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ABSTRACT

We establish the Hurwicz-Uzawa integrability of the broad class of discrete-choice additive random-utility models of individual consumer behavior with perfect substitutes (linear indifference) preferences and divisible goods. We derive the corresponding indirect utility function and then establish a representative consumer formulation for this entire class of models. The representative consumer is always normative, facilitating aggregate welfare analysis. These findings should be of interest to the literatures in macro, trade, industrial organization, labor and ideal price index measurement that use representative consumer models, such as CES and its variants. Our results generalize such representative consumer formulations to the broad, empirically-relevant class of models of behavior that are routinely used in the discrete-choice analysis of micro data, including specifications that do not suffer from the IIA property and that allow for heterogeneous consumer preferences and incomes. These flexible discrete-choice formulations also overcome many of the known limitations of CES and its variants for equilibrium prices and markups, trade liberalization effects and welfare analysis. We also discuss quasi-linear integrability in the case where products are indivisible.

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1 Introduction

Extensive literatures spanning macro, trade, industrial organization and labor have relied on models of imperfect competition with a CES representative consumer due to its computational tractability and its connection back to empirically-realistic discrete-choice models of individual-level logistic demand (e.g., Anderson et al., 1987, 1992; Verboven, 1996).[1] The microfoundation of the CES in discrete-choice models with corner solutions offers an appealing rebuttal of criticisms that the representative consumer consumes all the variety in the market. However, recent literature has noted the limitations of the CES formulation and some of the stylized implications it imposes on susbtitution patterns and equilibrium prices and quantities (e.g., Helpman, 2011; Zhelobodko et al., 2012; Melitz and Redding, 2014; Sheu, 2014; Adao et al., 2017; Matsuyama and Ushchev, 2017, 2021; Matsuyama, Forthcoming). Many of the quick-fixes used in the literature either do not fully resolve these stylized results or are impractical. The popular nested CES model still exhibits IIA within nests and requires imposing a nesting structure ex ante (e.g., Atkeson and Burstein, 2008; Berger et al., Forthcoming). Flexible functional forms like the translog and AIDS models (e.g., Bergin and Feenstra, 2000, 2001; Melitz and Ottaviano, 2008; Bilbiie et al., 2012; Feenstra and Weinstein, 2017; Redding and Weinstein, 2020) may be impractical as they may require calibrating a very large number of parameters, none of which can be connected back to consumers' deep structural taste parameters.[2]

We derive several new results that generalize the representative consumer formulation of aggregate demand corresponding to a broad class of additive random utility "discrete-choice" models of individual consumer demand. We start by establishing the Hurwicz and Uzawa (1971) integrability of the expected discrete-continuous demand function corresponding to a consumer with perfect substitutes preferences (i.e., linear indifference curves) and *any* absolutely-continuous, additive random utility component, hereafter the *additive random utility model* (ARUM). In the ARUM, individual demand exhibits corner solutions for all but at most one of the market goods (e.g., Deaton and Muellbauer, 1980; Hanemann, 1984). We then derive and characterize the indirect utility function for this broad class of ARUMs. We also derive necessary conditions on the functional form of each product's mean utility function in order for the ARUM discrete-continuous demand model to satisfy integrability. These necessary conditions for integrability exclude the use of arbitrary flexible functional forms for the ARUM indirect utility function, often used when deriving the demand function using duality (e.g., Dubin and McFadden, 1984; Hanemann, 1984; Chiang, 1991; Chintagunta, 1993).

[1]Applications include industrial organization (e.g., Judd, 1985; B. Curtis Eaton, 1989; Zhelobodko et al., 2012; Dhingra and Morrow, 2019), macroeconomics (e.g., Blanchard and Kiyotaki, 1987), monetary policy (e.g., Beck and Lein, 2020), economic growth (e.g., Romer, 1987), trade and economic geography (e.g., Krugman, 1995; Melitz, 2003, 2008; Fajgelbaum et al., 2011; Khandelwal, 2010; Bernini and Tomasi, 2015; Crin'o and Ogliari, 2017), labor (e.g., Berger et al., Forthcoming; Card, 2022) and ideal price index measurement (e.g., Redding and Weinstein, 2020).

[2]These specifications start with the representative consumers' dual problem, typically using a second-order flexible functional form to approximate the indirect utility or expenditure function. As a result, the parameters do not have a connection to deep structural parameters, such as preferences.

Finally, we establish that this broad class of ARUMs has a *positive* and *normative* representative consumer formulation with a direct linkage to McFadden (1981)'s Social Welfare Function.

To establish the link to a representative consumer theory, we study the aggregate demand from an underlying population of consumers making discrete choices. From the Gorman form of the indirect utility, it follows that the expected discrete-continuous aggregate demand system for the ARUM always has a positive and normative representative consumer formulation with a class of preferences that nests the popular CES model depending on the assumed distribution of the random utility. The characterization of the normative representative consumer's indirect utility function in terms of McFadden's Social Welfare Function facilitates a practical means for welfare analysis utilizing a broad class of ARUM discrete-choice models of demand. We present a number of examples, including the Becker-Lancaster "characteristics" models, GEV, random-coefficients Logit and multinomial Probit models that allow for flexible substitution patterns between products and for which the representative consumer's demand and indirect utility can be expressed in closed-form in most cases. Many of these formulations do not exhibit the IIA property or the corresponding unrealistic consumer substitution patterns associated with the popular Logit model and its CES formulation (e.g., Helpman, 2011; Adao et al., 2017). For instance, we use our results to demonstrate the structural interpretation of the mixed-CES representative consumer model as a normative representative consumer for a population of individual consumers with random-coefficients Logit demand, a popular empirical model of individual demand. The extant literature has shown that the aggregate demand system for the pure discrete-choice model with indivisible quantities is not integrable (Nocke and Schutz, 2017); although *quasi-linear* integrability holds for the special case of the simple logit (Nocke and Schutz, 2018). We derive conditions that generalize quasi-linear integrability of the pure discrete-choice demand system for any absolutely continuous random utility distribution.

Our results contribute to a literature on differentiated products markets dating back at least to Archibald et al. (1986) that seeks to establish a connection between the convenient CES representative consumer framework (e.g., Spence, 1976; Dixit and Stiglitz, 1977) and a natural, empirically-realistic class of individual demand systems. The extant literature has derived results only for simple Logit and One-level Nested Logit discrete-choice models (e.g., Anderson et al., 1987, 1992; Verboven, 1996). In their list of research priorities, Anderson et al. (1992, p.91) specifically indicate: "...it would be interesting to find the form of the representative consumer's utility function for other discrete choice models, such as the Probit, the nested MNL, and the GEV." Not only do our results nest these specifications, they generalize to a much broader class of ARUMs.

Our findings also contribute to the related question of whether the representative consumer's utility function is *normative* and can be treated as a social welfare function (e.g., Dow and da Costa Werlang, 1988). As discussed in Barnett and Serletis (2008), many of the flexible but ad hoc, local approximations of the representative consumer's demand, such as generalized Leontief, translog, AIDS and its variants fail to satisfy integrability without stringent parameter restrictions (Feenstra, 2003; Mehta, 2015). Nonintegrable demand specifications have uninterpretable welfare implications and may not be amenable to reliable counterfactual policy simulations. Parameter restrictions imposed to ensure integrability can be difficult to interpret since, in general, those restrictions cannot be linked directly to the consumer's utility function and preferences.

Substantively, our results contribute to the trade and economic geography literatures, where recent work has identified several limitations with the popular CES model and its implications for demand elasticities, mark-ups, and substitution patterns (Melitz and Redding, 2014; Adao et al., 2017; Matsuyama and Ushchev, 2017). The CES model implies that larger-share firms sustain higher equilibrium mark-ups (e.g., Krugman, 1980), which is inconsistent with the empirical regularity that small-share firms tend to gain the most from trade (e.g., Kehoe, 2005) and leading to constant elasticity of trade with respect to trade costs (e.g., Novy, 2013). Allowing elasticities to vary with the level of consumption can lead to pro-competitive effects of trade-liberalization that reduce mark-ups and increase welfare (e.g., Mayer et al., 2021). Our findings facilitate the use of integrable representative consumer models with more flexible elasticity functions. These more flexible specifications also generate more robust ideal price index and cost-of-living measurements (e.g., Redding and Weinstein, 2020) and have robust implications for equilibrium returns from product variety (e.g., Sheu, 2014; Adao et al., 2017).[3] Furthermore, our class of models allows for specifications that relax the IIA property, thereby allowing for more flexible substitution patterns, mark-ups, and pass-through rates than the *separable translog* or CES models.[4] Finally, in contrast with flexible functional forms like the translog and AIDS models, our results embody a class of microfounded models for which the parameters have a deep structural intepretation.

Our results should also have an important impact in the macro-labor literature, where the CES labor supply aggregator implies identical elasticity of substitution between different production factors, such as skilled labor versus capital and unskilled labor versus capital. The extant literature has used a one-level nested CES aggregator (Krusell et al., 2000; Berger et al., Forthcoming) which retains the limitations of CES within the nest. Our results facilitate the use of more flexible microfounded specifications, including tractable multi-level nested logit, GEV, and random coefficients models.

The remainder of the article is organized as follows. Section 2 presents the setup of ARUM individual demand with perfect substitutes preferences under divisibility and establishes the Hurwicz and Uzawa (1971) integrability of the individual expected demand. Section 3 studies the aggregation problem of the ARUM individual expected demand, establishes the existence of the positive and normative representative consumer associated with the demand system, and presents several

[3] For instance, the elasticity of substitution parameter in a mixed CES model, which has an ARUM microfoundation, has a direct connection with the elasticity of trade with respect to trade costs in a gravity model (e.g., Adao et al., 2017).

[4] Matsuyama and Ushchev (2017) show that both the CES and separable translog models are part of a broader class of homothetic-single-aggregator demand models that mechanically exhibit the restrictive and unrealistic IIA property since cross-price effects arise through a common price index. When the IIA property holds, the relative consumption of any pair of goods will be invariant to the prices and entry/exit of any third product in response to, for instance, a change in trade liberalization policy.

examples. Section 4 concludes.

