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IMPLICATIONS FOR COMPETITION AND GROWTH UNDER DECLINING SEARCH FRICTIONS

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ABSTRACT

As search frictions become smaller in the market for a consumer product, buyers are able to locate and access more sellers per unit of time. In response, sellers choose to design varieties of the product that are more specialized in order to exploit differences in the buyers' preferences. I find mild conditions on the fundamentals under which the decline in search frictions and the increase in specialization have exactly offsetting effects on the extent of competition in the market. Under these conditions, price dispersion remains constant over time even though search frictions are vanishing. Buyer's surplus and seller's profit, however, grow at a constant endogenous rate, as the endogenous increase in specialization allows sellers to cater better and better to the heterogeneous desires of buyers.

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1 Introduction

I study the effect of declining search frictions on competition, price dispersion and surplus in a version of the Burdett and Judd (1983) model of the consumer product market where sellers optimally decide how much to specialize their variety of the product to the desires of particular buyers. I find that under mild conditions on the fundamentals, the increase in competition caused by the decline in search frictions is exactly offset by the decrease in competition caused by the endogenous increase in the specialization of product varieties. Under these conditions, price dispersion remains constant over time, even though search frictions become vanishingly small. The buyers' surplus and the sellers' profits, however, grow over time at a constant rate, as the endogenous increase in specialization allows sellers to provide more and more value to buyers.

Price dispersion is a well-documented phenomenon. A variety of empirical studies have shown that the same good is sold at very different prices at the same point in time by different retailers operating in the same geographical market (see, among many others, Stigler 1961, Pratt, Wise and Zeckhouser 1979, Galenianos, Pacula and Persico 2012). Price dispersion is sizeable when the notion of a good is as narrow as a UPC code and becomes larger when the notion of a good is expanded to include different varieties of the product (see, e.g., Kaplan and Menzio 2015, Kaplan, Menzio, Rudanko and Trachter 2019). Price dispersion cannot be explained away by unobserved differences in the amenities provided by different retailers (see, e.g., Sorensen 2000, Kaplan and Menzio 2015).

The textbook theory of price dispersion is developed in Butters (1977), Varian (1980) and Burdett and Judd (1983). Because of search frictions, buyers come into contact only with a subset of sellers. Some buyers do not contact any sellers, some buyers contact only one seller, and some buyers contact multiple sellers. Hence, an individual seller faces both buyers who are captive (in the sense that they are not in contact with any other seller) and buyers who are not captive (in the sense that they are in contact with some other seller). And when a seller faces both captive and non-captive buyers and cannot price discriminate between the two groups, the only equilibrium features price dispersion.

The textbook theory of price dispersion predicts that, as search frictions decline and the fraction of buyers who are not captive increases, competition between sellers should strengthen, markups should fall, and price dispersion should decline and eventually disappear. These predictions of the theory, though, do not appear to be borne out in the data. Indeed, online retail does not seem to feature systematically lower markups nor systematically lower price dispersion than offline retail, even though search frictions are presumably lower online (see, e.g., Brynjolfsson and Smith 2000, Baye, Morgan and Sholten 2004, Ellison and Ellison 2014). Relatedly, the average coefficient of variation for prices of consumer products found by Pratt, Wise and Zeckhouser (1979) in the late 70's, the one found by Lach (2002) in 1993, and the one found by Kaplan and Menzio (2015, 2016) in 2010 are quite close, even though the period between 1979 and 2010 has witnessed improvements in information and communication technology that presumably have lowered search frictions (e.g. Internet access to locate stores, Smartphones with GPS for directions to stores, etc...).

So why did technological progress–which ostensibly reduced search frictions in the consumer product market–did not visibly enhance competition between sellers, lower markups, and drive down the extent of price dispersion? In this paper, I propose an answer to these questions based the insight, borrowed from Kyiotaki and Wright (1993), that the extent of product differentiation responds endogenously to the extent of search frictions. In developing the insight, I build a novel theory of industry dynamics, and a novel theory of economic growth through market deepening and specialization.[1]

I consider a dynamic version of Burdett and Judd (1983). On one side of the market for some product, there is a continuum of infinitely-lived firms. Each firm sells a variety of the product that is defined by its breadth, i.e. the fraction of buyers that like it. Broader (i.e. less specialized) varieties are liked by a larger fraction of buyers, but provide those buyers with lower utility. Narrower (i.e. less specialized) varieties are liked by a smaller fraction of buyers, but provide those buyers with higher utility. Each firm can pay a lumpy cost to design a new variety of the product, such that designing more specialized varieties requires more input than designing less specialized varieties. On the other side of the product market, there is a constant flow of short-lived buyers with unit demand. Each buyer searches the market and, as a result, comes into contact with n randomly-selected firms, where n is Poisson with coefficient λ_t. Each buyer inspects the variety sold and the price charged by each of the n firms with whom he comes into contact and decide whether and where to purchase the good. Search frictions decline over time, in the sense that the buyer's average number of contacts λ_t grows at the constant exogenous rate $g_\lambda > 0$.

I focus on a Balanced Growth Path (BGP) of the product market, i.e. an initial condition for the state of the market and an associated rational expectations equilibrium along which endogenous variables grow at some endogenous, constant rate. The crucial feature of a BGP is that the extent of price dispersion remains constant over time. In fact, in a BGP, the distribution of prices grows at a constant rate, in the sense that every quantile of the distribution grows at the same constant rate. Therefore, in a BGP, the distribution of prices normalized by their average remains constant over time and so does the extent of price dispersion. I focus on a BGP in the spirit of King, Plosser and Rebelo (1998). First, the BGP described a situation where improvements in communication and information technology that lower search frictions have no effect on competition and on price dispersion, which broadly speaking appears to be the case empirically. Second, the sufficient conditions for a BGP identify the properties of the fundamentals that one needs to relax in order for the economy to be on a non-balanced path, where competition and price dispersion change over time.

I derive conditions on the fundamentals that are necessary for the existence of a BGP.

[1] Of course, it could be the case that price dispersion has nothing to do with search frictions in the first place, a point made in Baye and Morgan (2004).

Specifically, I show that a BGP might exist only if: (i) the buyer's utility function has a constant elasticity alpha with respect to the degree of specialization of the consumed variety; (ii) the price of the input used by the firm to design a new variety grows at the same rate as the price of the output; (iii) the quantity of input used by the firm to design a new variety is a strictly increasing function of the variety's degree of specialization relative to the average degree of specialization of varieties in the market. I then show that the necessary conditions for the existence of a BGP are also sufficient, as long as a system of three time-invariant equations in three scalar time-invariant unknowns has a solution.

Why does a BGP exist? As search frictions decline, firms can reach more and more buyers per unit of time and, for this reason, they find it optimal to design varieties of the product that are more and more specialized. When the buyer's utility function is isoleastic in the degree of specialization of the consumed variety, the firm's design cost depends on the relative degree of specialization of the designed variety, and the firm's design cost grows at the same rate as the price of output, then firms find it optimal to increase the degree of specialization of their varieties at exactly the same rate at which search frictions decline. When this is the case, the probability that a buyer contacts k firms with a variety that he likes is a random draw from a constant Poisson distribution. And the firm meets a constant flow of buyers who are in contact with k competitors whose product they like. For this reason, the extent of competition in the market remains constant over time and, hence, so does the extent of price dispersion.

While the number of relevant contacts between buyers and firms remains constant, each contact involves trade of an increasingly specialized variety. For this reason, the buyer's average surplus, the firm's flow profits, the firm's value and the price distribution all grow over time. The common growth rate is αg_λ, i.e. the rate g_λ at which search frictions decline (and the rate at which specialization increases) multiplied by the elasticity α of the buyers' utility function with respect to the degree of specialization of the consumed variety. The mechanism behind growth is related to observations by Adam Smith. Smith argued that the extent of the market constraints the degree of specialization. Increases in the extent of the market allow further specialization of production based on technical advantages or increasing returns to scale and, in turn, engender economic growth. Similarly, here, search frictions determine the *depth of the market*. And declining search frictions engender growth by allowing firms to design products that are better and better tailored to a particular subset of buyers.

Behind the aggregate behavior of the market, there lie rich industry dynamics. In a BGP, every firm goes through a common cycle with constant periodicity, although the beginning of the cycle takes place at a different calendar time for different firms. The cycle begins with the firm designing a new variety of the product. At that moment in time, the firm is the best in the market—in the sense that the firm's variety is the most specialized, the firm offers the highest surplus to its customers, and the firm's profit and value are highest. Over the cycle, the firm loses ground, as more and more competitors design a new variety of the product and leapfrog the firm. Specifically, when the firm has gone through $z\%$ of the cycle, the firm's variety is at the z-th quantile of the specialization distribution, the firm offers to its customers a surplus at the z-th quantile of the surplus distribution, the firm's profit and value are at the z-th quantiles of the profit and value distributions. The cycle ends with the firm scrapping its variety. At that moment in time, the firm is the worst in the market.

The key mechanism highlighted in this paper is that declining search frictions induce firms to develop more specialized varities in order to take advantage of the heterogeneity in the preferences of different buyers. The decline in search frictions is a pro-competitive force, as it allows buyers to compare more sellers. The increase in the degree of specialization of the varieties sold by firms is an anti-competitive force, as it thins out the number of sellers that are relevant to each buyer. Under mild conditions, the decline in search frictions and the increase in specialization exactly offset each other, competition remains constant and so does price dispersion. The idea that lower search frictions tends to increase competition and that higher specialization tends to decrease competition seems pretty uncontroversial. The idea that lower search frictions triggers higher specialization is so commonplace to be the subject of best sellers. In "The Long Tail", Anderson argues that the Internet has led a proliferation of niche varieties in markets that used to be dominated by generic varieties. In his words "*When you dramatically lower the cost of connecting supply and demand, it changes the entire nature of the market.* [...] *Bringing niches within reach reveals latent demand for non-commercial content*" and "*The mass market is turning into a mass of niches.*"

Related literature The paper provides a novel explanation for the observation that declining search frictions do not appear to have reduced price dispersion in the consumer goods market. There are several papers on the related topic of why price dispersion in online markets is not lower than in offline markets. Ellison and Ellison (2014) consider a search model of the market for used books, in which buyers have heterogeneous valuation for a book, sellers have a single copy of the book and they meet buyers sequentially at some exogenous speed. Price dispersion obtains in equilibrium because different sellers meet captive buyers at different speed. They show that, when the arrival rate of captive buyers increases, the highest prices increase (as sellers can sample more buyers within the same amount of time). When the fraction of non-captive buyers increases, the lowest prices fall (as the competition for these buyers intensifies). Overall, price dispersion increases. Translated in the context of a Burdett-Judd model, Ellison and Ellison (2014) model online markets as markets where the buyer's valuation is higher and the fraction of non-captive buyers is higher than in offline markets. In my model, the only fundamental change is the average number of sellers contacted by each individual buyer. Ellison and Wolitsky (2012) argue that price dispersion need not disappear in online markets because sellers find it optimal to engage in obfuscation, i.e. actions that make it hard for buyers to observe a seller's price. My theory relates to Ellison and Wolitsky (2012) because the increase in specialization in varieties acts–in a mechanical sense–like obfuscation. Baye and Morgan (2001) argue that price dispersion need not disappear in online markets if the participation to these markets is costly to the sellers. They show that, in equilibrium, sellers randomize over participating to the online market and, conditional on participating, they randomize over the posted price.

The paper provides a novel theory of growth, where a deepening of the market allows firms to develop more specialized product and take advantage of the heterogeneity in the preferences of different buyers. There are several related theories of Smithian growth, where economic growth is caused by an increase in the size of the market, increase that allows production to become more specialized and take advantage of technological differences across locations or of increasing returns to scale. Kelly (1997) consider an economy with multiple locations, which are geographically separated. Each location has a technological advantage in producing a particular intermediate good. When locations are not connected, each location inefficiently produces all intermediates. As more and more locations become connected, each location specializes in the intermediate good that they produce most efficiently. Locay (1990) considers an economy in which market production features increasing returns to scale, but home-production avoids agency problems. He shows that, as the size of the market increases because of e.g. population growth, production moves from the home to the market and productivity increases. In contrast to these theories, my theory highlights the role of market depth—the facility with which buyers meet sellers within a market—in generating economic growth. Since a market can always grow deeper but cannot expand indefinitely (at least with a fixed population), my theory generates perpetual growth rather than a growth episode. My theory is also loosely related to Romer (1990) in the sense that in both theories growth manifests itself with an increase in the number of varieties.

