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A THEORY OF THE NOMINAL CHARACTER OF STOCK SECURITIES

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ABSTRACT

We construct recursive solutions for, and study the properties of the dynamic equilibrium of an economy with three types of agents: (i) house-hold/investors who supply labor with a finite elasticity, consume a large variety of goods that are not perfect substitutes and trade government bonds; (ii) firms that produce those varieties of goods, receive productivity shocks and set prices in a Calvo manner; (iii) a government that collects an income-driven fiscal surplus and acts mechanically, buying and selling bonds in accordance with a Taylor policy rule based on expected inflation. In this setting we show that stock market returns are much less than one-for-one related to inflation over a one-year holding period, which means that stock securities have a strong nominal character. We also show that their nominal character diminishes as the length of the stock-holding period increases, in accordance with empirical evidence.

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Because the underlying activity of firms is physical, barring money illusion, one might expect that stocks are a one-for-one hedge against inflation. But, the data is not consistent with this hypothesis so that an explanation is required. When one regresses nominal stock returns of a one-year holding period on the one-year rate of inflation, one typically obtains a slope coefficient that is not far from zero, which means that stock securities have a strong nominal character. However, over a five-year holding period, the slope coefficient is closer to 1. We seek to explain both stylized facts together, by means of an equilibrium model.

We build our model block after block, which means that the successive models we consider are progressively richer, starting with a simple “cashless” endowment economy (Section 2), turning then to a production economy with flexible prices (Section 3.1), then adding sticky prices (Section 3.2), collecting the results in Section 4, and finishing with an economy with demand for cash and a zero lower bound for the interest rate (Section 5.1).

The analysis suggests that our economic models have the ability to capture the co-movement between stock returns and inflation, with different elements of the model changing or amplifying the economic mechanisms behind this co-movement. The model with flexible prices explains the nominal character of stocks. The basic mechanism at play is that productivity shocks affect real stock returns and inflation in offsetting ways, so that nominal stock returns tend to be unrelated to inflation. The model with sticky prices explains the way in which the nominal character of stocks slowly vanishes as one lengthens the holding period. The model with money is developed for the sake of realism; it shows that our results are essentially unchanged when households hold money balances.

We underscore our contribution: (i) to our knowledge, there does not exist a theory of the relation between realized nominal stock returns and inflation, (ii) the theory we develop is entirely based on received macroeconomic theory, (iii) the theory matches not just the empirical slope over short holding periods, it also matches the way the slope changes as one increases the holding period, (iv) whether nominal government debt is in the form of nominal bonds or money makes little difference to the explanation.

The models are solved exactly (either analytically or numerically) to capture the effect of non-linearities. We use a technique developed in Dumas and Lyasoff (2012), which effectively adopts an event-tree approach with Markovian probabilities. Here, all of the uncertainty in the model is couched in terms of a simple binomial structure.

The question of whether stocks provide a one-for-one hedge against inflation is a long-standing important question in financial economics. We describe in Section 1 the extensive empirical work that has been done so far and the surprising empirical regularities that have been discovered. We are not aware of any existing equilibrium model that would have been proposed to account for the empirical evidence. Because the extant empirical work has also compared stocks to bonds in regard to inflation, we similarly analyze bonds in Section 5.2.

Following standard Asset Pricing theory with long-lived agents, we impose the basic “no-bubble” principle that the real value of outstanding nominal gov-

ernment debt is equal to the present discounted value of real primary government surpluses. That principle holds whether or not we make any of the three assumptions we now spell out and comment upon.

The first assumption is that outstanding government debt is contractually defined as being denominated in purely nominal terms. The nominal unit in question is only a measurement unit in Sections 2 to 4. It becomes money physically held in Section 5.1.

The second assumption is that the government (inclusive of the central bank), whose objective function is not spelled out, implements a mechanical monetary policy of inflation targeting by fixing the nominal rate of interest as a function of expected inflation. This form of monetary policy has been practised explicitly during the last thirty years, and implicitly during the previous twenty years, as even central banks that were controlling the quantity of money in circulation were actually using the nominal rate of interest as an intermediate target.¹ We show in Section 2.3 below that the inflation-targeting assumption is not critical to our results.²

The third assumption is about fiscal policy.³ We fix a monotonically increasing relation between real primary surplus and real national income. This assumption is crucial and makes ours a fiscal theory of the nominal character of stock securities: with some degree of persistence of productivity, a positive productivity shock raises real stock prices and simultaneously increases the prospect of higher government surpluses, thus raising the real value of government debt. For a given, contractual nominal value, this reduces the price level.⁴

In our applications, we simply assume that the primary budget surplus is proportional to national income. Our proportionality parameter τ can obviously be interpreted as a constant rate of income tax, with no government expenditures. Cochrane (2021) postulates a more sophisticated s-shaped process. And Jiang, Lustig, van Nieuwerburgh and Xiaolan (2019), as well as Kung (2021), take care to model separately the revenues and the expenditures of the government.

As was the case for monetary policy, the objective function of the government is not spelled out; it does not choose the amount of the primary surplus (taxes and expenditures), which means that the stochastic process for the surplus is specified as exogenous. In macroeconomic parlance, an exogenous primary surplus is referred to as a “non Ricardian” or “active” fiscal policy or regime.⁵ The

¹See, for instance, Friedman (1975).

²We show that, in the endowment economy or when prices are flexible, our result about the nominal character of stock returns is materially the same under the alternative assumption of a quantitative bond-supply policy.

³Following Sargent and Wallace (1975), the literature has stressed the unavoidable financial linkage between monetary and fiscal policies when the central bank intervenes in the money market, including the Treasury-bill market, to set the nominal interest rate. A recent, elaborately argued exposition of that view is to be found in Leeper and Leith (2016).

⁴Our theory presents some similarity to the line of argument of Geske and Roll (1983) but does not rely on monetization.

⁵The label “active” is due to Leeper (1991). The label “non Ricardian,” although commonly used, is somewhat misleading because the principle of Ricardian equivalence still holds: the exogenous stochastic process for the surplus that is postulated is a member of a whole

opposite, a so-called “Ricardian” or “passive” fiscal policy, would let the surplus be adjusted mechanically to satisfy at least some government budget constraints in a way that turns them into identities. In that case, an indeterminacy would open up.⁶ That is the key reason for which we posit a non Ricardian policy.⁷ The only workable alternative to our assumption would be to specify an objective function to be maximized by the government; the surplus would then be endogenous – just like the consumption of households is chosen by them under their budget constraint –, but not in a way that would leave open any Ricardian indeterminacy.

We make one more, purely simplifying assumption about the behavior of output, and later the behavior of productivity, which is the only shock in our model.⁸ In the endowment-economy model, we assume that the growth rate of the former is Identically, Independently Distributed (IID) over time. Then, in the more elaborate production-economy model, we assume that the growth rate of the latter is IID. That is meant to represent in a simple way a very persistent productivity level process. That assumption has been validated by many empirical studies, which we cite in Section 3.1.

A few macroeconomic models have incorporated the stock market: Marshall (1992), Challe and Giannitsarou (2014) and Swanson (2014). Challe and Giannitsarou (2014) study the response of the stock market to a monetary policy shock, whereas we focus exclusively on a productivity shock.

Some theoretical contributions that deal with asset prices in New Keynesian settings – not all of them in a general-equilibrium formulation, but with additional utility features such as habit formation – include Li and Palomino (2014),

equivalence class of exogenous stochastic processes that are Ricardo equivalent (more tax at one point in time, less tax at another) in present-value terms and in terms of equilibrium allocations.

⁶In the words of Nakajima and Polemarchakis (2005), “a fiscal policy is called ‘Ricardian’ if it guarantees that the public debt vanishes at each terminal node for all possible, equilibrium or non-equilibrium, values of price levels and other endogenous variables.” In that case, the fiscal surplus cannot be exogenous throughout.

Nakajima and Polemarchakis (2005) prove that, as long as fiscal policy is Ricardian, the value of government debt is indeterminate. See also Niepelt (2019), page 181. This is in conformity with Woodford (2003, page 125) and Cochrane (2011). As Definition 1 below makes clear, a zero public debt at each terminal node is part of the definition of equilibrium. There is no need to require that terminal condition to hold for all possible values of the price levels and other endogenous variables.

⁷To dispel a potential misunderstanding, Niepelt (2019, page 180) writes: “That a non-Ricardian policy regime may determine the initial price level and thereby revalue initially outstanding nominal debt does not mean that the government can choose primary surpluses [...] arbitrarily. Standard asset pricing and rational expectations imply that, when nominal debt is issued for the first time [...], the government cannot raise more resources in present value terms than it repays in the future.”

⁸With this single shock, the conditional correlation between all stochastic variables is equal to 1 and their unconditional correlation is also high. For that reason, the R^2 of the relation between stock return and inflation will be very high, which is not true in the data. When we succeed in obtaining a slope very much smaller than 1, it will follow mechanically that the volatility of stock returns will be smaller than the volatility of inflation, which is counterfactual. Our purpose is singlemindedly to explain the slope, not the R^2 . Additional shocks could be added that would reduce the R^2 .

Wei (2009), Castelnovo and Nistico (2010), Nistico (2012), Song (2017) and Campbell, Plueger and Viceira (2019).⁹

Several recent papers have discussed the relation between productivity, inflation and bond pricing and sought to explain exposure of bonds to inflation. Rudebusch and Swanson (2012) provide a calibration and apply it to bond prices. De Paoli, Scott and Weeken (2010) includes technology shocks in a numerical study of the influence of real and nominal frictions on risk premia. The negative correlation between productivity (with it, consumption) and inflation is at work in much of the production-based asset pricing literature that seeks to rationalize an upward sloping nominal term structure when the real term structure is downward sloping. Examples of such models are: De Paoli et al. (2010), Kung (2015), Hsu and Palomino (2019).

1 Empirical evidence

We refer to the empirical evidence that pertains to the relation between stock returns and inflation. Lintner (1975), Bodie (1976), Jaffee and Mandelker (1976), Nelson (1976), Fama and Schwert (1977), Fama (1981), Gultekin (1983), Boudoukh and Richardson (1993), Erb, Harvey, and Viskanta (1995), Goto and Valkanov (2000), and Bekaert and Wang (2010) all document on U.S. data – and Solnik (1983) documents on international data – a low or even negative correlation between nominal stock returns and either realized or expected inflation, at a monthly or annual frequency.

Boudoukh and Richardson (BR), whose dataset covers close to two hundred years of annual U.S. (as well as UK) data, introduce a distinction between the *ex ante* and the *ex post* forms of the correlation of stock returns with inflation. To capture the *ex post* correlation, BR simply regress one-year holding-period realized nominal stock returns on one-year realized inflation. They do the same for five-year holding-period realized nominal stock returns and five-year realized inflation. In both cases the slope coefficient is found to be significantly positive but, more importantly, significantly less than 1. Since that regression measures the exposure of stocks to inflation, it can be concluded that stocks are in large part “nominal” assets. And the slope is many times larger for the five-year data: stocks are less nominal for a five-year holding period than for a one-year holding period.^{10,11}

⁹More recently, Pflueger and Rinaldi (2021).

¹⁰Goto and Valkanov (2002) and Hagmann and Lenz (2005), using a vector autoregression, show an attenuation of the negative relation following the Volcker reform of monetary policy.

¹¹The *ex ante* (or “asset-pricing”) relation, otherwise called the “Fisher” hypothesis (applied to stocks as opposed to bonds or Treasury Bills), relates conditionally expected nominal stock returns to conditionally expected inflation. Under the null hypothesis, including the assumption of risk neutrality, the regression slope is expected to be equal to 1, reflecting a constant real rate of return. When anticipating inflation, agents have available an information set, which the econometrician treats as instrumental variables. BR use past inflation and past interest rates as instrumental variables. They do not reject the null hypothesis on five-year data but reject it on one-year data.

Katz, Lustig and Nielsen (2017) using a panel of countries confirm that stock markets are slow to incorporate news about future inflation, so that they do not qualify to be called “real” assets, whereas bond markets are not as slow. Their empirical investigation entertains three potential explanations of this phenomenon, two of which include behavioral features such as misperception of future inflation and one of which remain within the frame of rational-expectations. Our theory assumes rational expectations. Section 5.2 below returns to the Katz et al. evidence.

It would seem useful to start this paper with an update of the empirical work of BR over the last few decades. However, that is impossible to do. The volatility of inflation has been very low, making it impossible to estimate with precision the slope of the ex post relation. Furthermore, the requirement of observing five-year returns would leave only a small number of effective observations. A long history is required.

To provide fresh evidence on the relation between stock returns and inflation, we update to the year 2020 the long-history dataset, extending it as well to the nineteen countries for which a continuous record of returns exists, from 1900 onwards.¹² The database is that of Dimson, Marsh and Staunton (2021a).¹³ As in BR, the ex post regression being run is quite simply:¹⁴

$$R_{t \rightarrow t+i} = \alpha_i + \beta_i \times \pi_{t \rightarrow t+i} + \varepsilon_{t,i}; i \in \{1, 5\}; t = 1, \dots, 117 \quad (1)$$

where R is the annual, nominal log-rate of return on the equity index and π is the annual log-rate of inflation, with $i = 1$ for the one-year time interval and $i = 5$ for the five-year time interval. If the real rate of return on stock were independent of inflation, one would expect $\beta_i = 1$. Because the five-year rates of return are calculated every year, there is overlap in the data and the Generalized Method of Moments – with Newey-West adjustment over five years – is used to compute heteroskedasticity-and-autocorrelation consistent (HAC) standard errors of the estimates.

The results are displayed in Figure 1. The top panel of the figure indicates that five-year slopes, over the whole panel of countries, tend to be higher than the one-year slopes. The bottom-left panel shows that many more than 5% of the countries exhibit slopes that are significantly smaller than 1 at the 5% level of a one-sided significance test, and more so for one-year slopes than for five-year slopes. Finally, the bottom-right panel shows that for six countries the five-year slope is significantly greater than the one-year slope at the 10% level of significance, the same being true for eight countries at the 20% level, for eleven countries at the 30% level etc..

In short, the panel broadly confirms the pattern observed by BR on U.S. and UK data.

¹²The nineteen countries are: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Ireland, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal, South Africa, Spain, Sweden, Switzerland, the United Kingdom and the United States. Germany is not included because the humungous hyperinflation of 1922-1923 alone drives the result.

¹³The indices are described in Dimson, Marsh and Staunton (2021b).

¹⁴This is a purely descriptive exercise, for comparison later with simulated data. We are not assuming that inflation is exogenous relative to stock returns.

