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ABSTRACT

This paper develops a new model with heterogeneous firms under perfect competition in a Heckscher-Ohlin-Samuelson setting. We show that trade need not make selection in the comparative advantage sector stricter as suggested by earlier work. Selection is driven by the capital intensity in entry costs relative to production costs. If trade raises (reduces) the wage rental ratio, and entry costs are more labor intensive than production costs in a sector, then the ratio of entry cost to production costs will rise (fall) and selection will become weaker (stricter) in this sector. Moreover, we show that the central theorems of the HOS model (as well as the standard generalizations using duality) carry over in our setting.

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1 Introduction

This paper provides a completely new competitive general equilibrium apparatus that builds on the Heckscher-Ohlin-Samuelson (HOS) model, but with firms that are *heterogeneous* in terms of their costs. Our focus is on incorporating firm heterogeneity into a perfectly competitive setting and showing that the standard tools, including the insights from duality (Dixit and Norman, 1980; Woodland, 1982), can be extended to heterogeneous firms and understanding what drives selection in this simple and transparent setting.[1]

We first show how to adapt the standard techniques in HOS models of trade, using either the primal or dual approach, to incorporate firm heterogeneity. Second, we use the model to show that the main theorems (Stolper-Samuelson, Rybczynski and HOS) in the standard HOS setting go through with heterogeneous firms. Third, we provide novel and simple results on what drives selection. This is important as one of the predictions of heterogeneous firm models is that trade makes selection stricter, at least in the sector with a comparative advantage. In a Ricardian model with heterogeneous firms, see Melitz (2003), competition increases and selection becomes stricter with trade. In a Heckscher-Ohlin model, Bernard, Redding and Schott (2007) show that if production and entry costs have the same factor intensity, then with costly trade selection becomes stricter in the comparative advantage sector.[2] If selection in a sector becomes stricter with trade, the average firm in the sector is more productive, i.e., there are productivity gains from trade. We show trade need not make selection stricter in a sector. Selection is driven by the difference in the capital intensity in entry costs and production costs, and trade could make selection *stricter* or *weaker* in a sector.

Not only does our approach considerably simplify the understanding of what drives selection, in addition, our predictions on selection are empirically testable. The difference in factor intensities of entry and production can be proxied by differences in factor intensity of young and old firms in an industry since young firms will incur both entry and production costs, while old ones will likely incur mostly production costs. In a companion paper, see Bai, Chatterjee, Krishna and Ma (2021), we show that the richer predictions of our model on selection arising from a minimum wage are indeed very much supported by the data.

Our model is a competitive analogue of the monopolistically competitive heterogeneous firm setting common in international trade and industrial organization. To make the competitive model with heterogeneous firms internally consistent, we place limits on the extent of a firm's ability

[1] Under perfect competition, supply at the firm level is irrelevant to outcomes, only aggregate supply matters as it, together with demand pins down equilibrium prices. Our setup results in upward sloping supply curves at the firm and industry level due to firm heterogeneity and capacity constraints.

[2] They do point out in footnote 12 of Bernard et al. (2007) that if production and entry costs have different intensities, further selection effects occur.

to supply at its marginal costs, i.e., the firm has capacity constraints. In particular, each draw of marginal cost allows a firm to produce only one unit of the good at that marginal cost. There is, of course, free entry so that in equilibrium firms will be indifferent between adding a unit of capacity or not. Thus we do not pin down the size of each firm, as is to be expected in a competitive setting, only the potential capacity and the selection cutoff. Our approach to incorporating firm heterogeneity fits perfectly into the standard competitive HOS setting. Our hope is to provide a simple and intuitive way to highlight the role of heterogeneity in general equilibrium by leveraging the familiarity of the competitive setup.

The recent literature on firm heterogeneity follows one of two seminal papers: Melitz (2003) which builds on standard models of monopolistic competition in a single factor model with constant marginal costs, and Eaton and Kortum (2002) which assumes that costs are random and that the lowest cost firm making each variety is the supplier and can supply all that is needed. We offer a third alternative: heterogeneous firms with perfect competition and capacity constraints. Melitz (2003) highlights the cleansing effect of trade, namely that selection becomes stricter due to trade because competition becomes stronger. Bernard et al. (2007) extend the work of Melitz (2003) to a HOS setting. They show that the cleansing effect of trade (as in Melitz (2003)) always occurs in all sectors but that it is more highly concentrated in comparative advantage industries but only *in the presence of costly trade*. There is no such prediction if trade is costless. The intuition for their result is simple. Trade reduces prices overall since goods can be more widely sourced. In the presence of trade costs, trade liberalization reduces the *relative* price index of the comparative advantage sector since imported goods incur trade costs while domestically produced ones do not. Thus, competitive pressures rise by more in the comparative advantage sector, which makes selection stricter there.

A key assumption in Bernard et al. (2007) is that entry costs and production costs in a sector have the same factor intensity. As a result, they do not consider any selection effects arising from the channel we highlight, namely the difference in factor intensity in production and entry costs. In contrast, the selection effects that are at the heart of our model come from allowing factor intensity to differ in entry costs relative to production costs. As a consequence, our setting yields results that are richer than theirs and do not depend on there being costly trade.

Our results are also important when it comes to the effects of trade liberalization on industry productivity. The standard prediction, as in Bernard et al. (2007), is that selection becomes stricter and more so in the comparative advantage industry. We emphasize that this prediction is far from a general result. What happens to selection depends on the relative capital intensity of entry to production costs and on the effect of trade on factor prices. If trade raises (reduces) the wage rental ratio, and entry costs are *more* labor intensive than production costs in a sector, then the ratio of entry cost to production costs will rise (fall) and we show that as a result, selection will become weaker (stricter) in this sector. Our predictions on *trade liberalization* and *selection* are empirically testable and have not been tested to date. Chen, Imbs and Scott (2009) tests the implications of trade openness for productivity in the short versus long run in a Ricardian setting. Using a version of Melitz and Ottaviano (2008) they show, both theoretically, and empirically using data on EU countries, that in the short run a fall in trade costs at home shifts out the supply curve and reduces prices. However, these effects are reversed in the long run where the fall in domestic trade costs and increased competition in the short run results in firms relocating abroad and reducing competition at home in the long run. In contrast to Chen et al. (2009), we focus on selection effects in a Heckscher-Ohlin setting and suggest a new channel that drives selection effects in response to openness. As greater openness raises the relative price of the good the country has a comparative advantage in, it raises the real return to the factor used intensively in making that good. If production in this comparative advantage sector is more intensive (less intensive) in this factor than is entry, then selection will become stricter (weaker) in the comparative advantage sector.

An advantage of our approach is that it makes clear the links between product and factor markets and the channels through which trade operates in general equilibrium with heterogeneous firms. It is also likely to be useful for teaching purposes as it embeds firm heterogeneity into the workhorse multi-good multi-factor HOS setting making possible an intuitive understanding of how trade and selection operate together. It is worth emphasizing that there are no distortions in our setup. Consequently, there is no role for government policy to improve welfare. Selection becoming stricter means that higher cost capacity is shut down by all firms. Since productivity is measured as a residual at the firm level, it will look like the percentage change in output is smaller than the percentage change in inputs used for all firms, i.e., that firm productivity has risen.

The paper proceeds as follows. The setting is explained in Section 2. Section 3 characterizes the supply side of the model with heterogeneous firms. It extends the standard two-good two-factor HOS model to allow for heterogeneous firms and proves our new results regarding selection. It also shows that the main theorems: the Stolper-Samuelson, Rybczynski, and HOS theorems generalize to our setting. Section 4 concludes. The Appendix has two parts. The first part shows how the problem can be extended to many goods and many factors elegantly using duality. The second part contains details of the proofs in the body of the paper.

2 The Setting

In this section, we look at the different modules that make up the supply side of the model. As this is a competitive setting, we can use the standard tools, albeit with a slight twist. To build our intuition about the model we will focus on the two-good and two-factor case. Each country j has capital endowment K^j and labor endowment L^j. These factors earn rental rate r^j and wage w^j.

We will suppress j for simplicity from here onwards till needed. Firms in a country can make the two manufactured goods, x and y. These goods are homogeneous and all countries can make all goods. It is worth emphasizing that within an industry, all manufacturing firms make the *same* variety of the manufactured good.

Producers of manufactured goods do not know their production costs before entering and discover them *ex-post*. First they pay the fixed entry costs in an industry of $f^e c^e(w, r)$. This entitles them, should they wish, to produce a *single* unit of the good at the cost $c(w, r)\theta$ where $c(\cdot)$ is the base unit cost and θ is the inverse of its realized productivity. Firms draw their θ from the distribution $f(\theta)$. A higher θ denotes higher costs and lower productivity. As will be made clear below, the goods differ in their factor intensities. We will assume that good x is labor intensive relative to good y in terms of its *production* costs and *overall costs*, i.e., in terms of its total costs (entry plus production costs).

