, the international reserve currency. Investors are willing to buy dollar-denominated bonds. Maggiori et al. [2020] attribute this home currency bias to the costs of currency hedges and behavioral bias.

Sbrancia, 2011, Chari, Dovis, and Kehoe, 2020]. An example would be financial regulation that requires some participants to hold home bonds.

Finally, in models of frictional financial intermediation, the wedges can also be interpreted as the shadow cost of participating via intermediaries in foreign markets. Gabaix and Maggiori [2015], Itskhoki and Mukhin [2021], Fukui, Nakamura, and Steinsson [2023] all consider models in which investors cannot access currency markets directly. In these models, domestic investors only invest in the domestic bond market and only intermediaries can trade foreign currencies. Similarly, Gourinchas, Ray, and Vayanos [2020], Greenwood, Hanson, Stein, and Sunderam [2020] study models with domestic preferred-habitat investors and global arbitrageurs. This class of models retains only two of the four Euler equations we consider. Crucially, complete market segmentation of this kind corresponds to an *infinite* Euler equation wedge — domestic investors face an unbounded cost of accessing foreign bond markets. Our quantitative analysis shows that this is far more than necessary: a constant wedge of around 40 basis points per annum suffices to match the data. The seminal Gabaix and Maggiori [2015] model and all the other models that followed thus seem to impose a restriction that is larger than what the data require. We do not need market segmentation to resolve these exchange rate puzzles — only a modest friction.

When we calibrate the exchange rate predictability, volatility, the Fama regression coefficient, and the SDF variance to match the data, the cross-currency bond Euler equation wedges have to be at least 40 basis points. Thus, a small departure from frictionless markets, either in the form of transaction costs, convenience yields, or intermediation frictions, can account for the facts, thereby ruling in this plausible class of models as a resolution of the exchange rate puzzles. As mentioned earlier, the key to generating pro-cyclical exchange rates is the cross-currency bond Euler equation wedge's level, as opposed to its covariance with the SDF. In fact, a constant Euler equation wedge, which may be interpreted as a constant transaction cost for foreign currency bonds, suffices to address the exchange rate puzzles. This is because the wedge's first moment directly enters the Euler equations, so that even a constant wedge triggers an endogenous adjustment in the exchange rate cyclicality. In the case of our Euro/dollar example, we can simply choose the European home currency bias for Eurozone bonds and the U.S. bias for dollar bonds to be constant and add up to the variance of the Euro/dollar exchange rate, roughly 100 basis points. In this case, investors in the Euro/dollar market no longer have to believe that their currency appreciates in bad states of the world — the Euro/dollar exchange rate can be perfectly a-cyclical.

Importantly, the wedges do not have to drive exchange rate fluctuations. A constant wedge suffices to satisfy the necessary conditions. The role of the wedges is to break the tight link between exchange rate cyclicality and predictability that frictionless bond markets impose. Once that link is severed, exchange rate fluctuations can be driven by other shocks in the model — productivity, monetary policy, demand — without the model being forced into counter-cyclical exchange rates.

Like the Hansen and Jagannathan [1991] bound, our results offer a model-free diagnostic tool for the necessary conditions needed to explain exchange rate disconnect and predictability, and clarify the role that financial frictions play in equilibrium models of exchange rates. Using this tool, we *rule in* a class of models involving home bias/transaction costs, convenience yields, or intermediation frictions. In comparison, Lustig and Verdelhan [2019] examine the joint restrictions on exchange rate cyclicality/volatility and currency risk premia. They show that frictionless bond trading imposes counterfactually strong restrictions on these moments, which cannot be relaxed by market incompleteness. Jiang, Krishnamurthy, Lustig, and Sun [2024] show that convenience yields on dollar-denominated safe assets can relax these restrictions for the dollar, allowing the model to simultaneously resolve the cyclicality/volatility puzzles and the risk premium puzzle. While Lustig and Verdelhan [2019] study general currency pairs, Jiang et al. [2024]'s mechanism focuses on the dollar.

By contrast, our paper focuses on all bilateral exchange rates and adds another restriction — between exchange rate cyclicality and predictability — and, to our knowledge, is the first one to study this connection. We show that frictionless bond trading also imposes a counterfactually strong restriction on these two moments, while wedges in Euler equations, even small and constant ones, are able to relax these restrictions. Hence, the key ingredient needed to make progress on the exchange rate disconnect and Backus-Smith puzzles for general currency pairs is not market incompleteness, but rather frictions that give rise to wedges in the cross-country bond Euler equations.

Literature. Chari, Kehoe, and McGrattan [2002] analyze a complete-markets model of exchange rates with sticky prices and identify the Backus-Smith puzzle as the key failure of their model. Corsetti, Dedola, and Leduc [2008], Pavlova and Rigobon [2012] consider incomplete market models of exchange rates. In their model, domestic and foreign investors only invest in a risk-free bond that pays off in a global numéraire, implicitly dropping all 4 Euler equations. This implicitly introduces wedges in one of the four bond Euler equations. They report progress on the Backus-Smith puzzle. Chari et al. [2002] and Corsetti et al. [2008] use first-order log-linear approximations to solve their model, and thus approximate away the covariance terms that give rise to these puzzles.

A different strand of the literature segments the markets and thus introduces wedges into the bond Euler equations. Alvarez, Atkeson, and Kehoe [2002a, 2009a] consider a Baumol-Tobin style model in which investors pay a cost to transact in currency and bond markets. Relatedly, Jiang, Krishnamurthy, and Lustig [2018], Jiang et al. [2024] explore the dollar exchange rate implications of convenience yields earned on dollar safe assets, another type of the Euler equation wedges. However, these convenience yields on dollar-denominated assets do not help to resolve the bilateral exchange rate puzzles. Finally, a third strand of the literature, starting with the seminal work by Gabaix and Maggiori [2015], Itskhoki and Mukhin [2021], imputes a central role to financial intermediaries, drawing on insights from the literature on intermediary asset pricing.

Other recent work by Hassan [2013], Dou and Verdelhan [2015], Chien, Lustig, and Naknoi [2020], Jiang and Richmond [2023] takes a different tack by introducing heterogeneity in household trading technologies. Active households can freely trade bonds and other state contingent claims, whereas the inactive households have no access to the asset market. In this case, while the four Euler equations we consider in this paper hold for the active households without any wedges, their marginal utilities have different cyclicality than the country-level aggregate marginal utilities. As a result, the model disconnects aggregate consumption from the SDF of the Euler equation to which the model applies.

The paper is organized as follows. We start by discussing the benchmark complete-market case in section 2. Section 3 discuss the conditional Backus-Smith puzzle in the incomplete-market case. Section 4 analyzes the unconditional Backus-Smith puzzle in the incomplete-market case, and presents the main impossibility result. Finally, section 5 inserts bond Euler equation wedges, and characterizes the restrictions these wedges need to satisfy to make progress on the unconditional Backus-Smith puzzle. The Appendix contains the proof of all propositions.

2 Complete Markets and Exchange Rate Puzzles

Let Δs_{t+1} denote the real exchange rate movement. A positive value means that the home currency appreciates. Let $m_{t,t+1}$ and $m^*_{t,t+1}$ denote the home and foreign investors' one-period pricing kernels in log. In the case of complete markets, exchange rates act as shock absorbers for the shocks to the pricing kernels:

$$\Delta s_{t+1} = m_{t,t+1} - m^*_{t,t+1}.$$

This expression for the log change in the real exchange rate has puzzling implications.

Counter-cyclicality/Backus-Smith puzzle. When markets are complete, the unconditional exchange rate cyclicality, which we define as the covariance between the exchange rate movement Δs_{t+1} and the SDF differential $m_{t,t+1} - m^*_{t,t+1}$, satisfies

$$cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var(\Delta s_{t+1}) > 0.$$

We obtain a very general result: in any complete-market models, the unconditional exchange rate cyclicality is always positive because a higher marginal utility growth in the home country is associated with a home currency appreciation. The model can only generate exchange rate disconnect by shrinking the variance of the exchange rate to zero. In the Breeden-Lucas case with $m_{t,t+1} = \log \delta - \gamma \Delta c_{t+1}$, the implied changes in the log exchange rates are perfectly negatively correlated with consumption growth differences $corr(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta s_{t+1}) = -1$, which is strongly counterfactual [Kollmann, 1991, Backus and Smith, 1993].

Volatility puzzle. As was noted by Brandt et al. [2006b], the implied volatility of the exchange rate will be too high if the market price of risk clears the Hansen and Jagannathan [1991] bounds, unless the pricing kernels are highly correlated across countries.

$$var_t(\Delta s_{t+1}) = var_t(m^*_{t,t+1}) + var(m_{t,t+1}) - 2corr_t(m_{t,t+1}, m^*_{t,t+1}) std_t(m_{t,t+1}) std_t(m^*_{t,t+1}).$$

We would need a correlation of the pricing kernels $corr_t(m_{t,t+1}, m^*_{t,t+1})$ close to one. In the case of the standard Breeden-Lucas SDF, this would imply close to perfectly correlated consumption growth across countries: $corr_t(\Delta c_{t+1}, \Delta c^*_{t+1}) \approx 1$. This prediction is counterfactual [see Backus, Kehoe, and Kydland, 1992]. Complete markets models with time-additive utility cannot match the observed volatility of exchange rates of around 10% to 15% per annum without generating a counterfactually high correlation of consumption growth across countries. Exchange rates are much less volatile than the stochastic discount factors we back out from asset market data (stylized fact 5). That explains why researchers have resorted to incomplete market models. These models mitigate the volatility puzzle by allowing the exchange rate to be less volatile than the SDF differential, but they do not resolve the Backus-Smith puzzle, as we show in the next section.

These puzzles are partially governed by the specific nature of the pricing kernel. The Breeden-Lucas SDF assumes time-additive utility. Colacito and Croce [2011] impute a preference for early resolution to uncertainty to the stand-in investor in an endowment economy with long-run risks [Bansal and Yaron, 2004a]. In this long-run risks economy, it is feasible to make progress on the volatility puzzle by choosing highly correlated persistent components of consumption growth, while still matching the low correlation of consumption growth observed in the data. The long-run risks can push the correlation of the pricing kernels $corr_t(m_{t,t+1}, m^*_{t,t+1})$ to one by choosing perfectly correlated long-run consumption growth $corr_t(x_{t+1}, x^*_{t+1}) = 1$. However, in their benchmark calibration, this mechanism reduces the exchange rate correlation with current consumption growth to zero, but does not overturn the sign. In closely related work, Verdelhan [2010] explores the habit model's exchange rate implications, and concludes that this model cannot entirely resolve the Backus-Smith puzzle. More broadly, even if the SDF is not aggregate consumption growth, we are not aware of any evidence that currencies tend to appreciate when the marginal utility growth of the stand-in investor is higher at home than abroad. The Backus-Smith puzzle applies to any SDF, not just the Breeden-Lucas one.

3 Conditional Exchange Rate Cyclicality and Incomplete Markets

Next, we examine the exchange rate cyclicality when we shut down some asset markets. To do so, we use the covariance of the exchange rate with marginal utility growth as the yardstick of exchange rate cyclicality. For most currencies, there is no evidence that they tend to appreciate when the local investors experience a bad shock resulting in higher marginal utility growth, regardless of how one measures marginal utility.

