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ABSTRACT

Financial variables in third countries explain a significant fraction of bilateral exchange rate movements, even after local variables and common factors are controlled for. This paper proposes an explanation of this exchange rate spill-over pattern based on market incompleteness, which arises when agents in different countries bear different exposures to foreign risks, while the asset space is not complete enough to hedge out these risks. This mechanism weakens the link between exchange rates and local fundamentals, and generates additional exchange rate variations and comovements. To study bilateral exchange rates, it is not sufficient to consider only the local conditions.

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When the international financial markets are incomplete, agents in different countries do not fully share their risks and, as a result, do not equalize their marginal utility growth state by state. As Backus et al. (2001) show, incomplete markets introduce a wedge η between the exchange rate movement and the home and foreign agents' marginal utility growth differential. While this view has helped the literature make progress in understanding exchange rate puzzles (Brandt et al., 2006; Lustig and Verdelhan, 2019), it leaves open a question: what drives the η wedge, and what is its economic interpretation? Could it be the shocks that also drive the home and foreign agents' marginal utility growth? Could it be other local shocks that are not necessarily spanned by the marginal utility growth? What else could it be?

In this paper, I make a case that the incomplete-market η wedge in the bilateral exchange rate between two countries, home and foreign, could come from shocks from a third country. This happens when home and foreign agents choose different portfolios and bear different exposures to the third country's shocks. As a result, a shock in the third country moves the strength of the home currency differently from that of the foreign currency, generating a movement in their bilateral exchange rate that is not fully explained by the home and foreign agents' marginal utility growth. In comparison, when markets are complete, perfect international risk-sharing implies that home and foreign agents choose to bear the same amount of risk that comes from any third country. As a result, the shocks in the third country affect the home and foreign countries equally and have a symmetric effect on their currency strength.

This *international spill-over* in exchange rates can help us understand two salient features of exchange rate dynamics. First, each country's exchange rate movements appear to be disconnected from its own economic fundamentals, exhibiting a lack of correlation with macro variables that should drive marginal utility growth (Meese and Rogoff, 1983; Backus and Smith, 1993; Engel and West, 2005); second, exchange rate movements are highly correlated across countries, exhibiting a strong factor structure (Verdelhan, 2018). The international spill-over weakens the link between exchange rates and local fundamentals on the one hand, and propagates a country's shocks to the bilateral exchange rate movements between two other countries and gives rise to additional exchange rate comovements on the other hand.

This explanation of exchange rate disconnect and comovements complements the prior works, which largely focus on settings in which agents in different countries are exposed to correlated latent shocks. For example, if the marginal utilities are driven by long-run consumption risks that are correlated across countries, then, even in complete markets, the equilibrium exchange rate movements do not closely follow short-run consumption growth patterns, while they share a common factor from the correlated long-run risks. On top of the latent marginal utility shocks, my paper shows that the international spill-over generates an additional layer of exchange rate disconnect and comovements, even when the marginal utilities happen to be not so correlated across countries.

To make this argument, this paper proceeds in three steps. First, I present reduced-form evidence that the bilateral exchange rate movements are correlated with the third country's shocks. I conduct an exchange rate accounting exercise similar to Verdelhan (2018), who shows that two common exchange rate factors, dollar and carry, explain a large fraction of variations in the exchange rate movements. In my empirical exercise, I focus on financial variables that proxy for stochastic discount factors (SDFs) as the explanatory variables, and test if the third countries' SDF proxies have additional explanatory power for the bilateral exchange rate movements.

A natural question for the implementation of this test is which country to choose as the third country. In the spirit of letting the data speak for itself, I consider all possible third countries and focus on their common variations. For example, to study whether the changes in interest rates explain the bilateral exchange rate movement between the U.S. and the U.K., I first construct the *third-country principal components* which describe the common variations in the interest rates of all other countries such as Australia, Japan, Germany, and so on. Then, I regress the U.S.-U.K. exchange rate movement on their own interest rate changes and the third-country principal components. I find that third-country principal components help explain the bilateral exchange rate movements, even after the local interest rate changes are controlled for. Quantitatively, introducing the third-country principal components roughly doubles the explanatory power of the interest rate changes, and similar pattern holds for other financial variables such as stock returns and financial crisis indicators. When I use stock returns and 3-month and 10-year interest rates as explanatory variables, the local variables and the third-country principal components together explain 26.39% of the variations in the exchange rate movements between developed countries.

Moreover, the explanatory power of the third-country variables survives even after controlling for the common exchange rate factors. On average, the dollar and the carry factors explain 26.73% of the variations in the exchange rates. If I introduce the third-country principal components of stock returns and interest rates, the adjusted R^2 improves to 36.65%. Introducing the local stock returns and interest rates further improves the adjusted R^2 to 43.11%. This result is in stark contrast to the lack of explanatory power of fundamental variables such as consumption growth for exchange rate movements, and suggests that maybe the exchange rates are not so disconnected from the economy after all, if we use financial variables to proxy for the SDFs and pay attention to the third countries.

Having established the economic significance of the international spill-over in the exchange rate data, I turn to studying the theoretical mechanism that generates this effect. In the second part of the paper, I develop a stylized model that instantiates this international spill-over effect and highlights the role played by market incompleteness. The model is an exchange economy in which the agents from different countries trade financial assets because of different time preferences. In this model, there are three countries: home, foreign, and the third country. The only shocks are endowment shocks, which are transmitted internationally because of cross-country holdings in financial claims and fluctuations in the terms of trade.

I begin with the complete-market case, in which perfect international risk-sharing implies that the home and foreign countries' agents choose the same degree of exposures to the risks from the third country. As a result, the shocks in the third country affect the home and foreign countries equally and have a symmetric effect on their currency strength, which insulate the bilateral exchange rate between the home and foreign countries from the third-country shocks.

The equilibrium outcomes are very different when markets are incomplete. For simplicity, I assume that the agents in different countries can only trade each other's risk-free bonds denominated in the local numéraires. Because of the difference in time preferences, the home and foreign agents choose to have bond portfolios with different sizes and compositions. In particular, their portfolios have different exposures to the third country's endowment shock. As a result, a shock in the third country could move the strength of the home currency differently from that of the foreign currency, generating a movement in their bilateral exchange rate. This is the key mechanism that generates the international spill-over under incomplete markets, which propagates the third country's shock to the bilateral exchange rate between the home and foreign countries.

Interestingly, while the exchange rate spill-over arises when we deviate from the complete-market benchmark, it disappears again when we approach the most extreme case of market incompleteness: financial autarky. When the agents cannot trade any financial assets across countries, their consumption is still exposed to foreign endowment shocks due to variations in the terms of trade. However, the bilateral exchange rate movement is not exposed to the third-country shocks. As such, the international spill-over effect is a feature of an intermediate degree of market incompleteness, in which case the agents can trade financial assets across countries and bear different exposures to foreign risks, but the asset space is not complete enough to hedge out these foreign risks they bear.

This simple model illustrates the economic mechanism that gives rise to the international spill-over effect in exchange rates through cross-country asset holdings and incomplete risk-sharing. As with any models, it takes a specific set of assumptions and simplifications. In the third part of the paper, I develop a more general argument for the international spill-over effect that does not rely on these specific assumptions. In particular, the Euler equations for cross-country bond holdings alone are informative about the restrictions that the equilibrium exchange rate movements have to satisfy in the presence of imperfect international risk-sharing.

My general argument begins with the following observation. The Euler equations for cross-country bond holdings impose additional restrictions on the exchange rate dynamics when we consider multiple foreign countries. For example, let us fix the U.S. as the home country, and consider two foreign countries: U.K. and Japan. On the one hand, Triangular arbitrage rules out any discrepancies in spot exchange rates: once I know the bilateral exchange rate between the home and each foreign country (U.S.-U.K. and U.S.-Japan), the bilateral exchange rate between the two foreign countries (U.K.-Japan) is directly implied.

On the other hand, when markets are incomplete, the Euler equations between the two foreign countries are *not* implied from the Euler equations between the home and each foreign country. As a result, when we extend the analysis from two to three countries, the number of independent exchange rates increases from 1 to 2 (U.S.-U.K. and U.S.-Japan), while the number of unique sets of Euler equation restrictions increases from 1 to 3 (U.S.-U.K., U.S.-Japan, and U.K.-Japan). The Euler equations between the two foreign countries, which are usually ignored in two-country models, impose an additional constraint on the joint distribution of exchange rate movements and SDFs. As a result, even if our goal is to understand the bilateral exchange rate between the U.S. and the U.K., it is no longer sufficient to only study the local shocks in these two countries: the shocks from Japan could also matter, resulting in an international spill-over of Japanese shocks to the U.S.-U.K. exchange rate.

To interpret the economic meaning of the additional constraint, I build on the no-arbitrage approach of Backus, Foresi, and Telmer (2001) and derive a general decomposition of exchange rate movements. I project any bilateral exchange rate movement (e.g., U.S.-U.K.) on its local (i.e., U.S. and U.K.) SDFs and obtain a residual. The international spill-over effect is defined as a non-zero correlation between this residual and a third country's SDF (e.g., Japan's). When markets are complete, the residual is zero as the exchange rate movement is spanned by the local SDFs, which guarantees this covariance to be zero.

When markets are incomplete, I ask what the Euler equations' restriction implies if there is no spill-over effect. I show that it simplifies to a specific restriction that connects the exchange rate covariance between the home and any two foreign countries to a linear combination of their SDF covariances. If we fix a home country and consider N foreign countries, then, we obtain one such restriction for each pair of foreign countries, which gives us $N(N-1)/2$ restrictions in total. In comparison, there are only $1 + N$ structural parameters that come from the exchange rate projection, one for each country. Even if we treat these parameters as free parameters, when we consider $N = 10$ foreign developed countries that we tend to think have low-cost access to international bond markets, we need to fit $N(N-1)/2 = 45$ restrictions with only $1 + N = 11$ parameters. Therefore, the no-spill-over assumption requires a high degree of coincidence between the exchange rate covariance and the SDF covariance.

However, the exchange rate covariance and the SDF covariance have weak connections, reflecting a general disconnect between exchange rate dynamics and economic fundamentals that has been noted in the literature. I specifically test if any choice of parameter values can satisfy these restrictions in the data. To do so, I need to again take a stance on what proxies for the SDFs and hence the SDF covariance structure. I use the consumption growth, the changes in 10-year and 3-month interest rates, and the stock market return. Using the Generalized Method of Moments (GMM), I reject the null that all moment restrictions are satisfied in an overidentifying restrictions test (i.e., the J-test), which suggests that the no-spill-over null imposes too tight and too many restrictions that are not consistent with the data.

By rejecting this necessary condition of the no-spill-over null, this result makes a general case for the presence of the international spill-over in exchange rates, which allows the bilateral exchange rate movement (e.g., U.S.-U.K.) to load on a foreign SDF (e.g., Japan's), even after the U.S. and U.K. SDFs are controlled for. Intuitively, this comovement introduces additional degrees of freedom to satisfy the cross-country restrictions arising from the Euler equations. This characterization is general and does not rely on specific assumptions about preferences, frictions, and shocks. It can be applied to a large set of equilibrium international macroeconomic models. The only economic restriction is imposed by assuming investors in each country can freely trade domestic and foreign risk-free bonds, which is consistent with the fully specified model I presented in the second part of the paper. For other contingent claims such as equity and long-term bonds, the financial markets *may or may not* be open, which allows me to characterize the exchange rate dynamics under a flexible degree of market incompleteness.

In conclusion, the results in this paper support a novel channel through which market incompleteness propagates one country's SDF to other countries' bilateral exchange rate movements. This international spill-over effect gives rise to common variations in exchange rates, even when the SDFs happen to be uncorrelated across countries. By exposing bilateral exchange rate movements to shocks from a third country, this spill-over effect also offers a reason why the link between exchange rate movements and local fundamentals might be weak. In this way, this effect helps us tie together the exchange rate comovements and disconnect under one mechanism, which complements the prior works that rely on complete markets and correlated SDFs.

Literature Review. Market incompleteness is a salient feature of the international financial markets. Prior works have proposed various models with specific preferences and frictions and shown how market incompleteness impacts the dynamics of business cycles and exchange rates (see Alvarez, Atkeson, and Kehoe (2002, 2009); Kehoe and Perri (2002); Chari, Kehoe, and McGrattan (2002); Corsetti, Dedola, and Leduc (2008); Pavlova and Rigobon (2010, 2012); Hassan (2013); Bruno and Shin (2015); Favilukis, Garlappi, and Neamati (2015); Maggiori (2017)). Much of the focus has been given to the two-country setting, in which the bilateral exchange rate movement is driven by shocks in these two countries. This paper studies the international spill-over effect in a multi-country setting, which propagates shocks from a third country to the bilateral exchange rate movement.

The empirical results in this paper build on the "third-country effect" found in the previous literature. Hodrick and Vassalou (2002); Berg and Mark (2015) show that inflation, output gap, and interest rates in a third country predict future bilateral exchange rate movements. In comparison, this paper focuses on the contemporaneous relationship between financial variables and bilateral exchange rate movements, and shows a significant increase in the explanatory power for concurrent exchange rate movements when the third-country variables are introduced. This paper also proposes a different theoretical mechanism for the spill-over effect based on market incompleteness.[1]

[1] More broadly, Miranda-Agrippino and Rey (2022) also identify a global financial cycle that drives foreign exchange

The general characterization of the spill-over effect builds on the preference-free approach to modeling incomplete-market exchange rate dynamics developed by Backus, Foresi, and Telmer (2001); Brandt, Cochrane, and Santa-Clara (2006).[2] The closest paper in this literature is Lustig and Verdelhan (2019), who characterize the exchange rate volatility, cyclicality and risk premium in a two-country setting. I further develop this approach in two ways. First, generalizing this approach to a multi-country setting, I show that incomplete markets give rise to additional constraints that are not present in the two-country setting, and these constraints are crucial for understanding the exchange rate spill-over effect. Second, I propose a projection-based exchange rate decomposition, which is convenient for characterizing exchange rate covariances in this paper, and potentially for other applications in future works.[3]

To my best knowledge, this paper is the first to show how market incompleteness can generate exchange rate comovements in the cross-section of currencies. The prior literature explains these patterns by various mechanisms that generate common factors in SDFs (Farhi and Gabaix, 2015; Corte, Riddiough, and Sarno, 2016; Colacito, Croce, Gavazzoni, and Ready, 2018; Mueller, Stathopoulos, and Vedolin, 2017; Jiang and Richmond, 2019). In contrast, this paper shows that incomplete markets give rise to an international spill-over effect and generate exchange rate comovements in exchange rates even when SDFs are uncorrelated.