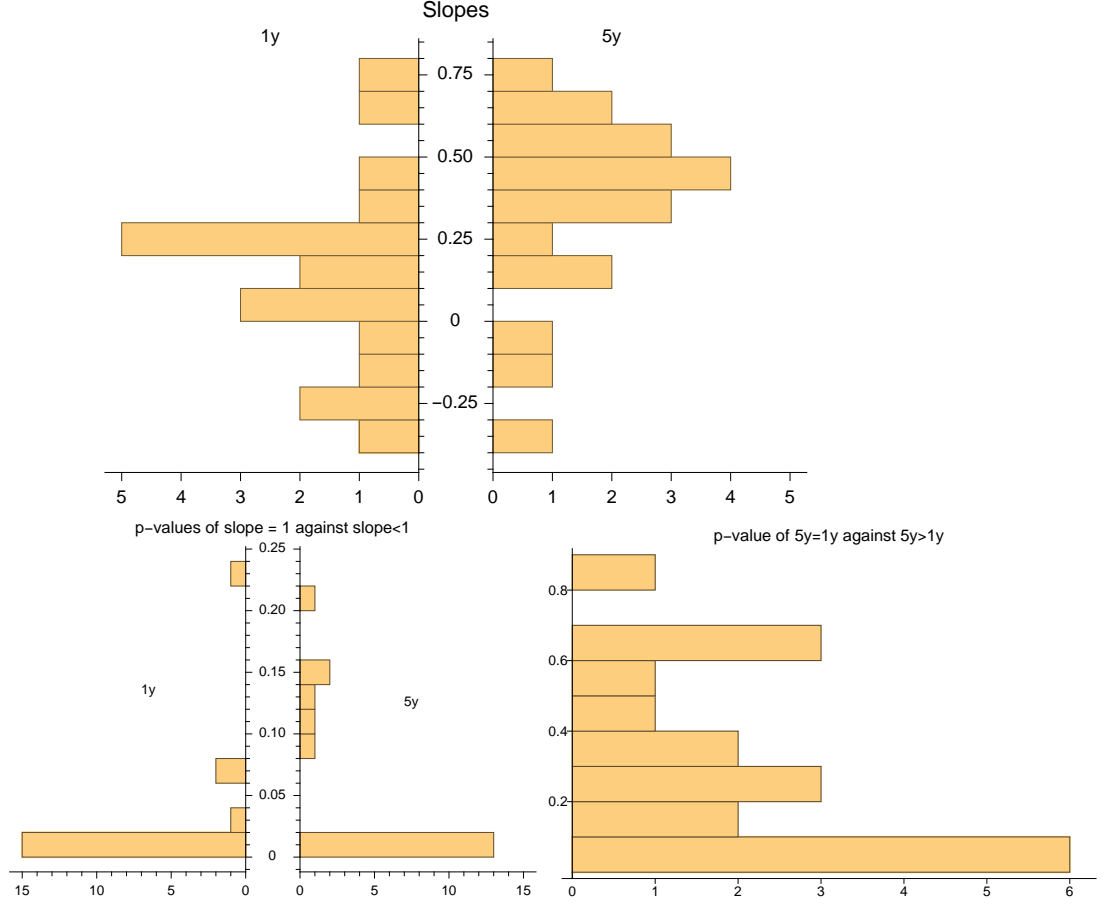


Figure 1: **Comparing slopes against inflation of one-year holding-period returns to five-year ones.** Numbers of countries (out of 19 countries) are on the horizontal axes. The paired histograms of the top panel compare the estimated slopes. The paired histogram of the bottom-left panel indicates to what extent these slopes are smaller than the number 1. The p-value shown is $1 - \text{CDF}[\mathcal{N}(0, 1), (1 - \hat{\beta}) / \text{std}(\hat{\beta})]$. The simple histogram of the bottom-right panel indicates to what extent the five-year slope is larger than the one-year one. The p-value shown is $1 - \text{CDF}[\mathcal{N}(0, 1), (\hat{\beta}_5 - \hat{\beta}_1) / \text{std}(\hat{\beta}_5 - \hat{\beta}_1)]$.

2 A simple endowment-economy model

To make clear the mechanisms at work, we begin with a stripped-down version of the model. We postulate an endowment economy in which competitive economic agents need no money to transact and in which prices of goods and services are fully flexible. This is a “cashless” economy in which money is only a conventional unit of account in terms of which the contractual face value of the debt of the government is stipulated (Woodford (1995)).

2.1 Description of the economy

We consider a financial market populated with one (or a continuum of identical) household(s), for which we use a subscript 1, and one government (inclusive of the central bank), subscripted 2. A set of exogenous time sequences of individual income (or output)¹⁵ received by households $\{y_t \in \mathbb{R}_{++}; t = 0, \dots, T\}$ are placed on an event tree or lattice, y_0 being fixed. These are received by the households only. For simplicity, we consider a binomial tree so that a given node $j \in \{d, u\}$ at time t is followed by two new nodes $\{d, u\}$ at time $t + 1$ at which the two values of income are denoted $\{y_{t+1,d}, y_{t+1,u}\}$.¹⁶ The transition probabilities are equal to $1/2$. Notice that the tree accommodates the exogenous state variables only.

There are at least two securities available in the financial market: a claim on output (called “equity”) and one nominally riskless (which means default-free) one-period nominal bond (with a face value equal to 1 unit of account). By contract, the payoff of the bond is set in nominal terms. The household trades all securities to maximize some lifetime utility. From their standpoint, the market is effectively complete.¹⁷

The government’s primary surplus (taxes in excess of expenditures) is denoted s_t in real terms, S_t in nominal terms. The number of units (measured by the nominal face value of the future payoff) of the one-period bond with which the private sector *exits* time t is denoted $\theta_{1,t}$ and its exiting financial wealth, equity not included, $F_{1,t} \triangleq \theta_{1,t} / (1 + i_t)$, is the market value of the nominally riskless bond holdings, i_t being obviously the nominal rate of interest at time t .

Households: we assume that the utility function of the private sector is time-additive and isoelastic. Let the relative risk aversion of the household be $1 - \gamma$ and their impatience factor be $\rho < 1$ (so that $u(c, t) = \rho^t c^\gamma / \gamma; \gamma < 1$). The private sector (agent carrying a subscript 1) maximizes:

$$\sup_{\{c_t, \theta_{1,t}\}} \mathbb{E}_0 \sum_{t=0}^T u(c_t, t) \quad (2)$$

subject to:

¹⁵We use these two words interchangeably.

¹⁶The new nodes $\{d, u\}$ that succeed state j should really be written $\{d_j, u_j\}$.

¹⁷Households have access to the equity but, being identical, they effectively do not trade it. The government does not trade equity by assumption.

- terminal conditions:

$$\theta_{1,T,j} = 0; j \in \{d, u\} \quad (3)$$

- a sequence of flow budget constraints:¹⁸

$$\begin{aligned} P_{t,j} \times c_{t,j} + \frac{\theta_{1,t,j}}{1 + i_{t,j}} + s_{t,j} \times P_{t,j} &= \theta_{1,t-1,j-} + P_{t,j} \times y_{t,j}; \\ t &= 0, \dots, T-1; j \in \{d, u\} \end{aligned}$$

- and given initial holdings:

$$\theta_{1,-1,j} = \bar{\theta}_1; j \in \{d, u\} \quad (4)$$

The initial condition (4) at $t = 0$ is given in terms of a nominal outstanding claim $\bar{\theta}_1 = -\bar{\theta}_2$ of the public on the government.

Government: the objective function of the government (agent carrying a subscript 2) is not spelled out. It trades the one-period nominal bond in a mechanical way following a Taylor rule specified below. It raises taxes yielding a primary surplus. It uses the proceeds not to consume goods but to pay back its initial outstanding debt and, on the way, to trade bonds.

It does this under the budget constraints:¹⁹

$$\frac{\theta_{2,t,j}}{1 + i_{t,j}} = \theta_{2,t-1,j-} + s_{t,j} \times P_{t,j}; t = 0, \dots, T-1; j \in \{d, u\} \quad (5)$$

with given initial holdings:

$$\theta_{2,-1,j} = \bar{\theta}_2; j \in \{d, u\}$$

and terminal condition:

$$\theta_{2,T,j} = 0; j \in \{d, u\} \quad (6)$$

We make the following assumptions:

Assumption 1 *The growth rate of output is IID over time:*²⁰

$$\frac{y_{t+1,u}}{y_{t,j}} = 1 + u; \frac{y_{t+1,d}}{y_{t,j}} = 1 + d; u > d; t = 0, \dots, T-1; j \in \{d, u\}$$

Assumption 2 *The Taylor rule is:*²¹

$$1 + i_{t,j} = (1 + \bar{i}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u} + \frac{1}{2}P_{t+1,d}}{P_{t,j}}}{1 + \bar{\pi}} \right)^\phi; \phi \geq 0; \phi \neq 1; t = 0, \dots, T-1; j \in \{d, u\} \quad (7)$$

¹⁸The node $j-$ of time $t-1$ is the node that precedes node j at time t .

¹⁹Please, bear in mind that $\theta_{2,t}$ is a negative number in case of debt.

²⁰For simplicity, the notations u and d are simultaneously subscripts and quantities.

²¹See also Henderson and McKibbin (1993).

Assumption 3 *The real budget surplus is specified to be, at all times and in all states, proportional to real income:*

$$s_{t,j} = \tau \times y_{t,j}; t = 0, \dots, T; j \in \{d, u\}$$

Assumption 1 means that the process $\{y_t\}$ of the level of income is permanent or infinitely persistent. Assumption 2 gives the Taylor rule. It creates an infinitely elastic supply of bonds at the nominal rate of interest i_t . When setting the nominal rate of interest, the principal aim of the government is to anchor inflationary expectations. In this paper, we write the Taylor rule as a forward-looking formula relating the nominal rate of interest to the rationally-expected rate of inflation, as in Clarida, Galí and Gertler (2000), Bernanke and Boivin (2000) and Svensson and Woodford (2009) who refer to this implementation as “Inflation-Forecast Targeting.” By doing this, the government sets expected inflation.²² Monetary policy does not respond to realized inflation. As we shall see, realized inflation will follow from output shocks and fiscal policy.²³

Assumption 3 means that the government surplus s_t is exogenously fixed in real terms. It is “non Ricardian.” We have given in the introduction the reasons for making that assumption: if fiscal policy were “Ricardian,” the equilibrium would be indeterminate.

2.2 Equilibrium

We are ready to derive the equilibrium behavior of the economy:

Definition 1 *An equilibrium is defined as a joint process for the allocation of consumption c_t , the price level P_t , the amount of government bonds outstanding $\theta_{2,t}$ and the nominal rate of interest i_t such that the supremum of the private sector’s objective function (2), subject to (3) to (4), is reached for all t , the government abides by its period budget constraints (5) and terminal conditions (6) and follows the mechanical rule (7), and the market-clearing conditions,*

$$\theta_{1,t,j} + \theta_{2,t,j} = 0 \tag{8}$$

are also satisfied with probability 1 at all times $t = 0, \dots, T - 1$ and states j .

Let the government consume no goods. There exists an obvious analytical equilibrium solution for which

$$c_{t+1,u} = y_{t+1,u}; c_{t+1,d} = y_{t+1,d}$$

and for which, for that reason, the market clears.

²²We would call *monetary shock* a deviation from the exact rule. The specification (7) means that throughout this paper there are no monetary shocks.

²³In most models of monetary economics, the Taylor rule is backward looking in that it captures the central bank’s reaction to *realized inflation*. Realized inflation is really a proxy for rationally expected inflation, a proxy that a central bank would rely on when it has access to incomplete information. In this paper, we make the assumption that the central bank has access to the same full information as the private sector.

The solution functions for the nominal variables, $\theta_{1,t,j}$, $\theta_{2,t,j}$, $F_{1,t,j}$ and $F_{2,t,j}$, are homogeneous of degree 1 with respect to P_t , reflecting absence of money illusion. Once consumption is known, and given that the government surplus is exogenous, the system determines the real value of government debt as a discounted present value. The only endogenous state variable is the current price level P_t , which is determined at time zero by the government flow budget constraint (5) written at $t = 0$:

$$f_{2,0} \times P_0 = \bar{\theta}_2 + s_0 \times P_0 \quad (9)$$

where $f_{2,0}$ is (minus) the real present value of future real surpluses, $\bar{\theta}_2$ is a given (negative) amount of nominal claim outstanding and $s_0 = \tau \times y_0$ a given time-0 primary surplus.

In Appendix A we prove by backward induction the following results:

Proposition 1 *Under Assumptions 1 to 3, the realized rates of inflation are:*

$$\begin{aligned} \frac{P_{t+1,u}}{P_{t,j}} &= \frac{k}{1+u} \times (1+i_{t,j}) \\ \frac{P_{t+1,d}}{P_{t,j}} &= \frac{k}{1+d} \times (1+i_{t,j}) \end{aligned} \quad (10)$$

where

$$k \triangleq \rho \times \left[\frac{1}{2} (1+u)^\gamma + \frac{1}{2} (1+d)^\gamma \right] \quad (11)$$

The nominal rate of interest is constant:

$$1+i_{t,j} = \left(\frac{1+\bar{i}}{(1+\bar{\pi})^\phi} \right)^{\frac{1}{1-\phi}} \times \left[k \times \left(\frac{1}{2} \frac{1}{1+u} + \frac{1}{2} \frac{1}{1+d} \right) \right]^{\frac{\phi}{1-\phi}} \quad (12)$$

The price level at time 0 is equal to

$$P_0 = \frac{-\bar{\theta}_2}{\tau \times y_0 \times \left[1 + \frac{k \times (-1+k^T)}{-1+k} \right]} \quad (13)$$

$k < 1$ is assumed. Inflation is lower in the u state than in the d state. That result is very much in line with the Fiscal Theory of the Price Level (FTPL).²⁴ When output growth is higher than expected, the government, because of persistence, runs larger real surpluses in the indefinite future. As a result, for the given outstanding nominal debt, the price level can readjust to a lower level, which means lower realized inflation.²⁵

²⁴See Leeper (1991), Sims (1994), Woodford (1995, 1998, 2001), Cochrane (2005a) and Niepelt (2004). A recent, comprehensive survey is Leeper and Leith (2016).

²⁵In the remarks that follow Proposition 5 below, we comment on the empirical validity of a generalized version of Proposition 1.