We define a firm to be just a collection of draws. A firm with many draws will pay the fixed cost for each draw and have a higher potential capacity. How much of its capacity it chooses to use depends on the price and the draws it has. Given price, only capacity that is low enough in terms of costs will be utilized. In other words, firms use all their profitable capacity. Firms with low-cost draws will be willing to supply even at a low price, while firms with high-cost draws will not. In this way, each firm will have a weakly upward-sloping supply function as they will only use the capacity such that price exceeds costs. Firms are by nature heterogeneous: Some firms will have many draws and some will have few. Some will have good draws of costs and some will have bad draws. Firms with many good cost draws will have elastic supply at low prices, while firms with only a few bad draws will be willing to supply a small quantity at relatively high prices. Firms with no draws have no supply at any price.[3] In addition, in equilibrium, free entry requires that the expected profits from entry are zero.[4]

Our simple model predicts many of the patterns seen in data. Bernard, Jensen, Redding and Schott (2012) summarizes many of these. Firms with many draws will have a better best draw and a worse worst draw. Thus, by the logic of order statistics, larger firms, i.e., those with many draws, will tend to be willing to supply at lower prices as they are more likely than smaller firms to have some capacity that is worth using. As a result, at any price, larger firms will look like they are more productive — a common feature of the data. In addition, in the presence of trade costs, firms whose best θ draws put their costs above the trade cost adjusted price will not produce and hence not export. This will give rise to another feature often pointed out in the data — larger and more

[3] Note that we need to assume that each firm has a finite number of draws for our interpretation to work. With a continuum of draws, each firm's supply would just be a scaled-down version of industry supply. Firms will differ from one another only when there are a finite number of draws.

[4] Note that a firm is an artificial construct in perfect competition as the distribution of capacity across firms is irrelevant to outcomes in this setting. Only total capacity matters.

productive firms tend to export. However, as there are no fixed costs of exporting in our model, this feature is not the result of larger firms selecting into exporting by being able to cover this fixed cost.

The cutoff, $\tilde{\theta}$, will determine which of its capacity draws a firm will choose to use in a given equilibrium. This will provide a model of firm level heterogeneity in supply functions that will allow us to make firm level predictions that can be aggregated to the industry level, as industry supply is merely the horizontal sum of the firm supply functions. With this view of a firm, it follows that selection at the industry level will be mirrored in selection at the firm level. There is free entry so that the mass of firms (supply) in the sector is endogenously determined. Each firm in a country is competitive and takes the price for the country's variety as given. However, as firms are ex-post heterogeneous, some firms earn quasi-rents.[5]

There are no financing frictions or credit constraints. The unit input requirement in production, the amount of input $\ell = L, K$ needed to produce a unit of good $i = x, y$, is denoted by $a_{\ell i}(\omega)$ where $\omega = \frac{w}{r}$ denotes the wage rental ratio. By Shephard's Lemma this is also equal to the derivative of unit costs of production of i with respect to the price of input ℓ. The unit input requirement in entry, the amount of input $\ell = L, K$ needed in entry so that a unit of good $i = x, y$ can be produced is denoted by $a^e_{\ell i}(\omega)$. Let p_x and p_y denote the price of the goods x and y. The analogue to the standard unit input requirements of input ℓ for good i, denoted by $A_{\ell i}$, includes both the requirement in entry and in production.

We will first analyze what happens in a small country and then look at the effects of trade with the rest of the world. We show how prices (p_x, p_y) define selection cutoffs for the *ex post* heterogeneous firms making each good. This is the novel part of the paper. Then we show how to solve for factor prices and outputs and show that the standard results (the Stolper-Samuelson and Rybcynski theorems) hold. Once we have this, we are able to write down supply and together with demand to solve for equilibrium prices and show that the HOS theorem also holds. We find that under some conditions, trade does makes selection stricter in the good the country has a comparative advantage in and weaker in the other good, though this is not always the case. If a country is capital abundant, then the good that is capital intensive in production and overall, will have its price rise when the country opens up to trade. This will raise the real rewards of capital. It will make selection stricter in the comparative advantage good if the capital intensity in entry is less than that in production and make selection weaker if the capital intensity in entry is more than that in production.

[5]This gives a standard competitive model with upward sloping supply curves at the firm and industry level, and hence, quasi-rents in the short run.

3 The $2 \times 2 \times 2$ Model

The two-good, two-factor, two-country (home and the rest of the world) model is the workhorse model in the HOS setting. In order to build our intuition, we will focus on this in the body of the paper.

Throughout we will make the assumption that good x is the labor intensive good and good y is the capital intensive one, that there are no factor intensity reversals or specialization. Since we have factor intensity in production and in entry, we will further assume that good x is labor intensive both in terms of production and overall. The capital labor ratio in production (entry) in sector i is defined as the ratio of the unit input requirement of capital to that of labor in production (entry), i.e., $\frac{a_{Ki}}{a_{Li}}$ ($\frac{a^e_{Ki}}{a^e_{Li}}$). The overall unit input requirement of factor i in sector j is defined as the expected unit input requirement of the factor in production plus entry and is denoted by A_{ij}. The overall capital intensity in sector j is $\frac{A_{Kj}}{A_{Lj}}$.

Assumption 1. There are no factor intensity reversals and endowments are such that there is no specialization in both countries. Good y is capital intensive in terms of production costs so $\frac{a_{Kx}}{a_{Lx}} < \frac{a_{Ky}}{a_{Ly}}$ and overall so $\frac{A_{Kx}}{A_{Lx}} < \frac{A_{Ky}}{A_{Ly}}$.

Producers can be thought of as expected profit maximizers under perfect competition. Thus the production side of the economy can be modelled in the usual way. Expected producer surplus in a sector is defined by

$$\pi(p, w, r) = \int_0^{\tilde{\theta}} (p - \theta c(w, r)) f(\theta) d\theta. \tag{1}$$

Integrating by parts and using the definition of $\tilde{\theta}$ yields that

$$\pi(p, w, r) = c(w, r) \left[\int_0^{\tilde{\theta}} F(\theta) d\theta \right]. \tag{2}$$

3.1 Free Entry

Free entry implies that expected producer surplus in sector i be exactly equal to the cost of entry, $c_i^e(w,r)f_i^e$ for $i = x, y$. Therefore, for our two-sector economy free entry dictates that:[6]

$$\left[\int_0^{\tilde{\theta}_x} F_x(\theta)d\theta\right] = \frac{c_x^e(w,r)f_x^e}{c_x(w,r)} \tag{3}$$

$$\left[\int_0^{\tilde{\theta}_y} F_y(\theta)d\theta\right] = \frac{c_y^e(w,r)f_y^e}{c_y(w,r)} \tag{4}$$

This free entry condition can be interpreted as the expected quasi rents, $\int_0^{\tilde{\theta}} F(\theta)d\theta$, being equal to the real cost of entry $\frac{c^e(w,r)}{c(w,r)}f^e$. With a unit mass of firms entering, supply at a cutoff θ is just $F(\theta)$.[7] The area under $F(\cdot)$ up to $\tilde{\theta}$ is analogous to producer surplus (the area between the price and the supply curve) in a competitive setting. Free entry requires that this area equals $\frac{c^e(w,r)}{c(w,r)}f^e$ which can be thought of as the cost of entry in terms of the units of the good being made or the real cost of entry. This is what drives cutoffs. Given (w, r) we can get $\left(\tilde{\theta}_x, \tilde{\theta}_y\right)$ from the free entry conditions (3) and (4). This then fixes the position of the price equal cost curves of the marginal firm. Let us call these cutoffs $\left(\tilde{\theta}_x(w,r), \tilde{\theta}_y(w,r)\right)$. Note that these cutoffs are unaffected by the *absolute level* of factor prices. Doubling factor prices would leave cutoffs unaffected as the right hand side of equations (3) and (4) would not change. In other words, $\tilde{\theta}_i(w,r), i = x, y$ is homogeneous of degree zero in factor prices[8]. Theorem 1 shows how selection is affected by the wage rental ratio.

Theorem 1 (**Selection and factor prices**). (i) An increase in the relative price of a factor makes selection stricter, that is $\tilde{\theta}_i(w,r)$ falls, (weaker, that is, $\tilde{\theta}_i(w,r)$ rises) when entry in the sector uses this factor less (more) intensively than does production.

Proof: If ω, the wage rental ratio rises, what happens to the right hand side of equations (3) and (4) depends on the labor intensity of entry costs relative to production costs. For example, if entry was more labor intensive than production, $c^e(w,r)$ would rise by more than $c(w,r)$ in percentage terms so that $\frac{c^e(w,r)}{c(w,r)}$ would rise, and to keep equality between the two sides, $\tilde{\theta}(w,r)$ would have to rise, i.e., selection would have to become weaker. Thus, the difference in the factor intensity in production and entry plays a key role in driving selection. A more formal proof is in the Appendix.

[6] We allow the distribution of draw of costs to differ across sectors.

[7] If a mass N enters, both costs and returns increase proportionately.

[8] Thus, we should really write $\left(\tilde{\theta}_x(\omega), \tilde{\theta}_y(\omega)\right)$ where ω is the wage rental ratio, but we use the current notation for simplicity.

Two examples of this result are worth noting.

Example 1: Entry costs in a sector are paid solely in terms of the good being made in that sector.

If this is the case, then entry costs and production costs have the same factor intensity so that the identity of the marginal firm, $\tilde{\theta}$, is fixed and there are no selection effects due to changes in factor prices. In this case, it can be shown, see Lemma 1 in the Appendix, that factor intensity in production $\left(\frac{a_{Ki}}{a_{Li}}\right)$ and overall $\left(\frac{A_{Ki}}{L_{Li}}\right)$ are the same and the model basically boils down to the standard HOS setting.

Example 2: Entry into both sector requires the use of a common entry good which is made using both good x and y. As a result, the capital intensity of entry is in between that of the two goods. Hence, if $k_x < k_y$ as we assume, $k_y < k_x^e = k_y^e < k_y$ (where $k_i$is the capital intensity in production in i and k_i^e denotes the capital intensity in entry in i for $i = x, y$). As a result, an increase in the wage rental ratio would make selection stricter in x and weaker in y[9].

