NBER WORKING PAPER SERIES

NEAR-RATIONAL EQUILIBRIA IN HETEROGENEOUS-AGENT MODELS: A VERIFICATION METHOD

Leonid Kogan
Indrajit Mitra

Working Paper 30111
http://www.nber.org/papers/w30111

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 2022, Revised May 2024

We thank Santiago Bazdresch (discussant), Jack Favilukis (discussant), Ken Judd, Aubhik Khan, Jonathan Parker, Tony Smith (discussant), Julia Thomas, and seminar participants at City University of Hong Kong, Hong Kong University, INSEAD, MIT, Nanyang Technological University, Ohio State University, UC Berkeley, UCLA, University of Michigan, and at the Duke AP conference, Minnesota Macro-Asset Pricing conference, MIT Capital Markets Workshop, Society for Economic Dynamics Annual Meeting, SITE Stanford, and the Western Finance Association Meeting for comments. We thank Anton Petukhov for excellent research assistance. The views expressed here are ours and not necessarily those of the Federal Reserve Bank of Atlanta, the Federal Reserve System, or the National Bureau of Economic Research.

Near-Rational Equilibria in Heterogeneous-Agent Models: A Verification Method
Leonid Kogan and Indrajit Mitra
NBER Working Paper No. 30111
June 2022, Revised May 2024
JEL No. C02,C18,C63,C68,E00,E37,G1

ABSTRACT

We propose a general simulation-based procedure for estimating quality of approximate policies in heterogeneous-agent equilibrium models, which allows to verify that such approximate solutions describe a near-rational equilibrium. Our procedure endows agents with superior knowledge of the future path of the economy, while imposing a suitable penalty for such foresight. The relaxed problem is more tractable than the original, and results in an upper bound on agents' welfare. Our method applies to various solution algorithms. We illustrate our approach in two applications: the incomplete-markets model of Krusell and Smith (1998) and the heterogeneous firm model of Khan and Thomas (2008).

Leonid Kogan
MIT Sloan School of Management
100 Main Street, E62-636
Cambridge, MA 02142
and NBER
lkogan@mit.edu

Indrajit Mitra
Federal Reserve Bank of Atlanta
1000 Peachtree Street N.E.
Atlanta, GA 30309
indrajit.mitra@atl.frb.org

1 Introduction

Cross-sectional heterogeneity among households and firms is at the heart of many important economic phenomena. In general, it is impossible to aggregate a cross-section of agent characteristics in dynamic heterogenous-agent economies, especially in the presence of constraints (e.g., financial constraints) or un-hedgeable sources of risk (e.g., idiosyncratic labor income shocks). In such models, endogenous quantities, such as risk premia, depend on the cross-sectional distribution of agent characteristics, such as household wealth or firm capital. Since the cross-sectional distribution is an infinite dimensional state-variable, it is typically impossible to solve exactly for equilibrium in this class of models. Under the common solution approach, introduced in the highly influential paper by Krusell and Smith (1998), agents follow relatively simple approximate policies that avoid the burden of solving a dynamic optimization problem with a high-dimensional state space. Specifically, agents summarize the state of the economy by a low-dimensional state vector, typically keeping track of only a few cross-sectional moments.[1] The approximate solution of the original model can be viewed as an exact equilibrium in a near-rational economy, in which agents pursue suboptimal policies (see Krusell and Smith (1998), page 874). If agents suffer small welfare losses from failing to fully optimize, expanding further resources on improving the policies is unproductive, and approximate policies are plausible as a description of near-rational behavior. This argument is in the spirit of modeling economic agents as satisficing rather than optimizing, as in Simon (1978).

Do approximate solutions thus constructed describe near-rational equilibria? A significant limitation of the commonly used approaches, including the one by Krusell and Smith, is that currently there are no reliable general methods for verifying the degree of welfare loss under candidate near-rational equilibria in heterogeneous-agent models. We discuss the limitations of one common approach, based on Euler equation errors, in Section 2. In this paper we

[1]This approximation method is widely used, see e.g., Heathcote, Storesletten, and Violante (2009) and Guvenen (2011) for a survey of the solution methodology and applications.

propose a general method for bounding welfare losses due to suboptimality of policies under the approximate solution. Our technique allows one to compute provable bounds on the degree of welfare loss under approximate policies. It is straightforward to implement and has general applicability, being usable in conjunction with various approximation algorithms.

The key to tractability of our approach is that it establishes an upper bound on the agents' welfare loss without computing the optimal policies. This is essential when dealing with infinite-dimensional models, where optimal individual policies are infeasible to compute, even in a candidate near-rational equilibrium. The main idea of our approach is the following. We alter the original problem of an agent by enlarging her information set to allow for perfect knowledge of the future path of prices (more generally, the aggregate state process of the economy), while simultaneously penalizing the agent's objective for such foresight. The modified problem is much more tractable than the original problem, because the aggregate state of the economy in the modified problem follows a deterministic process. Moreover, if the penalty for perfect foresight is chosen properly (we discuss the precise requirement in the main body of the paper), the value function of the modified problem is always higher than the value function of the original problem. We thus obtain an upper bound on the agent's welfare. The lower bound results from following the sub-optimal policy prescribed by the approximate solution. The difference between the two bounds limits the agent's welfare loss from above. A small difference indicates that the degree of sub-optimality is economically small, and the approximate equilibrium is indeed near-rational. A large difference does not necessarily imply that the sub-optimal policy is grossly inefficient, as it may result from the value function of the modified problem being significantly higher than the value function of the original problem.

To illustrate the potential of our method, we apply it to two well-known models, which feature an approximate equilibrium with aggregate uncertainty. First, we consider the incomplete markets model of Krusell and Smith (1998). This is a stochastic growth model in which individual agents face uninsurable labor income risk as well as aggregate shocks to the productivity of capital. Krusell and Smith compute an approximate equilibrium by summarizing the cross-sectional distribution of wealth among the agents using only the average per capita level of wealth. Our second application of the information relaxation approach is to the model of Khan and Thomas (2008). Their model features a heterogeneous cross-section of firms in general equilibrium. In the approximate equilibrium, Khan and Thomas summarize the cross-sectional distribution by the mean capital stock of all firms in the economy. We provide accompanying code which shows our approach applied to these two models and to the simpler illustrative examples in Section 2 and Section 3 of this paper.[2]

We quantify the degree of sub-optimality of agents' policies under both models. We establish that in both settings the original solutions imply relatively low individual welfare losses for most initial configurations of the economy. Thus, for the calibrated models under consideration, our method confirms that their approximate solutions describe a near-rational equilibrium. This is especially important for the model of Khan and Thomas (2008), because we are able to show that the key finding of that paper that non-linearities in individual firm policies do not have a quantitatively large effect on aggregate dynamics is not a result of firms adopting grossly sub-optimal policies.

Next, we stress-test the approximation algorithms in the above applications by introducing transitional dynamics in an economy perturbed away from its steady state. Since the standard solution methods, such as that of Krusell-Smith, are not intended to approximate equilibrium dynamics accurately when the economy is away from its steady state, it is not clear a-priori how well such methods may perform. For both models, we consider the following two transitional dynamics experiments. Starting from the steady-state of the model, we consider: (i) an unanticipated permanent increase in the volatility of aggregate productivity shocks (we consider two-fold and five-fold increases); or, (ii) an unanticipated 50% reduction in capital stock of all agents in the Krusell-Smith economy and a similar reduction in capital stock of every firm in the Khan-Thomas economy. In the first case, the economy transitions to a new

[2]See https://www.dropbox.com/sh/rqe859kstso6vk0/AACBZNNuxCIqUY7BZRJfqti4a?dl=0.

steady state following a permanent regime shift. In the second case, the economy reverts to the original steady state following a large transient shock. Our methodology shows the welfare bound in each experiment to be larger than in the steady-state case, in one instance rising by more than an order of magnitude, thus indicating potentially large welfare losses.

As an additional methodological contribution, we extend the information relaxation approach to settings in which agents have Epstein-Zin preferences (henceforth "EZ"). The original information-relaxation formulation does not apply non-time separable preferences, such as EZ. Under non-time separable preferences, the agent's objective cannot be evaluated over individual paths under the relaxed information set, which negates the benefit of information relaxation. Another intuitive way to see this is to note that EZ-agent have preference over timing of the resolution of uncertainty, hence one cannot evaluate the agent's objective by averaging the realized utility of a given consumption policy across independent alternative future paths, which is an essential element of the original information-relaxation approach. We overcome this difficulty by re-stating the original problem using the variational utility formulation of Geoffard (1996) and Dumas, Uppal, and Wang (2000), which expresses the decision-making problem of agents with EZ problems as a time-separable problem. This allows us to apply the information relaxation approach to settings where agents have EZ preferences.

Our objective in this paper is to establish near-rationality of individual policies under approximate solutions of equilibrium models, as measured by the associated welfare loss. A small welfare loss implies that individual agents have little to gain by refining their strategies. However, there is no guarantee that price dynamics in such a near-rational economy is similar to that in an exact equilibrium of the original model. Small mistakes by individual agents may potentially lead to large differences in equilibrium outcomes (e.g., Akerlof and Yellen (1985), Jones and Stock (1987), Naish (1993), Krusell and Smith (1996), Hassan and Mertens (2011)). An important and challenging task for future research is to develop general quantitative tools for evaluating the effect of small deviations from individual rationality on equilibrium price dynamics.

Related literature

The basic idea of using information relaxations and martingale multipliers to formulate a dual stochastic optimization problem can be traced back to Bismut (1973) (in a continuous-time setting) and Rockafellar and Wets (1976) and Pliska (1982) (in the discrete-time finite horizon setting). Back and Pliska (1987) apply this technique to single-agent problems in financial economics. Most of the existing applications of information relaxations deal with the optimal stopping problems, typically in the context of pricing American or Bermudan options, e.g., Davis and Karatzas (1994), Rogers (2002), Haugh and Kogan (2004), and Andersen and Broadie (2004). Rogers (2007) and Brown, Smith, and Sung (2010) extend the information-relaxation idea to general dynamic optimization problems. We use the formulation in Brown et al. (2010), which incorporates both perfect and partial information relaxations and derives penalty processes from the value function of the original problem. Our paper is the first to apply the information relaxation approach to approximate solutions of heterogeneous-agent equilibrium models.

The existing literature on approximate solutions of equilibrium models uses several approaches to evaluating approximate solutions. One common approach is to compare forecasts of aggregate states of the economy with actual realizations from the simulation, and to judge the approximation quality by the accuracy of the forecasts, e.g., their R^2. A well-known limitation of this approach is that a high forecast R^2 does not guarantee that the approximation quality is high (den Haan, 2010). Krusell and Smith (1998) evaluate multi-period forecasts as a more stringent test, and den Haan (2010) develops a yet more stringent procedure for comparing the law of motion used to formulate agents' policy functions to the true law of motion implied by the approximate solution of the model. These approaches have two main limitations relative to the method we propose in this paper: they do not provide a guarantee of approximation quality and do not describe the welfare cost of approximation errors.

Another popular approach, due to den Haan and Marcet (1994), evaluates Euler equation errors of the approximate solution along the simulated path of the economy. Under the null hypothesis that the agent's policies are optimal, the $\mathcal{L}^2$ norm of the Euler equation errors is distributed as a χ^2 random variable, and a standard hypothesis test can be carried out. The limitation of this method is that small Euler equation errors do not imply low welfare loss. As we show in Section 2, Euler equation errors can be small while sub-optimal policies are infinitely costly in welfare terms. The inadequacy of Euler equation errors as a measure of the approximation quality of equilibrium solutions is also highlighted in Kubler and Schmedders (2003).

Santos (2000) shows that small Euler equation errors do imply small policy function errors for a restricted class of models – importantly, this result is limited to the models in which equilibria correspond to the solution of the central planner's problem. In more general models, theoretical guarantees on the accuracy of policy functions are not available. In such situations, our approach allows one to compute a generally applicable bound on the approximation accuracy of agents' policies. Kubler and Schmedders (2005) propose a method of error analysis where the quality of the approximation to the equilibrium is judged by its proximity to an exact equilibrium in a close-by economy. In contrast to this approach, our method establishes an upper bound on the welfare loss in the original economy, which is due to agents following suboptimal policies.

In the language of numerical analysis, estimates of the errors in equilibrium policies, as in Santos (2000), represent forward error analysis – where the quality of the approximation is judged by how close the approximate solution (including agents' policies and endogenous processes, such as prices) is to the exact solution. In comparison, Kubler and Schmedders (2005) use the logic of backward error analysis, where one evaluates how much the inputs of the model need to be modified to make the approximate solution satisfy all equilibrium conditions exactly. One can view our method as a form of forward error analysis with provable guarantees of approximation quality, where the distance between the approximate and exact solution is measured in economic terms – in terms of individual welfare loss under approximate policies. Judd, Maliar, and Maliar (2017) provide a complementary view of the solution quality. They establish a lower bound on the (forward) error of an approximate solution to an equilibrium model. While our approach provides a sufficient condition for the accuracy of an approximate solution, the lower bound provides a necessary condition since the true errors are larger than the lower bound.

The rest of the paper is organized as follows. In Section 2 we show that the method of Euler equation errors can fail to detect sub-optimal policies with large utility losses. In Section 3 we formulate the relaxed problem and outline the construction of penalty functions. To illustrate our approach, we apply it to a model in which the state space is finite-dimensional and the optimal policy can be computed with high accuracy. In Sections 4 and 5, we apply our method to the Krusell-Smith model and the model of Khan-Thomas, respectively, which feature infinite-dimensional state spaces. In Section 6, we show how information relaxation can be applied to settings in which agents have EZ preferences. Section 7 concludes.

2 Shortcomings of the Euler equation errors approach

We use two examples to show that the method of Euler equation errors can fail to detect sub-optimal policies with large utility losses. In the first example, the agent incurs infinite loss in expected utility from adopting a sub-optimal policy; however, the Euler equation errors remain finite. In the second example, a finite sample test based on Euler equation errors fails to reject the null hypothesis of policy optimality, even though the welfare loss associated with the policy is substantial. We describe the setting for both of our examples next.

Consider an investor with time-0 wealth w_0 and log utility over consumption . The agent has access to a risk-free bond with a constant rate of return R^B and a stock whose distribution of time $t+1$ return R^S_{t+1} depends on the time t value of a state variable X. To rule out arbitrage, we assume that the risk-free rate is strictly bounded by the maximum and minimum possible value of the stock return each period. Assume that X follows an n state Markov process and takes n possible values $X_1, X_2, \cdots, X_n$ with a time-independent transition probability $P_{ij} \equiv \text{Prob}\left(X_{t+1} = X_j | X_t = X_i\right)$. The investor solves

$$\sup_{c_t>0,\phi_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t \, , \tag{1}$$

subject to the budget constraint:

$$w_{t+1} = (w_t - c_t)\left(\phi_t R^S_{t+1} + (1-\phi_t)R^B\right), \tag{2}$$

and a credit constraint,

$$w_t \geq 0, \tag{3}$$

which, in particular, rules out Ponzi schemes. In (1), β is the time-preference parameter, c_t is time-t consumption, and ϕ_t is the share of the investor's wealth in the stock at time-t.

The optimal consumption policy $c^\star_t$ is to consume a constant fraction of wealth:

$$c^\star_t = (1-\beta)w_t \, . \tag{4}$$

The optimal portfolio policy is to choose ϕ_t to maximize the certainty equivalent of the one-period return on wealth, that is,

$$\phi^\star_t = \arg\sup_{\phi_t} B(\phi_t, X_t), \tag{5}$$

where $B(\phi_t, X_t)$ is the certainty equivalent of the one-period return on wealth:

$$B(\phi_t, X_t) \equiv \mathbb{E}\left[\log\left(\phi_t R^S_{t+1} + (1-\phi_t)R^B\right) | X_t\right] \, . \tag{6}$$

The expectation operator in equation (6) is over the distribution of stock returns R^S_{t+1} conditional on the time-t realization of the state variable X. We derive the optimal policies (4) and (5) in Appendix A.1.

Next, consider the suboptimal policy resulting from the investor having incorrect beliefs about the distribution of stock returns. Specifically, instead of the true transition probabilities P_{ij}, the investor supposes that the transition probability is $\widehat{P}_{ij} = \text{Prob}\left(X_{t+1} = X_j | X_t = X_i\right)$. The investor solves

$$\sup_{(\phi_u, c_u)_{u\geq 0}} \widehat{\mathbb{E}}_0 \sum_{t=0}^{\infty} \beta^t \log c_t \,, \tag{7}$$

subject to the budget constraint (2), where the expectation $\widehat{\mathbb{E}}_0$ in (7) is taken under the investor's beliefs. Since the agent's consumption policy is independent of portfolio returns (the agent has log-utility), the investor still consumes the same fraction of wealth as in equation (4), that is,

$$\widehat{c}_t = (1-\beta) w_t. \tag{8}$$

The investor's portfolio policy $\widehat{\phi}_t$ is

$$\widehat{\phi}_t = \arg\sup_{\phi} \widehat{\mathbb{E}} \left[\log\left(\phi R^S_{t+1} + (1-\phi) R^B\right) | X_t \right] \,. \tag{9}$$

Euler equation errors. To quantify deviations from optimality, den Haan and Marcet (1994), henceforth DM, proposed a hypothesis test based on deviations from the first-order optimality conditions. In our example, the investor's first-order optimality equations are $1 = \beta \mathbb{E}\left[R^B \left(c_{t+1}/c_t\right)^{-1} | X_t\right]$ and $1 = \beta \mathbb{E}\left[R^S_{t+1} \left(c_{t+1}/c_t\right)^{-1} | X_t\right]$. Deviations from these first-order conditions, that is, the Euler equation errors, are defined as:

$$\epsilon^B_t \equiv 1 - E\left[R^B / \left(\widehat{\phi} R^S_{t+1} + (1-\widehat{\phi}) R^B\right) | X_t\right], \quad \epsilon^S_t \equiv 1 - E\left[R^S_{t+1} / \left(\widehat{\phi}_t R^S_{t+1} + (1-\widehat{\phi}_t) R^B\right) | X_t\right], \tag{10}$$

where we used equation (2) to express consumption growth in terms of the portfolio return in the Euler equation. DM show that under the null hypothesis that the policy is optimal, the test statistics constructed from the $\mathcal{L}^2$ norm of the Euler equation errors

$$s^B \equiv \sum_{t=1}^{T} \left(\epsilon_t^B\right)^2 /T, \quad s^S \equiv \sum_{t=1}^{T} \left(\epsilon_t^S\right)^2 /T, \tag{11}$$

approach a χ^2 distribution with one degree of freedom as T approaches infinity.

We follow DM in implementing their test: we simulate N paths of stock returns (each of length T) and compute the values of the statistics s^B and s^S along each sample path. Using many paths minimizes the likelihood of not rejecting the null hypothesis due to luck. Finally, we compute the fraction of times these statistics fall in the upper and lower critical 5% region of a χ^2 distribution with one degree of freedom. If these realized fractions are substantially different from 5%, we have evidence that the policy being examined is not optimal.