Finally, in the broader literature, this paper contributes to our understanding of the asset pricing dynamics in incomplete markets. First, my approach is complementary to Bakshi, Cerrato, and Crosby (2018) and Sandulescu, Trojani, and Vedolin (2021); Korsaye, Trojani, and Vedolin (2023), who use asset return data to discipline the SDF dynamics and characterize the roles of incompleteness and segmentation in international asset markets, respectively. Second, my approach

rate comovements. This cycle is related to the U.S. monetary policy and the global financial conditions, which could directly impact foreign exchange rates through its effects on foreign SDFs. That said, heterogeneous portfolio exposures and imperfect risk-sharing that this paper studies provide an additional reason why different foreign currencies can have different exposures to the global financial cycle.

[2] Other works based on this approach include Maurer and Tran (2021); Sandulescu et al. (2021); Lewis and Liu (2022); Chernov et al. (2023).

[3] For example, Appendix B.7 uses this decomposition to characterize the exchange rate volatility and risk premium, which replicates the dilemma in Lustig and Verdelhan (2019) in a more transparent way. In concurrent work, Jiang, Krishnamurthy, Lustig, and Sun (2021) also apply a similar approach to study the Euler equation wedges in a two-country setting.

also builds on a large literature including Hansen and Jagannathan (1991); Alvarez and Jermann (2005) that uses asset price data to understand the properties of SDFs through the lens of the Euler equations. In a multi-country economy, Euler equations impose joint restrictions on the properties of the SDFs and exchange rates, and my results suggest that these restrictions are useful for understanding exchange rate disconnect and comovements.

The rest of this paper is organized as follows. Section 1 presents empirical evidence for the international spill-over effect in exchange rates. Section 2 presents a simple model that instantiates the international spill-over effect and highlights the role played by market incompleteness. Section 3 further develops a general theoretical argument for its presence. Section 4 concludes. Appendices contain the proof and data sources.

1 Reduced-Form Evidence of Exchange Rate Spill-Over

In the main empirical analysis, the sample contains 11 developed countries: Australia, Canada, Switzerland, Germany, Denmark, the U.K., Japan, Norway, New Zealand, Sweden, and the U.S. Data are quarterly, 1988Q1 to 2023Q4. Appendix A provides details of data construction.

1.1 Third-Country Variables in Bilateral Exchange Rates

To begin with, I examine whether the international spill-over is an economically important feature of the exchange rate data. In an ideal world, if the SDFs are observable, we can directly test the presence of the spill-over by regressing the bilateral exchange rate movement $\Delta e_t^{i/j}$ between countries i and j on the SDFs of these countries and the SDF of a third country k:

$$\Delta e_t^{i/j} = \alpha^{i/j} + \gamma^{(i)} m_t^{(i)} + \gamma^{(j)} m_t^{(j)} + \beta^{(k)} m_t^{(k)} + u_t^{k,i/j}. \tag{1}$$

This exchange rate accounting exercise is similar to Verdelhan (2018), except we use SDFs instead of common exchange rate factors as explanatory variables. The coefficient $\beta^{(k)}$ captures the extent to which the third country's SDF explains the bilateral exchange rate movement between countries i and j, even after their own SDFs are controlled for. In this way, we can ensure that the correlation between the bilateral exchange rate and the third country's SDF is not driven by the correlations in the SDFs.

Unfortunately, the SDFs are not directly observable, and their estimation is in general a difficult task especially in the presence of incomplete markets. In this paper, I pursue a much more modest objective: I use observable variables as proxies for the SDFs, and test whether bilateral exchange rates comove with these proxies from third countries. More precisely, I extract common variations in third countries using the principal component analysis (PCA) and test whether the bilateral exchange rates comove with these common variations.

For example, let us consider the change in the 10-year interest rate $\Delta r_t^{(i)}$ as a plausible proxy for the SDFs. For any country pair (i, j), I construct the first three principal components of the changes in the 10-year interest rates in all other countries except i and j, which I denote as $PC_{1,t}^{-i,-j}$, $PC_{2,t}^{-i,-j}$, and $PC_{3,t}^{-i,-j}$. Then, I regress the bilateral exchange rate movement between countries i and j on their local interest rate changes as well as the third-country principal components:

$$\Delta e_t^{i/j} = \alpha^{i/j} + \gamma^{(i)} \Delta r_t^{(i)} + \gamma^{(j)} \Delta r_t^{(j)} + \beta^{(1)} PC_{1,t}^{-i,-j} + \beta^{(2)} PC_{2,t}^{-i,-j} + \beta^{(3)} PC_{3,t}^{-i,-j} + u_t^{i/j}. \quad (2)$$

For example, if i is the U.S. and j is the U.K., then, the principal components describe the common variations in other countries such as Australia, Japan, Germany, and so on. We are interested in knowing whether these foreign variations explain the bilateral exchange rate movement between the U.S. and the U.K. after controlling for their own interest rate changes.

I run this regression for each unique country pair (i, j), and report the average adjusted R^2 across all country pairs in Table (1). Column (1) reports the average adjusted R^2 of the regression with only the local variables in countries i and j. Column (2) reports the average R^2 with only the third-country principal components. Column (3) reports the average R^2 with both the local and the third-country variables, and Column (4) reports the increase in the adjusted R^2 from (1) to (3).

The first row shows that the local changes in 10-year yields in countries i and j explain 6.46% of the variations in their bilateral exchange rate movements. When we introduce the principal components of the interest rate changes in other countries, the adjusted R^2 roughly doubles to 12.44%. In other words, the third-country variations in the 10-year interest rates contain additional information that helps explain the bilateral exchange rate movements.

I also consider other variables to proxy for the SDFs. In addition to the change in the 10-year bond yield, I also use the change in the 3-month bond yield, the cum-dividend national stock market return, the change in Jordà et al. (2017)'s financial crisis indicator, and the real consumption growth. Table (1) shows that the third-country principal components based on these variables help explain the bilateral exchange rate movements, and in some cases have greater explanatory power than the local variables. Introducing the third-country principal components in addition to the local variables also improves the adjusted R^2. In particular, the row "Stock+Bond" includes both stock returns and the changes in 10-year and 3-month yields. In this case, 26.39% of the variations in exchange rate movements between developed countries can be explained by the local and third-country variables.

The only exception is the consumption growth, which has low explanatory power in general.

Table 1: Adjusted R^2 of the Third-Country Principal Components.

Variable	(1) Local Variables Only	(2) Third-Country PCs Only	(3) Both	(4) Improvement from (1)
Change in 10Y Bond Yield	6.46	10.38	12.44	5.98
Change in 3M Bond Yield	6.84	8.09	11.94	5.09
Stock Return	12.86	11.20	18.03	5.17
Stock+Bond	20.05	17.86	26.39	6.34
JST Financial Crisis	8.77	14.77	18.36	9.58
Consumption Growth	-0.54	-0.49	-0.81	-0.26

Notes: The regression equation is Eq. (2). Column (1) reports the mean adjusted R^2 of the regression without controlling for the third-country principal components. Column (2) reports the mean adjusted R^2 with only the third-country principal components. Column (3) reports the mean adjusted R^2 with both the local variables and the third-country principal components. Column (4) reports the increase in the adjusted R^2 between Columns (3) and (1). Data are quarterly, 1988Q1–2023Q4; except for JST Financial Crisis, which is annual, 1988–2020.

As such, perhaps the bilateral exchange rates are not so disconnected from the economy after all, if we use financial variables to proxy for the SDFs and pay attention to other foreign countries.

1.2 Horse Race with Common Exchange Rate Factors

We can further put this result in the context of the factor structure in exchange rate movements. A salient fact about exchange rate movements is that the correlation structure of bilateral exchange rates can be summarized by a small number of factors, and by the dollar factor and the carry factor in particular (Lustig et al., 2011; Verdelhan, 2018). Certainly, these common exchange rate factors could be themselves driven by the common variations in SDFs and by the international spill-over effect. That said, it would be interesting to know whether the third-country variables explain exchange rate variations after controlling for the common exchange rate factors. For example, using the interest rate changes as SDF proxies, I run the following regression:

$$\begin{aligned}\Delta e_t^{i/j} &= \alpha^{i/j} + \omega^{dol} f_{dol,t} + \omega^{carry} f_{carry,t} \\ &\quad + \gamma^{(i)} \Delta r_t^{(i)} + \gamma^{(j)} \Delta r_t^{(j)} + \beta^{(1)} PC_{1,t}^{-i,-j} + \beta^{(2)} PC_{2,t}^{-i,-j} + \beta^{(3)} PC_{3,t}^{-i,-j} + u_t^{i/j},\end{aligned} \tag{3}$$

which additionally controls for the dollar factor $f_{dol,t}$ and the carry factor $f_{carry,t}$. The dollar factor is constructed as the equal-weighted average of foreign currency returns against the dollar, and the carry factor is constructed as the return differential between high-interest rate and low-interest rate foreign currencies.

Table (2) reports the average adjusted R^2 of the regression Eq. (3). Column (1) reports the average adjusted R^2 of the regression with only the common exchange rate factors. Consistent with the literature, the dollar and carry factors explain a large fraction of the exchange rate movements. Column (2) additionally includes the local variables, and Column (3) includes the third-country principal components. Column (4) includes the exchange rate factors, the local variables, and the third-country principal components.

The table shows that both the local variables and the third-country principal components improve the explanatory power, and their information contents are different so that the adjusted R^2

Table 2: Adjusted R^2 of the Third-Country Principal Components, with Common Exchange Rate Factors

Variable	(1) Exchange Rate Factors	(2) Factors+Local	(3) Factors+Third-Country	(4) All
Change in 10Y Bond Yield	26.73	30.13	32.63	34.42
Change in 3M Bond Yield	26.73	31.47	32.46	35.71
Stock Return	26.73	33.43	33.13	36.21
Stock+Bond	26.73	38.27	36.65	43.11
JST Financial Crisis	33.95	37.37	39.47	41.72
Consumption Growth	26.73	26.44	26.22	26.07

Notes: The regression equation is Eq. (3). Column (1) reports the mean adjusted R^2 of the regression with the common exchange rate factors *dollar* and *carry*. Column (2) reports the mean adjusted R^2 with both the common factors and the local variables. Column (3) reports the mean adjusted R^2 with both the common factors and the third-country principal components. Column (4) reports the adjusted R^2 with the common factors, the local variables and the third-country principal components. Data are quarterly, 1988Q1–2023Q4; except for JST Financial Crisis, which is annual, 1988–2020.

increases when we include both. In the case in which we use equity returns and bond yields, the adjusted R^2 increases from 26.73% as explained by the common exchange rate factors alone to 43.11%. As such, the third-country variables can account for a significant fraction of the exchange rate movements, even after the common exchange rate factors are controlled for.

In the remainder of this paper, I will provide a theoretical explanation for this international spill-over effect. Having said that, a conclusive empirical test of Eq. (1) is left for future work, which requires a full-scale estimation of the SDF dynamics across countries, and should be ideally connected to the agents' preferences and consumption processes.

1.3 Statistical Significance

To assess the statistical significance of the coefficients associated with the third-country principal components, I conduct two Wald tests. First, in the horse-race regression (2) between the local variables and the third-country principal components, I jointly test the null hypothesis that the coefficients $\beta^{(1)} = \beta^{(2)} = \beta^{(3)} = 0$. Second, in the horse-race regression (3) with the common exchange rate factors, I jointly test the null hypothesis that the coefficients $\beta^{(1)} = \beta^{(2)} = \beta^{(3)} = 0$.

I conduct these Wald tests for each country pair (i, j), and report the fraction of the country pairs for which the null hypothesis is rejected at the 5% level in Table (3). If I use stock returns, interest rate changes, or both, the third-country principal components are statistically significant for at least half of the observations. In other words, the third-country principal components contain information that helps explain the bilateral exchange rate movements, even after the local variables and the common exchange rate factors are controlled for. The statistical significance of the JST Financial Crisis is lower, as this variable is annual and has a much smaller sample size.

1.4 More Countries

Moreover, I expand the sample to include 25 economies: Argentina, Australia, Brazil, Bulgaria, Canada, Chile, China, Croatia, Czech Republic, Denmark, Germany, Hong Kong, Hungary, India, Indonesia, Japan, Korea, Malaysia, Mexico, New Zealand, Norway, Philippines, Poland, Romania, Russia, Singapore, South Africa, Sweden, Switzerland, Taiwan, Thailand, Turkey, United Kingdom, United States. The sample is shorter, 1998Q1 to 2023Q4.

Appendix A.1 reports the results of regressions (2) and (3) for this extended sample. The results are similar. The third-country variables help explain the bilateral exchange rate movements, and their explanatory power is robust to the inclusion of the common exchange rate factors.