The notion that market deepening—modelled as a decline in search frictions—leads to economic growth by leveraging heterogeneity can also be found in Martellini and Menzio (2020, 2021). Martellini and Menzio (2020, 2021) consider a frictional labor market in the spirit of Mortensen and Pissarides (1994) where firm-worker matches are heterogeneous with respect to their quality. They find conditions under which the unemployment, vacancy, and the workers' transition rates are constant over time in the face of declining search frictions, as they appear to be in the data. Under these conditions, they show that market deepening leads to growth by allowing workers and firms to sort into better matches. In this paper, I consider a frictional product market in the spirit of Burdett and Judd (1983). I find conditions under which price dispersion and competition remain constant over time in the face of declining search frictions, as they appear to be in the data. I then show that, under these conditions, market deepening leads to economic growth by allowing firms to design more specialized products. The key difference between the two theories is related to the focus on price dispersion. In a product market analogue of Martellini and Menzio (2020, 2021), a buyer's valuation for the variety of a seller would be a draw from a continuous, unbounded distribution. Given this type of heterogeneity, the seller would always face a continuous demand, and price dispersion would not obtain as an equilibrium outcome. In my model, a buyer's valuation for the variety of a seller is a draw from a binary random variable (i.e. like, dislike). Given this type of heterogeneity, the firm faces a demand that is discontinuous wherever the price distribution has a mass point and, hence, there must be price dispersion in equilibrium. And, since discrete heterogeneity in buyers' valuations is needed to generate price dispersion, sustained growth can only be achieved by allowing firms to redesign their varieties.

Lastly, the paper provides a novel theory of industry dynamics. The firms in the market design more specialized products as search frictions decline. Each individual firm follows an (S, s) cycle, which starts with the design of a new variety and ends with the scrapping of the variety. The timing of the (S, s) cycle is different for different firms so that the cross-sectional distribution of varieties evolves in a balanced fashion. Mechanically, the industry dynamics resembles those in a stationary equilibrium of menu cost model (see, e.g., Caplin and Spulber 1986, Benabou 1988, Diamond 1993). Substantially, though, the industry dynamics in menu cost models are driven by inflation and the (S, s) strategy is about nominal prices, while in my model the industry dynamics are driven by market deepening and the (S, s) strategy is about specialization of varieties. There are some papers that, like mine, study the effect of declining trading frictions on the equilibrium of an industry. Guthmann (2020) studies the industry dynamics in a market where buyers accumulate information about sellers over time. He shows that prices and markups fall towards the competitive benchmark. Perla (2019) studies the industry dynamics in a market where buyers become progressively aware of sellers. He shows that buyer awareness tends to lower industry markups. In contrast, I show that declining frictions do not necessarily lower markups when firms can choose the degree of specialization of their variety.

2 Environment and definition of a BGP

In this section, I describe the physical environment of the market for some consumer product and then I formally define a Balanced Growth Path (BGP).

2.1 Environment

Consider the market for a consumer product that can be produced in different varieties. On one side of the market, there is a measure 1 of infinitely-lived firms. A firm pays a lumpy cost $C_t(x)$ to design a variety of the good with *breadth* $x \in [0, 1]$, where x denotes the probability that an individual buyer likes that particular variety of the good and $C_t(\cdot)$ is a strictly positive and decreasing function. Varieties with a high x are generic: they are liked by a large fraction of buyers and they are relatively cheap to design. Varieties with a low x are niche: they are liked by a small fraction of buyers and they are relatively expensive to design. I will sometimes refer to $1/x$ as the *specificity* of a variety. After paying the design cost $C_t(x)$, a firm can produce its variety at a constant unit cost, which, for the sake of simplicity, is assumed to be zero. A firm sets the price $p_t \geq 0$ of its variety at date t taking as given the joint distribution of its competitors over the breadth of their varieties and their prices.

The design cost function $C_t(x)$ is equal to the price of the input in the design process, q_t, times the quantity of input $c(x/X_t^\beta)$, where $c(\cdot)$ is a decreasing function, X_t is the average breadth of varieties in the market and $\beta \geq 0$ is a coefficient capturing production externalities. For $\beta = 0$, the breadth of the varieties produced by other firms does not affect the design cost of an individual firm. For $\beta > 0$, a decline in the breadth of the varieties produced by other firms leads to a decline in the design cost of an individual firm.

On the other side of the market, there is a flow of short-lived buyers. Specifically, a measure $b \cdot dt$ of buyers enters the market during each interval of time of length dt. A buyer demands one unit of the good. A buyer searches the market and contacts $n \in \{0, 1, 2, ...\}$ firms, where n is the realization of a Poisson distribution with coefficient λ_t. A buyer observes the variety sold and the price charged by every firm that he contacts and, then, he decides whether and where to purchase the good. If a buyer purchases a variety of the good with breadth x at the price p and he likes that variety, he enjoys a utility of $u(x) - p$, where $u(\cdot)$ is a strictly positive and strictly decreasing function. If a buyer purchases a variety of the good with breadth x at the price p and he does not like the variety, he enjoys a utility of $-p$. That is, many buyers like a generic variety, but they get little utility from consuming it. Few buyers like a niche variety, but, if they do, they get more utility from consuming it. A buyer exits the market whether he purchases the good or not.

The environment is non-stationary. The number n of firms contacted by a buyer follows a Poisson distribution with coefficient λ_t, where λ_t is equal to the average number of firms contacted by a buyer. I will sometimes refer to $1/\lambda_t$ as the extent of search frictions. The average number of firms contacted by a buyer λ_t grows at some constant, exogenous rate $g_\lambda > 0$. The growth in the number of firms contacted by a buyer is meant to capture declining search frictions due to developments in communication technology that make it easier for buyers to locate sellers (e.g., telephone, mobile phone, Internet, smart phone, etc. . .). The price q_t of the input used in the design of new varieties grows at some exogenous rate g_q.

Some comments about the environment are in order. First, let me comment on how I model specialization and product differentiation. I assume that a buyer has a probability x of liking a variety of the good with breadth $x \in [0, 1]$. If the buyer likes the variety, he enjoys a utility of $u(x)$ from consuming it. If the buyer does not like the variety, he enjoys a utility of 0 from consuming it. This highly stylized approach to modelling specialization and product differentiation is borrowed from Kyiotaki and Wright (1993) and it is reminiscent of Diamond (1982).

Second, let me comment on how I model production and retailing. I assume that the same agent (the firm) designs a variety of the product and retails it. The assumption makes the analysis simple. The assumption, however, is not always realistic, as product design (choosing the breadth of the variety) and retailing (choosing the retail price) are decisions that are often made by different agents (the producer and the retailer). Moreover, I assume that every producer/retailer carries only one variety. Again, the assumption makes the analysis simple even though, in reality, producers have a portfolio of products and retailers an assortment of varieties.

Third, let me comment on how I model the search process. I assume that buyers search non-sequentially and contact a number n of firms that is distributed as a Poisson with coefficient λ_t. This specification of the search process guarantees that there is a positive fraction of buyers who is in contact with a single firm as well as a positive fraction of buyers who is in contact with more than one firm. As explained in Butters (1977), Varian (1980) and Burdett and Judd (1983), the coexistence of buyers with one contact and buyers with multiple contacts is necessary to generate price dispersion in equilibrium.[2] When firms sell products that are liked by all buyers (as in Butters 1977, Varian 1980 and Burdett and Judd 1983), the extent of competition in the market is determined by the fraction of buyers with n contacts, $n = 0, 1, 2, \ldots$. When firms sell products that are liked by a subset of buyers (as in my model), the extent of competition is determined by the fraction of buyers who are in contact who are in contact with n-squiggle firms selling a good that they like, $k = 0, 1, 2....$ This is where the Poisson assumption comes into play. Indeed, if n is distributed as a Poisson with coefficient λ_t and the average breadth of a variety is X_t, then k is distributed as a Poisson with coefficient $\lambda_t X_t$. Therefore, if $\lambda_t X_t$ remains constant over time, the distribution of n-squiggle is constant and, for that reason, the extent of competition is stable.[3]

2.2 Definition of a BGP

To formally define a BGP, it is useful to first lay out some definitions and some notation. The initial state of the market is the cumulative distribution function $H_0(x)$ of the breadth of the variety of the good sold by different firms. A path for the market is a list of: (i) policy functions $\{x_{\ell,t}, T_t(x), s_t(x), p_t(x)\}$, where $x_{\ell,t}$ denotes the breadth of new varieties, $T_t(x)$ denote the time until a variety with breadth x is scrapped, $p_t(x)$ denotes the price charged by firms with a variety of breadth x, and $s_t(x) \equiv u(x) - p_t(x)$ is the surplus offered by firms with a variety of breadth x to buyers who like that variety; (ii) value functions $\{R_t(x), V_t(x)\}$, where $R_t(x)$ denotes the flow profit for a firm selling a variety of breadth x and $V_t(x)$ denotes the value of a firm selling a variety of breadth x; (iii) distributions $\{F_t(s), G_t(p), H_t(x)\}$, where $F_t(s)$ is the fraction of firms offering a surplus non-greater

[2] Menzio and Trachter (2015) show that price dispersion may emerge when all buyers are captive, but only if some sellers are large.

[3] The Poisson distribution may not be the only one with the property that the number of relevant contacts for a buyer is a distribution that depends on λ_t and X_t only through their product $\lambda_t X_t$. Lester, Visschers and Wolthoff (2015) list other distributions with the same property. I conjecture that a BGP would also exist for any of those distribution.

than s, and $G_t(p)$ is the fraction of firms charging a price less than p. The path for the market is an equilibrium if the value and policy functions maximize the agents' problems and the distributions are consistent with the agents' policy and the initial state of the market. A BGP is an initial state of the market and an equilibrium path associated with that initial state such that endogenous variables grow at some endogenous, constant rate.

Consider a buyer visiting the market at date t. The buyer comes into contacts with n firms with probability

$$\lambda_t^n \frac{e^{-\lambda_t}}{n!}, \text{ for } n = 0, 1, 2, ... \tag{2.1}$$

The buyer obtains a negative surplus from purchasing any variety of the product that he does not like, as the utility from consuming such variety is 0 and the disutility from paying the price p is positive. Hence, the buyer restricts attention to the n firms that sell a variety of the good that he likes. The probability that buyer contacts k firms that sell a variety that he likes is given by

$$\sum_{n=k}^{\infty} \lambda_t^n \frac{e^{-\lambda_t}}{n!} \frac{n!}{k!(n-k)!} X_t^k (1-X_t)^{n-k}, \text{ for } k = 0, 1, 2, ..., \tag{2.2}$$

where X_t denotes the average breadth of varieties

$$X_t = \int x dH_t(x). \tag{2.3}$$

The expression in (2.2) is easy to understand. The buyer comes into contact with $n \geq k$ firms with probability given in (2.1). Conditional on contacting n firms, the buyer likes the varieties of k of them with probability $X_t^k(1-X_t)^{n-k} \cdot n!/\,k!(n-k)!$. Among these firms, the buyer purchases the good from the firm that offers him the highest surplus $u(x) - p$, as long as this surplus is non-negative.