We start by assuming that investors can invest in risk-free bonds denominated in domestic and foreign currency. Our analysis is silent on the rest of the market structure, such as whether other assets like equities or long-term bonds can be traded.

We assume that the exchange rate and pricing kernel innovations are jointly normally distributed. Then, the four risk-free bond Euler equations imply:

$$
\begin{aligned}
0 &= \mathbb{E}_t[m_{t,t+1}] + \frac{1}{2}var_t(m_{t,t+1}) + r_t, \\
0 &= \mathbb{E}_t[m_{t,t+1}] + \frac{1}{2}var_t(m_{t,t+1}) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) + cov_t(m_{t,t+1}, -\Delta s_{t+1}) + r^*_t, \\
0 &= \mathbb{E}_t[m^*_{t,t+1}] + \frac{1}{2}var_t(m^*_{t,t+1}) + r^*_t, \\
0 &= \mathbb{E}_t[m^*_{t,t+1}] + \frac{1}{2}var_t(m^*_{t,t+1}) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) + cov_t(m^*_{t,t+1}, \Delta s_{t+1}) + r_t.
\end{aligned}
$$

Reorganizing terms, we obtain two expressions that relate the expected excess return of a strategy that goes long in foreign currency and borrows at the domestic risk-free rate to the riskiness of the exchange rate:

$$(r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) = -cov_t(m_{t,t+1}, -\Delta s_{t+1}), \tag{1}$$

$$(r_t - r_t^*) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) = -cov_t(m^*_{t,t+1}, \Delta s_{t+1}). \tag{2}$$

The first expression takes the home investors' perspective. If the foreign currency tends to appreciate (i.e., higher $-\Delta s_{t+1}$) in the home investors' high marginal utility states (i.e., higher $m_{t,t+1}$), then, the foreign currency is a good hedge and the home investors demand a lower expected return to hold it, which leads to a lower $(r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1})$ on the left-hand side. Similarly, the second expression takes the foreign investors' perspective, and relates the currency excess return to the covariance between the foreign investors' SDF and the exchange rate movement.

We sum (1) and (2) to obtain the following proposition.

Proposition 1. *In the log-normal case, the conditional exchange rate cyclicality satisfies*

$$cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1}) > 0. \tag{3}$$

The intuition for Proposition 1 rests on a symmetry embedded in equations (1) and (2). Consider a U.S. investor long in Euro bonds and a Eurozone investor long in dollar bonds. Their expected excess returns are mirror images in logs: the interest rate differential $r_t^* - r_t$ and the expected exchange rate movement $\mathbb{E}_t[\Delta s_{t+1}]$ appear with *opposite* signs for the two investors, so they cancel exactly when the two equations are summed. What does *not* cancel is the variance term $\frac{1}{2} var_t(\Delta s_{t+1})$, which enters with the *same* sign for both. As a result, the aggregate FX premium — the sum of the two investors' expected excess returns — always equals the conditional variance of the exchange rate. Since this aggregate premium is strictly positive whenever there is exchange rate risk, the average investor must perceive their foreign currency position as risky: foreign currency must tend to depreciate in bad states for the average investor. In the log-normal setting, this is equivalent to $cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) > 0$.

This does not rule out one currency being a safe haven. For example, if the dollar is safe from the Eurozone perspective ($-cov_t(m^*_{t,t+1}, \Delta s_{t+1}) < 0$), then the Euro must be perceived as risky by U.S. investors ($-cov_t(m_{t,t+1}, -\Delta s_{t+1}) > 0$) by an even larger amount, so that their sum equals the positive exchange rate variance. One currency can be the safe haven, but the counter-cyclical force from the other side always dominates.

Proposition 1 implies that exchange rate innovations are conditionally positively correlated with relative marginal utility growth. Equivalently, the home currency depreciates in good states for the home investor, or appreciates in good states for the foreign investor. This is a conditional version of the Backus-Smith puzzle. Lustig and Verdelhan [2019] derive a more restrictive version of this result assuming incomplete market wedges that are jointly log-normal with the SDF and the exchange rate. Our derivation does not use incomplete market wedges.[4]

Moreover, we do not need to consider the Euler equations for any assets beyond the four risk-free bonds to reach this conclusion, in contrast to, e.g., Chernov et al. [2023b]. The bond Euler equations alone are sufficiently informative.

The result calls into question the approach of introducing "risk premium shocks" that drive U.I.P. deviations into international finance models. In the literature, both risk premium shocks and wedges that represent deviations from cross-currency bond Euler equations are often referred to collectively as U.I.P. or financial shocks [see, e.g., Farhi and Werning, 2014, Itskhoki and Mukhin, 2021]. However, accounting for risk explicitly and tying risk premia to covariance with the SDF still implies a strong restriction: as long as the bond Euler equations hold without wedges, the exchange rate cyclicality puzzle reappears. Financial shocks that only drive risk premium variations cannot overturn this result.

Finally, the SDF and exchange rate dynamics might not be conditionally Gaussian, e.g., as in Rietz [1988], Longstaff and Piazzesi [2004], Barro [2006], Farhi and Gabaix [2016]. We show in Appendix A.5 that the result extends to settings with rare disasters and jump risk, but also that under certain non-Gaussian distributions the covariance condition can be reversed. We discuss the conditions under which the result holds in those settings.

4 Unconditional Exchange Rate Cyclicality and Incomplete Markets

In international real business cycle models that link the SDFs to aggregate consumption shocks, we are interested in understanding how the exchange rate moves in response to relative consumption growth. Backus and Smith [1993] summarize this relationship by regressing the exchange rate movement on relative consumption growth, and find pro-

[4]While Lustig and Verdelhan [2019] find that market incompleteness helps with the Brandt, Cochrane, and Santa-Clara [2006a] puzzle in reducing volatility, it cannot change the sign of the conditional covariance. We have not assumed that markets are complete to derive this result. In the case of complete markets, state-contingent interest rate parity obtains $m_{t,t+1} - m^*_{t,t+1} = \Delta s_{t+1}$ and this covariance result is directly obtained. Our proposition shows that this result is much more general, as long as investors can freely trade home and foreign risk-free bonds. In other words, the risk-free bond Euler equations discipline the joint dynamics of the exchange rates and marginal utility growth to imply counter-cyclical exchange rates.

cyclical exchange rates. To relate our result to this Backus-Smith coefficient, we need to characterize the unconditional exchange rate cyclicality. To do so, we use the law of total covariance:

$$cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = \mathbb{E}[cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})] + cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]).$$

Our previous result shows the conditional exchange rate cyclicality $cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})$ on the right-hand side is always positive. To generate a negative unconditional cyclicality $cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})$ on the left-hand side of this equation, we need a negative $cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}])$ on the right-hand side that is greater in magnitude than the average conditional cyclicality $\mathbb{E}[cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})]$:

$$-cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) > \mathbb{E}[cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})]. \tag{4}$$

In other words, the exchange rate movement needs to be strongly predictable by the expected SDF differential $\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}]$. In this way, this equation connects the unconditional exchange rate cyclicality to exchange rate predictability.

Recall the case of complete markets, $\Delta s_{t+1} = m_{t,t+1} - m^*_{t,t+1}$. The exchange rate has to absorb all of the shocks to the pricing kernels in each state of the world. When markets are complete, the unconditional exchange rate cyclicality satisfies

$$cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var(\Delta s_{t+1}) > 0.$$

Thus, in any complete-market models (which by definition allows home and foreign agents to trade risk-free bonds), the unconditional exchange rate cyclicality is always positive: a higher marginal utility growth in the home country is associated with a home currency appreciation. The model can only generate exchange rate disconnect by shrinking the variance of the exchange rate to zero.

Next, we consider a more general case of incomplete markets in which we shut down some trade in non-bond asset markets. We begin by introducing some concepts.

The Fama Regression Coefficient b. We use f_t to denote the log of the one-period forward exchange rate in units of foreign currency per dollar. The log excess return on buying foreign currency forward is

$$rx_{t+1} = f_t - s_{t+1} = -\Delta s_{t+1} + f_t - s_t,$$

where $f_t - s_t$ denotes the forward discount and Δs_{t+1} denotes the appreciation of the home currency. When the Covered Interest Rate Parity holds, we further obtain $f_t - s_t = r_t^* - r_t$ and, as a result, we can restate the log excess return on a long position in foreign currency as $rx_{t+1} = -\Delta s_{t+1} + r_t^* - r_t$.

Now, consider the standard Tryon [1979], Hansen and Hodrick [1980], Fama [1984] time-series regression:

$$\Delta s_{t+1} = a + b(f_t - s_t) + \varepsilon_{t+1}. \tag{5}$$

In the data, the slope coefficient b tends to be negative: a higher-than-usual foreign interest rate predicts further appreciation of the foreign currency, or, equivalently, a further depreciation of the home currency.

Following Fama [1984], the regression yields

$$b = \frac{cov(f_t - s_t, \mathbb{E}_t[\Delta s_{t+1}])}{var(f_t - s_t)} = \frac{cov(\mathbb{E}_t[rx_{t+1}], \mathbb{E}_t[\Delta s_{t+1}]) + var(\mathbb{E}_t[\Delta s_{t+1}])}{var(f_t - s_t)}.$$

To get a negative slope coefficient b, we need $cov(\mathbb{E}_t[rx_{t+1}], \mathbb{E}_t[\Delta s_{t+1}]) < -var(\mathbb{E}_t[\Delta s_{t+1}])$, which means that risk premia have to be more volatile than the expected change in the spot rate. Backus, Foresi, and Telmer [2001] analyze sufficient conditions for these U.I.P. violations in a large class of affine asset pricing models.

The Meese-Rogoff R^2. The Meese and Rogoff [1983] puzzle states that exchange rates are hard to forecast. Put differently, the R^2 in a forecasting regression is low. In a recent survey of exchange rate predictability, Rossi [2013] concludes that the Meese-Rogoff findings have not been conclusively overturned.[5] There is some limited evidence of exchange rate predictability, but the evidence usually is specific to certain countries and horizons, and the predictability is not stable.

We assume the linear predictor yields the best forecast. In this case, we can obtain this R^2 from a projection of the exchange rate changes on its predictors. Let R^2 denote the fraction of the predictable variation in the exchange rate:

$$R^2 = \frac{var(\mathbb{E}_t[\Delta s_{t+1}])}{var(\Delta s_{t+1})},$$

[5] At higher frequencies ranging from one day to one month, order flow seems to predict changes in the spot exchange rate [see Evans and Lyons, 2002a, 2005]. This data is proprietary and may not be available in real-time to all investors. Gourinchas and Rey [2007] report evidence that the net foreign asset position predicts changes in the exchange rate out-of-sample.

and let R^2_{Fama} denote the R^2 of the Fama regression (5):

$$R^2_{Fama} = \frac{var(b(f_t - s_t))}{var(\Delta s_{t+1})}.$$

Characterizing Unconditional Exchange Rate Cyclicality. Our main result characterizes the unconditional exchange rate cyclicality. Without loss of generality, we assume that the covariance between the home SDF's conditional variance and the exchange rate movement from the home perspective is higher than the covariance between the foreign SDF's conditional variance and the exchange rate movement from the foreign perspective:

$$cov\left(var_t(m_{t,t+1}), \mathbb{E}_t[\Delta s_{t+1}]\right) \geq cov\left(var_t(m^*_{t,t+1}), -\mathbb{E}_t[\Delta s_{t+1}]\right).$$

If this condition is not satisfied, we simply need to swap the labeling of the home and foreign countries.