2 Individual Demand with Perfect Substitutes Preferences and Divisible Goods

In this section, we start with the general formulation of the ARUM for an individual consumer with perfect substitutes preferences. We first derive the corresponding expected discrete-continuous demand system. We then prove the Hurwicz and Uzawa (1971) integrability of the expected demand system and characterize the analytic form of the corresponding indirect utility function. There is broad empirical support for such discrete-choice models of individual consumer demand. In a broad study of consumer packaged goods shopping behavior in the U.S., Dubé (2019) finds that consumers select a single brand alternative in a category for over 90% of the observed transactions.

2.1 Perfect Substitutes and Discrete-Continuous Choice

We focus on a market that supplies J variants of a differentiated product – *brands*. An individual consumer consumes at most one of the brands. To obtain this *discrete-choice* behavior, we use a random-utility formulation in which consumers perceive the J brands as perfect substitutes as in the *simple re-packaging model with varieties* (e.g., Deaton and Muellbauer, 1980; Hanemann, 1984).[5] This formulation involves linear indifference curves, leading to corner solutions for all but one alternative – "discrete choice."

Formally, a consumer with total income y chooses a consumption bundle $\mathbf{q} = (q_1, ..., q_J)'$ to solve the following utility-maximization problem:

$$\mathbf{q}^* = \underset{\mathbf{q}\in\mathbb{R}_+^J}{\text{argmax}} \sum_{j=1}^{J} q_j \psi_j (\epsilon_j) \text{ s.t. } \mathbf{q}'\mathbf{p} \leq y \tag{2.1}$$

where $\boldsymbol{\psi} = (\psi_1, ..., \psi_J)'$, is the vector of constant marginal utilities for each product, or "brand qualities," with $\psi_j \geq 0$, $\mathbf{p} = (p_1, ..., p_J)'$ are the strictly positive brand prices, and $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_J)' \sim F_{\boldsymbol{\epsilon}} (\boldsymbol{\epsilon})$ is a vector of random component to utility with absolutely-continuous distribution function $F_{\boldsymbol{\epsilon}} (\boldsymbol{\epsilon})$ and full support on $\mathbb{R}^J$.[6]

Due to the linearity of the indifference curves associated with the perfect substitutes preferences, the utility maximization problem (2.1) leads to a corner solution in which only a single brand is

[5] We do not consider the alternative "mutual exclusivity" formulation of discrete-choice herein (e.g., Hanemann, 1984).

[6] The random utility shocks are typically included for mathematical purposes, to smooth the probabilistic demand function, and for substantive purposes, to rationalize consumer switching even in the absence of price changes.

chosen.[7] The consumer chooses brand j (WLOG) if

$$\frac{p_j}{\psi_j(\epsilon_j)} = \min_{k\in\{1,2,\dots,K\}}\left\{\frac{p_k}{\psi_k(\epsilon_k)}\right\}.$$

Assumption 1. $\log(\psi_j(\epsilon_j)) = \theta_j + \epsilon_j, \forall j$, *where* $\theta_j \in \mathbb{R}$ *is the deterministic component of product* $j's$ *quality.*

Under Assumption 1, the consumer chooses brand j if,

$$\begin{aligned}\epsilon_j - \epsilon_k &\geq [\theta_k - \log(p_k)] - [\theta_j - \log(p_j)], \forall k \\ &= -(\delta_j - \delta_k),\end{aligned} \tag{2.2}$$

where $\delta_j := \theta_j - \log(p_j)$. The corresponding probability that the consumer chooses brand j is

$$\begin{aligned}\pi_j(\boldsymbol{\delta}(\mathbf{p})) &\equiv \Pr(\epsilon_j - \epsilon_k \geq \delta_k - \delta_j, \forall k) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\delta_j-\delta_1+\tilde{\epsilon}}\cdots\int_{-\infty}^{\delta_j-\delta_{j-1}+\tilde{\epsilon}}\int_{-\infty}^{\delta_j-\delta_{j+1}+\tilde{\epsilon}}\cdots\int_{-\infty}^{\delta_j-\delta_J+\tilde{\epsilon}} \\ &\quad f_{\boldsymbol{\epsilon}}(\epsilon_1,\dots,\epsilon_{j-1},\tilde{\epsilon},\epsilon_{j+1},\dots,\epsilon_J)\,d\epsilon_J\dots d\epsilon_{j+1}d\epsilon_{j-1}\dots d\epsilon_1 d\tilde{\epsilon}\end{aligned} \tag{2.3}$$

which does not depend on the consumer's income.

Therefore, the consumer has the following *expected* discrete-continuous demand for brand j:

$$q_j(\mathbf{p}, y) \equiv \mathbb{E}_{\boldsymbol{\epsilon}}\left[q_j^*|\mathbf{p}\right] = \pi_j(\boldsymbol{\delta}(\mathbf{p}))\frac{y}{p_j}. \tag{2.4}$$

[8]

The following analysis will help with the integrability results derived in the sections below. The expected indirect brand utility per unit purchased with expectations taken against the distribution of $\boldsymbol{\epsilon}$ is

$$\mathcal{G}(\boldsymbol{\delta}) := \mathbb{E}_{\boldsymbol{\epsilon}}\left[\max_{j\in\{1,2,\dots,J\}}\{\delta_j + \epsilon_j\}\right]. \tag{2.5}$$

In a pure discrete-choice model with totally-inelastic quantity demanded, $\mathcal{G}(\boldsymbol{\delta})$ is the social surplus function (e.g., McFadden, 1981; Small and Rosen, 1981). From the Williams-Daly-Zachary (WDZ) theorem (Williams, 1977; Daly and Zachary, 1978; Fosgerau et al., 2013, 2021, WDZ thoeorem henceforth), we know that:

$$\nabla_{\boldsymbol{\delta}}\mathcal{G}(\boldsymbol{\delta}) = \boldsymbol{\pi}' \tag{2.6}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_J)'$ is the vector of choice probabilities. Therefore, we can re-write the

[7] Note that the probability of ties is zero because $F_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})$ is absolutely continuous on $\mathbb{R}^J$.

[8] There is no loss of generality in the fact that $\delta_j = \theta_j - \log(p_j)$ has a coefficient equal to -1 on log-price. This property is a result, not an assumption, and will be true for any random utility distribution, $F_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})$. From an empirical perspective, this result also does not imply that log-price has a coefficient equal to -1 in the demand system $q_j(\mathbf{p}, y)$. As we show in Section 3.3 below, the scale of the random utility distribution enters the demand model $q_j(\mathbf{p}, y)$ as the price coefficient.

consumer's choice probability for brand j (2.3) as

$$\pi_j(\boldsymbol{\delta}(\mathbf{p})) = \frac{H_j(\boldsymbol{\delta})}{\sum_{k=1}^{J} H_k(\boldsymbol{\delta})} \tag{2.7}$$

where $H(\boldsymbol{\delta}) := \exp(\mathcal{G}(\boldsymbol{\delta}))$ is a *choice probability generating function* (Fosgerau et al., 2013).

For the remainder of the paper, we let $H_j(\cdot)$ denote the partial derivative of $H(\cdot)$ with respect to its jth element δ_j: $H_j(\boldsymbol{\delta}) := \{\nabla_{\boldsymbol{\delta}} H(\boldsymbol{\delta})\}_j$. Similarly, we let $H_{jk}(\boldsymbol{\delta})$ denote the cross partial derivative with respect to δ_j and δ_k, which we assume exist and are continuous on $\mathbb{R}^J$ as in McFadden (1981). We have thus far not found a way to relax this assumption, which ensures the Slutsky matrix is well defined during our integrability proof. As is well-known in the literature, the formulation of the choice probability in (2.7) is consistent with any absolutely-continuous distribution of the random utilities, $F_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})$, including popular empirical implementations like the simple Logit, GEV, Probit and random coefficients models.

It follows immediately that since $\mathcal{G}(\boldsymbol{\delta})$ is convex in $\boldsymbol{\delta}$ (Rust, 1994; Chiong et al., 2016; Chiong and Shum, 2019), $H(\boldsymbol{\delta})$ is also convex in $\boldsymbol{\delta}$ because $\exp(\cdot)$ is a nondecreasing convex function on $\mathbb{R}$. $H_j(\boldsymbol{\delta})$ is non-negative for any $\boldsymbol{\delta} \in \mathbb{R}^J$ and for any j because $H(\boldsymbol{\delta})$ and π_j are all strictly positive.

2.2 Integrability of the Individual's Expected Demand System

We now state and prove the main results of the paper. We first establish the integrability of the *expected* individual discrete-continuous demand function (2.4). We then derive the analytic form of the corresponding expenditure function and indirect utility function.

The following Theorem 1 establishes the integrability of the demand function (2.4) by showing that it satisfies the necessary and sufficient conditions from Hurwicz and Uzawa (1971).

Theorem 1. *The demand function $\mathbf{q}(\mathbf{p}, y)$ of the form (2.4) is integrable in the Hurwicz and Uzawa (1971) sense as, for any $\mathbf{p} \in \mathbb{R}_+^J$ and for any income level $y \in \mathbb{R}_+$, it satisfies the following conditions:*

(i) (Differentiability of Demand) $\mathbf{q}(\mathbf{p}, y)$ is a continuously differentiable function with bounded partial derivatives,

(ii) (Budget Balancedness) $\mathbf{p}'\mathbf{q} = y$,

(iii) (Slutsky Symmetry) The associated Slutsky matrix $\mathbf{S} := \left\{\frac{\partial q_k}{\partial p_j} + q_k \frac{\partial q_k}{\partial y}\right\}_{j,k \in \{1,2,\ldots,J\}}$ is symmetric, and

(iv) (Negative Semidefiniteness of Slutsky Matrix) $\mathbf{S}$ is negative semidefinite.

Proof. See Appendix A. □

An interesting component of the proof is that a necessary and sufficient condition for the Slutsky Symmetry and Negative Semidefiniteness of the ARUM demand system is that the mean utility per unit consumed of a brand has the form: $\delta_j = c_j - \eta \log(p_j)$ where $c_j \in \mathbb{R}$ and $\eta \geq 0$. This condition holds mechanically for our ARUM, where $\eta = 1$ and $c_j = \theta_j$. However, it also shows why alternative, ad hoc formulations of the discrete-continuous demand system might not be integrable. For instance, some researchers have used the dual approach to derive the discrete-continuous model of demand from a flexible functional form assumption for the indirect utility (e.g., Hanemann, 1984; Chiang, 1991; Chintagunta, 1993).

