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ABSTRACT

Techniques of regulated Brownian motion are used to analyze the behavior of the exchange rate when official policy reaction functions are subject to future stochastic changes. We examine exchange-rate dynamics in alternative cases where the authorities promise (i) to confine a floating rate within a predetermined range and (ii) to peg the currency once it reaches a predetermined future level. Similarities between these and several related examples of regime switching are stressed.

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1. Introduction

The typical forward-looking variable in an economist's model is driven by an exogenous forcing process whose form is fixed for all time. Yet in the real world, there are many examples in which the forcing process is subject to change once a certain event occurs. When variables such as interest rates, current accounts, inflation or exchange rates reach extreme values, authorities may not only change their policies – they may also change their policy reaction *functions*.

A number of papers examine the behavior of forward-looking variables when an otherwise passive policy maker intervenes to keep the variables from moving out of a predetermined range. In this spirit, Krugman (1988a) and Miller and Weller (1988) study exchange rate target zones, and Dixit (1988, 1987), Dumas (1988) and Krugman (1988b) study the allocation of capital. In all these models, the authority acts in a way to keep the forward-looking variable within a desired range or band. These papers also share a useful technical approach to these problems, that of regulated Brownian motion, which often gives simple and intuitive answers.

A related literature studies the effects on forward-looking variables of once-and-for-all changes in regime. Flood and Garber (1983), for example, study a case in which the exchange rate floats freely until it reaches a pre-announced level, at which time the government intervenes to keep the exchange rate fixed thereafter. Because rational investors anticipate the transition to the peg, the level and dynamics of the rate will be different than under a permanent float. Flood and Garber

(1983) apply a first-stopping-time methodology to this regime-switching problem. Unfortunately, they are unable to derive a closed-form solution using that mathematically cumbersome approach.

In this paper we apply techniques of regulated Brownian motion to (i) clarify the relationship between the newer target-zone results of Krugman (1988a) and Miller and Weller (1988), and the process-switching model of Flood and Garber (1983); and (ii) present simple and intuitive results to a number of interesting cases in stochastic process switching including the one posed (but not solved in closed form) by Flood and Garber (1983). In section 2 below, we lay out the general exchange rate model. Section 3 contains the solutions to several specific process-switching examples. Section 4 concludes.

2. The Model

To keep the analysis simple, we use the standard flexible-price monetary model of the exchange rate.[1] In this framework, the (log) spot exchange rate at time t, $x(t)$, is the sum of a set of macroeconomic fundamentals, $k(t)$, plus a speculative term proportional to the expected percentage change in the exchange rate:

$$x(t) = k(t) + \alpha E\Big(dx(t)/dt \mid \phi(t)\Big). \tag{1}$$

Above, the parameter α can be interpreted as the semi-elasticity of domestic money demand with respect to the interest rate, E is the expectations operator, and $\phi(t)$ is the time-t information set, which includes the current value of fundamentals, $k(t)$, as well as any explicit or implicit restrictions the authorities have placed on the future evolution of fundamentals. (For example, the authorities may have announced that they intend to keep the exchange rate from moving outside certain limits, or that they will fix the exchange rate once it reaches a certain level. This information about future policies would be incorporated into $\phi(t)$.) Included among the fundamental factors that affect $k(t)$ is a variable measuring relative national money supplies, which are assumed to be controlled directly by monetary authorities. Included as well are other, exogenous determinants of

[1] The discussion adopts a continuous-time approach, which allows a neater characterization of the solutions than comparable discrete-time methods. No essentials of the results, however, depend on the continuous-time assumption.

exchange rates that the authorities cannot influence.[2]

The monetary authorities may intervene to influence exchange rates by altering the stochastic process governing (relative) money-supply growth. This in turn will alter the process driving the fundamentals, $k(t)$. A regime of freely floating exchange rates is said to be in effect when the authorities refrain from intervening to offset shocks to fundamentals. Under a free float, we assume the fundamentals to evolve according to the process:

$$dk(t) = \eta dt + \sigma dz(t), \tag{2}$$

where η is the (constant) expected change in k, dz is a standard Wiener process, and σ is a constant. Equation (2) is just the continuous-time version of a random walk with trend.[3] As noted above, the authorities can control k through intervention, so k need not follow (2) under regimes other than a free float.

In a rational-expectations equilibrium with no speculative bubbles, there is a unique exchange-rate path that satisfies (1). This path has the integral representation:

$$x(t) = E\Big(\alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}k(s)ds \mid \phi(t)\Big) = \alpha^{-1}\int_t^{\infty} e^{(t-s)/\alpha}E\Big(k(s) \mid \phi(t)\Big)ds, \tag{3}$$

[2] On the monetary model of exchange rates, see, for example, Frenkel (1978) and Mussa (1978). The model consists of four equations. First, there is a domestic money demand equation for the country we study:

$$m(t) - p(t) = \alpha_0 + \alpha_1 y(t) - \alpha i(t) + \nu(t), \quad \alpha_1, \alpha > 0, \tag{i}$$

where m is the log of the domestic money supply, p is the log of the domestic price level, y is the log of real income, i is the nominal interest rate, and ν is a random money-demand shock. Money demand by the rest of the world is given by:

$$m^*(t) - p^*(t) = \alpha_0 + \alpha_1 y^*(t) - \alpha i^*(t) + \nu^*(t), \tag{ii}$$

where the asterisks denote the rest-of-the-world counterparts to the variables in (1). The model assumes that purchasing power parity holds up to an exogenously varying real exchange rate shock q, so the log of the nominal exchange rate, x, is:

$$x(t) = p(t) - p^*(t) + q(t). \tag{iii}$$

The model also assumes that domestic and foreign assets are perfect substitutes up to an exogenously varying risk premium on domestic-currency assets, ρ. Expected depreciation is thus the sum of the nominal interest-rate differential and the risk premium:

$$E\Big(dx(t)/dt \mid \phi(t)\Big) = i(t) - i^*(t) - \rho(t). \tag{iv}$$

Subtracting (ii) from (i), and using (iii) and (iv) gives equation (1) in the text with

$$k(t) = \alpha_1(y^*(t) - y(t)) + m(t) - m^*(t) + q(t) + \alpha\rho(t) + \nu^*(t) - \nu(t). \tag{v}$$

[3] The propositions below can be rederived using more complex forcing processes, such as the mean-reverting process:

$$dk(t) = (\eta - \theta k(t))dt + \sigma dz.$$

See Section 3.5 below.

a representation valid under *any* policy regime or sequence of policy regimes. In words, (3) equates the current exchange rate to the present discounted value of expected future fundamentals (the discount rate is $1/\alpha$). Below, the equilibrium exchange-rate value given by the present-value formula (3) is called the exchange rate's *saddlepath* value.