To test the reliability of the DM approach, we explicitly compute the investor's loss in expected utility from adopting the policies (8) and (9) relative to adopting the optimal policies (4) and (5). We report this welfare loss as a fractional certainty equivalent loss, which we define as follows. Let $U\left((\phi_t, c_t)_{t\geq 0}; W_0, X_0\right)$ be the investor's expected utility from adopting policies ϕ_t and c_t for $t \geq 0$, for initial wealth W_0 and initial state X_0. For each W_0 and X_0, we first compute the expected utility under the optimal policy $U\left((\phi_t^\star, c_t^\star)_{t\geq 0}; W_0, X_0\right)$. Next, we compute the initial wealth $\widehat{W}_0$ that is needed to achieve this utility from adopting the potentially suboptimal policy. That is, we solve for $\widehat{W}_0$ from the equation $U\left((\phi_t^\star, c_t^\star)_{t\geq 0}; W_0, X_0\right) = U\left((\widehat{\phi}_t, \widehat{c}_t)_{t\geq 0}; \widehat{W}_0, X_0\right)$. We define the fractional certainty equivalent loss as:

$$\eta \equiv \frac{\widehat{W}_0 - W_0}{W_0}\,. \tag{12}$$

2.1 Finite Euler equation residuals, unbounded welfare loss

Consider the special case in which stock returns are independent and identically distributed. In this case, the certainty equivalent of the single period ahead portfolio return is a constant $B(\phi) = \mathbb{E}\log\left(\phi R_{t+1}^S + (1-\phi)R^B\right)$. Therefore, the optimal portfolio is a constant $\phi^\star$, where

$\phi^\star = \arg\sup_\phi B(\phi)$. The suboptimal portfolio policy (9) also implies a constant $\widehat{\phi}$ where

$$\widehat{\phi} = \arg\sup_{\phi} \widehat{\mathbb{E}}\left[\log\left(\phi R^S_{t+1} + (1-\phi)R^B\right)\right]. \tag{13}$$

We prove in Appendix A.2, that the fractional certainty equivalent loss from adopting the suboptimal policy is

$$\eta = \exp\left(\frac{\beta}{1-\beta}\left(B(\phi^\star) - B(\widehat{\phi})\right)\right) - 1, \tag{14}$$

where, importantly, $B(\phi^\star)$ and $B(\widehat{\phi})$ are independent of the time-preference parameter β.

Equation (14) shows that if $\widehat{\phi} \neq \phi^\star$, then the welfare loss becomes arbitrarily large as the time preference parameter β approaches unity. However, we see from equation (10) that the magnitude of Euler equation errors does not increase as β approaches one — these errors are independent of β! The Euler equation errors do not grow because these errors are based on deviations from the one-period ahead Euler equations, which fail to aggregate the effect of such deviations over multiple periods. In terms of utility loss, however, a sufficiently patient investor who puts non-negligible weight to utility loss far into the future suffers a very large utility loss, even though the single-period deviation from the suboptimal policy appears small.

2.2 Low-probability persistent disasters

In this section we use the setting from Section 2 to show that the finite-sample test based on Euler equation errors may fail to reject suboptimal policies. The key feature of our example is the presence of a rare but persistent disaster state. The investor underestimates the persistence of this state. Since the state is rare, the investor's mistake is infrequently realized, and therefore a finite-sample test may fail to detect a policy associated with significant welfare loss.

Specifically, we consider the case in which the state variable X takes three values X_1, X_2, and X_3; the realized stock returns in these states are 1.3, 0.91, and 0.7, respectively. The true transition probability matrix is shown in Panel A of Table 1. The investor believes (incorrectly) that the transition probability matrix is the one shown in Panel B of Table 1. Comparing the transition matrices, we see that the investor underestimates the persistence of the disaster state X_3. Note that even though the investor makes large errors in the conditional dynamics of the state variable X, these errors are much milder unconditionally. For example, while the true relative frequencies of occurrence of X_1, X_2, and X_3 are 49.45%, 49.45%, and 1.1%, respectively, the investor believes these frequencies to be 49.5%, 49.5%, and 1%, respectively. Similarly, while the true average stock return and volatility are 10.05% and 19.84%, respectively, the investor believes these quantities to be 10.09% and 19.82%, respectively. Finally, we choose the investor's time preference parameter $\beta = 0.99$ and the risk-free rate $R^B = 1.04$.

As a result of underestimating the persistence of X_3, the investor overinvests in the stock in this state relative to the optimal portfolio by a significantly large amount. While the optimal policy is to invest $\phi^\star(X_3) = 0.52$, the investor chooses $\widehat{\phi}(X_3) = 1.73$. The first row of Table 2 shows the investor's welfare loss in each state as a result of adopting the suboptimal policy (13). We see that the fractional certainty equivalent loss is substantial: 4.09% in states X_1 and X_2 and 8.39% in the state X_3.

Next, we use the DM test using Euler equation errors to assess evidence against the null hypothesis (that the investor's policy is optimal). The stringency of the DM test depends on the length of the time series T used. As DM point out, a low value of T increases the likelihood of a Type II error (i.e., the test fails to detect a suboptimal policy), whereas if T is sufficiently large, every approximate solution will be rejected. DM suggest choosing T to be "substantially bigger than the length of the empirical series...". In their example, they choose the series to be 20 times the length of the empirical series. We assume that the model of Section 2 is being used to analyze consumption-portfolio choices using the post-War data. Similar to DM, we choose $T = 6000$, which is close to twenty times the length of the quarterly consumption series in post-War data. We simulate $N = 10,000$ paths of stock returns and we compute the values of s^B and s^S along each sample path. Finally, we compute the fraction of times s^B and s^S fall in the upper and lower critical 5% region of a χ^2 distribution with one degree of freedom.

Rows 2 through 4 in Panel A of Table 2 show the results. We see that no entry is substantially different from 5%; hence according to this test, we would fail to reject the null in spite of the fact that the investor incurs substantial welfare loss from adopting these policies. In order to detect evidence against the null, we have to choose a much longer time series. This can be seen from Panel B of Table 2, where we use $T = 100,000$. In this case, both s^B and s^S appear in the 5% critical region 43.12% of the time when the initial state is X_3.

Our example highlights a weakness of the approach of using Euler equation errors (see also the related discussion in den Haan and Marcet (1994)). Both examples in Section 2.1 and Section 2.2 underscore the need for a reliable test to detect suboptimal policies. We propose and discuss such a test next.

3 Information relaxation: The main idea

In our analysis of approximate equilibria, we apply the information relaxation method proposed in Brown et al. (2010). We introduce the main ideas of this method in this section first through an illustrative example and then list the general algorithm. We refer the readers to Brown et al. (2010) for full technical details.

3.1 Illustrative Example

3.1.1 The environment and the formulation of the relaxed problem

Consider a standard finite-horizon consumption-savings problem. Time is discrete, $t = (0, \cdots, T)$. Each period the agent receives a random labor income which takes two possible values $\{y_H, y_L\}$ with $y_H > y_L$.[3] The probability of receiving y_H is p each period. The agent chooses consumption c_t and stores the rest in a risk-free asset with constant total return R.

[3]We assume a binomial distribution for income shocks for clarity of exposition, our analysis goes through without any meaningful changes with a continuous distribution of income shocks.

We denote the agent's feasible consumption policy by $C = (c_0, c_1, ..., c_T)$.

At each date t, the agent observes the history of income shocks realized up to and including this date (but not the future shocks). We denote this history by $y^t = (y_0, y_1, ..., y_t)$. All feasible consumption policies must be adapted to the information structure of the agent, i.e., consumption choices are functions of the observed past histories of income shocks. Thus, making the agent's information structure explicit, $C = (c_0(y^0), c_1(y^1), ..., c_T(y^T))$.

The agent has a time-separable constant relative risk aversion utility function with curvature γ. To illustrate the information relaxation approach in an environment featuring constraints, we assume that the agent faces a borrowing constraint—she can borrow up to a fraction $b > 0$ of her wealth (this implies, in particular, that the agent always satisfies a credit constraint $w_t \geq 0$). Let w_t denote the agent's wealth at the beginning of period t, including the realized income in the current period. The agent solves the dynamic optimization problem:

$$\sup_{\{C\colon c_t \leq (1+b)w_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \right], \tag{15}$$

where the agent's wealth and consumption satisfy the dynamic budget constraint:

$$w_t = (w_{t-1} - c_{t-1})R + y_t. \tag{16}$$

We denote the value function of the above problem by $V_t(w_t)$:

$$V_t(w_t) = \mathbb{E}_t \left[\sum_{s=t}^{T} \beta^{s-t} \frac{(c_s^\star)^{1-\gamma}}{1-\gamma} \right], \tag{17}$$

where $c_s^\star$ is the optimal consumption policy at time s.

We formulate a relaxed problem by allowing the agent to have access to information about the future realizations of income shocks. The name "information relaxation" reflects the notion that this formulation relaxes information constraints placed on the agent. Specifically, consider a *complete information relaxation*, whereby we allow the agent to condition her consumption choices on the knowledge of the entire future sequence of income shocks. To distinguish the feasible policies of the relaxed problem from those of the original problem, we denote the former by $C^{\mathcal{R}} = (c_0^{\mathcal{R}}(y^T), c_1^{\mathcal{R}}(y^T), ..., c_T^{\mathcal{R}}(y^T))$.

While providing the agent with such informational advantage compared to that in the original problem, we impose a penalty on the objective function, designed to offset the effect of information relaxation. The penalty is a stochastic process λ_t, which depends on the consumption policy and the entire path of income shocks, y^T: $\lambda_t(C^{\mathcal{R}}, y^T)$. The only requirement we impose on the penalty process is that if the consumption process is chosen to depend only on the information available to the agent, the resulting penalty is non-positive in expectation, i.e.,

$$\mathbb{E}_0 \left[\lambda_t \left(C^{\mathcal{R}}, y^T \right) \right] \leq 0. \tag{18}$$

It is easy to see that the value function of the relaxed problem:

$$V_0^{\mathcal{R}}(w_0) = \sup_{\{C^{\mathcal{R}}:\, c_t^{\mathcal{R}} \leq (1+b) w_t^{\mathcal{R}}\}} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \left(\frac{\left(c_t^{\mathcal{R}}\right)^{1-\gamma}}{1-\gamma} - \lambda_t(C^{\mathcal{R}}, y^T) \right) \right], \tag{19}$$

subject to the dynamic budget constraint:

$$w_t^{\mathcal{R}} = (w_{t-1}^{\mathcal{R}} - c_{t-1}^{\mathcal{R}}) R + y_t, \tag{20}$$

is at least as high as the value function of the original problem. The reason is that the consumption policy $C^{\star}$, optimal under the agent's original problem (15-16), is also a feasible policy for the relaxed problem (19-20), and the expected penalty under such policy adds a non-negative term to the agent's expected utility, according to (18). Thus, we establish a *weak duality* result: the value function of the relaxed problem is always at least as large as the value function of the original decision problem,

$$V_0^{\mathcal{R}}(w_0) \geq V_0(w_0). \tag{21}$$

We refer to $V_0^{\mathcal{R}}(w_0)$ as an upper bound on the value function of the original problem and the difference $V_0^{\mathcal{R}}(w_0) - V_0(w_0)$ as the *duality gap*.

Next, consider a feasible but potentially suboptimal consumption policy $\widehat{C}$. Under this policy, the expected utility of the agent is given by:

$$V_0^{\mathcal{L}}(w_0) = \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \frac{(\widehat{c}_t)^{1-\gamma}}{1-\gamma} \right]. \tag{22}$$

$V_0^{\mathcal{L}}(w_0)$ is a lower bound on the value function of the original problem and results in a welfare loss of $V_0(w_0) - V^{\mathcal{L}}(w_0)$. For time-separable problems, such as the one we are considering here, the lower bound is simple to estimate: simulate many paths of income shocks and compute the realized utility $\sum_{t=0}^{T} \beta^t \frac{(\widehat{c}_t)^{1-\gamma}}{1-\gamma}$ along each path. The sample mean of the realized utilities for all the simulated paths is an unbiased estimate of $V_0^{\mathcal{L}}(w_0)$. The computation of the lower bound for non-time separable problems is more involved and we discuss it in Section 6.

The inequality (21) implies that the agent's (true) welfare loss is bounded above by the difference between the value function of the relaxed problem (19-20) and the expected utility under the suboptimal policy $\widehat{c}_t(y^t)$:

$$V_0(w_0) - V_0^{\mathcal{L}}(w_0) \leq V_0^{\mathcal{R}}(w_0) - V_0^{\mathcal{L}}(w_0)\,. \tag{23}$$

We thus have a framework for computing an upper bound on welfare loss resulting from sub-optimal strategies: define a valid penalty process for the relaxed problem, and then compare the expected utility of the agent under information relaxation with her expected utility under the suboptimal policy of interest. This approach is especially useful where it is infeasible to solve for the optimal policies of the original problem.[4] Note that this approach estimates the right-hand side of (23), $V_0^{\mathcal{R}}(w_0) - V_0^{\mathcal{L}}(w_0)$, which includes both the actual welfare loss $V_0(w_0) - V_0^{\mathcal{L}}(w_0)$ and the duality gap $V_0^{\mathcal{R}}(w_0) - V_0(w_0)$. Clearly, the duality gap has to be small for the approach to be useful in estimating the actual welfare loss. The duality gap depends on the penalty λ_t, and we discuss its choice next.

[4]For complete information relaxation, the relaxed problem is deterministic and hence easy to solve.

3.1.2 Constructing the penalty for foresight

Brown et al. show that it is possible to make the duality gap, arbitrarily small. In particular, they show (using backwards induction) that under an *ideal* penalty, $V_0^{\mathcal{R}} - V_0 = 0$.[5] The definition of an ideal penalty by Brown et al. adopted to our example is as follows. The penalty is defined for each possible sequence of income shocks, and each possible sequence of consumption choices, without requiring the consumption policy to be non-anticipating. Specifically, consider an arbitrary path of income shocks y^T, and a budget-feasible positive sequence of consumption choices $c^T = (c_0, c_1, \cdots, c_T)$. Note that c^T is not a consumption policy, it denotes a sequence of positive real numbers representing a particular path of consumption. The corresponding values of agent's wealth $(w_0, w_1, \cdots, w_T)$ satisfy the dynamic budget constraint equation (20).

To develop intuition for how the ideal penalty of Brown et al. adopted to our example may look, we first consider a somewhat restricted version of information relaxation: suppose that at time t, the decision maker is able to anticipate perfectly the state next period, and must use the original information structure starting from the following period, i.e., perfect foresight is there for a single period only. In this case, the Bellman equation of the relaxed problem at time t takes the form:

$$V_t^{\mathcal{R}}(w_t) = \sup_{\{c_t^{\mathcal{R}}:\, c_t^{\mathcal{R}} \leq (1+b)w_t\}} \frac{(c_t^{\mathcal{R}})^{1-\gamma}}{1-\gamma} - \lambda_t^{\star}(c_t^{\mathcal{R}}, y^{t+1}) + \beta V_{t+1}((w_t - c_t^{\mathcal{R}})R + y_{t+1}). \tag{24}$$

In the above expression, V_{t+1} is the value function of the original problem, which coincides with the value function of the relaxed problem starting at $t+1$, given our assumption of a single-period relaxation at time t. There is no expectation operator in the Bellman equation, because the agent anticipates income at time $t+1$, y_{t+1}, perfectly. $\lambda_t^{\star}(c_t^{\mathcal{R}}, y^{t+1})$ is the penalty

[5] This strong duality result is more demanding than the weak duality result in (21). Weak duality is typically straightforward to establish under various dual formulations, a familiar example is the Lagrangean duality framework in linear and nonlinear programming. Strong duality places stricter conditions on the original optimization problem. For instance, in nonlinear optimization, strong duality may fail to hold if the feasible set of the original problem is not convex, e.g., Bertsekas (1999).

term, to be determined, which depends on the realization of income next period, and the current choice of consumption under the single-period perfect foresight, $c_t^{\mathcal{R}}$, which is also allowed to dependent on next-period income. Compare the above expression to the Bellman equation of the original consumption choice problem:

$$V_t(w_t) = \sup_{\{c_t:\, c_t \le w_t\}} \frac{(c_t)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\big[V_{t+1}((w_t - c_t)R + \tilde{y}_{t+1})|y^t\big]. \tag{25}$$

The second term on the right in (25) is the standard expectation of V_{t+1} over the possible values of $\tilde{y}_{t+1}$, taking the realizations of income shocks $y_0, \cdots, y_t$ as given. In order for the optimal consumption policy of the relaxed problem to be the same as for the original problem, we set the penalty to equalize expressions in equations (24) and (25):

$$\lambda_t^\star(c_t^{\mathcal{R}}, y^{t+1}) = \beta V_{t+1}((w_t - c_t^{\mathcal{R}})R + y_{t+1}) - \beta \mathbb{E}\big[V_{t+1}((w_t - c_t^{\mathcal{R}})R + \tilde{y}_{t+1})|y^t\big]. \tag{26}$$

Brown et al. (2010) show, using mathematical induction, that the above expression for the ideal penalty is valid in general, and not just for single-period information relaxations (for completeness, we reproduce their proof in the context of our problem in Appendix A.3). Given the current portfolio value of the relaxed problem, $w_t^{\mathcal{R}}$, the ideal penalty is

$$\lambda_t^\star(C^{\mathcal{R}}, y^T) = \beta V_{t+1}((w_t^{\mathcal{R}} - c_t^{\mathcal{R}})R + y_{t+1}) - \beta \mathbb{E}\big[V_{t+1}((w_t^{\mathcal{R}} - c_t^{\mathcal{R}})R + \tilde{y}_{t+1})|y^t\big]. \tag{27}$$

y_{t+1} in the above expression is not a random variable. It is the value of the income shock at time $t+1$ from the particular path of shocks y^T for which we are defining the ideal penalty. In contrast, the labor income shock at time $t+1$, i.e., $\tilde{y}_{t+1}$, is a random variable. The second term on the right in (27) is the conditional expectation of V_{t+1} over the possible values of $\tilde{y}_{t+1}$, taking the realizations of income shocks $y_0, \cdots, y_t$ as given. Thus, the second term depends only on $\left(c^{\mathcal{R}}\right)^t$ and y^t, and so the penalty $\lambda_t^\star$ depends on $\left(c^{\mathcal{R}}\right)^t$ and y^{t+1}. In particular, $\lambda_t^\star(C^{\mathcal{R}}, y^T)$ depends on the contemporaneous consumption choice $c_t^{\mathcal{R}}$ and the future income shock y_{t+1} explicitly, and on the earlier consumption choices $\left(c^{\mathcal{R}}\right)^{t-1}$ and income shocks y^t

implicitly, through $w_t^{\mathcal{R}}$ and the dynamic budget constraint. Going forward, we use more concise notation for the penalty:

$$\lambda_t^\star(C^{\mathcal{R}}, y^T) = \beta\left(V_{t+1}(w_{t+1}^{\mathcal{R}}) - \mathbb{E}_t\big[V_{t+1}(w_{t+1}^{\mathcal{R}})\big]\right) . \tag{28}$$

To visualize how the penalty affects the solution of the relaxed problem, consider the dependence of the ideal penalty on the contemporaneous consumption choice $c_t^{\mathcal{R}}$. The time-1 penalty $\lambda_1^\star$, is shown in Panels A and B of Figure 1. It depends on $c_1^{\mathcal{R}}$, the income in the following period y_2, and current wealth w_1, where w_1 captures the dependence of the penalty on prior consumption choices and income shocks. We plot the penalty for two different levels of current wealth, $w_1 = 2$ and $w_1 = 3$, in Panels A and B, respectively. Each line in these figures shows the penalty as a function of $c_1^{\mathcal{R}}$ and the two possible values of y_2: the solid line corresponds to $y_2 = y_H$, while the dash-dot line corresponds to $y_2 = y_L$.

To see how the penalty discourages the agent from using information about future income, consider a relaxed problem with the agent observing the time-2 income shock in advance and using this information in his time-1 consumption decision. Without the penalty, the agent can take advantage of his knowledge of the future—if the agent knows that next period's income will be high, (i.e., $y_2 = y_H$), it is optimal to choose higher time-1 consumption and save less than if $y_2 = y_L$. Panel C shows that this is the case. For instance, an agent with $w_1 = 2$ borrows to the maximum extent allowed when she knows that next period's income $y_2 = y_H$ but saves 20% of her wealth when $y_2 = y_L$.