Proposition 2. *Each of the following is a necessary condition for a negative unconditional exchange rate cyclicality, i.e.,* $cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$*:*

(a)

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}} + \sqrt{R^2}\left(\frac{1}{b}\frac{R^2_{Fama}}{R^2} - 1\right). \tag{6}$$

If the Fama regression yields the best predictor of the exchange rate movement, then, we can simplify this formula to

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}} + \sqrt{R^2}\left(\frac{1}{b} - 1\right) = \frac{1}{\sqrt{R^2}} - \sqrt{R^2} + sign(b)\frac{std(f_t - s_t)}{std(\Delta s_{t+1})}. \tag{7}$$

(b)

$$\sqrt{\frac{std(\mathbb{E}_t[rx_{t+1}])}{std(\mathbb{E}_t[\Delta s_{t+1}])} + \frac{std\left(var_t(m_{t,t+1})\right)}{std(\mathbb{E}_t[\Delta s_{t+1}])}} > \frac{std(\Delta s_{t+1})}{std(\mathbb{E}_t[\Delta s_{t+1}])} = \frac{1}{\sqrt{R^2}}. \tag{8}$$

Between these conditions,

- *(a)* $\Rightarrow$ *(b).*
- *If Fama regression yields the best predictor and* $b \notin (0, 1)$*, (b)* $\Leftrightarrow$ *(a); otherwise (b) is a weaker condition.*

To provide some intuition behind condition (a), let us begin with the inequality (4) that must be satisfied to generate a negative unconditional exchange rate cyclicality:

$$-cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) > \mathbb{E}[cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})].$$

The expected SDF drift term $\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}]$ on the left-hand side is closely related to the interest rate differential. Thus, its covariance with the expected currency movement is reminiscent of the Fama regression, which is why the Fama b-coefficient enters the bound.

On the right-hand side, given (3), it can be rewritten as $\mathbb{E}[cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1})] = \mathbb{E}[var_t(\Delta s_{t+1})] = var(\Delta s_{t+1}) - var(\mathbb{E}_t[\Delta s_{t+1}])$, which compares the unconditional variance of the exchange rate movement to the variance of the expected exchange rate movement. This comparison reflects the maximal amount of exchange rate predictability, hence the role of the Meese-Rogoff R^2 in the bound. A low R^2 means that the exchange rate is largely unpredictable, i.e., $var(\Delta s_{t+1})$ is much larger than the variance of the predicted component $var(\mathbb{E}_t[\Delta s_{t+1}])$, which makes the right-hand side of (4) larger and the bound harder to satisfy.

In this way, (4) relates the Fama b-coefficient on the left-hand side to the exchange rate predictability R^2 on the right. It is useful to see the following intermediate step in our proof for this proposition:

$$\underbrace{-b \times \frac{var(f_t - s_t)}{var(\mathbb{E}_t[\Delta s_{t+1}])}}_{\text{Fama Regression Term}} + \underbrace{\frac{std\,(var_t(m_{t,t+1}))}{std(\mathbb{E}_t[\Delta s_{t+1}])}}_{\text{Jensen Term}} > \underbrace{\frac{var(\Delta s_{t+1}) - var(\mathbb{E}_t[\Delta s_{t+1}])}{var(\mathbb{E}_t[\Delta s_{t+1}])} = \frac{1}{R^2} - 1}_{\text{Exchange Rate Predictability}}.$$

Reorganizing this expression further yields condition (a) in Proposition 2. Condition (b) is derived through additional algebraic manipulation of condition (a).

Moreover, conditions (a) and (b) are necessary, but not sufficient conditions for a negative unconditional exchange rate cyclicality. In our derivation, we use the fact that the correlation is bounded between -1 and 1. Therefore, our bounds can be thought of as an international version of the Hansen-Jagannathan bound. While the Hansen-Jagannathan bound relates the volatility of the SDF to the Sharpe ratio of any asset, our bounds relate the volatility of the conditional variance of the SDF to the Fama regression coefficient and the exchange rate predictability.

Notably, our bounds tighten as the R^2 decreases: as exchange rates become less predictable, we need more variations in the conditional risk premia and the conditional price of risk to generate a negative unconditional exchange rate cyclicality. In the limit, as we approach the Meese and Rogoff [1983]'s benchmark random walk case in which exchange rate movements are not predictable, $1/\sqrt{R^2}$ on the right-hand side approaches infinity. The model simply cannot deliver pro-cyclical exchange rates. As such, these bounds deliver an impossibility result: if we take Meese and Rogoff [1983] random walk result regarding *exchange rate predictability* at face value, then we cannot make progress on the Backus and Smith [1993] puzzle regarding *exchange rate cyclicality*.

On the other hand, while it is enticing to conclude that, holding R^2 constant, a small but negative Fama coefficient b can lower the right-hand side of Eq. (6) and make this condition more likely to hold, we note that R^2_{Fama} and b are closely related. When the b shrinks towards zero, the R^2_{Fama} shrinks towards zero as well. In fact, Eq. (7) shows that, holding R^2 constant, a small but negative b means a high forward premium volatility $std(f_t - s_t)$ relative to the exchange rate volatility $std(\Delta s_{t+1})$, which is also rejected by the data.

Before we turn to more realistic cases, we make two more observations. First, in the case of U.I.P., $b = 1$ and Eq. (7) becomes:

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}}.$$

A natural case to consider under the U.I.P. is the case of constant market prices of risk. Then, we get an impossibility result: $0 > 1/\sqrt{R^2}$, so the unconditional exchange rate cyclicality cannot be negative.

Second, in the case of a fully predictable exchange rate movement, the R^2 tends to one and Eq. (7) becomes:

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{b}.$$

As long as the slope coefficient b is negative, then the bound is trivially satisfied, even when the R^2 is very high but not equal to 1.

In summary, the intuition behind the bound in Proposition 2 is a race between two forces. On one side, Proposition 1 establishes a floor: the average conditional covariance is always positive and equal to the average conditional variance of the exchange rate. On the other side, the predictable component of exchange rate movements — captured by the expected SDF differential, and proxied in the data by the interest rate differential — could in principle generate a negative covariance of conditional expectations large enough to overcome this floor. Proposition 2 characterizes exactly how large that predictable component would need to be.

However, given the observed Fama regression coefficient and the near-zero Meese-

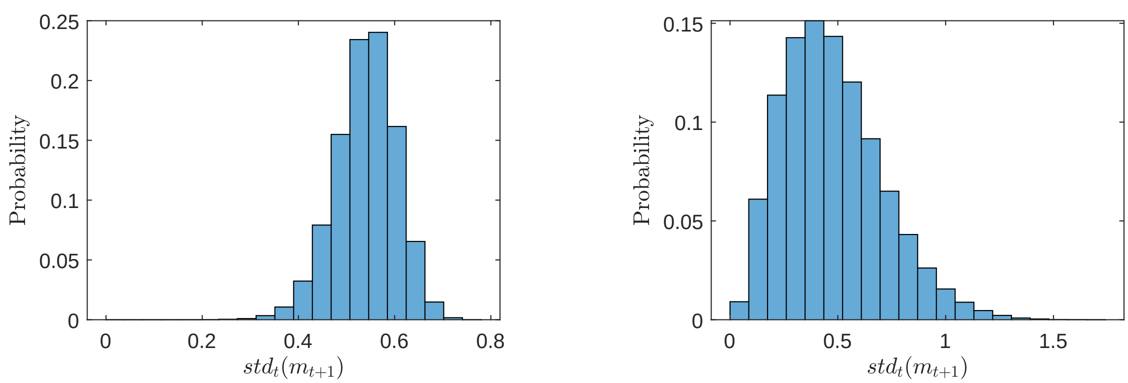
Rogoff R^2, the predictable component falls far short. In the limit, as the R^2 approaches zero, the required offset approaches infinity — the model simply cannot deliver procyclical exchange rates. The exchange rate cyclicality and predictability puzzles are therefore not independent empirical facts. They are tied together through the bond Euler equations, and jointly they form a nearly inescapable trap for frictionless models.

4.1 Calibrated Examples

Let us compute the bound using some empirically plausible values. In the data, the exchange rate movements are only moderately predictable. Suppose the Fama regression yields the best predictor of the exchange rate movement, $R^2 = 5\%$ at the one-year horizon, the Fama regression coefficient is $b = -1$, and the annualized exchange rate volatility is $std(\Delta s_{t+1}) = 10\%$. Then, Eq. (7) implies that the log SDF needs to have a very high variability in its conditional variance:

$$std\left(var_t(m_{t,t+1})\right) > 0.4. \tag{9}$$

For comparison, the unconditional standard deviation of the log SDF variance is only 0.067 in the long-run risk model in Bansal and Yaron [2004b], and 0.26 in the external habit



(A) LONG-RUN RISK MODEL (B) EXTERNAL HABIT MODEL

FIGURE 1. DISTRIBUTIONS OF THE CONDITIONAL LOG SDF VOLATILITY IN DIFFERENT ASSET PRICING MODELS

Notes: The figure plots the distributions of the conditional log SDF volatility in the long-run risk model in Bansal and Yaron [2004b] and the external habit model in Campbell and Cochrane [1999]. We simulate 1,000,000 monthly periods in each model and report the histogram of the annualized log SDF volatility.

model in Campbell and Cochrane [1999], even though both models manage to generate empirically plausible levels of the equity risk premium. Figure (1) plots the distributions of the conditional SDF volatility in these two models, which are related to the maximum Sharpe ratio in the economy according to the Hansen and Jagannathan [1991] bound. In the long-run risk model, the conditional log SDF volatility is between 0.3 and 0.7, a narrow range that gives rise to a small $std\left(var_t(m_{t,t+1})\right)$. In the external habit model, the conditional log SDF volatility is more variable, at the expense of having some states in which the SDF is not very volatile and the maximum Sharpe ratio is very small, and some states in which the SDF is extremely volatile and the maximum Sharpe ratio is very high.

If we take either model as a quantitatively accurate representation of the SDF, then, Eq. (9) implies that the model cannot generate a negative unconditional exchange rate cyclicality. In other words, the home and foreign investors' Euler equations governing their risk-free bond holdings impose important restrictions on the equilibrium exchange rate dynamics, such that a negative unconditional exchange rate cyclicality requires a very volatile SDF conditional variance that is not in line with standard asset pricing models.

Alternatively, suppose that the forward premium has no predictive power for the exchange rate movement, which implies $R^2_{Fama} = 0$ and $b = 0$. Suppose some other predictor generates an $R^2 = 5\%$ and the annualized exchange rate volatility is $std(\Delta s_{t+1}) = 10\%$.