Table 3: Wald Tests for the Third-Country Principal Components

Regression	(1) Eq. (2), Local+Third-Country	(2) Eq. (3), Factors+Local+Third-Country
Change in 10Y Bond Yield	58.18	61.82
Change in 3M Bond Yield	49.09	58.18
Stock Return	61.82	47.27
Stock+Bond	56.36	47.27
JST Financial Crisis	28.89	20.00
Consumption Growth	5.45	1.82

Notes: We report the fraction of country pairs for which the null hypothesis that the coefficients associated with the third-country principal components are all zero, i.e., $\beta^{(1)} = \beta^{(2)} = \beta^{(3)} = 0$, is rejected at the 5% level. Column (1) reports the result for Eq. (2), and Column (2) reports the result for Eq. (3). Data are quarterly, 1988Q1–2023Q4; except for JST Financial Crisis, which is annual, 1988–2020.

2 A Motivating Example

Next, I develop a concrete example to illustrate the international spill-over effect in the currency market. We consider a model with two periods, $\tau \in \{t, t+1\}$, and three countries, $i \in \{0, 1, 2\}$. Each country i has a unique variety of local consumption goods, and a representative agent. In period τ, the agent in country i receives an endowment which is equal to $y_\tau^{(i)}$ units of the local consumption goods. The endowment in period t is $y_t^{(i)} = 1$. The endowment in period $t+1$ is stochastic.

The representative agent in country i has a log preference over consumption $c_\tau^{(i)}$:

$$\log c_t^{(i)} + \rho^{(i)} \mathbb{E}\left[\log c_{t+1}^{(i)}\right],$$

and the time preference $\rho^{(i)}$ can be heterogeneous across countries. The aggregate consumption $c_\tau^{(i)}$ is a Cobb-Douglas aggregate of the three countries' local goods. Let $c_{j,\tau}^{(i)}$ denote country i's agent's consumption of country j's goods in period τ, then,

$$\log c_\tau^{(i)} = \alpha \log c_{i,\tau}^{(i)} + \sum_{j \neq i} \frac{1-\alpha}{2} \log c_{j,\tau}^{(i)}.$$

Reflecting home bias towards domestic goods, $\alpha > 1/3$.

I use $m_{t+1}^{(i)}$ to denote the log marginal utility growth of country i's agent from period t to period $t+1$, which also coincides with the agent's SDF. The agent's first-order condition implies that

$$\exp(m_{t+1}^{(i)}) \stackrel{\text{def}}{=} \rho^{(i)} \frac{(c_{t+1}^{(i)})^{-1}}{(c_t^{(i)})^{-1}}.$$

Let $e_\tau^{i/j}$ denote the log bilateral exchange rate between countries i and j, which is the relative price between the consumption baskets in these two countries. Our convention is that $e_\tau^{i/j}$ takes a higher value if the consumption basket in country i is more expensive.[4]

[4]This model does not have a nominal layer, and the SDFs, exchange rates, and bonds are defined in real terms. Currency appreciation means an increase in the price of the local consumption basket relative to the foreign consumption

2.1 The Complete-Market Case

When markets are complete, I obtain the standard international risk-sharing condition. For example, between countries 1 and 2,

$$\Delta e_{t+1}^{1/2} = m_{t+1}^{(1)} - m_{t+1}^{(2)}. \tag{4}$$

In this case, I can solve the equilibrium exchange rate and SDFs based on the social planner's optimization problem. Let $\pi^{(i)}$ denote country i's Pareto weight. I obtain the following result.

Proposition 1. *The SDF of country i's agent can be expressed as*

$$m_{t+1}^{(1)} = \bar{m}^{(1)} - \alpha \Delta \log y_{t+1}^{(1)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(2)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(0)}, \tag{5}$$

and the bilateral exchange rate between countries 1 and 2 can be expressed as

$$\Delta e_{t+1}^{1/2} = \bar{\mu}^{1/2} + \frac{3\alpha - 1}{2} (\Delta \log y_{t+1}^{(2)} - \Delta \log y_{t+1}^{(1)}), \tag{6}$$

where the constants $\bar{m}^{(1)}$ and $\bar{\mu}^{1/2}$ are independent of the endowment shocks $y_{t+1}^{(i)}$.

The proof and the expressions of the constants are presented in Appendix B.1. Eq. (5) shows that the equilibrium SDF in each country loads on all three countries' endowment shocks. In complete markets, perfect international risk-sharing implies that the representative agents in countries 1 and 2 choose to bear the same amount of country 0's endowment risk. Then, Eq. (4) implies that country 0's endowment risk cancels out in the bilateral exchange rate between countries 1 and 2, which is confirmed by Eq. (6): the bilateral exchange rate between countries 1 and 2 is only a function of the local endowments $y_{t+1}^{(1)}$ and $y_{t+1}^{(2)}$. A shock to country 0's endowment $y_{t+1}^{(0)}$ has no impact on $e_{t+1}^{1/2}$.

basket.

2.2 The Incomplete-Market Case

Next, we consider a setting with incomplete financial markets, in which agents can only trade the risk-free bonds in different countries. The risk-free bond in country i pays 1 unit of the local consumption basket in period $t+1$, which in period t has a log interest rate of $r^{(i)}$ and a price of $\exp(-r^{(i)})$ in the unit of the local consumption basket.

Let $p_\tau^{(i)}$ denote the price of country i's local goods in the unit of country i's consumption basket. Let $b_j^{(i)}$ denote country i's agent's holding of country j's bond. The budget constraints of country i's agent are

$$p_t^{(i)} y_t^{(i)} = \sum_j b_j^{(i)} + c_t^{(i)}$$

$$p_{t+1}^{(i)} y_{t+1}^{(i)} + \sum_j b_j^{(i)} \exp(r^{(j)} + \Delta e_{t+1}^{j/i}) = c_{t+1}^{(i)}.$$

The market clearing conditions for goods are

$$\sum_i c_{j,\tau}^{(i)} = y_\tau^{(j)}.$$

The bonds are in zero net supply. The market clearing conditions for bonds are

$$\sum_i b_j^{(i)} \exp(e_t^{i/j}) = 0.$$

While the model has no simple closed-form solution in the presence of stochastic endowments, it can be solved numerically. To illustrate the international spill-over effect, I consider a simple case in which each country's endowment in period $t+1$ has two possible values, $y_{t+1}^{(i)} \in \{0.95, 1.05\}$, with equal probability. The endowment shocks are also i.i.d. across countries. To motivate international trade in the bond market, we assume that the agents have different time preferences: $\rho^{(0)} = 1$, $\rho^{(1)} = 2$, and $\rho^{(2)} = 0.5$, so that the agent in country 1 is more patient while the agent in country 2 is less patient. I assume a consumption home bias of $\alpha = 0.8$.

Table (4) reports the equilibrium outcomes. As country 1's agent is more patient, it is a net saver. It takes long positions on all three countries' bonds to benefit from diversification. Conversely, as country 2's agent is less patient, it is a net borrower and takes short positions on all three countries' bonds. The consumption allocations and the equilibria exchange rates are also consistent with the agents' time preferences: country 1's agent consumes less in period 1 and more in period 2, while country 2's agent consumes more in period 1 and less in period 2. Moreover, in period 1, since there is a weaker demand for country 1's goods from its own agent who has the strongest demand, the price of country 1's goods is lower and weakens its exchange rate relative to country 0's goods. In period 2, there is a stronger demand for country 1's goods, which strengthens its exchange rate relative to country 0.

Motivated by the empirical exercise, I regress the bilateral exchange rate movement $\Delta e_{t+1}^{1/2}$ between countries 1 and 2 on the three countries' SDFs:

$$\Delta e_{t+1}^{1/2} = \alpha^m + \beta_0^m m_{t+1}^{(0)} + \beta_1^m m_{t+1}^{(1)} + \beta_2^m m_{t+1}^{(2)} + \varepsilon_{t+1}.$$

Table 4: Model Solution

	Country 0	Country 1	Country 2
Bond Portfolio			
$b_0^{(i)}$	-0.00	0.02	-0.01
$b_1^{(i)}$	-0.00	0.01	-0.01
$b_2^{(i)}$	-0.00	0.03	-0.02
Consumption			
$c_t^{(i)}$	0.53	0.43	0.61
$\mathbb{E}[c_{t+1}^{(i)}]$	0.53	0.61	0.43
Exchange Rates			
$e_t^{0/i}$	0.00	0.19	-0.15
$\mathbb{E}[\Delta e_{t+1}^{0/i}]$	0.00	-0.33	0.33
Regression Coefficients			
β_i^m	0.20	1.15	-0.71
β_i^y	-0.10	-0.78	0.57

Notes: This table reports the equilibrium outcomes of the incomplete-market model.

I find a non-zero β_0^m coefficient, which means that country 0's SDF explains the bilateral exchange rate between countries 1 and 2, even after their own SDFs are controlled for. This result exemplifies the international spill-over effect in the currency market, as well as the deviation from the international risk-sharing condition $\Delta e_{t+1}^{1/2} = m_{t+1}^{(1)} - m_{t+1}^{(2)}$. Specifically, the positive sign of β_0^m indicates that country 1's currency appreciates against country 2's currency when country 0's SDF increases, or, equivalently, when country 0's consumption declines. To understand this result, note that country 1's agent is a net saver, and country 2's agent is a net borrower. When country 0's endowment declines, its currency appreciates and increases the value of country 1's holdings of country 0's bond. This wealth effect further increases country 1's consumption relative to country 2. Given the home bias in consumption, country 1's agent bids up the price of their own local consumption goods, which leads to an appreciation of country 1's currency against country 2's currency.

I also regress the bilateral exchange rate movement $\Delta e_{t+1}^{1/2}$ between countries 1 and 2 on the three countries' log endowment growth rates:

$$\Delta e_{t+1}^{1/2} = \alpha^y + \beta_0^y \Delta \log y_{t+1}^{(0)} + \beta_1^y \Delta \log y_{t+1}^{(1)} + \beta_2^y \Delta \log y_{t+1}^{(2)} + \varepsilon_{t+1}.$$

I obtain a negative β_0^y coefficient, which is consistent with a positive β_0^m given the inverse relationship between local endowment and marginal utility growth. Moreover, the magnitude of the β_0^y coefficient associated with the third country's endowment shock is a non-trivial fraction of the β_2^y coefficient associated with the local endowment shock, even when the bond positions are relatively small compared to consumption. While this setting is stylized, it suggests that the exchange rate spill-over effect could be economically significant in the real world, especially when the agents' foreign bond positions are substantial.

In this way, this example illustrates a stark difference between the complete-market case and the incomplete-market case: when markets are complete, Proposition 1 shows that the third country's endowment shock has no impact on the bilateral exchange rate. When markets are incomplete, it has a significant impact.

2.3 The Financial Autarky Case

It is also interesting to consider the other extreme case of asset space specification: financial autarky. In this case, agents from different countries cannot trade any financial assets with each other. In each period, they receive their endowments and trade consumption goods. I obtain the following characterization of the equilibrium SDFs and exchange rates.

Proposition 2. *The SDF of country i's agent can be expressed as*

$$m_{t+1}^{(i)} = \log \rho^{(i)} + \alpha \Delta \log y_{t+1}^{(i)} + \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(j)} + \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(k)},$$

and the bilateral exchange rate between countries 1 and 2 can be expressed as

$$\Delta e_{t+1}^{1/2} = \frac{3\alpha - 1}{2} (\Delta \log y_{t+1}^{(2)} - \Delta \log y_{t+1}^{(1)}).$$

The proof is presented in Appendix B.2. In this financial autarky case, we again obtain a clear separation between the bilateral exchange rate movement between countries 1 and 2 and country 0's endowment shock as in the complete-market case. If we think of this financial autarky case and the complete-market case as two ends of a continuum in the asset space specification, the incomplete-market model is somewhere in between. However, the incomplete-market model generates an international spill-over effect between the bilateral exchange rate movement and the third country's fundamental shock, which is absent in both ends of the continuum. The intuition is that the spill-over effect requires international trading in the asset markets to pass shocks from one country to other countries through financial profits and losses, while the asset markets are not complete enough to hedge out these shocks.

A natural question is whether this international spill-over effect transcends the specific setting considered in this example. In the next section, I develop a framework to characterize this spill-over effect, and show it is not only possible but also general. However, multi-country models with incomplete markets are generally difficult to solve in closed form, which motivates a different, no-arbitrage approach.

3 Exchange Rate Spill-Over in a General Economy

In this section, I derive the main theoretical results in a general setting. It suffices to consider any given period. I use the subscript t to denote the variables known before the shocks in the period are realized, and I use the subscript $t+1$ to denote the variables known after the shocks are realized. I assume all shocks are drawn from a multivariate normal distribution. The results hold much more generally in a dynamic, infinite-horizon, continuous-time setting (see Appendix B.8).

3.1 More Is Different: from Two to Three Countries

Exchange rate models predominantly feature two countries. Implicitly, we regard adding a third country as a natural but trivial extension. This intuition is straightforward when markets are complete. For example, consider three countries denoted as $0, 1, 2$. I refer to country 0 as the home country, and countries 1 and 2 as the two foreign countries. Each country is populated with a representative agent, who derives utility from consumption. Let $m_{t+1}^{(i)}$ denote the log marginal utility growth of country i's agent, which also coincides with its stochastic discount factor (SDF).

Let $\Delta e_{t+1}^{0/i}$ denote the log bilateral exchange rate movement, which takes a higher value if the currency in country 0 is stronger. When markets are complete, the SDF in each country is unique, and the bilateral log exchange rate movement between any two countries is equal to the differential between their SDFs:

$$\begin{aligned} \Delta e_{t+1}^{0/1} &= m_{t+1}^{(0)} - m_{t+1}^{(1)}, \\ \Delta e_{t+1}^{0/2} &= m_{t+1}^{(0)} - m_{t+1}^{(2)}. \end{aligned} \tag{7}$$

By Triangular arbitrage,

$$\Delta e_{t+1}^{1/2} = \Delta e_{t+1}^{0/2} - \Delta e_{t+1}^{0/1} = (m_{t+1}^{(0)} - m_{t+1}^{(2)}) - (m_{t+1}^{(0)} - m_{t+1}^{(1)}) = m_{t+1}^{(1)} - m_{t+1}^{(2)},$$

which means that, once we know the SDF differential between the home country and each foreign country, we can directly pin down the bilateral exchange rate movement between any two countries. As such, it suffices to study the bilateral dynamics from the perspective of the home country.