3.2 The Stolper Samuelson Theorem

Profit maximization under perfect competition requires that price be equal to the marginal (and average) cost of production of the *marginal* unit of capacity used. This gives:

$$p_x = \tilde{\theta}_x(\omega)c_x(w,r) \tag{5}$$

$$p_y = \tilde{\theta}_y(\omega)c_y(w,r) \tag{6}$$

Equations (5) and (6) show that given product prices, we can solve for w, r in a manner analogous to that in the simple HOS setting.[10] Note that as $p_i = \tilde{\theta}_i(w,r)c_i(w,r)$ for $i = x, y$, a fall in $\tilde{\theta}_i(w,r)$ occurs because price rises by less than costs or falls by more than costs. Also note that only relative prices matter: doubling all product prices would only double all factor prices as $\tilde{\theta}$ is unaffected by the price *level*, only by *relative* prices.

3.2.1 An Illustrative Example

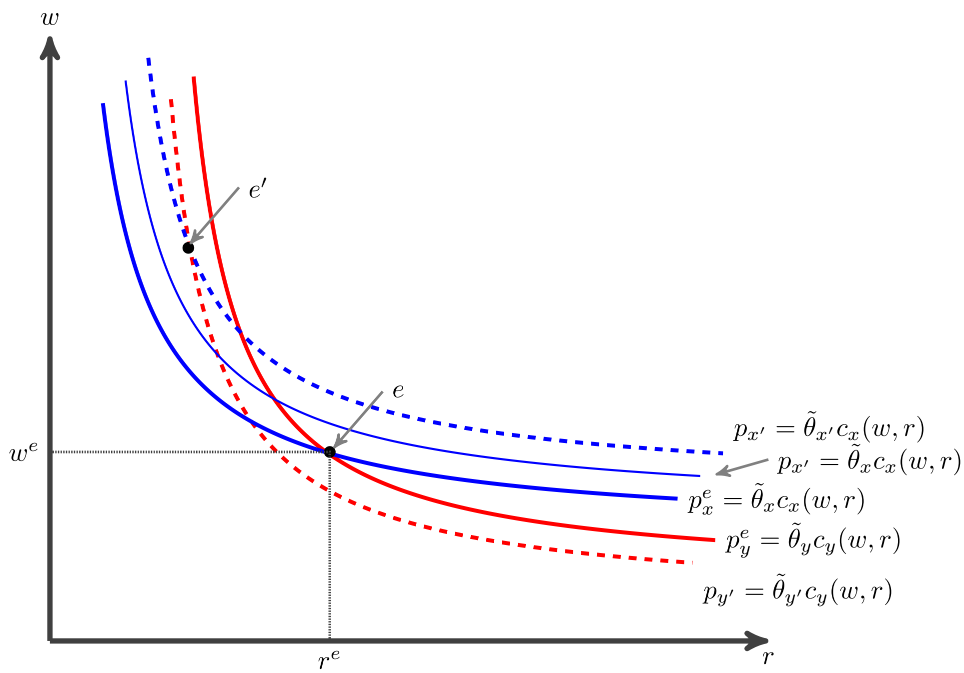
We are now in a position to see how product prices drive factor prices, i.e., see if the Stolper-Samuelson Theorem holds.

We will begin with a simple case where the argument can be made diagramatically. Suppose that

[9]Note that this means that $\tilde{\theta}_x(\overset{-}{w},\overset{+}{r})$, $\tilde{\theta}_y(\overset{+}{w},r)$ where the sign above the variable reflects the sign of the partial derivative.

[10]If there are no selection effects, as is the case when entry costs in each sector are in terms of the good made in that sector, then $\tilde{\theta}_x$ and $\tilde{\theta}_y$ are fixed.

Figure 1: Product Prices and Factor Prices



$k_x < k_x^e$ and $k_y > k_y^e$. This would be the case if for example, entry in both goods used goods x and y to make the common entry input, i.e., Example 2 above.

Stolper Samuelson Theorem for Example 2: An intuitive proof for this case is as follows. We can also think of cutoffs as a function of product prices. In particular, we can show that $\tilde{\theta}_x(\overset{-}{p_x}, \overset{+}{p_y}), \tilde{\theta}_y(\overset{+}{p_x}, \overset{-}{p_y})$. A plus (minus) sign above a variable means that the function is increasing (decreasing) in that variable. It is useful to think of this as happening in two steps. In the first step, keep the cutoffs fixed. This is the direct effect of price changes without incorporating selection. In the second step, let cutoffs change to reflect the selection effect. In our illustrative case, x uses capital more intensively than does production, while entry in y uses capital less intensively than production, so that both steps work in the same direction.

Consider Figure 1. The blue curves depict the price equal to cost loci for good x and the red ones depict those for y. An increase in the price of a good shifts the locus outwards. As x is labor intensive, and as the slope of the curve gives the capital intensity, the curve for good x is flatter than that for good y at any factor price ratio as drawn. The original equilibrium is at e where the thick blue and red curves intersect. Cutoffs at this equilibrium are $\tilde{\theta}_x$ and $\tilde{\theta}_y$. Suppose we keep the selection cutoff fixed at $\tilde{\theta}_x$ and $\tilde{\theta}_y$. Then the increase in the price of x from p_x to $p_{x'}$ shifts out the price equal to cost curve for x from the curve labelled $p_x^e = \tilde{\theta}_x c_x(w, r)$, i.e., the thick blue curve, to the thin blue curve labelled $p_{x'} = \tilde{\theta}_x c_x(w, r)$. Note that this step results in w rising and r falling *á la* Stolper-Samuelson.

Now for the second step. If entry costs are more capital intensive than production costs in x but less capital intensive than production costs in y, then this increase in w and fall in r from the first step reduces the cost of entry relative to production in good x, and this hence makes selection stricter in good x and weaker in good y. Thus, in the second step, the price increase in x reduces $\tilde{\theta}_x$ to $\tilde{\theta}_{x'}$ and raises $\tilde{\theta}_y$ to $\tilde{\theta}_{y'}$ which shifts the curve for x further out and for y further in as shown in Figure 1 by the dashed blue and red curves. In effect, the selection effects further increase $\frac{p_x}{\tilde{\theta}_x}$ while reducing $\frac{p_y}{\tilde{\theta}_y}$. Note, both the first and second steps, i.e., direct and indirect effects work in the same direction. So the Stolper-Samuelson theorem remains, but is *magnified* due to the selection effect. The total effect through both steps of the increase in the price of x, from p_x^e to $p_{x'}$, results in the equilibrium moving from e to e'. Of course, in an analogous manner, an increase in the price of the capital intensive good raises the rental rate and reduces the wage. It also reduces the "real cost" of entry ($\frac{c_y^e(w,r)}{c_y(w,r)}$) in the capital intensive good, making selection tighter there, and as it raises the "real cost" of entry ($\frac{c_x^e(w,r)}{c_x(w,r)}$) in the labor intensive good, making selection looser there.

The above illustrates what happens when prices change exogenously in the presence of selection and gives an example where the Stolper Samuelson Theorem holds even with selection effects. However, the Stolper-Samuelson theorem holds much more generally than this. The two steps of the argument could move factor prices in opposite directions and the theorem could still hold. If both steps of the argument above do not move cutoffs in the same direction, it is hard to say which effect will dominate. For example, suppose entry costs are more capital intensive than production costs in both sectors. Then an increase in the price of x raises the wage rental ratio. This reduces the ratio of entry to production costs in *both* sectors and makes selection stricter in both sectors. This further shifts out the price equal to cost curves in x *and* in y. While the former further raises $\frac{w}{r}$, the latter lowers it. For this reason, a more formal proof is needed to better understand the conditions needed for the Stolper-Samuelson theorem to hold. Assumption 1 is maintained from here onwards.

Assumption 1: $\frac{a_{Lx}(w,r)}{a_{Kx}(w,r)} > \frac{a_{Ly}(w,r)}{a_{Ky}(w,r)}$ and $\frac{A_{Lx}(w,r)}{A_{Kx}(w,r)} > \frac{A_{Ly}(w,r)}{A_{Ky}(w,r)}$.

In words, this means that is, not only is good x labor intensive in terms of production costs, i.e., $\frac{a_{Lx}(w,r)}{a_{Kx}(w,r)} > \frac{a_{Ly}(w,r)}{a_{Ky}(w,r)}$, but that in addition, it remains labor intensive overall, i.e., $\frac{A_{Lx}(w,r)}{A_{Kx}(w,r)} > \frac{A_{Ly}(w,r)}{A_{Ky}(w,r)}$. The assumption that x is labor intensive in terms of total costs is not needed if there is no selection. Without selection, though $A_{\ell i} \neq a_{\ell i}$, for $\ell = L, K$ and $i = x, y$, it can be shown, see **Lemma 1** in the Appendix, that $\frac{A_{Li}}{A_{Ki}} = \frac{a_{Li}}{a_{Ki}}$ so that if a good is labor intensive in terms of its production costs, it remains so in terms of its production plus entry costs.

Lemma 2 is a necessary stepping stone. $\bar{S}_{ji}$ for $j = K, L$ and $i = x, y$ denotes the weighted value share of factor j in good i. They play a key role in the proof of the Stolper Samuelson theorem.

Lemma 2:
(i) $\bar{S}_{Ki} + \bar{S}_{Li} = 1$ for $i = x, y$.
(ii) $\frac{A_{Ki}}{A_{Li}} = \frac{w}{r}\frac{\bar{S}_{Ki}}{\bar{S}_{Li}}$ for $i = x, y$.
(iii) $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}} \Leftrightarrow \frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$ and $\bar{S}_{Lx} > \bar{S}_{Ly}$, $\bar{S}_{Kx} < \bar{S}_{Ky}$.
Proof: In the Appendix.

Theorem 2 (Stolper-Samuelson). Under Assumption 1, the Stolper-Samuelson theorem remains valid. An increase in the price of a good raises the real return to the factor used intensively in its production and reduces the real return of the factor used intensively in the other good.

Proof: In Appendix.