Since the government only trades bonds and private agents are homogeneous, it is not traded at all and its price is virtual. Since the equity security is defined – *for the time being* – as paying the output (revenues), its price x_t in real terms (the current output not included) is equal to²⁶

$$\begin{aligned} x_{t,j} &= \rho \frac{\frac{1}{2}(c_{t+1,u})^{\gamma-1} \times (y_{t+1,u} + x_{t+1,u}) + \frac{1}{2}(c_{t+1,d})^{\gamma-1} \times (y_{t+1,d} + x_{t+1,d})}{(y_t)^{\gamma-1}}; \\ x_T &= 0 \end{aligned} \tag{14}$$

Proposition 2 *Under Assumptions 1 to 3, the real gross rates of return on the stock market are:*

$$\begin{aligned} &\frac{1+u}{k} \text{ in a } u \text{ state} \\ &\frac{1+d}{k} \text{ in a } d \text{ state} \end{aligned}$$

Proof. The stock-market price is proportional to output:

$$x_{t,j} = \hat{x}_t \times y_{t,j}; t < T$$

Based on (14), \hat{x}_t (a form of dividend-price ratio) is *deterministic*.^{27,28}

$$\begin{aligned} \hat{x}_t &= k \times (1 + \hat{x}_{t+1}) \\ \hat{x}_t &= \frac{k \times (-1 + k^{T-t})}{-1 + k} \end{aligned}$$

The real gross rate of return is

$$\frac{1 + \hat{x}_{t+1}}{\hat{x}_t} \frac{y_{t+1}}{y_{t,j}}$$

Hence the result. ■

The model generates a positive relation between real returns and output growth. On a u node, the real stock market return is higher than in the d node with the same predecessor while inflation is lower but that is just a “proxy” result of a common cause, namely the output shock, which acts both on the stock market and on tax collection. The real rate of return on equity being low in a state in which inflation is high, it is conditionally negatively correlated with the rate of inflation and the stock market is not a one-for-one hedge against inflation. In fact, for their product, which is the realized gross *nominal* rate of return on stocks, we have:

²⁶Had we included equity holdings in the budget of the household, this equation would have been the first-order condition with respect to the number of shares held.

²⁷Not surprisingly, because of the proportional tax, there exists a systematic relation between the real stock market price per unit of output \hat{x}_t (the price-dividend ratio) and the real discounted value of government debt per unit of output $\hat{f}_{2,t}$ (defined in Appendix A): $-\hat{f}_{2,t}/\tau = \hat{x}_t$. Over time, they both decline deterministically.

²⁸If the horizon were infinite:

$$\hat{x}_t = \frac{k}{1-k}$$

Parameter	Value
ρ	0.99
$\bar{\pi}$	2%/year
$\bar{\tau}$ (in figures) or $\bar{\tau}$ (in simulations)	$1/\rho - 1 + \bar{\pi}$ “neutral” as per Footnote 44
$1 - \gamma$	1
tax rate τ	1/3
σ	4
η	2
volatility of z growth	1%/year
expected value of z growth	0
probability u and d	1/2
ω (price stickiness)	0.6

Table 1: **Parameter values for the numerical illustrations; one-year periods**

Proposition 3 *Under Assumptions 1 to 3, the realized nominal rate of return on stocks is equal to the nominal interest rate, which is constant.*

Realized, nominal stock market returns are unrelated to realized inflation.

In the current section, the *output* growth rates u and d are given parameters, which cannot be varied. To prepare for the situation in which output is endogenous, however, we derive for the two states the *aggregate-demand schedules* (or *IS* curves) that relate the future prices P_u, P_d to the future outputs y_u, y_d . We draw Figure 2 which shows these relations in the two states at time $t + 1$ for the two output shocks at time $t + 1$ for a fixed level of output and a fixed price level at time t . Proposition 1 shows that, for a given rate of interest, inflation is decreasing in output. But the relations displayed in the figure also incorporate the influence of future output on the current rate of interest as per Equation (12). In that way, the schedules of the left-hand and right-hand panels are interdependent (in a symmetric way, in the sense that $P_u(y_u, y_d)$ and $P_d(y_d, y_u)$ are the same function).

These schedules satisfy the following property:

Proposition 4 *Under Assumptions 1 to 3, as long as $\phi < 1 + (1 + d) / (1 + u)$, the aggregate demand schedules (inclusive of policy rule) are increasing functions of income y when $\phi > 1$ and decreasing functions when $\phi < 1$.*

The proof is in Appendix B.

2.3 Discussion

It is obviously not true empirically that the nominal rate of return on equity is constant. We have reached that conclusion because the assumptions we have

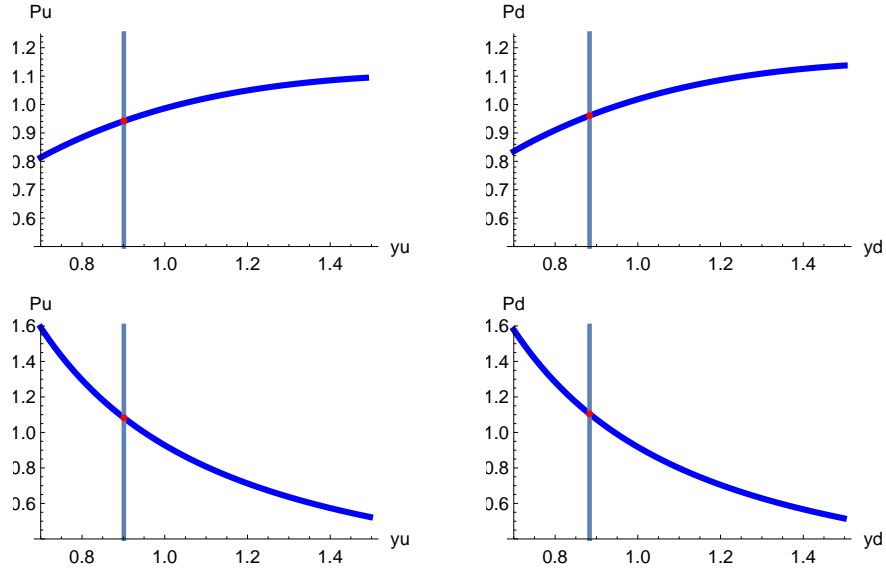


Figure 2: **Aggregate-demand curve (inclusive of policy rule) with fixed outputs, $\phi = 1.5$ (top panels) and $\phi = 0.5$ (bottom panels) at $t = T - 1$.** In each row, the left-hand panel shows $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the given value of $y_{t+1,d}$. The given value of $y_{t+1,u}$ is shown as a vertical line. The right-hand panel shows $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the given value of $y_{t+1,u}$. The given value of $y_{t+1,d}$ is shown as a vertical line. Parameter values are as in Table 1. The time- t price level is set at 1. The time- t level of output y_t is set at 0.92386.

made are simple. We indicate now to what degree the relation between nominal stock returns and inflation would have been different if some of the assumptions made were altered.

In the coming sections, two of the assumptions made so far are changed or relaxed. First, we modify Assumption 1 in the next section and replace it with the assumption that the growth rate of productivity is IID. In that case, the rate of growth of output will no longer be IID so that the (log of) output will no longer be 100% persistent. Results will be amended accordingly and we shall cite empirical evidence showing that productivity is, indeed, highly persistent. For the time being, suffice it to point out that, in the course of the proof of Proposition 1 in Appendix A, we reach an intermediate and more general proposition 7 that does not depend on the IID assumption and says that, if it is true for whatever reason that the real return on debt is higher when a u shock occurs than when a d one does, then inflation is lower in the u state than in the d state. And, if the same is true for real returns on equity, then there will tend to exist an offsetting effect between them and inflation, making nominal returns less than one-for-one responsive to inflation.

Second, the cashless-economy assumption that we have made is relaxed in Section 5.1 below. There we show an isomorphy between the cashless economy and an economy with money, obtained by simply lumping together money balances and the outstanding amount of government debt. Hence our results will still hold true.

Finally, we can also explore alternatives to the Taylor-rule policy assumption (7). Instead of targeting inflation by setting the nominal rate of interest, the government could follow a quantitative policy. Instead of the supply of bonds being infinitely elastic at the rate set by the Taylor formula, the supply of bonds could be infinitely inelastic with respect to the rate of interest, while it could still be responsive to expected inflation. For the sake of discussion, suppose that the supply, which is the face value of the debt of the government, is constant in nominal terms. In that case, the debt is settled in part by the taxes raised and in part by being inflated away. The inflation is just what is needed to achieve that balance. In Appendix C, we prove that the expression (10) remains the same but the nominal rate of interest, while still deterministic, is no longer constant (unless $T \rightarrow \infty$):²⁹

$$\frac{1}{1 + i_t} = \frac{k \times (-1 + k^{T-t})}{-1 + k^{T-t+1}}$$

The formula for equity prices is unchanged. It follows that the value of the nominal rate of return on equity in a state u is still equal to its value in a state d , and equal to the nominal rate of interest, the only difference being that the rate of interest now gradually rises over time (unless the end date T of the economy is infinitely far). Our proposition 3 is basically unchanged. The additional contribution of endogenizing bond supply through the Taylor rule is only to keep the rate of interest constant over time. Thus the Taylor-rule

²⁹For $T \rightarrow \infty$, $1/(1 + i_t) = k$.

assumption does not play an important role in the explanation of the nominal character of stock securities.

Given the isomorphy announced above between the cashless economy and the economy with money, the same result would hold true if monetary policy were quantitative and fiscal policy non existent, that is, if there were zero debt and a constant supply of money.

3 Production-economy models

The model of Section 2 combines the policy rule with the aggregate-demand (or “*IS*” side) of the economy. The solution obtained is complete when output is exogenous. We now introduce firms and endogenize output, produced out of labor only. We develop the aggregate-supply side, productivity z being now the exogenous state variable, which is assigned to the nodes of the tree.

That is done for the sake of realism. More importantly, that allows us to show that, when prices at which goods are sold are flexible, there is still no link over one or several periods between realized nominal stock returns and realized inflation. Thereafter, we introduce sticky prices to show how nominal stock returns then react to inflation, and how the nominal character of stocks disappears gradually over longer holding periods.

3.1 Flexible prices

In this section, we assume that firms are free to adjust their prices.

Households: There exists now a continuum $\iota \in [0, 1]$ of differentiated varieties of the good.³⁰ The argument c_t of the households’ utility is a composite defined as³¹

$$c_t \triangleq \left(\int_0^1 c_{\iota,t}^{\frac{\sigma-1}{\sigma}} d\iota \right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma > 1$ is the elasticity of substitution between the separate varieties. As a result, their demand for each separate variety ι is

$$c_{\iota,t} = \left(\frac{P_{\iota,t}}{P_t} \right)^{-\sigma} c_t$$

where $P_{\iota,t}$ is the nominal price of variety ι and P_t is the general price index, which is defined generally as

$$P_t \triangleq \left(\int_0^1 P_{\iota,t}^{1-\sigma} d\iota \right)^{\frac{1}{1-\sigma}} \quad (15)$$

but will be particularized below. In addition, the utility function of households now contains a separate, additive term for the dis-utility of labor. The full utility

³⁰Here, we follow Chapter 8 in Walsh (2010) and Challe (2005).

³¹We suppress the subscript $j \in \{d, u\}$ at time t in subsequent equations.

function that households optimize is

$$\sup_{\{c,l,\theta_1\}} \mathbb{E}_0 \sum_{t=0}^T u(c_t, t) - v(l_t, t)$$

subject to terminal conditions (3), a sequence of flow budget constraints:

$$P_t \times c_t + \frac{\theta_{1,t}}{1 + i_t} + \theta_{X,t} \times P_t \times x_t + s_t \times P_t = \theta_{1,t-1} + \theta_{X,t-1} \times P_t \times (\delta_t + x_t) + W_t \times l_t$$

and given initial holdings:

$$\begin{aligned} \theta_{1,-1} &= \bar{\theta}_1 \\ \theta_{X,-1} &= 1 \end{aligned}$$

where W_t is the nominal wage rate, l_t the number of hours worked, $\theta_{X,t}$ equity holdings, x_t the real price of equity and δ_t real dividends distributed. Since households alone hold the stock, it will be the case at equilibrium that $\theta_{X,t} = \theta_{X,t-1} = 1$. We have in mind, however, that the first-order condition for equity holdings will serve to price the equity.

We assume an isoelastic dis-utility of work: $v(l, t) = \rho^t \times l^\eta / \eta; \eta > 1$. The households' first-order condition for hours worked is

$$\frac{l_t^{\eta-1}}{c_t^{\gamma-1}} = \frac{W_t}{P_t} \quad (16)$$

Firms: The production function for variety ι of the good is

$$y_{\iota,t} = z_t \times l_{\iota,t} \quad (17)$$

where z_t is a productivity shock, the same for all firms and $l_{\iota,t}$ is the amount of labor utilized for the production of good ι .

Firms are free to adjust their prices at will. Firms producing variety ι that choose price $P_{\iota,t}$ sell an amount of goods equal to $(P_{\iota,t}/P_t)^{-\sigma} c_t$, for which they will have to hire an amount of labor equal to $(P_{\iota,t}/P_t)^{-\sigma} c_t / z_t$. Their profits are:

$$P_{\iota,t} \left(\frac{P_{\iota,t}}{P_t} \right)^{-\sigma} \times c_t - W_t \times \left(\frac{P_{\iota,t}}{P_t} \right)^{-\sigma} \frac{c_t}{z_t}$$

Optimizing the selling price:

$$(1 - \sigma) \left(\frac{P_{\iota,t}}{P_t} \right)^{-\sigma} \times c_t - \frac{W_t}{P_t} \times (-\sigma) \left(\frac{P_{\iota,t}}{P_t} \right)^{-\sigma-1} \frac{c_t}{z_t} = 0$$

so that:

$$\frac{P_{\iota,t}^*}{P_t} = \frac{\sigma}{\sigma - 1} \varphi_t$$

where

$$\varphi_t \triangleq \frac{W_t}{z_t \times P_t} \quad (18)$$

We interpret φ_t as the real marginal cost of labor. Profits are maximized by setting a mark up and an optimal price P^* related to the price-elasticity of demand, giving *the same value of P_t^* for all varieties*.

In the aggregate, firms produce $(P_t^*/P_t)^{-\sigma} \times y_t$. Total labor employed is:

$$\left(\frac{P_t^*}{P_t}\right)^{-\sigma} \times \frac{y_t}{z_t}$$

Letting l_t stand for the labor supplied by households, the clearing of the labor market requires:

$$l_t = \left(\frac{P_t^*}{P_t}\right)^{-\sigma} \times \frac{y_t}{z_t} \quad (19)$$

Equilibrium: By Walras' law, the equilibrium in the financial market and the equilibrium in the labor market imply the equilibrium in the goods market: $c_t = y_t$. Furthermore, since all the firms behave the same way, (15) implies that $P_t = P_t^*$. Equations (18), (16) and (19) imply that the flexible-price level of output is:

$$y_{f,t} = \left(\frac{\sigma-1}{\sigma} z_t^\eta\right)^{\frac{1}{\eta-\gamma}} \quad (20)$$

and that the supply price is indeterminate. The determination of the price level is then left entirely to the aggregate demand side (inclusive of the policy rule) exactly as in Section 2. Since $\sigma > 1$, $\eta > 1$ and $\gamma < 1$, output is an increasing function of productivity.

The IID-growth case described in Section 2 can be recast in terms of productivity shocks. From this point on, we replace Assumption 1 with the following:

Assumption 4 *The growth rate of productivity z is IID.*

Equation (20) shows that, since $\sigma > 1$, $\eta > 1$ and $\gamma < 1$, the resulting equilibrium diagrams remain identical to Figure 2, reinterpreted as showing *endogenous* values of the flexible-price output $\{y_d, y_u\}$.