The following corollary characterizes the analytic forms of the expenditure function and indirect utility functions, respectively, corresponding to the expected demand system (2.4).

Corollary 1. *The expenditure function corresponding to the demand function (2.4) is*

$$e(\mathbf{p}, u) = \frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}, \ u \geq 0,$$

and the corresponding indirect utility function is

$$v(\mathbf{p}, y) = yH(\boldsymbol{\delta}(\mathbf{p})). \tag{2.8}$$

Proof. See Appendix A. □

Theorem 1 and Corollary 1 characterize the preferences of a broad class of ARUM models with choice probabilities, $\{\pi_j(\mathbf{p})\}_{j=1}^{J}$, spanning many well-known discrete-choice models from the empirical literature including Logit, random coefficients ("mixed") Logit, GEV and Probit.

An immediate consequence of Corollary 1 is that the indirect utility function for our expected aggregate discrete-continuous-choice demand system (2.4) has the Gorman form. We will use this property extensively in the aggregation results below.

3 Aggregate-Discrete-Continuous-Choice Demand and the Representative Consumer

We now derive the analytic form of aggregate demand and characterize the class of preferences for the representative consumer corresponding to the ARUM from Section 2. The Gorman form of the indirect utility function leads to the well-known result that aggregate demand depends only on aggregate income and is independent of the distribution of income across consumers. Furthermore, from the Gorman form, it follows that a positive and normative representative consumer exists and can be used for welfare analysis.

3.1 Aggregate Demand

Consider a population of N consumers indexed $i = 1, ..., N$ each with ARUM preferences as in Section 2.1 and incomes $\{y^i\}_{i=1}^{N}$. The corresponding expected aggregate discrete-continuous-choice demand system is:

$$Q_j(\mathbf{p}, Y) = \sum_{i=1}^{N} q_j^i(\mathbf{p}, y^i) = \frac{Y}{p_j}\pi_j(\boldsymbol{\delta}(\mathbf{p})) \qquad \text{for } j = 1, ..., J \tag{3.1}$$

where $Y = \sum_{i=1}^{N} y^i$ is the aggregate income.

The demand system (3.1) can also be derived as the aggregation over a continuum of consumers making deterministic choices, as in Anderson et al. (1992) for the case of pure discrete choice with indivisible goods. Suppose the population consists of a continuum of consumers with mass N and income distribution $F_y(\cdot)$. The corresponding aggregate demand system is:

$$\begin{aligned} Q_j(\mathbf{p}, Y) &= N \iint \mathbf{1}(\delta_j + \epsilon_j \geq \delta_k + \epsilon_k \, \forall k \neq j) \tfrac{y^i}{p_j} dF_y(y^i)\, dF_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) \\ &= \tfrac{Y}{p_j}\pi_j(\boldsymbol{\delta}(\mathbf{p})) \end{aligned} \tag{3.2}$$

where $\pi_j(\boldsymbol{\delta}(\mathbf{p})) = \frac{H_j(\boldsymbol{\delta})}{\sum_{k=1}^{J} H_k(\boldsymbol{\delta})}$ as before in (2.7) and $Y = N \int y^i dF_y(y^i)$ is aggregate income.

3.2 The Representative Consumer

We now characterize the representative consumer formulation of the aggregation of the ARUM. Of particular interest is the ability to use the representative consumer for welfare analysis.

Consider a Bergson-Samuelson social welfare function (Samuelson, 1956) $\mathcal{W}: \mathbb{R}^N \to \mathbb{R}$ that maps vectors of individual utilities, $(u^1, u^2, ..., u^N)$, into a "social utility" and is increasing, continuously differentiable, and concave. We can define the social indirect utility function as the solution to the following income allocation problem:

$$\begin{aligned} v^{\mathcal{W}}(\mathbf{p}, Y) = \max_{\{y^i\}_{i=1}^{N}} \quad & \mathcal{W}\left(v^1(\mathbf{p}, y^1), ..., v^N(\mathbf{p}, y^N)\right) \\ & \text{s.t.} \textstyle\sum_{i=1}^{N} y^i = Y, \end{aligned} \tag{3.3}$$

where $v^i(\mathbf{p}, y^i)$ is consumer i's indirect utility function. A positive representative consumer with demand $\mathbf{Q}(\mathbf{p}, Y) = \sum_{i=1}^{N} \mathbf{q}^i(\mathbf{p}, y^{i,\mathcal{W}})$ is also a *normative representative consumer* with respect to the social welfare function $\mathcal{W}(\cdot)$ when her indirect utility is $v(\mathbf{p}, Y) = v^{\mathcal{W}}(\mathbf{p}, Y)$.

The following proposition establishes that a positive and normative representative consumer for our ARUM exists and has an analytic form for her indirect utility.

Proposition 1. *(Integrability of the Aggregate Discrete-Continuous-Choice Demand System) The demand system $\mathbf{Q}(\mathbf{p}, Y)$ characterized by (3.1) corresponds to a representative consumer with indirect utility*

$$v(\mathbf{p}, Y) = Y \cdot H(\boldsymbol{\delta}(\mathbf{p})) \tag{3.4}$$

where

$$v(\mathbf{p}, Y) = v^{\mathcal{W}}(\mathbf{p}, Y)$$

for any Bergson-Samuelson social welfare function, $\mathcal{W}(\cdot)$*.*

Proof. See Appendix B. □

The proof of Proposition 1 relies on the Gorman form of the indirect utility in the ARUM (2.8). The proposition establishes that a population of consumers with perfect substitutes preferences, divisible consumption and an absolutely-continuous distribution of random utility always has a positive and normative representative consumer.

This characterization of the representative consumer formulation spans many popular empirical ARUM models such as random coefficients models (e.g., mixed-Logit), multinomial Probit and GEV. Below we provide some examples. However, for many popular empirical models like mixed-Logit and multivariate Probit, the function $H(\cdot)$ may not have a closed-form representation and would need to be computed numerically.

Proposition 1 generalizes Anderson et al. (1987) and Anderson et al. (1992, Section 3.7), who study the CES representative consumer with aggregate Logit discrete-choice demand, as well as Verboven (1996, Section 4) who studies the extension to a Nested Logit model. Our results are also related to Anderson et al. (1992, Section 3.4) who prove the existence of a representative consumer for the pure discrete-choice model under indivisible (not divisible) goods and quasi-linear preferences. However, Nocke and Schutz (2017) show that the quasi-linear pure discrete-choice model can generate negative consumption, leading to a non-binding budget constraint. This problem does not arise in our model where the quantities consumed will be strictly positive due to divisibility. We discuss the integrability of the demand system corresponding to the pure discrete-choice model below in section 3.4.

The following corollary indicates how to conduct welfare analysis using the normative representative consumer formulation in 1.

Corollary 2. *(Welfare Analysis with the Aggregate Discrete-Continuous-Choice Demand System) Consider the representative consumer characterized by (3.4). The represenative consumer's compensating variation for a change in price from* $\mathbf{p}^0$ *to* $\mathbf{p}^1$*is*

$$CV = Y^0 \frac{H(\boldsymbol{\delta}(\mathbf{p}^1)) - H(\boldsymbol{\delta}(\mathbf{p}^0))}{H(\boldsymbol{\delta}(\mathbf{p}^1))}.$$

Proof. See Appendix B for derivation. □

Below we explore several commonly-used example of ARUMs from the empirical literature for which $H(\cdot)$ can be derived in closed form in most cases.

Proposition 1 can be extended in several ways to connect various assumptions about the representative consumer to other popular implementations of the discrete-choice model.

3.2.1 Adding a Numeraire

Many empirical applications include an additional, divisible numeraire good to allow for the case that some consumers do not purchase any of the brands in the commodity group. Consider the following extension of our model in Section 2 to include the consumption $q_z \in \mathbb{R}_+$ of a numeraire good (all other goods) with price normalized to one. We redefine the consumer's utility-maximization problem (2.1) as follows

$$\left(\mathbf{q}^*, q_z^*\right) = \underset{(\mathbf{q}, q_z) \in \mathbb{R}_+^{J+1}}{\operatorname{argmax}} \left(\sum_{j=1}^{J} q_j \psi_j\left(\epsilon_j\right) \right) \left(q_z\right)^\alpha \text{ s.t. } q_z + \mathbf{q}'\mathbf{p} \leq y. \tag{3.5}$$

The consumer's expected demand is as follows

$$q_j\left(\mathbf{p}, y\right) = \frac{1}{1+\alpha} \cdot \frac{y}{p_j} \cdot \pi_j \tag{3.6}$$

$$q_z\left(\mathbf{p}, y\right) = \frac{\alpha}{1+\alpha} \cdot y \tag{3.7}$$

where the choice probabilities $\left\{\pi_j\right\}_{j=1}^{J}$ are identical to (2.2) and the aggregate demand system is

$$Q_j\left(\mathbf{p}, Y\right) = \sum_{i=1}^{I} q_j^i = \frac{1}{1+\alpha} \cdot \frac{Y}{p_j} \pi_j\left(\mathbf{p}\right) \qquad \text{for } j = 1, ..., J. \tag{3.8}$$

Appendix B.3 shows that Proposition 1 still holds for the aggregate demand system (3.8) and that the corresponding representative consumer still has preferences with indirect utility: $v\left(\mathbf{p}, Y\right) = \frac{Y}{1+\alpha}\left\{H\left(\boldsymbol{\delta}\left(\mathbf{p}\right)\right)\right\}\left(\frac{\alpha Y}{1+\alpha}\right)^\alpha$. Once again, this result generalizes for any absolutely-continuous distribution of random utility.

3.2.2 Characteristics Models

Proposition 1 also nests characteristics models in the spirit of Lancaster (1966). We can re-write the deterministic component of consumer's quality as $\theta_j = \mathbf{x}_j'\boldsymbol{\beta}$ where $\mathbf{x}_j$ is a vector of product characteristics and $\boldsymbol{\beta}$ is a vector of tastes. We then obtain

$$\delta_j\left(p_j, \mathbf{x}_j\right) = -\log p_j + \mathbf{x}_j'\boldsymbol{\beta} \tag{3.9}$$

and the proof of Proposition 1 remains the same.