The ideal penalty discourages such behavior. We see from panel A of Figure 1 (which corresponds to $w_1 = 2$) that if the agent chooses higher consumption in the $y_2 = y_H$ state relative to the $y_2 = y_L$ state, the expected penalty is positive (shown by the solid line). An ideal penalty has the property that the benefit of perfect foresight is exactly offset by the negative effect of the penalty, and the agent finds it optimal to chose a non-anticipative consumption policy while knowing future realizations of income shocks. As long as the consumption choice is non-anticipating, i.e., $c_1^{\mathcal{R}}$ does not depend on y_2, the expected value of the penalty is zero (shown by the dash line), and the welfare of the agent is not impaired by the penalty.

Next we show how the ideal penalty discourages the agent from conditioning the time-0 consumption choice on knowledge of y_2 to achieve a smoother consumption path. To see this compare panels A and B of Figure 1. From these figures we see that for both realizations of y_2, the gradient of the penalty $\lambda_1^\star$ is larger in absolute value for $w_1 = 2$ than for $w_1 = 3$. Selecting higher c_0 in the $y_2 = y_H$ state relative to the $y_2 = y_L$ state raises the expected penalty term $\lambda_1^\star$, making it positive even if the consumption choice at time 1 is non-anticipative. This illustrates the inter-temporal connections between various penalty terms and consumption choices.

An ideal penalty is as difficult to compute as the solution of the original problem. We therefore define the penalty based on an approximation to the value function:

$$\lambda_t(C^{\mathcal{R}}, y^T) = \beta \left(\widehat{V}^{\mathcal{L}}_{t+1}(w^{\mathcal{R}}_{t+1}) - \mathbb{E}_t \left[\widehat{V}^{\mathcal{L}}_{t+1}(w^{\mathcal{R}}_{t+1}) \right] \right) , \tag{29}$$

where $V_t^{\mathcal{L}}(w_t)$ is the expected utility resulting from the agent's consumption policy choice defined in (22), but for general t. This is a feasible penalty because it satisfies equation (18). However, this penalty choice results in an upward biased estimate of the actual welfare loss of the agent. We see this from Panel A of Figure 2. In this figure, the value function of the relaxed problem $V_0^{\mathcal{R}}(w_0)$ (shown by the dot-dash line), is greater than the value function $V_0(w_0)$ (shown by the dashed line). The difference between the two value functions (i.e., the duality gap) arises from a sub-optimal choice of penalty; had we used the ideal penalty, $\lambda_t^\star$, the two value functions would have coincided. The solid line in this figure is the expected utility of the agent $V_0^{\mathcal{L}}(w_0)$, from adopting the sub-optimal policy. The information relaxation approach provides us with an estimate of the difference $V_0^{\mathcal{R}}(w_0) - V_0^{\mathcal{L}}(w_0)$; this difference is an upper bound on the actual welfare loss $V_0(w_0) - V_0^{\mathcal{L}}(w_0)$ (see equation (23)).

3.1.3 Estimating the upper bound on the agent's welfare loss

We estimate the upper bound on welfare loss from adopting a potentially sub-optimal consumption policy $\widehat{C}$ in three steps. First, we simulate N paths of income shocks. Next, we solve the relaxed problem (19)-(20) along each path y^T using the penalty process (29) and obtain $V_0^{\mathcal{R}}(w, y^T)$ along each path. Finally, we take the sample mean, $V_0^{\mathcal{R}}(w_0)$ over the N paths. In our experiments, we use $N = 100$ paths of income shocks, each of length $T = 100$.[6]

Panel B of Figure 2 shows an estimate of the upper bound on welfare loss of an agent whose consumption policy is sub-optimal because she mistakenly believes that the probability of the high state $\widehat{p} = 0.89$ when the true probability $p = 0.9$. We report the upper bound on welfare loss as a fractional certainty equivalent loss:

$$\overline{\eta}(w_0) = \frac{w_0' - w_0}{w_0'}\,, \tag{30}$$

where w_0' is computed by solving:

$$V_0^{\mathcal{L}}(w_0') = V_0^{\mathcal{R}}(w_0)\,. \tag{31}$$

The numerator on the right-hand side of equation (30) is therefore the additional amount of time-0 wealth, w_0', needed by an agent following a sub-optimal policy to obtain the level of expected utility equal to the value function of the relaxed problem with time-0 wealth equal to w_0.

The solid line in Figure 2 shows an estimate of $\overline{\eta}$ computed using the information relaxation approach. For this simple, partial equilibrium example, since we are able to solve for the actual welfare loss, we plot it in the same figure (the dot-dash line) alongside the upper-bound in panel B of Figure 2. We define the actual welfare loss, η, using the value function $V_0(w)$ instead of $V_0^{\mathcal{R}}(w)$, i.e. w_0' in equation (30) is the root of the equation $\widehat{V}_0(w_0') = V_0(w_0)$ instead

[6]We do not show the 5% and 95% confidence intervals for the estimated upper bound on welfare loss in this example because this estimate is so precise that the confidence band is not visible separately from the estimated mean.

of equation (31). In this example, the agent's welfare loss relative to the optimal policy is less than 0.27% in certainty equivalent terms. Information relaxation bounds the maximum welfare loss at less than 0.3%.

The upper bound offers an accurate approximation to the true welfare loss when the difference between $V_0^{\mathcal{R}}(w)$ and $V_0(w)$ is small. Using a penalty function that is far from the ideal penalty may result in a large difference between the value functions of the relaxed problem and the original problem. To see this, consider as an example a penalty function that is identically zero. In other words, the agent is not penalized for foresight. Note that zero penalty is a feasible choice since it satisfies equation (18). We see from Panel C of Figure 2 that the estimate of the upper bound on welfare loss with a zero penalty is quite inflated. With this sub-optimal penalty choice, information relaxation bounds the maximum welfare loss at less than 65%, whereas the actual welfare loss is less than 0.27%.

Next, we vary the degree of sub-optimality of policies adopted by the agent and we estimate the welfare loss for each of these policies. In particular, we compare the welfare loss to the agent from adopting policies corresponding to $\widehat{p} = 0.87$, $\widehat{p} = 0.88$, and $\widehat{p} = 0.89$ and report both the upper bound (solid line) and the actual welfare loss (dot-dash line) in panels A, B, and C, respectively of Figure 3. These figures show two properties of the upper bound on welfare loss. First, the upper bound declines as the agent's policy approaches the optimal policy. For example, the maximum value of the upper bound on welfare loss is less than 2.3% when $\widehat{p} = 0.87$, while it is less than 0.3% when $\widehat{p} = 0.89$. Second, the difference between $V_0^{\mathcal{R}}(w_0)$ and $V_0(w_0)$ decreases as the agent's policy approaches the optimal policy. Our numerical example illustrates the general continuity result established in Brown et al. (2010), Proposition 2.3(ii): as the penalty function converges to the ideal penalty, the upper bound must converge to the true value function. Intuitively, as the approximate policy gets closer to the optimal policy, the expected utility $V_t^{\mathcal{L}}(w_t)$ improves and approaches the value function $V_t(w_t)$. This, in turn, implies that the penalty λ_t approaches the ideal penalty $\lambda_t^{\star}$, and therefore, the duality gap $V_t^{\mathcal{R}}(w_t) - V_t(w_t)$ shrinks. In other words, the estimated upper bound on welfare loss declines as the agent's policy approaches the optimal policy both because of a reduction in the actual welfare loss and also because of a decline in the duality gap.

3.2 The general algorithm

The general information relaxation approach follows the same logic as the basic example above. We describe how to construct upper and lower bounds on welfare loss from adopting a potentially sub-optimal policy. In this section, we restrict our attention to models with time-additive preferences. The setting with non-time separable preferences introduces a conceptual challenge whose resolution involves additional steps—we therefore postpone its discussion to Section 6.

Original problem. Consider a discrete-time setting $t = 0, \cdots, T$. Let X be the set of possible values for the endogenous state vector that summarizes past information relevant for decision making by the agent (e.g., including wealth or the capital stock). Let Z be the set of possible values for the exogenous shock vector and Q be the transition function that describes its evolution. The exogenous shocks may include both primitive shocks, such as productivity, as well as equilibrium prices which are taken by the agent as given. The structure of the agent's information is described by the filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, ...\mathcal{F}_T\}$, which is an increasing sequence of σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$ for all $t < T$. Consider an agent (e.g., a consumer or a firm) who solves a time-additive problem

$$\sup_{a \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t F\left(x_t, a_t, z_t\right) \right], \tag{32}$$

where x_t and z_t are the time-t values of the endogenous state vector X and the exogenous shock Z, respectively. $F\left(x_t, a_t, z_t\right)$ in (32) is the agent's time-t objective. The agent chooses action $a = (a_0, \cdots, a_T)$ from the set of possible choices Y such that the time-t action $a_t(x_t, z^t)$ depends on x_t and the history of past shocks $z^t = (z_0, \cdots, z_t)$ up to time t. The constraints on these choices (e.g., budget constraint, credit constraints faced by the agent, etc.) are described by the correspondence $\mathcal{A}^{\mathbb{F}} : X \times Z \to X$, where the superscript on $\mathcal{A}$ explicitly indicates that the time-t choices depends only on information available to the agent up to time t. Finally, let the law of motion of the state vector be

$$x_{t+1} = \phi_t \left(x_t, a_t, z_{t+1}\right). \tag{33}$$

The agent's Bellman equation is

$$V_t(x_t, z_t) = \sup_{a_t \in \mathcal{A}(x_t, z^t)} F\left(x_t, a_t, z^t\right) + \beta \mathbb{E}\left[V_{t+1}\left(x_{t+1}, z_{t+1}\right) | \mathcal{F}_t\right], \tag{34}$$

where the expectation operator in (34) is conditional on the time-t information set of the agent.

Next, we describe the steps involved in constructing a lower and upper bound on the welfare loss of an agent at time-0, when the state vector has value x_0, the exogenous shock has value z_0, and the agent adopts a potentially sub-optimal policy $\widehat{a}_t(x_t, z^t)$.

Lower bound. We construct the lower bound on welfare loss $V_0^{\mathcal{L}}(x_0, z_0)$ using a simulation-based approach in three steps:

1. Simulate multiple paths of the exogenous shock z^T according to the given transition function Q.

2. For each path, start with (x_0, z_0) and use the sequence of z_t along this path together with the policy $\widehat{a}_t(x_t, z^t)$ to compute a realized time-series of the state vector x_t using its law of motion in equation (33).

3. Use the time-series of x_t, a_t, and z_t to compute the discounted sum of rewards (32). This is an estimate of $V_0^{\mathcal{L}}(x_0, z_0, z^T)$ along the shock sequence z^T.

Repeat steps (2)—(3) above for all paths of z^T. The sample mean of $V_0^{\mathcal{L}}(x_0, z_0, z^T)$ along all these paths is an estimate of the lower bound of the value function $V_0^{\mathcal{L}}(x_0, z_0)$.

The computation of the lower bound is fast because this step does not involve any optimization. Furthermore, because the computations across paths are independent of each other, we compute them in parallel. This results in a significant reduction of computing time.[7] Also, because the lower bound is computed using a simulation-based approach, it is robust to whether the exogenous shocks have discrete or continuous support.

Upper bound. We construct the upper bound on welfare loss $V_0^{\mathcal{R}}(x_0, z_0)$ in two steps. First, relax the information set of the agent. In the illustrative example of this section, we reveal the entire future sequence of realizations of the exogenous shock to the agent. The advantage of full information relaxation is that the relaxed problem is deterministic. The general information relaxation approach, however, allows for partial information relaxations, where the agent receives some partial information about the future. Let the information set of the relaxed problem for a general information relaxation be the finer filtration $\mathbb{G} = \{\mathcal{G}_0, \mathcal{G}_1, ...\mathcal{G}_T\}$, where $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_T$, and let $y_t^{\mathcal{R}} \in \mathcal{A}^{\mathbb{G}}$ be the set of policies that are adapted to $\mathbb{G}$.

Second, simulate many future paths of the exogenous shock z^T. Along each path, solve the relaxed Bellman equation

$$\begin{aligned} V_t^{\mathcal{R}}(x_t, z_t) = \sup_{a_t^{\mathcal{R}} \in \mathcal{A}^{\mathbb{G}}} & F\left(x_t, a_t^{\mathcal{R}}, z_t\right) + \beta \mathbb{E}\left[V_{t+1}^{\mathcal{R}}\left(x_{t+1}, z_{t+1}\right) | \mathcal{G}_t\right] \\ & - \beta \left(\mathbb{E}\big[\widehat{V}_{t+1}(x_{t+1}, z_{t+1}) | \mathcal{G}_t\big] - \mathbb{E}\big[\widehat{V}_{t+1}(x_{t+1}, z_{t+1}) | \mathcal{F}_t\big]\right), \end{aligned} \tag{35}$$

using backward induction, starting with the terminal condition $V_T^{\mathcal{R}}(x_T, a_T, z_T) = F(x_T, a_T, z_T)$. The last two terms on the right-hand side of (35) represent the penalty for foresight:

$$\lambda_t = \beta \left(\mathbb{E}\big[V_{t+1}^{\mathcal{L}}(x_{t+1}, z_{t+1}) | \mathcal{G}_t\big] - \mathbb{E}\big[V_{t+1}^{\mathcal{L}}(x_{t+1}, z_{t+1}) | \mathcal{F}_t\big]\right), \tag{36}$$

where we have suppressed the dependence of the penalty process on the choice variables and

[7] For example, estimating the lower bound for the illustrative example in Section 3.1 by averaging over 10^4 paths took 1.8 seconds using a single Matlab worker with the computation performed using an Intel 3 GHz processor. Using multiple Matlab workers reduced the computation time in proportion to the number of workers employed. In this example, we used $T = 100$ and approximated the agent's wealth (which is a continuous state variable) by a discrete grid with 40 points.

the exogenous variables to avoid introducing more notation.

In the case of full information relaxation, computing the terms $\mathbb{E}\left[V_{t+1}^{\mathcal{R}}\left(x_{t+1}, z_{t+1}\right)|\mathcal{G}_t\right]$ and $\mathbb{E}\left[\widehat{V}_{t+1}(x_{t+1}, z_{t+1})|\mathcal{G}_t\right]$ are straightforward since they do not involve an expectation operator. Each of these terms is a function of the choice variable $a_t^{\mathcal{R}}$ and the functions are determined using the actual realization of the shock vector z_{t+1} and the law of motion of the state variable equation (33). In the case of partial information relaxation, three terms on the right-hand side of (35) involve an expectation conditional on the information sets $\mathcal{F}_t$ and $\mathcal{G}_t$. These three terms are estimated using the three-step procedure: (i) discretize the choice variable a_t, (ii) simulate many realizations of z_{t+1} (given current z_t) and for each realization of z_{t+1}, compute x_{t+1} for each point on the discrete grid for a_t using (33), and (iii) compute the sample mean of the corresponding value functions $V_{t+1}^{\mathcal{L}}(x_{t+1}, z_{t+1})$ and $\widehat{V}_{t+1}(x_{t+1}, z_{t+1})$ for each grid-point of a_t. Because the conditional expectations are computed using simulations, this approach is robust to whether the exogenous shock has discrete or continuous support. The solution to the relaxed Bellman equation (35) is repeated for all paths of z^T. The sample mean over all paths is an unbiased estimate of the upper bound $V_0^{\mathcal{R}}(x_0, z_0)$.

To get an estimate of the time needed to compute the upper bound, note that the time needed to solve the Bellman equation (35) is generally less than the time needed to solve the Bellman equation of the original problem (32), because of the simplified information structure of the relaxed problem. Let us denote the solution time of (35) by τ_0. Ignoring parallel overhead computational costs, the time required to estimate the upper bound $V^{\mathcal{R}}(x_0, z_0)$ is equal to $\tau_0 N_p/N_{\|}$, where N_p and $N_{\|}$ are the number of sample paths used and the number of parallel workers used in the computation, respectively. Similar to the computation of the lower bound, significant reduction in computation times are obtained by employing a large number of workers $N_{\|}$.

Before we apply the information relaxation approach to specific models, we note that there are two distinct sources of approximation errors involved in our analysis. Various intermediate steps involved in solving the original problem and the relaxed problem under our approach, such as numerical integration, numerical optimization, function interpolation, and, in general, using finite-precision arithmetic for computation, also generate approximation errors. Such steps are carried out using conventional numerical methods, which possess asymptotic performance guarantees. We do not focus on these sources of errors. While relying on asymptotic accuracy results is not ideal, associated issues are not specific to heterogeneous-agent models and, in applications, are typically controlled by refining relevant approximation schemes. Our focus in this paper is instead on specific challenges caused by high dimensionality of the state space of heterogeneous-agent equilibrium models. Approximation methods used to handle this curse of dimensionality typically do not have theoretical performance guarantees, even asymptotically. Our analysis aims to fill this gap on the solution quality by using a numerical algorithm.

4 Application 1: imperfect insurance with aggregate uncertainty

We demonstrate the potential of the information relaxation methodology by computing bounds on the welfare loss of individual agents in the incomplete market model of Krusell and Smith (1998) (henceforth, KS). This model is a canonical example of a model with an infinite dimensional state space. We review the model, the equilibrium concept, and their solution approach briefly. We refer the reader to the original paper for details.

4.1 The model

The KS model is the Aiyagari model (Aiyagari, 1994) with aggregate uncertainty. Time is discrete, $t = (0, 1, \cdots, \infty)$. There is a continuum of agents of unit measure with identical constant relative risk aversion preferences:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{{c_t}^{1-\gamma}}{1-\gamma} \right]. \tag{37}$$

There is a single consumption good produced using a Cobb-Douglas production function:

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}\,, \tag{38}$$

where the capital share parameter, $0 < \alpha < 1$, k and l are capital and labor inputs, respectively, and z is aggregate productivity. All agents are exposed to the aggregate shock z, which takes one of two values $z = \{z_h, z_l\}$ (with $z_h > z_l$) and follows a Markov chain. Each period's output is partly used for consumption and partly added to the next-period capital stock, resulting in the capital accumulation constraint:

$$k_t = (1-\delta)k_{t-1} + y_{t-1} - c_{t-1}\,. \tag{39}$$

where δ is the capital depreciation rate.

Households collect capital rent and labor income each period. Individual labor income is exposed to idiosyncratic employment shocks, ε_t. We assume that each agent supplies $\bar{l}$ units of labor if employed ($\varepsilon_t = 1$), and zero units if unemployed ($\varepsilon_t = 0$). Employment shocks are cross-sectionally independent, conditionally on the aggregate productivity shock. Thus, based on the law of large numbers, the unemployed fraction of the population depends only on the aggregate state. We denote the equilibrium unemployment rate conditional on z_h and z_l by u_h and u_l, respectively. Then the aggregate labor supply in the two states u_h and u_l are given by $L_h = (1-u_h)\bar{l}$ and $L_l = (1-u_l)\bar{l}$, respectively.