Now, let us consider the case of incomplete markets. I assume that investors can freely trade both home and foreign risk-free bonds. Let $r_t^{(i)}$ denote the risk-free rate in country i. Between countries 0 and 1, opening these basic markets implies the following four Euler equations:

$$\begin{aligned} 1 &= \mathbb{E}_t[\exp(m_{t+1}^{(0)} + r_t^{(0)})] = \mathbb{E}_t[\exp(m_{t+1}^{(1)} + r_t^{(1)})], \\ 1 &= \mathbb{E}_t[\exp(m_{t+1}^{(0)} - \Delta e_{t+1}^{0/1} + r_t^{(1)})] = \mathbb{E}_t[\exp(m_{t+1}^{(1)} + \Delta e_{t+1}^{0/1} + r_t^{(0)})]. \end{aligned} \tag{8}$$

In this setting, the financial markets for other contingent claims, such as equities and long-term bonds, may or may not open, which allows me to characterize the exchange rate dynamics under a flexible degree of market incompleteness. If other contingent claims are tradable, they impose additional restrictions on the equilibrium exchange rate dynamics.

Assuming all shocks are jointly normal, the Euler equations imply the following restrictions on the equilibrium exchange rate movement:

$$-cov_t(m_{t+1}^{(0)}, -\Delta e_{t+1}^{0/1}) - \frac{1}{2} var_t(\Delta e_{t+1}^{0/1}) = cov_t(m_{t+1}^{(1)}, \Delta e_{t+1}^{0/1}) + \frac{1}{2} var_t(\Delta e_{t+1}^{0/1}),$$

which has a simple interpretation: the log currency risk premium from the home country's perspective (i.e., $cov_t(m_{t+1}^{(0)}, -\Delta e_{t+1}^{0/1})$) should match the inverse of the log currency risk premium from the foreign country's perspective (i.e., $cov_t(m_{t+1}^{(1)}, \Delta e_{t+1}^{0/1})$), after adjusting for second-order Jensen's terms. Rearranging terms, this restriction can be expressed as

$$cov_t(m^{(0)}_{t+1} - m^{(1)}_{t+1} - \Delta e^{0/1}_{t+1}, \Delta e^{0/1}_{t+1}) = 0. \tag{9}$$

Similarly, between countries 0 and 2, the Euler equations for risk-free bonds imply

$$cov_t(m^{(0)}_{t+1} - m^{(2)}_{t+1} - \Delta e^{0/2}_{t+1}, \Delta e^{0/2}_{t+1}) = 0. \tag{10}$$

The key point in this section is that, while the bilateral exchange rate movements $\Delta e^{0/1}_{t+1}$ and $\Delta e^{0/2}_{t+1}$ between the home country and each foreign country directly imply the bilateral exchange rate movement $\Delta e^{2/1}_{t+1}$ between the two foreign countries by Triangular arbitrage, the Euler equations (9) and (10) between the home country and each foreign country *do not* imply the Euler equations between the two foreign countries. Specifically, the Euler equations between the two foreign countries additionally imply the following restriction.

Proposition 3. *When agents can freely trade risk-free bonds, the Euler equations imply*

$$cov_t(m^{(0)}_{t+1} - m^{(1)}_{t+1} - \Delta e^{0/1}_{t+1}, \Delta e^{0/2}_{t+1}) + cov_t(m^{(0)}_{t+1} - m^{(2)}_{t+1} - \Delta e^{0/2}_{t+1}, \Delta e^{0/1}_{t+1}) = 0. \tag{11}$$

The proof is presented in Appendix B.3. This proposition shows that, when we extend our analysis from two to three countries, the number of independent exchange rate movements increases from 1 to 2, while the number of unique sets of Euler equations for cross-country bond holdings increases from 1 to 3. The additional Euler equations impose an additional restriction on the exchange rate dynamics that are absent in the two-country setting.

The complete-market case is an exception: Eq. (7) implies $m^{(0)}_{t+1} - m^{(i)}_{t+1} - \Delta e^{0/i}_{t+1} = 0$, which satisfies conditions (9), (10), and (11) trivially. Perhaps this is why we are used to studying only the bilateral dynamics in the complete-market setting. When markets are incomplete, this is no longer sufficient, as the Euler equations between the foreign countries impose an additional restriction on the exchange rate dynamics.

What does this additional restriction (11) imply for the exchange rate dynamics? As we saw in the motivating example in Section 2, international bond investments give rise to a novel international spill-over effect in the currency market, defined as one country's SDF shock directly affecting the bilateral exchange rate between two other countries, *even after the covariances between their SDFs are accounted for*. We can already see some hints of this effect by rearranging Eq. (11):

$$2cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2}) = cov_t(m_{t+1}^{(0)} - m_{t+1}^{(1)}, \Delta e_{t+1}^{0/2}) + cov_t(m_{t+1}^{(0)} - m_{t+1}^{(2)}, \Delta e_{t+1}^{0/1}), \tag{12}$$

which relates the exchange rate covariance $cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2})$ on the left-hand side to the covariances between each foreign SDF (e.g., $m_{t+1}^{(1)}$) and the home exchange rate movement against the other foreign country (e.g., $\Delta e_{t+1}^{0/2}$). For a given exchange rate covariance $cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2})$, the exchange rate movement $\Delta e_{t+1}^{0/2}$ might need to comove with the other foreign country's SDF $m_{t+1}^{(1)}$ in a particular fashion to satisfy this restriction.

To derive a more precise characterization of the exchange rate spill-over effect, I next introduce a decomposition of the exchange rate movement that helps isolate the spill-over effect and derive its economic interpretation.

3.2 Exchange Rate Decomposition and the Exchange Rate Spill-Over

Next, let us formally define the exchange rate spill-over in this setting. A given exchange rate movement $\Delta e_{t+1}^{0/1}$ can be decomposed as

$$\Delta e_{t+1}^{0/1} = x_t^{(1)} + y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)} + w_t^{(1)} \varepsilon_{t+1}^{(1)}, \tag{13}$$

such that the last term $\varepsilon_{t+1}^{(1)}$ is orthogonal to the SDFs $m_{t+1}^{(0)}$ and $m_{t+1}^{(1)}$, and has an expectation of zero. This decomposition is unique with the coefficients satisfying

$$\begin{aligned}
y_t^{(1)} &= \frac{cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(1)})cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})}{var_t(m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})^2}, \\
z_t^{(1)} &= \frac{cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(1)})var_t(m_{t+1}^{(0)}) - cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(0)})cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})}{var_t(m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})^2}, \\
x_t^{(1)} &= \mathbb{E}_t[\Delta e_{t+1}^{0/1}] - y_t^{(1)}\mathbb{E}_t[m_{t+1}^{(0)}] - z_t^{(1)}\mathbb{E}_t[m_{t+1}^{(1)}], \\
w_t^{(1)}\varepsilon_{t+1}^{(1)} &= \Delta e_{t+1}^{0/1} - x_t^{(1)} - y_t^{(1)}m_{t+1}^{(0)} - z_t^{(1)}m_{t+1}^{(1)}.
\end{aligned}$$

The derivation is presented in Appendix B.4. Intuitively, the terms $y_t^{(1)}m_{t+1}^{(0)}$ and $z_t^{(1)}m_{t+1}^{(1)}$ describe how the bilateral exchange rate movement loads on the SDFs in the two countries, with the coefficients $y_t^{(1)}$ and $z_t^{(1)}$ interpreted as the *SDF-FX pass-through*. When the markets are complete, the exchange rate movement comoves with the SDFs one-for-one (i.e., $\Delta e_{t+1}^{0/1} = m_{t+1}^{(0)} - m_{t+1}^{(1)}$), implying a full pass-through of $y_t^{(1)} = 1$ and $z_t^{(1)} = -1$. As we will see later, when the markets are incomplete, $|y_t^{(1)}|$ and $|z_t^{(1)}|$ can take values lower than 1, reflecting a partial pass-through from the SDFs to the exchange rate.

The last term $w_t^{(1)}\varepsilon_{t+1}^{(1)}$ captures the residual in the bilateral exchange rate movement that is not explained by the two countries' SDFs. Given the unique mapping, we can also define the operator $\mathcal{R}^{0,1}[\Delta e_{t+1}^{0/1}] \stackrel{\text{def}}{=} w_t^{(1)}\varepsilon_{t+1}^{(1)}$, which projects $\Delta e_{t+1}^{0/1}$ on the SDFs of countries 0 and 1 and extracts the residual.

This decomposition allows us to derive an economic interpretation for the effect that market incompleteness has on the pass-through from SDFs to exchange rates in subsequent analysis. Under joint normality, the exchange rate dynamics can be represented by $(x_t^{(1)}, y_t^{(1)}, z_t^{(1)}, w_t^{(1)})$.[5]

Now, consider another foreign country 2. The international spill-over effect with respect to the

[5] Alternatively, they can also be represented by the expected exchange rate movement $\mathbb{E}_t[\Delta e_{t+1}^{1/2}]$, the bilateral exchange rate movement's variance $var_t(\Delta e_{t+1}^{1/2})$, and its covariances with the two countries' SDFs $cov_t(\Delta e_{t+1}^{1/2}, m_{t+1}^{(1)})$ and $cov_t(\Delta e_{t+1}^{1/2}, m_{t+1}^{(2)})$.

bilateral exchange rate movement $\Delta e_{t+1}^{0/1}$ and the foreign SDF $m_{t+1}^{(2)}$ is defined as

$$cov_t(\mathcal{R}^{0,1}[\Delta e_{t+1}^{0/1}], m_{t+1}^{(2)}) = cov_t(w_t^{(1)} \varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)}) \neq 0.$$

In other words, the bilateral exchange rate movement between countries 0 and 1 comoves with the SDF of country 2, even after we account for the correlation between the SDFs.

Let us consider the complete-market case as a benchmark. The bilateral exchange rate movement satisfies $\Delta e_{t+1}^{0/1} = m_{t+1}^{(0)} - m_{t+1}^{(1)}$, which implies that $\varepsilon_{t+1}^{(1)} = 0$, and there is no spill-over effect from any third-country SDF: $cov_t(\varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)}) = 0$.

In contrast, when markets are incomplete, $cov_t(\varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)})$ could be non-zero. If so, the covariance between the bilateral exchange rate and the third country's SDF can be expressed as

$$cov_t(m_{t+1}^{(2)}, \Delta e_{t+1}^{0/1}) = cov_t(m_{t+1}^{(2)}, y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)}) + cov_t(m_{t+1}^{(2)}, w_t^{(1)} \varepsilon_{t+1}^{(1)}), \tag{14}$$

which can be non-zero for two separate reasons: (i) if the SDFs are correlated, the first term on the right-hand side can be non-zero; (ii) in addition, a non-zero correlation between the exchange rate residual $\varepsilon_{t+1}^{(1)}$ and the third-country SDF $m_{t+1}^{(2)}$ can also generate a non-zero second term on the right-hand side.

I am interested in understanding whether the latter term $cov_t(m_{t+1}^{(2)}, w_t^{(1)} \varepsilon_{t+1}^{(1)})$ capturing the international spill-over effect has a non-trivial contribution. If so, market incompleteness gives rise to an additional channel that generates exchange rate comovements, even when the SDFs happen to be uncorrelated. This new channel could generate additional exchange rate comovements in the cross-section, while disentangling bilateral exchange rate movements from the local SDFs. The motivating model in the previous Section 2 suggests that this is possible, as the third country's SDF shock directly enters the bilateral exchange rate movement.

This exchange rate decomposition so far is just accounting. Eq. (13) remains a general description of the exchange rate dynamics. Economic restrictions are imposed by the Euler equations (8) arising from cross-country bond holdings, which we next study in a multi-currency setting.

3.3 Three Countries and the Main Result

Next, I consider three countries. Applying the exchange rate decomposition to country pairs $(0, 1)$ and $(0, 2)$, I obtain

$$\Delta e_{t+1}^{0/1} = x_t^{(1)} + y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)} + w_t^{(1)} \varepsilon_{t+1}^{(1)}, \quad (15)$$

$$\Delta e_{t+1}^{0/2} = x_t^{(2)} + y_t^{(2)} m_{t+1}^{(0)} + z_t^{(2)} m_{t+1}^{(2)} + w_t^{(2)} \varepsilon_{t+1}^{(2)}, \quad (16)$$

where $\varepsilon_{t+1}^{(i)}$ is orthogonal to $m_{t+1}^{(0)}$ and $m_{t+1}^{(i)}$ by construction. Like the two-country case, the exchange rate dynamics are mapped uniquely to these expressions, and the coefficients $x_t^{(i)}$, $y_t^{(i)}$, $z_t^{(i)}$, and $w_t^{(i)}$ are uniquely defined by the exchange rate moments.

To understand the constraints that the cross-country Euler equations impose on the exchange rate movements, I analyze what happens if there is no international spill-over effect, that is

Condition NS. *For any countries i, j, k, the residual term in the bilateral exchange rate movement $\Delta e_{t+1}^{i/j}$ is not correlated with the third country's SDF:*

$$cov_t(\mathcal{R}^{i,j}[\Delta e_{t+1}^{i/j}], m_{t+1}^{(k)}) = 0.$$

Under this condition, the cross-country Euler equations that we characterize in Section 3.1 imply the following results.