Proposition 5 *Under Assumptions 2 to 4, and flexible prices, the equilibrium is unique and Propositions 1 to 3 remain true with $1+u$ replaced by $1+\hat{u} \triangleq (1+u)^{\frac{\eta}{\eta-\gamma}}$, $1+d$ replaced by $1+\hat{d} \triangleq (1+d)^{\frac{\eta}{\eta-\gamma}}$ and k replaced by*

$$\hat{k} \triangleq \rho \times \left[\frac{1}{2} (1+\hat{u})^\gamma + \frac{1}{2} (1+\hat{d})^\gamma \right] \quad (21)$$

($\hat{k} < 1$ being now assumed anew). In particular, the nominal rate of return on stocks remains equal to the nominal interest rate, which is constant.

Proof. The result follows directly from Equation (20). ■

The empirical evidence is very much in line with this generalization of Proposition 1.³² Smets and Wouters (2007) fit to seven US macroeconomic time series

³²We are very grateful to Rafael Wouters for these references.

both a DSGE model and a VAR specification, both producing very similar results with a highly persistent total factor productivity (TFP) shock (our Assumption 4). Their Figure 7 indicates clearly that the DSGE model produces a negative impact response of inflation to a positive TFP shock (our Proposition 1). Altig, Christiano, Eichenbaum and Lindé (2011) after performing a VAR analysis on US data comment their Figure 2 saying that: “Finally notice that a neutral technology shock leads to an initial sharp fall in the inflation rate.” Furthermore, the VAR fit performed by Alves (2004, Figure 2) on six OECD countries leads uniformly to the same conclusion.

The interpretation often given for these empirical results is that, in production economies with sticky prices, positive productivity shocks relax price pressure. In our paper, however, that negative correlation is present already in the flexible-price version of the model. The stickiness of prices is not responsible for the results.

Since in our model productivity is a (geometric) random walk, it is highly persistent. For that reason the shock is also news about future productivity. The empirical literature confirms a negative impact of news shocks about TFP on prices and inflation. See, for instance, Kurmann and Sims (2021), Barsky and Sims (2011), and Miranda-Agrippino, Hacıoglu Hoke and Bluwstein (2019).

Dividends and stock prices: As the firms enjoy market power, they generate positive profits. *We now re-define the aggregate stock security as paying corporate profits* (as opposed to paying output, which it was in Section 2). The real, future profits, assumed to be distributed as dividends, are:

$$\delta_t = \left[\left(\frac{P_t^*}{P_t} \right)^{1-\sigma} - \frac{W_t}{P_t} \times \left(\frac{P_t^*}{P_t} \right)^{-\sigma} \frac{1}{z_t} \right] \times y_t = \frac{1}{\sigma} y_t$$

The value of the stock market, in real terms, not including current profits, is:

$$\begin{aligned} x_t = & \rho \frac{1}{\left(\left(\frac{\sigma-1}{\sigma} z_t^\eta \right)^{\frac{1}{\eta-\gamma}} \right)^{\gamma-1}} \left[\frac{1}{2} \left(\left(\frac{\sigma-1}{\sigma} z_{t+1,u}^\eta \right)^{\frac{1}{\eta-\gamma}} \right)^{\gamma-1} \right. \\ & \times \left(\frac{1}{\sigma} \left(\frac{\sigma-1}{\sigma} z_{t+1,u}^\eta \right)^{\frac{1}{\eta-\gamma}} + x_{t+1,u} \right) \\ & \left. + \frac{1}{2} \left(\left(\frac{\sigma-1}{\sigma} z_{t+1,d}^\eta \right)^{\frac{1}{\eta-\gamma}} \right)^{\gamma-1} \times \left(\frac{1}{\sigma} \left(\frac{\sigma-1}{\sigma} z_{t+1,d}^\eta \right)^{\frac{1}{\eta-\gamma}} + x_{t+1,d} \right) \right]; \\ x_T = & 0 \end{aligned}$$

Therefore,

Proposition 6 *Under Assumptions 2 to 4 and flexible prices, the stock-market price (current profits not included) is*

$$x_t = \hat{x}_t \times \left(\frac{\sigma-1}{\sigma} z_t^\eta \right)^{\frac{1}{\eta-\gamma}}; x_T = 0$$

where \hat{x}_t is deterministic.³³

$$\begin{aligned}\hat{x}_t &= \frac{1}{\sigma} \times \hat{k} \times (1 + \hat{x}_{t+1}); \hat{x}_T = 0 \\ \hat{x}_t &= \frac{1}{\sigma} \times \frac{\hat{k} \times (-1 + \hat{k}^{T-t})}{-1 + \hat{k}}\end{aligned}$$

and where \hat{k} is as defined in (21).

The stock-market price (current profits not included) is proportional to productivity taken to a power. It follows that Proposition 3, saying that realized, nominal stock market returns are unrelated to realized inflation, still holds.

In the next section, we introduce sticky prices. They will explain the next fact that we are trying to understand, i.e., that the link between inflation and nominal stock returns varies depending on the length of the holding period.

3.2 Sticky prices

We now develop in standard New Keynesian fashion (see, for instance, Galí (2008), Walsh (2010) or Challe (2005)), the case in which firms are not free to set their prices, thus generating the Phillips curve, which endogenizes total income y .³⁴ The Phillips curve relates the price level to output contemporaneously. Taking a cue from Dumas and Lyasoff (2012), we later shift it to time $t + 1$, so that, in our rendition, it will relate the *future* price level to *future* income. Here again, productivity growth is IID but income growth, inflation and stock returns are no longer IID as they depend on an endogenous state variable reflecting path dependence.

Firms: they are not free to adjust their prices at will. Instead, as in Calvo (1983), each firm at each point in time is allowed, with a probability $1 - \omega$, to adjust its price to an optimal level P_t^* (which will be the same for all firms). By the Law of Large Numbers, a fraction $1 - \omega$ do so, so that the price index, or general price level, P_t particularizes to:³⁵

$$P_t \triangleq \left[(1 - \omega) \times (P_t^*)^{1-\sigma} + \omega \times (P_{t-1})^{1-\sigma} \right]^{\frac{1}{1-\sigma}}; t = 1, \dots, T \quad (22)$$

³³If the horizon were infinite,

$$\hat{x}_t = \frac{1}{\sigma} \times \frac{\hat{k}}{1 - \hat{k}}$$

³⁴Gorodnichenko and Weber (2016) show empirically that, after monetary policy announcements, the conditional volatility of stock market returns rises more for firms with stickier prices than for firms with more flexible prices and that sticky prices are, indeed, costly for firms.

³⁵This equation should really be:

$$P_t \triangleq \left[(1 - \omega) \times \int_0^1 (P_{\iota,t}^*)^{1-\sigma} d\iota + \omega \times \int_0^1 (P_{\iota,t-1})^{1-\sigma} d\iota \right]^{\frac{1}{1-\sigma}}$$

We are going to find that $P_{\iota,t}^*$ is the same for all ι but that is not true for $P_{\iota,t-1}$. The index of price dispersion across firms should really be present in the derivations below. We ignore it, as does most of the literature. For more details on this, see the appendix of Challe and Giannitsarou (2014). We thank Edouard Challe for confirmation.

Firms maximize their market value on the equity market. With regard to setting the current price $P_{\iota,t}$ of variety ι , the part of each firm's objective function that depends on it is:³⁶

$$\sup_{P_{\iota,t}} \sum_{i=0}^{T-t} \mathbb{E}_t \left[(\rho\omega)^i \frac{(c_{t+i})^{\gamma-1}}{(c_t)^{\gamma-1}} \left(\frac{P_{\iota,t}}{P_{t+i}} - \varphi_{t+i} \right) \left(\frac{P_{\iota,t}}{P_{t+i}} \right)^{-\sigma} y_{t+i} \right]; t = 1, \dots, T$$

(where: $y_t \triangleq \left(\int_0^1 y_{\iota,t}^{\frac{\sigma-1}{\sigma}} d\iota \right)^{\frac{\sigma}{\sigma-1}}$), with a solution $P_{\iota,t} = P_t^*$ which is:³⁷

$$\frac{P_t^*}{P_t} = \frac{\sigma}{\sigma-1} \frac{\mathbb{E}_t \sum_{i=0}^{T-t} (\rho\omega)^i (c_{t+i})^{\gamma-1} y_{t+i} \varphi_{t+i} \left(\frac{P_{t+i}}{P_t} \right)^{\sigma}}{\mathbb{E}_t \sum_{i=0}^{T-t} (\rho\omega)^i (c_{t+i})^{\gamma-1} y_{t+i} \left(\frac{P_{t+i}}{P_t} \right)^{\sigma-1}}; t = 1, \dots, T \quad (23)$$

a function of y_t for which the numerator and the denominator will be computed by backward induction. To that aim, we restate Equation (23) in recursive form (for $t = 1, \dots, T$):

$$\begin{aligned} \frac{P_t^*}{P_t} &= \frac{\sigma}{\sigma-1} \frac{c_t^{\gamma-1} y_t \varphi_t + A(t, y_t)}{c_t^{\gamma-1} y_t + B(t, y_t)} \\ A(t, y_t) &\triangleq \mathbb{E}_t \rho \omega \left(\frac{P_{t+1}}{P_t} \right)^{\sigma} \left[(c_{t+1})^{\gamma-1} y_{t+1} \varphi_{t+1} + A(t+1, y_{t+1}) \right] \\ A(T, y_T) &= 0 \\ B(t, y_t) &\triangleq \mathbb{E}_t \rho \omega \left(\frac{P_{t+1}}{P_t} \right)^{\sigma-1} \left[(c_{t+1})^{\gamma-1} y_{t+1} + B(t+1, y_{t+1}) \right] \\ B(T, y_T) &= 0 \end{aligned} \quad (24)$$

Equilibrium in the goods and labor markets: As a result of their choice of price, a proportion ω of firms produce $(P_{t-1}/P_t)^{-\sigma} \times y_t$ on an average and employ $(P_{t-1}/P_t)^{-\sigma} \times y_t/z_t$ units of labor and a proportion $1 - \omega$ of firms produce $(P_t^*/P_t)^{-\sigma} \times y_t$ and employ $(P_t^*/P_t)^{-\sigma} \times y_t/z_t$ units of labor.³⁸ Total labor employed is:

$$\left[\omega \times \left(\frac{P_{t-1}}{P_t} \right)^{-\sigma} + (1 - \omega) \times \left(\frac{P_t^*}{P_t} \right)^{-\sigma} \right] \times \frac{y_t}{z_t}$$

³⁶The overall objective function is the maximization of equity value, which includes additional terms not dependent on $P_{\iota,t}$. See the value of the stock market (26) and (27) below. For this calculation, the price P_0 is arbitrary.

³⁷The proof is standard. See Galí (2008), Walsh (2010) or Challe (2005).

³⁸Because of (22):

$$y_t \equiv \left\{ \omega \times \left[\left(\frac{P_{t-1}}{P_t} \right)^{-\sigma} \times y_t \right]^{\frac{\sigma-1}{\sigma}} + (1 - \omega) \times \left[\left(\frac{P_t^*}{P_t} \right)^{-\sigma} \times y_t \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

i.e., the amounts produced by the two categories of firms add up to y_t .

Letting l_t stand for the labor supplied by households, the clearing of the labor market requires:

$$l_t = \left[\omega \times \left(\frac{P_{t-1}}{P_t} \right)^{-\sigma} + (1 - \omega) \times \left(\frac{P_t^*}{P_t} \right)^{-\sigma} \right] \times \frac{y_t}{z_t} \quad (25)$$

Substitution of Equations (18), (16), (25) and (22) into (24) gives the equilibrium time- t Phillips curve $P_t/P_{t-1} = \text{Phill}_t(y_t)$ in implicit form:^{39, 40}

$$\begin{aligned} & \left(\frac{1 - \omega \times \left(\frac{P_{t-1}}{P_t} \right)^{1-\sigma}}{1 - \omega} \right)^{\frac{1}{1-\sigma}} = \frac{\sigma}{\sigma - 1} \frac{1}{y_t^\gamma + B(t, y_t)} \\ & \times \left\{ \left(\frac{y_t}{z_t} \right)^\eta \left[\omega \times \left(\frac{P_{t-1}}{P_t} \right)^{-\sigma} + (1 - \omega) \times \left(\frac{1 - \omega \times \left(\frac{P_{t-1}}{P_t} \right)^{1-\sigma}}{1 - \omega} \right)^{-\frac{\sigma}{1-\sigma}} \right]^{\eta-1} \right. \\ & \left. + A(t, y_t) \right\}; t = 1, \dots, T \end{aligned}$$

where:

$$\begin{aligned} A(t, y_t) &= \rho \omega \mathbb{E}_t \left(\frac{P_{t+1}}{P_t} \right)^\sigma \\ & \times \left\{ \left(\frac{y_{t+1}}{z_{t+1}} \right)^\eta \left[\omega \times \left(\frac{P_t}{P_{t+1}} \right)^{-\sigma} + (1 - \omega) \times \left(\frac{P_{t+1}^*}{P_{t+1}} \right)^{-\sigma} \right]^{\eta-1} + A(t+1, y_{t+1}) \right\} \end{aligned}$$

and:

$$B(t, y_t) = \rho \omega \mathbb{E}_t \left(\frac{P_{t+1}}{P_t} \right)^{\sigma-1} [y_{t+1}^\gamma + B(t+1, y_{t+1})]$$

The shapes of the Phillips curves are illustrated in Figures 3 and 4, along with the accompanying aggregate-demand curves derived according to Section 3.1.

The time $t+1$ Phillips curves and the general-equilibrium system to be solved: because the time- t aggregate-demand relations established in Section 3.1 relate time- $t+1$ prices to time- $t+1$ output, it is convenient to shift the Phillips curves to time $t+1$, for both states u and d . In this way, we are left with a system of four equations in four unknowns: $\{P_{t+1,u}/P_t, P_{t+1,d}/P_t, y_{t+1,u}, y_{t+1,d}\}$

³⁹ As we saw in the previous section, in the case of full price flexibility ($\omega = 0$), the Phillips curve is vertical at the flexible-price level of output: $y_t = ((\sigma - 1) z_t^\eta / \sigma)^{1/(\eta - \gamma)}$.

⁴⁰ There exists an explicit, approximate form for the Phillips function, as suggested in Galí (2015).