3.3 Examples

We now work out a number of representative consumer formulations that correspond to popular specifications of discrete-choice demand at the individual consumer level. In several cases, we show that models with flexible substitution patterns can still generate a tractable and analytic form for the representative consumer. However, our results generalize to a much broader class of discrete-choice models, such as the multinomial Probit, which does not exhibit the IIA property. While a positive and normative representative consumer exists for the multinomial Probit, $H(\boldsymbol{\delta})$ and $v(\mathbf{p}, Y)$ can only be derived numerically due to the non-analytic form of the integration over a multivariate Normal distribution.

Example 1. (Simple Logit and One-level Nested Logit Demand) As an illustrative example, we re-work the example of Verboven (1996, Section 4) to determine the representative consumer's preferences for a population of underlying consumers whose product choices are drawn from the Nested Logit model. Suppose aggregate demand (3.1) has the the one-level aggregate discrete-continuous-choice Nested Logit demand functional form:

$$Q_j(\mathbf{p}, Y) = \frac{Y}{p_j} \frac{(\exp(\sigma\delta_j))^{\frac{1}{\lambda_l}} \left(\sum_{k \in B_l} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_l}}\right)^{\lambda_l - 1}}{\sum_{1 \leq K \leq \mathcal{K}} \left(\sum_{k \in B_K} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_K}}\right)^{\lambda_K}} \qquad \text{for } j = 1, ..., J,$$

where B_l denotes the group to which product j belongs and $\sigma = 1/\mu > 0$, where μ is the scale parameter of the GEV random utility distribution. The corresponding function $H(\boldsymbol{\delta})$ is then as follows:

$$H(\boldsymbol{\delta}) = \left\{ \sum_{1 \leq K \leq \mathcal{K}} \left(\sum_{k \in B_K} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_K}} \right)^{\lambda_K} \right\}^{\frac{1}{\sigma}}. \tag{3.10}$$

From Proposition 1, the representative consumer's indirect utility function is

$$v(\mathbf{p}, Y) = Y \left\{ \sum_{1 \leq K \leq \mathcal{K}} \left(\sum_{k \in B_K} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_K}} \right)^{\lambda_K} \right\}^{\frac{1}{\sigma}} \tag{3.11}$$

and the corresponding direct utility function is

$$u(\mathbf{Q}) = \sum_{1 \leq K \leq \mathcal{K}} \left(\sum_{k \in B_K} C_k^{\frac{\sigma}{\sigma + \lambda_K}} Q_k^{\frac{\sigma}{\sigma + \lambda_K}} \right)^{\frac{\sigma + \lambda_K}{\sigma + 1}}, \tag{3.12}$$

where $C_k = \exp(\theta_k)$. We can easily conduct welfare analysis by substituting the formulation of $H(\boldsymbol{\delta})$ from (3.10) into the CV formula given in Corollary 2 to obtain the compensating differential for a price change.

If we let $C_k = 1\ \forall k$, $\rho = \frac{\sigma}{\sigma+1}$, and $\rho_K = \frac{\sigma}{\sigma+\lambda_K}$, the utility function (3.12) reduces to the Nested Logit model of demand in Verboven (1996, Section 4). If we also let λ_K =1, the model further reduces to to the simple Logit model of demand studied by Anderson et al. (1987, 1992) with standard CES direct utility function.

Example 2. ("Product Differentiation" Logit (GEV)) We now derive the indirect utility function for a representative consumer with demand equivalent to the "Product Differentiation" version of the aggregate discrete-continuous GEV Logit demand system (e.g., Bresnahan et al., 1997). Suppose aggregate demand (3.1) has the the aggregate discrete-continuous-choice "Product Differentiation Logit" demand functional form:

$$Q_j(\mathbf{p}, Y) = \frac{Y}{p_j} \frac{(\exp(\sigma\delta_j))^{\frac{1}{\lambda_g}} \sum_g a_g \left(\sum_{k \in B_{gl}(j)} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_g}}\right)^{\lambda_g - 1}}{\sum_g a_g \left(\sum_{B_{gl} \in g} \left(\sum_{k \in B_{gl}} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_g}}\right)^{\lambda_g}\right)} \qquad \text{for } j = 1, ..., J,$$

where $B_{gl}(j)$ denotes the group to which product j belongs. The corresponding probability generating function $H(\boldsymbol{\delta})$ is then as follows:

$$H(\boldsymbol{\delta}) = \left\{\sum_g a_g (H^g(\boldsymbol{\delta}))^\sigma\right\}^{\frac{1}{\sigma}} = \left\{\sum_g a_g \left(\sum_{B_{gl} \in g} \left(\sum_{k \in B_{gl}} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_g}}\right)^{\lambda_g}\right)\right\}^{\frac{1}{\sigma}}, \tag{3.13}$$

where $H^g(\boldsymbol{\delta})$ is the same as in the Nested Logit and for each g, $a_g \in (0, 1]$ such that $\sum_g a_g = 1$, and $\lambda_g \in (0, 1)$. From Proposition 1, the representative consumer's indirect utility function is

$$v(\mathbf{p}, Y) \quad = \quad Y \left\{\sum_g a_g \left(\sum_{B_{gl} \in g} \left(\sum_{k \in B_{gl}} (\exp(\sigma\delta_k))^{\frac{1}{\lambda_g}}\right)^{\lambda_g}\right)\right\}^{\frac{1}{\sigma}}. \tag{3.14}$$

This formulation allows for more flexible underlying substitution patterns at the individual consumer level when the nests are overlapping. At the same time, this formulation still generates an analytic formulation of the representative consumer's demand and indirect utility. We can easily conduct welfare analysis by substituting the formulation of $H(\boldsymbol{\delta})$ from (3.13) into (2) to obtain the compensating differential for a price change.

Example 3. (Random Coefficients Logit) As another illustrative example, we derive the indirect utility function for a representative consumer with demand equivalent to the aggregate discrete-continuous random coefficients Logit demand system (e.g., Nair et al., 2005). We show that the corresponding representative consumer formulation takes the familiar *mixed CES* formulation used in the recent trade literature to overcome some of the stylized implications of the basic CES model, namely due to the underlying logit model's IIA property. (e.g., Sheu, 2014; Adao et al., 2017; Redding and Weinstein, 2020). For instance, the mixed CES produces markedly different measurements of the gains from product variety (e.g., Sheu, 2014; Adao et al., 2017). Unlike the Nested Logit and GEV specifications above, the random coefficients "mixed logit" does not require pre-classifying products into nests. Finally, the results from Proposition 1 and Corollary 2 above confirm that the mixed CES representative consumer is normative and can be used for valid welfare analysis.

For the case of a discrete distribution of heterogeneity, we have:

$$Q_j(\mathbf{p}, Y) = \frac{Y}{p_j} \sum_{i=1}^{I} \frac{\exp\left(\sigma^i\left(\delta_j + \nu_j^i\right)\right)}{\sum_{k=1}^{J} \exp\left(\sigma^i\left(\delta_k + \nu_k^i\right)\right)} \lambda^i \qquad \text{for } j = 1, ..., J \tag{3.15}$$

where $\delta_j = \bar{\gamma}_j - \ln p_j$,

$$\boldsymbol{\nu} := \begin{cases} \boldsymbol{\gamma}^1 - \bar{\boldsymbol{\gamma}}, & \text{with probability } \lambda^1 \\ \vdots & \\ \boldsymbol{\gamma}^I - \bar{\boldsymbol{\gamma}}, & \text{with probability } \lambda^I \end{cases}, \ \sigma^i := \begin{cases} \sigma^1, & \text{with probability } \lambda^1 \\ \vdots & \\ \sigma^I, & \text{with probability } \lambda^I \end{cases},$$

and σ^i is the inverse of the scale parameter of the Type I Extreme Value distribution for mass point $i \in \{1, ..., I\}$. An interesting result herein is that the expenditure-share system corresponding to the demand system (3.15) is the mixed CES model (e.g., Adao et al., 2017; Redding and Weinstein, 2020).

From the WDZ theorem, we know that:

$$\nabla_{\boldsymbol{\delta}} \mathcal{G}(\boldsymbol{\delta}) = (\pi_1, \pi_2, ..., \pi_J)$$

where $\pi_j = \sum_{i=1}^{I} \frac{\exp\left(\sigma^i\left(\delta_j + \nu_j^i\right)\right)}{\sum_k \exp\left(\sigma^i\left(\delta_k + \nu_k^i\right)\right)} \lambda^i$, $\forall j$ and $\mathcal{G}(\boldsymbol{\delta}) := \ln(H(\boldsymbol{\delta})) = \sum_i \frac{\lambda^i}{\sigma^i} \ln\left(\sum_k \exp\left(\sigma^i\left(\delta_k + \nu_k^i\right)\right)\right)$. Hence, the formulation of $H(\boldsymbol{\delta})$ is as follows:

$$H(\boldsymbol{\delta}) = \exp\left(\sum_i \frac{\lambda^i}{\sigma^i} \ln\left(\sum_{k=1}^{J} \exp\left(\sigma^i\left(\delta_k + \nu_k^i\right)\right)\right)\right). \tag{3.16}$$

From Proposition 1, the representative consumer has the following indirect utility function:

$$v(\mathbf{p}, Y) \quad = Y \prod_i \left(\sum_k \exp\left(\sigma^i\left(\delta_k + \nu_k^i\right)\right)\right)^{\frac{\lambda^i}{\sigma^i}}. \tag{3.17}$$

Equations (3.15) and (3.17) indicates that the richer underlying behavior of a heterogeneous population of consumers making discrete choices can be captured using a representative consumer model without losing analytic tractability. Once again, we can easily conduct welfare analysis by substituting the formulation of $H(\boldsymbol{\delta})$ from (3.15) into the CV formula given in Corollary 2 to obtain the compensating differential for a price change.