Proposition 4. *The no-spill-over Condition NS together with the cross-country Euler equations* (9), (10), (11) *imply*

$$y_t^{(1)} = y_t^{(2)} \overset{def}{=} y_t,$$

and

$$cov_t(\Delta e^{0/1}_{t+1}, \Delta e^{0/2}_{t+1}) = \frac{cov_t(y_t m^{(0)}_{t+1} + z^{(1)}_t m^{(1)}_{t+1}, m^{(0)}_{t+1} - m^{(2)}_{t+1}) + cov_t(y_t m^{(0)}_{t+1} + z^{(2)}_t m^{(2)}_{t+1}, m^{(0)}_{t+1} - m^{(1)}_{t+1})}{2}. \tag{17}$$

The proof is given in Appendix B.5. Eq. (17) imposes a restriction on the joint distribution of the exchange rate covariances and the SDF covariances under Condition NS. It connects the exchange rate covariance on the left-hand side to a linear combination of SDF covariances on the right-hand side.

For a given home country 0, this restriction applies to any foreign country pair (i, j). If we have N foreign countries, then, we have $N(N-1)/2$ unique foreign country pairs and hence $N(N-1)/2$ such moment restrictions. Even if we take $z^{(i)}_t$ and y_t as free parameters, we only have $N+1$ parameters to fit these $N(N-1)/2$ moment restrictions. For $N = 10$ foreign developed countries who can pairwise trade risk-free bonds, we have $N(N-1)/2 = 45$ moment restrictions and only $N+1 = 11$ parameters. As a result, these conditions based on the cross-country Euler equations imply tight constraints between the exchange rate covariances and the SDF covariances.

3.4 A Case for International Spill-Over

Ultimately, whether Eq. (17), a necessary condition for the no-spill-over Condition NS, holds or not is an empirical question. To evaluate this condition empirically, I first offer a broad discussion of the recent literature, and then conduct a more formal test under specific assumptions.

First, the literature has noted a general disconnect between exchange rate movements and local fundamental variables that proxy for the SDFs (Meese and Rogoff, 1983; Backus and Smith, 1993; Engel and West, 2005). More recently, Chernov et al. (2023) find weak correlations between exchange rates and local financial market variables such as bond returns, which they frame as the financial exchange rate disconnect. Given these results, it is not surprising that the covariance structure of exchange rate movements is also disconnected from the covariance structure of the

SDFs.

Moreover, international portfolio holdings exhibit large home bias, another major puzzle in international macroeconomics (Lewis, 1995). For example, in 2019 among developed countries, the share of domestic assets is 70% in equity portfolio and 71% in bond portfolio (Jiang et al., 2022). In models with standard preferences and heterogeneous country-level income processes, the high degree of home bias implies poor international risk-sharing through holdings of external financial assets and therefore a low correlation in the marginal utility growth across countries. This different approach of assessing international risk-sharing also suggests that the SDF correlations could be lower than the observed exchange rate correlations. In the extreme case of uncorrelated SDFs, the right-hand side of Eq. (17) is zero regardless of the values of y_t and $z_t^{(i)}$, which will fail to match the exchange rate covariance on the left-hand side.

These empirical observations suggest that the SDFs might not have a corresponding covariance structure that matches the exchange rate covariances in Eq. (17). To test this condition formally, I need to take a stance on what the SDFs are. For example, the SDFs could be driven by consumption growth: $m_{t+1}^{(i)} = \gamma \Delta c_{t+1}^{(i)}$. Using this SDF proxy, I can test whether Eq. (17) holds for all foreign country pairs (i, j) and for arbitrary choices of y_t and $z_t^{(i)}$.

Specifically, I evaluate this condition using the Generalized Method of Moments (GMM). The moment condition for the foreign country pair (i, j) is

$$\begin{aligned} 0 = & \frac{\gamma^2}{2} cov_t(y_t \Delta c_{t+1}^{(0)} + z_t^{(1)} \Delta c_{t+1}^{(1)}, \Delta c_{t+1}^{(0)} - \Delta c_{t+1}^{(2)}) \\ & + \frac{\gamma^2}{2} cov_t(y_t \Delta c_{t+1}^{(0)} + z_t^{(2)} \Delta c_{t+1}^{(2)}, \Delta c_{t+1}^{(0)} - \Delta c_{t+1}^{(1)}) - cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2}), \end{aligned}$$

with the following empirical counterpart:

$$\begin{aligned} 0 = & (\hat{y}_t \Delta \tilde{c}_{t+1}^{(0)} + \hat{z}_t^{(1)} \Delta \tilde{c}_{t+1}^{(1)})(\Delta \tilde{c}_{t+1}^{(0)} - \Delta \tilde{c}_{t+1}^{(2)}) \\ & + (\hat{y}_t \Delta \tilde{c}_{t+1}^{(0)} + \hat{z}_t^{(2)} \Delta \tilde{c}_{t+1}^{(2)})(\Delta \tilde{c}_{t+1}^{(0)} - \Delta \tilde{c}_{t+1}^{(1)}) - \Delta \tilde{e}_{t+1}^{0/1} \Delta \tilde{e}_{t+1}^{0/2}, \end{aligned} \tag{18}$$

where the variables with tilde denote their demeaned values in the data, and $\hat{y}_t$ and $\hat{z}_t^{(i)}$ denote y_t

and $z_t^{(i)}$ scaled by $\gamma^2/2$.[6] This equation system has $N(N-1)/2 = 45$ moment conditions, one for each unique foreign country pair, and $N+1 = 11$ parameters for the home and each foreign country.

I run both the first-stage and the two-stage GMM tests, and report the p-values of the overidentifying restrictions test (the J-test) in Table (5). A small p-value rejects the null hypothesis that there exist some parameter values for y_t and $z_t^{(i)}$ such that all moment conditions are satisfied. Appendix A.2 presents more details. In our context, this means that no matter how we specify the structural parameters $(\gamma, y_t, z_t^{(i)})$, we can never satisfy all the 45 moment conditions under the no-spill-over Condition NS.

I also consider other SDF proxies such as the stock market return and the 3-month and 10-year interest rate changes. The p-values in the J-test using these alternative SDF proxies are also reported in Table (5). The results are similar, which suggest that the no-spill-over Condition NS imposes too many and too strict restrictions on the cross-country covariances between the exchange rates and the SDFs, and these restrictions are not consistent with the data. If these SDF proxies are valid, we can reject the no-spill-over condition and conclude that the international spill-over effect is a general phenomenon in the incomplete-market world we live in.

[6] Our moment conditions cannot separately identify γ and the magnitudes of $(y_t, z_t^{(i)})$, but this is not a problem for our purpose of testing whether any choice of the parameter values can fit the moment conditions.

Table 5: J-Test Statistics of GMM.

Variable	p-Value, 1st-Stage GMM	p-Value, Two-Stage GMM
Change in 10Y Bond Yield	0.00	0.00
Change in 3M Bond Yield	0.00	0.00
Stock Return	0.00	0.00
Consumption Growth	0.00	0.00

Notes: The table reports the p-value of the J-test against the null that all moment conditions are satisfied. The GMM moment conditions are given in Eq. (18). The sample includes 11 developed countries listed in Appendix A. Data are quarterly, 1988Q1–2023Q4.

3.5 Discussions

In this last subsection, I offer some discussions to further clarify the interpretation of the results.

(a) Do we need representative agents? It is not necessary to assume that each country has a representative agent. As long as each country has some investors who can trade home and foreign risk-free bonds with negligible transaction costs, which is a reasonable assumption for developed countries, then, their Euler equations will hold and comprise the main object that this paper studies.

In comparison, the theoretical results do not apply to the case in which cross-country trade in risk-free bonds has to be intermediated by a common financier, as in Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021). In this case, the home and foreign agents' SDFs do not price each other's risk-free bonds, which gives rise to additional wedges in the Euler equations. That said, in this case, it is very plausible that shocks from third countries can affect the common financier's risk-taking capacity and hence impact the bilateral exchange rates, which could give rise to a similar international spill-over effect that I characterize in this paper.

(b) Relationship between the international spill-over effect and global factors? While my characterization of the international spill-over effect shows that shocks to a country's SDF appear in the bilateral exchange rate movement between two other countries, it does not restrict the spilled-over shocks to be *global* or *idiosyncratic*, which describe the factor structure in exchange rates. The U.S. is a good example: we tend to think that the U.S. SDF is a global factor that affects foreign exchange rates and asset prices. As such, the presence of the U.S. SDF in the bilateral exchange rate between two foreign countries gives rise to additional loadings on a global factor. Alternatively, if we consider a peripheral country whose SDF is unlikely to covary with global factors, then, the spill-over effect could pass its idiosyncratic shock to foreign exchange rates. That said, the international spill-over result suggests that exchange rates can be more correlated than the economic fundamentals, which could contribute to the strong common factor structure in exchange rates.

(c) $\Delta e = m - m^*$ in incomplete markets? When markets are incomplete, the SDFs are not unique. For any given home SDF $m_{t+1}^{(0)}$, $m_{t+1}^{(1),*} \stackrel{\text{def}}{=} m_{t+1}^{(0)} - \Delta e_{t+1}^{0/1}$ is always a valid foreign SDF that correctly prices all asset returns. Moreover, Sandulescu et al. (2021) show that the minimum-dispersion SDFs constructed from asset returns always satisfy the relationship $\Delta e_{t+1}^{0/1} = m_{t+1}^{(0)} - m_{t+1}^{(1)}$. Therefore, if we only focus on the asset market equilibrium, we can always find the ideal case with full pass-through and zero residual.

The decomposition proposed in this paper takes a different perspective. To the extent that we are interested in understanding the relationship between the exchange rate dynamics and the underlying economy, we would like to focus on the home and foreign SDFs that reflect the agents' marginal utility growth. These consumption-based SDFs are uniquely defined, and they do not necessarily satisfy $\Delta e_{t+1}^{0/1} = m_{t+1}^{(0)} - m_{t+1}^{(1)}$. For example, in the fully specified model we considered in Section 2, $\Delta e_{t+1}^{0/1} \neq m_{t+1}^{(0)} - m_{t+1}^{(1)}$, which makes it economically meaningful to study the nonzero residual from the exchange rate decomposition.

(d) Generality. I considered a discrete-time setting for expositional clarity. The results hold more generally in an infinite-horizon economy with general diffusion processes in continuous time. Appendix B.8 provides a brief discussion.

(e) The magnitude of the pass-through coefficients. The pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ are important equilibrium objects that describe exchange rate dynamics. The Euler equations from cross-country bond holdings also have important implications for their magnitude. The following result characterizes the bounds on the pass-through coefficients imposed by the volatility of the bilateral exchange rate movement.

Proposition 5. *The Euler equations impose the following constraints on the pass-through coeffi-*

cients $y_t^{(1)}$ *and* $z_t^{(1)}$*:*

$$\begin{aligned} y_t^{(1)}(y_t^{(1)}-1)var_t(m_{t+1}^{(0)})+z_t^{(1)}(z_t^{(1)}+1)var_t(m_{t+1}^{(1)})+ \\ [y_t^{(1)}(z_t^{(1)}+1)+z_t^{(1)}(y_t^{(1)}-1)]cov_t(m_{t+1}^{(0)},m_{t+1}^{(1)})\leq 0, \end{aligned} \tag{19}$$

$$y_t^{(1)}var_t(m_{t+1}^{(0)})-z_t^{(1)}var_t(m_{t+1}^{(1)})+(z_t^{(1)}-y_t^{(1)})cov_t(m_{t+1}^{(0)}.m_{t+1}^{(1)})=var_t(\Delta e_{t+1}^{0/1}). \tag{20}$$

The proof is presented in Appendix B.6. This result shows that, for given SDF variances and covariances, we can reject some values of the pass-through coefficients by noting that they generate too much exchange rate volatility. For example, if $y_t^{(1)}=1$ and $z_t^{(1)}=-1$ as in the complete-market case, Eq. (19) becomes $0\geq 0$ and Eq. (20) becomes $var_t(m_{t+1}^{(0)}-m_{t+1}^{(1)})=var_t(\Delta e_{t+1}^{0/1})$, which recovers the key result in Brandt et al. (2006). If the cross-country correlation between the SDFs is low, the exchange rate variance has to be in the same order of magnitude as the SDF variance, which is not consistent with the empirical evidence.

This result generalizes Brandt et al. (2006) by asking how much lower the pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ have to be in order to match the observed exchange rate volatility. Quantitatively, at annual frequency, the exchange rate volatility between developed countries is in the order of 10%, implying $var_t\left(\Delta e_{t+1}^{0/1}\right)=1\%$. In comparison, given the existence of financial assets with Sharpe ratios above 0.6, the Hansen and Jagannathan (1991) bound implies $var_t(m_{t+1}^{(0)})\geq 0.6^2$. Together, these empirical observations imply the following restriction:

$$var_t\left(\Delta e_{t+1}^{0/1}\right)\leq \frac{1}{36}\cdot var_t(m_{t+1}^{(0)}). \tag{21}$$

If we additionally assume that the home and foreign SDFs have the same variance, i.e., $\sigma_t^2=var_t(m_{t+1}^{(0)})=var_t(m_{t+1}^{(1)})$, and that the cross-country SDF correlation is $corr_t(m_{t+1}^{(0)},m_{t+1}^{(1)})=0.3$, which is close to the average cross-country correlation in consumption growth among developed countries, then, we can graphically represent the two equations in Proposition 5 in Figure (1): Eq. (19) requires the coefficients to lie within the red ellipse, and Eq. (20) requires the coefficients to lie above the blue line. The intersection of these two regions is very small and close to $(0,0)$,

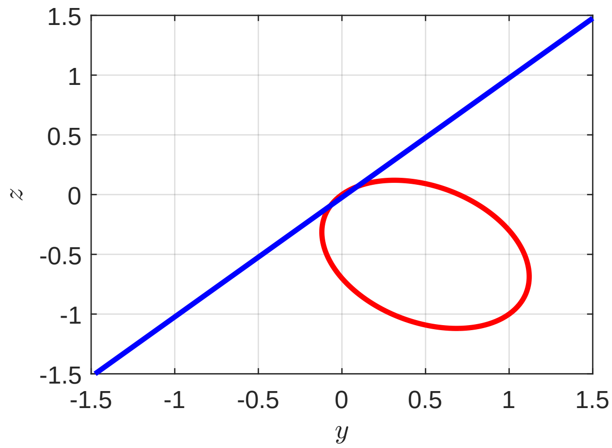

Figure 1: Bounds on the Magnitude of the Pass-Through Coefficients

Notes: This figure shows the bounds on the pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ imposed by Proposition 5.

which implies that the pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ have to be very close to zero in order to produce an empirically plausible exchange rate volatility. In this numerical example, the greatest possible magnitude of either coefficient is 0.09, which is much smaller than the complete-market benchmark of 1.