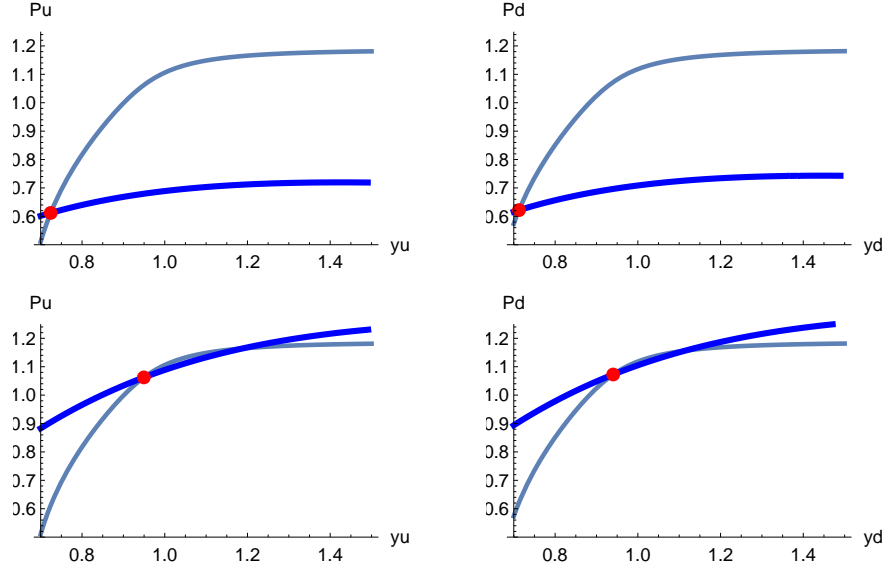


Figure 3: **Aggregate-demand (inclusive of policy rule) and aggregate-supply curves with sticky-price equilibrium output and $\phi = 1.5$ at $t = T - 1$. Top panels: low-output equilibrium. Bottom panels: high-output equilibrium (leftmost intersection).** In each row, left-hand panel: $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the *equilibrium* sticky-price value of $y_{t+1,d}$. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the *equilibrium* sticky-price value of $y_{t+1,u}$. The lighter (finer) solid line is the Phillips or aggregate-supply curve; the darker (wider) solid line is the aggregate-demand (inclusive of policy rule) curve. Parameter values are as in Table 1. The time- t price level is set at 1. The time- t level of output y_t is set at 0.92386, which is 3.551% above the flexible-price level.

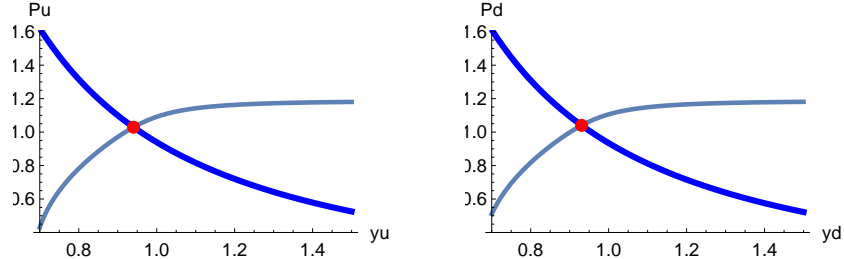


Figure 4: **Aggregate-demand (inclusive of policy rule) and aggregate-supply curves with sticky-price equilibrium output and $\phi = 0.5$ at $t = T - 1$; single equilibrium.** Left-hand panel: $P_{t+1,u}$ plotted against $y_{t+1,u}$ for the *equilibrium* sticky-price value of $y_{t+1,d}$. Right-hand panel: $P_{t+1,d}$ plotted against $y_{t+1,d}$ for the *equilibrium* sticky-price value of $y_{t+1,u}$. The lighter (finer) solid line is the Phillips or aggregate-supply curve; the darker (wider) solid line is the aggregate-demand (inclusive of policy rule) curve. Parameter values are as in Table 1. The time- t price level is set at 1. The time- t level of output y_t is set at 0.92386, which is 3.551% above the flexible-price level.

which must be solved numerically for each node of the tree (each capturing exogenous state variable z_t) and for each value of the endogenous state variable y_t , recursively for $t = T - 1, \dots, 0$. Equivalently, since productivity is exogenous, the *output gap* – defined as the ratio of the actual, sticky-price output y_t to the flexible-price output (20) minus 1 – can be recognized as the endogenous state variable. The current general price level P_t is also an endogenous state variable but, in the absence of nominal illusion by firms, it can be factored out in this calculation on grounds of homogeneity.^{41,42}

The shapes of the aggregate-demand and Phillips curves are such that there may not exist solutions, that there may be multiple solutions and that gradient-based solvers do not find them easily. We find our way towards the solutions by starting with the flexible-price solution ($\omega = 0$) and by gradually increasing ω in small increments, and by starting at no price adjustment ($\omega = 1$) and by gradually decreasing ω (an approach by so-called homotopy).

When $\phi > 1$, there can be two solutions,⁴³ which are shown in the two panels of Figure 3, for the point in time $t = T - 1$. In such a case, it is impossible

⁴¹In addition, when household utility is isoelastic and the production function satisfies the property of constant returns to scale, a scale-invariance property can be exploited: we need not do the calculation for every node of each point in time t , which differ only in the level of productivity z_t . For the several nodes of time t , the functions that are carried backward (f_1 or f_2 , A and B) can be deduced from a single one of them.

⁴²To prime a sticky-price path, productivity z_0 being given, the output gap at time 0 can be set at 0 and the price P_0 can be set at a flexible-price level similar to that of Section 3.1 (but with a market value of government debt that reflects the subsequent stickiness of prices).

⁴³And, for low enough values of current output y_t there are no solutions.

to continue the recursion to earlier points in time. This difficulty would not have even been spotted by the large number of researchers who work not with the exact system of equations but with a system that is linearized around the flexible-price solution.

When $\phi < 1$, the equilibrium, if it exists, is surely unique as shown in Figure 4. The reason is that, in that case, as we have seen under Proposition 4, the aggregate-demand functions (inclusive of the policy rule) are decreasing, while the Phillips curve, of course, is increasing. In what follows, so that we can obtain a unique solution at any point in time, we make the following assumption:

Assumption 5 $\phi < 1$.

In other words, monetary policy is “passive,” in the terminology of Leeper (1991). In association with the active (i.e., non Ricardian) fiscal policy already assumed, we are considering Regime *F* in the vocabulary of Leeper and Leith (2018).

There is evidence in Clarida et al. (2000) that the response to inflation has been above one after the Volcker reform. This evidence has been questioned by Cochrane (2011) who shows that the response coefficient is not identified empirically in the simultaneous equations system. To show that Cochrane’s argument applies entirely in our context, we use our simulation data (see below) to run on each of the 10,000 paths a time-series regression of the rate of interest on realized inflation, as Clarida et al. have done. While the value of the Taylor coefficient ϕ in our model is 0.5, we find a median regression slope of 0.311, with an upper quintile of 0.313 and a lower quintile of 0.31. An empirical regression clearly does not estimate the Taylor coefficient correctly.

Stationary functions: As mentioned, we solve the system for each node of the tree (each node capturing exogenous state variable z_t) and for each value of the endogenous state variable, recursively for $t = T - 1, \dots, 0$. With the impatience parameter set at $\rho = 0.99$, the value $T = 270$ years is sufficiently large for functions carried backward to be unchanging by the time we get to time 0. The stationary functions capture the equilibrium of an economy with an horizon that has been increased indefinitely.

Simulation: After solving all the equations of all times in a backward sequence, we use the stationary functions to simulate the economy, drawing at random the event of a u or a d productivity shock z over 200 time steps of one year each. Ten thousand paths are drawn. The behavior of the output gap over time is described in Figure 5 and formulated in the following⁴⁴

⁴⁴We set \bar{r} to be equal to the neutral rate of interest, where we define “neutral” as follows:

Definition 2 *Under IID growth of productivity, the neutral rate of interest of an economy is the value of the interest rate that would prevail in a flexible-price economy when \bar{r} is equal to the equilibrium interest rate (12) (with $1 + u$, $1 + d$ and k replaced as in Proposition 5).*

The value of the neutral rate is

$$1 + i = \frac{1 + \bar{\pi}}{\hat{k} \times \left(\frac{1}{2} \frac{1}{1+u} + \frac{1}{2} \frac{1}{1+d} \right)}$$

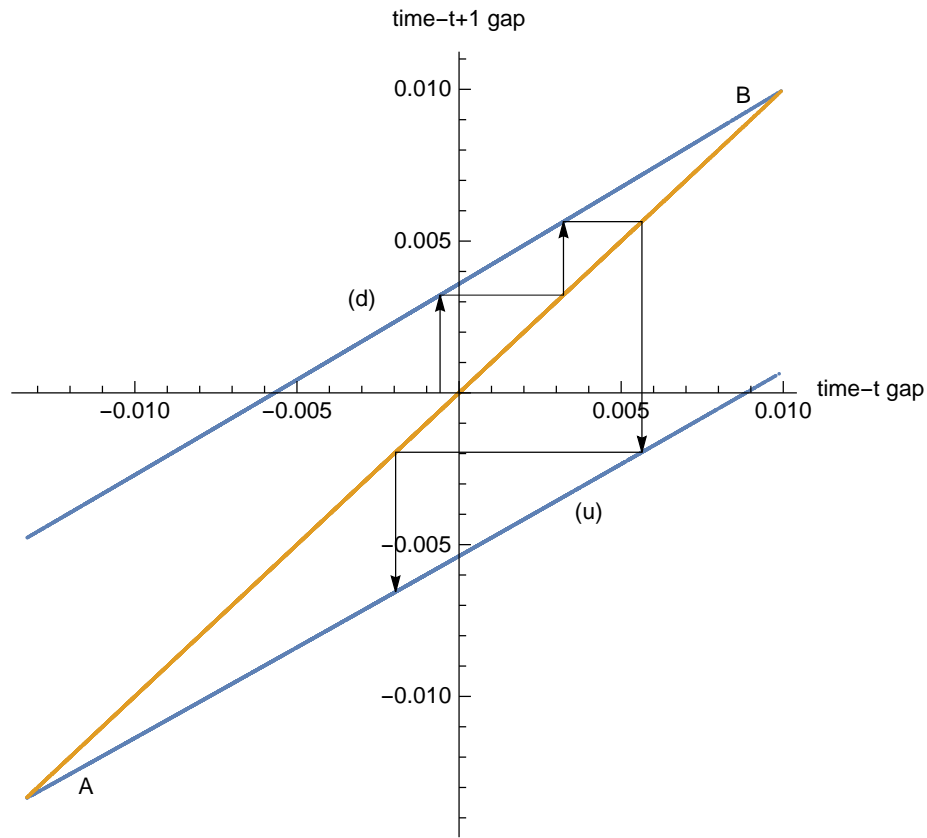


Figure 5: **Relation between output gap at time t and at time- $t + 1$ conditional on a u productivity shock (lower line) and a d shock (upper line)**, across 10,000 paths at a fixed date. The 45° line is also shown. Parameters are as in Table 1 with $\bar{\tau}$ set at a neutral rate (see definition in Footnote 44). A fragment of a sample path is drawn for illustration; it contains two d shocks followed by two u shocks. The upper “anchor” point is labelled B, the lower one A.

Observation 1 *Under Assumptions 2 to 5 and Calvo pricing,*

1. *The output gap is bounded above and below. When it is at its lower bound, a u productivity shock leaves it unchanged. When it is at its upper bound, a d productivity shock leaves it unchanged.*
2. *Except at the lower bound, a u productivity shock at time $t + 1$ decreases the output gap relative to its value at t . Except at the upper bound, a d shock increases it:*

$$\frac{y_{u,t+1}}{y_{f,u,t+1}} < \frac{y_t}{y_{f,t}} < \frac{y_{d,t+1}}{y_{f,d,t+1}}$$

3. *The lower the time- t output gap, the smaller (in absolute value) the decrease in case of u shock. The higher the time- t output gap, the smaller the d increase:*

$$\frac{d\left(\frac{y_{u,t+1}}{y_{f,u,t+1}}\right)}{d\left(\frac{y_t}{y_{f,t}}\right)} < 1; \frac{d\left(\frac{y_{d,t+1}}{y_{f,d,t+1}}\right)}{d\left(\frac{y_t}{y_{f,t}}\right)} < 1$$

When prices are flexible (output gap being equal to zero by definition), a u productivity shock brings the price level down and induces firms to increase equilibrium output. But, when prices are sticky, firms are not able to increase prices as much as they would with flexible prices, which means that the output gap decreases. However, since the gap admits a lower bound, when it is there, a u shock does not decrease it. Mechanically, the lower bound is an “anchor point” for the response to u shocks (point A in Figure 5). Furthermore, the closer the gap is to the lower bound, the smaller is the decrease. Similarly, the upper bound is an anchor point (point B) for the response to d shocks. These effects are clearly visible on the figure.

4 Stock returns and inflation

The real, future profits, assumed to be distributed as dividends, are:⁴⁵

$$\begin{aligned} \delta_{t+1} \triangleq & \left[\omega \times \left(\frac{P_t}{P_{t+1}} - \varphi_{t+1} \right) \left(\frac{P_t}{P_{t+1}} \right)^{-\sigma} \right. \\ & \left. + (1 - \omega) \times \left(\frac{P_{t+1}^*}{P_{t+1}} - \varphi_{t+1} \right) \left(\frac{P_{t+1}^*}{P_{t+1}} \right)^{-\sigma} \right] \times y_{t+1} \end{aligned} \quad (26)$$

With the parameter values of Table 1, the neutral rate is equal to 3.02%. With that value of \bar{z} , the rate of interest that prevails in the sticky-price economy at an output gap equal to zero is equal to 3.109%.

⁴⁵Current profit δ_t differs from one firm to the other, depending on which firm is allowed currently to change its price. For future profits, we neglect current price dispersion, as explained in Footnote 35.

The value of the stock market, in real terms, current profits not included, is:

$$x_t = \rho \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{\gamma-1} (\delta_{t+1} + x_{t+1}) \right] \quad (27)$$

Numerical illustrations will indicate the responsiveness of the rate of return on the stock market to inflation. We study the role of productivity shocks over one vs. several periods. As we saw in Sections 2 and 3.1, if prices set by firms were fully flexible, there would be no relation whatever between the nominal return on stocks and the rate of inflation.

When the prices set by firms are sticky, the output depends on the previous-period output gap, thus generating richer dynamics for stock returns and inflation. There can occur many values for inflation and many values for nominal stock returns depending on the value of the preexisting output gap. But, over a single time-step, productivity can only be increased by a factor u or d , obviously. Figure 6, left-hand panel, displays the two variables in a cross-section of paths.⁴⁶ We discover that *the cloud of simulated points is organized into rays*. The two rays that appear are segments of increasing, near-straight lines. Of the two rays, the upper (lower) one portrays the relation conditioning upon productivity growth being u (d). On a given ray, the points that plot farther from the origin correspond to higher values of the output gap prior to the shock, as the labelling of selected points indicates. We observe:

Observation 2 *Under Assumptions 2 to 5 and Calvo pricing,*

1. *Across different values of the time- t output gap, conditional upon productivity growth being u or d at time $t + 1$, the time- $t + 1$ realized inflation and realized nominal stock returns are near-linearly, positively related.*
2. *For the same time- t value of the output gap, when productivity is increased by a factor u , inflation is lower and the nominal stock return is higher than when the factor is d .*
3. *Upon a u move, the lowest values of inflation and nominal stock return are reached when the gap is at its lower bound (anchor point A on the figure). Upon a d move, the highest values of inflation and nominal stock return are reached when the gap is at its upper bound (anchor point B on the figure).*

These effects can be understood as follows. When the productivity shock is u , inflation is lower and the real return on stocks is higher than with a d shock, just like they were with flexible prices. But, with sticky prices, prices cannot move as much as they did with flexible ones so that they no longer fully offset the change in real returns, leaving a higher nominal return. And, as we have

⁴⁶When drawn along one path, the picture is nearly identical, as it should be under a stationary-growth equilibrium.