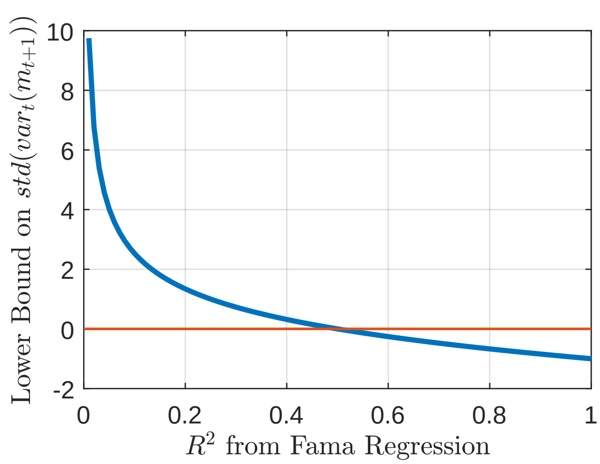


FIGURE 2. LOWER BOUND ON THE VARIABILITY OF THE CONDITIONAL LOG SDF VARIANCE AS A FUNCTION OF EXCHANGE RATE PREDICTABILITY

Notes: The figure plots the lower bound on $std\left(var_t(m_{t,t+1})\right)$ as a function of the Fama regression R^2 as implied by Eq. (7). We take $b = -1$ and $std(\Delta s_{t+1}) = 10\%$.

Then, Eq. (6) implies

$$std\left(var_t(m_{t,t+1})\right) > std(\Delta s_{t+1})\left(\frac{1}{\sqrt{R^2}} - \sqrt{R^2}\right) = 0.42,$$

which provides a similar bound as in the previous case.

We can also generalize the bound in Eq. (9) to a wider range of exchange rate predictability. In Figure (2), we vary the R^2 from the Fama regression and report the bound on the volatility of the conditional log SDF variance. As the exchange rate predictability increases, the lower bound on the unconditional volatility of the conditional log SDF variance declines. When R^2 is 20% in the Fama regression, the lower bound becomes 0.13, compared to 0.40 when R^2 is 5%. In other words, as the exchange rate becomes more predictable, the Euler equations can rationalize a negative unconditional exchange rate cyclicality without requiring a higher degree of variability in the conditional SDF variance.

Finally, Proposition 2 offers a loose bound in the sense that we used the property $corr\left(var_t(m_{t,t+1}), \mathbb{E}_t[\Delta s_{t+1}]\right) \leq 1$ to derive the inequality, just like Hansen and Jagannathan [1991], which makes the bounds in this proposition necessary but not sufficient conditions. The bounds are only sufficient conditions when $corr\left(var_t(m_{t,t+1}), \mathbb{E}_t[\Delta s_{t+1}]\right) = 1$. If, instead, the correlation is 0.5, Eq. (9) in the numerical example becomes

$$std\left(var_t(m_{t,t+1})\right) > 0.4/(1/2) = 0.8,$$

which further sharpens the result by doubling the lower bound on $std\left(var_t(m_{t,t+1})\right)$.

4.2 The Role of the Time Horizon

We consider a Fama regression with horizon k periods:

$$\Delta s_{t,t+k} = a_k + b_k(f_t^k - s_t) + \varepsilon_{t+k}.$$

Similarly, we use $R^2_{Fama,k}$ to denote the R^2 of this regression, and we define

$$R_k^2 = \frac{var(\mathbb{E}_t[\Delta s_{t,t+k}])}{var(\Delta s_{t,t+k})}.$$

Using these definitions, we can generalize the bounds in Proposition 2 to a multi-period setting.

Proposition 3. *Each of the following is a necessary condition for a negative unconditional exchange rate cyclicality, i.e., $cov(m_{t,t+k} - m^*_{t,t+k}, \Delta s_{t,t+k}) < 0$:*

(a)

$$\frac{std\left(var_t(m_{t,t+k})\right)}{std(\Delta s_{t,t+k})} > \frac{1}{\sqrt{R_k^2}} + \sqrt{R_k^2}\left(\frac{1}{b_k}\frac{R^2_{Fama,k}}{R_k^2} - 1\right).$$

If the Fama regression yields the best predictor of the exchange rate movement, then, we can simplify this formula to

$$\frac{std\left(var_t(m_{t,t+k})\right)}{std(\Delta s_{t,t+k})} > \frac{1}{\sqrt{R_k^2}} + \sqrt{R_k^2}\left(\frac{1}{b_k} - 1\right) = \frac{1}{\sqrt{R_k^2}} - \sqrt{R_k^2} + sign(b_k)\frac{std(f_t^k - s_t)}{std(\Delta s_{t,t+k})}$$

(b)

$$\sqrt{\frac{std(\mathbb{E}_t[rx_{t,t+k}])}{std(\mathbb{E}_t[\Delta s_{t,t+k}])} + \frac{std\left(var_t(m_{t,t+k})\right)}{std(\mathbb{E}_t[\Delta s_{t,t+k}])}} > \frac{std(\Delta s_{t,t+k})}{std(\mathbb{E}_t[\Delta s_{t,t+k}])} = \frac{1}{\sqrt{R_k^2}}.$$

Among these conditions,

- *(a) $\Rightarrow$ (b).*
- *If Fama regression yields the best predictor and $b \notin (0,1)$, (b) $\Leftrightarrow$ (a); otherwise (b) is a weaker condition.*

As we increase the horizon, $std\left(var_t(m_{t,t+k})\right)$ increases faster than $\sqrt{k}$, while $std(\mathbb{E}_t[\Delta s_{t,t+k}])$ converges to zero if a long-run version of PPP holds and the real exchange rate is stationary.[6] On the other hands, as $k \to \infty$, a long-run version of U.I.P kicks in and $b_k \to 1$. Lustig, Stathopoulos, and Verdelhan [2019] show that long-run U.I.P. is implied by no arbitrage when real exchange rates are stationary. Finally, there is evidence that exchange rates are more predictable as k rises, indicating that R_k^2 rises. Thus, as the horizon increases, the bound is easier to satisfy. We return to this observation in the next section on wedges.

5 Bond Euler Equation Wedges

Proposition 1 shows that IRBC models with the four bond Euler equations cannot simultaneously generate a negative Backus-Smith coefficient and replicate the Fama regression

[6] As noted by Rogoff [1996], the real exchange rate's rate of convergence to its long-run mean is slow.

coefficient and the Meese-Rogoff puzzle. As a result, we need to entertain models that break these four Euler equations in one way or another.

One approach to doing so is in Corsetti et al. [2008], Pavlova and Rigobon [2012], who consider incomplete-market settings in which only one type of bond is traded. When the bond is denominated in a given country's numéraire, this set-up drops two of the four Euler equations we consider. When the bond is denominated in a basket of country-level numéraires, this set-up drops all of our four Euler equations and replaces them with two new ones. Investors around the world clearly do trade bonds in different currencies, so that we will not entertain this possibility as a resolution to the puzzle.

We instead outline two other approaches that can work: introducing convenience yields on home bonds, and introducing transaction costs when investing in foreign bonds. We study these cases next and explain how they implicitly insert cross-currency Euler equation wedges into the model we have considered.

5.1 Home Currency Bias, Convenience Yields, and Financial Repression

First, we can break the Euler equations by considering the case where investors have an extra preference towards purchasing the bonds in their own currency. We denote these home wedges for the domestic and foreign investor respectively as (ϕ_t, ϕ_t^*). Then, the four Euler equations can be expressed as follows:

$$\begin{aligned} \exp(\phi_t) &= \mathbb{E}_t\left[\exp(m_{t,t+1} + r_t)\right], \\ 1 &= \mathbb{E}_t\left[\exp(m_{t,t+1} - \Delta s_{t+1} + r_t^*)\right], \\ \exp(\phi_t^*) &= \mathbb{E}_t\left[\exp(m_{t,t+1}^* + r_t^*)\right], \\ 1 &= \mathbb{E}_t\left[\exp(m_{t,t+1}^* + \Delta s_{t+1} + r_t)\right]. \end{aligned} \tag{10}$$

Assuming log-normality, we obtain two expressions that relate the expected currency excess return to the perceived risks from the home and foreign perspectives as well as the wedges:

$$\begin{aligned} (r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) + \phi_t &= -cov_t(m_{t,t+1}, -\Delta s_{t+1}), \\ (r_t - r_t^*) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) + \phi_t^* &= -cov_t(m_{t,t+1}^*, \Delta s_{t+1}). \end{aligned}$$

These two equations clarify how wedges modify the key intuition from Proposition 1. Recall that without wedges, summing the home and foreign Euler equations causes the interest rate differential and expected exchange rate movement to cancel, leaving only the variance term. The aggregate FX premium — the sum of the two investors' expected excess returns — has to equal the conditional variance $var_t(\Delta s_{t+1})$, which is strictly positive and forces exchange rates to be counter-cyclical.

With wedges, summing the two equations above yields an aggregate risk premium equal to $var_t(\Delta s_{t+1}) + \phi_t + \phi_t^*$. When both home and foreign investors have a preference for their home currency assets ($\phi_t < 0$ and $\phi_t^* < 0$), the aggregate risk premium falls below the exchange rate variance. If the wedges are sufficiently negative, the aggregate risk premium can even turn negative, which is consistent with pro-cyclical exchange rates from both home and foreign perspectives.

We formalize this intuition in the following proposition.

Proposition 4. *In the presence of Euler equation wedges, the conditional exchange rate cyclicality is given by:*

$$cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1}) + (\phi_t + \phi_t^*), \tag{11}$$

and the conditional correlation is given by:

$$corr_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = \frac{std_t(\Delta s_{t+1})}{std_t(m_{t,t+1} - m^*_{t,t+1})} \left(1 + \frac{(\phi_t + \phi_t^*)}{var_t(\Delta s_{t+1})}\right).$$

In order to obtain a conditionally pro-cyclical exchange rate with $cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$, we need

$$var_t(\Delta s_{t+1}) < -(\phi_t + \phi_t^*). \tag{12}$$

In other words, to generate an exchange rate disconnect from SDF innovations, there is no need to shrink the conditional exchange rate variance to zero. Negative wedges ϕ and ϕ^* can mitigate the conditional version of the Backus-Smith puzzle.

It is instructive to understand why these wedges help, and why time variation in the wedges is not needed. When discussing Proposition 1, we explained the key to the failure to rationalize a counter-cyclical exchange rate is that if both home and foreign investors are on their Euler equations, then the foreign currency must be risky on average for each currency pair, because the FX investors jointly earn back the variance in expectation. The wedges break this connection between the variance of exchange rates and the cyclicality. To see why, suppose we chose the wedges to be minus one half of the variance in each country. Then, the adjusted exchange rate variance $var_t(\Delta s_{t+1}) + (\phi_t + \phi_t^*)$ on the right-hand side of (11) is exactly zero: it is as if we have eliminated currency risk and reset the exchange rate variance to zero, so that the exchange rate no longer needs to be countercyclical for the currency pair.