While Theorem 1 links factor prices to to selection, and Theorem 2 links product prices to factor prices, Lemma 3 links product prices directly to selection. We do so in order to highlight the selection effects of product price changes and for completeness.

Lemma 3 (**Selection and product prices**). (i) If entry costs have the same factor intensity as production costs, then selection will not be affected by product price changes. (ii) If, as in Example 2 above, the factor intensity of entry costs are in between the factor intensity of production costs in the two sectors, then an increase in the price of a good makes selection in this sector stricter while making selection in the other sector weaker[11]. In other words: $\frac{\partial \tilde{\theta}_i(p_i, p_{-i})}{\partial p_i} < 0$, $\frac{\partial \tilde{\theta}_i(p_i, p_{-i})}{\partial p_{-i}} > 0$. (iii) If Assumption 1 holds, an increase in the price of a product makes selection stricter (weaker) if entry costs in the sector are less (more) intensive in the factor used intensively in production.

Proof: (i) follows from the fact that in this case, $\frac{c_x^e(w,r)f_x^e}{c_x(w,r)}$ is independent of factor prices. (ii) Consider an increase in the price of good x, the labor intensive good. Both direct and indirect effects move in the same direction so the Stolper Samuelson Theorem holds and in fact is magnified. w/r rises and $\frac{c_x^e(w,r)f_x^e}{c_x(w,r)}$ falls as entry costs rise by less than production costs. As a result, selection becomes stricter. The increase in w/r analogously makes selection in good y weaker. Analogous arguments work for an increase in the price of good y. An increase in price of good y reduces w and raises r, so selection in x weakens and selection in y tightens. (iii) If Assumption 1 holds, then the Stolper Samuelson theorem holds and an increase in the price of a product raises the relative price of the factor used intensively by it. This makes selection stricter (weaker) if entry costs in the sector are less (more) intensive in the factor used intensively in production.

[11] This is the case when, for example, entry cost are common across goods and use both goods as inputs.

3.3 Factor Market Clearing

There is an endowment L of labor and K of capital in the country. Given prices, we can get factor prices and hence total income which is the value of factor payments:

$$I = wL + rK \tag{7}$$

Let $\bar{\theta}(\tilde{\theta}) = \int_0^{\tilde{\theta}} \theta f(\theta) d\theta$. Clearly, $\bar{\theta}(\tilde{\theta})$ moves in the same direction as $\tilde{\theta}$. From Shephard's Lemma, a firm in industry i ($i = x, y$) with one unit of capacity of type θ demands $\theta c_{wi}(w, r) + f^e c^e_{wi}(w, r)$ units of labor and $\theta c_{ri}(w, r) + f^e c^e_{ri}(w, r)$ units of capital, where $c_{wi}(w, r) = \frac{\partial c_i(w,r)}{\partial w} = a_{Li}(w, r)$ and $c_{ri}(w, r) = \frac{\partial c_i(w,r)}{\partial r} = a_{Ki}(w, r)$, $i = x, y$. All firms that enter incur f^e, but only firms with θ below $\tilde{\theta}$ are active. Let N_x and N_y denote the mass of firms which enter in sector x and y respectively. Let $A_{\ell i}$ be the total unit input requirement in equilibrium of factor ℓ in sector i. The factor market clearing (FMC) gives:

$$N_x A_{Lx}(w, r) + N_y A_{Ly}(w, r) = L \tag{8}$$

$$N_x A_{Kx}(w, r) + N_y A_{Ky}(w, r) = K \tag{9}$$

where,

$$A_{Li}(w, r) = c_{wi}(w, r)\bar{\theta}_i(\tilde{\theta}_i(w, r)) + f^e_i c^e_{wi}(w, r) \tag{10}$$

$$A_{Ki}(w, r) = c_{ri}(w, r)\bar{\theta}_i(\tilde{\theta}_i(w, r)) + f^e_i c^e_{ri}(w, r). \tag{11}$$

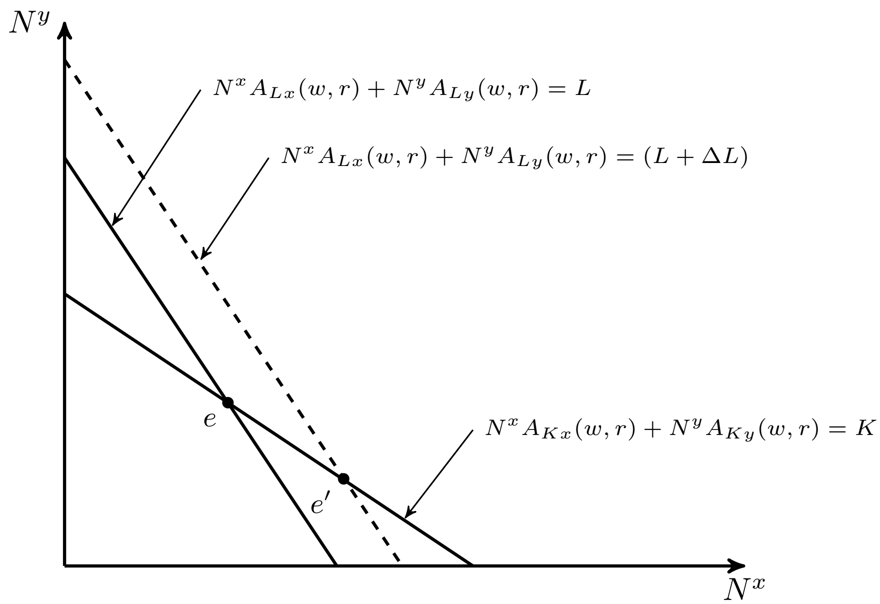
At given product prices, and hence at given factor prices, only N_x, N_y are unknown.

Given product prices, we have given factor prices and cutoffs and so $A_{\ell i}$'s are given. As a result, we can solve the factor market clearing conditions (equations (8) and (9)) for entry. Factor market clearing is depicted in Figure 2 by the solid lines that represent equations (8) and (9). As $\frac{A_{Lx}(w,r)}{A_{Kx}(w,r)} > \frac{A_{Ly}(w,r)}{A_{Ky}(w,r)}$, the factor market clearing condition for labor is steeper than that for capital. We now turn to look at the effects of endowment changes on entry.

Theorem 3 (**Rybczynski**). If a good x is labor intensive overall, i.e., $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}}$, then an increase in labor endowment L raises entry in the overall labor intensive product, while reducing it in the overall capital intensive one. Similarly, an increase in K raises entry in the overall capital intensive product and reduces it in the overall labor intensive one.

Proof: Given product prices, we have factor prices and these pin down the $A_{\ell i}(\cdot)$ values. Thus the factor market clearing conditions (8) and (9) are just straight lines. These are depicted in Figure 2. With N_y on the vertical axis and N_x on the horizontal axis, the labor market clearing line given

Figure 2: Entry and Endowments



by equation (8) is steeper than the capital market clearing one given by equation (9) as long as $\frac{A_{Lx}(\cdot)}{A_{Ly}(\cdot)} > \frac{A_{Kx}(\cdot)}{A_{Ky}(\cdot)}$. An increase in L, at given prices, shifts the labor market clearing curve outwards in a parallel manner. This raises the mass of entry in the labor intensive sector and reduces it in the capital intensive one. Similarly, an increase in K raises the mass of entry in the capital intensive sector and reduces it in the labor intensive one. A more formal proof is in the Appendix

In other words, entry responds to endowments just as output does in the HOS model. As a result, supply responds to endowments as denoted by $S_x(p_x, p_y; \overset{+}{L}, \overset{-}{K})$, $S_y(p_x, p_y; \overset{-}{L}, \overset{+}{K})$. A plus (minus) above a variable means the function is increasing (decreasing) in that variable.

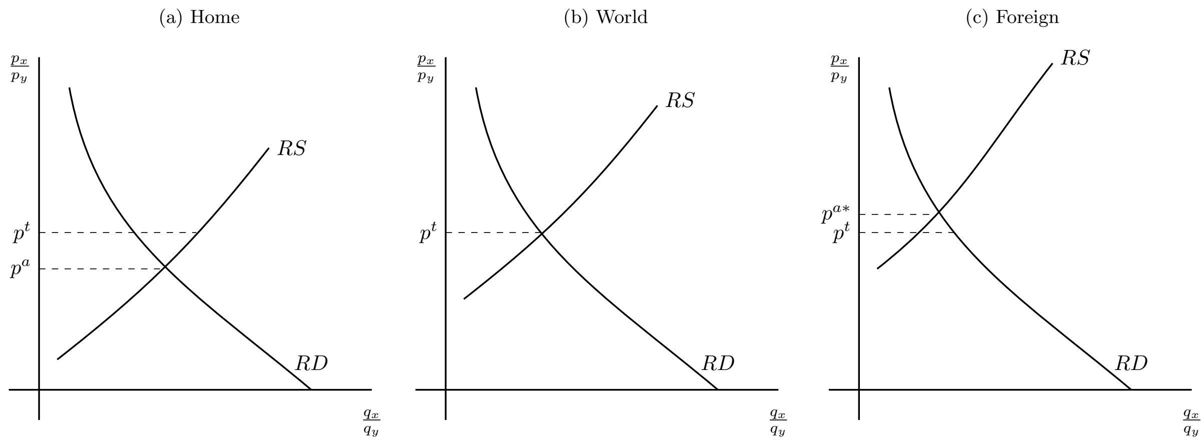
3.4 Supply, Demand and Equilibrium Prices

Theorem 4 (**HOS Theorem**). A country exports the good that uses intensively (in terms of production and overall factor intensity) the good it is relatively abundant in.