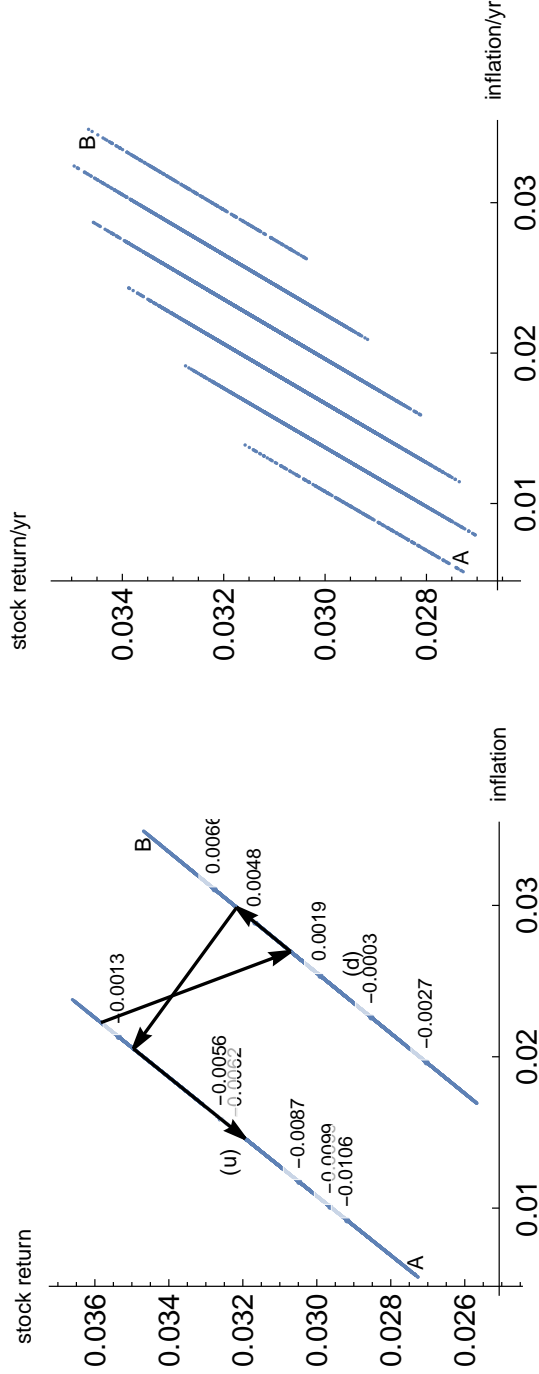


Figure 6: **Relation between one-period nominal stock return and one-period inflation (left-hand panel) and relation between the same two variables measured over five periods (right-hand panel)**, across 10,000 paths at a fixed date. Parameters are as in Table 1 with $\bar{\pi}$ set at a neutral level (see definition in Footnote 44). In the left-hand panel, the labelling of some of the points indicates the level of the output gap. A fragment of a sample path is drawn for illustration; it contains two d shocks followed by two u shocks. Points A and B are the anchor points of the output gap.

seen, for any given value of the output gap at time t , a u productivity shock causes the output gap to be down at time $t + 1$, the more so as the time- t gap is higher, or farther from its lower bound.

Not conditioning on the productivity growth, i.e., across the two rays of the entire cloud of points, the slope coefficient of a regression of the nominal stock return on inflation is as small as 0.0966.

Observation 2 invites empirical tests that would be conducted conditioning on observed or, at least, estimated productivity shocks.

When measuring returns over a longer holding period, the relation is similar but several combinations of u and d productivity moves are possible. For instance, over five periods, six orderless combinations with repetition are possible. The six corresponding rays are shown in Figure 6, right-hand panel: the highest-slope ray reflects realizations in which all five productivity moves are u ; the straight line supporting that ray is identical to that of a single u move in the left-hand panel. The ray second from the top contains realizations for which four moves are u and one is d in any order etc.⁴⁷ The lowest-slope ray reflects realizations in which all five productivity moves are d ; the line supporting that ray is identical to that of a single d move in the left-hand panel.

Across all the paths, i.e., not conditioning on the productivity growth combinations, the slope coefficient of an across-paths regression of the nominal stock return on inflation is equal to 0.2348, much higher than the single-move number 0.0966.⁴⁸ This result about the slope is produced by the following:

Observation 3 *Under Assumptions 2 to 5 and Calvo pricing,*

1. *For the same time- t value of the output gap, when productivity is increased five times by a factor u , inflation per period and the nominal stock return per period are lower than they would be with a single u productivity move over one period. That differential effect is strongest when the output gap is at its upper bound.*
2. *If, however, the gap is at its lower bound, inflation per period and the stock return per period are equal to what they would be with a single u move (anchor point effect)*
3. *Symmetric statements are true for d moves.*
4. *Hybrid combinations of productivity moves have an intermediate effect, over a wider range of values for inflation and the nominal stock return.*

These statements follow from Observation 1 above, which implies that, for the same time- t value of the output gap, when productivity is increased five

⁴⁷Except for the highest-slope and lowest-slope rays, the rays are actually a set of rays with very similar supporting straight lines, one for each order in which the productivity shocks occur. In the 3D space of (inflation, stock return, starting output gap), they would be separate rays.

⁴⁸For a ten-year holding period, the slope would be slightly larger: 0.2639.

<i>Statistic</i>	α_1	β_1	α_5	β_5
Median	0.029	0.091	0.14	0.233
Upper quintile	0.03	0.127	0.142	0.242
Lower quintile	0.028	0.048	0.139	0.22
<i>Std error</i>				
Median	0.00	0.013	0.001	0.012
Upper quintile	0.00	0.016	0.002	0.014
Lower quintile	0.00	0.011	0.001	0.01

Table 2: **Stock Returns and Contemporaneous Inflation:** the regressions are those of Equation (1). Parameters are as in Table 1 with $\bar{\tau}$ set at a neutral level (see Definition 2 in Footnote 44). The table is obtained from 10,000 paths drawn at random.

times by a factor u , the output gap decreases less per period than it would with a single u productivity move over one period. Because successive decreases in the gap decrease in size as each one puts the economy closer to the lower bound, five moves in the same direction have less effect per period than a single move over one period. One can also imagine on the basis of the same observation and Figure 5 that hybrid combinations of productivity moves cause the gap to move more widely.

The observation implies that the anchor points (at the lower bound for u moves and at the upper bound for d moves) become more attractive for multiple moves, and that the unconditional regression slope is higher over five periods than it is over one period. We stress that the direction of the increase in the regression slope is not the mechanical result of frictions slowly working themselves out, so that one would approach over time the flexible-price outcome. In fact, the increase is produced by a modification away from the flexible-price outcome, which, as the reader will recall, was a zero slope.

We ask whether the message conveyed by Figure 6 is confirmed by *time-series* regressions. Can the model fit the facts listed in Section 1? Having simulated 10,000, 200-period long paths of the economy of Section 3.2, we run on each an *ex post* regression in the manner of BR.⁴⁹

The results of running on simulated data the same *ex post* regression as (1) are shown in Table 2.⁵⁰

The results are exactly in conformity with the intuition conveyed above, in that the five-year regression slope is higher than the one-year slope. The results are also in close conformity with the empirical results of BR. Recall that both their slope coefficients were positive, with the exact same disparity between

⁴⁹We drop the first ten periods of the paths to ensure that statistical results do not depend on the initial condition, which is just the nominal amount $\bar{\theta}_2$ of government debt outstanding at $t = 0$.

⁵⁰Here again, because the five-year rates of return are calculated every year, there is overlap in the data and the Generalized Method of Moments is used to compute heteroskedasticity- (and autocorrelation-) consistent standard errors.

them.⁵¹

Basically, therefore, we have discovered the reason for which BR (and we in Section 1) found very different slopes for different lengths of stock holding period.

5 Extensions

5.1 Money balances and the zero lower bound

In the interest of simplicity, we have so far considered a cashless economy à la Woodford (1995). We extend the model to the case in which there exist actual money balances. Now there is cash explicitly in the economy, side by side with government bonds. To do that, we build an equilibrium model of money demand along the lines of Allais (1947), Baumol (1952) and Tobin (1956).⁵² We must observe at the outset that, when the nominal rate of interest approaches zero, money demand grows towards infinity thereby creating *a natural lower bound* on the rate of interest. Meanwhile, the demand of the private sector for the government bond drops steadily.⁵³ The government cum central bank, as noted by J. M. Keynes, falls into a “liquidity-trap” regime that is akin to Quantitative Easing.⁵⁴

Calling M monetary claims, money supply at time t is: $M_{2,t}$ (a negative number because, like $\theta_{2,t}$, it is a liability of the government cum central bank); money demand is $M_{1,t}$; the seignorage, an indirect tax, collected at time t and measured in nominal terms of that date is: $M_{1,t} \times (1 - 1/(1 + i_t))$. Households receive an income of a single good and no income in cash. At time t , the financial wealth available for consumption is:

$$P_t \times y_t + \theta_{1,t-1} + M_{1,t-1} - F_{1,t} - S_t$$

The proceeds $P_t \times y_t$ from the sale of the physical income are in the form of a deposit at a bank. Cash on hand $M_{1,t-1}$ and the other financial items are assumed to be readily available in cash. Cash can be withdrawn by taking trips to the bank. Each trip costs a fixed real amount ν .⁵⁵ The smaller the number of trips $N_{1,t}$ the household decides to take to the bank, the more cash the household holds on an average over the time period $[t, t + 1]$:⁵⁶

$$M_{1,t} = \frac{P_t \times y_t}{2 \times N_{1,t}}$$

⁵¹Had we included in our model a monetary shock, we could also have increased both our simulated slope coefficients at will.

⁵²Please, see Baumol and Tobin (1989).

⁵³The sum of the demands for money and bonds remains determinate and finite.

⁵⁴On the zero lower bound, a very active topic of research during the Great Recession, see the following papers: McCallum (2000), Krippner (2012), Wright (2012), Gavin et al. (2013), Pribsch (2013), Greenwood et al. (2014), Swanson and Williams (2014).

⁵⁵To preserve scale invariance (see footnote 41), we do not take ν to be a constant; we assume it proportional to output.

⁵⁶We could have assumed instead that all the financial wealth except cash on hand is deposited with a bank.

so that the cost of the trips at current prices is:⁵⁷

$$\nu \times P_t \times N_{1,t} = \nu \times P_t \times \frac{P_t \times y_t}{2 \times M_{1,t}}$$

At time T , money balances, unlike debt, do not terminate at 0. Even without a refund, the private sector holds money till the end because it has to. We set $1/(1+i_T) = 0$.

In Appendix D, we derive the set of equations (41) to be solved at each node of the tree. A change of unknown variables θ :

$$\hat{\theta}_{1,t} \triangleq \theta_{1,t} + P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}}; \quad \hat{\theta}_{2,t} \triangleq \theta_{2,t} - P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}}$$

along with a change of backward iterates:⁵⁸

$$\hat{F}_{1,t} \triangleq F_{1,t} + \frac{P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}}}{1 + i_t}; \quad \hat{F}_{2,t} \triangleq F_{2,t} - \frac{P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}}}{1 + i_t}$$

transforms the system of equations into one that is identical to the system (29) of Appendix A, which we solved in the absence of money. We thus demonstrate that, *for a given value of the endogenous variable y_t* , money is simply added to government bonds and is *otherwise irrelevant*. The government surplus being exogenous anyway, seignorage being refunded and inflation targeting being an infinitely elastic central-bank reaction function, money demand only serves to determine money supply, as has been pointed out by many authors.⁵⁹

We point out that, with $M_{2,-1} \neq 0$, the initial level of government debt $\theta_{2,-1}$ could be equal to zero. In the cashless economy, the initial condition of the Fiscal theory with zero debt would leave the price level of goods indeterminate at all times. But, in the economy with money, the initial condition does determine the initial price level even then, the present value of future government surpluses being compared to the outstanding stock of money $M_{2,-1}$. This shows that our results are not predicated on the validity of the strict, debt-based Fiscal theory.

⁵⁷That cost is truly a deadweight loss; no one gets the benefit of it. For the sake of computational simplicity we imagine that it is refunded to the private sector in the form of a lumpsum transfer $\zeta_{1,t} = P_t \times y_t \times \nu \times P_t / (2 \times M_{1,t})$ coming from the outside, thus keeping in our equation system only the distortionary effect of the cost but not its wealth effect. Without that assumption, the trips to the bank being deadweight losses, $c_t \neq y_t$.

⁵⁸Note: $\hat{\theta}_{1,t} / (1 + i_t) = \hat{F}_{1,t}$

⁵⁹Note, however, that the change of variables is valid only for strictly positive nominal interest rates. If we used it blindly in the numerical implementation, the nominal rate of interest could become negative, despite the natural lower bound. To prevent that error in the computation, we superimpose on the Taylor rule an artificial zero lower bound on the nominal rate of interest (we actually implement a smooth variant of that relation.):

$$1 + i_t = \max \left[1, (1 + \bar{i}) \times \left(\frac{\frac{1}{2} P_{t+1,u} + \frac{1}{2} P_{t+1,d}}{P_t} \right)^\phi \right] \quad (28)$$

<i>Statistic</i>	α_1	β_1	α_5	β_5
Median	0.003	0.02	0.02	0.117
Upper quintile	0.004	0.045	0.02	0.13
Lower quintile	0.003	-0.006	0.019	0.104
<i>Std error</i>				
Median	0.00	0.009	0.00	0.008
Upper quintile	0.00	0.011	0.001	0.011
Lower quintile	0.00	0.008	0.00	0.006

Table 3: **Stock Returns and Contemporaneous Inflation in the presence of money:** the regressions are those of Equation (1). Parameters are as in Table 1 but with \bar{i} set at a neutral level (see Definition 2 in Footnote 44). The table is obtained from 10,000 paths drawn at random.

We amend the “aggregate demand” subsystem of equations of Section 2 to reflect the modified policy rule (28) of Footnote 59, leaving intact the “aggregate supply” subsystem of Section 3.2 and we solve by backward induction exactly as we did before (with the additional parameter $\nu = 1\%$ of output).⁶⁰ Under the parameter and state variable combinations considered so far, the result is identical to that of the previous Figure 4, simply because the cashless economy itself never produced a negative value for the rate of interest. In order to make liquidity-trap episodes possible, the Taylor-rule interest parameter \bar{i} is now set 1.5% below the neutral rate of the flexible-price economy (as defined in Footnote 44). The change of assumption highlights the way in which the Taylor-rule interest parameter \bar{i} affects the probability of approaching the zero lower bound.