We look for a competitive recursive equilibrium. Let $\psi_t(k, \epsilon)$ denote the cross-sectional distribution function at time t, defined over individual capital stock and employment status. Aggregate output depends on the aggregate capital stock, $K_t = \int \psi(k, \epsilon) dk d\epsilon$, and the aggregate supply of labor. Input prices in competitive equilibrium are determined by their marginal product, hence the capital rent r and the wage rate w are given by:

$$r(K_t, L_t, z_t) = \alpha z_t \left(\frac{K_t}{L_t}\right)^{\alpha-1}, \qquad w(K_t, L_t, z_t) = (1-\alpha) z_t \left(\frac{K_t}{L_t}\right)^{\alpha}. \tag{40}$$

Individuals optimize their consumption-investment policies under rational expectations about market prices, i.e., we assume that they correctly forecast the law of motion of the equilibrium cross-sectional distribution of agents, denoted by:

$$\psi_t = \mathcal{H}(\psi_{t-1}, z_{t-1})\,. \tag{41}$$

Thus, optimal individual policies depend on the cross-sectional distribution of capital.

The value function of the agents satisfies the Bellman equation:

$$\begin{aligned} V_t(k_t, \varepsilon_t, z_t, \psi_t) &= \sup_{c_t \geq 0, k_{t+1} \geq 0} \left[\frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \left[V_{t+1}(k_{t+1}, \varepsilon_{t+1}, z_{t+1}, \psi_{t+1}) \right] \right] \\ \text{where} \quad k_{t+1} &= (1 - \delta + r_t) k_t + w_t \bar{l} \varepsilon_t - c_t, \\ \psi_{t+1} &= \mathcal{H}(\psi_t, z_t), \end{aligned} \tag{42}$$

The main difficulty in computing the competitive equilibrium arises because of the dependence of equilibrium prices on the cross-sectional distribution of agents. Thus, to solve for the equilibrium, we must determine the law of motion, $\psi_{t+1} = \mathcal{H}(\psi_t, z_t)$.

Solution approach. To make the problem tractable, KS use a low-dimensional approximation to the infinite-dimensional cross-sectional distribution ψ_t. This approach, introduced in Krusell and Smith (1998), approximates all relevant information about the cross-sectional distribution by its first K moments, $\{m_1, m_2, \cdots, m_K\}$. This results in a low-dimensional fixed-point problem. In particular, in their analysis, KS restrict their attention to the cross-sectional mean m_1. For notational simplicity, from now on, we omit the sub-script in m_1 and denote the distribution's mean simply by m. To speed up computation further, KS posit an approximate law of motion for m that is log-linear:

$$\widehat{\mathcal{H}} : \log m_{t+1} = a^z + b^z \log m_t\,, \qquad z = \{z_g, z_b\}\,. \tag{43}$$

In an approximate equilibrium, the equilibrium dynamics of the cross-sectional distribution of capital in the economy must conform closely to the assumed law of motion, equation (43).

To solve the fixed-point problem, we start with an initial guess for the four constants $\{a^z, b^z\}$. The individual optimization problem, i.e., equation (42), is solved for optimal policies $k_{t+1}(k_t, \epsilon_t, z_t, m_t)$. With these policies, we simulate a long time series of the cross-sectional distribution using a large sample cross-section of agents and compute the time-series of the cross-sectional mean, m.[8] Next, we compute the ordinary least squares regression estimates of $\{a^z, b^z\}$ based on equation (43). We re-solve the individual problem, i.e., equation (42), with these updated estimates of $\{a^z, b^z\}$, and the new optimal policies are used to simulate a new time-series of m. This is used to update $\{a^z, b^z\}$, and this process is continued till the system converges.[9] In solving for the approximate equilibrium, we use parts of the code in Maliar, Maliar, and Valli (2010).

4.2 The relaxed problem

We apply the approach that we described in Section 3 to compute an upper bound on the welfare loss of individual agents by relaxing the information set of the agents. In particular, we start with an initial cross-sectional distribution of capital across agents and draw a sequence of aggregate shocks $z_0, \cdots, z_{T-1}$ and a panel of idiosyncratic employment shocks. Given the aggregate and individual shocks in each period, we use the approximate equilibrium policies of the agents and compute their choice of capital stock for the following period, which, in turn, gives us the cross-sectional mean of the distribution of capital stock in the following period, m_{t+1}. Using equation (40), we compute the prices, i.e., capital returns r and wages w, corresponding to the realized sequence of aggregate shocks. To minimize the gap between the value function of the relaxed problem and the value function of the original problem of

[8]See also, Young (2010), who uses a histogram over the capital grid to obtain the time-series of m, instead of the simulating the path of m using a cross-section. While this approach speeds up the computation, the equilibrium is approximate since agents approximate the cross-sectional distribution by its mean.

[9]We stop when the maximum change in the policy function k_{t+1} between successive iterations is less than 10^{-8}, and the change in the norm of the vector $\{a^z, b^z\}$ between successive iterations is less than 10^{-8}.

the agent, we apply a partial relaxation, revealing only future aggregate shocks but not the agent's idiosyncratic employment shocks.

We define the penalty according to equation (36):

$$\lambda_t = \beta \left(\mathbb{E}_t \big[\widehat{V}_{t+1}(k_{t+1}, \tilde{\varepsilon}_{t+1}, z_{t+1}, m_{t+1}) | \mathcal{G}_t \big] - \mathbb{E}_t \big[\widehat{V}_{t+1}(k_{t+1}, \tilde{\varepsilon}_{t+1}, \tilde{z}_{t+1}, m_{t+1}) | \mathcal{F}_t \big] \right) , \qquad (44)$$

where $\mathcal{G}_t$ denotes the information set of the agent. In equation (44), $\mathcal{G}_t$ contains the following period's realization of the aggregate shock z_{t+1} and the cross-sectional mean of capital m_{t+1}, but not $\tilde{\varepsilon}_{t+1}$. Therefore, we average over possible employed and un-employed future states. Knowledge of the transition probabilities for z and ε allows us to compute both the terms in (44) above explicitly as a function of the decision variables of the relaxed problem, $(k^{\mathcal{R}}_{t+1}, c^{\mathcal{R}}_t)$. Finally, we use the budget constraint to eliminate $k^{\mathcal{R}}_{t+1}$. Along a particular path, the penalty λ_t is then a function of consumption $c^{\mathcal{R}}_t$ only.

4.3 Results

We carry out our baseline analysis using the same parameters as in Krusell and Smith (1998). The preference parameters are $\beta = 0.99$, and $\gamma = 1$. On the production side, the parameters are: the capital share $\alpha = 0.36$, the depreciation rate $\delta = 0.025$, aggregate productivity shocks take values $z_h = 1.01$, $z_l = 0.99$, and the corresponding aggregate unemployment rates are $u_h = 0.04$, $u_l = 0.10$. The transition probability matrix for (z, ε) is in Table 4.

All of our simulation results use a sample cross-section of $N = 10^5$ agents. The length of each sample path T used in solving the relaxed problem must be the same as that used in solving the individual agent's problem equation (42); we choose $T = 10^3$. We average over 500 paths to compute unbiased estimates of the value function of the relaxed problem, $V^{\mathcal{R}}$. In choosing the number of paths, we face a trade-off—using more paths provides more precise estimates of $V^{\mathcal{R}}$, but increases the computational time since the relaxed problem is solved path-by-path.

We use a simulation-based approach to estimate the lower bound $V^{\mathcal{L}}$ under the sub-optimal policy of the agent in the approximate equilibrium. In particular, we simulate 10^5 future paths of aggregate and idiosyncratic shocks. For each path, we compute realized prices r and w, as well as the realized utility. The sample mean of the realized utilities is our estimate of $V^{\mathcal{L}}$.

As in Section 3, we report the upper bound on the welfare loss as a fractional certainty equivalent. The definition is analogous to equation (12):

$$\overline{\eta}(k_0) = \frac{k_0' - k_0}{k_0'}\,, \tag{45}$$

where k_0' is the root of the equation $V^{\mathcal{L}}(k_0') = V_0^{\mathcal{R}}(k_0)$. The numerator of $\overline{\eta}$ is the extra capital needed by an agent following a sub-optimal policy to obtain the level of expected utility equal to the value function of the relaxed problem with initial capital k_0, keeping all other state variables the same.

4.3.1 Baseline results

The welfare loss of an agent depends on the current state: the agent's capital stock, employment status, and the state of the aggregate economy captured by the realization of the aggregate shock and the cross-sectional distribution of capital across the agents. In our baseline results, we report welfare loss for an agent in a typical state of the economy, i.e., where the cross-sectional distribution of capital corresponds to the stochastic steady state. [10]

Figure 4 summarizes the results. Panels A and B correspond to the state in which the agent is unemployed (i.e., $\epsilon = 0$) and employed (i.e., $\epsilon = 1$), respectively. The aggregate

[10] The computational time involved in estimating the fractional certainty loss is in line with our general estimates in Section 3.2. We implemented the Krusell-Smith model using a discrete state space with 6×10^4 grid points—a discrete 60 point grid over capital, a 250 point grid over aggregate capital stock, and two states each for the idiosyncratic and aggregate shocks. Our code was not optimized for computational efficiency, hence only the relative computing times for different elements of the problem are informative. It took close to 6 minutes to solve the Bellman equation of the original problem (42) using un-vectorized code with a single Matlab worker on a 3 GHz processor. It took slightly less time to estimate the upper bound using a single path of aggregate shock realizations. To generate our final results, we used 500 paths and employed 40 Matlab workers. This took about 80 minutes, which is in line with the general estimates of computating times in Section 3.2. It took 10 seconds to estimate the lower bound over a single path. We used 10^5 paths and 40 Matlab workers to obtain the final estimate of the lower bound. This took 7 hours.

shock z is low. We see from both figures that individual welfare losses are small, especially for high levels of initial capital. For example, an agent who is unemployed (Panel A) and has capital stock equal to the bottom 5% of the distribution of capital stock, suffers a welfare loss that is at most 0.13% in fractional certainty equivalent terms. This number drops to 0.04% for an agent whose capital stock is equal to the distribution's median. These numbers are similar for an agent who is currently employed (Panel B). Thus, our results verify that the approximation of Krusell and Smith based on moment truncation, produces an approximate equilibrium in which agents come very close to fully optimizing their objectives when the economy is in a typical initial state.

4.3.2 Alternate calibrations.

Next, we analyze the accuracy of the approximate solution of Krusell and Smith for alternate calibrations of their model. Table 6 shows the results, with panels I and II showing results for the case in which the agent is unemployed and employed, respectively. Rows (2) and (3) show results for a lower value of the time-preference parameter β and a higher value of the volatility of aggregate productivity σ_z, respectively. We see that the upper bound on welfare loss of agents remain low. Row (4) shows results for the case in which agents have a higher risk aversion $\gamma = 2$ compared to $\gamma = 1$ in the baseline. While the upper bound on welfare loss is twice as large as in the baseline calibration, they are still small. For instance, the upper bound on welfare loss of an agent who is unemployed ($\epsilon_0 = 0$) and whose capital stock is at the bottom 10% of the cross-sectional distribution is only 0.22% in fractional certainty equivalent terms (compared to 0.11% in the baseline). Thus, our information relaxation method establishes that the moment truncation approximation generates relatively low individual welfare loss in these cases.

Finally, in rows (5) through (7) we show results for as we vary the number of agents in the economy. Once again, the upper bound on welfare loss remains low.

4.3.3 Transitional dynamics

Next, we consider how accurately the approximate solutions describe the transitional dynamics of the economy following an unanticipated aggregate shock. The transitional dynamics of equilibrium models is often of great interest. Yet the standard solution methods, such as that of KS, are not intended to approximate equilibrium dynamics accurately when the economy is away from its steady state. It is therefore unclear a priory how well such approximation approaches perform under such experiments. This becomes an important issue when drawing conclusions about transitional dynamics of the economy based on approximate numerical solutions. The information relaxation approach is useful in this context because it provides a provable upper bound which quantifies the approximation accuracy of numerical solutions in such experiments.

We illustrate this approach using two examples in which we compute the transitional dynamics of the KS economy following two kinds of unanticipated shocks. In our first experiment, the economy experiences a large transitory shock: a sudden loss of 50% of capital stock of every agent. Such large shocks are considered in the disaster risk literature (see e.g., Gourio 2012). The second shock is a regime change: the economy gradually transitions from its baseline equilibrium to the new stochastic steady state following an unanticipated permanent increase in the volatility of the aggregate shock z. We consider two values for this increase: a two-fold and a five-fold increase in volatility.

Figure 5 shows the result for the scenario in which all agents suddenly lost half of their capital stock. We assume that every agent knows that the economy has experienced the shock which has depleted the aggregate capital stock of the economy to half its value. In other words, they observe the mean m decline. Accordingly, they adopt the policy corresponding to the new value of m, in the period in which the shock is realized. Since all structural parameters of the economy have stayed unchanged, agents rebuild their capital stock over the next several periods. Panels A and B correspond to the welfare loss of an agent who is unemployed and employed, respectively, in the period in which the unanticipated shock arrives. From these two figures we see that the upper bound to the welfare loss is larger relative to the baseline scenario. For example, an agent who is unemployed (Panel A) and has capital stock equal to the bottom 5% of the distribution, suffers a welfare loss that is at most 0.30% in fractional certainty equivalent terms. The corresponding value was 0.13% in the stochastic steady-state. Comparing Panels A and B, we see that the welfare losses are similar in magnitude. Thus, our information relaxation method establishes that, following the shock to capital stock, the moment truncation approximation generates relatively low individual welfare loss.

Next we use the information-relaxation algorithm to quantify how well the Krusell-Smith algorithm performs following a permanent increase in the volatility of the aggregate shock, z. We assume that agents re-optimize and immediately switch to new policies following the regime change. Figure 6 shows the result. Panels A and B correspond to the welfare loss of an agent following a two-fold and five-fold increase in aggregate volatility, respectively. In both cases the agent is initially employed and the aggregate state of the economy is low. Panel A shows that for the two-fold increase in volatility, the upper bound to the welfare loss is not much larger than the baseline scenario. However, panel B shows that the welfare loss is potentially larger by an order of magnitude for the five-fold increase in volatility compared to the welfare loss in the stochastic state. For example, an agent who has capital stock equal to the bottom 5% of the distribution, suffers a welfare loss that is at most 3.0% in fractional certainty equivalent terms. The corresponding value was 0.13% in the stochastic steady-state. Thus, our information relaxation method establishes that the quality of the KS approximation approach continues to be good following an increase in aggregate volatility, unless the shock is extremely large.

5 Application 2: heterogeneous firms with aggregate uncertainty

In this section we apply our method to the general equilibrium model of Khan and Thomas (2008). This model features a heterogeneous cross-section of firms facing non-convex adjustment costs and exposed to persistent aggregate and firm-specific productivity shocks. We use information relaxation to estimate the upper bound to the loss in firm value from following suboptimal investment policies. We briefly outline the model using the notation of Khan and Thomas (2008). We refer the reader to the original paper for more details.

5.1 The model

A continuum of firms of unit mass use a decreasing returns to scale technology with production function:

$$y_t = z_t \varepsilon_t k_t^{\alpha} n_t^{\nu} \,, \tag{46}$$

where $0 < \alpha < 1$ and $\alpha + \nu < 1$. Productivity has an aggregate component z and a firm-specific component ε. Both follow a Markov process, where $z_t \in \{z^1, ..., z^{N_z}\}$ and $P(z_{t+1} = z^j | z_t = z^i) = \pi_{ij}$, $\varepsilon_t \in \{\varepsilon^1, ..., \varepsilon^{N_\varepsilon}\}$ and $P(\varepsilon_{t+1} = \varepsilon^j | \varepsilon_t = \varepsilon^i) = \pi_{ij}^{\varepsilon}$. The firm hires labor n_t in period t after observing that period's productivity. The capital accumulation constraint is:

$$\gamma k_{t+1} = (1 - \delta) k_t + i_t \,, \tag{47}$$

where i_t is investment, δ is the depreciation rate of capital, and γ is a constant which captures the growth rate of labor-augmenting technological progress. Firms face non-convex adjustment cost of capital: there are no adjustment costs if investment is within a small range $i_t \in [ak_t, bk_t]$, where the parameters $a \leq 0 \leq b$. However, if investment is outside this range, then the adjustment cost is equal to $\xi_t \omega_t$, where ω_t is the real wage rate and ξ_t is a uniformly distributed random variable $\xi \sim \mathcal{U}[0, \bar{\xi}]$ that is independent across firms and over time.

Each period a firm chooses labor and investment to maximize its net present value:

$$\begin{aligned}
V^1(\varepsilon_t, k_t, \xi_t; z_t, \mu_t) &= \sup_{i_t \in \mathbb{R},\, n_t \geq 0} \Bigg[\left(z_t \varepsilon_t k_t^\alpha n_t^\nu - \omega_t n_t - i_t - \omega_t \xi_t \, I\{i_t \notin [a k_t, b k_t]\} \right) p_t \\
&+ \beta \sum_{i=1}^{N_z} \pi_{ij} \sum_{l=1}^{N_\varepsilon} \pi_{lm}^\varepsilon V(\varepsilon^m, k_{t+1}; z^j, \mu_{t+1}) \Bigg] \\
\mu_{t+1} &= \mathcal{H}(\mu_t, z_t) \\
V(\varepsilon_t, k_t; z_t, \mu_t) &= \int_0^{\bar{\xi}} \bar{\xi}^{-1} \, V^1(\varepsilon_t, k_t, \xi; z_t, \mu_t) \, d\,\xi \,, \qquad (48)
\end{aligned}$$

where $I\{\}$ is the indicator function, $V^1(\varepsilon_t, k_t, \xi_t; z_t, \mu_t)$ is the present value of a firm with idiosyncratic productivity ε_t and realized adjustment cost ξ_t, $V(\varepsilon_t, k_t; z_t, \mu_t)$ is the present value of the firm prior to the realization of the adjustment cost ξ_t, p_t is the price at which current output is valued, and μ is the cross-sectional distribution over individual firms' capital stock and idiosyncratic shocks. The presence of non-convex adjustment costs and persistent differences in firm-productivity lead to non-linear firm policies. This prevents aggregation of the cross-section of firms into a representative firm. Equilibrium prices, therefore, depend on the cross-sectional distribution of capital stock and idiosyncratic shocks μ_t of all firms in the economy, in addition to current aggregate productivity z_t. Firms optimize under rational expectations about market prices, i.e., we assume that firms correctly forecast the equilibrium law of motion of the distribution, $\mu_{t+1} = \mathcal{H}(\mu_t, z_t)$.

The model is closed by assuming that a representative household owns all firms in the economy. The household maximizes expected lifetime utility over consumption C_t and labor L_t:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \right] = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t (\log C_t - \varphi L_t) \right]. \qquad (49)$$

An exact solution of equilibrium policies is infeasible in this model because of the dependence of prices on the cross-sectional distribution of capital μ. Khan and Thomas (2008) adopt the solution approach of Krusell and Smith (1998) and approximate the cross-sectional distribution by its mean. To solve the model, they make two more approximations. First, they assume that the mean m follows a log-linear law of motion:

$$\widehat{H} : \log m_{t+1} = a^z + b^z \log m_t \,, \qquad z = \{z^1, ..., z^{N_z}\} \,. \tag{50}$$

Second, they assume that the dependence of the price p_t on the mean takes the form:

$$\log \widehat{p}_t = a_p^z + b_p^z \log m_t \,, \qquad z = \{z^1, ..., z^{N_z}\} \,. \tag{51}$$

The solution algorithm iteratively solves for optimal firm policies and the constants $\{a^z, b^z, a_p^z, b_p^z\}_{z=1}^{N_z}$ such that the decision rules of individual firms are consistent with the aggregate savings and leisure decisions of the representative household. For exact details see Khan and Thomas (2008).