Other solutions to (1) exist, of course, but these involve extraneous bubble terms, which are driven by self-fulfilling expectations. In the present model, a bubble $\beta(t)$ is a random variable that can be added to the saddlepath solution in (3) to produce a new solution. The bubble, $\beta(t)$, is expected to grow over time at rate $1/\alpha$ because it necessarily satisfies the homogeneous part of (1):

$$1/\alpha = E\Big(d\beta(t)/dt \mid \phi(t)\Big)/\beta(t).$$

We assume that such bubble solutions are ruled out by market forces, so that the exchange rate is on the saddlepath defined by equation (3) in equilibrium.[4]

Given (2) and the types of regime changes we will consider, it is reasonable to suppose that the saddlepath solution for the exchange rate can be written as a twice continuously differentiable function of a single variable, the current fundamental:

$$x(t) = S(k(t)). \tag{4}$$

Naturally, the precise form of the function $S(k)$ depends (as is demonstrated below) on the types of regime shifts (if any) that the market thinks are possible.[5] A well-known special case is the one in which the authorities are committed to a *permanent* exchange-rate float, so that fundamentals are expected to follow process (2) forever. In this case, the conditional expectations in (3) are easy to evaluate, since they depend exclusively on current fundamentals, and not on possible future regime shifts:

$$x(t) = S(k(t))$$

[4] A deep issue set aside here is whether the saddlepath solution is the unique equilibrium solution in the absence of some government contingency plan for intervening in case of an extreme deviation between the exchange rate and its saddlepath level. The analysis in Obstfeld and Rogoff (1983) suggests a negative answer. Such contingent intervention arrangements would, however, have exchange-rate effects similar to those of the target-zones analyzed below. In particular, the relevant saddlepath would become nonlinear.

[5] We will show later that $S(k)$ is monotonically increasing in k in the cases analyzed in this paper.

$$= \alpha^{-1} \int_t^\infty e^{(t-s)/\alpha} E(k(s) \mid k(t)) ds = \alpha^{-1} \int_t^\infty e^{(t-s)/\alpha} (k(t) + (s-t)\eta) ds = k(t) + \alpha\eta. \quad (5)$$

If, however, the market expects the authorities to intervene in the future, fundamentals will not always follow (2), so the exchange rate will not follow (5), even while floating freely. In such cases, direct computation of the sequence of conditional expectations in the present-value formula (3) that defines $S(k)$ is likely to be burdensome. We therefore follow an alternative, two-step approach to determine $S(k)$ when a regime switch from (2) to some other process is possible. *First,* we characterize the family of functions of form $x = G(k)$ that satisfy the equilibrium condition (1) so long as fundamentals evolve according to (2). *Second,* we find the member of this family that satisfies boundary conditions appropriate to the stochastic regime switch under consideration. This last function is the saddlepath solution, $S(k)$.

A more detailed justification for this two-step procedure is offered in context below. The method is, however, analogous to one commonly used for diagrammatically analyzing a one-time step change in fundamentals in deterministic asset-price models. In those models, one first observes that before the disturbance occurs, the asset price is given by some member of the class of general solutions to the differential equation defining asset-market equilibrium. (This general solution corresponds to the solution $G(k)$ mentioned in the previous paragraph.) Next, one pins down the *economically relevant* pre-disturbance solution, which generally differs from the saddlepath, by two boundary conditions: one of these forces the asset price to be continuous at the moment the disturbance occurs, and the other forces the asset price to be on the saddlepath afterward.[6]

To implement step one of the procedure outlined above – finding the general solution $x = G(k)$ – use Itô's lemma and equation (1) to express expected depreciation during the float as:[7]

$$E\Big(dx/dt \mid \phi\Big) = E\Big(dG(k)/dt \mid \phi\Big) = \eta G'(k) + \frac{\sigma^2}{2} G''(k), \quad (6)$$

where we have assumed $G(k)$ is twice continuously differentiable. Combining (1) and (6) yields a

[6] For an exposition, see Obstfeld and Stockman (1985).

[7] Where it does not create confusion, we drop the time-dependence notation. It is worth noting that while we refer to $G(k)$ as a "general" solution, it is general only if attention is restricted to solutions that depend on current fundamentals alone. In fact, (1) has even more general solutions, for example, solutions that are functions not only of current fundamentals, but also of variables extraneous to the model. Such solutions, discussed by Froot and Obstfeld (1988), are not considered in this paper.

second-order differential equation that the exchange rate in (1) and (3) must satisfy:

$$G(k) = k + \alpha\eta G'(k) + \frac{\alpha\sigma^2}{2} G''(k). \tag{7}$$

The general solution to (7) is:

$$G(k) = k + \alpha\eta + A_1 e^{\lambda_1 k} + A_2 e^{\lambda_2 k}, \tag{8}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$ are the roots to the quadratic equation in λ,

$$\lambda^2 \alpha\sigma^2/2 + \lambda\alpha\eta - 1 = 0, \tag{9}$$

and A_1, A_2 are constants of integration.[8] Equation (8) forms the basis of our analysis below; as just discussed, a single member of the family that (8) defines will turn out to be equivalent to the present-discounted-value formula for x in (3). This is just the function $S(k)$.

Notice that there are two parts to the general solution (8): one linear, and the other nonlinear, in k. The linear portion, $k + \alpha\eta$, would be the standard linear saddlepath solution if no change in the fundamentals process (2) were possible, so that a free float were permanently in effect (see equation (5)). This part of the solution would be expected to grow linearly with time under a permanent free float.

The nonlinear part of (8) defines additional solutions to (1), all of which depend only on current fundamentals. When the market expects the exchange rate to float freely forever, these solutions can be viewed as bubble paths, since along them, the exchange rate differs from the present discounted value of expected future fundamentals, $k + \alpha\eta$. The nonlinear term in (8), like any other bubble, is expected to grow exponentially at the rate $1/\alpha$ if a switch from the free-float regime is a probability-zero event. To see why, consider how the term $A_1 e^{\lambda_1 k}$ in (8) affects the exchange rate's behavior. Under a permanent free float, the time-t expected value of this term at time $\tau > t$ relative to its value at t is:

$$E\left(e^{\lambda_1(k(\tau)-k(t))} \mid \phi(t)\right) = E\left(e^{\lambda_1\eta(\tau-t)+\lambda_1\sigma\int_t^\tau dz(s)} \mid k(t)\right) = e^{\lambda_1\eta(\tau-t)} e^{\lambda_1^2\sigma^2(\tau-t)/2}. \tag{10}$$

[8] For a discussion of the techniques used to solve closely related examples, see Dixit (1988). Equation (8) can be shown to be the unique general solution to (7) using the method of Wronksian determinants, provided that $\lambda_1 \neq \lambda_2$ and σ^2 is finite.