For the case of an absolutely continuous distribution of heterogeneity, $F_{\sigma,\boldsymbol{\nu}}(\sigma,\boldsymbol{\nu})$, we have:

$$Q_j(\mathbf{p},Y) = \frac{Y}{p_j}\int \frac{\exp(\sigma(\delta_j+\nu_j))}{\sum_k \exp(\sigma(\delta_k+\nu_k))} dF_{\sigma,\boldsymbol{\nu}}(\sigma,\boldsymbol{\nu}) \qquad \text{for } j=1,...,J$$

where $H(\boldsymbol{\delta})$ is as follows:

$$H(\boldsymbol{\delta}) = \exp\left(\int \frac{1}{\sigma}\ln\left(\sum_k \exp(\sigma(\delta_k+\nu_k))\right) dF_{\sigma,\boldsymbol{\nu}}(\sigma,\boldsymbol{\nu})\right).$$

The representative consumer's indirect utility function is:

$$v(\mathbf{p},Y) \quad = \quad Y\left\{\exp\left[\int \frac{1}{\sigma}\ln\sum_k \exp(\sigma(\delta_k+\nu_k))\, dF_{\sigma,\boldsymbol{\nu}}(\sigma,\boldsymbol{\nu})\right]\right\}. \tag{3.18}$$

However, this specification requires numerical integration and does not yield closed-form expressions for demand, indirect utility and $H(\boldsymbol{\delta})$.

Example 4. (Simple Logit with a Continuum of Products) We now consider the typical differentiated products model used in the macro and trade literatures which assumes a continuum of products distributed uniformly over the choice set Ω (or the "continuous Logit" (Ben-Akiva et al., 1985)). Although this example is technically not nested in Proposition 1, which only considers the case with a finite number of products, we show that the corresponding indirect utility function still has the form of (3.4). As always, a consumer chooses brand j if

$$\epsilon(j)-\epsilon(\omega) \geq \delta(\omega)-\delta(j), \quad \forall\omega\in\Omega.$$

The expected demand for brand j across the population of consumers with measure N is

$$Q_j(\mathbf{p},Y) = \frac{Y}{p(j)}\frac{\exp(\sigma\theta(j))\,p(j)^{-\sigma}}{\int_{\omega\in\Omega}\exp(\sigma\theta(\omega))\,p(\omega)^{-\sigma}\,d\omega} \qquad \text{for } j\in\Omega. \tag{3.19}$$

Now consider the usual representative consumer model with CES utility defined over a continuum of brands and with income Ny. The utility function is given by

$$u = \left(\int_{\omega\in\Omega}\exp(\sigma\theta(\omega))^{\frac{\sigma}{\sigma+1}}\,Q(\omega)^{\frac{\sigma}{\sigma+1}}\,d\omega\right)^{\frac{\sigma+1}{\sigma}}.$$

It is straightforward to show that the corresponding demand function is

$$\begin{aligned} Q_j(\mathbf{p},Y) &= Y\exp(\theta(j))\,p(j)^{-\sigma-1}\,P^{-1} \\ &= \frac{Y}{p(j)}\frac{\exp(\sigma\theta(j))\,p(j)^{-\sigma}}{\int_{\omega\in\Omega}\exp(\sigma\theta(\omega))\,p(\omega)^{-\sigma}\,d\omega}, \end{aligned} \tag{3.20}$$

where $P = \int_{\omega\in\Omega} \exp(\theta(\omega)) p(\omega)^{-\sigma} d\omega$ is the Dixit-Stiglitz price index. The equivalence of the aggregate demand (3.19) and the representative consumer's demand (3.20) establishes the representative consumer formulation for a population of consumers making discrete choices from a continuum of product alternatives. It is also straightforward to show that the representative consumer's indirect utility function is:

$$v(\mathbf{p}, Y) \quad = Y \left(\int_{\omega\in\Omega} \exp(\delta(\omega))\, d\omega \right)^{\frac{1}{\sigma}}$$

which has the same form as (3.4).

Example 5. (Multinomial Probit) We now explore the implementation of our results for a population of consumers making *Probit* discrete choices with correlated random utilities. Since a multinomial Probit demand system cannot be derived in closed form, our analysis herein will be numeric. We use parameter values for the trinomial Probit demand system for competing Chinese movie theaters in Dubé et al. (2017):[9]

$$\begin{array}{ll} u_1 = \delta_1 + \epsilon_1 = & -0.0486 - 0.5474 \log p_1 + \epsilon_1 \\ u_2 = \delta_2 + \epsilon_2 = & -0.1460 - 0.5474 \log p_2 + \epsilon_2 \\ u_0 = \delta_0 + \epsilon_0 = & \epsilon_0 \end{array}, \qquad \Delta\boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 - \epsilon_0 \\ \epsilon_2 - \epsilon_0 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -0.1739 \\ -0.1739 & 1.1745 \end{bmatrix} \right) \tag{3.21}$$

where alternative $j = 0$ is a no-purchase alternative.

We now illustrate our findings by computing the aggregate welfare change associated with a change in the movie theater price level from $\mathbf{p}^0 = (\$75, \$75)$ to $\mathbf{p}^1 = (\$20, \$20)$. In Dubé et al. (2017), the latter price represented the equilibrium prices at an off-peak hour of the day whereas the former prices represented the typical "regular" box office prices. We assume that the aggregate income level of the representative consumer is $Y^0 = 1{,}000$. We compute $G(\boldsymbol{\delta}(\mathbf{p}))$ using Monte Carlo simulation to evaluate the integral over the Gaussian random utility terms:

$$\begin{aligned} H(\boldsymbol{\delta}(\mathbf{p})) &\equiv \exp(\mathcal{G}(\boldsymbol{\delta}(\mathbf{p}))) \\ &= \exp\left(\mathbb{E}_{\boldsymbol{\epsilon}} \left[\max_{j\in\{1,2,0\}} \{\delta_j + \Delta\epsilon_j\} \right] \right) \\ &\approx \exp\left(\frac{1}{N_s} \sum_{s=1}^{N_s} \left\{ \max_{j\in\{1,2,0\}} \{\delta_j + \Delta\epsilon_j^s\} \right\} \right), \end{aligned}$$

where $\Delta\epsilon_0 = \epsilon_0 - \epsilon_0 = 0$ and $N_s = 5{,}000$ simulation draws. The compensating differential, CV, can then be calculated using Corollary 2. The simulation gives $H(\boldsymbol{\delta}(\mathbf{p}^0)) = 1.005$, $H(\boldsymbol{\delta}(\mathbf{p}^1)) = 1.046$, and $CV = \$39$ as the compensating variation associated with the price decrease.

[9] Dubé et al. (2017) look at demand between two competing movie theaters along with a "no purchase" outside alternative.

3.4 The Aggregate Pure-Discrete-Choice Demand System with Perfect Substitutes and Indivisibility

We now briefly consider the aggregate demand system corresponding to a *pure discrete-choice model* with unit-inelastic consumer-level quantity demanded, which has become a popular model in the industrial organization literature (e.g., Berry, 1994; Berry et al., 1995). As above, the consumer's choice-specific values have the additive random utility form

$$v_j \quad = \delta_j + \epsilon_j \qquad j = 1, 2, ..., J \tag{3.22}$$

where $\delta_j \equiv \theta_j + g\left(y - p_j\right)$, we normalize $\delta_J \equiv g\left(y\right)$ and $g\left(\cdot\right)$ is the consumption utility function of the numeraire good.[10] The consumer has the following *expected* demand for the numeraire good:

$$q_z\left(\mathbf{p}, y\right) \equiv \mathbb{E}_{\boldsymbol{\epsilon}}\left[q_z^*|\mathbf{p}\right] = y - \sum_{j=1}^{J} \pi_j\left(\boldsymbol{\delta}\left(\mathbf{p}\right), y\right) p_j \tag{3.23}$$

where $\pi_j\left(\boldsymbol{\delta}\left(\mathbf{p}\right), y\right)$ is still the consumer's probability of choosing alternative j as in (2.3). If there are N consumers in the market with aggregate income $Y = Ny$, then the expected aggregate demand system is given by:

$$\begin{aligned} Q_j\left(\mathbf{p}, Y\right) &= N\pi_j\left(\boldsymbol{\delta}\left(\mathbf{p}\right), y\right) \qquad \text{for } j = 1, ..., J \\ Q_z\left(\mathbf{p}, Y\right) &= N\left(y - \sum_{j=1}^{J} \pi_j\left(\boldsymbol{\delta}\left(\mathbf{p}\right), y\right) p_j\right). \end{aligned} \tag{3.24}$$

Nocke and Schutz (2017) show that the aggregate demand system (3.24) is not integrable in the sense of Hurwicz and Uzawa (1971). Inutitively, under indivisibility, the budget balancedness condition no longer holds for the aggregate demand system over the entire domain of prices and income levels.[11] Therefore, the representative consumer formulation from Proposition 1 no longer holds.

However, Nocke and Schutz (2018) show that *quasi-linear* integrability, proposed by Nocke and Schutz (2017), holds when (3.24) is derived from the special case of (3.22) with i.i.d. Type I extreme-value distributed random utility shocks, ϵ_j, and linear consumption utility for the numeraire good: $g\left(y - p_j\right) = \alpha y - \alpha p$, $\alpha > 0$.[12] It is straightforward to extend this implication of Nocke and Schutz

[10] We now need to include a numeraire good comprising expenditure on other goods and services to satisfy the *adding-up* condition. The normalization $p_J = \theta_j = 0$ corresponds to the case where alternative J is an "outside" or non-market good.

[11] The Hurwicz and Uzawa integrability of the demand system (3.24) requires $Q_j\left(\mathbf{p}, Y\right) \geq 0$ and $Q_z\left(\mathbf{p}, Y\right) \geq 0$ over the entire price-income space, $\mathbb{R}_{++}^J \times \mathbb{R}_+$. To see that demand is not defined over the entire domain, consider the case where $Y < \min\left\{p_j, j = 1, ..., J\right\}$ and, hence, $Q_z\left(\mathbf{p}, Y\right) < 0$. Indeed, empirical applications of the pure discrete-choice model must restrict the domain to ensure $Y \geq \max\left\{p_j, j = 1, ..., J\right\}$ to ensure the empirical demand model is well defined.

[12] Quasi-linear integrability requires that the demand system $Q_j\left(\mathbf{p}, Y\right)$ for $j = 1, ..., J$ be independent of income.

(2018) in the case of linear utility for the numeraire to show that *quasi-linear* integrability holds under any absolutely continuous random utility shock distribution. See Appendix B.4 for a proof.