Proof: It is clear that supply is homogeneous of degree zero in prices.[12] This follows from $R(P, V)$ being convex in P and V and homogeneous of degree one (HD1) in P and in Vas shown in the Appendix. Doubling prices would double factor prices and leave selection and all unit input requirements and hence outputs, unaffected. Alternatively, it follows from the definition of the revenue function. As $R(P, V)$ is HD1 in prices, $R_P(P, V)$ which equals expected supply, is

[12]In the Appendix, we define the revenue function, $R(P, V)$, for this problem. We show that it has the usual properties associated with the revenue function as derived in the literature, see Dixit and Norman (1980).

Figure 3: Supply, Demand and Equilibrium Prices



homogeneous of degree zero (HD0) in prices. $R_P(P, V)$ gives supply and as $R_{p_i p_i} = \partial^2 R / \partial p_i^2 \geq 0$ due to convexity, this supply is increasing in own price. Moreover, as $R_P(P, V)$ is HD0 in P, $R_{p_i p_i} p_i + R_{p_i p_{-i}} p_{-i} = 0$. Hence, $R_{p_i p_{-i}} \leq 0$. In other words, there is a positive own price effect on supply and a negative cross price effect on supply so we can write $S_x(\overset{+}{p_x}, \overset{-}{p_y}; \overset{+}{L}, \overset{-}{K})$, $S_y(\overset{-}{p_x}, \overset{+}{p_y}; \overset{-}{L}, \overset{+}{K})$ where the sign above the variable denotes the sign of the partial derivative.

We assume identical homothetic demand for the two goods x and y. Demand is decreasing in its own price and increasing in the price of the other good as goods are substitutes. Supply rises in own price and falls with the other price as shown. Thus, relative demand is downward sloping and relative supply is upward sloping. There is an integrated domestic market for each good. Equilibrium is given by the intersection of relative demand and supply. This is depicted in Figure 3 (a) and (c) for the home and foreign countries, respectively.

From the Rybczynski theorem, at any given price, the labor abundant country (home) will have a relative supply of the labor intensive good (good x) to the right of that of the other country, and hence have a low autarky relative price of the labor intensive good. Analogously, the foreign country will have a high autarky relative price of good x. The trade price is depicted by the intersection of world relative demand and world relative supply which, as a convex combination of the two country's relative supplies, lies in between the two as in Figure 3(b). As a result, the trade price must lie between the autarky prices of the two countries so that the relative price of the labor intensive good will rise (fall) in the labor (capital) abundant country which will export (import) the overall labor intensive good.

End of Proof

We have thus shown that the standard theorems in the HOS model carry over to our setting with selection. Moreover, we have shown what drives selection. In some cases, as in our illustrative example, when entry costs are common across sectors and use both goods as inputs, and the country is labor abundant (and so has a comparative advantage in the labor intensive sector), trade will makes selection stronger in the comparative advantage sector and weaker in the other. As the trade occurs, the relative price of good x will rise. This in turn will raise w/r (through the Stolper-Samuelson theorem) and hence, make selection stricter in x and weaker in y. In other words, trade makes selection stricter in the comparative advantage sector and weaker in the other sector. However, this need not always be the case as is evident from Theorem 1.

It is also worth noting that in our Example 2, an increase in the price of x will make selection stricter in x and weaker in y.As selection is stricter in x, and as there must be a positive supply response, the mass of firms in x must rise, so that entry will rise. Note that trade will raise both exit of existing firms (selection becomes stricter) and entry of new firms (the mass of firms rises) in the comparative advantage sector, that is, there will be more churning in this sector. In the other sector, selection will become easier, but fewer firms will enter. In other words, there will be less churning in the comparative disadvantaged sector. This result is much like that in Bernard et al. (2007). As selection could either rise or fall in the comparative advantage sector in general as in Theorem 1, our model has a richer set of predictions.

4 Conclusion

This paper develops a new perfectly competitive heterogeneous firm model of supply in general equilibrium in a Heckscher-Ohlin setting. The model extends the HOS model to allow for selection effects. Firm heterogeneity co-exists with competition because firms are capacity constrained which prevents the more productive firms from taking over. Entry is free and firms enter in search of quasi rents that accrue to more productive firms. We show that the standard HOS results continue to hold in this setting with minimal additional assumptions. Our model predicts when trade will make selection in a sector stricter and when it will make it weaker. We hope the paper will help empirical researchers to test novel predictions as is done in Bai et al. (2021).

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A Appendix

This section has two parts. The first part contains generalizations of the model using duality and the second has the details of the proofs in the body of the paper.

A.1 Generalizations Using Duality

The revenue function is the value function to a maximization problem as well as a minimization problem.

A.1.1 The Revenue Function as a Minimization Problem

We can define $R(P, V)$ as the value function for a minimization problem. Prices are given by the vector P and endowments by the vector W. V is the factor price vector. There are n goods and m

factors. We use i to denote good, and ℓ to denote factor. We define the revenue function as the value function for the problem of minimizing factor payments ($W \cdot V$) subject to *expected* price, (the price times the probability the unit of capacity is active) being less than or equal to expected unit cost of production plus the cost of entry, $f_i^e c_i^e(W)$. We need to use the expected price as investing in a unit of capacity only gives a unit of output when one chooses to use the capacity installed, i.e., when the cost drawn is low enough to warrant production, that is, when $\theta < \tilde{\theta}$. This is also when the production costs will be incurred though these depend on the cost draw. Entry costs are incurred whether or not there is production. Let

$$\bar{\theta}_i(\tilde{\theta}_i) = \int_0^{\tilde{\theta}_i} \theta f_i(\theta) d\theta. \tag{A.1}$$

Thus, we minimize WV subject to $F_i(\tilde{\theta}_i)p_i \leq \bar{\theta}_i(\tilde{\theta}_i)c_i(W) + f^e c^e(W)$ for $i = 1..n$. with respect to the vectors W and $\tilde{\Theta}$ (of cutoffs), given P and V. The Lagrangian is

$$L(W, \lambda, \tilde{\Theta} : P, V) = \sum_{\ell=1}^{m} w_\ell V_\ell - \sum_{i=1}^{n} \Big(\lambda_i (F_i(\tilde{\theta}_i)p_i - (\bar{\theta}_i(\tilde{\theta}_i)c_i(W) + f_i^e c_i^e(W)) \Big) \tag{A.2}$$

This gives the revenue function, $R(P, V)$, as the value function for this problem. The first order conditions with respect to $\tilde{\theta}_i$ give the familiar price equal to cost conditions, but with a twist as this holds only for the marginal unit of capacity.

$$p_i - \tilde{\theta}_i c_i(W) \leq 0, \quad \text{with C.S. w.r.t. } \tilde{\theta}_i \quad \text{for } \ i = 1, ..., n, \tag{A.3}$$

where C.S. is short for Complementary Slackness.

The first order conditions with respect to W gives

$$V_\ell - \sum_{i=1}^{n} N_i \left[\bar{\theta}_i(\tilde{\theta}_i) a_{\ell i}(W) + f_i^e a_{\ell i}^e(W) \right] \leq 0, \text{ with C.S. w.r.t. } W \quad for \ \ell = 1, .., m. \tag{A.4}$$

where $\bar{\theta}_i(\tilde{\theta}_i)a_{\ell i}(W)$ is the expected requirement of factor ℓ needed to produce a unit of good i, while $f_i^e a_{\ell i}^e(W)$ is the requirement of factor ℓ needed for the entry cost that needs to be paid in order to draw a cost (θ) of production for a unit in good i. We can interpret λ_i as the mass of firms in i.[13] The above is the standard factor market clearing condition, albeit with a twist, as production only occurs if costs are below the cutoff. We denote by $A_{\ell i}(W)$ the unit expected production plus

[13] This is manifested in the first order conditions for the dual maximization problem: maximize the value of expected output subject to X, V being feasible, being identical to that for the current minimization problem. This is just a consequence of the saddle point nature of these two problems. See Dixit and Norman (1980).

entry costs. Thus

$$A_{\ell i}(W) = \bar{\theta}_i(\tilde{\theta}_i)a_{\ell i}(W) + f_i^e a_{\ell i}^e(W).$$

This allows us to rewrite the factor market clearing conditions for interior solutions as

$$V_\ell - \sum_{i=1}^{n} N_i \left[A_{\ell i}(W)\right] = 0 \text{ for } \ell = 1, .., m. \tag{A.5}$$

The first order conditions with respect to λ_i are

$$F_i(\tilde{\theta}_i)p_i - \bar{\theta}_i(\tilde{\theta}_i)c_i(W) - f_i^e c_i^e(W) = 0, \text{for } i = 1, ..n. \tag{A.6}$$

Integrating by parts and using equation (A.3), it follows that

$$\begin{aligned}
F_i(\tilde{\theta}_i)p_i - \bar{\theta}_i(\tilde{\theta}_i)c_i(W) &= F_i(\tilde{\theta}_i)p_i - \left(\int_0^{\tilde{\theta}_i} \theta_i f_i(\theta)d\theta\right) c_i(W) \\
&= F_i(\tilde{\theta}_i)p_i - \left(\tilde{\theta}_i F_i(\tilde{\theta}_i) - \int_0^{\tilde{\theta}_i} F_i(\theta)d\theta\right) c_i(W) \\
&= F_i(\tilde{\theta}_i)\left[p_i - \tilde{\theta}_i c_i(W)\right] + \left[\int_0^{\tilde{\theta}_i} F_i(\theta)d\theta\right] c_i(W) \\
&= c_i(W)\int_0^{\tilde{\theta}_i} F_i(\theta)d\theta.
\end{aligned}$$

Thus, equation (A.6) can be rewritten as

$$\int_0^{\tilde{\theta}_i} F_i(\theta)d\theta = \frac{f_i^e c_i^e(W)}{c_i(W)}, \text{ for } i = 1, ..., n. \tag{A.7}$$

We call these conditions the free entry conditions. They can be interpreted as expected quasi rents being equal to the real cost of entry. Equations (A.3), (A.5), and (A.7) give the supply side of the problem.