Notice that (10) is the product of two components. The first component, $e^{\lambda_1\eta(\tau-t)}$, grows with τ at a rate dependent on the deterministic trend growth rate of fundamentals, η; the second component, $e^{\lambda_1^2\sigma^2(\tau-t)/2}$, grows with τ because of the uncertainty in the growth in fundamentals, measured by σ. If k were a completely deterministic process, with $\sigma = 0$, then from (9), $\lambda_1 = 1/\alpha\eta$, and the second component of (10) would remain constant at 1 forever. Under these circumstances, it is easy to see that (10) would grow at rate $1/\alpha$, the rate at which the conditional means of all bubble solutions to (1) grow. When $\sigma > 0$, however, (10) still must grow at rate $1/\alpha$ (because λ_1 is a root of the quadratic equation (9)); but if $\sigma > 0$, then $\lambda_1 < 1/\alpha\eta$. This implies that the variance of fundamentals (through Jensen's inequality) contribute more and the deterministic growth of fundamentals contributes less to the overall growth rate of (10). The same reasoning applies to the second nonlinear term in (8).

Although the nonlinear terms in (8) can be viewed as defining bubble solutions to (1) under a permanent free float, we do not want to throw them away in solving for the saddlepath exchange rate under a free float that could terminate. When there is some type of regime-switching, fundamentals may not remain permanently a random walk, and the present-value formula (3) therefore is *not* equal to the simple linear expression (5). Under the possibility of a regime switch, the exchange rate's saddlepath value prior to the switch will necessarily depend on the nonlinear term in (8). Just which initial conditions A_1 and A_2 are appropriate depends on the boundary conditions associated with the regime switch, conditions to be determined in step two of the two-step solution procedure outlined above.

Before proceeding to this second step in the next section, however, we introduce a diagram that will aid in visualizing the various boundary conditions that are derived there. Figure 1 shows the family of paths given by (8) for the symmetric case in which $A_1 = -A_2$, and where $\eta > 0$. On the horizontal axis is the value of the fundamental, k, and on the vertical axis is the value of the exchange rate, x. The line labelled FF indicates the linear solution given by (5), which corresponds to the case $A_1 = A_2 = 0$. (FF is also the saddlepath under a permanent free-float regime. It has a 45-degree slope and passes through the point $k = 0$, $x = \alpha\eta$.) As described above, the curvature of each path is supported by the expected growth rate of fundamentals and Jensen's inequality. The apparent asymmetry in the paths reflects the trend in the growth of fundamentals, η: as k increases, the paths converge more quickly toward the saddlepath and diverge more slowly away from it than they would in the symmetric case without trend growth, $\eta = 0$.

3. Examples

This section carries out the second step of the solution method outlined in section 2. The discussion takes up sequentially the boundary conditions implied by several possible regime-switching scenarios. In terms of the mathematics, all that is involved is the appropriate choice of the two arbitrary constants in (8), A_1 and A_2. A key implication of the discussion is that the same unifying principle leads to solutions for all of the problems considered.

3.1. Exchange Rate Target Zones

Suppose the authorities want to keep the exchange rate from penetrating the upper and lower levels, $\underline{x}$ and $\overline{x}$. When the exchange rate reaches one of these boundaries, the authorities do to fundamentals what is necessary to keep x from moving outside of its target zone. However, they do not prevent a movement of x back into the interior of the zone. Exchange-rate behavior within a target zone was studied originally by Krugman (1988a), whose analysis has been extended by Miller and Weller (1988).[9]

One way for the authorities to enforce the target zone is to place lower and upper limits, $\underline{k}$ and $\overline{k}$, on the fundamentals. If the fundamentals are prevented from moving outside the range $[\underline{k}, \overline{k}]$, and if (as turns out to be true in equilibrium) $S(k)$ is monotonically increasing in k, the exchange rate will be confined between the lower and upper values $\underline{x} = S(\underline{k})$ and $\overline{x} = S(\overline{k})$. (As usual, $S(k)$ is the saddlepath value of the exchange rate within the target zone.) For the exchange rate to be free to move back within the zone after it has touched one of its edges, the bounds $[\underline{k}, \overline{k}]$ must be *reflecting* barriers on the fundamentals process. Clearly, the authorities can enforce any desired exchange-rate target zone by choosing appropriate reflecting barriers on the fundamentals.

[9] For present purposes, it does not matter whether it is the home or foreign government that manages the exchange rate (or a committee representing both governments). The term "authorities" should be understood as encompassing all of these possible controllers.

Before solving the model under the arrangements just described, it is worth considering just how the authorities' policy should be conveyed to the market. One possibility is that the authorities announce, "We will let the exchange rate float freely within the range $[\underline{x}, \overline{x}]$, but when it reaches the band's edge we will intervene to keep it from going further." While this may be the most intuitive way to announce an exchange-rate target zone, it introduces a potential for multiple equilibria by not giving market participants enough information about the future behavior of fundamentals. For example, at the moment the authorities first announce that the exchange rate will be confined within bands, the market could set $x = \overline{x}$ if the authorities respond by setting $k = \overline{k}$ immediately. An initial exchange rate of $\overline{x}$ is an equilibrium when $k = \overline{k}$ provided the authorities subsequently allow k to move downward according to (2), but not upward. Similar reasoning, but based on different market anticipations about the management of fundamentals, shows that an initial exchange rate of $x = \underline{x}$ could also occur. The point is that multiple values of x can be supported as equilibria through accommodating adjustments of the fundamentals.[10]

Because announcing exchange-rate objectives without specifying the accompanying policies may not be enough to determine a unique equilibrium exchange rate, we stay with the initial formulation offered above: Momentary intervention prevents *fundamentals* from rising above the upper bound $\underline{k}$ or below the lower bound $\overline{k}$, but no intervention occurs otherwise. As we will show in a moment, a policy of credibly announcing that $\underline{k}$ and $\overline{k}$ are reflecting barriers on the fundamentals accomplishes the objective of keeping the exchange rate within the range $[\underline{x} = S(\underline{k}), \overline{x} = S(\overline{k})]$. And because the policy does not allow market expectations to influence the evolution of fundamentals inside the band, problem of multiple equilibria does not arise.