5.2 The relaxed problem

To compute the value function $V_0^{\mathcal{R}}(\varepsilon, k; z_0, \mu_0)$ of the relaxed problem for a particular initial state of the economy (z_0, μ_0), we simulate paths of the economy starting from this state. The relaxed information set contains the realization of all future aggregate productivity shocks; however, it does not include future realizations of idiosyncratic productivity and adjustment costs shocks. We define the penalty in the relaxed problem according to equation (36):

$$\begin{aligned} \lambda_t &= \beta \left(\mathbb{E}_t \big[\widehat{V}_{t+1}(\varepsilon_{t+1}, k_{t+1}; z_{t+1}, \widehat{\mu}_{t+1}) | \mathcal{G}_t \big] - \mathbb{E}_t \big[\widehat{V}_{t+1}(\varepsilon_{t+1}, k_{t+1}; z_t', \widehat{\mu}_{t+1}) | \mathcal{F}_t \big] \right) \\ &= \beta \left(\sum_{m=1}^{N_\varepsilon} \pi_{lm}^{\varepsilon} \widehat{V}(\varepsilon^l, k_{t+1}; z_{t+1}, m_{t+1}) - \sum_{j=1}^{N_z} \pi_{ij} \sum_{m=1}^{N_\varepsilon} \pi_{lm}^{\varepsilon} \widehat{V}(\varepsilon^l, k_{t+1}; z^i, m_{t+1}) \right) . \end{aligned} \tag{52}$$

Along a particular path $z_0, z_1, \cdots, z_T$ of the aggregate shock z, the time-t penalty λ_t is therefore a function of the control $k_{t+1}^{\mathcal{R}}$. We solve for the optimal control of this relaxed problem and obtain a single realization of the value function of the relaxed problem. We repeat this over many simulated paths of aggregate shocks; the sample mean over these paths is our estimate of $V_0^{\mathcal{R}}(\varepsilon, k; z_0, \mu_0)$.

5.3 Results

We carry out our baseline analysis using the same parameters as in Khan and Thomas (2008) (see Table 5 of this paper). We choose the number of states for the aggregate and idiosyncratic shocks equal $N_z = 11$ and $N_\varepsilon = 15$, respectively, as in their original paper. All of our simulation results use a sample cross-section of $N = 10^5$ firms. Sample paths are $T = 10^3$ long, and we average over 15000 paths to compute unbiased estimates of the value function of the relaxed problem, $V^{\mathcal{R}}(\varepsilon, k; z_0, \psi_0)$. Similar to our approach for the KS model in Section 4, we use a simulation-based approach to estimate the expected utility, $\widehat{V}(\varepsilon, k; z_0, \psi_0)$, under the sub-optimal policy of the firm in the approximate equilibrium. In particular, we simulate 10^5 future paths of aggregate and idiosyncratic shocks, and compute the realized utility along each path. The sample mean of the realized utilities is our estimate of $\widehat{V}$. Since there are no optimizations involved, this step is fast (the overall distribution of computing times is very similar to that reported in footnote 10).

As in Section 3, we report the welfare loss as a fractional certainty equivalent. The definition is analogous to equation (30):

$$\eta = \frac{k_0' - k_0}{k_0'}\,, \tag{53}$$

where k_0' is the root of the equation $\widehat{V}_0(k_0') = V_0^{\mathcal{R}}(k_0)$. The numerator of η in equation (53), is the extra capital needed by a firm following a sub-optimal policy to obtain the level of expected utility equal to the value function of the relaxed problem with initial capital k_0, keeping all other state variables the same.

Baseline Results

The loss in firm value depends on the aggregate state of the economy, i.e. the realization of the aggregate shock and the cross-sectional distribution μ, as well as the firm's current state, i.e. the capital stock k and the idiosyncratic shock ϵ. In our baseline results, we report the loss in firm value in a typical state of the economy, i.e., where the cross-sectional distribution corresponds to the stochastic steady state. In Figure 7 we present upper bounds on the loss in firm value in three different idiosyncratic productivity states: the lowest state $\varepsilon = \varepsilon^1$ in Panel A, the middle state $\varepsilon = \varepsilon^8$ in Panel B, and the highest state $\varepsilon = \varepsilon^{15}$ in Panel C, all as functions of the firm's capital stock, k.

We see from Figure 7 that information relaxation bounds the welfare losses from following sub-optimal policies in the near-rational equilibrium to be economically negligible. For example, comparing across Panels A through C, we see that the upper bound is largest in Panel A, corresponding to the state $\varepsilon = \varepsilon^1$. Even in this state, a firm with capital stock equal to the bottom 5% of the distribution suffers a welfare loss that is at most 0.02% in fractional certainty equivalent terms. In the same figure we see that a firm with capital stock equal to the median of the distribution suffers a welfare loss that is less than 0.01%. For higher firm-specific shocks, the upper bound is even smaller. In Panel C which corresponds to $\varepsilon = \varepsilon^{15}$, for example, the loss is essentially zero. In sum, this shows that firms do not incur significant loss by following suboptimal policies under circumstances they typically encounter in the near-rational equilibrium.

Transitional Dynamics

Next, we test the efficiency of the suboptimal policies by considering transitional dynamics of the economy away from the stochastic steady-state. As in Section 4, we consider two unanticipated shocks. In our first experiment, the economy experiences a large transitory shock: a sudden destruction of 50% of capital stock of every firm in the economy. The second shock is a regime change: the economy gradually transitions from its baseline equilibrium to the new stochastic steady state following an unanticipated permanent increase in the volatility of the aggregate shock z. We consider two shock sizes—a two-fold increase and also an extremely large shock corresponding to a five-fold increase in aggregate volatility.

Figure 8 shows the upper bound on welfare loss for the unanticipated destruction of 50%

of capital stock of firms. We see from these figures that the upper bound is now larger than in the stochastic steady-state. For example, in Panel A which corresponds to the state $\varepsilon = \varepsilon^1$, a firm with capital stock equal to the bottom 5% of the distribution suffers a welfare loss that is bounded at 1% in fractional certainty equivalent terms. The corresponding value in the stochastic steady-state was 0.02%. The potential losses to firm value are even larger in Panels B and C of Figure 8.

Figure 9 shows the upper bound on welfare loss for the unanticipated increase in the volatility of aggregate productivity. Panel A corresponds to a doubling of aggregate volatility; we see that the upper bound is similar to the stochastic steady-state in this case. Panel B corresponds to a five-fold increase in aggregate volatility. We see from this figures that the upper bound is much larger than in the stochastic steady-state. For example, a firm with capital stock equal to the bottom 5% of the distribution suffers a welfare loss which is bounded from above by our information relaxation approach at 0.5% in fractional certainty equivalent terms. The corresponding value in the stochastic steady-state is about an order of magnitude smaller at 0.07%. The results of these two transitional dynamics experiments therefore indicate potentially large losses in firm value in the Khan and Thomas economy when the economy experiences very large, unanticipated shocks.

6 Non-separable preferences

Macrofinance models often feature non-time separable preferences, such as Epstein-Zin (henceforth "EZ") preferences. However, the information relaxation approach, in the form discussed so far, cannot bound welfare loss of agents in settings where agents have non-time separable preferences because such agents care about the timing of resolution of uncertainty (e.g., late versus early resolution of uncertainty). Put differently, their utility depends directly on the structure of the information set. In this section, we extend the information relaxation approach to settings featuring agents with EZ preferences. We do so by utilizing the variational utility formulation of Geoffard (1996) and Dumas et al. (2000), which converts the problem with non-time separable preferences into a time-separable problem.

6.1 The main idea

We illustrate our approach using the example of the consumption-savings problem discussed in Section 3, except, for simplicity, we assume that the agent cannot borrow. The common formulation of EZ preferences uses a non-linear time aggregator to define an agent's objective recursively:

$$J_t(w_t) = \sup_{\{C:c_t \le w_t\}} \frac{1}{\gamma}\left[c_t^{\rho} + \beta\big(\gamma \mathbb{E}_t[J_{t+1}(w_{t+1})]\big)^{\frac{\rho}{\gamma}}\right]^{\frac{\gamma}{\rho}}, \tag{54}$$

where w_t is the agent's wealth at time t. In equation (54), β is the time-preference parameter, the parameter ρ is related to the elasticity of inter-temporal substitution (EIS) by EIS $= 1/(1-\rho)$, and the parameter γ is related to the coefficient of relative risk-aversion (RRA) by RRA $= 1-\gamma$. The agent's consumption and saving decision is subject to the budget constraint

$$w_{t+1} = (w_t - c_t)R + y_{t+1}\,. \tag{55}$$

The variational utility formulation of EZ preferences by Geoffard (1996) and Dumas et al. (2000) converts (54) to a time-separable problem by introducing an endogenous discount rate process. The agent's problem under their formulation takes one of two forms, depending on whether the agent prefers early or late resolution of uncertainty. We discuss each case separately since the approaches to construct upper and lower bounds are different.

6.1.1 Early resolution of uncertainty

Under the variational utility formulation, an agent who prefers early resolution of uncertainty (i.e., who has $\rho > \gamma$) solves the time-separable problem

$$V(w_0,\theta_0) = \sup_{\{C:c_t \le w_t\}} \sup_{\nu_t \in \Upsilon} \mathbb{E}_0\left[\sum_{t=0}^{T} \theta_t f(c_t,\nu_t)\right], \tag{56}$$

where the *felicity function* $f(c, \nu)$ is given by

$$f(c_t, \nu_t) = \frac{c_t^\gamma}{\gamma}\left[1 - \beta^{-\frac{\gamma}{\rho-\gamma}}\left(1-\nu_t\right)^{\frac{\rho}{\rho-\gamma}}\right]^{\frac{\rho-\gamma}{\rho}}. \tag{57}$$

In equation (56), ν_t is a stochastic discount rate process constrained to lie in the region Υ such that $f(c_t, \nu_t)$ is real-valued. If $\rho > 0$, then $\Upsilon = \{\nu_t | 1 - \beta^{\gamma/\rho} \leq \nu_t \leq 1\}$, while if $\rho < 0$, then $\Upsilon = \{\nu_t | -\infty < \nu_t \leq 1 - \beta^{\gamma/\rho}\}$. Finally, θ_t in (56) follows

$$\theta_{t+1} = \theta_t(1-\nu_t). \tag{58}$$

The time-t value function of the above problem, $V(w_t, \theta_t)$, is linear in θ_t. Linearity implies that the optimal c_t and ν_t do not depend on θ_t, and the Bellman equation for (56) is

$$j_t(w_t) = \sup_{\{C: c_t \leq w_t\}} \sup_{\nu_t \in \Upsilon} f\left(c_t, \nu_t\right) + (1-\nu_t)\mathbb{E}_t\left[j_{t+1}(w_{t+1})\right], \tag{59}$$

where $j_t(w_t) = V_t(w_t, \theta_t)/\theta_t$. Equations (54) and (59) imply identical optimal consumption policies; furthermore, $J_t(w_t) = j_t(w_t)$.

Upper bound. Because (56) is of the max-max form, we construct the upper bound on welfare loss of an agent who adopts a potentially sub-optimal consumption policy $\widehat{c}_t(w_t)$ by following the same logic as in Section 3. The only difference here is that there are now two controls, c_t and ν_t, instead of just c_t in the setting with time-separable preferences. To operationalize the construction of the upper bound on the agent's value function, we first relax the information set by fully revealing the entire sequence of income shock realizations y^T along each path. Next, we solve the relaxed problem path-by-path, after applying a penalty $\lambda_t(C^{\mathcal{R}}, \nu^{\mathcal{R}}, y^T)$ for foresight. The penalty λ_t must satisfy $\mathbb{E}_0\left[\lambda_t\left(C^{\mathcal{R}}, \nu^{\mathcal{R}}, y^T\right)\right] \leq 0$, where $C^{\mathcal{R}} = \left(c_0^{\mathcal{R}}(y^T), c_1^{\mathcal{R}}(y^T), ..., c_T^{\mathcal{R}}(y^T)\right)$ and $\nu^{\mathcal{R}} = \left(\nu_0^{\mathcal{R}}(y^T), \nu_1^{\mathcal{R}}(y^T), ..., \nu_T^{\mathcal{R}}(y^T)\right)$ denote feasible consumption and discount rate policies of the relaxed problem. The value function of the relaxed problem is

$$V_0^{\mathcal{R}}(w_0, \theta_0) = \sup_{\{C^{\mathcal{R}}: c_t^{\mathcal{R}} \leq w_t^{\mathcal{R}}\}} \sup_{\nu^{\mathcal{R}} \in \Upsilon} \mathbb{E}_0 \left[\sum_{t=0}^{T} \theta_t^{\mathcal{R}} f\left(c_t^{\mathcal{R}}, \nu_t^{\mathcal{R}}\right) - \lambda_t(C^{\mathcal{R}}, \nu^{\mathcal{R}}, y^T) \right], \tag{60}$$

where $\theta_{t+1}^{\mathcal{R}} = \theta_t^{\mathcal{R}}(1 - \nu_t^{\mathcal{R}})$.

We solve (60) on a path-by-path basis for multiple paths and take the sample mean of the value functions over all paths to obtain an unbiased estimate of $V_0^{\mathcal{R}}(w_0, \theta_0)$. This sample mean of the relaxed problem's value function is an upper bound on the value function of the original problem (56):

$$V_0^{\mathcal{R}}(w_0, \theta_0) \geq V_0(w_0, \theta_0). \tag{61}$$

The reason is that the policies $C_t^\star$ and $\nu_t^\star$ that are optimal under the agent's original problem (56), are also feasible policies for the relaxed problem (60), and the expected penalty under these policies adds a non-negative term to the agent's expected utility. Dividing both sides of (61) by θ_0, noting that $V_0^{\mathcal{R}}(w_0, \theta_0)$ is linear in θ_0, and using $V_0(w_0, \theta_0)/\theta_0 = J_0(w_0)$, we get $V_0^{\mathcal{R}}(w_0, 1) \geq J_0(w_0)$.

While a large class of penalties are feasible, we choose the time-t penalty in (60) based on the expression for the ideal penalty, $\lambda_t(c_t^{\mathcal{R}}, \nu_t^{\mathcal{R}}, y^{t+1}) = \theta_{t+1}\left(\widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}}) - \mathbb{E}_t\left[\widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}})\right]\right)$, where $\widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}})$ is an approximate value function for (54). The Bellman equation for the relaxed problem (60) is then

$$j_t^{\mathcal{R}}(w_t^{\mathcal{R}}) = \sup_{\{c_t^{\mathcal{R}}: c_t^{\mathcal{R}} \leq w_t^{\mathcal{R}}\}} \sup_{\nu_t^{\mathcal{R}} \in \Upsilon} f\left(c_t^{\mathcal{R}}, \nu_t^{\mathcal{R}}\right) + \left(1 - \nu_t^{\mathcal{R}}\right)\left(j_{t+1}^{\mathcal{R}}(w_{t+1}^{\mathcal{R}}) - \widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}}) + \mathbb{E}_t\left[\widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}})\right]\right), \tag{62}$$

where $w_{t+1}^{\mathcal{R}}$ satisfies the budget constraint

$$w_{t+1}^{\mathcal{R}} = (w_t^{\mathcal{R}} - c_t^{\mathcal{R}})R + y_{t+1}. \tag{63}$$

We solve (62) by backward induction, starting with the terminal condition $j_T^{\mathcal{R}}(w_T^{\mathcal{R}}) = \left(w_T^{\mathcal{R}}\right)^\gamma/\gamma$.

Lower bound. To construct the lower bound for the agent's value function, we start with the right-hand side of the Bellman equation (59), but, instead of optimally choosing the consumption policy and the discount rate process, we use the potentially sub-optimal policy $\widehat{c}_t(w_t)$. Any choice of a discount rate process $\widehat{\nu}_t \in \Upsilon$ that is adapted to the agent's information structure results in a valid lower bound. We compute the lower bound $j_t^{\mathcal{L}}(w_t)$ as

$$j_t^{\mathcal{L}}(w_t) = f\left(\widehat{c}_t, \widehat{\nu}_t\right) + (1 - \widehat{\nu}_t)\mathbb{E}_t\left[j_{t+1}^{\mathcal{L}}(w_{t+1})\right], \tag{64}$$

with $\widehat{\nu}_t$ defined as

$$\widehat{\nu}_t(w_t) = 1 - \beta\left(\gamma \widehat{u}_t(w_t)\right)^{\frac{\rho-\gamma}{\gamma}}\left[\widehat{c}_t^{\rho}(w_t) + \beta(\gamma \widehat{u}_t(w_t))^{\frac{\rho}{\gamma}}\right]^{\frac{\gamma-\rho}{\rho}}, \tag{65}$$

where $\widehat{u}_t(w_t) = \mathbb{E}_t\left[\widehat{j}_{t+1}(w_{t+1})\right]$, and $\widehat{j}_{t+1}(w_{t+1})$ is the approximate value function of the original problem (54). Our choice for $\widehat{\nu}_t$ is motivated by the fact that the optimal discount rate for the original problem (59) has the same form as (65), with $\widehat{u}_t(w_t)$ replaced by $u_t(w_t) = \mathbb{E}_t\left[j_{t+1}(w_{t+1})\right]$. The function $j_0^{\mathcal{L}}(w_0)$ is a lower bound for the original problem because both the consumption policy $\widehat{c}_t$ and the discount rate process $\widehat{\nu}_t$ are potentially sub-optimal.

Panel A of Figure 10 shows the result of a numerical example where we compute the upper bound of certainty equivalent loss $\overline{\eta}$ from adopting policies with varying degrees of sub-optimality. While the true probability of a high income shock y_H is $p = 0.9$, the solid line corresponds to the optimal policy where the agent believes this probability to be $\widehat{p} = 0.89$ while the dot-dash line shows results for $\widehat{p} = 0.87$. The values of other parameters are: high and low income realization levels $y_H = 4$ and $y_L = 1$, respectively, the risk-free rate $R = 1.02$, the agent's time preference parameter $\beta = 0.9$, RRA$= 2$, EIS $= 1.5$, and the agent's horizon $T = 100$. We use 100 paths in computing the upper bound. From this figure, we see that the upper bound on welfare loss of the agent is greater for a more sub-optimal policy: $\overline{\eta} = 0.2\%$

for an agent with wealth $w = 10$ who believes $\widehat{p} = 0.87$, while $\overline{\eta} = 0.1\%$ for an agent who has the same wealth, but who believes $\widehat{p} = 0.89$.

6.1.2 Late resolution of uncertainty

If the agent prefers late resolution of uncertainty (i.e., $\rho < \gamma$ for this agent) then the variational utility formulation of (54) is represented as a solution to the time-separable problem[11]

$$V(w_0, \theta_0) = \sup_{\{C: c_t \leq w_t\}} \inf_{\nu_t \in \Upsilon} \mathbb{E}_0 \left[\sum_{t=0}^{T} \theta_t f(c_t, \nu_t) \right], \tag{66}$$

where $f(c_t, \nu_t)$ is given by (57) and $\nu_t \in \Upsilon$. If $\rho > 0$, then $\Upsilon = \{\nu_t : -\infty < \nu_t \leq 1 - \beta^{\gamma/\rho}\}$, while if $\rho < 0$, then $\Upsilon = \{\nu_t : 1 - \beta^{\gamma/\rho} \leq \nu_t \leq 1\}$.