A.1.2 The Revenue Function as the Value Function of a Maximization Problem

The revenue function can alternatively be defined as the value function for the following problem. Maximize $PF(\tilde{\theta})N$ subject to $AN \leq V$.

The Lagrangian is

$$L(N, \tilde{\theta}, W : P, V) = \sum_{i=1}^{n} p_i F_i(\tilde{\theta}_i) N_i - \sum_{\ell=1}^{m} \mu_\ell \left[\sum_{i=1}^{n} (A_{\ell i} N_i - V_\ell) \right] \tag{A.8}$$

The first order condition with respect to μ_ℓ is

$$\sum_{i=1}^{n} (A_{\ell i} N_i - V_\ell) = 0 \text{ for } \ell = 1, ..m. \tag{A.9}$$

or the factor market clearing conditions.

The first order condition with respect to $\tilde{\theta}_i$ is

$$p_i f_i(\tilde{\theta}_i) N_i - \sum_{\ell=1}^{m} \mu_\ell \left[\tilde{\theta}_i f(\tilde{\theta}_i) a_{\ell i} N_i \right] = 0 \tag{A.10}$$

$$p_i N_i - \sum_{\ell=1}^{m} \tilde{\theta}_i \mu_\ell a_{\ell i} \left[(N_i) \right] = 0 \tag{A.11}$$

$$\left[p_i - \tilde{\theta}_i \sum_{\ell=1}^{m} c_i(W) \right] N_i = 0 \tag{A.12}$$

when we realize that μ can be interpreted as W since $A_{ij} = \left[\bar{\theta}_i(\tilde{\theta}_i) a_{\ell i} + a^e_{\ell i} \right]$ .This says that price equals marginal cost of production for the marginal firm. If not, then N_i is zero.

The first order condition with respect to N_i is

$$p_i F_i(\tilde{\theta}_i) - \sum_{\ell=1}^{m} \mu_\ell A_{\ell i} = 0 \tag{A.13}$$

$$p_i F_i(\tilde{\theta}_i) - \sum_{\ell=1}^{m} \mu_\ell \left[\bar{\theta}_i(\tilde{\theta}_i) a_{\ell i} + a^e_{\ell i} \right] = 0 \tag{A.14}$$

$$p_i F_i(\tilde{\theta}_i) - \bar{\theta}_i(\tilde{\theta}_i) c_i(W) - f^e_i c^e_i(W) = 0. \tag{A.15}$$

Integrating by parts and using $p_i - \tilde{\theta}_i c_i(W) = 0$ gives

$$\int_0^{\tilde{\theta}_i} F_i(\theta) d\theta = \frac{f^e_i c^e_i(W)}{c_i(W)}, \text{for } i = 1, ..n \tag{A.16}$$

as shown earlier. This is the free entry condition.

Note that the two problems have the same first order conditions when we interpret λ as N and μ as W. This is a manifestation of duality. All the usual properties of the revenue function hold. $R(P,V)$ is convex in P and concave in V.[14] It is homogeneous of degree one in P and in V. $R_V(P,V) = W$, and $R_{VV}(p,V) = W_V(P)$ and equals zero in the case where $n = m$, so that we have what is called local Factor Price Equalization (FPE). $R_{p_i}(P,V) = -\lambda_i F_i(\tilde{\theta}_i) = N_i F_i(\tilde{\theta}_i)$ from the envelope theorem on the Lagrangian so that $R_P(P,V) = F(\tilde{\theta}(P))N(P,V)$ or expected supply. As $R(P,V)$ is convex in P, supply is increasing in own price. As $R(P,V)$ is homogenous of degree one in P, $R_P(P,V)$ is homogeneous of degree zero in P, so that supply depends only on relative prices. Moreover, by Euler's rule, $R_{p_i}(P,V)p_i = -\sum_{j\neq i}^{n} R_{p_j}(P,V)p_j$. $R_{PV}(P,V) = R_{VP}(P,V)$ which connects the Rybczynski and Stolper-Samuelson theorems. The standard generalizations as in Dixit and Norman (1980) could also be shown.

A.2 Proofs

Lemma 1 If entry costs are in terms of the good being made, so that there is no selection, then $\frac{A_{Lx}(\cdot)}{A_{Kx}(\cdot)} = \frac{a_{Lx}(\cdot)}{a_{Kx}(\cdot)} > \frac{A_{Ly}(\cdot)}{A_{Ky}(\cdot)} = \frac{a_{Ly}(\cdot)}{a_{Ky}(\cdot)}$.

Proof:

$$\begin{aligned}
A_{Lx} &= \left[c_{wx}(w,r)\bar{\theta}_x(\tilde{\theta}_x(w,r)) + f_x^e c_{wx}^e(w,r)\right] \\
&= c_{wx}(w,r)\left[\tilde{\theta}_x F(\tilde{\theta}_x) - \int_0^{\tilde{\theta}_x} F(\theta)d\theta\right] + f_x^e c_{wx}^e(w,r) \\
&= c_{wx}(w,r)\left[\tilde{\theta}_x F(\tilde{\theta}_x) - \frac{c_x^e(w,r)f_x^e}{c_x(w,r)} + \frac{f_x^e c_{wx}^e(w,r)}{c_{wx}(w,r)}\right]
\end{aligned}$$

where the second line follows from integrating by parts, the third follows from the free entry condition. If entry costs are in terms of the good being made, then $c_x^e(w,r) = c_x(w,r)$. Thus $\frac{c_x^e(w,r)f_x^e}{c_x(w,r)} = f_x^e = \frac{f_x^e c_{wx}^e(w,r)}{c_{wx}(w,r)}$. So $A_{Lx} = c_{wx}(w,r)\tilde{\theta}_x F(\tilde{\theta}_x)$ and as a result, $\frac{A_{Lx}(\cdot)}{A_{Kx}(\cdot)} = \frac{a_{Lx}(\cdot)}{a_{Kx}(\cdot)}$. The analogous proof works for other total input requirements.

End of proof.

[14]Recall $R(P,V)$ is the value function for both the minimization problem above and the maximization problem where the expected value of output is maximized subject to feasibility. As $R(P,V)$ is the minimized value of $W \cdot V$ in the minimization problem, it must must be concave in V. As $R(P,V)$ is the maximized value of expected output subject to feasibility, it must be convex in P.

Lemma 2:

(i) $\bar{S}_{Ki} + \bar{S}_{Li} = 1$ for $i = x, y$.

(ii) $\frac{A_{Ki}}{A_{Li}} = \frac{w}{r}\frac{\bar{S}_{Ki}}{\bar{S}_{Li}}$ for $i = x, y$.

(iii) $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}} \Leftrightarrow \frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$ and $\bar{S}_{Lx} > \bar{S}_{Ly}$, $\bar{S}_{Kx} < \bar{S}_{Ky}$.

Proofs:

(i) $\bar{S}_{Ki} + \bar{S}_{Li} = 1$ for $i = x, y$.

Consider $i = x$.

$$\begin{aligned} \bar{S}_{Kx} + \bar{S}_{Lx} &= [(1 - v^x)S_{Kx} + v^x(S^e_{Kx})] + [(1 - v^x)S_{Lx} + v^x(S^e_{Lx})] && \text{(A.17)} \\ &= (1 - v^x)\left[S_{Kx} + S_{Lx}\right] + v^x\left[S^e_{Kx} + S^e_{Lx}\right] && \text{(A.18)} \\ &= (1 - v^x) + v^x && \text{(A.19)} \\ &= 1 && \text{(A.20)} \end{aligned}$$

(ii) $\frac{A_{Ki}}{A_{Li}} = \frac{\bar{S}_{Ki}w}{\bar{S}_{Li}r}$ for $i = x, y$.

Consider A_{Lx}. All firms that enter and get the option to produce a unit of x incur entry costs $f^e_x c_x(w, r)$ so that expected unit labor costs of entry are $f^e_x a^e_{Lx}(\omega)$ However, only firms with θ below $\tilde{\theta}$ are active and incur production costs. Thus, expected unit labor costs in production are $a_{Lx}(\omega)\int_0^{\tilde{\theta}} \theta f(\theta)d\theta$. Let $\bar{\theta}(\tilde{\theta}(w, r)) = \int_0^{\tilde{\theta}} \theta f(\theta)d\theta$. Hence,

$$A_{Lx} = a_{Lx}(\omega)\int_0^{\tilde{\theta}_x} \theta f(\theta)d\theta + f^e_x a^e_{Lx}(\omega). \quad \text{(A.21)}$$

Integrating by parts gives

$$\begin{aligned} A_{Lx} &= a_{Lx}(\omega)\left[\tilde{\theta}_x F(\tilde{\theta}_x) - \int_0^{\tilde{\theta}_x} F(\theta)d\theta\right] + f^e_x a^e_{Lx}(\omega) && \text{(A.22)} \\ &= \tilde{\theta}_x F(\tilde{\theta}_x)a_{Lx}(\omega)\left[1 - \frac{\int_0^{\tilde{\theta}_x} F(\theta)d\theta}{\tilde{\theta}_x F(\tilde{\theta})}\right] + \frac{f^e_x c^e_x(w, r)}{c_x(w, r)}\frac{c_x(w, r)a^e_{Lx}(\omega)}{c^e_x(w, r)} && \text{(A.23)} \\ &= \tilde{\theta}_x F(\tilde{\theta}_x)a_{Lx}(\omega)\left[1 - v^x\right] + \left[\int_0^{\tilde{\theta}_x} F(\theta)d\theta\right]\frac{c_x(w, r)a^e_{Lx}(\omega)}{c^e_x(w, r)} && \text{(A.24)} \end{aligned}$$

where we use the free entry condition $\frac{f^e_x c^e_x(w,r)}{c_x(w,r)} = \int_0^{\tilde{\theta}_x} F(\theta)d\theta$.