To determine exchange-rate behavior within a target zone, we therefore solve for the exchange-rate path that satisfies (1) given fundamentals evolving according to (2) within reflecting barriers $\underline{k}$ and $\overline{k}$. The solution can be expressed as a special case of (3):

$$x(t) = S(k(t)) = E\Big(\alpha^{-1} \int_t^{\infty} e^{(t-s)/\alpha} k(s) ds \mid k(t), k(s) \in [\underline{k}_r, \overline{k}_r]\Big), \tag{11}$$

where the r subscript indicates that the barriers on fundamentals are reflecting. As noted in section

[10] See Obstfeld and Stockman (1985, section 2.3) for a more detailed discussion of related indeterminacies.

2, direct evaluation of the conditional expectation in (11) is much more difficult than in (5) (the case of a permanent free float): the bounds on fundamentals imply that the saddlepath exchange rate $S(k)$ will no longer be a purely linear function of k.

We have already taken the first step in solving the problem by deriving the general nonlinear solution $x = G(k)$ given by (8). Some member of this family of solutions must characterize exchange-rate behavior when k is *in the interior of* $[\underline{k}_r, \overline{k}_r]$, where (1) and (2) simultaneously hold. A nontrivial logical gap must be bridged, however, before concluding that equation (8) is also relevant *at the boundary* of this interval, that is, at the barriers $k = \underline{k}_r$ and $k = \overline{k}_r$. The needed bridge is supplied by the fact that the saddlepath solution $S(k)$ is *continuous* on the entire interval $[\underline{k}_r, \overline{k}_r]$. Continuity ensures that if $S(k)$ coincides with a function of form $G(k)$ on the interior of that interval, $S(k)$ coincides with the same function at the edges.

All that remains, then, is to determine the boundary conditions on $G(k)$ implied by the reflecting barriers. These conditions deliver unique values for the undetermined coefficients A_1 and A_2 in (8), and therefore tie down uniquely the member of the class $G(k)$ that coincides with $S(k)$ when k lies between the reflecting barriers.

The appropriate boundary conditions on $G(k)$ are the "smooth-pasting" conditions discussed in Dixit (1987), Dumas (1988), and Krugman (1988a,b):

$$G'(\underline{k}_r) = 0, \tag{12}$$

$$G'(\overline{k}_r) = 0. \tag{13}$$

The intuition behind these conditions is straightforward, as the following argument shows. Consider equation (12), for example. If $(\underline{k}_r, \underline{x})$ is an equilibrium point under the target zone, equilibrium condition (1) must hold at that point. At the boundary, however, investors have a one-sided bet on fundamentals. Investors know that from $k = \underline{k}_r$, k can move only upward. Suppose (contrary to (12)) that the function $G(k)$ describing the saddlepath under a zone passes through $(\underline{k}_r, \underline{x})$ with a *negative* slope, that is, with $G'(\underline{k}_r) < 0$. Point 1 in Figure 2 shows a function $G(k)$ for which this is the case. The inequality $G'(\underline{k}_r) < 0$ states that under a hypothetical free float with the exchange rate given by $x = G(k)$, the positive-probability event that k falls below $\underline{k}_r$ would cause x to rise above $\underline{x}$. But then $(\underline{k}_r, \underline{x})$ cannot also be an equilibrium point (and thus satisfy (1)) when $\underline{k}_r$ is a reflecting lower barrier: in the latter case, the downside risk is exactly the same as under a free float (the exchange rate can still move down along $G(k)$ if k rises), but there is no longer any upside exchange-rate risk (from a fall in k). A similar argument disposes of the possibility that $G'(\underline{k}_r) > 0$ (imagine that point 2 in Figure 2 is vertically below $\underline{k}_r$, and repeat the line of reasoning just presented). The contradiction is avoided only if $G'(\underline{k}_r) = 0$, so that under the free float described by $G(k)$, a small downward move in k from $\underline{k}_r$ doesn't affect the exchange rate. At the lower barrier, the saddlepath must therefore look like $S(k)$ in the figure, that is, it must be at a local minimum. A parallel argument establishes the second smooth-pasting condition (13), which states that the saddlepath has a local maximum at $\bar{k}_r$.[11]

Using (8), we can write the smooth-pasting conditions as:

$$1 + A_1\lambda_1 e^{\lambda_1 \underline{k}} + A_2\lambda_2 e^{\lambda_2 \underline{k}} = 0, \tag{14}$$

$$1 + A_1\lambda_1 e^{\lambda_1 \bar{k}} + A_2\lambda_2 e^{\lambda_2 \bar{k}} = 0. \tag{15}$$

Equations (14) and (15) yield the following proposition:[12]

Proposition 1. When fundamentals follow (2) within the reflecting barriers $\underline{k}$ and $\bar{k}$, the saddlepath solution (11) is:

$$x = S(k) = k + \alpha\eta + \left(\frac{\lambda_2 e^{\lambda_2\bar{k}+\lambda_1 k} - \lambda_2 e^{\lambda_2\underline{k}+\lambda_1 k} + \lambda_1 e^{\lambda_1\underline{k}+\lambda_2 k} - \lambda_1 e^{\lambda_1\bar{k}+\lambda_2 k}}{\lambda_1\lambda_2 e^{\lambda_2\underline{k}+\lambda_1\bar{k}} - \lambda_1\lambda_2 e^{\lambda_2\bar{k}+\lambda_1\underline{k}}}\right). \tag{16}$$

If we let the lower barrier, $\underline{k}$, go to minus infinity, (16) simplifies to:

$$x = S(k) = k + \alpha\eta - \lambda_1^{-1} e^{\lambda_1(k-\bar{k})}. \tag{17}$$

[11] Harrison (1985) contains a formal derivation of the smooth-pasting condition. Dixit (1988) offers a discrete-time motivation of those results.

[12] Krugman (1988a) first derived this result. He actually assumes that the authorities announce exchange-rate bands, and then prevent movements of the fundamentals that would push the equilibrium price outside those bands. The smooth-pasting logic underlying his solution implies, however, that the authorities are enforcing a two-sided reflecting barrier on the fundamentals. As a result, his solution is exactly the same as the one given in the text. Miller and Weller (1988), who work with a mean-reverting process, also assume that the authorities announce price bands and that the smooth-pasting conditions hold.

If in addition we let the upper barrier, $\bar{k}$, go to infinity, (17) becomes the linear saddlepath in (5):

$$x = k + \alpha\eta. \tag{18}$$

Only when both boundaries are infinitely distant is the exchange rate a linear function of fundamentals.