4 Conclusions

We have shown that the expected aggregate demand corresponding to a broad class of ARUMs is integrable and has a normative representative consumer formulation with a specific form of preferences. We provide several examples of representative consumer formulations that can be rationalized as populations of consumers with discrete-continuous demands generating flexible substitution patterns between products. While Hurwicz-Uzawa integrability does not hold for the pure discrete choice model with indivisible quantities, we derive broader conditions than in the extant literature under which quasi-linear integrability holds. An interesting direction for future research would establish the conditions under which the general class of ARUMs with non-integrable random utility components studied by Sorensen and Fosgerau (2022) is also integrable and has a representative consumer formulation.

$g(y - p_j) = \alpha y - \alpha p$ is the only specification where the corresponding demand system does not depend on income, as y cancels out.

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Appendix

A Appendix of Section 2: Proof of Theorem 1 and Corollary 1

This Appendix proves that the demand system (2.4) satisfies the four necessary and sufficient conditions for Hurwicz and Uzawa (1971) integrability and derives the exact form of the corresponding indirect utility function. We break this proof into three sub-sections. We prove the budget balancedness and differentiability conditions in Section A.1, and the Slutsky Symmetry condition in Section A.2. In Section A.3, we first characterize the expenditure function and indirect utility functions corresponding to the demand system (2.4) and use them to establish the negative semi-definiteness condition, which completes the proof of integrability.

Throughout the proof, we omit the argument of $H(\cdot), H_j(\cdot), H_{jk}(\cdot)$, and $\boldsymbol{\delta}(\cdot)$ when it is clear from the context for simplicity in the notation.

A.1 Proof of the Differentiability and Budget Balancedness

To establish differentiability, we substitute the discrete-choice probabilities (2.7) into (2.4):

$$q_j(\mathbf{p}, y) = \frac{y}{p_j}\frac{H_j}{H} \qquad \text{for } j \in \{1, 2, ..., J\}. \tag{A.1}$$

Differentiability follows from the smoothness assumptions for $H(\boldsymbol{\delta})$ and $\delta_j(p_j, \cdot, \cdot)$. The boundedness of the partial derivatives of $\mathbf{q}(\mathbf{p}, Y)$ follows from the assumption that $H(\cdot) > 0$, the mixed partial derivatives $H_{jk}(\cdot)$ exist, and $H_{jk}(\cdot)$ is continuous on $\mathbb{R}^J$ $\forall\, j, k$.

To show the budget balancedness, we show that for any $y \geq 0$,

$$y = \sum_{k=1}^{J} p_k q_k(\mathbf{p}, y).$$

The right-hand side can be written as

$$\sum_{k=1}^{J} p_k q_k(\mathbf{p}, y) = y \sum_{k=1}^{J} \pi_k(\boldsymbol{\delta}(\mathbf{p})) = y,$$

where equality to y follows from the fact that $\sum_{k=1}^{J} \pi_k(\mathbf{p}) = 1$.

A.2 Proof of the Slutsky Symmetry Condition

We now establish the Slutsky symmetry of the individual demand system (2.4). Recall $\delta_j = \theta_j - \log(p_j)$, which is continuously differentiable in p_j. For $k \neq j$,

$$
\begin{aligned}
\frac{\partial q_k(\mathbf{p}, y)}{\partial p_j} &= \frac{y}{p_k}\left\{\frac{\delta_j'(p_j) H_{jk}}{H} - \frac{\delta_j'(p_j) H_j H_k}{H^2}\right\} \\
\frac{\partial q_k(\mathbf{p}, y)}{\partial y} &= \frac{1}{p_k}\frac{H_k}{H}
\end{aligned}
$$

The Slutsky symmetry condition is:

$$
\frac{\partial q_k(\mathbf{p}, y)}{\partial p_j} + q_j(\mathbf{p}, y)\frac{\partial q_k(\mathbf{p}, y)}{\partial y} = \frac{\partial q_j(\mathbf{p}, y)}{\partial p_k} + q_k(\mathbf{p}, y)\frac{\partial q_j(\mathbf{p}, y)}{\partial y} \tag{A.2}
$$

The left-hand side can be re-written as follows:

$$
\begin{aligned}
&\frac{y}{p_k}\left\{\frac{\delta_j'(p_j) H_{jk}}{H} - \frac{\delta_j'(p_j) H_j H_k}{H^2}\right\} + \frac{y}{p_j p_k}\left\{\frac{H_j H_k}{H^2}\right\} \\
=&\frac{1}{H^2}\left\{\frac{y}{p_k}\delta_j'(p_j) H H_{jk} - \frac{y}{p_k}\delta_j'(p_j) H_j H_k + \frac{y}{p_j p_k} H_j H_k\right\}.
\end{aligned}
$$

The symmetry condition becomes:

$$
\begin{aligned}
&\frac{1}{p_k}\left[\delta_j'(p_j) H H_{jk} - \delta_j'(p_j) H_j H_k\right] \\
=&\frac{1}{p_j}\left[\delta_k'(p_k) H H_{jk} - \delta_k'(p_k) H_j H_k\right]
\end{aligned}
$$

which further simplifies to:

$$
\frac{\delta_j'(p_j)}{p_k}\left[H H_{jk} - H_j H_k\right] = \frac{\delta_k'(p_k)}{p_j}\left[H H_{jk} - H_j H_k\right].
$$

The Slutsky symmetry condition (A.2) can therefore be characterized as a system of ordinary differential equations for each j, k pair:

$$
p_j \delta_j'(p_j) = p_k \delta_k'(p_k).
$$

This yields the form of the solution:

$$
\delta_j(p_j) = -\eta \log p_j + c_j, \forall j \tag{A.3}
$$

for $\eta \in \mathbb{R}$ and some $c_j \in \mathbb{R}$, which is sufficient for Slutsky symmetry. Condition (A.3) is also necessary for Slutsky symmetry due to the uniqueness of the solution of an ordinary differential equation (up to initial conditions determined by c_j). This condition holds mechanically for the ARUM where $\eta = 1$ and $c_j = \theta_j$.

A.3 Proof of the Negative Semidefiniteness of the Slutsky Matrix and the Integrability of the Expected Discrete-Continuous-Choice Demand System

In this section, we first characterize the expenditure function and indirect utility function corresponding to the individual demand system (2.4). We then prove the negative semidefiniteness of the Slutsky matrix which both completes our proof of integrability and characterizes the corresponding preferences.

The strategy for our proof is *guess-and-verify*. We start with the *guess* that the indirect utility function corresponding to the ARUM individual demand system (2.4) has the form:

$$v(\mathbf{p}, y) = yH(\boldsymbol{\delta}(\boldsymbol{p})) \tag{A.4}$$

with corresponding expenditure function

$$e(\mathbf{p}, u) = \frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}, \tag{A.5}$$

where the function $H(\cdot)$ is defined as in Section 2.1 and $u \geq 0$ WLOG.[13] Our intuition for this guess comes from the following observation. The log-indirect utility function for the consumer problem (2.1) where $\eta = 1$ is:

$$\begin{aligned} \log\{\tilde{v}(\mathbf{p}, y, \boldsymbol{\epsilon})\} &= \log\left\{\max_{\mathbf{q}\in\mathbb{R}^J_+} \sum_{j=1}^{J} q_j \psi_j(\epsilon_j) \text{ s.t. } \mathbf{q}'\mathbf{p} \leq y\right\} \\ &= \max_{j\in\{1,2,\ldots,J\}} \{\log y + \delta_j + \epsilon_j\} \end{aligned} \tag{A.6}$$

because of the corner solution in which the chosen brand k has demand: $q_k = \frac{y}{p_k}$. Taking the exponentiated form of the expectation of (A.6) with respect to the random utility, $\boldsymbol{\epsilon}$, gives:

$$\exp(\mathbb{E}_{\boldsymbol{\epsilon}}[\log \tilde{v}(\mathbf{p}, y, \boldsymbol{\epsilon})]) = \exp(\log y + \mathcal{G}(\boldsymbol{\delta}(\mathbf{p}))) = yH(\boldsymbol{\delta}(\mathbf{p})). \tag{A.7}$$

We verify that the candidate $e(\mathbf{p}, u)$ is a valid expenditure function associated with the demand system (2.4) by showing that it satisfies the two following sufficient conditions (for sufficiency, see, e.g., Mas-Colell et al., 1995, p.80): (i) $\nabla_{\mathbf{p}} e(\mathbf{p}, u)$ evaluated at $e(\mathbf{p}, u) = y$ generates the individual demand system (2.4), and therefore, the Hessian of $e(\mathbf{p}, u)$ lines up with the Slutsky matrix we examined in Section A.2, and, (ii) $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$.

[13] See, e.g., Jackson (1986) which imposed similar restrictions.

We first verify that $e(\mathbf{p}, u)$ is the expenditure function associated with the individual demand system (2.4).

Lemma 1. *Let $\delta_j = -\log p_j + \theta_j$. For the expenditure function (A.5),*

$$\nabla_{\mathbf{p}} e(\mathbf{p}, u)\big|_{e(\mathbf{p},u)=y} = \left(\frac{y}{p_1}\pi_1, ..., \frac{y}{p_J}\pi_J\right),$$

where u is the utility level that satisfies $e(\mathbf{p}, u) = y$.

Proof. By differentiating the candiate expenditure function once, we obtain:

$$\begin{aligned}\frac{\partial \left\{\frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}\right\}}{\partial p_j} &= \frac{uH_j(\boldsymbol{\delta}(\mathbf{p}))}{p_j H(\boldsymbol{\delta}(\mathbf{p}))^2}, \; j = 1, ..., J \\ &= \frac{u\pi_j}{p_j H(\boldsymbol{\delta}(\mathbf{p}))}.\end{aligned} \tag{A.8}$$

Evaluating (A.8) at $y = e(\mathbf{p}, u) \equiv \frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}$ gives

$$\frac{\partial \left\{\frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}\right\}}{\partial p_j}\Bigg|_{u=yH(\boldsymbol{\delta}(\mathbf{p}))} = \frac{y}{p_j}\pi_j,$$

which is the ARUM discrete-continuous-choice demand system (2.4). This result also confirms that the Slutsky matrix in Section A.2 is the Hessian of our candidate expenditure function $e(\mathbf{p}, u)$. $\square$

Next, we prove that our candiate expenditure function $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$, and therefore, it is a legitimate expenditure function.