Upper bound. We construct an upper bound to the value function $V_0^{\mathcal{R}}(w_0)$ using the following two-step process: (1) use any discount rate process $\widehat{\nu}_t \in \Upsilon$ that is adapted to the agent's information structure and (2) perform information relaxation on the problem $\sup_{\{C: c_t \leq w_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{T} \theta_t f(c_t, \widehat{\nu}_t) \right]$. In step (1), we choose $\widehat{\nu}_t$ to be

$$\widehat{\nu}_t(w_t, c_t) = 1 - \beta \left(\gamma \widehat{u}_t(w_t)\right)^{\frac{\rho-\gamma}{\gamma}} \left[c_t^{\rho} + \beta (\gamma \widehat{u}_t(w_t))^{\frac{\rho}{\gamma}} \right]^{\frac{\gamma-\rho}{\rho}}, \tag{67}$$

where $\widehat{u}_t(w_t) = \mathbb{E}_t \left[\widehat{j}_{t+1}(w_{t+1}) \right]$. Our choice is motivated by the fact that the optimal discount rate for the original problem (66) has the same form as (67), with $\widehat{u}_t(w_t)$ replaced by $u_t(w_t) = \mathbb{E}_t \left[j_{t+1}(w_{t+1}) \right]$.

The second step (i.e., information relaxation) is identical to that discussed in Section 3. Specifically, we reveal the entire sequence of income shock realizations y^T along each path and solve the relaxed problem path-by-path, after applying a penalty $\lambda_t(C^{\mathcal{R}}, y^T)$ for foresight.

[11] Agents with $\rho = \gamma$ have CRRA preferences and are indifferent about the timing of resolution of uncertainty. We discussed this case in Section 3.

The penalty must satisfy $\mathbb{E}_0\left[\lambda_t\left(C^{\mathcal{R}}, y^T\right)\right] \leq 0$. The value function of the relaxed problem is

$$V_0^{\mathcal{R}}(w_0, \theta_0) = \sup_{\{C^{\mathcal{R}}:\, c_t^{\mathcal{R}} \leq w_t^{\mathcal{R}}\}} \mathbb{E}_0\left[\sum_{t=0}^{T} \widehat{\theta}_t f\left(c_t^{\mathcal{R}}, \widehat{\nu}_t(w_t, c_t^{\mathcal{R}})\right) - \lambda_t(C^{\mathcal{R}}, y^T)\right], \tag{68}$$

where $\widehat{\theta}_{t+1} = \widehat{\theta}_t\left(1 - \widehat{\nu}_t(w_t, c_t)\right)$.

The Bellman equation for (68) is

$$j_t^{\mathcal{R}}(w_t) = \sup_{\{c_t^{\mathcal{R}}:\, c_t^{\mathcal{R}} \leq w_t\}} f\left(c_t^{\mathcal{R}}, \widehat{\nu}_t(w_t, c_t)\right) + \left(1 - \widehat{\nu}_t(w_t, c_t)\right)\left(j_{t+1}^{\mathcal{R}}(w_{t+1}^{\mathcal{R}}) - \lambda_t(c_t^{\mathcal{R}}, y^{t+1})\right), \tag{69}$$

where the penalty $\lambda_t(c_t^{\mathcal{R}}, y^{t+1}) = \widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}}) - \mathbb{E}_t\left[\widehat{j}_{t+1}(w_{t+1}^{\mathcal{R}})\right]$ and $w_{t+1}^{\mathcal{R}} = (w_t - c_t^{\mathcal{R}})R + y_{t+1}$. Finally, we average over many paths to obtain an unbiased estimate of $V_0^{\mathcal{R}}(w_0, \theta_0)$. We prove in Appendix B.1, that

$$j_0^{\mathcal{R}}(w_0) \geq J_0(w_0)\,. \tag{70}$$

Lower bound. Constructing the lower bound in this case is not as straightforward as the construction in Section 6.1.1. Once again, the idea of information relaxation turns out to be useful in constructing a lower bound. In this construction, we fix a potentially sub-optimal consumption policy $\widehat{c}_t$ and relax the agent's information set in (66) to solve the relaxed problem

$$V^{\mathcal{L}}(w_0, \theta_0) = \inf_{\nu_t^{\mathcal{L}} \in \Upsilon} \mathbb{E}_0\left[\sum_{t=0}^{T} \theta_t f(\widehat{c}_t, \nu_t^{\mathcal{L}}) + \lambda_t^{\mathcal{L}}(\nu^{\mathcal{L}}, y^T)\right], \tag{71}$$

where the penalty $\lambda_t^{\mathcal{L}}(\nu^{\mathcal{L}}, y^T)$ in (71) must satisfy $\mathbb{E}_0\left[\lambda_t^{\mathcal{L}}\left(\nu^{\mathcal{L}}, y^T\right)\right] \leq 0$. We prove in Appendix B.2, that

$$V_0^{\mathcal{L}}(w_0, \theta_0) \leq V_0(w_0, \theta_0). \tag{72}$$

Dividing both sides of the inequality (72) by θ_0, using linearity of the value functions with respect to θ_0, and using $V_0(w_0, \theta_0)/\theta_0 = J(w_0)$, we get $V_0^{\mathcal{L}}(w_0, 1) \leq J_0(w_0)$.

We choose the penalty $\lambda_t^{\mathcal{L}}(\nu_t^{\mathcal{L}}, y^{t+1}) = (1 - \nu_t^{\mathcal{L}})\left(\mathbb{E}_t\left[\widehat{j}_{t+1}(w_{t+1}^{\mathcal{L}})\right] - \widehat{j}_{t+1}(w_{t+1}^{\mathcal{L}})\right)$. Panel B of Figure 10 shows the result of a numerical example with identical parameters as those used in panel A, except that now the agent has EIS = 0.4 (this value of the EIS implies that the agent prefers late resolution of uncertainty because the agent's RRA = 2). From the figure we see the same pattern as panel A: the upper bound on welfare loss of the agent is greater for a more sub-optimal policy.

7 Conclusion

Our analysis shows that information relaxation techniques can be effectively used to establish the accuracy of approximate solutions for equilibria in heterogeneous-agent models. This methodology is general, easy to implement, and has a wide range of potential applications beyond the scope of this paper. For instance, information relaxation could be used to evaluate the accuracy of solutions obtained using perturbation techniques. The latter approach is widely used for DSGE models (for a recent application, see Mertens and Judd (2012) and Mertens (2011)) because of its ability to handle models with high-dimensional state vectors, and is supported by the computational software Dynare. More recently, Boppart, Krusell, and Mitman (2018) and Auclert, Bardóczy, Rognlie, and Straub (2020) introduce an efficient method to solve for the approximate equilibrium of a large class of heterogenous agent models using a first-order perturbation around the stochastic steady-state. They assume that the equilibrium is well approximated as a linear system in the space of perfect-foresight shock sequences of finite length. Our method can be used to guarantee the quality of the linearization assumption when their method is applied to solve for the equilibrium of a particular model. Yet another natural application of our approach is to evaluating welfare loss resulting from heuristic policies motivated by behavioral biases of the agents.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics 109*(3), 659 – 684.

Akerlof, G. A. and J. L. Yellen (1985). Can small deviations from rationality make significant differences to economic equilibria? *The American Economic Review 75*, 708 – 720.

Andersen, L. and M. Broadie (2004). Primal-dual simulation algorithm for pricing multidimensional American options. *Management Science 50*(9), 1222 – 1234.

Auclert, A., B. Bardóczy, M. Rognlie, and L. Straub (2020). Using the sequence-space jacobian to solve and estimate heterogeneous-agent models. Working Paper 26123, National Bureau of Economic Research.

Back, K. and S. R. Pliska (1987). The shadow price of information in continuous time decision problems. *Stochastics 22*(2), 151 – 186.

Bertsekas, D. (1999). *Nonlinear Programming.* Athena Scientific.

Bismut, J. M. (1973). Conjugate convex functions in optimal stochastic control. *Journal of Math. Anal. Appl. 44*, 384 – 404.

Boppart, T., P. Krusell, and K. Mitman (2018). Exploiting mit shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control 89*, 68–92.

Brown, D., J. E. Smith, and P. Sung (2010). Information relaxations and duality in stochastic dynamic programs. *Operations Research 58*(4), 785 – 801.

Davis, M. H. A. and I. Karatzas (1994). A deterministic approach to optimal stopping. In F. P. Kelly (Ed.), *Probability, Statistics and Optimisation*, pp. 455–466. John Wiley & Sons Ltd.

den Haan, W. J. (2010). Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents. *Journal of Economic Dynamics and Control 34*(1), 79 – 99.

den Haan, W. J. and A. Marcet (1994). Accuracy in simulations. *Review of Economic Studies 61*(1), 3 – 17.

Dumas, B., R. Uppal, and T. Wang (2000). Efficient intertemporal allocations with recursive utility. *Journal of Economic Theory 93*, 240 – 259.

Geoffard, P. (1996). Discounting and optimizing: Capital accumulation problems as variational minmax problems. *Journal of Economic Theory 69*, 53 – 70.

Gourio, F. (2012, May). Disaster risk and business cycles. *American Economic Review 102*(6), 2734–66.

Guvenen, F. (2011). Macroeconomics with heterogeneity: A practical guide. Working Paper 17622, National Bureau of Economic Research.

Hassan, T. A. and T. M. Mertens (2011, May). The social cost of near-rational investment. NBER Working Papers 17027, National Bureau of Economic Research, Inc.

Haugh, M. B. and L. Kogan (2004). Pricing American options: A duality approach. *Operations Research 52*(2), 258 – 270.

Heathcote, J., K. Storesletten, and G. L. Violante (2009). Quantitative macroeconomics with heterogeneous households. *Annual Review of Economics 1*(1), 319–354.

Jones, S. R. G. and J. H. Stock (1987, March). Demand disturbances and aggregate fluctuations: The implications of near rationality. *Economic Journal 97*(385), 49–64.

Judd, K. L., L. Maliar, and S. Maliar (2017). Lower bounds on approximation errors to numerical solutions of dynamic economic models. *Econometrica 85*(3), 991 – 1012.

Khan, A. and J. K. Thomas (2008). Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics. *Econometrica 76*(2), 395–436.

Krusell, P. and A. A. Smith (1996). Rules of thumb in macroeconomic equilibrium: A quantitative analysis. *Journal of Economic Dynamics and Control 20*, 527–558.

Krusell, P. and A. A. Smith (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867 – 896.

Kubler, F. and K. Schmedders (2003). Stationary equilibria in asset-pricing models with incomplete markets and collateral. *Econometrica 71*(6), 1767–1793.

Kubler, F. and K. Schmedders (2005). Approximate versus exact equilibria in dynamic economies. *Econometrica 73*(4), 1205–1235.

Maliar, L., S. Maliar, and F. Valli (2010). Solving the incomplete markets model with aggregate uncertainty using the krusell-smith algorithm. *Journal of Economic Dynamics and Control 34*(1), 42–49.

Mertens, T. M. (2011). Excessively volatile stock markets: Equilibrium computation and policy analysis. *Working Paper*.

Mertens, T. M. and K. L. Judd (2012). Equilibrium existence and approximation for incomplete market models with substantial heterogeneity. *Working Paper*.

Naish, H. F. (1993, January). The near optimality of adaptive expectations. *Journal of Economic Behavior & Organization 20*(1), 3–22.

Pliska, S. R. (1982). Duality theory for some stochastic control models. *Lecture Notes in Control and Information Sciences 43*(2), 329 – 337.

Rockafellar, R. T. and R. J. B. Wets (1976). Nonanticipativity and l^1 martingales in stochastic optimization problems. *Mathematical Programming Studies 6*, 170 – 187.

Rogers, L. C. G. (2002). Monte Carlo valuation of American options. *Mathematical Finance 12*(3), 271 – 286.

Rogers, L. C. G. (2007). Pathwise stochastic optimal control. *SIAM J. Control Optim. 46*(3), 1116–1132.

Santos, M. S. (2000). Accuracy of numerical solutions using the euler equation residuals. *Econometrica 68*(6), 1377–1402.

Simon, H. A. (1978). Rationality as process and as product of thought. *The American Economic Review 68*, 1 – 16.

Young, E. R. (2010). Solving the incomplete markets model with aggregate uncertainty using the krusell–smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics and Control 34*(1), 36 – 41.

Appendix

A Proofs related to Section 2

A.1 Optimal policies

The investor's problem (1) implies the Bellman equation:

$$V\left(w_t, X_t\right) = \sup_{c_t > 0, \phi_t} \log c_t + \beta \mathbb{E}_t V\left(w_{t+1}, X_{t+1}\right), \tag{A.1}$$

where $V\left(w_t, X_t\right)$ is the investor's value function, w_t is the time-t wealth of the investor, and the investor's wealth evolves according to equation (2). The value function satisfies

$$V\left(w_t, X_t\right) = \frac{1}{1-\beta} \log w_t + v(X_t). \tag{A.2}$$

Equations (A.1) and (A.2), together with the budget constraint (2), imply

$$V\left(w_t, X_t\right) = \sup_{c_t > 0, \phi_t} \log c_t + \beta \mathbb{E}_t \left(\frac{1}{1-\beta} \log\left(w_t - c_t\right) + \log\left(\phi_t R^S_{t+1} + (1-\phi_t) R^B\right) + v(X_{t+1}) \right). \tag{A.3}$$

The first order conditions for optimal consumption (4) and optimal portfolio choice (5) follow from (A.3).

A.2 Derivation of expected utility loss equation (14)

The optimal and suboptimal consumption policies in Section 2.1 both have the form

$$c_t = (1-\beta) w_t. \tag{A.4}$$

That is $c_t = c_t^\star = (1-\beta) w_t$ for the optimal policy and $c_t = \widehat{c}_t = (1-\beta) w_t$ for the suboptimal policy. This implies that the time-t consumption c_t is given by

$$c_t = (1-\beta) w_0 \prod_{s=0}^{t-1} \frac{w_{s+1}}{w_s}. \tag{A.5}$$

In the setting of Section 2.1, since stock returns are independent and identically distributed, both the optimal and the suboptimal portfolio are time-invariant. Let us denote the share of the investor's wealth in the stock for both the optimal and the suboptimal portfolio choices by ϕ (i.e., $\phi = \phi^\star$ for the optimal portfolio and $\phi = \widehat{\phi}$ for the suboptimal portfolio). Combining the budget constraint (2) with the consumption policy (A.4), the return on wealth under both the optimal and the suboptimal policies is given by

$$\frac{w_{s+1}}{w_s} = \beta \left(\phi R^S_{s+1} + (1-\phi) R^B\right). \tag{A.6}$$

Combining equations (A.5) and (A.6), taking logs we get

$$\log c_t = \log(1-\beta) + \log w_0 + t \log \beta + \sum_{s=0}^{t-1} \log\left(\phi R^S_{s+1} + (1-\phi) R^B\right). \tag{A.7}$$

Equation (A.7) implies that

$$\sum_{t=0}^{\infty}\beta^t \log c_t = \frac{\log(1-\beta)}{1-\beta} + \frac{\log w_0}{1-\beta} + \frac{\beta\log\beta}{(1-\beta)^2} + \sum_{t=0}^{\infty}\beta^t \sum_{s=0}^{t-1}\log\left(\phi R^S_{s+1} + (1-\phi)R^B\right) \tag{A.8}$$

where we used the identities $\sum_{t=0}^{\infty}\beta^t = 1/(1-\beta)$ and $\sum_{t=0}^{\infty} t\beta^t = \beta/(1-\beta)^2$. Finally, taking the time-0 expectation of both sides of equation (A.8), and using the definition of the certainty equivalent of the one-period return on wealth equation (6), we get

$$\begin{aligned}
\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \log c_t\right] &= \frac{\log(1-\beta)}{1-\beta} + \frac{\log w_0}{1-\beta} + \frac{\beta\log\beta}{(1-\beta)^2} + \sum_{t=0}^{\infty}\beta^t t B(\phi), \\
&= \frac{\log(1-\beta)}{1-\beta} + \frac{\log w_0}{1-\beta} + \frac{\beta\log\beta}{(1-\beta)^2} + \frac{\beta}{(1-\beta)^2}B(\phi),
\end{aligned} \tag{A.9}$$

where we used the identity $\sum_{t=0}^{\infty}\beta^t t = \beta/(1-\beta)^2$ in going from the first to the second equality.

Equation (A.9) implies that the expected utility from adopting the optimal consumption and portfolio policy is

$$U\left(\phi^\star, (c_t^\star)_{t\geq 0}; w_0, X_0\right) = \frac{\log w_0}{1-\beta} + \frac{\log(1-\beta)}{1-\beta} + \frac{\beta\log\beta}{(1-\beta)^2} + B(\phi^\star)\frac{\beta}{(1-\beta)^2}\,, \tag{A.10}$$

where $\phi^\star$ is given by equation (9). Similarly, the expected utility under the suboptimal policy is

$$U\left(\widehat{\phi}, (\widehat{c}_t)_{t\geq 0}; \widehat{W}_0, X_0\right) = \frac{\log w_0}{1-\beta} + \frac{\log(1-\beta)}{1-\beta} + \frac{\beta\log\beta}{(1-\beta)^2} + B(\widehat{\phi})\frac{\beta}{(1-\beta)^2}\,, \tag{A.11}$$

where $\widehat{\phi}$ is given by equation (13). Note that the true distribution of returns is used in computing $B(\widehat{\phi})$ in equation (A.11). Using the definition of the fractional certainty equivalent welfare loss equation (12), we get that the fractional certainty equivalent loss from adopting the suboptimal policy is

$$\eta = \exp\left(\frac{\beta}{1-\beta}\left(B(\phi^\star) - B(\widehat{\phi})\right)\right) - 1.$$

A.3 The ideal penalty

Proposition 1 (Ideal penalty) *Let $V_t(w_t)$ and $V_t^{\mathcal{R}}(w_t^{\mathcal{R}})$ be the time-t value functions of the original problem* (15) *and the relaxed problem* (19)*, respectively. Then,*

$$V_0(w_0) = V_0^{\mathcal{R}}(w_0), \tag{A.12}$$

if the penalty process λ_t in equation (19) *is chosen to be equal to $\lambda_t^\star$, where*

$$\lambda_t^\star(C^{\mathcal{R}}, y^T) = \beta \left(V_{t+1}((w_t^{\mathcal{R}} - c_t^{\mathcal{R}})R + y_{t+1}) - \mathbb{E}\big[V_{t+1}((w_t^{\mathcal{R}} - c_t^{\mathcal{R}})R + \tilde{y}_{t+1})|y^t\big]\right). \tag{A.13}$$

Proof. *We prove Proposition 1 by adapting the proof in Brown et al. (2010) in the context of the problem in Section 3. As in that section, we assume complete information relaxation.*

The Bellman equation for the original problem is

$$V_t(w_t) = \sup_{c_t:\, 0<c_t\leq(1+b)w_t} \frac{c_t^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}\left[V_{t+1}(w_{t+1})|y^t\right], \tag{A.14}$$

where w_{t+1} is given by equation (16)*. The Bellman equation for the relaxed problem is*

$$V_t^{\mathcal{R}}(w_t^{\mathcal{R}}) = \sup_{c_t^{\mathcal{R}}:\, 0<c_t^{\mathcal{R}}\leq(1+b)w_t^{\mathcal{R}}} \frac{\left(c_t^{\mathcal{R}}\right)^{1-\gamma}}{1-\gamma} - \lambda_t^\star(C^{\mathcal{R}}, y^T) + \beta V_{t+1}^{\mathcal{R}}(w_{t+1}^{\mathcal{R}}) \tag{A.15}$$

where $w_{t+1}^{\mathcal{R}}$ and $\lambda_t^\star(C^{\mathcal{R}}, y^T)$ are given by equations (20) *and* (A.13)*, respectively.*

The penalty (A.13) *is feasible since it trivially satisfies the condition* (18)*. The proof of equation* (A.12) *follows from using mathematical induction. To see this, note that the terminal value functions of the original and relaxed problems are both identical and equal to zero, that is, $V_{T+1}(w_{T+1}) = 0$ and $V_{T+1}^{\mathcal{R}}(w_{T+1}^{\mathcal{R}}) = 0$. For the inductive step, we show that if the value functions are the same function of wealth at time $t+1$, that is, if*

$$V_{t+1}(w_{t+1}) = V_{t+1}^{\mathcal{R}}(w_{t+1}), \tag{A.16}$$

then the time-t value functions for the original and relaxed problems must also be the same function of wealth: $V_t(w_t) = V_{t+1}^{\mathcal{R}}(w_t)$. Indeed, this follows from noting that the right-hand side of equation (A.15) *reduces to the right-hand side of* (A.14)*, after using equation* (A.16) *and the expression for the ideal penalty* (A.13) *in* (A.15). ■

B Proof related to Section 6

B.1 Proof of Equation (70)

For the upper bound, we start with the variational formulation of the original problem equation (66).