Notice that the saddlepath solution given in (16) is of the form hypothesized earlier: it is a function of the current state k and the two barriers. It is also straightforward to verify that $S(k)$ is monotonically increasing over its domain of definition, as claimed earlier.

Figure 3 illustrates two possible exchange rate paths described by (16). The paths share a common upper barrier, $\bar{k}$, but differ with respect to the lower barrier. Path 1 in the figure shows the behavior of the exchange rate when there are finite reflecting barriers at $\underline{k}$ and $\bar{k}$. This path has several noteworthy features. First, its shape reflects the influence of expected policy changes at the fundamental barriers. In the neighborhood of $\bar{k}$, for example, the exchange rate is below FF, the saddlepath under a hypothetical free float. This bending away from FF near $\bar{k}$ reflects a greater expected fall in x compared with a situation without boundaries. Second, the equilibrium solution behaves much like the saddlepath when the exchange rate is within the band but not close to either boundary. When the band is wider, the equilibrium solution is close to FF for a greater range of fundamentals. Path 2, for example, shows the case in which the lower boundary is infinitely distant (equation (17)). This graphical intuition is made precise in (18), which shows that the equilibrium solution converges to the saddlepath when both barriers are infinitely distant. The implication is that for a narrow band, the free-float solution FF will almost never be a good approximation to the true equilibrium path.

3.2. Dual Absorbing Barriers

Suppose now that the authorities wish to let the exchange rate float until it reaches a lower or an upper level, $\underline{x}$ or $\bar{x}$, at which time they plan to fix x permanently. To keep the spot rate fixed at one of these levels, the authorities must hold the fundamentals constant at $\underline{k} = S^{-1}(\underline{x})$

or $\bar{k} = S^{-1}(\bar{x})$, respectively. This class of problems is a generalization of that posed by Flood and Garber (1983), who are concerned with the behavior of a floating exchange rate when the authorities plan to switch to a fixed-rate regime at a single, predetermined level of the exchange rate, $\bar{x}$. We discuss the Flood and Garber example in more detail in the next section.

Once again, there is the issue of how the authorities' intentions are conveyed to the public. In order to avoid potential multiple equilibria, we assume that the authorities inform investors that fundamentals will follow (2) until k reaches $\underline{k}$ or $\bar{k}$. At that time the authorities will fix k, thereby fixing the exchange rate at $\underline{x} = S(\underline{k})$ or $\bar{x} = S(\bar{k})$, respectively.

Given the boundaries, $\underline{k}$ and $\bar{k}$, the saddlepath solution is:

$$x(t) = S(k(t)) = E\Big(\alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}k(s)ds \mid k(t), k(s) \in [\underline{k}_a, \bar{k}_a]\Big), \tag{19}$$

where the a subscript denotes that the barriers on fundamentals are absorbing. As before, direct evaluation of (19) is very cumbersome. The methods used above apply directly, however, and lead to a simple answer.

The first step once again is to examine the exchange rate's value at the boundaries. Fortunately, the boundary values of integral (19) are easy to evaluate. They are:

$$\underline{x} = S(\underline{k}_a) = E\Big(\alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}k(s)ds \mid k(t) = \underline{k}_a\Big) = \alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}\underline{k}_a ds = \underline{k}_a, \tag{20}$$

$$\bar{x} = S(\bar{k}_a) = E\Big(\alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}k(s)ds \mid k(t) = \bar{k}_a\Big) = \alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}\bar{k}_a ds = \bar{k}_a. \tag{21}$$

In words, once fundamentals are fixed permanently, the spot rate is just the capitalized value of current fundamentals, either $\underline{k}_a$ or $\bar{k}_a$.

At the boundaries, (8) and either (20) or (21) must hold. Together they imply:

$$\alpha\eta + A_1 e^{\lambda_1 \underline{k}} + A_2 e^{\lambda_2 \underline{k}} = 0, \tag{22}$$

$$\alpha\eta + A_1 e^{\lambda_1 \bar{k}} + A_2 e^{\lambda_2 \bar{k}} = 0. \tag{23}$$

These two equations lead to a unique solution for the two constants in (8), and to the following proposition:

Proposition 2. When fundamentals follow (2) within the absorbing barriers $\underline{k}$ and $\overline{k}$, the saddlepath solution (19) is:

$$x = k + \alpha\eta\Big(1 + \frac{e^{\lambda_2\overline{k}+\lambda_1 k} - e^{\lambda_2\underline{k}+\lambda_1 k} + e^{\lambda_1\underline{k}+\lambda_2 k} - e^{\lambda_1\overline{k}+\lambda_2 k}}{e^{\lambda_2\underline{k}+\lambda_1\overline{k}} - e^{\lambda_2\overline{k}+\lambda_1\underline{k}}}\Big). \tag{24}$$

If we let the lower bound, $\underline{k}$, go to minus infinity, (24) simplifies to:

$$x = k + \alpha\eta\Big(1 - e^{\lambda_1(k-\overline{k})}\Big). \tag{25}$$

If in addition the upper bound, $\overline{k}$, goes to infinity, we again get the familiar linear solution in (5).

Figure 4 illustrates equation (24). It shows two exchange rate paths that share the same upper bound, but that have different lower bounds. Path 1 shows the behavior of x when the absorbing barriers are the points $\underline{k}$ and $\overline{k}$ in the figure. Path 2 is drawn to correspond to the extreme case in (25), where the lower bound is at minus infinity. It is clear from the figure, as well as from (18), that the exchange rate must lie on the 45-degree line through the origin at both absorbing barriers. When both boundaries are infinitely distant, the saddlepath is just the free-float saddlepath, FF. Notice also from the proposition that if there is no trend growth in fundamentals, $\eta = 0$, *all* solutions correspond to the 45-degree line (which then coincides with FF), regardless of the boundary values.

The intuition behind the saddlepath solution in Figure 4 is as follows. On the saddlepath, the exchange rate is the present discounted value of fundamentals, and the evolution of fundamentals is governed in part by their deterministic trend growth rate, which depends on η. Suppose that $\eta > 0$ (the case shown in the figure). As k approaches either $\overline{k}$ or $\underline{k}$, the probability that the exchange rate will still be floating on any given future date declines; and since η is set permanently to zero at the moment of pegging, the expected rate of monetary growth on any future date also declines as either absorbing barrier is approached. As a result, there is a progressive currency appreciation relative to FF as k moves towards one of the barriers. For $\eta < 0$, FF would lie below the 45-degree line and the saddlepath solution would be the mirror image of the one in Figure 4. When $\eta = 0$ the bending effects are absent because absorption of k has no effect on the expected change in fundamentals (which remains zero). Think of the saddlepath as being trapped between FF and the 45-degree line, which, as noted in the last paragraph, coincide when $\eta = 0$.