Thus

$$A_{Lx} = \tilde{\theta}_x F(\tilde{\theta}_x) a_{Lx}(\omega) [1 - v^x] + \left[\frac{\int_0^{\tilde{\theta}_x} F(\theta) d\theta}{\tilde{\theta}_x F(\tilde{\theta}_x)} \right] \frac{c_x(w,r) a^e_{Lx}(\omega) \tilde{\theta}_x F(\tilde{\theta}_x)}{c^e_x(w,r)} \tag{A.25}$$

$$= \tilde{\theta}_x F(\tilde{\theta}_x) a_{Lx}(\omega) [1 - v^x] + [v^x] a^e_{Lx}(\omega) \frac{c_x(w,r) \tilde{\theta}_x F(\tilde{\theta}_x)}{c^e_x(w,r)} \tag{A.26}$$

$$= \frac{c_x(w,r) \tilde{\theta}_x F(\tilde{\theta}_x)}{w} \left[\frac{w a_{Lx}(\omega)}{c_x(w,r)} [1 - v^x] + [v^x] \frac{w a^e_{Lx}(\omega)}{c^e_x(w,r)} \right] \tag{A.27}$$

$$= \frac{c_x(w,r) \tilde{\theta}_x F(\tilde{\theta}_x)}{w} \left[S_{Lx} [1 - v^x] + [v^x] S^e_{Lx} \right] \tag{A.28}$$

$$= \frac{c_x(w,r) \tilde{\theta}_x F(\tilde{\theta}_x)}{w} \bar{S}_{Lx}. \tag{A.29}$$

Using the same arguments, we can show that

$$A_{Kx} = \frac{c_x(w,r) \tilde{\theta}_x F(\tilde{\theta}_x)}{r} \bar{S}_{Kx}. \tag{A.30}$$

so that,

$$\frac{A_{Kx}}{A_{Lx}} = \frac{w \bar{S}_{Kx}}{r \bar{S}_{Lx}}. \tag{A.31}$$

(iii) $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}} \Leftrightarrow \frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$.

If $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}}$, then by part (ii) above we know that $\frac{w\bar{S}_{Ky}}{r\bar{S}_{Ly}} > \frac{w\bar{S}_{Kx}}{r\bar{S}_{Lx}}$ so $\frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$. Similarly, if $\frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$, then $\frac{rA_{Ky}}{wA_{Ly}} > \frac{rA_{Kx}}{wA_{Lx}}$ so $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}}$.

It is easy to see that $\bar{S}_{Lx} > \bar{S}_{Ly}$. From (iii) above,

$$\frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}} \Longleftrightarrow \bar{S}_{Ky}\bar{S}_{Lx} > \bar{S}_{Kx}\bar{S}_{Ly}. \tag{A.32}$$

From (i)

$$(1 - \bar{S}_{Ly})\bar{S}_{Lx} > (1 - \bar{S}_{Lx})\bar{S}_{Ly} \tag{A.33}$$

$$\Longleftrightarrow \bar{S}_{Lx} > \bar{S}_{Ly}. \tag{A.34}$$

That $\bar{S}_{Kx} < \bar{S}_{Ky}$ follows from (i).

End of Proof

Theorem 1 (Selection). An increase in the relative price of a factor makes selection stricter (weaker) when entry in the sector uses this factor less (more) intensively than does production.

Proof: Differentiating the free entry conditions (3) and (4) gives (for $i = x, y$)

$$\begin{aligned}\tilde{\theta}_i F(\tilde{\theta}_i)\frac{d\tilde{\theta}_i}{\tilde{\theta}_i} &= \frac{f_i^e\left[c_{wi}^e(w,r)dw + c_{ri}^e(w,r)dr\right]}{c_i(w,r)} \\ &- \frac{c_i^e(w,r)f_i^e}{[c_i(w,r)]}\left[\frac{wc_{wi}(w,r)}{c_i(w,r)}\frac{dw}{w} + \frac{rc_{ri}(w,r)}{c_i(w,r)}\frac{dr}{r}\right]\end{aligned}$$

so that

$$\tilde{\theta}_i F(\tilde{\theta}_i)\frac{d\tilde{\theta}_i}{\tilde{\theta}_i} = \frac{c^e(w,r)f_i^e}{c_i(w,r)}\left[(S_{Li}^e\hat{w} + S_{Ki}^e\hat{r}) - (S_{Li}\hat{w} + S_{Ki}\hat{r})\right] \tag{A.35}$$

where $S_{\ell i}$ and $S_{\ell i}^e$ are factor cost shares in production and entry respectively of factor ℓ in good i.

Using the fact that $\left[\int\limits_0^{\tilde{\theta}} F(\theta)d\theta\right] = \frac{c^e(w,r)f_e}{c(w,r)}$, we get

$$\frac{d\tilde{\theta}_i}{\tilde{\theta}_i} = \frac{\left[\int\limits_0^{\tilde{\theta}_i} F(\theta)d\theta\right]}{\tilde{\theta}_i F(\tilde{\theta}_i)}\left[\left(S_{Li}^e - S_{Li}\right)\hat{w} + \left(S_{Ki}^e - S_{Ki}\right)\hat{r}\right]$$

Consequently:

$$\begin{aligned}\widehat{\left(\tilde{\theta}_x\right)} &= v_x\left[(S_{Lx}^e - S_{Lx})\hat{w} + (S_{Kx}^e - S_{Kx})\hat{r}\right] &\text{(A.36)}\\ \widehat{\left(\tilde{\theta}_y\right)} &= v_y\left[(S_{Ly}^e - S_{Ly})\hat{w} + (S_{Ky}^e - S_{Ky})\hat{r}\right]. &\text{(A.37)}\end{aligned}$$

where $\left[\frac{\int\limits_0^{\tilde{\theta}_x} F(\theta)d\theta}{F(\tilde{\theta}_x)\tilde{\theta}_x}\right] = v_x$, $\left[\frac{\int\limits_0^{\tilde{\theta}_y} F(\theta)d\theta}{F(\tilde{\theta}_y)\tilde{\theta}_y}\right] = v_y$. Note that $v_x, v_y \in (0,1)$.

Rearranging terms and using the fact that $S_{Li} = 1 - S_{Ki}$ gives

$$\begin{aligned}\widehat{\left(\tilde{\theta}_x\right)} &= v_x\left[\left(S_{Lx}^e - S_{Lx}\right)\left(\hat{w} - \hat{r}\right)\right] &\text{(A.38)}\\ \widehat{\left(\tilde{\theta}_y\right)} &= v_y\left[\left(S_{Ly}^e - S_{Ly}\right)\left(\hat{w} - \hat{r}\right)\right]. &\text{(A.39)}\end{aligned}$$

Note that $k_x^e > k_x$ implies

$$\frac{a_{Kx}^e}{a_{Lx}^e} > \frac{a_{Kx}}{a_{Lx}} \Leftrightarrow \frac{ra_{Kx}^e}{wa_{Lx}^e} > \frac{ra_{Kx}}{wa_{Lx}} \Leftrightarrow \frac{S_{Kx}^e}{S_{Lx}^e} > \frac{S_{Kx}}{S_{Lx}}. \tag{A.40}$$

Using the fact that $S^e_{Kx} = 1 - S^e_{Lx}$ and simplifying shows

$$S_{Lx}\left[1 - S^e_{Lx}\right] > S^e_{Lx}\left[1 - S_{Lx}\right] \text{ or } S_{Lx} > S^e_{Lx}. \tag{A.41}$$

Thus, if $\frac{w}{r}$ rises, i.e., $(\hat{w} - \hat{r}) > 0$, then $\overset{\wedge}{\left(\tilde{\theta}_x\right)} > 0$ if capital intensity in entry of the good exceeds that in production as this implies $(S^e_{Lx} - S_{Lx}) > 0$. Similarly for $\overset{\wedge}{\left(\tilde{\theta}_y\right)}$. Thus, an increase in the wage rental rate makes selection weaker in a good if entry is more capital intensive than production in that good, or equivalently, entry is less labor intensive than production.

End of proof.

Theorem 2 (Stolper-Samuelson). Under Assumption 1, the Stolper-Samuelson theorem remains valid. An increase in the price of a good raises the real return to the factor used intensively in its production and reduces the real return of the factor used intensively in the other good.

Proof: We know that

$$p_x = \tilde{\theta}_x(w,r)c_x(w,r) \tag{A.42}$$
$$p_y = \tilde{\theta}_y(w,r)c_y(w,r) \tag{A.43}$$

Generically,

$$\begin{aligned} p &= \tilde{\theta}(w,r)c(w,r) \\ \ln p &= \ln\tilde{\theta}(w,r) + \ln c(w,r) \end{aligned}$$

Using hat notation and totally differentiating gives:

$$\hat{p}_x = S_{Lx}\hat{w} + S_{Kx}\hat{r} + \overset{\wedge}{\left(\tilde{\theta}_x\right)} \tag{A.44}$$
$$\hat{p}_y = S_{Ly}\hat{w} + S_{Ky}\hat{r} + \overset{\wedge}{\left(\tilde{\theta}_y\right)} \tag{A.45}$$
$$\hat{p}^e = S^e_L\hat{w} + S^e_K\hat{r}, \tag{A.46}$$

where $S_{\ell i}$ is the production cost share of ℓ in i for $\ell = L, K$ and $i = x, y$ while $\hat{p}^e_i = \frac{dc^e_i(w,r)}{c^e_i(w,r)}$ as "hats" denote percentage changes. $S^e_{\ell i}$ denotes the cost share of ℓ in sector i in entry cost. For example, $S_{Lx} = \frac{wa_{Lx}(\omega)}{c_x(w,r)}$ and $S^e_{Lx} = \frac{wa^e_{Lx}(\omega)}{c^e_x(w,r)}$.