Lemma 2. *The Hessian associated with the candidate expenditure function (A.5) is negative semidefinite, and therefore, the candidate expenditure function is concave in $\mathbf{p}$.*

Proof. Our goal in the proof below is to show the negative of the candidate expenditure function $-e(\mathbf{p}, u) = -\frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}$ is convex in $\mathbf{p}$, which implies $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$. Invoking $\delta_j(p_j) = -\log p_j + \theta_j$ and differentiating $-e(\mathbf{p}, u)$ with respect to $\mathbf{p}$ twice, we have:

$$-\nabla^2_{\mathbf{p}} e(\mathbf{p}, u) = \begin{bmatrix} \frac{y}{p_1^2}\left\{\frac{H_{11}}{H} + \frac{H_1}{H} - 2\left(\frac{H_1}{H}\right)^2\right\} & \frac{y}{p_1 p_2}\left\{\frac{H_{12}}{H} - 2\left(\frac{H_1}{H}\frac{H_2}{H}\right)\right\} & \cdots & \frac{y}{p_1 p_J}\left\{\frac{H_{1J}}{H} - 2\left(\frac{H_1}{H}\frac{H_J}{H}\right)\right\} \\ \frac{y}{p_2 p_1}\left\{\frac{H_{21}}{H} - 2\left(\frac{H_2}{H}\frac{H_1}{H}\right)\right\} & \frac{y}{p_2^2}\left\{\frac{H_{22}}{H} + \frac{H_2}{H} - 2\left(\frac{H_2}{H}\right)^2\right\} & \cdots & \frac{y}{p_2 p_J}\left\{\frac{H_{2J}}{H} - 2\left(\frac{H_2}{H}\frac{H_J}{H}\right)\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y}{p_J p_1}\left\{\frac{H_{J1}}{H} - 2\left(\frac{H_J}{H}\frac{H_1}{H}\right)\right\} & \frac{y}{p_J p_2}\left\{\frac{H_{J2}}{H} - 2\left(\frac{H_J}{H}\frac{H_2}{H}\right)\right\} & \cdots & \frac{y}{p_J^2}\left\{\frac{H_{JJ}}{H} + \frac{H_J}{H} - 2\left(\frac{H_J}{H}\right)^2\right\} \end{bmatrix}. \tag{A.9}$$

We show that (A.9) is positive semidefinite. The proof proceeds in the following four steps.

Step 1. We know that $\mathcal{G}(\boldsymbol{\delta})$ is convex in $\boldsymbol{\delta}$, and $H(\boldsymbol{\delta}) = \exp(\mathcal{G}(\boldsymbol{\delta}))$ is also convex in $\boldsymbol{\delta}$ because $\exp(\cdot)$ is an increasing convex function from $\mathbb{R}$ to $\mathbb{R}_+$. The Hessian of $H(\boldsymbol{\delta})$ is given by:

$$\nabla^2_{\boldsymbol{\delta}} H(\boldsymbol{\delta}) = \frac{1}{H}\begin{bmatrix} \frac{H_{11}}{H} - \frac{H_1 H_1}{H^2} & \frac{H_{12}}{H} - \frac{H_1 H_2}{H^2} & \cdots & \frac{H_{1J}}{H} - \frac{H_1 H_J}{H^2} \\ \frac{H_{21}}{H} - \frac{H_2 H_1}{H^2} & \frac{H_{22}}{H} - \frac{H_2 H_2}{H^2} & \cdots & \frac{H_{2J}}{H} - \frac{H_2 H_J}{H^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{H_{J1}}{H} - \frac{H_J H_1}{H^2} & \frac{H_{J2}}{H} - \frac{H_J H_2}{H^2} & \cdots & \frac{H_{JJ}}{H} - \frac{H_J H_J}{H^2} \end{bmatrix}.$$

The matrix $\nabla^2_{\boldsymbol{\delta}} H(\boldsymbol{\delta})$ is positive semidefinite because $H(\boldsymbol{\delta})$ is convex in $\boldsymbol{\delta}$.

Step 2. Let $\mathbf{p}$ be a positive price vector and let income $y \geq 0$. Consider the following matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1J} \\ a_{21} & a_{22} & \cdots & a_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ a_{J1} & a_{J2} & \cdots & a_{JJ} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{y}{p_1^2} a_{11} & \frac{y}{p_1 p_2} a_{12} & \cdots & \frac{y}{p_1 p_J} a_{1J} \\ \frac{y}{p_2 p_1} a_{21} & \frac{y}{p_2^2} a_{22} & \cdots & \frac{y}{p_2 p_J} a_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y}{p_J p_1} a_{J1} & \frac{y}{p_J p_2} a_{J2} & \cdots & \frac{y}{p_J^2} a_{JJ} \end{bmatrix}.$$

We claim that $\mathbf{A}$ is p.s.d. if and only if $\mathbf{B}$ is p.s.d.. To see why the claim is true, for $\mathbf{w} \in \mathbb{R}^J$, the quadratic form of $\mathbf{B}$ can be expressed as

$$\sum_{j=1}^{J}\sum_{k=1}^{J} \frac{y}{p_j p_k} w_j w_k a_{jk}.$$

For $\mathbf{v} \in \mathbb{R}^J$, the quadratic form of $\mathbf{A}$ can be expressed as

$$\sum_{j=1}^{J}\sum_{k=1}^{J} v_j v_k a_{jk}.$$

To deduce $\mathbf{A}$ p.s.d. from $\mathbf{B}$ being p.s.d., one can take $w_j = \frac{p_j}{\sqrt{y}} v_j$, and vice versa to deduce $\mathbf{B}$ p.s.d. from $\mathbf{A}$ being p.s.d..

Step 3. Applying the result from Step 2 above to $-\nabla^2_{\mathbf{p}} e(\mathbf{p}, u)$ given by (A.9), it now suffices to show

$$\begin{bmatrix} \frac{H_{11}}{H} + \frac{H_1}{H} - 2\left(\frac{H_1}{H}\right)^2 & \frac{H_{12}}{H} - 2\left(\frac{H_1}{H}\frac{H_2}{H}\right) & \cdots & \frac{H_{1J}}{H} - 2\left(\frac{H_1}{H}\frac{H_J}{H}\right) \\ \frac{H_{21}}{H} - 2\left(\frac{H_2}{H}\frac{H_1}{H}\right) & \frac{H_{22}}{H} + \frac{H_2}{H} - 2\left(\frac{H_2}{H}\right)^2 & \cdots & \frac{H_{2J}}{H} - 2\left(\frac{H_2}{H}\frac{H_J}{H}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{H_{J1}}{H} - 2\left(\frac{H_J}{H}\frac{H_1}{H}\right) & \frac{H_{J2}}{H} - 2\left(\frac{H_J}{H}\frac{H_2}{H}\right) & \cdots & \frac{H_{JJ}}{H} + \frac{H_J}{H} - 2\left(\frac{H_J}{H}\right)^2 \end{bmatrix} \tag{A.10}$$

is p.s.d.. We decompose (A.10) as the sum of the two matrices as follows:

$$\begin{bmatrix} \frac{H_{11}}{H}-\left(\frac{H_1}{H}\right)^2 & \frac{H_{12}}{H}-\left(\frac{H_1}{H}\frac{H_2}{H}\right) & \cdots & \frac{H_{1J}}{H}-\left(\frac{H_1}{H}\frac{H_J}{H}\right) \\ \frac{H_{21}}{H}-\left(\frac{H_2}{H}\frac{H_1}{H}\right) & \frac{H_{22}}{H}-\left(\frac{H_2}{H}\right)^2 & \cdots & \frac{H_{2J}}{H}-\left(\frac{H_2}{H}\frac{H_J}{H}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{H_{J1}}{H}-\left(\frac{H_J}{H}\frac{H_1}{H}\right) & \frac{H_{J2}}{H}-\left(\frac{H_J}{H}\frac{H_2}{H}\right) & \cdots & \frac{H_{JJ}}{H}-\left(\frac{H_J}{H}\right)^2 \end{bmatrix} + \begin{bmatrix} \frac{H_1}{H}-\left(\frac{H_1}{H}\right)^2 & -\frac{H_1}{H}\frac{H_2}{H} & \cdots & -\frac{H_1}{H}\frac{H_J}{H} \\ -\frac{H_2}{H}\frac{H_1}{H} & \frac{H_2}{H}-\left(\frac{H_2}{H}\right)^2 & \cdots & -\frac{H_2}{H}\frac{H_J}{H} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{H_J}{H}\frac{H_1}{H} & -\frac{H_J}{H}\frac{H_2}{H} & \cdots & \frac{H_J}{H}-\left(\frac{H_J}{H}\right)^2 \end{bmatrix}. \tag{A.11}$$

Notice that the first term of (A.11) is $H(\boldsymbol{\delta}) \cdot \left\{\nabla^2_{\boldsymbol{\delta}} H(\boldsymbol{\delta})\right\}$, which is p.s.d. because $H(\boldsymbol{\delta}) > 0$ and $\left\{\nabla^2_{\boldsymbol{\delta}} H(\boldsymbol{\delta})\right\}$ is p.s.d. by Step 1.

Invoking the Williams-Daly-Zachary theorem, the second term of (A.11) can be expressed in terms of choice probabilities $\{\pi_j\}_{j=1}^J$ as follows:

$$\begin{bmatrix} \pi_1-\pi_1^2 & -\pi_1\pi_2 & \cdots & -\pi_1\pi_J \\ -\pi_2\pi_1 & \pi_2-\pi_2^2 & \cdots & -\pi_2\pi_J \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_J\pi_1 & -\pi_J\pi_2 & \cdots & \pi_J-\pi_J^2 \end{bmatrix}. \tag{A.12}$$

(A.12) is (weakly) diagonally dominant with non-negative diagonal entries. To see why, note first that all the π_j's are nonnegative, and therefore, $|\pi_j\pi_k| = \pi_j\pi_k\ \forall j,k$. Without losing generality, take $j=1$, and we now examine whether $\left|\pi_1-\pi_1^2\right| - \sum_{k=2}^J |\pi_1\pi_k|$ is nonnegative:

$$\begin{aligned} \left|\pi_1-\pi_1^2\right| - \sum_{k=2}^J |\pi_1\pi_k| &= \pi_1-\pi_1^2-\pi_1\left(\sum_{k=2}^J \pi_k\right) \\ &= \pi_1-\pi_1^2-\pi_1(1-\pi_1) \\ &= 0. \end{aligned}$$

Note that we invoked $\pi_1 \geq \pi_1^2$ because $\pi_1 \in [0,1]$ in the first equality, and $\left(\sum_{k=2}^J \pi_k\right) = 1-\pi_1$ in the second equality because probabilities sum to one. Therefore, the matrix (A.12) is weakly diagonally dominant with non-negative diagonal entries, implying that (A.12) is p.s.d..