The negative bond wedges ϕ and ϕ^* can be interpreted in a number of ways. First, they can reflect a home bias by investors, which has been documented extensively in the literature [see, e.g. Lewis, 1995b], and particularly a home currency bias in bond holdings as documented by Maggiori et al. [2020]. In this case, investors are willing to accept lower expected returns on their home bonds.

Second, they can be a symptom of high convenience yields that home investors receive on their home bond holdings. Krishnamurthy and Vissing-Jorgensen [2012] present such evidence for U.S. Treasury bonds, while Diamond and Van Tassel [2021] present evidence for risk-free government bonds around the world. Note that the negative ϕ and ϕ^* can capture home investors' preference for home bonds, and not foreign investors' preferences for U.S. dollar bonds as argued for in Jiang et al. [2018]. We return to this later in the case below.

Third, the negative ϕ and ϕ^* can be seen as a symptom of financial repression. Governments routinely adopt measures to allow themselves to borrow at below-market rates. This is usually referred to as financial repression [see Reinhart et al., 2011, Chari et al., 2020]. During the Great Financial Crisis, banks were induced by their national governments to buy the sovereign debt of their countries [Acharya and Steffen, 2015, De Marco and Macchiavelli, 2016, Ongena, Popov, and Van Horen, 2019]. Since the 2008 GFC, central banks in advanced economies have increased the size of their balance sheets to purchase government bonds, a new wave of financial repression [see Hall and Sargent, 2022, for a comparison of the pandemic and two World Wars]. Financial repression come in other forms, including macro-prudential regulation that favors government bonds, direct lending to the government by domestic pension funds and banks, and moral suasion used to increase domestic bank holdings of government bonds [see Acharya and Steffen, 2015, De Marco and Macchiavelli, 2016, Ongena et al., 2019, for examples from Europe during the global financial crisis]. Japan's yield curve control policy is another textbook example of financial repression. In these situations, financial institutions also behave as if they have a preference for holding government bonds.

5.2 Cross-Currency Wedges and Costs of Foreign Bond Investment

We next consider wedges only in the Euler equations of investors buying foreign currency risk-free bonds. These wedges for the home and foreign investors respectively are denoted as ξ_t and ξ_t^*. Then, the four Euler equations can be expressed as follows:

$$1 = \mathbb{E}_t\left[\exp(m_{t,t+1} + r_t)\right], \tag{13}$$

$$\exp(\xi_t) = \mathbb{E}_t\left[\exp(m_{t,t+1} - \Delta s_{t+1} + r_t^*)\right], \tag{14}$$

$$1 = \mathbb{E}_t\left[\exp(m^*_{t,t+1} + r_t^*)\right], \tag{15}$$

$$\exp(\xi_t^*) = \mathbb{E}_t\left[\exp(m^*_{t,t+1} + \Delta s_{t+1} + r_t)\right]. \tag{16}$$

These wedges are security-specific: they only apply to the bonds denominated in a currency different from the domestic currency. If the wedges are positive, investors effectively apply a higher discount rate to the foreign-currency bond payoffs and therefore require a higher expected return. Intuitively, these wedges help because they break symmetry and imply a preference for the home currency even if the home currency is not a hedge.

Reorganizing terms, we obtain two expressions that relate the expected currency excess return to the perceived risks from the home and foreign perspectives as well as the wedges:

$$(r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) = -cov_t(m_{t,t+1}, -\Delta s_{t+1}) + \xi_t,$$
$$(r_t - r_t^*) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) = -cov_t(m^*_{t,t+1}, \Delta s_{t+1}) + \xi_t^*.$$

Combining these expressions, we obtain the following characterization of the conditional exchange rate cyclicality in the presence of Euler equation wedges.

Proposition 5. *In the presence of Euler equation wedges, the conditional exchange rate cyclicality is given by:*

$$cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1}) - (\xi_t^* + \xi_t),$$

and the conditional correlation is given by:

$$corr_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = \frac{std_t(\Delta s_{t+1})}{std_t(m_{t,t+1} - m^*_{t,t+1})}\left(1 - \frac{(\xi_t^* + \xi_t)}{var_t(\Delta s_{t+1})}\right).$$

In order to obtain a conditionally pro-cyclical exchange rate with $cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$, we need positive wedges that exceed the exchange rate variance:

$$var_t(\Delta s_{t+1}) < (\xi_t^* + \xi_t). \tag{17}$$

Consider an equilibrium in which agents hold a positive quantity of foreign bonds, as is the case in the data, then the ξ wedges are effectively a tax on the return obtained by investors on the foreign bond holdings. Thus, this case can reflect home currency bias as well as the transaction cost models of Alvarez, Atkeson, and Kehoe [2002b, 2009b], in which agents need to pay a cost to access foreign securities and currency markets.[7]

In Proposition 3, we showed that the k-horizon bound is easier to satisfy as k rises, in part because there is stronger exchange rate predictability in the long run than in the short run. The wedge counterpart of this result is that the wedges required for making long-horizon bond investments are lower than the wedges required for making short-horizon investments. When the wedges are interpreted in terms of transaction costs, this result is plausible, as a long-horizon investor will effectively pay a smaller per-period cost for cross-currency investments.

5.3 Four Wedges and Foreign Demand for Dollar Bonds

Finally, we consider the case with all four wedges:

$$\begin{aligned}
\exp(\phi_t) &= \mathbb{E}_t\left[\exp(m_{t,t+1} + r_t)\right], \\
\exp(\xi_t) &= \mathbb{E}_t\left[\exp(m_{t,t+1} - \Delta s_{t+1} + r_t^*)\right], \\
\exp(\phi_t^*) &= \mathbb{E}_t\left[\exp(m_{t,t+1}^* + r_t^*)\right], \\
\exp(\xi_t^*) &= \mathbb{E}_t\left[\exp(m_{t,t+1}^* + \Delta s_{t+1} + r_t)\right].
\end{aligned} \tag{18}$$

Assuming log-normality, we obtain two expressions that relate the expected excess return of a strategy that goes long the foreign bond to the perceived risks from the home and foreign perspectives as well as the wedges:

$$\begin{aligned}
(r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) &= -cov_t(m_{t,t+1}, -\Delta s_{t+1}) + \xi_t - \phi_t, \\
(r_t - r_t^*) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) &= -cov_t(m_{t,t+1}^*, \Delta s_{t+1}) + \xi_t^* - \phi_t^*.
\end{aligned}$$

Combining these equations yields the following characterization of the conditional exchange rate cyclicality.

[7] These cross-country wedges can also be interpreted as the product of subjective belief mistakes in the joint dynamics of the foreign bond return and the exchange rate. In this case, while the Euler equations hold without wedges under investors' subjective expectations, their biased beliefs implimply Euler equation wedges for cross-country bond holdings under the econometrician's information set — for example, if investors are systematically pessimistic about the foreign bond returns or they overestimate the risks, their belief bias can lead to positive ξ_t and ξ_t^* in the Euler equations.

Proposition 6. *In the presence of Euler equation wedges, the conditional exchange rate cyclicality is given by:*

$$cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1}) - (\xi^*_t + \xi_t) + (\phi_t + \phi^*_t),$$

and the conditional correlation is given by:

$$corr_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = \frac{std_t(\Delta s_{t+1})}{std_t(m_{t,t+1} - m^*_{t,t+1})}\left(1 + \frac{(\phi_t + \phi^*_t) - (\xi^*_t + \xi_t)}{var_t(\Delta s_{t+1})}\right).$$

In order to obtain conditionally pro-cyclical exchange rates with $cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$, we need

$$var_t(\Delta s_{t+1}) < (\xi^*_t + \xi_t) - (\phi_t + \phi^*_t),$$

which shows that both home bond bias and cross-currency transaction cost can help to mitigate the conditional version of the Backus-Smith puzzle.

Unconditional Cyclicality. Next, we derive restrictions on the wedges that are needed to change the sign of the unconditional exchange rate cyclicality.

Proposition 7. *Let q_t denote $\mathbb{E}_t[\Delta s_{t+1}]$, and let*

$$\omega = \mathbb{E}[-(\xi^*_t + \xi_t) + (\phi_t + \phi^*_t)] - cov\left(\phi^*_t, q_t\right) + cov\left(\phi_t, q_t\right).$$

In the presence of Euler equation wedges, each of the following is a necessary condition for a negative unconditional exchange rate cyclicality, i.e., $cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$:

(a)

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}}\left(1 + \frac{\omega}{var(\Delta s_{t+1})}\right) + \sqrt{R^2}\left(\frac{1}{b}\frac{R^2_{Fama}}{R^2} - 1\right).$$

If the Fama regression yields the best predictor of the exchange rate movement, then, we can simplify the formula to

$$\begin{aligned}\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} &> \frac{1}{\sqrt{R^2}}\left(1 + \frac{\omega}{var(\Delta s_{t+1})}\right) + \sqrt{R^2}\left(\frac{1}{b} - 1\right) \\ &= \frac{1}{\sqrt{R^2}}\left(1 + \frac{\omega}{var(\Delta s_{t+1})}\right) - \sqrt{R^2} + sign(b)\frac{std(f_t - s_t)}{std(\Delta s_{t+1})}.\end{aligned}$$

(b)

$$\sqrt{\frac{std(\mathbb{E}_t[rx_{t+1}])}{std(\mathbb{E}_t[\Delta s_{t+1}])}+\frac{std\left(var_t(m_{t,t+1})\right)}{std(\mathbb{E}_t[\Delta s_{t+1}])}} > \frac{1}{\sqrt{R^2}}\sqrt{1+\frac{\omega}{var(\Delta s_{t+1})}}.$$

Among these conditions,

- *(a) $\Rightarrow$ (b).*
- *If Fama regression yields the best predictor and $b \notin (0,1)$, (b) $\Leftrightarrow$ (a); otherwise (b) is a weaker condition.*

In the presence of the Euler equation wedges, these conditions can now be satisfied even if the exchange rate is close to a random walk with small R^2. If ω is negative enough to flip the sign on the right-hand side, then, we do not need to rely on a highly volatile market price of risk and/or large U.I.P. deviations. This being the case, the necessary condition of a negative unconditional exchange rate cyclicality can be satisfied with $b \leq 0$, even with a constant market price of risk.

It is important to clarify what role the wedges play in driving exchange rate fluctuations. Since a *constant* wedge suffices to satisfy the necessary conditions, the wedges need not be a source of exchange rate variation at all. Their role is not to drive the time-series variations of exchange rates, but rather to break the tight link between exchange rate cyclicality and predictability that frictionless bond markets impose. Once that link is broken, exchange rate fluctuations can be driven by any other shocks in the model — productivity, monetary policy, demand — without the model being forced into counter-cyclical exchange rates. In this sense, the wedges act as a precondition for a broader class of exchange rate models to be viable, not as the proximate cause of exchange rate movements.