Next, consider a firm selling a variety of the product with breadth x. During an interval of time of length dt, the firm meets an average of $b_0 \cdot dt$ buyers who do not have any contact with any other firm whose product they like, where b_0 is given by

$$\begin{aligned} b_0 &= b \sum_{n=1}^{\infty} n \frac{\lambda_t^n e^{-\lambda_t}}{n!} (1-X_t)^{n-1} \\ &= b\lambda_t e^{-\lambda_t} \sum_{n=0}^{\infty} \frac{(1-X_t)^n \lambda_t^n}{n!} \\ &= b \frac{1}{X_t} e^{-\lambda_t X_t} \lambda_t X_t. \end{aligned} \tag{2.4}$$

The firm meets an average of $bn\lambda_t^n \exp(-\lambda_t)/n!$ buyers with n contacts, with $n = 1, 2, \ldots$. One of the buyer's contacts is the firm. The probability that none of the other $n-1$ contacts sells a product that is liked by the buyer is $(1-X_t)^{n-1}$, where X_t is the average of x in the population of firms. The second line is obtained by collecting $\lambda_t \exp(-\lambda_t)$ in the first line. The third line is obtained by recognizing that the summation in the second line is equal to $\exp(-\lambda_t(1-X_t))$.

Similarly, during an interval of time of length dt, the firm meets an average of $b_k \cdot dt$ buyers who are in contact with exactly $k = 1, 2, \ldots.$ other firms whose product they like, where b_k is given by

$$\begin{aligned} b_k &= b \sum_{n=k+1}^{\infty} n \frac{\lambda_t^n e^{-\lambda_t}}{n!} \frac{(n-1)!}{k!(n-k-1)!} X_t^k (1-X_t)^{n-k-1} \\ &= b\lambda_t^{k+1} e^{-\lambda_t} \frac{X_t^k}{k!} \sum_{n=0}^{\infty} \frac{(1-X_t)^n \lambda_t^n}{n!} \\ &= b \frac{1}{X_t} e^{-\lambda_t X_t} \frac{\lambda_t^{k+1} X_t^{k+1}}{k!}. \end{aligned} \tag{2.5}$$

The firm meets an average of $bn\lambda_t^n \exp(-\lambda_t)/n!$ buyers with n contacts, with $n = k+1, k+2, \ldots.$ One of the buyer's contacts is the firm. The probability that exactly k of the other $n-1$ contacts of the buyer has a product that he likes is $(n-1)!/[k!(n-k-1)!]X_t^k(1-X_t)^{n-k-1}$. The second line is obtained by collecting $\lambda_t^{k+1} X_t^k \exp(-\lambda_t)/k!$ in the first line. The third line is obtained by recognizing that the summation in the second line is equal to $\exp(-\lambda_t(1-X_t))$.

The firm chooses its price, p, so as to maximize its flow profit, taking as given the number of buyers that it meets and the distribution of its competitors over the breadth of their varieties and the price that they charge. Since buyers make their purchasing decision based on the surplus that they are offered by firms whose variety they like, it is easier to formulate the firm's pricing problem as choosing the surplus $s = u(x) - p$ offered to buyers who like the firm's variety so as to maximize the flow profit, taking as given the number of buyers that the firm meets, the distribution of competitors over varieties $H_t(x)$, and the surplus $s_t(x)$ offered by firms with a variety x.

Formally, the pricing problem of the firm is

$$R_t(x) = \max_{s \geq 0} \left\{ \sum_{k=0}^{\infty} b_k x F_t(s)^k \right\} (u(x) - s), \tag{2.6}$$

where $F_t(s)$ is defined as

$$F_t(s) = \frac{\int_{x:s_t(x)\leq s} x dH_t(x)}{\int x dH_t(x)} \tag{2.7}$$

First, let me explain (2.7). The function $F_t(s)$ denotes the probability that a firm offers a surplus smaller than s to a buyer, conditional on the buyer liking its variety. Next, let me explain (2.6). The firm meets a flow b_k of buyers who are in contact with k other firms whose product they like. The probability that one of these buyers chooses to purchase from the firm rather than from one of the other n competitors is $xF_t(s)^k$, where x is the probability that the buyer likes the firm's variety, and $F_t(s)^k$ is the probability that all of the other k sellers offer a surplus smaller than s to the buyer. The profit that the firm enjoys for every unit sold is p, which is equal to $u(x) - s$.

Expanding the objective function of the firm in (2.6), I obtain

$$\begin{aligned} R_t(x) &= \max_{s\geq 0}\left\{\sum_{k=0}^{\infty} b\frac{x}{X_t}\lambda_t X_t e^{-\lambda_t X_t}\frac{\lambda_t^k X_t^k F_t(s)^k}{k!}\right\}(u(x)-s) \\ &= \max_{s\geq 0}\left\{\left[\sum_{k=0}^{\infty}\frac{\lambda_t^k X_t^k F_t(s)^k}{k!}\right] b\frac{x}{X_t}\lambda_t X_t e^{-\lambda_t X_t}\right\}(u(x)-s) \\ &= \max_{s\geq 0}\left\{b\frac{x}{X_t}\lambda_t X_t \exp^{-\lambda_t X_t(1-F_t(s))}\right\}(u(x)-s), \end{aligned} \tag{2.8}$$

where the third line makes use of the fact that the summation in the second line is equal to $\exp(\lambda_t X_t F_t(s))$. The search process is such that buyers contact a number n of firms distributed as a Poisson with coefficient λ_t. Then, buyers choose where to purchase the good based on which of the n firms' varieties they like, how specialized these varieties are, and at what price they are sold. This complicated search process leads to the remarkably simple expression for the profit of the firm in (2.8). Indeed, the profit of the firm is the same that would obtain if buyers sampled a number k of firms distributed as a Poisson with coefficient $\lambda_t X_t$, they liked the firm's variety with probability x/X_t and other sellers' varieties with probability 1, and the distribution of the surplus offered by other sellers was given by $F_t(s)$.

Finally, consider the firm's product design problem. The present value of profits $V_t(x)$ for a firm selling a variety of the good with breadth x at date t is given by

$$V_t(x) = \max_{T\geq 0, x_n\in[0,1]} \int_0^T e^{-\rho\tau} R_{t+\tau}(x)d\tau + e^{-\rho T}\left[V_{t+T}(x_n) - q_{t+T}c(x_n/X_{t+T}^{\beta})\right]. \tag{2.9}$$

The above expression is easy to understand. At all dates $t+\tau$ between t and $t+T$ the firm sells its existing variety of the good, and enjoys a flow profit of $R_{t+\tau}(x)$. At date $t+T$, the firm designs a new variety of the product with breadth x_n, pays the lumpy cost $q_{t+T}c(x_n/X_{t+T}^{\beta})$, and enjoys the continuation present value of profits $V_{t+T}(x_n)$.

Along a BGP, some variables remain constant over time and some variables growth at some constant, endogenous rate. Specifically, along a BGP, the distribution H_t of firms across the breadth of their varieties grows at some constant rate g_x, in the sense that every quantile of the distribution grows at that rate.[4] Formally,

$$H_t(x\exp(g_x t)) = H_0(x). \tag{2.10}$$

In a BGP, a firm that designs a new variety chooses the breadth $x_{\ell,t}$, which grows at the constant rate g_x, and a firm scraps a variety when its breadth reaches $x_{h,t}$, a cutoff that

[4] This definition of balanced growth for a distribution is standard in the recent literature studying growth in model with heterogeneity (see, e.g., Lucas and Moll 2014, Perla and Tonetti 2014, or Martellini and Menzio 2020).

also grows at the constant rate g_x. That is, in a BGP, an individual firm follows an (S, s) rule for redesigning its variety of the product, where the (S, s) bands grow at the constant rate g_x. Formally,

$$\begin{aligned} x_{\ell,t} &= x_{\ell,0}\exp(g_x t), \quad x_{h,t} = x_{h,0}\exp(g_x t), && (2.11) \\ T_t(x) &= \max\{0, \log(x/x_{h,t})/g_x\}, \text{ for } x \geq x_{\ell,t}. && (2.12) \end{aligned}$$

In a BGP, the price charged by a firm with a variety at any fixed quantile of the breadth distribution grows at some constant rate g_p, and the surplus offered by a firm with a variety at any fixed quantile of the breadth distribution grows at the constant rate g_s. Formally,

$$\begin{aligned} s_t(x\exp(g_x t)) &= s_0(x)\exp(g_s t), && (2.13) \\ p_t(x\exp(g_x t)) &= p_0(x)\exp(g_p t). && (2.14) \end{aligned}$$

Lastly, along a BGP, the present value of profits for a firm with a variety at any fixed quantile of the breadth distribution grows at some constant rate g_V, i.e.

$$V_t(x\exp(g_x t)) = V_0(x)\exp(g_V t). \tag{2.15}$$

It is important to point out that the balanced growth conditions above guarantee that the extent of price dispersion remains constant over time. In fact, conditions (2.10) and (2.14) imply that the price distribution G_t grows at the constant rate g_p, in the sense that every quantile of G_t grows at the rate g_p. Since G_t grows at a constant rate along a BGP, the distribution of normalized prices $p/\int p dG_t(p)$—which is the standard way to compare price distribution across goods or over time—is constant over time and, in this sense, the extent of price dispersion is constant. Similarly, conditions (2.10) and (2.13) imply that the surplus distribution F_t grows at the constant rate g_s, in the sense that every quantile of F_t grows at the rate g_s.

Before moving on to the existence and properties of a BGP, it is useful to reformulate the balanced growth condition (2.10). A quantile x of the firm distribution grows at the rate g_x between t and $t+dt$ if and only if $H_t(x)$ is equal to $H_{t+dt}(x\exp(g_x t))$. Analogously, a quantile x of the firm distribution grows at the rate g_x between t and $t+dt$ if and only if the measure of firms that are below x at t and above $x\exp(g_x t)$ at $t+dt$ is equal to the measure of firms that are above x at t and below $x\exp(g_x t)$ at $t+dt$. Anticipating that g_x is negative, this inflow-outflow condition can be written as

$$H_t(x) - H_t(x\exp(g_x dt)) = 1 - H_t(x_{h,t}\exp(g_x dt)), \text{ for } x \in [x_{\ell,t}, x_{h,t}]. \tag{2.16}$$

The left-hand side of (2.16) is the measure of firms that are below x at t and above $x\exp(g_x t)$ at $t+dt$. These are the firms with a variety between $x\exp(g_x t)$ and x. The right-hand side is the measure of firms that are above x at t and below $x\exp(g_x t)$ at $t+dt$. These are the firms that are above $x_{h,t}\exp(g_x dt)$ and in the next dt units of time pay the redesign cost and start producing a variety with breadth $x_{\ell,t+dt}$. Dividing by dt and taking limits for $dt \to 0$, (2.16) becomes

$$H_t'(x)x = H_t'(x_{h,t})x_{h,t}, \text{ for } x \in [x_{\ell,t}, x_{h,t}]. \tag{2.17}$$

I am now in the position to define a BGP.

Definition 1. *A BGP is a path for distributions $\{F_t, G_t, H_t\}$, a path for policies $\{s_t, p_t, x_{\ell,t}, x_{h,t}\}$, and a path for values $\{V_t, R_t\}$ such that:*

(i) The policies $\{s_t, p_t, x_{\ell,t}, x_{h,t}\}$ are optimal, in the sense that $s_t(x)$ solves the pricing problem (2.8) and $p_t = u(x) - s_t(x)$, and $x_{\ell,t}$ and $x_{h,t}$ solve the design problem (2.9);

(ii) The values V_t and R_t are maxima, in the sense that they are given by (2.8)-(2.9);

(iii) The distributions are consistent with agents' behavior;

(iv) The balanced growth conditions (2.11)-(2.15) and (2.17) are satisfied.

The above definition of a BGP may appear standard, but there are a few points worth discussing. First, I restrict attention to equilibria in which firms use pure strategies in the pricing decision, i.e. $p_t(x)$ is a function. When firms are identical, the only equilibrium of the Burdett-Judd competition is such that firms mix over prices. When firms are heterogeneous and distributed according to an atomless function, as in this paper, the mixed pricing strategy equilibrium is purified and, hence, the restriction is without loss in generality. Second, I restrict attention to equilibria in which the surplus distribution F_t is atomless. This is also without loss in generality as it is straightforward to rule out equilibria in which F_t has a mass point. Third, I restrict attention to equilibria in which firms follow a pure (S, s) strategy for the redesign of their varieties. This restriction is an afterthought in menu cost models (see, e.g., Benabou 1988, 1992, Caplin and Spulber 1986, Caplin and Leahy 1997). Burdett and Menzio (2017, 2018) show, however, that, in menu cost models with Burdett-Judd competition, there may also/only exist equilibria in which firms mix with respect to the (S, s) strategy. While, I do not entertain such equilibria here, it is important to notice that the forces behind randomization on the (S, s) strategy in Burdett and Menzio (2017, 2018) are not at work here as firms can adjust prices freely.