3.3. Single Barrier Problems

The path given by (25) has additional significance in the exchange rate literature: it is the unique path for the stochastic-process-switching problem posed by Flood and Garber (1983). Flood and Garber attempt to solve directly the integral representation for the exchange rate under a single absorbing barrier at $\bar{k}$:

$$x(t) = E\Big(\alpha^{-1}\int_t^\infty e^{(t-s)/\alpha}k(s)ds \mid k(t), k(s) \leq \bar{k}_a\Big). \tag{26}$$

Since they do not arrive at a closed-form solution, it is worth some additional discussion to see why the simple formula (25) is the solution to this problem.

With a single absorbing barrier at $\bar{k}$, we know from (8) and (21) that the exchange rate must satisfy:

$$x = k + \alpha\eta\Big(1 - e^{\lambda_1(k-\bar{k})}\Big) + A_2\Big(e^{\lambda_2 k} - e^{(\lambda_2-\lambda_1)\bar{k}+\lambda_1 k}\Big), \tag{27}$$

where A_2 is an arbitrary constant to be determined. Clearly, there will be a unique value for A_2 which makes (27) equal to the integral representation in (26).

Figure 5 shows the family of solutions given by (27). We graph six different paths by setting A_2 in (27) to positive (paths 1-3) and negative (paths 5 and 6) values. It is clear that for $k < \bar{k}$, the spot rate can take on any value below $\bar{x}$. (In keeping with the spirit of this model, we show in Figure 5 only paths along which the exchange rate and fundamentals strike the peg for the first time from below and the left: $x(0) < \bar{x}$ and $k(0) < \bar{k}$.[13])

It is easy to see that the solution to (26) must be (27) with $A_2 = 0$. Note first that as k becomes infinitely small, the presence of the barrier $\bar{k}_a$ has a negligible effect on the conditional expectation of future levels of k in (26). For such small k, the exchange rate should therefore be

[13] This translates into a restriction on the free parameter A_2. We consider only those cases in which:

$$A_2 < \frac{1-\alpha\eta\lambda_1}{(\lambda_1-\lambda_2)e^{\lambda_2\bar{k}}}.$$

From (9), the right-hand-side expression is positive.

approximately linear in the fundamentals (as in the no-boundary solution, equation (5)). Next, note that (27) becomes linear in k as $k \longrightarrow -\infty$ if and only if $A_2 = 0$. Thus, equation (26) and (27) can be equivalent only when $A_2 = 0$; but setting $A_2 = 0$ just gives solution (25), which was found by letting the lower bound on k go to minus infinity. Path 4 in Figure 5 thus gives the value of (26).

For finite values of $\underline{k}_a$, the price-fundamentals relationship must lie below path 4 in Figure 5. (This can be seen in Figure 4, where path 2 lies everywhere below path 1.) Figure 6 shows that a path such as 5 corresponds to the equilibrium for $\underline{k}_a$ and $\overline{k}_a$. Some paths, such as 6, lie everywhere below the 45-degree line, and therefore cannot be solutions to (3) for any value of $\underline{k}_a$. Similarly, the paths that lie above 4 in Figures 4 and 5 cannot be solutions for any $\underline{k}_a$.

The logic of this section can of course be applied directly to the case of a single reflecting barrier at $\overline{k}$. In that case it is straightforward to show that the exchange rate follows equation (17) above, so that:

$$x(t) = E\Big(\alpha^{-1} \int_t^{\infty} e^{(t-s)/\alpha} k(s) ds \mid k(t), k(s) \leq \overline{k}_r\Big) = k + \alpha\eta - \lambda_1^{-1} e^{\lambda_1(k-\overline{k})}. \tag{28}$$

3.4. Mixed Barriers

Suppose that the authorities expect eventually to peg the exchange rate at the depreciated value, $\overline{x}$, but that meanwhile they will not permit arbitrarily large *appreciations*. They might agree on a lower bound, $\underline{x}$, at which they would intervene, selling domestic assets in order to prevent the exchange rate from appreciating further. They would not, however, peg the exchange rate at the appreciated level, $\underline{x}$. This is a case where it is sensible to think of policy as providing a lower reflecting barrier, $\underline{k}_r$ as well as an upper absorbing barrier $\overline{k}_a$. Using the techniques above, the solution to this more complex problem is immediate.

With these mixed barriers, the exchange rate is given by the integral expression:

$$x(t) = E\Big(\alpha^{-1} \int_t^{\infty} e^{(t-s)/\alpha} k(s) ds \mid k(t), k(s) \in [\underline{k}_r, \overline{k}_a]\Big). \tag{29}$$

Using (8) and the boundary conditions (12) and (21), we have the proposition:

Proposition 3. When fundamentals follow (2) between a reflecting barrier $\underline{k}$ and an absorbing barrier $\bar{k}$, the saddlepath solution (29) is:

$$x = k + \alpha\eta + \frac{e^{\lambda_2 \bar{k} + \lambda_1 k} - \alpha\eta\lambda_2 e^{\lambda_2 \underline{k} + \lambda_1 k} + \alpha\eta\lambda_1 e^{\lambda_1 \underline{k} + \lambda_2 k} - e^{\lambda_1 \bar{k} + \lambda_2 k}}{\lambda_2 e^{\lambda_2 \underline{k} + \lambda_1 \bar{k}} - \lambda_1 e^{\lambda_2 \bar{k} + \lambda_1 \underline{k}}}. \tag{30}$$

Figure 7 shows that the paths satisfying (30) correspond to those *above* Path 4. Path 2, for example, traces out (30) with a lower reflecting barrier at $\underline{k}$. Naturally, the path "pastes" at $\underline{k}$ and meets the upper absorbing barrier according to (21). A path such as 1 would be the equilibrium for a higher reflecting barrier, while a path such as 3 would be the equilibrium for a lower one. As the point of intervention $\underline{k} \longrightarrow -\infty$, the exchange rate path converges to path 4 (given again by equation (25)). Paths such as 5 and 6 cannot satisfy these boundary conditions, since along them $G'(k)$ is strictly positive at all $k > \underline{k}$.

