NBER WORKING PAPER SERIES

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Working Paper No. 2730

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1988

This research is part of NBER's research program in Economic Fluctuations. Any opinions expressed are those of the author not those of the National Bureau of Economic Research.

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ABSTRACT

This paper presents calculations of the utility cost to consumers of following alternative decision rules in the environments specified by tests of the intertemporal allocation of consumption on aggregate data. The alternatives include excess and inadequate sensitivity to income and interest rate changes and ignoring information. The calculations find that the costs of large deviations from the optimal decision rule--consumption equal to current income, for example--are on the order of 1¢ to $1 per quarter. They are interpreted to suggest that the theory does not make predictions that are robust to small inaccuracies of modelling, including small costs of transactions and information, and that those small costs can account for rejections of the theory as it is applied to aggregate US data.

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The Sensitivity of Tests of the Intertemporal Allocation of Consumption to Near-Rational Alternatives

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The theory of the intertemporal allocation of consumption is at the heart of macroeconomics and finance. Many studies have tested the theory using aggregate data, in particular as tests of the permanent income hypothesis and the consumption based capital asset pricing model, and they often reject the versions of the theory that they specify. However, it's not clear whether these statistical rejections imply a robust rejection of the basic theory (in favor of, say, "liquidity constraints") or whether they are driven by the many simplifying assumptions of tractable and empirically useful models. The tests have also been criticized (among other reasons) for exploiting "too fine" predictions of the theory, for example that all individuals adjust their consumption on a weekly or monthly basis in response to changes in prospective returns on the stock market.

As one way to address and quantify these doubts, this paper presents calculations of the utility cost to consumers of following alternative decision rules in the environments specified by the tests. For example, one calculation finds the utility loss suffered by an individual who sets consumption equal to income in each period rather than following the optimal decision rule specified by the permanent income hypothesis in the environment of Flavin's (1981) test.

These utility costs are typically 10¢ to $1 per quarter (or 3¢ to 30¢ per month), that is, a steady stream of 10¢ to $1 per quarter would compensate the consumer for the utility loss he incurs by following the alternative decision rule. The utility costs are small because cyclical changes in consumption are small and because the utility costs of deviations from an optimum are an order of magnitude smaller than the deviation itself. For example, the standard deviation of the growth rate of quarterly real per capita nondurable consumption in postwar US data is 0.86 percentage points, and its level in 1986 was about $3500 per year, implying a change of about $7.50 each quarter. Now suppose the representative consumer makes a mistake, and consumes $7.50 too little this quarter and (1+r) x $7.50 too much next quarter, thereby washing out the phenomenon of cyclical consumption changes. A simple calculation given below shows that this "mistake" implies at most a 6.5¢ utility loss if the consumer's relative risk aversion coefficient is 1 and a 65¢ utility loss if his relative risk aversion coefficient is 10.

Why do we care about the utility costs of alternative decision rules? Suboptimal decision rules that cost a trivial amount of utility or profit are called *near-rational*. Near-rational behavior can be most easily interpreted as small mistakes: people don't literally maximize, they follow heuristic decision processes that we model by maximization. Their actual decisions may deviate from the optimal decision rules if the utility costs of doing so are trivial. Using this interpretation, Akerlof and Yellen (1985a, 1985b and 1987) argued for the principle that the predictions of a theory should be robust to near-rational behavior, and Akerlof (1979) applied this idea in the same way as in this paper to show that large deviations form optimal money holdings carry trivial costs.

In a second interpretation, the small mistakes are made by economists in modelling the world rather than by the agents we study. Empirically useful forms of economic theory gloss over many complexities of the decision problem that consumers actually face. There are small costs of transactions, information acquisition, decision, attention, etc., as well as the (hopefully) small effects of modelling simplifications to one consumption good, known forms for the distributions of stochastic processes, simple depreciation schedules for durable goods, etc. We can't know precisely what effect including these small corrections would have on the predictions of the theory until we work out a theory that includes them, but we can use the range of alternate decision rules that cost the consumer (say) $1 per quarter of utility as a guide to the range of behavior we might expect the theory to predict if a small (fixed) cost of $1 per quarter were properly included.

More precisely, suppose we calculate the achieved level of utility as a function of decision rule parameters. Then we can use this (indirect) *utility* function to measure the *economic* power, just as we use the *likelyhood* function to measure its *statistical* power. The range of alternative decision rule parameters that generate utility within (say) $1 of the optimum is the range against which the theory has little *economic* power, just as the range of alternative decision rule parameters within a given fraction of the maximum likelihood is the range against which the theory has little *statistical* power.

It may happen that a test can *statistically* reject the optimal decision rule in favor of alternatives with small utility costs, or that the likelihood function is more curved than the utility function. This situation indicates that a statistical rejection might be driven by modelling simplifications rather than by a failure of the basic theory. Though macroeconomics is often accused of not having enough data to statistically reject any model, such a situation indicates the opposite: that tests are able to *statistically* distinguish alternatives that are not well distinguished *economically*.