3 Necessary Conditions for a BGP

In this section, I derive some restrictions on the fundamentals of the model (i.e. the buyer's utility function, the firm's cost function, the exogenous growth rate of the price of inputs) that are necessary for the existence of a BGP. These restrictions are the analogue for our model of the restrictions on the utility function and on the production function that King, Plosser and Rebelo (1988) showed to be necessary for the existence of a BGP in the neoclassical growth model. Along the way, I will derive some properties of endogenous objects (i.e. the endogenous growth rate of the breadth of varieties, the endogenous distribution of varieties, etc...) that must hold in a BGP.

3.1 Distribution of varieties

For $t = 0$, the balanced growth condition (2.17) for the distribution of varieties reads

$$H_0'(x)x = H_0'(x_{h,0})x_{h,0}, \text{ for } x \in [x_{\ell,0}, x_{h,0}]. \tag{3.1}$$

The expression above is a differential equation for the initial distribution of varieties, H_0. The solution to the differential equation must satisfy the boundary condition $H_0(x_{\ell,0}) = 0$ because for $x < x_{\ell,0}$ the inflow of varieties is zero. The solution to the differential equation must also satisfy the boundary condition $H_0(x_{h,0}) = 1$ because all varieties $x > x_{h,0}$ are scrapped. The unique solution to the differential equation (3.1) that satisfies the two boundary conditions is

$$H_0(x) = \frac{\log x - \log x_{\ell,0}}{\log x_{h,0} - \log x_{\ell,0}}. \tag{3.2}$$

The initial distribution H_0 in (3.2) is the only distribution consistent with balanced growth between date 0 and dt.

Now suppose that the balanced growth (2.17) is satisfied for all $\tau \in [0, t]$. If that is the case, the distribution of varieties at date t is an H_t such that $H_t(x) = H_0(x \exp(-g_x t))$. Hence, H_t is given by

$$H_t(x) = \frac{\log x e^{-g_x t} - \log x_{\ell,0}}{\log x_{h,0} - \log x_{\ell,0}} = \frac{\log x - \log x_{\ell,t}}{\log x_{h,t} - \log x_{\ell,t}}, \tag{3.3}$$

where the second step makes use of the fact that $x_{h,t} = x_{h,0}\exp(g_{xt})$ and $x_{l,t} = x_{l,0}\exp(g_x t)$. It is immediate to verify that the distribution H_t in (3.3) satisfies the balanced growth condition (2.17) at date t and, hence, the distribution grows at a constant rate also between date t and $t + dt$. Moreover, the distribution H_t in (3.3) is the only one satisfying the balanced growth condition (2.17) at date t.

The above findings establish that, in any BGP, the distribution of varieties H_t must be log-uniform over the interval $[x_{\ell,t}, x_{h,t}]$. Conversely, the distribution of varieties H_t satisfies the constant growth condition (2.17) only if the initial distribution of varieties H_0 is log-uniform over the interval $[x_{\ell,0}, x_{h,0}]$ and and the firm's (S, s) thresholds $x_{\ell,t}$ and $x_{h,t}$ grow at the constant rate g_x. These findings are closely related to those in Caplin and Spulber (1986) and Benabou (1988) showing that the only stationary distribution of real prices in a menu cost model where firms follow a constant (S, s) rule is log-uniform.

3.2 Firm's pricing function

The firm's pricing problem is

$$R_t(x) = \max_{s \geq 0} \left\{ b \frac{x}{X_t} \lambda_t X_t \exp^{-\lambda_t X_t (1-F_t(s))} \right\} (u(x) - s). \tag{3.4}$$

Following the same arguments as in Burdett and Judd (1983), we can use (3.4) to establish some equilibrium properties of the F_t distribution. First, the surplus $s_t(x)$ must belong to the interval $[0, u(x))$. In fact, $R_t(x)$ is strictly positive as the firm can offer a surplus of 0 and sell its product at the price $p = u(x) > 0$ to the buyers who have not met any other firm with a product that they like. Since $R_t(x)$ is strictly positive, it follows that $s_t(x)$ must be strictly smaller than $u(x)$. Second, the support of F_t must be an interval. Suppose the support of F_t had a gap between s_0 and s_1, with $s_0 < s_1$. If that were the case, a firm that offered the surplus s_1 could achieve a strictly higher profit by offering s_0 instead, since it would sell the same quantity and enjoys a strictly higher profit per unit sold. Third, the lower bound on the support of F_t must be zero. Suppose that the lower bound were strictly positive. If this were the case, a firm that offered the surplus at the lower bound could achieve a strictly higher profit by offering 0 instead, since it would sell the same quantity and enjoys a strictly higher profit per unit sold. While the notation in (3.4) implies a restriction to distributions F_t without mass points, it would be straightforward to show that the restriction is without loss in generality.

The surplus $s_t(x)$ offered by a firm is strictly decreasing in the breadth of the firm's variety x. To this aim, take any x_0 and x_1, with $x_0 < x_1$, and let s_0 denote $s_t(x_0)$ and s_1 denote $s_t(x_1)$. Since x_1 chooses s_1 over s_0, it must be the case that

$$\exp(-\lambda_t X_t (1 - F_t(s_0))) \left[u(x_0) - s_0\right] \geq \exp(-\lambda_t X_t (1 - F_t(s_1))) \left[u(x_0) - s_1\right]. \tag{3.5}$$

Since x_0 chooses s_0 over s_1, it must be the case that

$$\exp(-\lambda_t X_t (1 - F_t(s_1))) \left[u(x_1) - s_1\right] \geq \exp(-\lambda_t X_t (1 - F_t(s_0))) \left[u(x_1) - s_0\right]. \tag{3.6}$$

The above inequalities together imply

$$\left[u(x_0) - u(x_1)\right] \left[\exp(-\lambda_t X_t (1 - F_t(s_0))) - \exp(-\lambda_t X_t (1 - F_t(s_1)))\right] \geq 0. \tag{3.7}$$

Since $u(x_0) > u(x_1)$, (3.7) implies that $s_0 \geq s_1$ and, hence, the surplus $s_t(x)$ offered by a firm must be weakly decreasing in the breadth of a firm's variety. Further, the surplus $s_t(x)$ must be strictly decreasing, or else the distribution $F_t(s)$ would have a mass point. These findings are intuitive. The gains from trade between a firm with a broad variety and an interested buyer are smaller than the gains from trade between a firm with a narrow variety and an interested buyer. Therefore, firms with a broad variety find it optimal to offer strictly less surplus than firms with a narrow variety.

3.3 Buyer's utility function

The first-order condition for the solution to the firm's pricing problem is

$$1 = \lambda_t X_t F_t'(s_t(x))(u(x) - s_t(x)). \tag{3.8}$$

The left-hand side of (3.8) is the (per-buyer) cost of a marginal increase in the surplus offered by the firm, which is given by the decline in profits per sale. The right-hand side is the (per-buyer) benefit of a marginal increase in the surplus offered by the firm, which is given by the increase in the number of sales, $\lambda_t X_t F_t'(s_t(x))$, times the profit per sale, $u(x) - s_t(x)$. Condition (3.8) states that the solution to the firm's pricing problem is such that the marginal cost and the marginal benefit of an increase in the amount of surplus offered by the firm are equated.

Consider a firm producing the broadest variety $x_{h,t}$ in the market. Since the lowest surplus offered by a firm is 0 and the surplus $s_t(x)$ offered by a firm is strictly decreasing in x, it follows that $s_t(x_{h,t}) = 0$ and, hence,

$$1 = \lambda_t X_t F_t'(0) u(x_{h,t}). \tag{3.9}$$

The average number of contacts per buyer λ_t grows at the rate g_λ. In a BGP, the average breadth of varieties, X_t, and the highest breadth of a variety, $x_{h,t}$, grow at the rate g_x. Moreover, in a BGP, the distribution F_t of surplus grows at the rate g_s and, hence, $F_t'(s\exp(g_s t))$ is equal to $F_0'(s)\exp(-g_s t)$. In light of these observations, (3.9) can be written as

$$1 = e^{(g_\lambda + g_x - g_s)t} \lambda_0 X_0 F_0'(0) u(x_{h,0} e^{g_x t}). \tag{3.10}$$

Differentiating (3.10) with respect to t gives

$$0 = (g_\lambda + g_x - g_s) u(x_{h,t}) + g_x u'(x_{h,t}) x_{h,t}. \tag{3.11}$$

The expression in (3.11) is a differential equation for the buyer's utility function u, as $x_{h,t}$ grows over time at the rate g_x. The unique solution to the differential equation is

$$u(x) = u_0 \left(\frac{1}{x}\right)^\alpha, \tag{3.12}$$

for some $u_0 > 0$ and $\alpha = (g_\lambda + g_x - g_s)/g_x$. The expression in (3.12) states that a necessary condition for the existence of a BGP is that the buyer's utility is a strictly increasing and isoelastic function of the degree of specificity $1/x$ of a variety. It is important to notice here that the condition $\alpha = (g_\lambda + g_x - g_s)/g_x$ should not be interpreted as a restriction on the elasticity of the buyer's utility function. The correct interpretation of the condition is that, for an arbitrary elasticity $\alpha > 0$ of the buyer's utility function, the growth rate g_s of surplus must equal $g_\lambda - (\alpha - 1) g_x$.

3.4 Growth rates

A firm producing a variety with breadth $x\exp(g_x t)$ offers the surplus $s_t(x\exp(g_x t))$ to its buyers. Therefore, the optimality condition (3.8) implies

$$1 = \lambda_t X_t F_t'(s_t(xe^{g_x t}))\left(u(xe^{g_x t}) - s_t(xe^{g_x t})\right). \tag{3.13}$$

The average number of contacts per buyer, λ_t, grows at the rate g_λ. The average breadth of varieties, X_t, grows at the rate g_x. The surplus $s_t(x\exp(g_x t))$ offered by a firm at a fixed quantile of the breadth distribution grows at the rate g_s. The surplus distribution F_t grows at the rate g_s and, hence, $F_t'(s\exp(g_s t))$ is equal to $F_0'(s)\exp(-g_s t)$. The utility $u(x\exp(g_x t))$ grows at the rate αg_x. In light of these observations, (3.13) can be written as

$$\begin{aligned} 1 &= e^{(g_\lambda + g_x - g_s)t}\lambda_0 X_0 F_0'(s_0(x))\left(u(x)e^{-\alpha g_x t} - s_0(x)e^{g_s t}\right) \\ &= e^{(g_\lambda + (1-\alpha)g_x - g_s)t}\lambda_0 X_0 F_0'(s_0(x))\left(u(x) - s_0(x)e^{(g_s+\alpha g_x)t}\right). \end{aligned} \tag{3.14}$$

The optimality condition (3.14) must hold for all t. Since the left-hand side of (3.14) is constant, the right-hand side must be constant as well. The first term on the right-hand side is constant because we showed that g_s must equal $g_\lambda - (\alpha-1)g_x$. The second term on the right-hand side is constant if $\alpha g_x + g_s$ is equal to 0 The solution to the two equations $g_s = g_\lambda - (\alpha - 1)g_x$ and $\alpha g_x + g_s = 0$ is such that the growth rate g_x of the breadth of varieties is equal to $-g_\lambda$, and the growth rate gs of the surplus offered by firms is equal to αg_λ. That is, in any BGP, the breadth of the varieties produced by firms must fall at the same rate at which the number of buyer's contacts grows. Moreover, in any BGP, the surplus offered by firms to buyers must grow at a rate equal to the rate at which the number of buyer's contacts grows multiplied by the elasticity of the buyer's utility function with respect to the specificity of a variety.