Appendix B.6 plots this figure using different SDF correlations. The results are similar: the pass-through coefficients have to be very small in order to match the exchange rate volatility, unless the SDF correlation is as high as 0.99. The pass-through coefficients are also related to the currency risk premium. Appendix B.7 provides a more detailed discussion.

4 Conclusion

In this paper, I show that cross-country bond investments give rise to a new channel of exchange rate spill-over effect when markets are incomplete. I develop a fully specified general equilibrium model to illustrate this channel, as well as a general argument for its existence. These results offer a new way of understanding the exchange rates' disconnect and comovement patterns. Empirically, third countries' financial variables indeed have economically meaningful explanatory power for the bilateral exchange rate movements, even after the local variables and the common exchange rate factors are controlled for. As a result of this international spill-over effect, even when we study the bilateral exchange rate between two countries, we need to pay attention to shocks from other countries as well.

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Appendix

A Empirical Appendix

The sample contains 11 developed countries: Australia, Canada, Switzerland, Germany, Denmark, the U.K., Japan, Norway, New Zealand, Sweden, and the U.S. The exchange rates are from Datastream, the interest rates are government debt yields from Global Financial Database. The real consumption data are from Oxford Economics. The cum-dividend equity returns are from MSCI. These data are quarterly, 1988Q1 to 2023Q4.

The financial crisis indicator is from Jordà et al. (2017). The sample is annual, 1988 to 2020, and does not contain data for New Zealand. It is merged with the end-of-year exchange rate data.

A.1 Exchange Rate Accounting in the Extended Sample

I also broaden the sample to include developing countries. The extended sample includes 25 economies: Argentina, Australia, Brazil, Bulgaria, Canada, Chile, China, Croatia, Czech Republic, Denmark, Germany, Hong Kong, Hungary, India, Indonesia, Japan, Korea, Malaysia, Mexico, New Zealand, Norway, Philippines, Poland, Romania, Russia, Singapore, South Africa, Sweden, Switzerland, Taiwan, Thailand, Turkey, United Kingdom, United States. Given the relatively poor coverage of short-term sovereign yield data in this sample, I construct the foreign 3-month interest rates from the U.S. 3-month interest rates and the currency spot and 3-month forward rates based on the covered interest rate parity.

The sample is quarterly. For regressions using stock returns, 3-month interest rates, and consumption growth, the sample is 1998Q1 to 2023Q4. For regressions using 10-year bond returns, 5 countries (Czech Republic, Indonesia, Mexico, Poland, and Turkey) are dropped due to limited data availability, and the sample is shorter, 1999Q2 to 2023Q2. I drop the financial crisis indicator as an explanatory variable because it is not available for many countries in the extended sample.

Table (A.1) repeats the regression in Table (1) in the extended sample. Similar to the results in the developed country sample, introducing the third-country principal components improves the explanatory power for the bilateral exchange rate movements.

Table (A.2) repeats the regression in Table (2) in the extended sample. Similarly, introducing the third-country principal components improves the explanatory power for the bilateral exchange rate movements, even after controlling for the common exchange rate factors. For example, when we use stock returns and 3-month and 10-year bond yields as SDF proxies, the adjusted R^2 increases from 27.81% to 36.33% when we introduce the third-country principal components. The adjusted R^2 further increases to 46.20% when we include both the local variables and the third-country principal components.

Finally, Table (A.3) reports the Wald tests for the extended sample of economies. Again, the null that the $\beta^{(i)}$ coefficients associated with the third-country principal components are jointly zero is rejected for a non-trivial fraction of country pairs.

Table A.1: Adjusted R^2 of the Third-Country Principal Components.

Variable	(1) Local Variables Only	(2) Third-Country PCs Only	(3) Both	(4) Improvement from (1)
Change in 10Y Bond Yield	9.82	7.40	14.41	4.59
Change in 3M Bond Yield	8.08	5.90	13.88	5.80
Stock Return	11.49	12.66	16.81	5.33
Stock+Bond	23.03	16.10	30.19	7.16
Consumption Growth	-0.21	-0.03	-0.02	0.19

Notes: The regression equation is Eq. (2). Column (1) reports the mean adjusted R^2 of the regression without controlling for the third-country principal components. Column (2) reports the mean adjusted R^2 with only the third-country principal components. Column (3) reports the mean adjusted R^2 with both the local variables and the third-country principal components. Column (4) reports the increase in the adjusted R^2 between Columns (3) and (1). Data are quarterly, 1988Q1–2023Q4; except for the 10-year bond, which is 1999Q2–2023Q2.

Table A.2: Adjusted R^2 of the Third-Country Principal Components, with Common Exchange Rate Factors

Variable	(1) Exchange Rate Factors	(2) Factors+Local	(3) Factors+Third-Country	(4) All
Change in 10Y Bond Yield	27.81	33.76	30.96	35.24
Change in 3M Bond Yield	22.40	29.50	27.22	34.51
Stock Return	22.40	28.92	30.33	33.67
Stock+Bond	27.81	41.99	36.33	46.20
Consumption Growth	22.40	22.31	23.07	22.99

Notes: The regression equation is Eq. (3). Column (1) reports the mean adjusted R^2 of the regression with the common exchange rate factors *dollar* and *carry*. Column (2) reports the mean adjusted R^2 with both the common factors and the local variables. Column (3) reports the mean adjusted R^2 with both the common factors and the third-country principal components. Column (4) reports the adjusted R^2 with the common factors, the local variables and the third-country principal components. Data are quarterly, 1988Q1–2023Q4; except for the 10-year bond, which is 1999Q2–2023Q2.

Table A.3: Wald Tests for the Third-Country Principal Components

Regression	(1) Eq. (2), Local+Third-Country	(2) Eq. (3), Factors+Local+Third-Country
Change in 10Y Bond Yield	37.89	22.63
Change in 3M Bond Yield	43.33	43.67
Stock Return	36.00	41.33
Stock+Bond	47.89	37.89
Consumption Growth	9.33	11.00

Notes: We report the fraction of country pairs for which the null hypothesis that the coefficients associated with the third-country principal components are all zero, i.e., $\beta^{(1)} = \beta^{(2)} = \beta^{(3)} = 0$, is rejected at the 5% level. Column (1) reports the result for Eq. (2), and Column (2) reports the result for Eq. (3). Data are quarterly, 1988Q1–2023Q4; except for the 10-year bond, which is 1999Q2–2023Q2.

A.2 GMM Moment Conditions

In Table (5), the first-stage GMM uses the identity weighting matrix, and the two-stage GMM uses the optimal weighting matrix. The optimal weighting matrix is based on a heteroscedasticity and autocorrelation consistent (HAC) estimation of the covariance matrix. The J-test tests the null hypothesis that all moment conditions are zero. Let $\bar{g}$ denote the sample average of the moment conditions, and let $\hat{V}$ denote the sample covariance matrix of the moment conditions. The J-test statistic is $T\bar{g}'\hat{V}^{-1}\bar{g}$, and has an asymptotic χ^2 distribution under the null.

B Proof

B.1 Proposition 1

Proof. Denote the Pareto weights for country i as $\pi^{(i)}$. The Lagrangian for social planner's problem is

$$\mathcal{L} = \mathbb{E}_t \left\{ \sum_{\tau=t,t+1} \left[\sum_i \rho_\tau^{(i)} \left(\alpha \log c_{i,\tau}^{(i)} + \sum_{j \neq i} \frac{1-\alpha}{2} \log c_{j,\tau}^{(i)} \right) + \sum_i \zeta_\tau^{(i)} \left(y_\tau^{(i)} - \sum_j c_{i,\tau}^{(j)} \right) \right] \right\}$$

where $\rho_\tau^{(i)} = 1$ if $\tau = t$, and $\rho_\tau^{(i)} = \rho^{(i)}$ if $\tau = t+1$. The consumption allocation rule is given by

$$c_{j,\tau}^{(i)} = \left(1_{\{i=j\}}\alpha + 1_{\{i \neq j\}} \frac{1-\alpha}{2} \right) \frac{\rho_\tau^{(i)} \pi^{(i)}}{\sum_k \left(1_{\{k=j\}}\alpha + 1_{\{k \neq j\}} \frac{1-\alpha}{2} \right) \rho_\tau^{(k)} \pi^{(k)}} y_\tau^{(j)},$$

which gives the aggregated consumption as

$$\begin{aligned}
c_\tau^{(1)} &= \prod_j (c_{j,\tau}^{(1)})^{1_{\{j=1\}}\alpha + 1_{\{j \neq 1\}} \frac{1-\alpha}{2}} \\
&= \alpha^\alpha \left(\frac{1-\alpha}{2} \right)^{1-\alpha} \rho_\tau^{(1)} \pi^{(1)} (y_\tau^{(1)})^\alpha (y_\tau^{(2)})^{\frac{1-\alpha}{2}} (y_\tau^{(0)})^{\frac{1-\alpha}{2}} \\
&\times \left(\alpha \rho_\tau^{(1)} \pi^{(1)} + \frac{1-\alpha}{2} \rho_\tau^{(0)} \pi^{(0)} + \frac{1-\alpha}{2} \rho_\tau^{(2)} \pi^{(2)} \right)^{-\alpha} \\
&\times \left(\alpha \rho_\tau^{(0)} \pi^{(0)} + \frac{1-\alpha}{2} \rho_\tau^{(1)} \pi^{(1)} + \frac{1-\alpha}{2} \rho_\tau^{(2)} \pi^{(2)} \right)^{-\frac{1-\alpha}{2}} \\
&\times \left(\alpha \rho_\tau^{(2)} \pi^{(2)} + \frac{1-\alpha}{2} \rho_\tau^{(0)} \pi^{(0)} + \frac{1-\alpha}{2} \rho_\tau^{(1)} \pi^{(1)} \right)^{-\frac{1-\alpha}{2}} .
\end{aligned}$$

The SDF of country 1 is given by

$$\begin{aligned}
m_{t+1}^{(1)} &= \log\left(\frac{\rho^{(1)} c_t^{(1)}}{c_{t+1}^{(1)}}\right) \\
&= \log \rho^{(1)} + \log c_t^{(1)} - \log c_{t+1}^{(1)} \\
&= -\alpha \Delta \log y_{t+1}^{(1)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(2)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(0)} \\
&+ \alpha \log\left(\frac{\alpha\rho^{(1)}\pi^{(1)} + \frac{1-\alpha}{2}\rho^{(0)}\pi^{(0)} + \frac{1-\alpha}{2}\rho^{(2)}\pi^{(2)}}{\alpha\pi^{(1)} + \frac{1-\alpha}{2}\pi^{(0)} + \frac{1-\alpha}{2}\pi^{(2)}}\right) \\
&+ \frac{1-\alpha}{2} \log\left(\frac{\alpha\rho^{(0)}\pi^{(0)} + \frac{1-\alpha}{2}\rho^{(1)}\pi^{(1)} + \frac{1-\alpha}{2}\rho^{(2)}\pi^{(2)}}{\alpha\pi^{(0)} + \frac{1-\alpha}{2}\pi^{(1)} + \frac{1-\alpha}{2}\pi^{(2)}}\right) \\
&+ \frac{1-\alpha}{2} \log\left(\frac{\alpha\rho^{(2)}\pi^{(2)} + \frac{1-\alpha}{2}\rho^{(1)}\pi^{(1)} + \frac{1-\alpha}{2}\rho^{(0)}\pi^{(0)}}{\alpha\pi^{(2)} + \frac{1-\alpha}{2}\pi^{(1)} + \frac{1-\alpha}{2}\pi^{(0)}}\right) \\
&= \bar{m}^{(1)} - \alpha \Delta \log y_{t+1}^{(1)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(2)} - \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(0)}.
\end{aligned}$$

Similarly, we obtain the SDF for country 2, hence the log exchange rate change, which is given by the difference of the SDFs

$$\begin{aligned}
\Delta e_{t+1}^{1/2} &= m_{t+1}^{(1)} - m_{t+1}^{(2)} \\
&= \frac{3\alpha-1}{2} \Delta \log \frac{y_{t+1}^{(2)}}{y_{t+1}^{(1)}} \\
&+ \frac{3\alpha-1}{2} \log\left(\frac{\alpha\rho^{(1)}\pi^{(1)} + \frac{1-\alpha}{2}\rho^{(0)}\pi^{(0)} + \frac{1-\alpha}{2}\rho^{(2)}\pi^{(2)}}{\alpha\pi^{(1)} + \frac{1-\alpha}{2}\pi^{(0)} + \frac{1-\alpha}{2}\pi^{(2)}}\right) \\
&- \frac{3\alpha-1}{2} \log\left(\frac{\alpha\rho^{(2)}\pi^{(2)} + \frac{1-\alpha}{2}\rho^{(1)}\pi^{(1)} + \frac{1-\alpha}{2}\rho^{(0)}\pi^{(0)}}{\alpha\pi^{(2)} + \frac{1-\alpha}{2}\pi^{(1)} + \frac{1-\alpha}{2}\pi^{(0)}}\right) \\
&= \bar{\mu}^{1/2} + \frac{3\alpha-1}{2}(\Delta \log y_{t+1}^{(2)} - \Delta \log y_{t+1}^{(1)}).
\end{aligned}$$