Jiang, Krishnamurthy, and Lustig [2021] present evidence that foreign investors obtain a higher convenience yield on dollar bonds than U.S. investors. They show that this possibility helps to explain why the dollar appreciates during global recessions. This case is one where foreigners prefer dollar bonds even more than U.S. investors, i.e., $\xi_t^* < 0$ and $\xi_t^* < \phi_t$, or $-\xi_t^* + \phi_t > 0$. Note that this case makes it harder, but not impossible, to satisfy the restriction in Proposition 7. That is, with,

$$\omega = \mathbb{E}[(-\xi_t^* + \phi_t) + (-\xi_t + \phi_t^*)] - cov\left(\phi_t^*, q_t\right) + cov\left(\phi_t, q_t\right),$$

if $(-\xi_t^* + \phi_t) > 0$, the rest of the terms need to be sufficiently negative to offset this positive term. This can occur if $\xi_t > 0$ and U.S. investors perceive an even higher transaction cost to invest in foreign bonds, or $\phi_t^* < 0$ and foreign investors derive a sufficiently high convenience yield on their own domestic currency holdings. As such, while foreign investors' bond convenience yield can generate the U.S. dollar appreciation during global recessions, a different type of the Euler equation wedge, which is closer to home bias, is needed to generate a negative unconditional exchange rate cyclicality.

5.4 Calibrated Examples

Let us return to the numerical example we considered in Section 4.1. If we assume that the Fama regression yields the best predictor, we can rearrange the condition in Proposition 7 to obtain

$$-\omega > var(\Delta s_{t+1})\left[1 + R^2\left(\frac{1}{b} - 1\right)\right] - \sqrt{R^2}std\left(var_t(m_{t,t+1})\right)std(\Delta s_{t+1}),$$

which provides a lower bound on the sum of the wedges. If we assume $var(\Delta s_{t+1}) \leq 0.1^2$, $R^2 \leq 0.05$ and $b = -1$ from data, and $std\left(var_t(m_{t,t+1})\right) = 0.067$ from the long-run risk model, then, we obtain

$$-\omega > 0.75\%.$$

For example, if we attribute ω all to the wedges ξ_t and ξ_t^* associated with the foreign transaction costs, then, the average cost $\mathbb{E}[\xi_t^* + \xi_t]/2$ that is necessary to generate a negative unconditional exchange rate cyclicality is about 40 basis points.

Moreover, given a negative Fama coefficient, i.e., $b < 0$, this bound becomes even tighter if the exchange rate is less predictable. In the limit, $R^2 = 0$ and the bound becomes $-\omega > var(\Delta s_{t+1}) = 1\%$, which is in line with our characterization of the conditional exchange rate cyclicality in Proposition 5 and Eq. (17) in particular. The bound is also tighter if $std\left(var_t(m_{t,t+1})\right)$ is smaller, i.e., if the conditional log SDF variance has a lower volatility.

We can also vary the value of $std\left(var_t(m_{t,t+1})\right)$ and plot the lower bound on the wedge $-\omega$. Figure (3) plots this relationship. Consistent with our results in Section 4.1, in the absence of the Euler equation wedges, $std\left(var_t(m_{t,t+1})\right)$ needs to be about 0.40 in order to generate a negative unconditional exchange rate cyclicality. This value, as we showed above, is much higher than the moments from standard asset pricing models such as the long-run risk model and the external habit model. However, with a small amount of wedges $-\omega$, the required $std\left(var_t(m_{t,t+1})\right)$ can be much smaller.

Alternatively, if we assume that the forward premium has no predictive power for

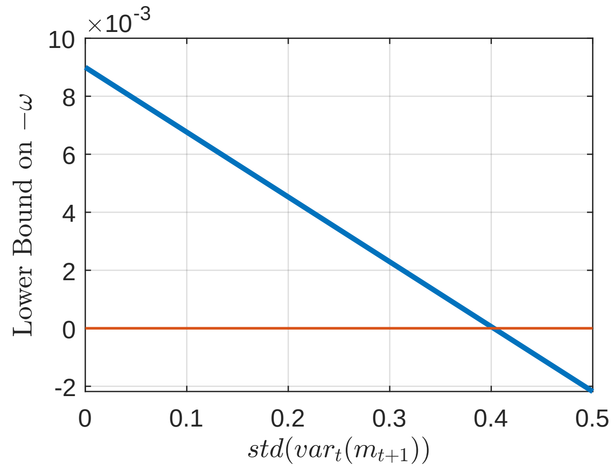


FIGURE 3. LOWER BOUND ON THE EULER EQUATION WEDGE AS A FUNCTION OF THE VARIABILITY OF THE CONDITIONAL LOG SDF VARIANCE

Notes: The figure plots the lower bound on $-\omega$ as a function of $std\,(var_t(m_{t,t+1}))$ as implied by Proposition 7. We take $b = -1$, $R^2 = 5\%$, and $std(\Delta s_{t+1}) = 10\%$.

the exchange rate movement with $b = 0$, while maintaining the other assumptions of $R^2 = 5\%$, $std(\Delta s_{t+1}) = 10\%$, and $std\,(var_t(m_{t,t+1})) = 0.067$, we obtain a slightly tighter bound

$$-\omega > 0.80\%.$$

In more recent research, new variables have been shown to predict exchange rate movements with corresponding increases in R^2 [Stavrakeva and Tang, 2020, Chahrour, Cormun, De Leo, Guerrón-Quintana, and Valchev, 2021, Dahlquist and Pénasse, 2022, Chernov, Dahlquist, and Lochstoer, 2023a]. For example, consider Kremens and Martin [2019], Kremens et al. [2023] who show that a combination of the quanto risk premium and interest rate differential can capture 16% of the exchange rate variation at the two-year horizon. If we assume that a similar explanatory power can be attained at the one-year horizon, then, Proposition 7(a) implies that, using the same calibration targets $var(\Delta s_{t+1}) = 0.1^2$, $R^2_{Fama} = 0.05$ and $b = -1$, and $std\,(var_t(m_{t,t+1})) = 0.067$, we obtain a lower bound of $-\omega$ at 0.52%. Engel and Wu [2024] report an R^2 close to 30% using macro fundamentals, much higher than previous estimates. Using $R^2 = 0.30$ in the same calibration, the lower bound on $-\omega$ drops to 0.28%, or about 28 basis points. Even under these highly optimistic predictability estimates, the required wedge remains non-trivial.

Compared to the bound of 0.75% we obtained above, these examples show that greater exchange rate predictability lowers the magnitude of the wedges required to generate a pro-cyclical exchange rate, but does not eliminate the need for them. However, we believe that the predictability estimates in the literature are likely to be upward biased due to data mining and publication bias, and therefore, the true R^2 is likely to be much lower than these estimates. In this case, the required wedges will be closer to the 40 basis points we obtained above.

5.5 Other Interpretations of the Euler Equation Wedges

Lastly, we discuss other specific models that nest our Euler equations and clarify the wedges they imply.

Bond Convenience Yields. The Euler equation wedges can also arise from bond convenience yields. For simplicity, we follow Jiang et al. [2024] and assume convenience yields arise from a bond-in-utility set-up, which can be microfounded by more structural models.

Specifically, the home households' utility is derived over consumption and the market value of home bond holdings:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \delta^t \left(u(c_t) + v(b_{H,t}) \right) \right],$$

and their budget can be expressed as

$$y_t + b_{H,t-1} \exp(r_{t-1}) + b_{F,t-1} \exp(r^*_{t-1} - s_t) = c_t + b_{H,t} + b_{F,t} \exp(-s_t).$$

Similarly, the foreign households also derive utility from holding the home bond, which leads to the following set of Euler equations:

$$\begin{aligned} 1 - \frac{v'(b_{H,t})}{u'(c_t)} &= \mathbb{E}_t \left[\exp(m_{t+1} + r_t) \right], \\ 1 &= \mathbb{E}_t \left[\exp(m_{t+1} - \Delta s_{t+1} + r^*_t) \right], \\ 1 &= \mathbb{E}_t \left[\exp(m^*_{t+1} + r^*_t) \right], \\ 1 - \frac{v'(b^*_{H,t})}{u'(c^*_t)} &= \mathbb{E}_t \left[\exp(m^*_{t+1} + \Delta s_{t+1} + r_t) \right]. \end{aligned}$$

In this setting, due to the non-pecuniary utility derived from holding the home bond, home and foreign households are willing to accept a lower return on the home bond, which is determined by the marginal utility of bond holding scaled by the marginal utility of consumption. The Euler equation wedges can be expressed as

$$
\begin{aligned}
\phi_t &= \log\left(1 - \frac{v'(b_{H,t})}{u'(c_t)}\right), \qquad \phi_t^* = 0, \\
\xi_t^* &= \log\left(1 - \frac{v'(b_{H,t}^*)}{u'(c_t^*)}\right), \qquad \xi_t = 0.
\end{aligned}
$$

This specification also has a simple interpretation: for example, when the foreign households derive a higher marginal utility $v'(b_{H,t}^*)$ from holding the home bond, they accept a lower risk-adjusted return on the home bond, which is associated with a lower wedge ξ_t^*.

Financial Repression. We consider a very stylized setting in which agents find domestic bonds more desirable due to regulatory reasons. This can be modeled as a shadow value $h(\cdot)$ from holding the domestic bond, which similarly enter the households' utility functions. This implies the following Euler equations:

$$
\begin{aligned}
1 - \frac{h'(b_{H,t})}{u'(c_t)} &= \mathbb{E}_t\left[\exp(m_{t+1} + r_t)\right], \\
1 &= \mathbb{E}_t\left[\exp(m_{t+1} - \Delta s_{t+1} + r_t^*)\right], \\
1 - \frac{h'(b_{F,t}^*)}{u'(c_t^*)} &= \mathbb{E}_t\left[\exp(m_{t+1}^* + r_t^*)\right], \\
1 &= \mathbb{E}_t\left[\exp(m_{t+1}^* + \Delta s_{t+1} + r_t)\right].
\end{aligned}
$$

In this case, the Euler equation wedges can be expressed as

$$
\begin{gathered}
\phi_t = \log\left(1 - \frac{h'(b_{H,t})}{u'(c_t)}\right), \qquad \phi_t^* = \log\left(1 - \frac{h'(b_{F,t}^*)}{u'(c_t^*)}\right), \\
\xi_t^* = 0, \qquad \xi_t = 0,
\end{gathered}
$$

which implies that the regulatory constraints effectively lower the required return on the domestic bond, which gives rise to home bias in bond holdings for both home and foreign households.

As long as $\phi_t < 0$ and $\phi_t^* < 0$, the ω term in the unconditional exchange rate cyclicality bound can be negative, which will make it easier to generate a pro-cyclical exchange rate.