Proof.

$$
\begin{aligned}
V_0(w_0,\theta_0) &= \sup_{\{C:c_t\le w_t\}} \inf_{\nu_t\in\Upsilon} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t f(c_t,\nu_t)\Big], \quad \theta_{t+1}=\theta_t(1-\nu_t) \\
&= \sup_{\{C:c_t\le w_t\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t^* f(c_t,\nu_t^*(c_t))\Big], \quad \theta_{t+1}^*=\theta_t^*(1-\nu_t^*(c_t)) \\
&\le \sup_{\{C:c_t\le w_t\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\widehat{\theta}_t(c_t) f(c_t,\widehat{\nu}_t(c_t))\Big], \quad \widehat{\theta}_{t+1}=\widehat{\theta}_t\,(1-\widehat{\nu}_t(c_t)) \\
&\le \sup_{\{C:c_t\le w_t\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\widehat{\theta}_t(c_t)\Big\{f(c_t,\widehat{\nu}_t(c_t))-\lambda_t(c_t,y^T)\Big\}\Big], \quad \widehat{\theta}_{t+1}=\widehat{\theta}_t\,(1-\widehat{\nu}_t(c_t)) \\
&\le \sup_{\{C^{\mathcal{R}}:c_t^{\mathcal{R}}\le w_t\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\widehat{\theta}_t(c_t^{\mathcal{R}})\Big\{f(c_t,\widehat{\nu}_t(c_t^{\mathcal{R}}))-\lambda_t(c_t^{\mathcal{R}},y^T)\Big\}\Big], \quad \widehat{\theta}_{t+1}=\widehat{\theta}_t\,(1-\widehat{\nu}_t(c_t)) \\
&= V^{\mathcal{R}}(w_0,\theta_0).
\end{aligned}
$$

The first inequality arises from a sub-optimal choice of the discount rate process $\widehat{\nu}_t(c_t)$, the second because of the super-martingale property of λ_t, and the final one because the choice set $C^{\mathcal{R}}$ is weakly greater than C since consumption choices C are functions of past histories of income shocks, while consumption choices $c^{\mathcal{R}}$ are conditioned on the entire (including the future) sequence of shocks.

Finally, dividing both sides of the inequality $V_0(w_0,\theta_0)\le V^{\mathcal{R}}(w_0,\theta_0)$ by θ_0, using linearity of the value functions with respect to θ_0, and using the definition $J(w_0)=V_0(w_0,\theta_0)/\theta_0$, we get the inequality (70). ■

B.2 Proof of Equation (72)

For the lower bound, we start with the expression for the lower bound equation (71):

Proof.

$$
\begin{aligned}
V^{\mathcal{L}}(w_0,\theta_0) &\equiv \inf_{\{\nu^{\mathcal{L}}:\nu_t^{\mathcal{L}}\in\Upsilon\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t^{\mathcal{L}} f(\widehat{c}_t,\nu_t^{\mathcal{L}})+\lambda_t(\nu^{\mathcal{L}},y^T)\Big], \quad \theta_{t+1}^{\mathcal{L}}=\theta_t^{\mathcal{L}}(1-\nu_t^{\mathcal{L}}) \\
&\le \inf_{\{\nu:\nu_t\in\Upsilon\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t f(\widehat{c}_t,\nu_t)+\lambda_t(\nu,y^T)\Big], \quad \theta_{t+1}=\theta_t(1-\nu_t) \\
&\le \sup_{\{C:c_t\le w_t\}} \inf_{\{\nu:\nu_t\in\Upsilon\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t f(c_t,\nu_t)+\lambda_t(\nu,y^T)\Big], \quad \theta_{t+1}=\theta_t(1-\nu_t) \\
&\le \sup_{\{C:c_t\le w_t\}} \inf_{\{\nu:\nu_t\in\Upsilon\}} \mathbb{E}_0\Big[\sum_{t=0}^{T}\theta_t f(c_t,\nu_t)\Big], \quad \theta_{t+1}=\theta_t(1-\nu_t) \\
&= V_0(w_0,\theta_0)\,.
\end{aligned}
$$

The first inequality arises because the choice set $\nu^{\mathcal{L}}$ is weakly greater than ν since discount rate choices ν are functions of past histories of income shocks, while $\nu^{\mathcal{L}}$ are conditioned on the entire (including the future) sequence of shocks. The second inequality arises because of the suboptimality of the consumption policy $\widehat{c}_t$. The third inequality arises because λ_t satisfies $\mathbb{E}_0\left[\lambda_t\left(\nu^{\mathcal{R}}, y^T\right)\right] \leq 0$.

■

Table 1: Transition matrix for the portfolio choice problem in Section 2.2.

A. Objective probabilities

	X_1	X_2	X_3
X_1	0.495	0.495	0.01
X_2	0.495	0.495	0.01
X_3	0.45	0.45	0.1

B. Subjective probabilities

	X_1	X_2	X_3
X_1	0.495	0.495	0.01
X_2	0.495	0.495	0.01
X_3	0.495	0.495	0.01

Table 2: Welfare loss and Euler equation errors for the portfolio choice problem in Section 2.2 for two different lengths of time series T.

A. $T = 6000$

	X_1	X_2	X_3
Fractional C.E. loss η	4.09	4.09	8.39
Lower 5%, Bond	4.76	5.31	4.59
Upper 5%, Bond	5.14	4.8	6.49
Lower 5%, Stock	4.76	5.31	4.59
Upper 5%, Stock	5.14	4.8	6.49

B. $T = 100,000$

	X_1	X_2	X_3
Fractional C.E. loss η	4.09	4.09	8.39
Lower 5%, Bond	4.12	4.54	1.02
Upper 5%, Bond	5.31	3.71	43.12
Lower 5%, Stock	4.12	4.54	1.02
Upper 5%, Stock	5.31	3.71	43.12

Table 3: Parameter values used in the consumption-saving problem of Section 3.

Parameter	Symbol	Value
Probability of high state	p	0.9
Labor income, high state	y_H	4
Labor income, low state	y_L	1
Maximum loan-to-wealth ratio	b	0.2
Risk-free interest rate	R	1.02
Agent's time-preference parameter	β	0.9
Agent's risk aversion	γ	5
Horizon	T	100

Table 4: Transition matrix for the model of Krusell and Smith in Section 4.

z, ϵ/z', ϵ'	$(z_b, 0)$	$(z_b, 1)$	$(z_g, 0)$	$(z_g, 1)$
$(z_b, 0)$	0.525000	0.350000	0.031250	0.093750
$(z_b, 1)$	0.038889	0.836111	0.002083	0.122917
$(z_g, 0)$	0.093750	0.031250	0.291667	0.583333
$(z_g, 1)$	0.009115	0.115885	0.024306	0.850694

Table 5: Baseline parameters of the Khan and Thomas (2008) model.

γ	β	δ	α	ν	ϕ	ρ_z	σ_{η_z}	ρ_ε	$\sigma_{\eta_\varepsilon}$	b	$\bar{\xi}$
1.016	0.977	0.069	0.256	0.640	2.400	0.859	0.014	0.859	0.022	0.011	0.0083

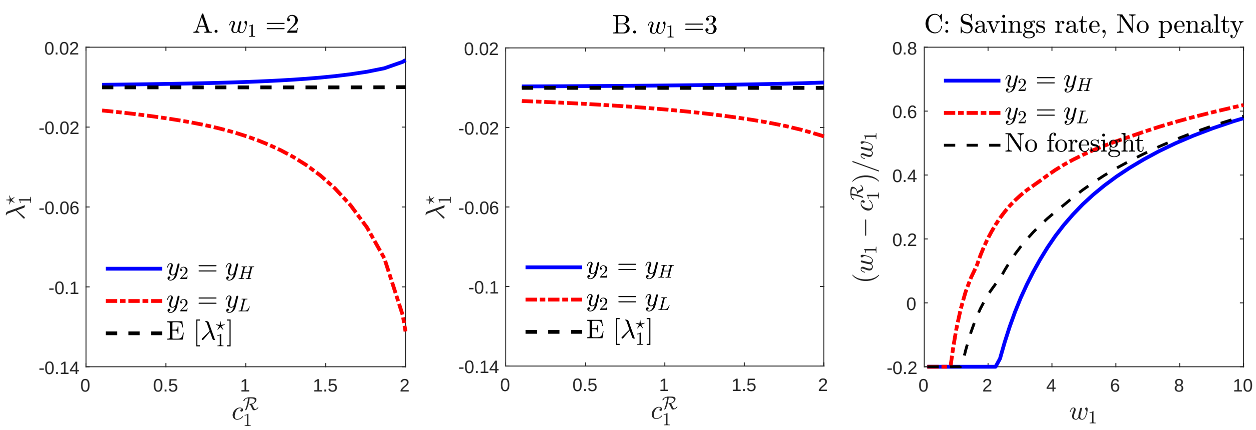

Figure 1: **Ideal penalty.** Panels A and B show the ideal penalty $\lambda_1^\star$ for the consumption-saving problem of Section 3 plotted as a function of time-1 consumption choice $c_1^{\mathcal{R}}$. The agent believes the probability of y_H is 0.89 when the true probability is $p = 0.9$. Panel A corresponds to time-1 wealth $w_1 = 2$, while Panel B corresponds to $w_1 = 3$. The solid and dash-dot lines are the penalty functions when the time-2 income shock is $y_2 = y_H$ and $y_2 = y_L$, respectively. The dash line shows the expected value of the penalty over the two possible realizations of y_2. This expectation is identically equal to zero. Panel C shows the agent's savings rate for the two possible realizations of time-2 income shocks (that are known to the agent). The dashed line shows this fraction without foresight. All other parameters are in Table 3.

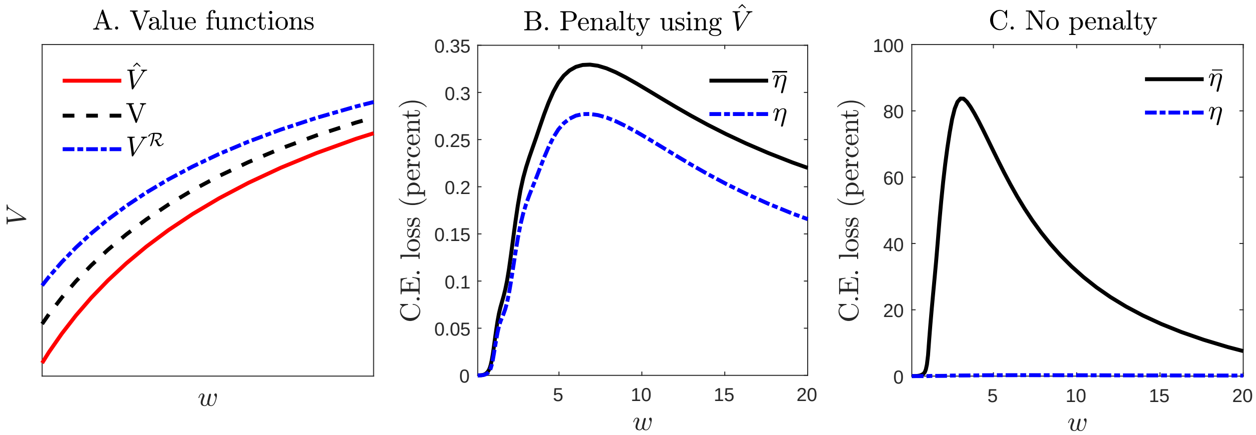

Figure 2: **Welfare loss, upper bound.** Panel A is an illustrative figure which shows the relative ordering of the upper bound $V^{\mathcal{R}}$, the agent's value function under the optimal policy V, and an unbiased estimate of the agent's expected utility from adopting sub-optimal policies $\widehat{V}$. While the true probability p of a high income shock, y_H is 0.9, the sub-optimal policies correspond to the optimal policy for realization of y_H that equals $\widehat{p} = 0.89$. The solid line in Panel B shows the upper bound of certainty equivalent loss from adopting the sub-optimal policy, while the dot-dash line shows the actual certainty equivalent loss. The upper bound is computed using $\widehat{V}$. Panel C shows the upper bound and the actual certainty equivalent loss with no penalty for foresight. We use the parameters shown in Table 3.

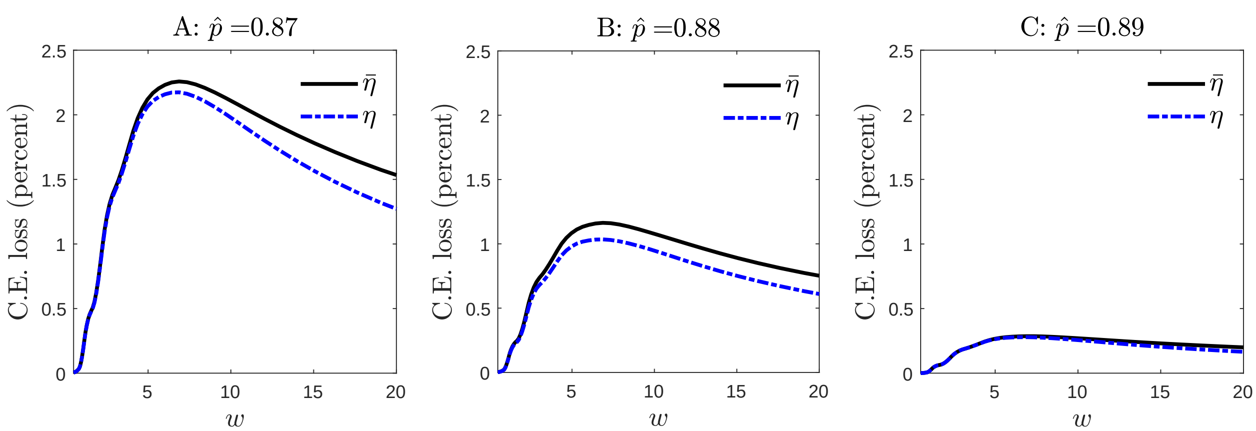

Figure 3: **Certainty equivalent loss for different sub-optimal policies.** Panels A, B, and C show the certainty equivalent loss from adopting policies with varying degrees of sub-optimality. While the true probability p of a high income shock, y_H is 0.9, the sub-optimal policies in panels A, B, and C correspond to the optimal policy where the agent's belief for realization of y_H equals $\widehat{p} = 0.87$, $\widehat{p} = 0.88$, and $\widehat{p} = 0.89$, respectively. We use the parameters shown in Table 3.

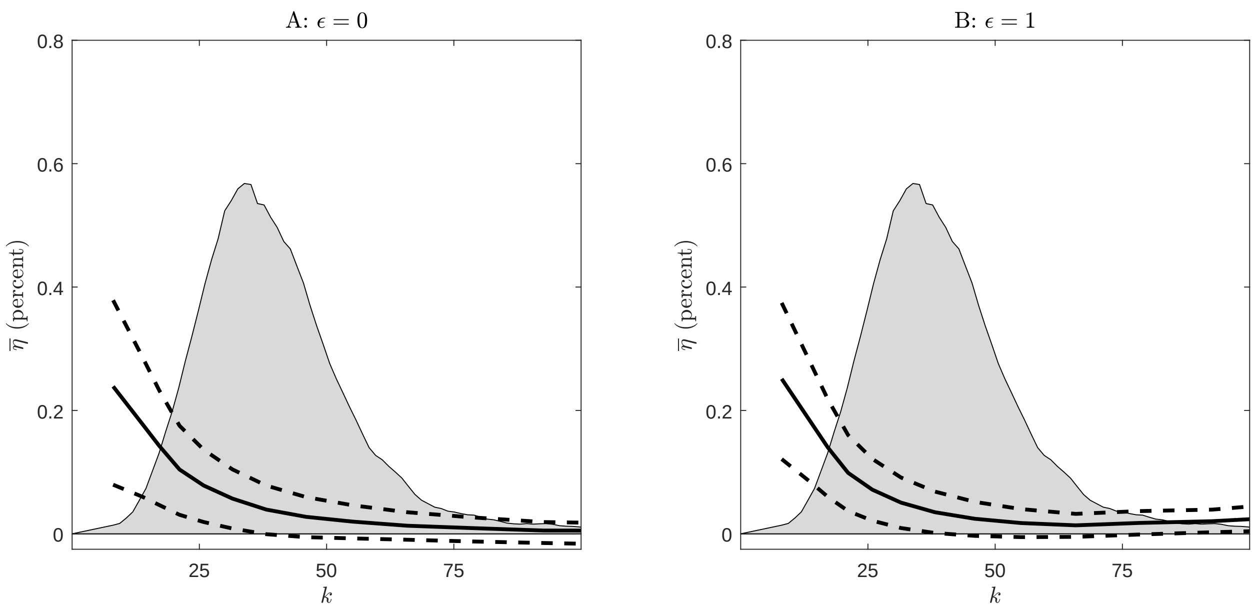

Figure 4: **Upper bound on welfare loss: Model of Krusell and Smith, stochastic steady-state.** Panels A and B show the upper bound of an agent's welfare loss as a function of his capital stock, k, when he is unemployed and employed, respectively. The upper bound on welfare loss is measured as a fractional certainty equivalent loss $\overline{\eta}$, which is defined in equation (45). The aggregate state of the economy is low. The value function of the relaxed problem is estimated by averaging over 500 paths of aggregate shocks. The shaded area shows the cross-sectional distribution of capital and corresponds to the stochastic steady-state distribution. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Krusell and Smith (1998).

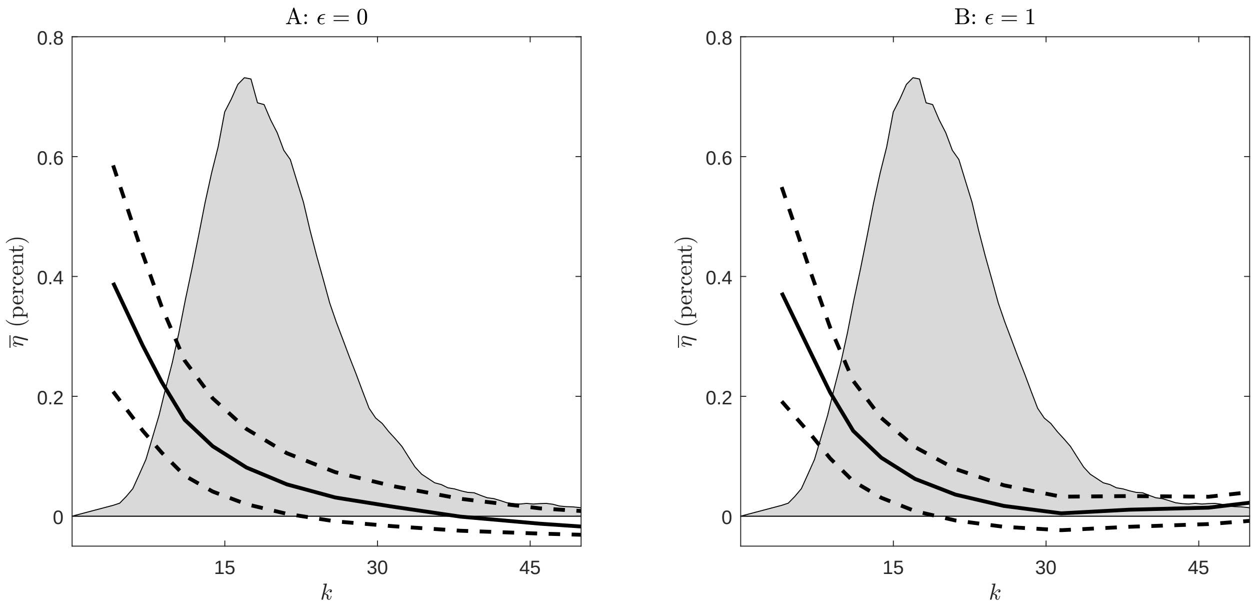

Figure 5: **Upper bound on welfare loss: Model of Krusell and Smith, transitional dynamics following 50% destruction of capital stock of all agents.** Panels A and B show the upper bound of an agent's welfare loss as a function of his capital stock, k, when he is unemployed and employed, respectively. The upper bound on welfare loss is measured as a fractional certainty equivalent loss $\overline{\eta}$, which is defined in equation (45). The aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital after the permanent shock is realized. The value function of the relaxed problem is estimated by averaging over 500 paths of aggregate shocks. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Krusell and Smith (1998).