$\square$

B.2 Proposition 2

Proof. The Lagrangian for households' problem in country i is

$$\mathcal{L} = \mathbb{E}_t \left\{ \sum_{\tau=t,t+1} \left[\rho_\tau^{(i)} \left(\alpha \log c_{i,\tau}^{(i)} + \sum_{j \neq i} \frac{1-\alpha}{2} \log c_{j,\tau}^{(i)} \right) + \lambda_\tau^{(i)} \left(p_\tau^{(i)} y_\tau^{(i)} - \sum_j p_\tau^{(j)} \exp(-e_\tau^{i/j}) c_{j,\tau}^{(i)} \right) \right] \right\}$$

The within-period solution implies

$$p_\tau^{(j)} \exp(-e_\tau^{i/j}) c_{j,\tau}^{(i)} = (1_{\{i=j\}}\alpha + 1_{\{i \neq j\}} \frac{1-\alpha}{2}) c_\tau^{(i)}.$$

On the other hand, since agents can not trade financial assets, the following budget constraint must hold

$$\sum_j p_\tau^{(j)} \exp(-e_\tau^{i/j}) c_{j,t}^{(i)} = p_\tau^{(i)} y_\tau^{(i)},$$

which, together with the within-period solutions, implies

$$\begin{aligned} c_{i,t}^{(i)} &= \alpha y_\tau^{(i)}, \\ c_{j,t}^{(i)} &= \frac{1-\alpha}{2} \exp(q_\tau^{i/j}) y_\tau^{(i)}, \quad i \neq j. \end{aligned} \tag{A.1}$$

The goods market clearing conditions require that

$$\sum_i c_{j,\tau}^{(i)} = y_\tau^{(j)},$$

which yields the following equations

$$\begin{aligned} \alpha y_\tau^{(0)} + \frac{1-\alpha}{2} \exp(q_\tau^{1/0}) y_\tau^{(1)} + \frac{1-\alpha}{2} \exp(q_\tau^{2/0}) y_\tau^{(2)} &= y_\tau^{(0)}, \\ \alpha y_\tau^{(1)} + \frac{1-\alpha}{2} \exp(q_\tau^{0/1}) y_\tau^{(0)} + \frac{1-\alpha}{2} \exp(q_\tau^{2/1}) y_\tau^{(2)} &= y_\tau^{(1)}, \\ \alpha y_\tau^{(2)} + \frac{1-\alpha}{2} \exp(q_\tau^{1/2}) y_\tau^{(1)} + \frac{1-\alpha}{2} \exp(q_\tau^{0/2}) y_\tau^{(0)} &= y_\tau^{(2)}. \end{aligned}$$

The equations can be restated as

$$\exp(q_\tau^{1/0})y_\tau^{(1)} + \exp(q_\tau^{2/0})y_\tau^{(2)} = 2y_\tau^{(0)},$$

$$2\exp(q_\tau^{1/0})y_\tau^{(1)} - \exp(q_\tau^{2/0})y_\tau^{(2)} = y_\tau^{(0)},$$

$$2\exp(q_\tau^{2/0})y_\tau^{(2)} - \exp(q_\tau^{1/0})y_\tau^{(1)} = y_\tau^{(0)},$$

which imply

$$\exp(q_\tau^{i/j}) = \frac{y_\tau^{(j)}}{y_\tau^{(i)}}.$$

Plug in Eq. (A.1) to obtain

$$c_{j,t}^{(i)} = \frac{1-\alpha}{2} y_\tau^{(j)}.$$

Hence,

$$c_\tau^{(i)} = \alpha^\alpha \left(\frac{1-\alpha}{2}\right)^{1-\alpha} (y_\tau^{(i)})^\alpha (y_\tau^{(j)})^{\frac{1-\alpha}{2}} (y_\tau^{(k)})^{\frac{1-\alpha}{2}}.$$

The intertemporal marginal rate of substitution is

$$\begin{aligned} m_{t+1}^{(i)} &= \log \rho^{(i)} + \log c_{t+1}^{(i)} - \log c_t^{(i)} \\ &= \log \rho^{(i)} + \alpha \Delta \log y_{t+1}^{(i)} + \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(j)} + \frac{1-\alpha}{2} \Delta \log y_{t+1}^{(k)}. \end{aligned}$$

The exchange rate is

$$e_t^{1/2} = \frac{3\alpha - 1}{2} \log \frac{y_t^{(2)}}{y_t^{(1)}},$$

$$\Delta e_{t+1}^{1/2} = \frac{3\alpha - 1}{2} \Delta \log \frac{y_{t+1}^{(2)}}{y_{t+1}^{(1)}}.$$

□

B.3 Proposition 3

Recast the third euler equation to obtain

$$\begin{aligned}
0 &= cov_t[\Delta e^{1/2}_{t+1}, \Delta e^{1/2}_{t+1} - (m^{(1)}_{t+1} - m^{(2)}_{t+1})] \\
&= cov_t[\Delta e^{0/2}_{t+1} - \Delta e^{0/1}_{t+1}, (\Delta e^{0/2}_{t+1} - (m^{(0)}_{t+1} - m^{(2)}_{t+1})) - (\Delta e^{0/1}_{t+1} - (m^{(0)}_{t+1} - m^{(1)}_{t+1}))] \\
&= cov_t[\Delta e^{0/1}_{t+1}, \Delta e^{0/1}_{t+1} - (m^{(0)}_{t+1} - m^{(1)}_{t+1})] + cov_t[\Delta e^{0/2}_{t+1}, \Delta e^{0/2}_{t+1} - (m^{(0)}_{t+1} - m^{(2)}_{t+1})] \\
&- cov_t[\Delta e^{0/2}_{t+1}, \Delta e^{0/1}_{t+1} - (m^{(0)}_{t+1} - m^{(1)}_{t+1})] - cov_t[\Delta e^{0/1}_{t+1}, \Delta e^{0/2}_{t+1} - (m^{(0)}_{t+1} - m^{(2)}_{t+1})],
\end{aligned}$$

where the first two terms are zero from the Euler equations (9) and (10). Hence,

$$0 = cov_t[\Delta e^{0/2}_{t+1}, \Delta e^{0/1}_{t+1} - (m^{(0)}_{t+1} - m^{(1)}_{t+1})] + cov_t[\Delta e^{0/1}_{t+1}, \Delta e^{0/2}_{t+1} - (m^{(0)}_{t+1} - m^{(2)}_{t+1})].$$

B.4 Decomposition of exchange rate movement

Since $cov_t(m^{(0)}_{t+1}, w^{(1)}_t \varepsilon^{(1)}_{t+1}) = cov_t(m^{(1)}_{t+1}, w^{(1)}_t \varepsilon^{(1)}_{t+1}) = 0$, we obtain

$$\begin{aligned}
cov_t(\Delta e^{0/1}_{t+1} - y^{(1)}_t m^{(0)}_{t+1} - z^{(1)}_t m^{(1)}_{t+1}, m^{(0)}_{t+1}) = 0, \\
cov_t(\Delta e^{0/1}_{t+1} - y^{(1)}_t m^{(0)}_{t+1} - z^{(1)}_t m^{(1)}_{t+1}, m^{(1)}_{t+1}) = 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
var_t(m^{(0)}_{t+1}) y^{(1)}_t + cov_t(m^{(0)}_{t+1}, m^{(1)}_{t+1}) z^{(1)}_t = cov_t(\Delta e^{0/1}_{t+1}, m^{(0)}_{t+1}), \\
cov_t(m^{(0)}_{t+1}, m^{(1)}_{t+1}) y^{(1)}_t + var_t(m^{(1)}_{t+1}) z^{(1)}_t = cov_t(\Delta e^{0/1}_{t+1}, m^{(1)}_{t+1}),
\end{aligned}$$

which imply

$$y_t^{(1)} = \frac{cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(1)})cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})}{var_t(m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})^2},$$
$$z_t^{(1)} = \frac{cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(1)})var_t(m_{t+1}^{(0)}) - cov_t(\Delta e_{t+1}^{0/1}, m_{t+1}^{(0)})cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})}{var_t(m_{t+1}^{(0)})var_t(m_{t+1}^{(1)}) - cov_t(m_{t+1}^{(0)}, m_{t+1}^{(1)})^2}.$$

Then, $x_t^{(1)}$ and $w_t^{(1)}\varepsilon_{t+1}^{(1)}$ are derived from $\mathbb{E}_t[w_t^{(1)}\varepsilon_{t+1}^{(1)}] = 0$.

B.5 Proposition 4

Plug the exchange rate decomposition into Proposition 3. We obtain

$$\begin{aligned}
0 &= -cov_t[\Delta e_{t+1}^{0/2}, \Delta e_{t+1}^{0/1} - (m_{t+1}^{(0)} - m_{t+1}^{(1)})] - cov_t[\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2} - (m_{t+1}^{(0)} - m_{t+1}^{(2)})], \\
2cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2}) &= cov_t[\Delta e_{t+1}^{0/1}, (m_{t+1}^{(0)} - m_{t+1}^{(2)})] + cov_t[\Delta e_{t+1}^{0/2}, (m_{t+1}^{(0)} - m_{t+1}^{(1)})] \\
&= cov_t[x_t^{(1)} + y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)} + w_t^{(1)}\varepsilon_{t+1}^{(1)}, (m_{t+1}^{(0)} - m_{t+1}^{(2)})] \\
&+ cov_t[x_t^{(2)} + y_t^{(2)}m_{t+1}^{(0)} + z_t^{(2)}m_{t+1}^{(2)} + w_t^{(2)}\varepsilon_{t+1}^{(2)}, (m_{t+1}^{(0)} - m_{t+1}^{(1)})] \\
&= cov_t(y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)}, m_{t+1}^{(0)} - m_{t+1}^{(2)}) + cov_t(y_t^{(2)}m_{t+1}^{(0)} + z_t^{(2)}m_{t+1}^{(2)}, m_{t+1}^{(0)} - m_{t+1}^{(1)}) \\
&- cov_t(w_t^{(1)}\varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)}) - cov_t(w_t^{(2)}\varepsilon_{t+1}^{(2)}, m_{t+1}^{(1)}),
\end{aligned}$$

where the last equality comes from the fact that by construction, $cov_t[w_t^{(1)}\varepsilon_{t+1}^{(1)}, m_{t+1}^{(0)}] = cov_t[w_t^{(2)}\varepsilon_{t+1}^{(2)}, m_{t+1}^{(0)}] = 0$.

Rearranging terms,

$$\begin{aligned}
cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2}) &= \frac{cov_t(y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)}, m_{t+1}^{(0)} - m_{t+1}^{(2)}) + cov_t(y_t^{(2)}m_{t+1}^{(0)} + z_t^{(2)}m_{t+1}^{(2)}, m_{t+1}^{(0)} - m_{t+1}^{(1)})}{2} \\
&- \frac{cov_t(w_t^{(1)}\varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)}) + cov_t(w_t^{(2)}\varepsilon_{t+1}^{(2)}, m_{t+1}^{(1)})}{2}.
\end{aligned} \tag{A.2}$$

Then, by plugging in $(i, j, k) = (0, 1, 2)$ and $(0, 2, 1)$, Condition NS directly implies

$$cov_t(w_t^{(1)}\varepsilon_{t+1}^{(1)}, m_{t+1}^{(2)}) = cov_t(\mathcal{R}^{0,1}[\Delta e_{t+1}^{0/1}], m_{t+1}^{(2)}) = 0,$$
$$cov_t(w_t^{(2)}\varepsilon_{t+1}^{(2)}, m_{t+1}^{(1)}) = cov_t(\mathcal{R}^{0,2}[\Delta e_{t+1}^{0/2}], m_{t+1}^{(1)}) = 0.$$

Therefore, Eq. (A.2) simplifies to

$$cov_t(\Delta e_{t+1}^{0/1}, \Delta e_{t+1}^{0/2}) = \frac{cov_t(y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)}, m_{t+1}^{(0)} - m_{t+1}^{(2)}) + cov_t(y_t^{(2)}m_{t+1}^{(0)} + z_t^{(2)}m_{t+1}^{(2)}, m_{t+1}^{(0)} - m_{t+1}^{(1)})}{2}.$$

Next, by plugging in $(i, j, k) = (1, 2, 0)$, Condition NS implies that $y_t^{(1)} = y_t^{(2)}$. To see this result, note that if $y_t^{(1)} \neq y_t^{(2)}$, then, by Triangular arbitrage,

$$\Delta e_{t+1}^{2/1} = \Delta e_{t+1}^{0/1} - \Delta e_{t+1}^{0/2} = (x_t^{(1)} - x_t^{(2)}) + (y_t^{(1)} - y_t^{(2)})m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)} - z_t^{(2)}m_{t+1}^{(2)} + (w_t^{(1)}\varepsilon_{t+1}^{(1)} - w_t^{(2)}\varepsilon_{t+1}^{(2)}).$$

Note that $\varepsilon_{t+1}^{(1)}$ is not only orthogonal to $m_{t+1}^{(0)}$ and $m_{t+1}^{(1)}$, but also orthogonal to $m_{t+1}^{(2)}$ under Condition NS. Similarly, $\varepsilon_{t+1}^{(2)}$ is also orthogonal to all three SDFs. Therefore,

$$\begin{aligned}\mathcal{R}^{1,2}[\Delta e_{t+1}^{2/1}] &= \mathcal{R}^{1,2}[(y_t^{(1)} - y_t^{(2)})m_{t+1}^{(0)} + (w_t^{(1)}\varepsilon_{t+1}^{(1)} - w_t^{(2)}\varepsilon_{t+1}^{(2)})]\\ &= (y_t^{(1)} - y_t^{(2)})\mathcal{R}^{1,2}[m_{t+1}^{(0)}] + (w_t^{(1)}\varepsilon_{t+1}^{(1)} - w_t^{(2)}\varepsilon_{t+1}^{(2)}).\end{aligned}$$

As long as $m_{t+1}^{(0)}$ is not fully spanned by the foreign SDFs of countries 1 and 2, the residual $\mathcal{R}^{1,2}[m_{t+1}^{(0)}]$ is non-zero, implying that

$$cov_t(\mathcal{R}^{1,2}[\Delta e_{t+1}^{2/1}], m_{t+1}^{(0)}) = (y_t^{(1)} - y_t^{(2)})cov_t(\mathcal{R}^{1,2}[m_{t+1}^{(0)}], m_{t+1}^{(0)}) \neq 0,$$

which generates spill-over from country 0's SDF to the bilateral exchange rate between countries 1 and 2 and thus violates Condition NS. To see why $cov_t(\mathcal{R}^{1,2}[m_{t+1}^{(0)}], m_{t+1}^{(0)}) \neq 0$, note that by construction, $\mathcal{R}^{1,2}[m_{t+1}^{(0)}]$ is orthogonal to $m_{t+1}^{(1)}$ and $m_{t+1}^{(2)}$. Therefore, $cov_t(\mathcal{R}^{1,2}[m_{t+1}^{(0)}], m_{t+1}^{(0)}) = var_t(\mathcal{R}^{1,2}[m_{t+1}^{(0)}])$ which is non-zero as long as $m_{t+1}^{(0)}$ is not fully spanned by the foreign SDFs.