3.5. More Complex Forcing Processes

The techniques above are practical only when the driving process in (2) is relatively simple; it is usually impossible to find closed-form general solutions to the analogues of (7) when k follows a more complicated forcing process. Nonetheless, some special cases do have solutions. Suppose, for example, that fundamentals are mean reverting:

$$dk(t) = (\eta - \theta k(t))dt + \sigma dz(t), \tag{31}$$

where η, θ and σ are known constants. Use of (1) and application of Itô's lemma lead to the differential equation:

$$G(k) = k + (\eta - \theta k)G'(k) + \frac{\sigma^2}{2}G''(k). \tag{32}$$

The following proposition gives the general solution to (32):

Proposition 4. When fundamentals follow (31), any solution to equation (1) must satisfy:

$$x = G(k) = \frac{k + \alpha\eta}{1+\theta} + A_1 M\Big(\frac{1}{2\theta}, \frac{1}{2}, \frac{2(\eta - \theta k)^2}{\theta\sigma^2}\Big) + A_2 M\Big(\frac{1+\theta}{2\theta}, \frac{3}{2}, \frac{(\eta - \theta k)^2}{\theta\sigma^2}\Big)\frac{(\eta - \theta k)}{\sqrt{\theta}\sigma}, \tag{33}$$

where A_1 and A_2 are arbitrary constants and $M(.,.,.)$ is the confluent hypergeometric function.[14]

Using the procedures discussed above, it is straightforward to rederive all the above propositions when fundamentals evolve according to (31). Naturally, for values of k such that $\theta k \approx \eta$ the mean-reversion component of (31) is unimportant, so that the solutions appear qualitatively very similar to those shown in the graphs. For values of k where mean reversion is important, the mean reversion introduces a new source of bending (toward the unconditional mean of k, η/θ) into the paths above.

4. Conclusions

This paper has shown how techniques of regulated Brownian motion can be applied to models of exchange-rate determination under a variety of possible future regime switches. The techniques used above are far simpler and more intuitive than the "brute-force" method of calculating the exchange rate directly as the expected present discounted value of fundamentals. In particular, our techniques yield an explicit, closed-form solution to the Flood and Garber (1983) example; and despite the complexities of solving the relevant conditional expectations directly, the answer derived here is surprisingly transparent. With such solutions in hand, the next step is to turn the data to determine the empirical importance of stochastic regime switches.

[14] See Slater (1965) for the properties of confluent hypergeometric functions.