Substituting (A.36) and (A.37) into (A.44) and (A.45) gives:

$$\begin{aligned}\hat{p}_x &= S_{Lx}\hat{w} + S_{Kx}\hat{r} + v_x\left[(S^e_{Lx} - S_{Lx})\hat{w} + (S^e_{Kx} - S_{Kx})\hat{r}\right] \\ \hat{p}_y &= S_{Ly}\hat{w} + S_{Ky}\hat{r} + v_y\left[(S^e_{Ly} - S_{Ly})\hat{w} + (S^e_{Ky} - S_{Ky})\hat{r}\right].\end{aligned}$$

Rearranging gives:

$$\begin{aligned}\hat{p}_x &= \left[(1 - v_x)S_{Lx} + v_x(S^e_{Lx})\right]\hat{w} + \left[(1 - v_x)S_{Kx} + v_x(S^e_{Kx})\right]\hat{r} \\ \hat{p}_y &= \left[(1 - v_y)S_{Ly} + v_y(S^e_{Ly})\right]\hat{w} + \left[(1 - v_y)S_{Ky} + v_y(S^e_{Ky})\right]\hat{r}\end{aligned}$$

or

$$\begin{aligned}\hat{p}_x &= \left[\bar{S}_{Lx}\right]\hat{w} + \left[\bar{S}_{Kx}\right]\hat{r} \\ \hat{p}_y &= \left[\bar{S}_{Ly}\right]\hat{w} + \left[\bar{S}_{Ky}\right]\hat{r}\end{aligned}$$

where

$$\begin{aligned}\left[(1 - v^x)S_{Lx} + v^x(S^e_{Lx})\right] &= \bar{S}_{Lx} \\ \left[(1 - v^x)S_{Kx} + v^x(S^e_{Kx})\right] &= \bar{S}_{Kx} \\ \left[(1 - v^y)S_{Ly} + v^y(S^e_{Ly})\right] &= \bar{S}_{Ly} \\ \left[(1 - v^y)S_{Ky} + v^y(S^e_{Ky})\right] &= \bar{S}_{Ky}\end{aligned}$$

As

$$\begin{bmatrix}\hat{p}_x \\ \hat{p}_y\end{bmatrix} = \begin{bmatrix}\bar{S}_{Lx} & \bar{S}_{Kx} \\ \bar{S}_{Ly} & \bar{S}_{Ky}\end{bmatrix}\begin{bmatrix}\hat{w} \\ \hat{r}\end{bmatrix}$$

$$\begin{bmatrix}\hat{w} \\ \hat{r}\end{bmatrix} = \begin{bmatrix}\bar{S}_{Lx} & \bar{S}_{Kx} \\ \bar{S}_{Ly} & \bar{S}_{Ky}\end{bmatrix}^{-1}\begin{bmatrix}\hat{p}_x \\ \hat{p}_y\end{bmatrix}$$

so that

$$\begin{bmatrix}\hat{w} \\ \hat{r}\end{bmatrix} = \frac{1}{DetA}\begin{bmatrix}\bar{S}_{Ky} & -\bar{S}_{Kx} \\ -\bar{S}_{Ly} & \bar{S}_{Lx}\end{bmatrix}\begin{bmatrix}\hat{p}_x \\ \hat{p}_y\end{bmatrix}.$$

where $DetA = \bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}$. Solving gives

$$\begin{aligned}
\hat{w} &= \frac{1}{\left(\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}\right)}\left[\bar{S}_{Ky}\hat{p}_x - \bar{S}_{Kx}\hat{p}_y\right] \\
\hat{r} &= \frac{1}{\left(\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}\right)}\left[-\bar{S}_{Ly}\hat{p}_x + \bar{S}_{Lx}\hat{p}_y\right].
\end{aligned}$$

It can be shown, see Lemma 1 in the Appendix, that $\frac{A_{Ki}}{A_{Li}} = \frac{\bar{S}_{Ki}}{\bar{S}_{Li}}$. Thus, since $\frac{A_{Ky}}{A_{Ly}} > \frac{A_{Kx}}{A_{Lx}}$ it follows that $\frac{\bar{S}_{Ky}}{\bar{S}_{Ly}} > \frac{\bar{S}_{Kx}}{\bar{S}_{Lx}}$ and as a result, $\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx} > 0$.

So,

$$\begin{aligned}
\hat{w} - \hat{r} &= \frac{1}{\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}}\left[\left(\bar{S}_{Ky} + \bar{S}_{Ly}\right)\hat{p}_x - \left(\bar{S}_{Kx} + \bar{S}_{Lx}\right)\right]\hat{p}_y & \text{(A.47)} \\
&= \frac{1}{\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}}\left[\hat{p}_x - \hat{p}_y\right] & \text{(A.48)} \\
&= \frac{1}{\bar{S}_{Lx}\bar{S}_{Ky} - \bar{S}_{Ly}\bar{S}_{Kx}}\left[\hat{p}_x - \hat{p}_y\right]. & \text{(A.49)}
\end{aligned}$$

In other words, an increase in the relative price of the labor (capital) intensive good raises (lowers) the wage rental ratio.

End of Proof

Theorem 3 (Rybczynski). An increase in the labor endowment, L, raises entry and output in the overall labor intensive product, while reducing it in the overall capital intensive one. Similarly, an increase in K raises entry in the overall capital intensive product and reduces it in the overall labor intensive one.

Proof: Using equation (9) and (8) gives:

$$\begin{aligned}
N_x A_{Lx} + N_y A_{Ly} &= L & \text{(A.50)} \\
N_x A_{Kx} + N_y A_{Ky} &= K & \text{(A.51)}
\end{aligned}$$

Product prices pin down factor prices and cutoffs, and hence the A_{ij}'s. Hence, at given product prices, entry mass changes satisfy

$$\begin{aligned}
A_{Lx} dN_x + A_{Ly} dN_y &= dL & \text{(A.52)} \\
A_{Kx} dN_x + A_{Ky} dN_y &= dK & \text{(A.53)}
\end{aligned}$$

so that

$$\begin{aligned} \lambda_{Lx}\hat{N}_x + \lambda_{Ly}\hat{N}_y &= \hat{L} \qquad (A.54) \\ \lambda_{Kx}\hat{N}_x + \lambda_{Ky}\hat{N}_y &= \hat{K} \qquad (A.55) \end{aligned}$$

where $\frac{A_{Lx}N_x}{L} = \lambda_{Lx},\ \frac{A_{Ly}N_y}{L} = \lambda_{Ly},\ \frac{A_{Kx}N_x}{K} = \lambda_{Kx}, \frac{A_{Ky}N_y}{K} = \lambda_{Ky}$.

Solving gives

$$\begin{bmatrix} \hat{N}_x \\ \hat{N}_y \end{bmatrix} = \begin{bmatrix} \lambda_{Lx} & \lambda_{Ly} \\ \lambda_{Kx} & \lambda_{Ky} \end{bmatrix}^{-1} \begin{bmatrix} \hat{L} \\ \hat{K} \end{bmatrix}$$

$$\begin{bmatrix} \hat{N}_x \\ \hat{N}_y \end{bmatrix} = \frac{1}{(\lambda_{Lx}\lambda_{Ky} - \lambda_{Kx}\lambda_{Ly})} \begin{bmatrix} \lambda_{Ky} & -\lambda_{Ly} \\ -\lambda_{Kx} & \lambda_{Lx} \end{bmatrix} \begin{bmatrix} \hat{L} \\ \hat{K} \end{bmatrix}$$

so that

$$\left\{ \left[\hat{N}_x = \frac{\hat{K}\lambda_{Ly} - \hat{L}\lambda_{Ky}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}}, \hat{N}_y = -\frac{\hat{K}\lambda_{Lx} - \hat{L}\lambda_{Kx}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}} \right] \right\}$$

Since

$$\begin{aligned} \lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx} &= \lambda_{Kx}\lambda_{Ky}\left(\frac{\lambda_{Ly}}{\lambda_{Ky}} - \frac{\lambda_{Lx}}{\lambda_{Kx}}\right) \\ &= \lambda_{Kx}\lambda_{Ky}\left(\frac{\frac{A_{Ly}N_y}{L}}{\frac{A_{Ky}N_y}{K}} - \frac{\frac{A_{Lx}N_x}{L}}{\frac{A_{Kx}N_x}{K}}\right) \\ &= \lambda_{Kx}\lambda_{Ky}\frac{K}{L}\left(\frac{A_{Ly}}{A_{Ky}} - \frac{A_{Lx}}{A_{Kx}}\right) < 0. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\hat{N}_x}{\hat{K}} &= \frac{\lambda_{Ly}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}} < 0, \quad \frac{\hat{N}_x}{\hat{L}} = \frac{-\lambda_{Ky}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}} > 0, \\ \frac{\hat{N}_y}{\hat{K}} &= \frac{-\lambda_{Lx}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}} > 0, \quad \frac{\hat{N}_y}{\hat{L}} = \frac{\lambda_{Kx}}{\lambda_{Kx}\lambda_{Ly} - \lambda_{Ky}\lambda_{Lx}} < 0. \end{aligned}$$

This proves that $N_x(.; \overset{+}{L}, \overset{-}{K})$, $N_y(.; \overset{-}{L}, \overset{+}{K})$.

End of proof.