Step 4. Each of the two summands in (A.11) is p.s.d., and therefore, the sum of the two terms is p.s.d. as well, establishing that $-\nabla^2_{\mathbf{p}} e(\mathbf{p},u)$ is p.s.d.. Therefore, $-e(\mathbf{p},u)$ is convex in $\mathbf{p}$, which is the desired conclusion. □

Note, as we proved that $e(\mathbf{p},u) = \frac{u}{H(\boldsymbol{\delta}(\mathbf{p}))}$ is the expenditure function for our ARUM discrete-continuous-choice demand system $\left\{\frac{y}{p_j}\pi_j\right\}_{j\in\{1,2,\ldots,J\}}$, the non-negativity of $H(\boldsymbol{\delta}(\mathbf{p}))$ and the unboundedness of income, y, ensures that the associated indirect utility function, $v(\mathbf{p},y) = yH(\boldsymbol{\delta}(\mathbf{p}))$, has the range $\mathbb{R}_+$.

B Proofs of Section 3

B.1 Proof of Proposition 1

The individual's indirect utility function

$$v^i\left(\mathbf{p}, y^i\right) = y^i H\left(\boldsymbol{\delta}\left(\mathbf{p}\right)\right),$$

has the Gorman form. Therefore, the positive representative consumer with income level $Y = \sum_{i=1}^{N} y^i$ has the indirect utility function

$$v\left(\mathbf{p}, Y\right) = Y H\left(\boldsymbol{\delta}\left(\mathbf{p}\right)\right),$$

and demand $\mathbf{q}\left(\mathbf{p}, Y\right) = \sum_{i=1}^{N} \mathbf{q}^i\left(\mathbf{p}, y^i\right)$ when Roy's identity is applied, for any $\mathbf{p} \in \mathbb{R}_+^J$ and for any allocation of income $\left\{y^1, y^2, ..., y^N\right\}$ (see, e.g., Varian, 1992, Section 9.4). The Gorman form also ensures that the representative consumer is normative, robust to any increasing, continuously differentiable, and concave social welfare function $\mathcal{W}\left(\cdot\right)$ (see, e.g., Mas-Colell et al., 1995, pp.119-120).

B.2 Proof of Corollary 2

The representative consumer's compensating variation is characterized by $v\left(\mathbf{p}^0, Y^0\right) = v\left(\mathbf{p}^1, Y^0 - CV\right)$. Substituting $v\left(\mathbf{p}^0, Y^0\right) = Y^0 H\left(\boldsymbol{\delta}\left(\mathbf{p}^0\right)\right)$ and $v\left(\mathbf{p}^1, Y^0 - CV\right) = \left(Y^0 - CV\right) H\left(\boldsymbol{\delta}\left(\mathbf{p}^1\right)\right)$ back and equating gives

$$Y^0 H\left(\boldsymbol{\delta}\left(\mathbf{p}^0\right)\right) = \left(Y^0 - CV\right) H\left(\boldsymbol{\delta}\left(\mathbf{p}^1\right)\right).$$

Rearranging gives

$$CV = Y^0 \frac{H\left(\boldsymbol{\delta}\left(\mathbf{p}^1\right)\right) - H\left(\boldsymbol{\delta}\left(\mathbf{p}^0\right)\right)}{H\left(\boldsymbol{\delta}\left(\mathbf{p}^1\right)\right)}.$$

B.3 Extension to Including the Numeraire

The remainder of the this subsection consists of extending the results above to include the numeraire in a manner similar to Anderson et al. (1992, Section 3.7) and Verboven (1996, Section 4). Specifically, we use the Cobb-Douglas bivariate utility over the numeraire and the products group.

Let $Y = \sum_{i=1}^{N} y^i$ be the representative consumer's total income. We have shown the existence of the direct utility function $u(Q_{1,...,}Q_J)$ such that the observed demand system $\mathbf{Q}\left(\mathbf{p}, Y\right)$ solves the usual budget-constrained utility maximization problem:

$$\max_{(Q_1,...,Q_J)} u\left(Q_1,...,Q_J\right) \qquad \text{s.t.} \qquad \sum_{1\leq j\leq J} p_j Q_j = Y.$$

We now modify the utility function $u(Q_1,...,Q_J)$ as $u(Q_1,...,Q_J)Q_z^{\alpha}$. The separability of the Cobb-Douglas utility $u(Q_1,...,Q_J)Q_z^{\alpha}$ between the consumption of the products, $(Q_1,...,Q_J)$, and of the numeraire, Q_z, ensures that the representative consumer always spends $Y/(1+\alpha)$ and $\alpha Y/(1+\alpha)$ on the inside goods and the numeraire, respectively. Therefore,

$$(Q_1,...,Q_J;Q_z) := \left(Q_1,...,Q_J;Y-\sum_{j=1}^{J} p_j Q_j\right)$$

solves

$$\max_{(Q_1,...,Q_J,Q_z)} u\left(Q_1,...,Q_J\right)Q_z^{\alpha} \qquad \text{s.t.} \qquad Q_z + \sum_{1\leq j\leq J} p_j Q_j = Y.$$

It is straightforward to verify that the resulting demand system has the following form:

$$\begin{aligned} Q_j\left(\mathbf{p},Y\right) &= \left(\frac{1}{1+\alpha}\right)\frac{Y}{p_j}\frac{H_j}{H} \qquad \text{for } j\in\{1,2,...,J\} \\ Q_z\left(\mathbf{p},Y\right) &= \left(\frac{\alpha}{1+\alpha}\right)Y. \end{aligned} \tag{B.1}$$

B.4 Proofs of Section 3.4

Nocke and Schutz (2017) define quasi-linear integrability as follows:

Definition. (Quasi-linear Integrability) A demand system $\mathbf{Q}\left(\mathbf{p}\right)$ is quasi-linear integrable if there exist a set $\mathcal{X}\subset\mathbb{R}^J$ and a function $u:\mathcal{X}\to\mathbb{R}$ such that for every $(\mathbf{p},Y)\in\mathbb{R}^J_{++}\times\mathbb{R}_+$ where $Q_z\left(\mathbf{p},Y\right)\geq 0$, $\mathbf{Q}\left(\mathbf{p}\right)$ is the unique solution of

$$\max_{(\mathbf{Q},Q_z)} \left(Q_z+u\left(\mathbf{Q}\right)\right) \qquad s.t. \qquad Q_z+\mathbf{p}'\mathbf{Q}\leq Y, Q_z\geq 0, \mathbf{q}\in\mathcal{X}.$$

Theorem 1 of Nocke and Schutz (2017) shows that a demand system is quasi-linear integrable if and only if the substitution matrix $\left(\frac{\partial Q_j(\mathbf{p},Y)}{\partial p_k}\right)_{1\leq j,k\leq J}$ is symmetric and negative semidefinite for every $\mathbf{p}\in\mathbb{R}^J_{++}$ and $Y\in\mathbb{R}_+$ for which the demand is well-defined. In our case, we need non-negative consumption of the numeraire: $Q_z\left(\mathbf{p},Y\right)\equiv Y-\sum_{j=1}^{J} p_j Q_j\left(\mathbf{p},Y\right)\geq 0$. We now show that the expected aggregate pure discrete-choice demand system (3.24) satisfies these conditions if and only if $g\left(Y-p_j\right)=\alpha\left(Y-p_j\right)$.

Consider the function $N\mathcal{G}\left(\boldsymbol{\delta}\left(\mathbf{p}\right)\right)$. The substitution matrix $\left(\frac{\partial Q_j(\mathbf{p},Y)}{\partial p_k}\right)_{1\leq j,k\leq J}$ is the Jacobian matrix of the demand system $\{Q_j(\mathbf{p},Y)\}_{1\le j\le J}$ with respect to $\mathbf{p}$. From

$$\frac{\partial Q_j(\mathbf{p},Y)}{\partial p_k}=\frac{\partial \delta_k}{\partial p_k}\frac{\partial Q_j(\mathbf{p},Y)}{\partial \delta_k}=-g'(y-p_k)\frac{\partial}{\partial \delta_k}\{N\pi_j\}=-g'(y-p_k)\frac{\partial^2}{\partial \delta_k\partial \delta_j}\{N\mathcal{G}(\boldsymbol{\delta}(\mathbf{p}))\}$$

for any $1\le j,k\le J$, the candidate substitution matrix is the product of the diagonal matrix $\mathbf{A}$ and the Hessian of $N\mathcal{G}(\boldsymbol{\delta}(\mathbf{p}))$ with respect to $\boldsymbol{\delta}$, where $\mathbf{A}$ is defined as follows:

$$a_{jk}=\begin{cases}-g'(y-p_k) & \text{if } j=k\\ 0 & \text{otherwise.}\end{cases}$$

Because both $\mathbf{A}$ and $N\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}$ are symmetric, they must commute for the candidate substitution matrix $\mathbf{A}N\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}$ to be symmetric: $\mathbf{A}N\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}=N\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}\mathbf{A}$. We need $-g'(y-p_j)=-g'(y-p_k)$ for any $1\le j,k\le J$ and for any price vector $\mathbf{p}$. The only functional form of $g(\cdot)$ that satisfies the symmetry condition is a linear function (up to a constant): $g(Y-p_j)=\alpha(Y-p_j)$.

When $g(Y-p_j)=\alpha(Y-p_j)$, the demand system is quasi-linear integrable if and only if the substitution matrix $\left(\frac{\partial Q_j(\mathbf{p},Y)}{\partial p_k}\right)_{1\le j,k\le J}=-\alpha N\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}$ is negative semidefinite. $\nabla^2_{\boldsymbol{\delta}}\{N\mathcal{G}(\boldsymbol{\delta})\}$ is positive definite because $\mathcal{G}(\boldsymbol{\delta}(\mathbf{p}))$ is strictly convex in $\boldsymbol{\delta}$ (Hofbauer and Sandholm, 2002). Therefore, the substitution matrix is concave in $\mathbf{p}$ because $-\alpha<0$.