Holding Cost. Moreover, if the agents face a holding cost or capital gain tax from holding foreign assets, the Euler equations also imply wedges. For example, if the home and foreign households both have positive positions on the cross-country bond holdings, the Euler equations can be expressed as

$$\begin{aligned}
1 &= \mathbb{E}_t \left[\exp(m_{t+1} + r_t)\right], \\
1 &= \mathbb{E}_t \left[\exp(m_{t+1} - \Delta s_{t+1} + r_t^* - \tau_t)\right], \\
1 &= \mathbb{E}_t \left[\exp(m_{t+1}^* + r_t^*)\right], \\
1 &= \mathbb{E}_t \left[\exp(m_{t+1}^* + \Delta s_{t+1} + r_t - \tau_t^*)\right],
\end{aligned}$$

where τ_t and τ_t^* are positive numbers denoting the reduction in returns from the cross-country positions. These wedges can be motivated by the capital gain tax on the returns r_t and r_t^* from foreign bond holdings, or by the management fee for international portfolios. In this case,

$$\begin{aligned}
\phi_t = 0, &\qquad \phi_t^* = 0, \\
\xi_t^* = \tau_t^*, &\qquad \xi_t = \tau_t.
\end{aligned}$$

As long as the transaction costs ξ_t and ξ_t^* are positive, the ω term in the unconditional exchange rate cyclicality bound can be negative, which will make it easier to generate a pro-cyclical exchange rate.[8]

6 Conclusion

We derive a simple and general characterization of the bilateral exchange rate's cyclicality with respect to the differential between home and foreign SDFs. If investors can freely trade the risk-free bonds in both countries, their optimality conditions impose strong restrictions on the relation between exchange rates and macro fundamentals that is difficult to reconcile with the data. In order to break this relation, we need wedges in the domestic or cross-currency Euler equations of investors, but we do not need to segment currency markets from other asset markets.

We regard our results as *ruling in* a class of models involving home bias/transaction costs, convenience yields, or intermediation frictions as resolutions to the exchange rate puzzles. In particular, the wedge of 40 basis points is plausible and in the range of fric-

[8] If the households hold negative positions on the foreign bond, the Euler equation wedges τ_t and τ_t^* need to be negative in order to reflect the reductions in their short positions.

tions measured in the literature, as for example in the literature on CIP deviations [Du, Tepper, and Verdelhan, 2018]. At a broader level, our approach offers a model-free diagnosis for the necessary conditions needed to explain exchange rate disconnect and predictability.

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Appendix

A Proof

A.1 Proposition 1

Proof. Combining

$$\mathbb{E}_t[\Delta s_{t+1}] + r_t - r_t^* = cov_t(m_{t,t+1}, -\Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}),$$
$$-(\mathbb{E}_t[\Delta s_{t+1}] + r_t - r_t^*) = cov_t(m^*_{t,t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}),$$

we directly obtain

$$cov_t(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = var_t(\Delta s_{t+1}) > 0.$$

□

A.2 Proposition 2

Proof. Our goal is to derive a necessary condition for a negative unconditional exchange rate cyclicality, i.e., $cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) < 0$. We start with the following decomposition:

$$\text{cov}\left(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}\right) = \mathbb{E}\left[\text{cov}_t\left(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}\right)\right] + \text{cov}\left(\mathbb{E}_t\left[m_{t,t+1} - m^*_{t,t+1}\right], \mathbb{E}_t\left[\Delta s_{t+1}\right]\right).$$

We start with the second term on the right-hand side. Using the definition of $p_t = \mathbb{E}_t[rx_{t+1}] = f_t - \mathbb{E}_t[s_{t+1}]$ and $q_t = \mathbb{E}_t[\Delta s_{t+1}]$, we can restate the covariance as follows:

$$cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) = cov(p_t + q_t, q_t) + \frac{1}{2} cov\left(var_t(m^*_{t,t+1}), q_t\right) - \frac{1}{2} cov\left(var_t(m_{t,t+1}), q_t\right).$$

Note $cov(p_t + q_t, q_t) = b \times var(p_t + q_t)$ by the construction of the Fama regression. Then,

$$cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) = b \times var(p_t + q_t) + \frac{1}{2} cov\left(var_t(m^*_{t,t+1}), q_t\right) - \frac{1}{2} cov\left(var_t(m_{t,t+1}), q_t\right).$$

Combining Proposition 1 and the above results, negative unconditional exchange rate cyclicality then implies

$$\begin{aligned}
&cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) \\
&= \mathbb{E}[var_t(\Delta s_{t+1})] + b \times var(p_t + q_t) + \frac{1}{2}cov\left(var_t(m^*_{t,t+1}), q_t\right) - \frac{1}{2}cov\left(var_t(m_{t,t+1}), q_t\right) < 0.
\end{aligned}$$

Rearranging terms,

$$\frac{1}{2}cov\left(var_t(m_{t,t+1}), q_t\right) + \frac{1}{2}cov\left(var_t(m^*_{t,t+1}), -q_t\right) > \mathbb{E}[var_t(\Delta s_{t+1})] + b \times var(p_t + q_t).$$

Without loss of generality,

$$\begin{aligned}
cov\left(var_t(m_{t,t+1}), q_t\right) &\geq \frac{1}{2}cov\left(var_t(m_{t,t+1}), q_t\right) + \frac{1}{2}cov\left(var_t(m^*_{t,t+1}), -q_t\right) \\
&> \mathbb{E}[var_t(\Delta s_{t+1})] + b \times var(p_t + q_t).
\end{aligned}$$

By the Cauchy-Schwarz inequality, a necessary (but not sufficient) condition is given by:

$$std\left(var_t(m_{t,t+1})\right) > \frac{\mathbb{E}[var_t(\Delta s_{t+1})] + b \times var(f_t - s_t)}{std(\mathbb{E}_t[\Delta s_{t+1}])}.$$

Using $\mathbb{E}[var_t(\Delta s_{t+1})] = var(\Delta s_{t+1}) - var(\mathbb{E}_t[\Delta s_{t+1}])$ and dividing both sides by $std(\mathbb{E}_t[\Delta s_{t+1}])$, this implies

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\mathbb{E}_t[\Delta s_{t+1}])} + 1 - b \times \frac{var(f_t - s_t)}{var(\mathbb{E}_t[\Delta s_{t+1}])} > \frac{var(\Delta s_{t+1})}{var(\mathbb{E}_t[\Delta s_{t+1}])} = \frac{1}{R^2}. \tag{A.1}$$

Now, notice that

$$b = \frac{cov(\Delta s_{t+1}, f_t - s_t)}{var(f_t - s_t)} = \frac{std(\Delta s_{t+1})}{std(f_t - s_t)}corr(\Delta s_{t+1}, f_t - s_t).$$

We obtain

$$\begin{aligned}
\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})\sqrt{R^2}} + 1 - b \times \frac{var(f_t - s_t)}{var(\Delta s_{t+1})R^2} &> \frac{1}{R^2}, \\
\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} + \sqrt{R^2} - b \times \frac{corr(\Delta s_{t+1}, f_t - s_t)^2}{b^2\sqrt{R^2}} &> \frac{1}{\sqrt{R^2}}, \\
\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} &> \frac{1}{\sqrt{R^2}} - \sqrt{R^2} + \frac{R^2_{Fama}}{b\sqrt{R^2}}.
\end{aligned}$$

When Fama Regression yields the best predictor, $R^2_{Fama} = R^2$, the formula is simplified to

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}} - \sqrt{R^2} + \frac{1}{b}\sqrt{R^2},$$

where

$$\frac{1}{b}\sqrt{R^2} = \frac{1}{b}\frac{|b|std(f_t - s_t)}{std(\Delta s_{t+1})} = sign(b)\frac{std(f_t - s_t)}{std(\Delta s_{t+1})}.$$

Hence, we arrive at condition (a).

Next, we show condition (b). By the linearity of covariance and imposing the symmetry assumption:

$$\begin{aligned} cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) &\geq cov(p_t + q_t, q_t) - cov\left(var_t(m_{t,t+1}), q_t\right) \\ &= var(q_t) + cov(p_t, q_t) - cov\left(var_t(m_{t,t+1}), q_t\right). \end{aligned}$$

Applying the Cauchy-Schwarz inequality twice, this covariance expression on the right-hand side can be bounded below as follows:

$$cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) \geq var(q_t) - std(p_t)std(q_t) - std(q_t)std\left(var_t(m_{t,t+1})\right).$$

This lower bound can be restated as:

$$cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) \geq var(q_t)\left(1 - \frac{std(p_t)}{std(q_t)} - \frac{std\left(var_t(m_{t,t+1})\right)}{std(q_t)}\right).$$

To get a negative unconditional Backus Smith coefficient, we need the following condition:

$$cov(m_{t,t+1} - m^*_{t,t+1}, \Delta s_{t+1}) = \mathbb{E}[var_t(\Delta s_{t+1})] + cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) < 0.$$

This can be restated as follows:

$$-\mathbb{E}[var_t(\Delta s_{t+1})] > cov(\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}], \mathbb{E}_t[\Delta s_{t+1}]) \geq var(q_t)\left(1 - \frac{std(p_t)}{std(q_t)} - \frac{std\left(var_t(m_{t,t+1})\right)}{std(q_t)}\right).$$

Rearranging terms, we obtain the following result:

$$\frac{std(p_t)}{std(q_t)} + \frac{std\left(var_t(m_{t,t+1})\right)}{std(q_t)} > 1 + \frac{\mathbb{E}[var_t(\Delta s_{t+1})]}{var(q_t)}.$$

By the definition of the unconditional variance, we have

$$1+\frac{\mathbb{E}[var_t(\Delta s_{t+1})]}{var(q_t)}=\frac{var(\mathbb{E}_t[\Delta s_{t+1}])+\mathbb{E}[var_t(\Delta s_{t+1})]}{var(\mathbb{E}_t[\Delta s_{t+1}])}=\frac{var(\Delta s_{t+1})}{var(\mathbb{E}_t[\Delta s_{t+1}])}.$$

Hence, we obtain the necessary condition (b):

$$\sqrt{\frac{std(p_t)}{std(q_t)}+\frac{std\left(var_t(m_{t,t+1})\right)}{std(q_t)}}>\frac{std(\Delta s_{t+1})}{std(q_t)}=\frac{1}{\sqrt{R^2}}.$$

To compare conditions (a) and (b), note that condition (a) can be written as Eq. (A.1), reproduced below,

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\mathbb{E}_t[\Delta s_{t+1}])}+1-b\times\frac{var(f_t-s_t)}{var(\mathbb{E}_t[\Delta s_{t+1}])}>\frac{1}{R^2},$$

it suffices to compare the term $std(\mathbb{E}_t[rx_{t+1}])/std(\mathbb{E}_t[\Delta s_{t+1}])=std(p_t)/std(q_t)$ in (b) with $1-bvar(f_t-s_t)/var(\mathbb{E}_t[\Delta s_{t+1}])=1-bvar(p_t+q_t)/var(q_t)$ in (a). Consider the general case, when the Fama regression does not necessarily yield the best predictor. Taking conditional expectations on both sides of the regression yields

$$q_t=a+b(p_t+q_t)+x_t,$$

where $x_t=\mathbb{E}_t[\varepsilon_{t+1}]$ satisfies

$$\begin{aligned}cov(x_t,p_t+q_t)&=cov(\varepsilon_{t+1},p_t+q_t)-cov(\varepsilon_{t+1}-x_t,p_t+q_t)=0,\\ cov(x_t,q_t)&=cov(x_t,b(p_t+q_t))+var(x_t)=var(x_t).\end{aligned}$$

Hence, $var(q_t)=b^2var(p_t+q_t)+var(x_t)$, and

$$\begin{aligned}1-b\frac{var(p_t+q_t)}{var(q_t)}&=1-\frac{1}{b}\left(1-\frac{var(x_t)}{var(q_t)}\right)\\&=\left(1-\frac{1}{b}\right)+\frac{1}{b}\frac{var(x_t)}{var(q_t)}.\end{aligned}$$

On the other hand, $p_t=-a/b+(1/b-1)q_t-x_t/b$, which implies

$$\frac{std(p_t)}{std(q_t)}=\frac{1}{std(q_t)}\sqrt{\left(\frac{1}{b}-1\right)^2var(q_t)+\frac{1}{b^2}var(x_t)-\frac{2}{b}\left(\frac{1}{b}-1\right)var(x_t)}$$

$$= \sqrt{\left(\frac{1}{b}-1\right)^2 + \left(\frac{2}{b}-\frac{1}{b^2}\right)\frac{var(x_t)}{var(q_t)}}.$$

Note that

$$\begin{aligned}\left(1-b\frac{var(p_t+q_t)}{var(q_t)}\right)^2 &= \left(\frac{1}{b}-1\right)^2 + \left(\frac{2}{b}-\frac{2}{b^2}\right)\frac{var(x_t)}{var(q_t)} + \frac{1}{b^2}\left(\frac{var(x_t)}{var(q_t)}\right)^2 \\ &= \left(\frac{std(p_t)}{std(q_t)}\right)^2 - \frac{1}{b^2}\left(1-\frac{var(x_t)}{var(q_t)}\right)\frac{var(x_t)}{var(q_t)},\end{aligned}$$

which implies that, when $var(x_t) > 0$, i.e., the Fama regression does not yield the best predictor, condition (a) is always tighter.