3.5 Firm's cost function

The profit for a firm selling a variety at a fixed quantile of the breadth distribution grows at the rate αg_λ. In fact,

$$\begin{aligned} R_t(xe^{g_x t}) &= \left\{ b\frac{xe^{g_x t}}{X_t}\lambda_t X_t e^{-\lambda_t X_t(1-F_t(s_t(x\exp(g_x t))))}\right\}\left(u(xe^{g_x t}) - s_t(xe^{g_x t})\right) \\ &= \left\{ b\frac{x}{X_0}\lambda_0 X_0 e^{-\lambda_0 X_0(1-F_t(s_0(x)))}\right\}\left(u(x)e^{-\alpha g_x t} - s_0(x)e^{g_s t}\right) \\ &= e^{\alpha g_\lambda t}R_0(x). \end{aligned} \tag{3.15}$$

The second line uses the fact that the average number of buyer's relevant contacts, $\lambda_t X_t$, remains constant over time, the buyer's utility from consuming a variety at a given quantile of the breadth distribution, $u(x\exp(g_x t))$, grows at the rate $-\alpha g_x$, and the surplus offered by a firm selling a variety at a given quantile of the breadth distribution, $s_t(x\exp(g_x t))$, as well as the surplus distribution, F_t, grow at the rate g_s. The third line uses the fact that $g_x = -g_\lambda$ and $g_s = \alpha g_\lambda$.

The value of a firm selling a variety at a fixed quantile of the breadth distribution is

$$V_t(xe^{g_x t}) = \int_0^T e^{-\rho\tau} R_{t+\tau}(xe^{g_x t}) d\tau + e^{-\rho T} \left[V_{t+T}(x_{\ell,t+T}) - q_{t+T} c\left(X_{t+T}^{-\beta} x_{\ell,t+T} \right) \right], \tag{3.16}$$
$$T = \log(x \exp(g_x t)/x_{h,t})/g_x.$$

In a BGP, the value of a firm selling a variety at a fixed quantile of the breadth distribution grows at the rate g_V and, hence, $V_t(x\exp(g_x t))$ equals $V_0(x)\exp(g_V t)$ and $V_{t+T}(x_{\ell,t+T})$ equals $V_T(x_{\ell,T})\exp(g_V t)$. In a BGP, the profit of a firm $R_{t+\tau}(x\exp(g_x t))$ equals $R_\tau(x)\exp(\alpha g_\lambda t)$. In a BGP, the breadth of a new variety $x_{\ell,t}$, the breadth of a scrapped variety $x_{h,t}$, and the average breadth X_t grow at the rate $g_x = -g_\lambda$. In light of these observations, (3.16) can be written as

$$e^{g_V t} V_0(x)$$
$$= e^{\alpha g_\lambda t} \int_0^T e^{-\rho\tau} R_\tau(x) d\tau + e^{-\rho T} \left[e^{g_V t} V_T(x_{\ell,T}) - q_T e^{g_q t} c\left(e^{-(1-\beta) g_\lambda t} X_T^{-\beta} x_{\ell,T} \right) \right], \tag{3.17}$$
$$T = \log(x/x_{h,0})/g_x.$$

Equation (3.17) must hold for all t and, hence, the right-hand side must grow at the same rate as the left-hand side, i.e. at the rate g_V. It is easy to see that the right-hand side grows at the same rate as the left-hand side only if all terms on the right-hand side do. The first term on the right-hand side grows at the rate g_V only if $g_V = \alpha g_\lambda$. The second term on the right-hand side grows at the rate $g_V = \alpha g_\lambda$ only if

$$e^{g_q t} c\left(e^{-(1-\beta) g_\lambda t} X_T^{-\beta} x_{\ell,T} \right) = e^{\alpha g_\lambda t} c\left(X_T^{-\beta} x_{\ell,T} \right). \tag{3.18}$$

The derivative of (3.18) with respect to t is

$$g_q c\left(X_{t+T}^{-\beta} x_{\ell,t+T} \right) - (1-\beta) g_\lambda c'\left(X_{t+T}^{-\beta} x_{\ell,t+T} \right) X_{t+T}^{-\beta} x_{\ell,t+T} = \alpha g_\lambda c\left(X_{t+T}^{-\beta} x_{\ell,t+T} \right). \tag{3.19}$$

There are two cases to consider. For $\beta \neq 1$, equation (3.19) is a differential equation for the firm's cost function c, as $X_t^{-\beta} x_{\ell,t}$ declines over time at the rate $g_\lambda(1-\beta)$. The unique solution to the differential equation is

$$c\left(\frac{x}{X^\beta} \right) = c_0 \left(\frac{X^\beta}{x} \right)^{\frac{\alpha g_\lambda - g_q}{(1-\beta) g_\lambda}}, \tag{3.20}$$

for some $c_0 > 0$. For $\beta = 1$, equation (3.19) holds if and only if

$$g_q = \alpha g_\lambda. \tag{3.21}$$

In the first case, the cost function c is strictly decreasing and isoelastic in the relative specificity of the variety X^β/x, with an elasticity of $(\alpha g_\lambda - g_q)/((1-\beta) g_\lambda)$. This is a knife-edge case, as there is no reason why the elasticity of the cost function should be tied to the elasticity α of the utility function. In the second case, the growth rate g_q of the price of the input used by firms in the design of a new variety grows at the rate αg_λ, the growth rate of the price of output. This is not a knife-edge case, as there are natural reasons why the price of the input used by firms should grow at the same rate as the price of the output sold by firms (e.g., the input used in the design of varieties is the output of firms). Therefore, I will focus on the second case.

3.6 Necessary conditions for a BGP

The following theorem summarizes the findings in this section.

Theorem 2. *(Necessary conditions for a BGP).*

1. *A BGP might exist only if:*
   - (a) *The buyer's utility function $u(x)$ has the form u_0/x^α for some $u_0 > 0$, $\alpha > 0$;*
   - (b) *The firm's cost function $C_t(x/X_t^\beta)$ has the form $q_t c(x/X_t)$, where the input price q_t grows at the rate $g_q = \alpha g_\lambda$ and the quantity of input $c(x/X_t)$ is strictly decreasing in x.*
2. *If a BGP exists, it must have the following properties:*
   - (a) *The distribution of varieties H_t is log-uniform over the interval $[x_{\ell,t}, x_{h,t}]$, and $x_{\ell,t}$ and $x_{h,t}$ decline at the constant rate $-g_x = g_\lambda$;*
   - (b) *The distribution of surplus F_t grows at the constant rate $g_s = \alpha g_\lambda$;*
   - (c) *The distribution of prices G_t grows at the constant rate $g_p = \alpha g_\lambda$;*
   - (d) *The value of a firm producing a variety at a fixed quantile of the H_t distribution grows at the constant rate $g_V = \alpha g_\lambda$.*

Part 1 of the theorem provides restrictions on the fundamentals of the model that are necessary for the existence of a BGP. The first restriction is that the buyer's utility function needs to be isoelastic with respect to the specificity of a variety. The restriction follows from the fact that—in order for the distribution of surplus to grow at a constant rate while the specificity of varieties grows at a constant rate—the elasticity of the buyer's utility function with respect to the specificity of a variety must be constant over time. The second restriction is that the price of the input used by a firm to design a variety must grow at the same rate as the price of the output and that the quantity of input to design a new variety with breadth $x_{\ell,t}$ must be constant. The second restriction follows from the fact that—in order for the value of a firm to grow at a constant rate—the price of an old variety and the cost of designing a new variety must grow at the same rate.

Part 2 of the theorem provides a characterization of the properties that any BGP must have. The most important property is that the breadth of the varieties sold by firms in the market must decline at the same rate at which the average number of firms contacted by a buyer increases. This property of a BGP is necessary to guarantee that the extent of competition in the market remains constant, where the extent of competition is captured by the number of firms contacted by a buyer who have a product that the buyer likes. As we shall see in the next section, the property is satisfied because, when search frictions decline, firms do indeed find it optimal to increase the specificity of their varieties at a rate that exactly offsets the decline in search frictions.

4 Sufficient Conditions for a BGP

In this section, I derive sufficient conditions for the existence of a BGP. I impose the restrictions on the fundamentals of the model that are necessary for the existence of a BGP and that are listed in the first part of Theorem 2. I also make use of the characterization of the necessary properties of the dynamics aggregate variables that must hold in any BGP and that are listed in the second part of Theorem 2. I show that the optimal policies for an individual firm conform to the requirements of a BGP, as described in Definition 1. This step of the analysis is carried out by showing that the problems of an individual firm have a time-invariant representation. I then show that the dynamics of aggregate variables are consistent with the policy of an individual firm. In the end, I show that the existence of a BGP boils down to the existence of a solution to a system of 3 time-invariant equations in 3 unknown scalars.

4.1 Distribution of varieties and surplus

Let $\hat{x}$ denote the breadth of a variety relative to the most specialized variety in the market, i.e. $\hat{x} = x/x_{\ell,t}$. I will refer to $\hat{x}$ as the *relative breadth* of the variety. Similarly, let $\hat{x}_\ell$ and $\hat{x}_h$ denote the variety with the lowest relative breadth and the variety with the highest relative breadth in the market, i.e. $\hat{x}_\ell = x_{\ell,t}/x_{\ell,t} = 1$ and $\hat{x}_h = x_{h,t}/x_{\ell,t}$. In any BGP, the breadth distribution $H_t(x)$ must be given by (3.3). Therefore, the relative breadth distribution is

$$\begin{aligned}\hat{H}(\hat{x}) &= H_t(\hat{x} \cdot x_{\ell,t}) \\ &= \frac{\log(\hat{x} \cdot x_{\ell,t}) - \log(\hat{x}_\ell \cdot x_{\ell,t})}{\log(\hat{x}_h \cdot x_{\ell,t}) - \log(\hat{x}_\ell \cdot x_{\ell,t})} = \frac{\log \hat{x} - \log \hat{x}_\ell}{\log \hat{x}_h - \log \hat{x}_\ell}.\end{aligned} \tag{4.1}$$

The relative breadth distribution is log-uniform over the interval $[\hat{x}_\ell, \hat{x}_h]$, and it remains constant over time.

Let $\hat{X}$ denote the average relative breadth, which is constant since the average breadth distribution is time-invariant. Using (4.1), $\hat{X}$ is given by

$$\hat{X} = \frac{\hat{x}_h - \hat{x}_\ell}{\log \hat{x}_h - \log \hat{x}_\ell}. \tag{4.2}$$

Also, let ϕ denote the buyer's average number of relevant contacts, i.e. $\phi = \lambda_t x_{\ell,t} \hat{X}$. In any BGP, $x_{\ell,t}$ must decline at the same rate at which the average number of buyer's contacts increases and hence ϕ must be constant. Using (4.2), it follows that

$$\phi = x_{\ell,0} \lambda_0 \frac{\hat{x}_h - \hat{x}_\ell}{\log \hat{x}_h - \log \hat{x}_\ell}. \tag{4.3}$$

Let $\hat{s}$ denote the surplus offered by a firm relative to the buyer's utility from consuming the most specialized variety in the market, i.e. $\hat{s} = s/(1/x_{\ell,t}^\alpha)$ which is equal to $s x_{\ell,t}^\alpha$. In any BGP, the surplus distribution F_t grows at the rate $g_s = \alpha g_\lambda$. Therefore, the relative surplus distribution is

$$\begin{aligned} \hat{F}(\hat{s}) &= F_t(\hat{s} x_{\ell,t}^{-\alpha}) \\ &= F_t(\hat{s} x_{\ell,0}^{-\alpha} \exp(\alpha g_\lambda)) = F_0(\hat{s} x_{\ell,0}^{-\alpha}). \end{aligned} \tag{4.4}$$

Hence, the relative surplus distribution remains constant over time. Also, since the surplus offered by a firm is strictly decreasing in the relative breadth of the firm's variety and the relative breadth distribution is log-uniform over the interval $[\hat{x}_\ell, \hat{x}_h]$, the fraction of firms offering a relative surplus below $\hat{s}(\hat{x}_0)$ is

$$\hat{F}(\hat{s}(\hat{x}_0)) = \frac{\int_{\hat{x}_0}^{\hat{x}_h} \hat{x} d\hat{H}(\hat{x})}{\int_{\hat{x}_\ell}^{\hat{x}_h} \hat{x} d\hat{H}(\hat{x})} = \frac{\hat{x}_h - \hat{x}_0}{\hat{x}_h - \hat{x}_\ell}. \tag{4.5}$$