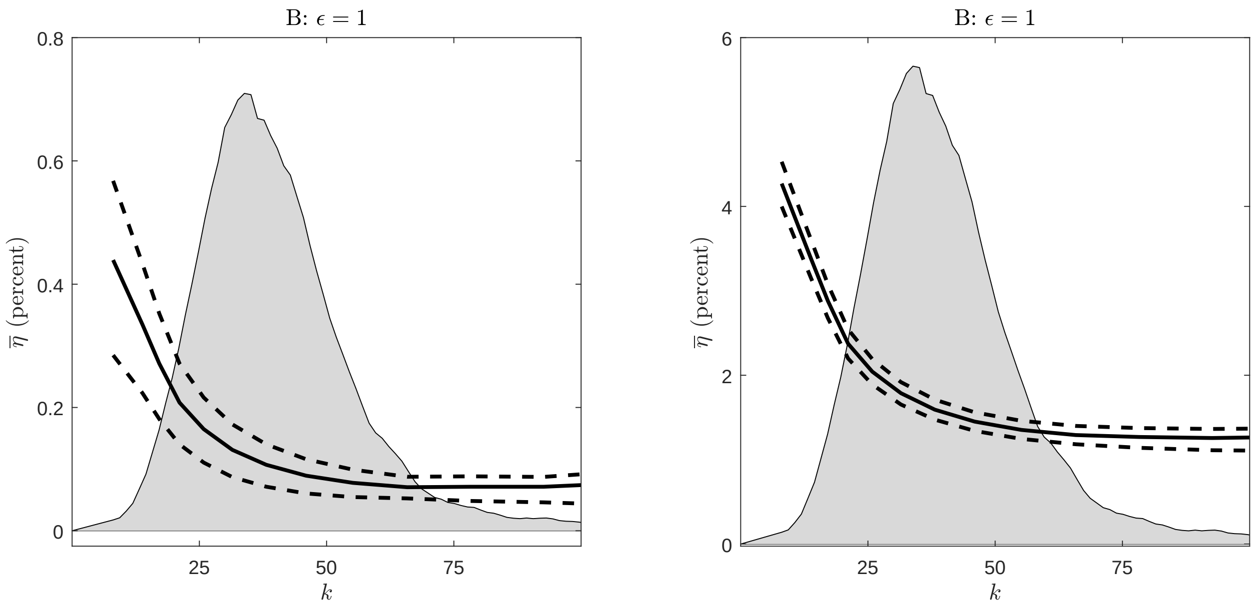

Figure 6: **Upper bound on welfare loss: Model of Krusell and Smith, transitional dynamics following a permanent increase in aggregate volatility.** Panels A and B show the upper bound of an agent's welfare loss as a function of his capital stock, k, following a two-fold and five-fold increase in aggregate volatility, respectively. In both cases the agent is initially employed and the aggregate state of the economy is low. The upper bound on welfare loss $\overline{\eta}$ is measured as a fractional certainty equivalent loss which is defined in equation (45). The area under the shaded curve shows the cross-sectional distribution of capital in the stochastic steady-state. The value function of the relaxed problem is estimated by averaging over 500 paths of aggregate shocks. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Krusell and Smith (1998).

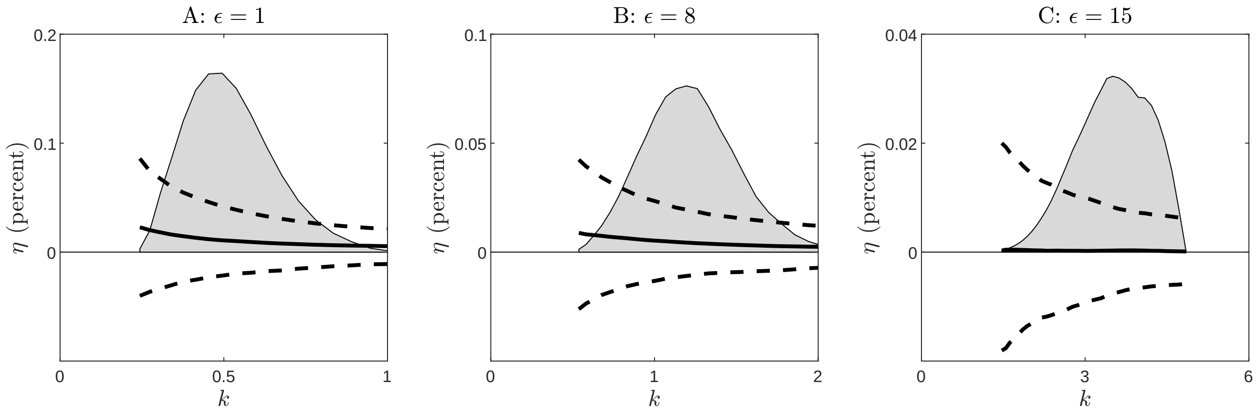

Figure 7: **Upper bound on welfare loss: Model of Khan and Thomas, stochastic steady-state.** Panels A, B, and C show the upper bound of loss in firm value as a function of the firm's capital stock, k, for three different idiosyncratic shocks, $\epsilon = 1$, $\epsilon = 8$, and $\epsilon = 15$, respectively. The loss is measured as a fractional certainty equivalent loss η, which is defined in equation (53). The aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital in the stochastic steady-state, conditional on ϵ. The value function of the relaxed problem is estimated by averaging over 15000 paths of aggregate shocks. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Khan and Thomas (2008) and are also reported in Table 5.

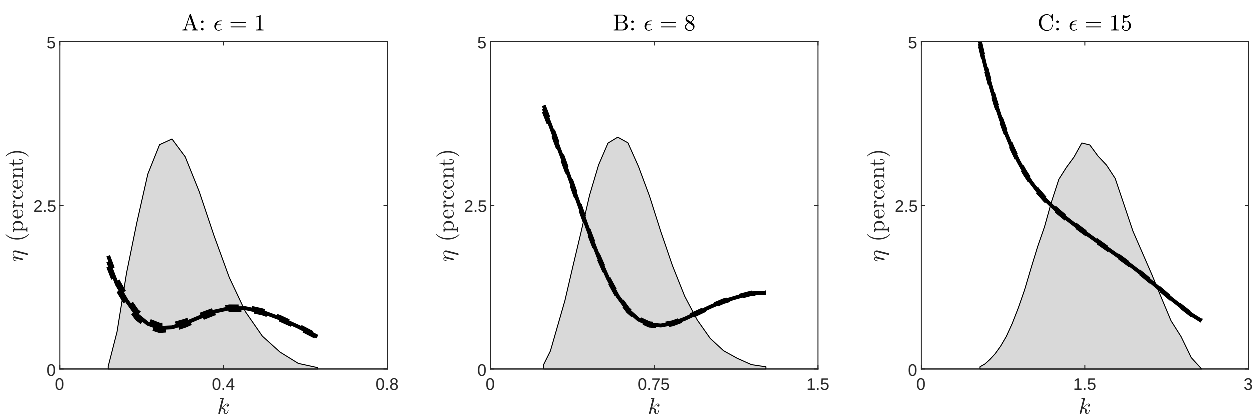

Figure 8: **Upper bound on welfare loss: Model of Khan and Thomas, transitional dynamics following 50% destruction of capital stock of all firms.** Panels A, B, and C show the upper bound of loss in firm value as a function of the firm's capital stock, k, for three different idiosyncratic shocks, $\epsilon = 1$, $\epsilon = 8$, and $\epsilon = 15$, respectively. The loss is measured as a fractional certainty equivalent loss η, which is defined in equation (53). The aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital immediately after the economy-wide capital loss is realized. The value function of the relaxed problem is estimated by averaging over 15000 paths of aggregate shocks. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Khan and Thomas (2008) and are also reported in Table 5.

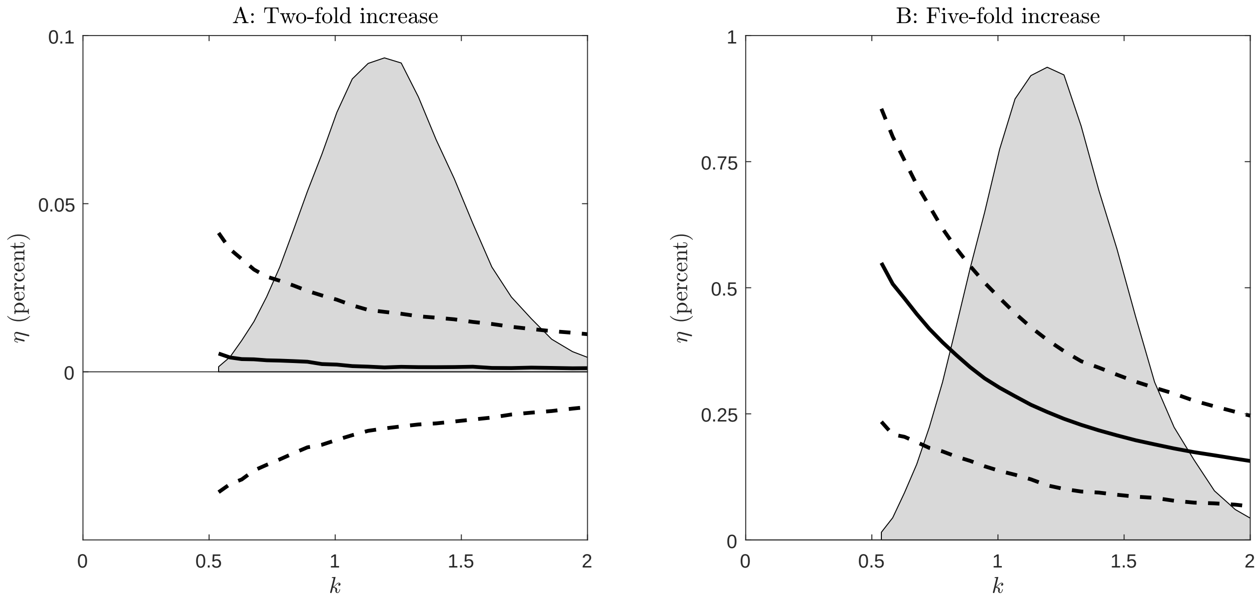

Figure 9: **Upper bound on welfare loss: Model of Khan and Thomas, transitional dynamics following a permanent increase in aggregate volatility.** Panels A and B show the upper bound of loss in firm value as a function of the firm's capital stock, k, for a two-fold and five-fold unanticipated increase in aggregate volatility, respectively. In both cases, the idiosyncratic shock $\epsilon = 8$. The loss is measured as a fractional certainty equivalent loss η, which is defined in equation (53). The aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital in the stochastic steady-state, conditional on ϵ. The value function of the relaxed problem is estimated by averaging over 15000 paths of aggregate shocks. Dashed lines are 95% Monte Carlo confidence bounds. All parameters values are identical to those in Khan and Thomas (2008) and are also reported in Table 5.

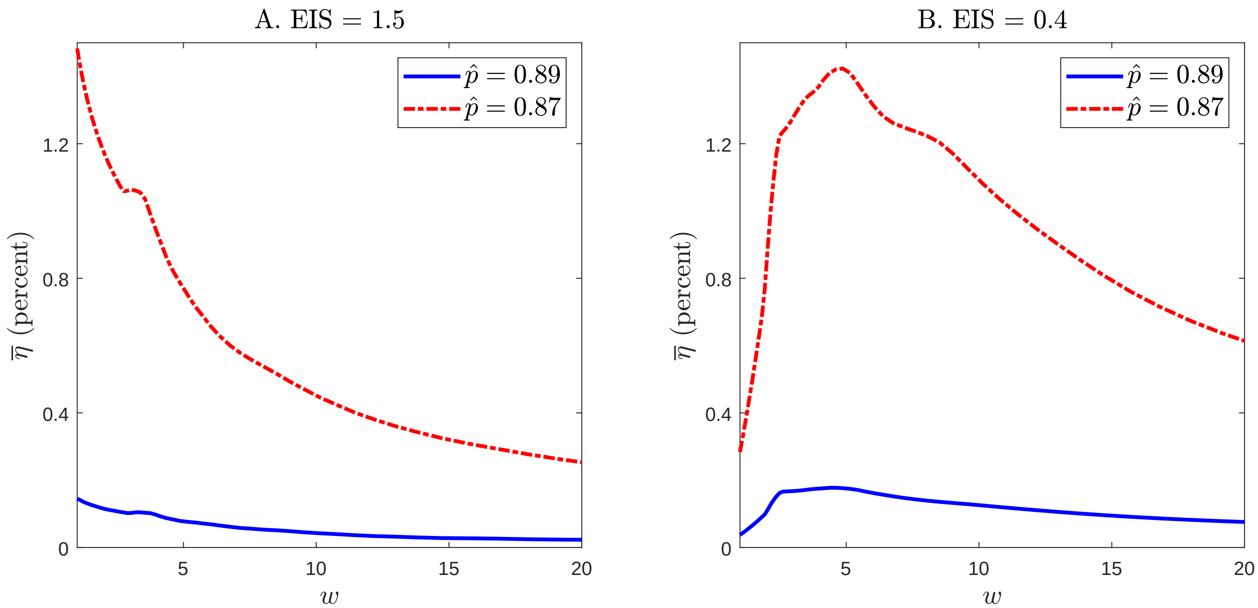

Figure 10: **Epstein Zin example.** Panels A and B show the upper bound of certainty equivalent loss $\overline{\eta}$ from adopting policies with varying degrees of sub-optimality. While the true probability of a high income shock y_H is 0.9, the solid line in both figures above correspond to the optimal policy where the agent's belief for realization of y_H equals $\widehat{p} = 0.89$ while the dot-dash line shows results for $\widehat{p} = 0.87$. The results in panel A are for an agent who prefers early resolution of uncertainty (EIS=1.5 and RRA=2), while those in panel B are for an agent who prefers late resolution of uncertainty (EIS=0.4 and RRA=2). Both agents have a time preference parameter of $\beta = 0.9$.

Table 6: Upper bound on welfare loss: Variations of the baseline Krusell and Smith calibration.

This table shows the upper bound on an agent's welfare loss for five different values of the agent's capital stock corresponding to the 10th through 90th percentiles of the capital stock distribution. Row (1) reports results for the baseline calibration, row (2) uses a lower value of β (baseline $\beta = 0.99$), row (3) uses a higher value of aggregate volatility σ_z (baseline $\sigma_z = 0.01$), and row (4) uses a higher value of risk aversion γ (baseline $\gamma = 1$). Rows (5) through (7) use the baseline parameter values, but vary the number of agents in the economy. The aggregate state of the economy is low. The value function of the relaxed problem is estimated by averaging over 500 paths. Numbers in parenthesis show the 95% Monte Carlo confidence interval.

	10%	25%	50%	75%	90%
			I. Idiosyncratic employment $\epsilon_0 = 0$		
(1) Baseline	0.11 $[0.04, 0.17]$	0.07 $[0.02, 0.12]$	0.04 $[0.00, 0.08]$	0.02 $[0.00, 0.05]$	0.02 $[-0.01, 0.04]$
(2) $\beta = 0.95$	0.05 $[-0.02, 0.13]$	0.03 $[-0.02, 0.09]$	0.02 $[-0.02, 0.06]$	0.02 $[-0.02, 0.05]$	0.01 $[-0.01, 0.04]$
(3) $\sigma_z = 0.02$	0.09 $[0.03, 0.15]$	0.06 $[0.01, 0.11]$	0.04 $[0.00, 0.07]$	0.02 $[-0.01, 0.05]$	0.01 $[-0.01, 0.03]$
(4) $\gamma = 2$	0.22 $[0.16, 0.29]$	0.16 $[0.11, 0.20]$	0.09 $[0.06, 0.13]$	0.05 $[0.02, 0.08]$	0.03 $[0.01, 0.06]$
(5) $N = 10^3$	0.14 $[0.02, 0.26]$	0.12 $[0.04, 0.20]$	0.07 $[0.01, 0.12]$	0.02 $[-0.02, 0.06]$	0.01 $[-0.03, 0.05]$
(6) $N = 10^4$	0.11 $[0.04, 0.18]$	0.06 $[0.01, 0.11]$	0.03 $[0.00, 0.07]$	0.03 $[-0.01, 0.06]$	0.01 $[-0.01, 0.04]$
(7) $N = 10^6$	0.09 $[0.03, 0.15]$	0.06 $[0.01, 0.11]$	0.04 $[0.00, 0.07]$	0.02 $[0.00, 0.05]$	0.01 $[-0.01, 0.03]$
			II. Idiosyncratic employment $\epsilon_0 = 1$		
(1) Baseline	0.10 $[0.04, 0.16]$	0.06 $[0.01, 0.10]$	0.03 $[0.00, 0.07]$	0.02 $[-0.01, 0.05]$	0.01 $[-0.01, 0.03]$
(2) $\beta = 0.95$	0.05 $[0.00, 0.09]$	0.03 $[0.00, 0.06]$	0.02 $[-0.01, 0.05]$	0.02 $[-0.01, 0.04]$	0.01 $[0.00, 0.03]$
(3) $\sigma_z = 0.02$	0.09 $[0.03, 0.14]$	0.05 $[0.01, 0.10]$	0.03 $[0.00, 0.07]$	0.02 $[-0.01, 0.04]$	0.01 $[0.00, 0.03]$
(4) $\gamma = 2$	0.20 $[0.14, 0.24]$	0.13 $[0.09, 0.17]$	0.07 $[0.04, 0.10]$	0.04 $[0.02, 0.06]$	0.02 $[0.00, 0.04]$
(5) $N = 10^3$	0.12 $[0.05, 0.19]$	0.09 $[0.03, 0.13]$	0.04 $[0.00, 0.08]$	0.03 $[0.00, 0.05]$	0.02 $[-0.01, 0.04]$
(6) $N = 10^4$	0.09 $[0.03, 0.14]$	0.06 $[0.02, 0.10]$	0.03 $[0.00, 0.06]$	0.02 $[-0.01, 0.05]$	0.01 $[-0.01, 0.03]$
(7) $N = 10^6$	0.09 $[0.03, 0.15]$	0.06 $[0.01, 0.10]$	0.04 $[0.00, 0.07]$	0.02 $[-0.01, 0.05]$	0.01 $[-0.01, 0.03]$