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Figure 1

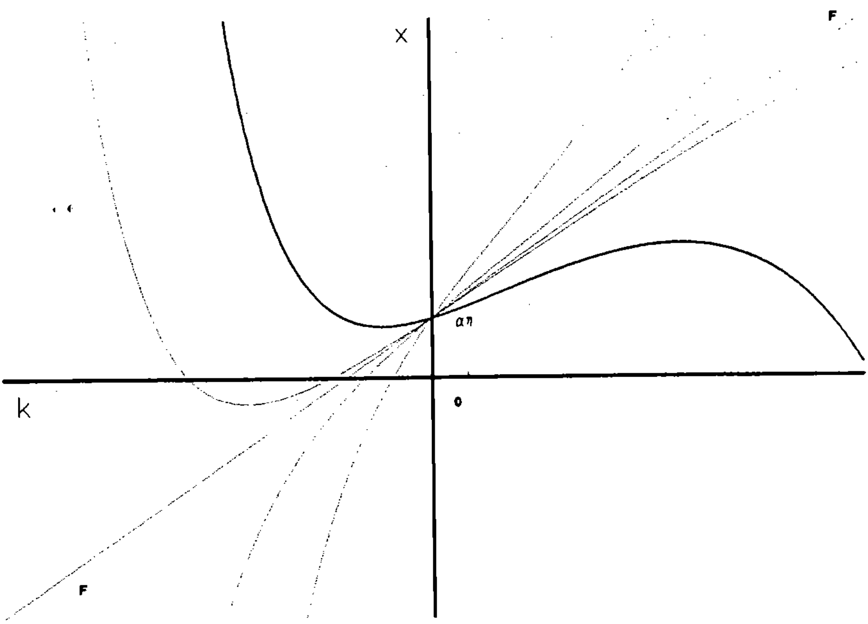

Figure 2

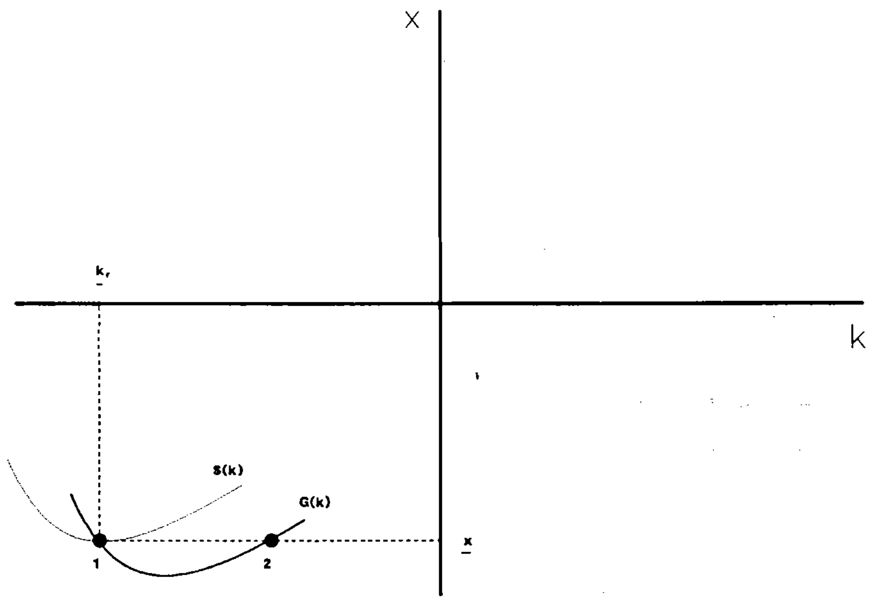

Figure 3

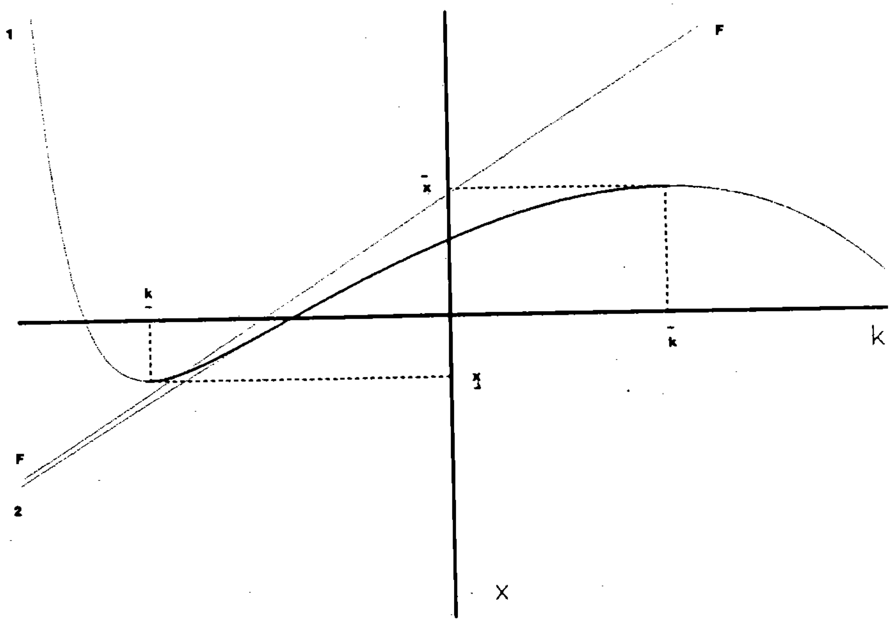

Figure 4

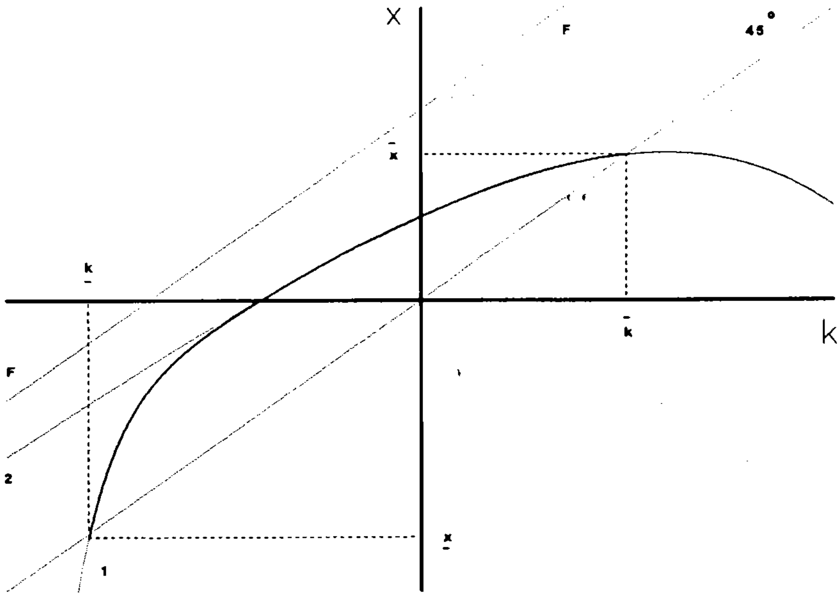

Figure 5

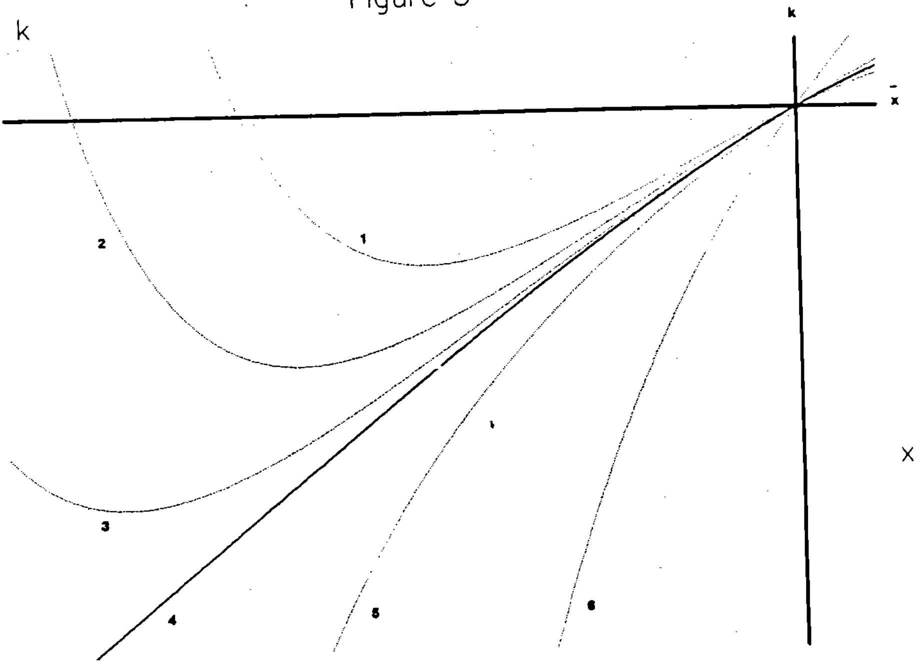

Figure 6

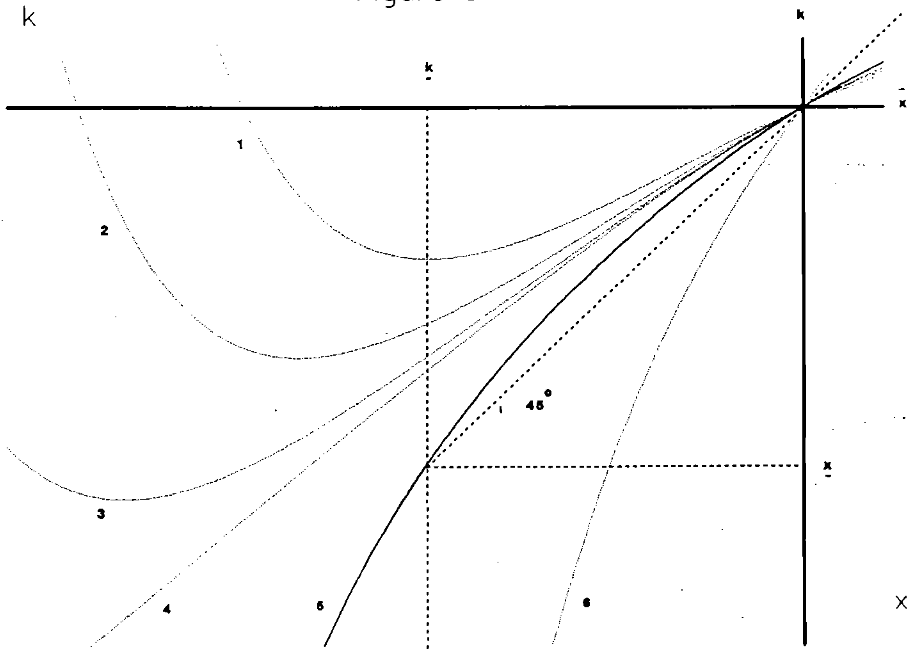

Figure 7