4.2 Firm's pricing problem

I now demonstrate that the firm's pricing problem has a time-invariant representation. The problem for a firm selling a variety with relative breadth $\hat{x}$ is

$$\begin{aligned} R_t(\hat{x} \cdot x_{\ell,t}) &= \max_{s \geq 0} \left\{ b \frac{\hat{x} \cdot x_{\ell,t}}{X_t} \phi e^{-\phi(1-F_t(s))} \right\} \left(\frac{u_0}{\hat{x}^\alpha \cdot x_{\ell,t}^\alpha} - s \right) \\ &= \max_{\hat{s} \geq 0} \left\{ b \frac{\hat{x}}{\hat{X}} \phi e^{-\phi(1-F_t(\hat{s} x_{\ell,t}^{-\alpha}))} \right\} \left(\frac{u_0}{\hat{x}^\alpha \cdot x_{\ell,t}^\alpha} - \frac{\hat{s}}{x_{\ell,t}^\alpha} \right) \\ &= e^{\alpha g_\lambda} x_{\ell,0}^{-\alpha} \cdot \max_{\hat{s} \geq 0} \left\{ b \frac{\hat{x}}{\hat{X}} \phi e^{-\phi(1-\hat{F}(\hat{s}))} \right\} \left(\frac{u_0}{\hat{x}^\alpha} - \hat{s} \right). \end{aligned} \tag{4.6}$$

In the first line of (4.6), I make use of the fact that the buyer's utility is isoelastic with respect to the specificity of the firm's variety. In the second line, I change the choice variable from the surplus s to the relative surplus $\hat{s} = s x_{\ell,t}^\alpha$ and make use of the fact that $g_x = -g_\lambda$. In the last line, I make use of (4.4).

Notice that the problem in the last line of (4.6) depends only on the firm's relative breadth $\hat{x}$ and not on calendar time t, i.e.

$$r(\hat{x}) = \max_{\hat{s}\geq 0} \left\{ b\frac{\hat{x}}{\hat{X}}\phi e^{-\phi(1-\hat{F}(\hat{s}))} \right\} \left(u_0\hat{x}^{-\alpha} - \hat{s}\right). \tag{4.7}$$

Denote with $\hat{s}(\hat{x})$ the optimal relative surplus that solves (4.7). Since a firm selling a variety with a constant relative breadth $\hat{x}$ finds it optimal to offer a constant relative surplus $\hat{s}(\hat{x})$, it follows that the surplus $s_t(\hat{x} \cdot x_{\ell,0} \exp(g_x t))$ offered by the firm is equal to $x_{\ell,0}^{-\alpha g_\lambda t}\hat{s}(\hat{x})$ and, hence, grows at a constant rate, as required by the definition of a BGP. Moreover, since the surplus offered by a firm with relative breadth $\hat{x}$ grows at the constant rate αg_λ and the distribution of relative breadths is stationary, the growth rate of the aggregate surplus distribution F_t is consistent with the individual behavior of firms.

Next, I solve for the optimal relative surplus $\hat{s}(\hat{x})$. The optimality condition for $\hat{s}(\hat{x})$ is

$$1 = \phi\hat{F}'(\hat{s}(\hat{x}))\left(u_0\hat{x}^{-\alpha} - \hat{s}(\hat{x})\right). \tag{4.8}$$

The relative surplus distribution (4.5) implies that $\hat{F}'(\hat{s}(\hat{x}))$ is equal to $-1/\left(\hat{x}_h - \hat{x}_\ell\right)\hat{s}'(\hat{x})$. Therefore, I can rewrite (4.8) as a differential equation for the optimal relative surplus

$$\hat{s}'(\hat{x}) = -\frac{\phi}{\hat{x}_h - \hat{x}_\ell}\left(u_0\hat{x}^{-\alpha} - \hat{s}(\hat{x})\right). \tag{4.9}$$

The unique solution to the differential equation (4.9) such that $\hat{s}(\hat{x}_h) = 0$, i.e. such that the relative surplus offered by a firm with the broadest variety in the market is zero, is

$$\hat{s}(\hat{x}) = \frac{u_0}{\hat{x}_h - \hat{x}_\ell}\phi\int_{\hat{x}}^{\hat{x}_h} z^{-\alpha}\exp\left(-\phi\frac{z - \hat{x}_\ell}{\hat{x}_h - \hat{x}_\ell}\right)dz. \tag{4.10}$$

The optimal surplus offered by a firm is strictly decreasing in the relative breadth of its variety for all $\hat{x}$ on the support $[\hat{x}_\ell, \hat{x}_h]$ of the equilibrium distribution. Off the equilibrium, the optimal surplus offered by a firm selling a variety with relative breadth $\hat{x} < \hat{x}_\ell$ is $\hat{s}(\hat{x}_\ell)$. Indieed, this firm has no incentive to offer surplus above $\hat{s}(\hat{x}_\ell)$ because it already beats all of its competitors. Conversely, a firm selling a variety with relative breadth $x > \hat{x}_h$ offers the surplus 0. This firm would like to offer less surplus but doing so would mean having no customers. Therefore, we have

$$\begin{aligned} &\hat{s}(\hat{x}) = \hat{s}(\hat{x}_\ell), \text{ for } \hat{x} \leq \hat{x}_\ell, \\ &\hat{s}(\hat{x}) = 0, \text{ for } \hat{x} \geq \hat{x}_h. \end{aligned} \tag{4.11}$$

Lastly, I characterize some properties of the firm's profit function $r(\hat{x})$. Substituting $1 - \hat{F}(\hat{s}(\hat{x}))$ with (4.5), I can write $r(\hat{x})$ as

$$r(\hat{x}) = \left\{ b\frac{\hat{x}}{\hat{X}}\phi e^{-\phi\frac{\hat{x}-\hat{x}_\ell}{\hat{x}_h-\hat{x}_\ell}} \right\}\left(u_0\hat{x}^{-\alpha} - \hat{s}(\hat{x})\right) \text{ for } \hat{x} \in [\hat{x}_\ell, \hat{x}_h], \tag{4.12}$$

and

$$r(\hat{x}) = \begin{cases} b\dfrac{\hat{x}}{\hat{X}}\phi\left(u_0\hat{x}^{-\alpha} - \hat{s}(\hat{x}_\ell)\right) & \text{for } x < x_\ell, \\ b\dfrac{\hat{x}}{\hat{X}}\phi e^{-\phi}u_0\hat{x}^{-\alpha} & \text{for } x > x_h. \end{cases}$$

It is easy to show that $r(\hat{x})$ is strictly decreasing in the relative breadth $\hat{x}$ if and only if $\alpha > 1$. Similarly, if $\alpha > 1$, it is easy to show that $r(\hat{x})$ is strictly convex in the relative breadth $\hat{x}$. I will assume that $\alpha > 1$ in what follows.

4.3 Firm's design problem

I now demonstrate that the firm's product design problem has a time-invariant representation. The problem for a with selling a variety with relative breadth $\hat{x}$ is

$$V_t(\hat{x}\cdot x_{\ell,t}) = \max_{T,x_n} \int_0^T e^{-\rho\tau}R_{t+\tau}(\hat{x}\cdot x_{\ell,t})d\tau + e^{-\rho T}\left[V_{t+T}(x_n) - q_{t+T}c\left(x_n/X_{t+T}\right)\right]. \quad (4.13)$$

I guess and verify that the solution to the functional equation in (4.13) is

$$V_t(\hat{x}\cdot x_{\ell,t}) = e^{\alpha g_\lambda}x_{\ell,0}^{-\alpha}\cdot v\left(\hat{x}\right). \quad (4.14)$$

The function $v\left(\hat{x}\right)$ in (4.14) is time-invariant and it is defined as

$$v\left(\hat{x}\right) = \max_{T,\hat{x}_n} \int_0^T e^{-(\rho-\alpha g_\lambda)\tau}r\left(\hat{x}e^{g_\lambda\tau}\right)d\tau + e^{-(\rho-\alpha g_\lambda)T}\left[v(\hat{x}_n) - \hat{q}c\left(\hat{x}_n/\hat{X}\right)\right], \quad (4.15)$$

in which $\hat{q}$ denotes the price of the input of the product design expressed as a ratio of the buyers' utility from consuming a variety with breadth $x_{\ell,t}$, i.e. $\hat{q} = q_t x_{\ell,t}^{\alpha}$.

In order to verify the guess, I substitute (4.15) in (4.14) and obtain

$$\begin{aligned} x_{\ell,t}^{-\alpha}\cdot v\left(\hat{x}\right) &= \max_{T,x_n}\int_0^T e^{-\rho\tau}R_{t+\tau}(\hat{x}\cdot x_{\ell,t})d\tau + e^{-\rho T}\left[x_{\ell,t+T}^{-\alpha}v\left(x_n/x_{\ell,t+T}\right) - q_{t+T}c\left(x_n/X_{t+T}\right)\right] \\ &= \max_{T,\hat{x}_n}\int_0^T e^{-\rho\tau}x_{\ell,t+\tau}^{-\alpha}r\left(\frac{\hat{x}\cdot x_{\ell,t}}{x_{\ell,t+\tau}}\right)d\tau + e^{-\rho T}\left[x_{\ell,t+T}^{-\alpha}v\left(\hat{x}_n\right) - \hat{q}x_{\ell,t+T}^{-\alpha}c\left(\hat{x}_n/\hat{X}\right)\right] \\ &= x_{\ell,t}^{-\alpha}\cdot\max_{T,\hat{x}_n}\int_0^T e^{-(\rho-\alpha g_\lambda)\tau}r\left(\hat{x}e^{g_\lambda\tau}\right)d\tau + e^{-(\rho-\alpha g_\lambda)T}\left[v\left(\hat{x}_n\right) - \hat{q}c\left(\hat{x}_n/\hat{X}\right)\right]. \end{aligned} \quad (4.16)$$

The second line makes use of the fact that $R_{t+\tau}(\hat{x}\cdot x_{\ell,t})$ is equal to $x_{\ell,t+\tau}^{-\alpha}r(\hat{x}\cdot x_{\ell,t}/x_{\ell,t+\tau})$ and that q_{t+T} is equal to $\hat{q}x_{\ell,t+T}^{-\alpha}$. The second line also changes the choice variable from the breadth x_n of the new variety to the relative breadth $\hat{x}_n$. The third line makes use of the fact that $x_{\ell,t}$ grows at the rate $-g_\lambda$ and, hence, $x_{\ell,t}^{-\alpha}$ grows at the αg_λ. Eliminating $x_{\ell,t}^{-\alpha}$ from the left-hand side of (4.16) and from the third line on the right-hand side of (4.16) verifies the conjectured solution to (4.13).

The product design problem in (4.15) depends only on the relative breadth $\hat{x}$ of the variety sold by the firm and not on calendar time. The problem has a simple interpretation.

The normalized value of a firm selling $\hat{x}$ is given by the sum between the present value of the normalized flow of profit between now and date T and the normalized present value of designing a new variety with relative breadth $\hat{x}_n$ at date T. The normalized flow profit depends only on the relative breadth of the firm's variety, which grows at the rate g_λ. The normalized present value of designing a new variety depends on $\hat{x}_n$. The effective rate at which the normalized profits are discounted is $\rho - \alpha g_\lambda$, where ρ is the rate of time preference and αg_λ is the growth rate of the normalization factor. The firm chooses T and $\hat{x}_n$ so as to maximize its normalized present value of profits. Clearly, the product design problem in (4.15) is well-defined if and only if the effective discount rate $\rho - \alpha g_\lambda$ is strictly positive.