Finally, we plug in $y_t \stackrel{\text{def}}{=} y_t^{(1)} = y_t^{(2)}$ into Eq. (A.2) and obtain

$$cov_t(\Delta e^{0/1}_{t+1}, \Delta e^{0/2}_{t+1}) = \frac{cov_t(y_t m^{(0)}_{t+1} + z^{(1)}_t m^{(1)}_{t+1}, m^{(0)}_{t+1} - m^{(2)}_{t+1}) + cov_t(y_t m^{(0)}_{t+1} + z^{(2)}_t m^{(2)}_{t+1}, m^{(0)}_{t+1} - m^{(1)}_{t+1})}{2}.$$

B.6 Proposition 5

The Euler equation Eq. (9) implies

$$\begin{aligned}
0 &= cov_t(\Delta e^{0/1}_{t+1}, \Delta e^{0/1}_{t+1} - (m^{(0)}_{t+1} - m^{(1)}_{t+1})) \\
&= cov_t(y^{(1)}_t m^{(0)}_{t+1} + z^{(1)}_t m^{(1)}_{t+1} + w^{(1)}_t \varepsilon^{(1)}_{t+1}, (y^{(1)}_t - 1) m^{(0)}_{t+1} + (z^{(1)}_t + 1) m^{(1)}_{t+1} + w^{(1)}_t \varepsilon^{(1)}_{t+1}) \\
&= y^{(1)}_t (y^{(1)}_t - 1) var_t(m^{(0)}_{t+1}) + z^{(1)}_t (z^{(1)}_t + 1) var_t(m^{(1)}_{t+1}) \\
&+ [y^{(1)}_t (z^{(1)}_t + 1) + z^{(1)}_t (y^{(1)}_t - 1)] cov_t(m^{(0)}_{t+1}, m^{(1)}_{t+1}) + (w^{(1)}_t)^2,
\end{aligned}$$

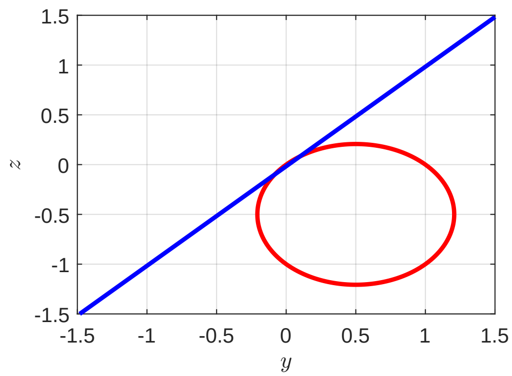
where we use the fact that by construction, $cov_t(w^{(1)}_t \varepsilon^{(1)}_{t+1}, m^{(0)}_{t+1}) = cov_t(w^{(1)}_t \varepsilon^{(1)}_{t+1}, m^{(1)}_{t+1}) = 0$. $(w^{(1)}_t)^2 \geq 0$ leads to constraint (a).

The Euler equation Eq. (9) also implies

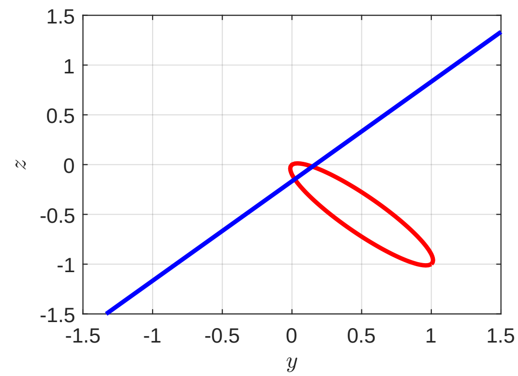
$$\begin{aligned}
var_t(\Delta e^{0/1}_{t+1}) &= cov_t(\Delta e^{0/1}_{t+1}, m^{(0)}_{t+1} - m^{(1)}_{t+1}) \\
&= cov_t(y^{(1)}_t m^{(0)}_{t+1} + z^{(1)}_t m^{(1)}_{t+1} + w^{(1)}_t \varepsilon^{(1)}_{t+1}, m^{(0)}_{t+1} - m^{(1)}_{t+1}) \\
&= y^{(1)}_t var_t(m^{(0)}_{t+1}) - z^{(1)}_t var_t(m^{(1)}_{t+1}) + (z^{(1)}_t - y^{(1)}_t) cov_t(m^{(0)}_{t+1} . m^{(1)}_{t+1}),
\end{aligned}$$

which leads to constraint (b).

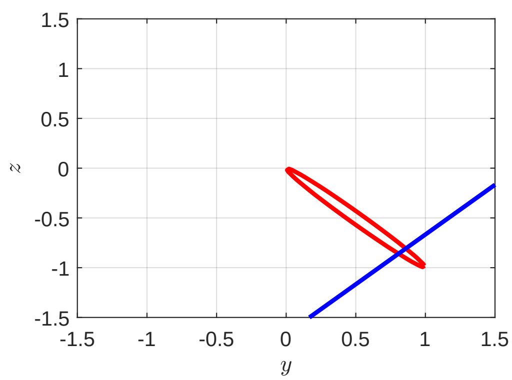
I also repeat the exercise in Figure (1) with different assumptions of the correlation between the home and foreign SDFs. Figure (A.1) plots the results. We can see that a very high SDF correlation, such as 0.99, is required to allow the pass-through coefficients to take values close to ± 1. Even when the SDF correlation is 0.9, the pass-through coefficients are bounded in a region very close to zero.



(a) SDF Correlation 0



(b) SDF Correlation 0.9



(c) SDF Correlation 0.99

Figure A.1: Bounds on the Magnitude of the Pass-Through Coefficients

Notes: This figure shows the bounds on the pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ imposed by Proposition 5.

B.7 Characterization of Exchange Rate Variance and Risk Premium

The exchange rate decomposition formula presented in the paper also provides an intuitive characterization of other exchange rate moments. For example, given Eq. (13), reproduced below,

$$\Delta e_{t+1}^{0/1} = x_t^{(1)} + y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)} + w_t^{(1)} \varepsilon_{t+1}^{(1)},$$

the exchange rate movement variance can be expressed as

$$var_t(\Delta e_{t+1}^{0/1}) = var_t(y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)}) + var_t(w_t^{(1)} \varepsilon_{t+1}^{(1)}), \tag{A.3}$$

and the currency risk premium can be expressed as

$$\begin{aligned} rp_t \stackrel{\text{def}}{=} \mathbb{E}_t \left[\Delta e_{t+1}^{0/1} + r_t^{(0)} - r_t^{(1)} \right] &= -\frac{1}{2} cov_t(m_{t+1}^{(1)}, \Delta e_{t+1}^{0/1}) - \frac{1}{2} cov_t(m_{t+1}^{(0)}, \Delta e_{t+1}^{0/1}) \\ &= -\frac{1}{2} cov_t(y_t^{(1)} m_{t+1}^{(0)} + z_t^{(1)} m_{t+1}^{(1)}, m_{t+1}^{(0)} + m_{t+1}^{(1)}). \end{aligned} \tag{A.4}$$

Note that the currency risk premium formula does not contain the residual term $w_t^{(1)} \varepsilon_{t+1}^{(1)}$. In this sense, the exchange rate residual is "not priced": a volatile residual increases the exchange rate variance, but it has no effect on the currency risk premium.

As a special case, when markets are complete, $y_t^{(1)} = 1$, $z_t^{(1)} = -1$ and $\varepsilon_{t+1}^{(1)} = 0$. The exchange rate moments become

$$\begin{aligned} var_t(\Delta e_{t+1}^{0/1}) &= var_t(m_{t+1}^{(0)} - m_{t+1}^{(1)}), \\ rp_t &= \frac{1}{2} var_t(m_{t+1}^{(1)}) - \frac{1}{2} var_t(m_{t+1}^{(0)}). \end{aligned}$$

The key argument in Lustig et al. (2011) is that, when there is a strong degree of market incompleteness which lowers the exchange rate variance, it also shrinks the risk premium towards zero. So it is difficult to find a calibration that simultaneously replicates both exchange rate variance and risk premium.

We can see this argument very clearly using our exchange rate decomposition: low SDF-FX pass-through coefficients $y_t^{(1)}$ and $z_t^{(1)}$ are required to reduce the variance of $y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)}$ and hence the exchange rate variance in Eq. (A.3). At the same time, a small variance of $y_t^{(1)}m_{t+1}^{(0)}+ z_t^{(1)}m_{t+1}^{(1)}$ also shrinks the currency risk premium in Eq. (A.4) to zero. By teasing out the SDF term $y_t^{(1)}m_{t+1}^{(0)} + z_t^{(1)}m_{t+1}^{(1)}$ from the residual $w_t^{(1)}\varepsilon_{t+1}^{(1)}$ in the exchange rate movement, our decomposition formula above shows how this SDF term ties together the exchange rate variance and the currency risk premium in the general case.

B.8 Exchange Rate Dynamics in Continuous Time

The results in this paper hold more generally in the following dynamic, infinite-horizon, and continuous-time economy. I fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a given filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions. I assume that all stochastic processes are adapted to this filtration.

Let $\mathcal{A}[X_t]$ denote the drift of the process X_t; the operator $\mathcal{A}$ is also known as the generator. With slight abuse of notation, let $[dX_t, dY_t]$ denote the instantaneous conditional covariance between two diffusion processes X_t and Y_t. Formally, it is defined as $[dX_t, dY_t] = d[X, Y]_t/dt$ where $[X, Y]_t$ is the standard quadratic covariation process between X_t and Y_t.

Countries are indexed by $i \in \{0, 1, \ldots, N\}$. Let $M_t^{(i)}$ and $m_t^{(i)}$ denote country i's cumulative SDF and its log. The log SDF follows a diffusion process:

$$dm_t^{(i)} = -\mu_t^{(i)}dt - \sigma_t^{(i)}dZ_t^{(i)},$$

and let $\rho_{i,j,t} = [dZ_t^{(i)}, dZ_t^{(j)}]$ denote the correlation between two countries' SDFs.

The mean, volatility and correlation parameters $\mu_t^{(i)}$, $\sigma_t^{(i)}$, and $\rho_{i,j,t}$ can be time-varying. I assume they follow $d\sigma_t^{(i)} = a_{\sigma,t}^{(i)}dt + b_{\sigma,t}^{(i)}dW_{\sigma,t}^{(i)}$, $d\mu_t^{(i)} = a_{\mu,t}^{(i)}dt + b_{\mu,t}^{(i)}dW_{\mu,t}^{(i)}$, $d\rho_{i,j,t} = a_{\rho,t}^{(i)(j)}dt + b_{\rho,t}^{(i)(j)}dW_{\rho,t}^{(i)(j)}$. All stochastic processes are adapted to the filtration $\mathcal{F}_t$ and the Brownian motions $W_{\sigma,t}^{(i)}, W_{\mu,t}^{(i)}, W_{\rho,t}^{(i)(j)}$ are allowed to be correlated with each other.

Without loss of generality, I fix country 0 as the base country. Let $\mathcal{E}_t^{0/i}$ denote the exchange rate between country 0 and country i, which increases when the currency in country 0 appreciates. Let $e_t^{0/i}$ denote the log exchange rate.

Following my derivation in the log-normal case, I can without loss of generality express the bilateral exchange rate movement as

$$de_t^{0/i} = x_t^{(i)} dt + y_t^{(i)} dm_t^{(0)} + z_t^{(i)} dm_t^{(i)} + w_t^{(i)} dY_t^{(i)},$$

where $y_t^{(i)}$ and $z_t^{(i)}$ describe the SDF-FX pass-through, and $dY_t^{(i)}$ is the residual. By construction, the residual is uncorrelated with the SDFs, i.e., $[dZ_t^{(0)}, dY_t^{(i)}] = [dZ_t^{(i)}, dY_t^{(i)}] = 0$.

I assume that investors can still trade both home and foreign risk-free bonds. Their risk-free bond prices follow

$$dP_t^{(i)} = r_t^{(i)} P_t^{(i)} dt.$$

As investors can hold domestic and foreign bonds, I have the following sets of Euler equations:

$$\begin{aligned} 0 &= \mathcal{A}[M_t^{(i)} P_t^{(i)}], \\ 0 &= \mathcal{A}[M_t^{(0)} (\mathcal{E}_t^{0/i})^{-1} P_t^{(i)}] = \mathcal{A}[M_t^{(i)} \mathcal{E}_t^{0/i} P_t^{(0)}]. \end{aligned}$$

Using these Euler equations, we can replicate all results in the main text. In fact, the continuous-time model bears strong similarity to the discrete-time, log-normal benchmark, so that the proof is almost identical. So, the results about the international spill-over effect also hold in this more general set-up with time-varying drifts, volatilities, and correlations.