The new version of Figure 6 is Figure 7, which shows that the lower bound on the rate of interest introduces a support from below for realized nominal stock returns. For that reason, the relation between inflation and stock returns described above in Section 4 is no longer nearly linear but is still positive, contingent on a given sequence of productivity shocks. Not conditioning on the productivity growth, the coefficient of an across-paths regression of the nominal stock return on inflation between the two variables is equal to 0.0743 over one period while it is equal to 0.1950 over five periods. As before, the slope is quite a bit larger over five periods than it is over one.

The new version of Table 2, which contained the results of *ex post* regressions across simulated paths, is Table 3.

The results are again in conformity with the empirical results of BR. Furthermore, we notice that the slope coefficients are lower than in Table 2. This is because the parameter \bar{i} of the Taylor rule has been set lower than before.

⁶⁰ Also, since we have assumed that money is not refunded, the terminal conditions, which were originally $F_{1,T,j} = F_{2,T,j} = 0$ must be replaced by: $\hat{F}_{1,T,j} = -\hat{F}_{2,T,j} = P_T \times \sqrt{\frac{1}{2}} y_T \times \nu$. We study the paths of the economy in a long-horizon situation. For that, the change of terminal condition is not very important.

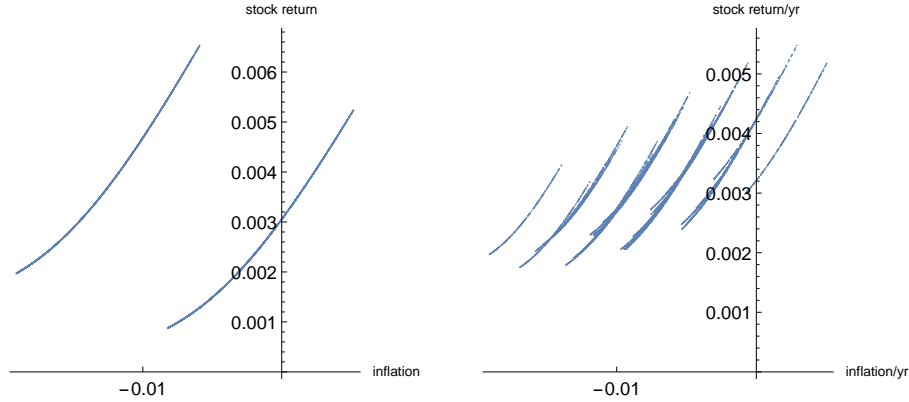


Figure 7: **Relation, in the presence of money, between one-period nominal stock return and one-period inflation (left-hand panel) and relation between the same two variables measured over five periods (right-hand panel), across 10,000 paths at a fixed date.** Parameters are as in Table 1, with $\bar{\tau}$ set 1.5% below the neutral level (see Definition 2 in Footnote 44).

5.2 Bond returns and inflation

In this section, we examine the relation between bond nominal returns and inflation to see if it is different from the relation between stock returns and inflation. We are motivated in doing this by Katz et al. (2017). Using a panel of countries they have confirmed empirically that stock markets are slow to incorporate news about future inflation so that they do not qualify to be called “real” assets, but they have found that the same is not true at all of bond markets. We now check whether our model can explain that difference between stocks and bonds. For that we return to the cashless economy and draw Figure 8, which relates the one-year and five-year nominal rates of return on a ten-year pure-discount nominal bond, across paths at a given point in time. The slope of an unconditional regression line of one-year returns on inflation is equal to 0.7202 (as opposed to 0.0966 for stocks) while the slope for five-year returns is equal to 1.1028 (as opposed to 0.2348 for stocks). The difference between long-term bonds and stocks is in the behavior of the payoff. In our model with sticky prices but flexible wages, dividends on stocks (Equation (26)) are adversely affected by inflation.

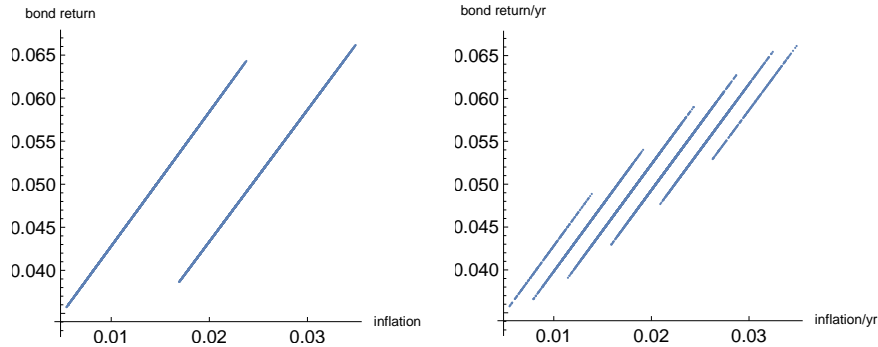


Figure 8: **Relation in a cashless economy between one-period nominal 10-year bond return and one-period inflation (left-hand panel) and relation between the same two variables measured over five years (right-hand panel)**, across 10,000 paths at a fixed date. Parameters are as in Table 1, with $\bar{\tau}$ set at a neutral level (see definition in Footnote 44).

6 Conclusion

Adopting a method that has been used to calculate dynamic financial-market equilibria, we have constructed the equilibrium of a cashless production economy with productivity shocks and with three types of agents: (i) household/investors who supply labor with a finite elasticity, consume a large variety of goods that are not perfect substitutes and trade government bonds; (ii) firms that produce those varieties of goods, setting prices in a Calvo manner; (iii) a government that collects an exogenous fiscal surplus and acts mechanically, buying and selling bonds in accordance with a Taylor policy rule based on expected inflation.

Merging the consumption-financial behavior of households with the policy rule, under IID productivity growth and no monetary shock, we have derived explicitly at each point in time and in each state of nature, aggregate-demand schedules (inclusive of policy rule) relating, at the next point in time and in each successor state, the price level to the level of output. We have shown that these schedules are decreasing if and only if the exponent of the Taylor rule that falls on expected inflation is less than 1. The aggregate supply schedules (or Phillips curve) that also apply to the next point in time are always increasing. The equilibrium is unique if the exponent is less than 1. Otherwise, because of the non linearities of the two types of schedules, two equilibria can exist.

In this equilibrium, we have priced the stock market, defined as the present discounted value of firms' profits and demonstrated that, in a flexible price version of our economy, the equilibrium nominal return on stocks is just equal to the riskless interest rate, which is constant, whereas inflation, for fiscal reasons, is higher when productivity growth is low: when output growth is higher than

expected, the government runs larger real surpluses in the future and as a result, the price level can readjust to a lower level, thus generating lower inflation. That explains a zero-slope relation between these rates and gives stocks a nominal character.

Moving to a sticky-price version of the economy, we have simulated the joint behavior of stock returns and inflation. That has allowed us to discover the reason for which Boudoukh and Richardson (1993) found different slopes for different holding-period lengths. The reason lies in the succession of productivity shocks that take place over several periods, and in the path dependence created by the output gap. We stress that the direction of the increase in the regression slope is not the mechanical result of frictions slowly working themselves out, so that one would approach over time the flexible-price outcome. In fact, the increase is produced by a modification away from the flexible-price outcome, which was a zero slope. When output growth is higher than expected, the government runs larger real surpluses in the future but, with sticky prices, the price level cannot readjust as much as it did with flexible prices. That is the reason for the positive slope between nominal returns and inflation.

The equilibrium has then been expanded to incorporate an explicit money demand à la Baumol and Tobin. The only effect of the zero lower bound thus created has been to support stock returns when they are low.

Finally, we turned to long-term bonds to observe that their behavior vis-à-vis inflation is more “real” than that of stocks, which explains the surprising empirical findings of Katz et al. (2017) that bond returns tend to move more one-for-one with inflation than do stock returns.

All the theoretical results of this paper invite empirical tests that would be conducted conditioning on observed (or estimated) productivity shocks.

Appendixes⁶¹

A Proof of Proposition 1

We write the first-order conditions of dynamic programming for the private sector. Then in order to “synchronize” the solution algorithm of the equations and allow recursivity, taking a cue from Dumas and Lyasoff (2012), we first shift all first-order conditions, except the Euler condition of portfolio choice, forward in time and, second, we no longer make explicit use of the investor’s position $\theta_{1,t-1}$ held when entering time t , focusing instead on the financial wealth: $F_{1,t} \triangleq \theta_{1,t}/(1+i_t)$ held when exiting time $t+1$. The existing-wealth functions are carried backward. Regrouping equations in that way leads to the equation system

$$\begin{aligned}
& \text{Flow budget constraints of private sector at time } t+1 \\
& P_{t+1,u} \times c_{t+1,u} + F_{1,t+1,u} + s_{t+1,u} \times P_{t+1,u} = \theta_{1,t} + P_{t+1,u} \times y_{t+1,u}; F_{1,T,u} = 0 \\
& P_{t+1,d} \times c_{t+1,d} + F_{1,t+1,d} + s_{t+1,d} \times P_{t+1,d} = \theta_{1,t} + P_{t+1,d} \times y_{t+1,d}; F_{1,T,d} = 0 \\
& \text{Flow budget constraints of government at time } t+1 \\
& F_{2,t+1,u} = \theta_{2,t} + s_{t+1,u} \times P_{t+1,u}; F_{2,T,u} = 0 \\
& F_{2,t+1,d} = \theta_{2,t} + s_{t+1,d} \times P_{t+1,d}; F_{2,T,d} = 0 \\
& \text{Portfolio-choice, or Euler, or Fisher condition at time } t \tag{29} \\
& \frac{1}{1+i_t} \frac{1}{P_t} = \rho \frac{\frac{1}{2}(c_{t+1,u})^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2}(c_{t+1,d})^{\gamma-1} \frac{1}{P_{t+1,d}}}{(c_t)^{\gamma-1}} \\
& \text{Taylor rule at time } t \\
& 1+i_t = (1+\bar{i}) \times \left(\frac{\frac{\frac{1}{2}P_{t+1,u} + \frac{1}{2}P_{t+1,d}}{P_t}}{1+\bar{\pi}} \right)^{\phi} \\
& \text{Market clearing at time } t \\
& \theta_{1,t} + \theta_{2,t} = 0
\end{aligned}$$

The exogenous state variable is y_t . There is also an endogenous state variable P_t but, in the absence of money illusion, it can be factored out on grounds of homogeneity. To reflect homogeneity with respect to the price level, let $\theta_{1,t} \equiv \vartheta_{1,t} \times P_t$; $\theta_{2,t} \equiv \vartheta_{2,t} \times P_t$; $F_{1,t+1,u} \equiv f_{1,t+1,u} \times P_{t+1,u}$; $F_{2,t} \equiv f_{2,t} \times P_t$; $F_{2,t+1,u} \equiv f_{2,t+1,u} \times P_{t+1,u}$ and postulate: $f_{1,t+1,u} = -f_{2,t+1,u}$, $f_{1,t+1,d} = -f_{2,t+1,d}$. The

⁶¹In the appendixes, we suppress the subscript $j \in \{d, u\}$ at time t .

system of equations simplifies to

Flow budget constraints of private sector

$$-f_{2,t+1,u} \times P_{t+1,u} + s_{t+1,u} \times P_{t+1,u} = \vartheta_{1,t} \times P_t$$

$$-f_{2,t+1,d} \times P_{t+1,d} + s_{t+1,d} \times P_{t+1,d} = \vartheta_{1,t} \times P_t$$

Flow budget constraints of government

$$f_{2,t+1,u} \times P_{t+1,u} = \vartheta_{2,t} \times P_t + s_{t+1,u} \times P_{t+1,u} \quad (30)$$

$$f_{2,t+1,d} \times P_{t+1,d} = \vartheta_{2,t} \times P_t + s_{t+1,d} \times P_{t+1,d} \quad (31)$$

Portfolio-choice, or Euler, or Fisher condition

$$\frac{1}{1+i_t} \frac{1}{P_t} = \rho \frac{\frac{1}{2} (y_{t+1,u})^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2} (y_{t+1,d})^{\gamma-1} \frac{1}{P_{t+1,d}}}{(y_t)^{\gamma-1}}$$

Taylor rule

$$1 + i_t = (1 + \bar{i}) \times \left(\frac{\frac{\frac{1}{2} P_{t+1,u} + \frac{1}{2} P_{t+1,d}}{P_t}}{1 + \bar{\pi}} \right)^\phi$$

Market clearing

$$\vartheta_{1,t} + \vartheta_{2,t} = 0$$

Government debt: Government debt is nominal and can be priced by means of the Fisher equation, which means that the financial wealth of the government can be obtained by the following backward induction:

$$\begin{aligned} \frac{F_{2,t}}{P_t} &\triangleq \frac{1}{P_t} \frac{\theta_{2,t}}{1+i_t} \\ &= \rho \frac{\frac{1}{2} (c_{t+1,u})^{\gamma-1} \left(-s_{t+1,u} + \frac{F_{2,t+1,u}}{P_{t+1,u}} \right) + \frac{1}{2} (c_{t+1,d})^{\gamma-1} \left(-s_{t+1,d} + \frac{F_{2,t+1,d}}{P_{t+1,d}} \right)}{(y_t)^{\gamma-1}} \end{aligned} \quad (32)$$

From (32), the backward dynamics of real government financial liabilities are provided by:

$$\begin{aligned} f_{2,t} &= \rho \frac{\frac{1}{2} (y_{t+1,u})^{\gamma-1} (-s_{t+1,u} + f_{2,t+1,u}) + \frac{1}{2} (y_{t+1,d})^{\gamma-1} (-s_{t+1,d} + f_{2,t+1,d})}{(y_t)^{\gamma-1}}; \\ f_{2,T} &= 0 \end{aligned} \quad (33)$$

The current real discounted value $f_{2,t}$ of government debt depends only on future income and future surpluses. It does not depend on interest-rate policy.

But the real *face value* $\vartheta_{2,t}$, which is the government's equilibrium portfolio choice or issuance decision, depends on the nominal rate of interest, which we now determine.

Inflation: Solving for inflation from the government flow budget constraints

(30), (31):

$$\begin{aligned}\frac{P_{t+1,u}}{P_t} &= \frac{\vartheta_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} \\ \frac{P_{t+1,d}}{P_t} &= \frac{\vartheta_{2,t}}{-s_{t+1,d} + f_{2,t+1,d}}\end{aligned}$$

so that the realized rates of inflation are:

$$\begin{aligned}\frac{P_{t+1,u}}{P_t} &= \frac{f_{2,t} \times (1 + i_t)}{-s_{t+1,u} + f_{2,t+1,u}} \\ \frac{P_{t+1,d}}{P_t} &= \frac{f_{2,t} \times (1 + i_t)}{-s_{t+1,d} + f_{2,t+1,d}}\end{aligned} \tag{34}$$

These relate the two levels of future inflation ($P_{t+1,u}/P_t, P_{t+1,d}/P_t$) to calendar time t , to the two levels of future real government debt ($-s_{t+1,u} + f_{2,t+1,u}, -s_{t+1,d} + f_{2,t+1,d}$) and to the current level of real government debt $f_{2,t}$. We call $f_{2,t}/(-s_{t+1} + f_{2,t+1})$ the “ex post *inverse* real gross rates of return on government debt”. It is also the ex post inverse real gross rates of return on *any* nominally riskless debt.