Denote as $\hat{x}_\ell$ the relative breadth of a new variety that solves (4.15). Notice that $\hat{x}_\ell$ is independent of calendar time t, since it solves a time-invariant problem, and it is independent of the relative breadth of the firm's variety $\hat{x}$, since it maximizes the firm's continuation value which is independent of $\hat{x}$. Therefore $\hat{x}_\ell$ is just a constant. Since $\hat{x}_\ell$ is a constant, the breadth of a new variety created at date t is simply $\hat{x}_\ell \cdot x_{\ell,t}$, which grows at the constant rate $-g_\lambda$. Denote as $T(\hat{x})$ the time when the firm scraps its variety that solves (4.15). It is easy to see that $T(\hat{x})$ is such that $\hat{x}\exp(g_\lambda T(\hat{x}))$ equals some $\hat{x}_h$, where $\hat{x}_h$ is a constant. Since $\hat{x}_h$ is a constant, a variety is scrapped at date t if its breadth is $\hat{x}_\ell \cdot x_{\ell,t}$, which grows at the constant rate $-g_\lambda$. The two thresholds for the firm's problem grow at the constant rate $-g_\lambda$, as required by the definition of a BGP. Moreover, given an initial distribution H_0 that is log-uniform over $[x_{\ell,0}, x_{h,0}]$, the growth rate of the aggregate breadth distribuiton H_t is constant and equal to $-g_\lambda$.

The first-order condition for $\hat{x}_\ell$ is

$$-\hat{q}c'\left(\hat{x}_\ell/\hat{X}\right)\Big/\hat{X} = -v'(\hat{x}_\ell). \tag{4.17}$$

The left-hand side of (4.17) is the cost from designing a marginally narrower variety, as $c(\hat{x}/\hat{X})$ is a strictly decreasing function. The right-hand side of (4.17) is the benefit from designing a marginally narrower variety. The marginal cost and the marginal benefit must be equal at the optimal relative breadth $\hat{x}$. For an appropriate choice of $c(\hat{x}/\hat{X})$, (4.17) is not only necessary for optimality but also sufficient.

The first-order condition for $T(\hat{x})$ is

$$r(\hat{x}_h) = (\rho - \alpha g_\lambda)\left[v(\hat{x}_\ell) - \hat{q}c\left(\hat{x}_\ell/\hat{X}\right)\right]. \tag{4.18}$$

The left-hand side of (4.18) is the benefit of scrapping the old variety an instant later, which is given by the flow profit $r(\hat{x}_h)$. The right-hand side of (4.18) is the cost of scrapping the old variety an instant later, which is given by the firm's value from producing a new variety with relative breadth $\hat{x}_\ell$, $v(\hat{x}_\ell)$, net of the cost of producing a variety $\hat{x}_\ell$, $\hat{q}c(\hat{x}_\ell/\hat{X})$, annuitized by the effective discount rate $\rho - \alpha g_\lambda$. Since $r(\hat{x})$ is strictly decreasing in $\hat{x}$, it follows that (4.18) is not only necessary but also sufficient for optimality.

It is useful to derive a continuous-time Bellman equation for $v(\hat{x})$. For all $\hat{x} > \hat{x}_h$ and $dt > 0$ small, $v(\hat{x})$ is equal to

$$v(\hat{x}) = r(\hat{x})dt + e^{-(\rho-\alpha g_\lambda)dt} v(\hat{x}e^{g_\lambda dt}). \tag{4.19}$$

Subtracting $v(\hat{x})\exp(-(\rho - \alpha g_\lambda)dt)$ from both sides of the equation, dividing by dt, and taking limits for $dt \to 0$, one recovers the standard continuous-time Bellman equation

$$(\rho - \alpha g_\lambda)\, v(\hat{x}) = r(\hat{x}) + v'(\hat{x})\hat{x}g_\lambda. \tag{4.20}$$

The left-hand side of (4.20) is the annuity value of the normalized present value of profits for a firm selling a variety with relative breadth $\hat{x}$. The first term on the right-hand side is the instantaneous normalized flow profit of the firm. The second term on the right-hand side is the change in the firm's normalized present value of profits, which is due to the fact that the relative breadth of the firm's variety grows at the rate g_λ.

4.4 Sufficient conditions for a BGP

The existence of a BGP boils down to the existence of a solution to a system of three equations: the first-order condition (4.17) for the relative breadth of a new variety, the first-order condition (4.18) for the relative breadth of a scrapped variety, and the condition (4.15) for the normalized value of a firm with a newly designed variety. These three equations have three unknowns: the buyer's average number of relevant contacts, ϕ, the relative breadth of a scrapped variety, $\hat{x}_h$, and the normalized value of a firm with a newly designed variety, $v(\hat{x}_\ell)$. Note that $\hat{x}_\ell$ is not an unknown because it equals 1 by construction. If a solution to the system of three equations in three unknows exists, then a BGP exists as demonstrated in the previous pages.

The first-order condition (4.17) for the relative breadth of a new variety can be written as

$$r(\hat{x}_\ell) = (\rho - \alpha g_\lambda) v(\hat{x}_\ell) - \hat{q}c'\left(\frac{\hat{x}_\ell}{\hat{X}}\right)\frac{\hat{x}_\ell g_\lambda}{\hat{X}}, \tag{4.21}$$

where $r(\hat{x}_\ell)$ is given by (4.12), $\hat{X}$ is given by (4.2), and $\hat{q}$ is given by

$$\hat{q} = q_0 \frac{\phi}{\lambda_0} \frac{\log \hat{x}_h - \log \hat{x}_\ell}{\hat{x}_h - \hat{x}_\ell}. \tag{4.22}$$

To obtain (4.21), I substitute out $v'(\hat{x}_\ell)$ using the continuous-time Bellman equation in (4.20). To obtain (4.22), I use the definition of $\hat{q}$ as $q_0 x_{\ell,0}^\alpha$ and then substitute out $x_{\ell,0}$ using the definition of ϕ in (4.3). Intuitively, condition (4.21) is an equation that requires ϕ to be such that an individual firm finds it optimal to design a variety $\hat{x}$, given that the relative breadth distribution is log-uniform over the interval $[\hat{x}_\ell, \hat{x}_h]$ and the normalized value of a firm with a newly designed variety is $v(\hat{x}_\ell)$.

The first-order condition (4.15) for the relative breadth of a scrapped variety is

$$r(\hat{x}_h) = (\rho - \alpha g_\lambda)\left[v(\hat{x}_\ell) - \hat{q}c\left(\hat{x}_\ell/\hat{X}\right)\right], \tag{4.23}$$

where $r(\hat{x}_h)$ is given by (4.12), $\hat{X}$ is given by (4.2), and $\hat{q}$ is given by (4.22). Intuitively, condition (4.23) is an equation that requires an individual firm to find it optimal to scrap its variety when its relative breadth reaches $\hat{x}_h$, given that the distribution of relative breadths is log-uniform over the interval $[\hat{x}_\ell, \hat{x}_h]$ and the normalized value of a firm with a newly designed variety is $v(\hat{x}_\ell)$.

The firm's value from selling a variety with relative breadth $\hat{x}_\ell$ is

$$v(\hat{x}_\ell) = \int_0^{\log(\hat{x}_h/\hat{x}_\ell)/g_\lambda} e^{-(\rho-\alpha g_\lambda)\tau} r\left(\hat{x}_\ell e^{g_\lambda \tau}\right) d\tau + (\hat{x}_h/\hat{x}_\ell)^{-\frac{\rho-\alpha g_\lambda}{g_\lambda}}\left[v(\hat{x}_\ell) - \hat{q}c\left(\hat{x}_\ell/\hat{X}\right)\right], \tag{4.24}$$

$r(\hat{x}_\ell \exp(g_\lambda \tau))$ is given by (4.12), $\hat{X}$ is given by (4.2), and $\hat{q}$ is given by (4.22). The expression above is derived from (4.15). I first use the fact that the firm finds it optimal to scrap a variety with relative breadth $\hat{x}_\ell$ after $\log(\hat{x}_h/\hat{x}_\ell)/g_\lambda$ units of time. I then use the fact that the firm finds it optimal to design a new variety with relative breadth $\hat{x}_\ell$. The expression in (4.24) is an equation that requires $v(\hat{x}_\ell)$ to be equal to the normalized present value of profits enjoyed by a firm following the optimal product design strategy, given that the relative breadth distribution is log-uniform over the interval $[\hat{x}_\ell, \hat{x}_h]$.

I am now in the position to state the sufficient conditions for the existence of a BGP.

Theorem 3. *(Sufficient conditions for a BGP) A BGP exists if:*

a. The buyer's utility function $u(x)$ has the form u_0/x^α for some $u_0 > 0$, $\alpha > 1$;

b. The firm's cost function $C_t(x/X_t^\beta)$ has the form $q_t c(x/X_t)$, where the input price q_t grows at the rate $g_q = \alpha g_\lambda$ and the quantity of input $c(x/X_t)$ is strictly decreasing in x and such that $v(\hat{x}) - qc(\hat{x}/\hat{X})$ is quasi-concave;

c. The discount rate ρ is greater than αg_λ;

d. The system of equations (4.21), (4.23)-(4.24) admits a solution with respect to ϕ, $\hat{x}_h$ and $v(\hat{x}_\ell)$.

The sufficient conditions for the existence of a BGP are more stringent than the necessary conditions. The elasticity α of the buyer's utility u function is required to be greater than 1 so as to guarantee that the necessary condition for the optimal choice of the relative breadth of a scrapped variety is also sufficient. The firm's design cost function c is required to be such that the product design problem is quasi-concave to guarantee that the necessary condition for the optimal choice of the relative breadth of a new variety is also sufficient. The discount rate ρ is required to be greater than αg_λ to guarantee that the firm's value function is well-defined. The system of equations (4.21), (4.23) and (4.24) are required to have a solution only because I was unable to prove analytically that there always is one. The difficulty in solving the system of equations is that the firm's optimal choices for $\hat{x}_\ell$ and $\hat{x}_h$ depend in a complicated way on the boundaries of the breadth distribution $\hat{x}_\ell$ and $\hat{x}_h$ and, hence, proving that the optimal choices can be lined up with the boudaries is difficult. In all my numerical examples, though, I did find a solution.

Let me provide some intuition for the existence of a BGP. As search frictions decline, firms meet more and more buyers per unit of time and, for this reason, they choose to design varieties of the product that are increasingly specialized. When the buyer's utility function is isoelastic with respect to the specificity of a variety, the firm's design cost is a function of the specificity of a variety relative to the market average, and the cost grows over time at the rate αg_λ, then firms find it optimal to increase the specificity of their varieties at exactly the same rate at which search frictions decline. When the increase in the specificity of product varieties exactly offsets the decline in search frictions, the probability that a buyer meets n firms selling a variety that he likes remains constant and, hence, the extent of competition among firms remains constant. For this reason, the price distribution, the buyer's surplus and the firm's value all grow along a balanced path at a common, constant rate.

5 Aggregate and Industry Dynamics

In this section, I describe some of the key properties of a BGP. In section 5.1, I describe the aggreate dynamics of the market along a BGP. In section 5.2, I describe the dynamics of individual firms along a BGP. At the aggregate level, the main findings are that, while declining search frictions do not generate any decline in price dispersion or any increase in volume, they do contribute to economic growth. At the industry level, the main finding is that every individual firm goes through the same cycle, which starts with the firm at the top of the market and ends with the firm at the bottom.

5.1 Aggregate dynamics

The initial distribution of firms across the breadth of their varieties is log-uniform over the interval $[x_{\ell,0}, x_{h,0}]$, where $x_{\ell,0}$ is given by $\phi(\log \hat{x}_h - \log \hat{x}_\ell)/\, \lambda_0(\hat{x}_h - \hat{x}_\ell)$ and $x_{h,0}$ is given by $\hat{x}_h \cdot x_{\ell,0}$. Firms scrap the variety of the product that they are selling when its breadth is reached by the cutoff $x_{h,t}$, where $x_{h,t}$ declines at the rate g_λ. Firms design new varieties of the product with a breadth of $x_{\ell,t}$, where $x_{\ell,t}$ declines at the rate g_λ. As a result, the breadth distribution falls over time at the constant rate g_λ. That is, the products sold on the market become more and more specialized, and they do so at a rate that is exactly equal to the rate at which search frictions decline.