When $var(x_t) = 0$, i.e., Fama regression yields the best predictor, we obtain

$$\left(1-b\frac{var(p_t+q_t)}{var(q_t)}\right)^2 = \left(\frac{std(p_t)}{std(q_t)}\right)^2,$$

and

$$1-b\frac{var(p_t+q_t)}{var(q_t)} = \left(1-\frac{1}{b}\right).$$

When $1-1/b > 0$, i.e., $b < 0$ or $b > 1$, conditions (a) and (b) are equivalent. Otherwise, $1-b\frac{var(p_t+q_t)}{var(q_t)} < 0 < \frac{std(p_t)}{std(q_t)}$, and condition (a) is tighter. □

A.3 Proposition 3

The proof is identical to the proof of Proposition 2. Just replace the one-period objects (e.g., $m_{t,t+1}$) with the multi-period objects (e.g., $m_{t,t+k}$).

A.4 Propositions 7

Proof. From

$$\begin{aligned}\phi_t &= \mathbb{E}_t[m_{t,t+1}] + \frac{1}{2}var_t(m_{t,t+1}) + r_t, \\ \xi_t &= \mathbb{E}_t[m_{t,t+1}] + \frac{1}{2}var_t(m_{t,t+1}) - \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) + cov_t(m_{t,t+1}, -\Delta s_{t+1}) + r_t^*, \\ \phi_t^* &= \mathbb{E}_t[m_{t,t+1}^*] + \frac{1}{2}var_t(m_{t,t+1}^*) + r_t^*, \\ \xi_t^* &= \mathbb{E}_t[m_{t,t+1}^*] + \frac{1}{2}var_t(m_{t,t+1}^*) + \mathbb{E}_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) + cov_t(m_{t,t+1}^*, \Delta s_{t+1}) + r_t,\end{aligned}$$

we obtain

$$\mathbb{E}_t[m_{t,t+1} - m^*_{t,t+1}] = \frac{1}{2}var_t(m^*_{t,t+1}) + r^*_t - \phi^*_t - \frac{1}{2}var_t(m_{t,t+1}) - r_t + \phi_t.$$

Compared to the proof of Proposition 2, the only difference is the additional wedge terms $-cov(\phi^*_t, q_t) + cov(\phi_t, q_t)$ in the covariance of conditional expectations. Following the same steps as in the proof of Proposition 2, a negative unconditional exchange rate cyclicality implies

$$\begin{aligned} &cov\left(var_t(m_{t,t+1}), q_t\right) - cov\left(\phi_t, q_t\right) + cov\left(\phi^*_t, q_t\right) \\ \geq &\mathbb{E}[var_t(\Delta s_{t+1}) - (\xi^*_t + \xi_t) + (\phi_t + \phi^*_t)] + b \times var(p_t + q_t). \end{aligned}$$

Let $\omega = \mathbb{E}[-(\xi^*_t + \xi_t) + (\phi_t + \phi^*_t)] - cov\left(\phi^*_t, q_t\right) + cov\left(\phi_t, q_t\right)$ denote the adjustment term that arises from the wedges. Applying the Cauchy-Schwarz bound $corr(var_t(m_{t,t+1}), q_t) \leq 1$ as before yields the necessary condition:

$$std\left(var_t(m_{t,t+1})\right) > \frac{\mathbb{E}[var_t(\Delta s_{t+1})] + \omega + b \times var(f_t - s_t)}{std(\mathbb{E}_t[\Delta s_{t+1}])}.$$

The remaining algebra to obtain condition (a) is identical to the proof of Proposition 2, with $var(\Delta s_{t+1})$ replaced by $var(\Delta s_{t+1}) + \omega$ throughout:

$$\frac{std\left(var_t(m_{t,t+1})\right)}{std(\Delta s_{t+1})} > \frac{1}{\sqrt{R^2}}\left(1 + \frac{\omega}{var(\Delta s_{t+1})}\right) - \sqrt{R^2} + \frac{R^2_{Fama}}{b\sqrt{R^2}}.$$

Condition (b) follows analogously, and the relation between conditions (a) and (b) is identical to that in Proposition 2. □

A.5 Non-Gaussian Case

We define conditional entropy as follows:

$$L_t(X_{t+1}) = \left(\log \mathbb{E}_t[X_{t+1}] - \mathbb{E}_t[x_{t+1}]\right),$$

where $x_{t+1} = \log X_{t+1}$. Note that $L_t(X_{t+1})$ is non-negative by Jensen's inequality, and it is zero if and only if X_{t+1} is degenerate. We can use μ_{it} to denote the i-th central conditional moment of x_{t+1}. Then, for the log moment generating function $k_t(x_{t+1}; s)$, we can state:

$$k_t(x_{t+1}; s) = \log \mathbb{E}_t \left[\exp(s x_{t+1})\right] = \sum_{j=1}^{\infty} s^j \kappa_{j,t} / j!$$

where $\kappa_{1t} = \mu_{1t}$, $\kappa_{2t} = \mu_{2t}$, $\kappa_{3t} = \mu_{3t}$, $\kappa_{4t} = \mu_{4t} - 3\mu_{2t}^2$, and μ_{it} denotes the i-th central conditional moment of x_{t+1}. This implies that the conditional entropy can be stated as the sum of the higher-order *cumulants*:

$$L_t(X_{t+1}) = \sum_{j=2}^{\infty} \kappa_{j,t} / j! = k_t(x_{t+1}; 1) - \kappa_{1t}.$$

The log of the currency risk premium (in levels) earned by domestic investors can be stated as:

$$(r_t^* - r_t) - \mathbb{E}_t[\Delta s_{t+1}] + L_t \left[\frac{S_t}{S_{t+1}}\right] = -C_t \left(M_{t,t+1}, \frac{S_t}{S_{t+1}} \right),$$

where co-entropy is defined as $C_t(x_{t+1}, y_{t+1}) = L_t(x_{t+1} y_{t+1}) - L_t(x_{t+1}) - L_t(y_{t+1})$ [Backus, Boyarchenko, and Chernov, 2018]. If x_{t+1} and y_{t+1} are independent, then $C_t(x_{t+1}, y_{t+1}) = 0$. If we define the cumulant generating function,

$$k_t(s_1, s_2) = \log \mathbb{E}_t \left[\exp(s_1 x_{t+1} + s_2 y_{t+1})\right]$$

then $C_t(x_{t+1}, y_{t+1}) = k_t(1,1) - k_t(1,0) - k_t(0,1)$. A long position in foreign currency is risky for the domestic investor when the foreign currency tends to depreciate (and the home currency appreciates) in worse states for the domestic investor, i.e. when

$$C_t \left(M_{t+1}, \frac{S_t}{S_{t+1}} \right) < 0$$

is more negative. Similarly, the log of the currency risk premium (in levels) earned by domestic investors can be stated as the co-entropy of the domestic SDF with the domestic currency's rate of appreciation:

$$(r_t - r_t^*) + \mathbb{E}_t[\Delta s_{t+1}] + L_t \left[\frac{S_{t+1}}{S_t}\right] = -C_t(M_{t,t+1}^*, \frac{S_{t+1}}{S_t}).$$

Proposition 8. *In the non-normal case, the conditional exchange rate cyclicality satisfies*

$$-C_t(M^*_{t,t+1}, \frac{S_{t+1}}{S_t}) - C_t\left(M_{t,t+1}, \frac{S_t}{S_{t+1}}\right) = L_t\left[\frac{S_t}{S_{t+1}}\right] + L_t\left[\frac{S_{t+1}}{S_t}\right] > 0. \tag{A.2}$$

In the normal case, we recover the covariance result in Proposition 1. When the domestic currency appreciates in worse states for the domestic investor, we have $C_t\left(M_{t,t+1}, \frac{S_t}{S_{t+1}}\right) < 0$. Similarly, when the foreign currency appreciates in worse states for the foreign investor, we have $C_t(M^*_{t,t+1}, \frac{S_{t+1}}{S_t}) < 0$.

Because the right-hand side is positive, as entropy is non-negative, we know that exchange rates will have to be counter-cyclical for at least one of the countries, and possibly both.

Proof. The U.S. Euler equations are $\mathbb{E}_t[\exp(m_{t,t+1} + r_t)] = 1$ and $\mathbb{E}_t[\exp(m_{t,t+1} + r^*_t - \Delta s_{t+1})] = 1$, where $m_{t,t+1} = \log M_{t,t+1}$. Taking logs and using the definitions of entropy and co-entropy, the first equation gives

$$0 = \mathbb{E}_t[m_{t,t+1}] + L_t(M_{t,t+1}) + r_t,$$

and the second gives

$$0 = \mathbb{E}_t[m_{t,t+1} - \Delta s_{t+1}] + r^*_t + C_t(M_{t,t+1}, \frac{S_t}{S_{t+1}}) + L_t(M_{t,t+1}) + L_t(\frac{S_t}{S_{t+1}}).$$

Subtracting the first from the second:

$$r^*_t - r_t - \mathbb{E}_t[\Delta s_{t+1}] + L_t\left(\frac{S_t}{S_{t+1}}\right) = -C_t\left(M_{t,t+1}, \frac{S_t}{S_{t+1}}\right).$$

Similarly, from the foreign Euler equations:

$$r_t - r^*_t + \mathbb{E}_t[\Delta s_{t+1}] + L_t\left(\frac{S_{t+1}}{S_t}\right) = -C_t\left(M^*_{t+1}, \frac{S_{t+1}}{S_t}\right).$$

Adding the two equations yields the proposition. The inequality follows because entropy is non-negative. $\square$