Proposition 7 *The ex post levels of inflation in the two states of nature are separately*

- *increasing functions of the ex post inverse real gross rates of return on nominally riskless debt*
- *increasing functions of the (ex ante) nominal gross rate of interest.*

To illustrate, assuming that debt returns more in real terms in a u state than in a d state, which is

$$\frac{-s_{t+1,u} + f_{2,t+1,u}}{f_{2,t}} > \frac{-s_{t+1,d} + f_{2,t+1,d}}{f_{2,t}}$$

then

$$\frac{P_{t+1,u}}{P_t} < \frac{P_{t+1,d}}{P_t}$$

Inflation is lower in the u state than in the d state.

Making use now of the IID assumption 1 and of the budget surplus assumption 3, the real discounted value of government debt $f_{2,t}$ is proportional to the level of income y_t at time t :

$$f_{2,t} = \hat{f}_{2,t} \times y_t \tag{35}$$

where $\hat{f}_{2,t}$ approaches zero *deterministically* as one approaches the terminal date. Indeed:

$$\begin{aligned}\frac{\hat{f}_{2,t}}{-\tau + \hat{f}_{2,t+1}} &= k; \hat{f}_{2,T} = 0 \\ \hat{f}_{2,t} &= -\tau \times \frac{k \times (-1 + k^{T-t})}{-1 + k}\end{aligned} \tag{36}$$

where we define k as (11), $k < 1$ being assumed for now.⁶²

The realized inverse real gross rates of return on government debt are:⁶³

$$\begin{aligned}\frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} &= \frac{k}{1+u} \text{ in a } u \text{ state} \\ \frac{f_{2,t}}{-s_{t+1,d} + f_{2,t+1,d}} &= \frac{k}{1+d} \text{ in a } d \text{ state}\end{aligned}$$

Substituting into (34) gives the result (10). Substituting into the Taylor rule (7) gives the result (12). Substituting into the government budget constraint (5) written at time 0 gives the result (13).

B Proof of Proposition 4

Combining (34) with (7), we get

$$\frac{P_{t+1,u}}{P_t} = \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} \times (1+\bar{i})^{\frac{1}{1-\phi}} \times \left(\frac{\frac{1}{2} \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} + \frac{1}{2} \frac{f_{2,t}}{-s_{t+1,d} + f_{2,t+1,d}}}{1+\bar{\pi}} \right)^{\frac{\phi}{1-\phi}}$$

and, therefore

$$\begin{aligned}\frac{\partial \frac{P_{t+1,u}}{P_t}}{\partial \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}}} &= (1+\bar{i})^{\frac{1}{1-\phi}} \times \left(\frac{\frac{1}{2} \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} + \frac{1}{2} \frac{f_{2,t}}{-s_{t+1,d} + f_{2,t+1,d}}}{1+\bar{\pi}} \right)^{\frac{\phi}{1-\phi}} \\ &\quad \times \left[1 + \frac{\frac{1}{-s_{t+1,u} + f_{2,t+1,u}}}{\frac{1}{-s_{t+1,u} + f_{2,t+1,u}} + \frac{1}{-s_{t+1,d} + f_{2,t+1,d}}} \times \frac{\phi}{1-\phi} \right]\end{aligned}$$

In view of (35),

$$\begin{aligned}\frac{\partial \frac{P_{t+1,u}}{P_t}}{\partial \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}}} &= (1+\bar{i})^{\frac{1}{1-\phi}} \times \left(\frac{\frac{1}{2} \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}} + \frac{1}{2} \frac{f_{2,t}}{-s_{t+1,d} + f_{2,t+1,d}}}{1+\bar{\pi}} \right)^{\frac{\phi}{1-\phi}} \\ &\quad \times \left[1 + \frac{\frac{1}{1+u}}{\frac{1}{1+u} + \frac{1}{1+d}} \times \frac{\phi}{1-\phi} \right]\end{aligned}$$

⁶²For an infinite horizon, with $k < 1$, we would get the Gordon formula and a constant real debt factor:

$$\hat{f}_{2,t} = \tau \times \frac{k}{-1+k}$$

⁶³The quantity:

$$k \times \left(\frac{1}{2} \frac{1}{1+u} + \frac{1}{2} \frac{1}{1+d} \right)$$

can be viewed as the expected inverse gross *real* rate of interest on *nominally* riskless claims, which is not equal to the inverse gross real rate on really riskless claims ($\rho \times [(1+u)^{\gamma-1} + (1+d)^{\gamma-1}] / 2$).

And, from (33), (36) and (11)

$$\frac{\partial \frac{f_{2,t}}{-s_{t+1,u} + f_{2,t+1,u}}}{\partial y_{t+1,u}} = \frac{1}{y_{t+1,u}} \times \left[-\frac{\rho \times \frac{1}{2} (1+d)^\gamma}{1+u} + \rho (\gamma-1) \frac{1}{2} (1+u)^{\gamma-1} \right] < 0$$

This shows that $\partial \frac{P_{t+1,u}}{P_t} / \partial y_{t+1,u}$ is positive if and only if

$$1 + \frac{\frac{1}{1+u}}{\frac{1}{1+u} + \frac{1}{1+d}} \times \frac{\phi}{1-\phi} > 0$$

The proposition follows.

C Derivation in the case of a quantitative fiscal policy

Suppose that the nominal supply of bonds (the face value of the debt of the government, as such written as a negative quantity) is fixed at $-\underline{B}$ ($\underline{B} > 0$). Then the only equation of the system (29) that is to be changed is the Taylor rule, which is replaced by:

Fixed supply

$$\theta_{2,t} = -\underline{B}$$

With notation: $\theta_{1,t} \equiv \vartheta_{1,t} \times P_t$; $\theta_{2,t} \equiv \vartheta_{2,t} \times P_t$; $F_{1,t+1,u} \equiv f_{1,t+1,u} \times P_{t+1,u}$; $F_{2,t} \equiv f_{2,t} \times P_t$; $F_{2,t+1,u} \equiv f_{2,t+1,u} \times P_{t+1,u}$) and postulating: $f_{1,t+1,u} = -f_{2,t+1,u}$, $f_{1,t+1,d} = -f_{2,t+1,d}$, some of the equations simplify to

Flow budget constraints of government

$$f_{2,t+1,u} \times P_{t+1,u} = -\underline{B} + s_{t+1,u} \times P_{t+1,u} \quad (37)$$

$$f_{2,t+1,d} \times P_{t+1,d} = -\underline{B} + s_{t+1,d} \times P_{t+1,d} \quad (38)$$

Fixed supply

$$\vartheta_{2,t} \times P_t = -\underline{B}$$

Government debt: The backward derivation of the present value of future surpluses is identical to what it was in Appendix A, leading to the result 36.

Price level: Solving for the price level from the government flow budget constraints (37), (38):

$$P_{t+1,u} = \frac{-\underline{B}}{-s_{t+1,u} + f_{2,t+1,u}}$$

$$P_{t+1,d} = \frac{-\underline{B}}{-s_{t+1,d} + f_{2,t+1,d}}$$

Using 36, the price levels are:

$$P_{t+1,u} = \frac{\underline{B}}{\tau \times y_{t+1,u}} \frac{-1+k}{-1+k^{T-t}}$$

$$P_{t+1,d} = \frac{\underline{B}}{\tau \times y_{t+1,d}} \frac{-1+k}{-1+k^{T-t}}$$

At date $t-1$, the government will have just enough real debt left that it gets it paid back with the time- T tax revenue.

Inflation is:

$$\frac{P_{t+1,u}}{P_t} = \frac{1}{1+u} \frac{-1+k^{T-t+1}}{-1+k^{T-t}}$$

$$\frac{P_{t+1,d}}{P_t} = \frac{1}{1+d} \frac{-1+k^{T-t+1}}{-1+k^{T-t}}$$

Nominal rate of interest: Given that, by definition:

$$f_{2,t+1,u} = \frac{1}{P_{t+1,u}} \frac{-\underline{B}}{1+i_{t+1,u}}$$

it follows that

$$\frac{k \times (-1+k^{T-t-1})}{-1+k^{T-t}} = \frac{1}{1+i_{t+1,u}}$$

or:

$$\frac{k \times (-1+k^{T-t})}{-1+k^{T-t+1}} = \frac{1}{1+i_t}$$

so that inflation is also written as:

$$\frac{P_{t+1,u}}{P_t} = \frac{k}{1+u} \times (1+i_t)$$

which is the same result as (10).

D Backward equation system for the Baumol-Tobin model of Section 5.1

In this appendix, the symbol \mathcal{W} stands for entering (or pre-trade) wealth.

$$\begin{aligned} \mathcal{L}_1(\mathcal{W}_{1,t}, \cdot, t) &= \sup_{c_t, \theta_{1,t}} \inf_{\phi_{1,t}} u_1(c_t, t) \\ &+ \frac{1}{2} \sum_{j=u,d} J_1(\theta_{1,t}, M_{1,t}, \cdot, t+1) \\ &+ \phi_{1,t} \times \left[\mathcal{W}_{1,t} - \frac{\theta_{1,t}}{1+i_t} - S_t + P_t \times y_t \times \left(1 - \frac{\nu \times P_t}{2 \times M_{1,t}} \right) - P_t \times c_t - M_{1,t} + \zeta_{1,t} \right] \end{aligned}$$

where: $\mathcal{W}_{1,t} \triangleq M_{1,t-1} + \theta_{1,t-1}$. The first-order condition with respect to $\theta_{1,t}$ is:

$$\frac{1}{2} \sum_{j=u,d} \frac{\partial}{\partial \mathcal{W}_{1,t}} J_1(\theta_{1,t}, M_{1,t}, \cdot, t+1) - \frac{\phi_{1,t}}{1+i_t} = 0$$

The first-order condition with respect to $M_{1,t}$ is:

$$\frac{1}{2} \sum_{j=u,d} \frac{\partial}{\partial \mathcal{W}_{1,t}} J_1(\theta_{1,t}, M_{1,t}, \cdot, t+1) + \phi_{1,t} \times \left[P_t \times y_t \times \frac{\nu \times P_t}{2 \times (M_{1,t})^2} - 1 \right] = 0$$

The envelope condition is:

$$\frac{\partial}{\partial \mathcal{W}_{1,t-1}} J_1(\theta_{1,t-1}, M_{1,t-1}, \cdot, t) = \phi_{1,t}$$

The Euler conditions are:

$$\begin{aligned} \frac{1}{2} \sum_{j=u,d} \phi_{1,t+1,j} - \frac{\phi_{1,t}}{1+i_t} &= 0; t = 0, \dots, T-1 \\ \frac{1}{2} \sum_{j=u,d} \phi_{1,t+1,j} &= \phi_{1,t} \times \left[1 - P_t \times y_t \times \frac{\nu \times P_t}{2 \times (M_{1,t})^2} \right]; t = 0, \dots, T \end{aligned}$$

The latter is simply:

$$\begin{aligned} \frac{1}{1+i_t} &= 1 - P_t \times y_t \times \frac{\nu \times P_t}{2 \times (M_{1,t})^2} \\ P_t \times y_t \times \frac{\nu \times P_t}{2 \times (M_{1,t})^2} &= 1 - \frac{1}{1+i_t} \\ P_t \times y_t \times \frac{\nu \times P_t}{1 - \frac{1}{1+i_t}} &= 2 \times (M_{1,t})^2 \\ M_{1,t} &= P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}} \end{aligned} \tag{39}$$

except at time $t = T$ where:

$$\begin{aligned} 1 - P_T \times y_T \times \frac{\nu \times P_T}{2 \times (M_{1,T})^2} &= 0 \\ M_{1,T} &= P_T \times \sqrt{\frac{1}{2} y_T \times \nu} \end{aligned}$$

Summing up, the set of equations to be solved at each node of the tree is:

Flow budget constraints of private sector

$$\begin{aligned} P_{t+1,j} \times c_{t+1,j} + F_{1,t+1,j} + M_{1,t+1,j} + S_{t+1,j} \\ = \theta_{1,t} + M_{1,t} + P_{t+1,j} \times y_{t+1,j}; \\ F_{1,T,j} = 0; j = u, d \end{aligned}$$

Flow budget constraints of government cum central bank

$$F_{2,t+1,j} + M_{2,t+1,j} = \theta_{2,t} + M_{2,t} + S_{t+1,j}; F_{2,T,j} = 0; j = u, d$$

Portfolio-choice or Euler conditions

(40)

$$\frac{1}{1+i_t} \frac{1}{P_t} = \rho \frac{\frac{1}{2} (c_{t+1,u})^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2} (c_{t+1,d})^{\gamma-1} \frac{1}{P_{t+1,d}}}{(c_t)^{\gamma-1}}; t = 0, \dots, T-1$$

$$M_{1,t} = P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}}; t = 0, \dots, T-1; M_{1,T} = P_T \times \sqrt{\frac{1}{2} y_T \times \nu}$$

Market clearing

$$\theta_{1,t} + \theta_{2,t} = 0; M_{1,t+1,u} + M_{2,t+1,u} = 0; M_{1,t+1,d} + M_{2,t+1,d} = 0; M_{1,t} + M_{2,t} = 0$$

Eliminating the money terms from this system and taking (39) into account:

$$\begin{aligned} P_{t+1,j} \times c_{t+1,j} + F_{1,t+1,j} + \frac{P_{t+1,j} \times \sqrt{\frac{1}{2} y_{t+1,j} \times \frac{\nu}{1 - \frac{1}{1+i_{t+1,j}}}}}{1 + i_{t+1,j}} + S_{t+1,j} \\ = \theta_{1,t} + P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}} + P_{t+1,j} \times y_{t+1,j}; \\ F_{1,T,j} = 0; j = u, d \\ F_{2,t+1,j} - \frac{P_{t+1,j} \times \sqrt{\frac{1}{2} y_{t+1,j} \times \frac{\nu}{1 - \frac{1}{1+i_{t+1,j}}}}}{1 + i_{t+1,j}} \\ = \theta_{2,t} - P_t \times \sqrt{\frac{1}{2} y_t \times \frac{\nu}{1 - \frac{1}{1+i_t}}} + S_{t+1,j}; F_{2,T,j} = 0; j = u, d \\ \frac{1}{1+i_t} \frac{1}{P_t} = \rho \frac{\frac{1}{2} (c_{t+1,u})^{\gamma-1} \frac{1}{P_{t+1,u}} + \frac{1}{2} (c_{t+1,d})^{\gamma-1} \frac{1}{P_{t+1,d}}}{(c_t)^{\gamma-1}} \\ \theta_{1,t} + \theta_{2,t} = 0 \end{aligned} \tag{41}$$

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