At date t, a firm selling a variety with breadth $\hat{x} \cdot x_{\ell,t}$ offers to its customers a surplus of $s_t(\hat{x} \cdot x_{\ell,t})$ equal to $\hat{s}(\hat{x}) x_{\ell,0}^{-\alpha} \exp(\alpha g_\lambda t)$, where $\hat{s}(\hat{x})$ is given by (4.10). The function $\hat{s}(\hat{x})$ is strictly decreasing in $\hat{x}$ and such that $\hat{s}(\hat{x}_h) = 0$. Since a firm selling a variety with breadth $\hat{x} \cdot x_{\ell,t}$ is at a constant quantile of the breadth distribution and since the surplus

$s_t(\hat{x} \cdot x_{\ell,t})$ offered by such a firm grows at the rate αg_λ, the surplus distribution F_t grows at the rate αg_λ.

At date t, a firm selling a variety with breadth $\hat{x} \cdot x_{\ell,t}$ charges to its customers a price of $p_t(\hat{x} \cdot x_{\ell,t})$ equal to $\hat{p}(\hat{x}) x_{\ell,0}^{-\alpha} \exp(\alpha g_\lambda t)$, where

$$\hat{p}(\hat{x}) = u_0 \hat{x}^{-\alpha} - \hat{s}(\hat{x}). \tag{5.1}$$

It is easy to verify that the function $\hat{p}(\hat{x})$ is such that $\hat{p}'(\hat{x}) = 0$ implies $\hat{p}''(\hat{x}) > 0$. Therefore, $\hat{p}(\hat{x})$ may be strictly decreasing in $\hat{x}$, strictly increasing in $\hat{x}$, or strictly decreasing over some interval $[\hat{x}_\ell, \hat{x}_c]$ and strictly increasing over the interval $[\hat{x}_c, \hat{x}_h]$. Irrespective of the shape of $\hat{p}(\hat{x})$, the price distribution G_t grows at the constant rate αg_λ, since a firm selling a variety with breadth $\hat{x} \cdot x_{\ell,t}$ is at a constant quantile of the breadth distribution and the price charged by such a firm grows at the constant rate αg_λ. Also irrespective of the shape of $\hat{p}(\hat{x})$, the price distribution G_t is non-degenerate because $\hat{p}'(\hat{x}) = 0$ implies $\hat{p}''(\hat{x}) > 0$.

At date t, a firm selling a variety with breadth $\hat{x} \cdot x_{\ell,t}$ trades with $b(\hat{x})$ buyers, where

$$b(\hat{x}) = b \frac{\hat{x}}{\hat{X}} \phi \exp\left[-\phi \frac{\hat{x} - \hat{x}_\ell}{\hat{x}_h - \hat{x}_\ell} \right]. \tag{5.2}$$

It is easy to verify that the function $b(\hat{x})$ is such that $b'(\hat{x}) = 0$ implies $b''(\hat{x}) < 0$. Therefore, $n(\hat{x})$ may be strictly decreasing in $\hat{x}$, strictly increasing in $\hat{x}$, or strictly increasing over some interval $[\hat{x}_\ell, \hat{x}_c]$ and strictly decreasing over the interval $[\hat{x}_c, \hat{x}_h]$. Irrespective of the shape of $n(\hat{x})$, the distribution of sales across firms remains constant over time, as the relative breadth distribution is constant over time.

Three aggregate properties of the BGP are worth highlighting. First, note that price dispersion remains constant in the face of declining search frictions. The price distribution $G_t(p)$ is non-degenerate and grows at the constant rate αg_λ. Hence, the distribution of normalized prices—i.e. prices divided by the average price—is non-degenerate and remains constant over time. In this sense, the extent of price dispersion remains unchanged even though search frictions become smaller and smaller. Intuitively, the extent of price dispersion remains constant because the exogenous decline in search frictions, which tends to increase in the extent of competition between firms, is undone by the endogenous increase in the specificity of the varieties of the product sold by firms.

Second, note that trade volume remains constant in the market, despite declining search frictions. Indeed, since a firm sells a quantity of output that only depends on the relative breadth of its variety and the relative breadth distribution is constant, it follows that trade volume remains constant over time. There is a simple intuition for this finding. On the one hand, the exogenous decline in search frictions increase the probability that a buyer locates a firm and the number of firms that he locates. On the other hand, the endogenous increase in the specificity of the varieties on the market lowers the probability that a firm has a variety of the product that the buyer likes to consume. In a BGP, these two opposing forces exactly offset each other.

Third, note that, while declining search frictions do not enhance competition nor increase volume, they do lead to economic growth. Indeed, the aggregate surplus captured by buyers as well as the aggregate profits enjoyed by firms grow over time at the rate αg_λ, i.e. the rate g_λ at which search frictions decline multiplied by the elasticity α of the buyers' utility function with respect to the specificity of the variety consumed. The economic growth generated by declining search frictions is related to Smithian growth. The view of Adam Smith is that the geographical extent of the market determines how much production can be specialized to exploit technological differences, input availability, or increasing returns to scale etc... As a result, as the size of the market increases, specialization increases and so does productivity. Here, declining search frictions cause a deepening of the market and, in response to such deepening, firms can design more specialized products. Specialization, in turn, leads to growth by exploting the heterogeneity in the preferences of different buyers. The key difference between Smithian growth and the type of market deepening growth illustrated in this paper is that the former is bounded by geography and population (a market can only expand so much), while the latter is not bounded by geography and population (a market can always grow deeper).

Lastly, it is useful to discuss the notion of price dispersion in the model. In the model, price dispersion is defined at the level of a good, where the notion of a "good" is defined by the collection of all the products sold in the market where buyers search. Given this notion of a good, the model generates price dispersion that remains constant over time. Yet, the varieties of sold by different firms are not identical, as they differ with respect to their degree of specialization. Therefore, a sensibly narrower definition of a good is the collection of all the varieties with a breadth in some interval $[x/(1+\epsilon), x(1+\epsilon)]$ for some $\epsilon > 0$. Also for this narrower definition of what a good is, the model still generates price dispersion that remains constant over time. It is only when a good is defined in its narrowest sense—i.e. varieties with the same specificity that appeal to the same subset of buyers—that the model generates no price dispersion. However, this is an artifact of the assumption that there is a continuous distribution of breadths. Indeed, if breadths were a discrete set, the model would generate price dispersion even among identical varieties.

I summarize the aggregate properties of a BGP in the theorem below.

Theorem 4. *(Aggregate dynamics in a BGP). In any BGP:*

a. *The breadth distribution H_t declines at the constant rate $-g_x = g_\lambda$, and is log-uniform over the interval $[x_{\ell,t}, x_{h,t}]$, with $x_{\ell,t} = x_{\ell,0}\exp(g_x t)$, $x_{h,t} = x_{h,0}\exp(g_x t)$, $\phi/\lambda_0 \cdot (\log \hat{x}_h - \log \hat{x}_\ell)/(\hat{x}_h - \hat{x}_\ell)$ and $x_{h,0} = \hat{x}_h \cdot x_{\ell,0}$.*
b. *The surplus function is $s_t(\hat{x} \cdot x_{\ell,t}) = \hat{s}(\hat{x}) x_{\ell,0}^{-\alpha} \exp(\alpha g_\lambda t)$, with $\hat{s}(\hat{x})$ given by (4.10). The surplus distribution F_t grows at the constant rate $g_s = \alpha g_\lambda$.*
c. *The price function is $p_t(\hat{x} \cdot x_{\ell,t}) = \hat{p}(\hat{x}) x_{\ell,0}^{-\alpha} \exp(\alpha g_\lambda t)$, with $\hat{p}(\hat{x})$ given by (5.1). The price distribution G_t is non-degenerate and grows at the constant rate $g_p = \alpha g_\lambda$.*

The distribution of normalized prices, $p/E_t[p]$, is non-degenerate and constant over time.

d. *Aggregate buyers' surplus and firms' profits grow at the constant rate αg_λ.*

5.2 Industry dynamics

As the market aggregates grow at a constant rate, individual firms repeatedly go through cycles of length $T = \log(\hat{x}_h/\hat{x}_\ell)/g_\lambda$. The nature of the cycle is identical at every repetition and for every firm, but the timing of the cycles is different for different firms.

A cycle begins when the firm designs and starts selling a new variety of the product. At this moment in time, the variety of the firm is the most specialized in the market, since it has a relative breadth of $\hat{x}_\ell$ which is the lower bound on the support of the relative breadth distribution $\hat{H}$. The surplus offered by the firm to its customers is the highest among all of its competitors, as the relative surplus $\hat{s}(\hat{x})$ is a strictly decreasing function of $\hat{x}$. The firm's flow profit and the firm's value are also the highest among all the firm's competitors, as the normalized profit $r(\hat{x})$ and the normalized value $v(\hat{x})$ are both strictly decreasing functions of $\hat{x}$. Hence, at the beginning of its cycle, the firm is the best in the market. The firm has the most specialized variety, it offers the highest surplus to its customers, it makes the highest profits, and it has the highest value.

Over the course of the cycle, the firm loses ground relative to the competition. The relative breadth of the firm's variety increases over the cycle, as more and more competitors redesign their product and leapfrog the firm. Specifically, after τ units of time since the beginning of the cycle, the firm's variety is at the $\hat{H}(\hat{x}_\ell \exp(g_\lambda \tau)) = \tau/T$ quantile of the relative breadth distribution. Over the cycle, the surplus offered by the firm decreases relative to the one offered by the competitors, as the firms who redesign their product offer higher surplus to their customers. Specifically, after τ units of time since the beginning of the cycle, the firm is at the $1 - \hat{H}(\hat{x}_\ell \exp(g_\lambda \tau)) = T/\tau - 1$ quantile of the surplus distribution. Over the cycle, the flow profit and the value of firm decline relative to the competition. Specifically, after τ units of time since the beginning of the cycle, the firm is at the $T/\tau - 1$ quantile of both the profit and value distributions.

After T units of time since the beginning of the cycle, the firm is the worst in the market. The firm produces the least specialized variety, since the variety has now a relative breadth of $\hat{x}_h$, which is the upper bound on the support of the relative breadth distribution $\hat{H}$. The firm offers to its customers a relative surplus of $\hat{s}(\hat{x}_h) = 0$, which is the lowest surplus among all its competitors .The firm's profit and the firm's value are both the lowest among all the firm's competitors. It is at this moment in time that the firm finds it optimal to scrap its variety, pay the lumpy cost, and design a new, more specialized variety of the product. The firm's cycle comes to an end and a new one begins.

I summarize the aggregate properties of a BGP in the theorem below.

Theorem 5. *(Industry dynamics in a BGP) In a BGP, every firm goes through the same cycle of length* $T = \log(\hat{x}_h/\hat{x}_\ell)/g_\lambda$.

a. The cycle begins with the firm designing a new variety with breadth $x_{\ell,t}$*;*

b. Over the cycle, the firm rises linearly from the bottom to the top of the breadth distribution, and the firm falls linearly from the top to the bottom of the surplus distribution, profit distribution and value distribution;

c. The cycle ends with the firm scrapping its variety and paying the redesign cost.

6 Conclusions

The central observation in this paper is that, as search frictions decline in the market for some consumer product, firms have an incentive to design more specialized varieties of the product because they are more likely to find buyers in their niche. While the decline in search frictions tends to increase the extent of competition in the market, by allowing buyers to locate and access more firms per unit of time, the increase in specialization tends to lower the extent of competition, by lowering the probability that the firm contacted by the buyer has a variety of the product that the buyer likes. These two countervailing effect exactly offset each other under relatively mild conditions on the buyer's utility function over varieties that are more or less specialized, and the firm's cost function for designing new varieties. When the two countervailing effects exactly offset each other, the extent of competition in the market remains constant. As a result, price dispersion and markups remain constant. The buyer's surplus and the firm's profits however grow over time at a constant rate as the increase in specialization allows firms to cater better and better to the heterogeneous desires of different buyers.

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