One limitation of this interpretation of utility loss calculations is that we should not expect near-rational decision rules to persist if there are institutions that can remove them. For example, consumers might be able to sign over their income streams to a firm, which then makes their consumption decisions for them and collects the surpluses available from reducing many consumers' small mistakes, or from reducing their small information costs if there are increasing returns in the activities corresponding to those costs. Pension plans, Christmas clubs, and mutual funds may in part perform these services for the problems of life-cycle and intra-year consumption allocations and for portfolio decisions. However, I know of no institutions that make *cyclical* allocations for the consumer-changes in consumption in response to changes in *aggregate* income or rates of return-which is the focus of the empirical literature and of this paper.

For this reason, propositions derived from dynamic optimization by firms alone[1] may be less sensitive to near-rational criticism. A suboptimal decision that costs IBM .1% of its profits is a small mistake from the firm's viewpoint, but quite valuable to a manager if he can improve the decision and capture some of the increased profit. Also, a firm that does not optimize can be taken over by a better set of managers. But there is no analogy to the market for corporate control at the level of the individual consumer.[2]

The body of this paper takes two approaches to argue that the range of decision rules that cost less than about $1 per quarter is in fact large, and encompasses alternative decision rules that are economically extreme and that can account for statistical rejections in the environments of common tests of the intertemporal allocation of consumption on aggregate data. Section 2 shows that first order deviations from an optimum carry only second order utility losses, so there are always alternative decision rules for which the ratio of utility losses to the magnitude of the deviation are as small as one likes. Section 3 calculates the exact utility losses of following a variety of specific alternative decision rules in environments that are typical of tests in the empirical literature.

2. Near Rationality and the Intertemporal Allocation of Consumption

One reason to suspect that the costs of alternative decision rules are small is that first order "mistakes" in decisions have second order consequences for utility, or that there are always decisions close to the optimal one for which the ratio of utility losses to the deviation from the optimum can be made as small as one wishes. These points have been most recently popularized by Akerlof and Yellen in essentially static contexts. This section extends them to the dynamic and stochastic case considered by the theory of the intertemporal allocation of consumption.[3]

The basic idea is most simply expressed in the context of the constrained maximization of a differentiable function $f(\underline{x})$

$$(2.1) \qquad \max_{\{\underline{x}\}} f(\underline{x}) \quad \text{s.t.} \quad \underline{p}_1'\underline{x} = w_1,\ \underline{p}_2'\underline{x} = w_2,\ \ldots\ \underline{p}_M'\underline{x} = w_M.$$

where $\underline{x}$ is a vector of choice variables. The first order conditions are

$$(2.2) \qquad Df(\underline{x}) = df(\underline{x}^*)/d\underline{x} = \sum_{i=1}^{M} \lambda_i \underline{p}_i$$

where $\underline{x}^*$ denotes an optimum and λ_i are Lagrange multipliers. Consider a deviation $\underline{x}^+ = \underline{x}^* + \Delta\underline{x}$, that satisfies the budget constraints, so $\underline{p}_i'\Delta\underline{x} = 0$ $i=1,\ldots M$. The effect of this deviation on the objective is

$$(2.3) \qquad f(\underline{x}^+) = f(\underline{x}^*) + 1/2\ \Delta\underline{x}'\ D^2 f\ \Delta\underline{x} + O(|\Delta\underline{x}|^3)\ .$$

In words, *(1) feasible first order deviations in choice variables have second order consequences.* The definition of derivative and limit in (2.2) imply that (2) *there are suboptimal feasible choices* $x^+ = \underline{x}^* + \Delta\underline{x}$ *for which the ratio of the size of the utility losses to the size of the deviation are as* *small as one wishes.* Formally stated, for all $\epsilon > 0$, there is a δ such that any $\Delta\underline{x}$ that satisfies the constraints $\underline{p}_i'\Delta\underline{x} = 0$ $i = 1,2,..M$, and is smaller than δ, $0 < |\Delta\underline{x}| < \delta$, has a ratio of utility losses to magnitude of deviation smaller than the given ϵ,

$$\text{(2.4)} \qquad \frac{f(\underline{x}^*) - f(\underline{x}^* + \Delta\underline{x})}{|\Delta\underline{x}|} < \epsilon \quad .$$

A version of the second statement holds near rather than precisely at the optimum: if f is twice differentiable, we can always choose a point near the optimum and a deviation from that point so that the ratio of losses to the deviation is arbitrarily small. This is shown in the appendix.

A simple and typical version of the consumer's problem is:

$$\text{(2.5)} \qquad \max_{\{c_0,c_1,..\}} U(\{c_0,c_1,...\}) = E\sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{(2.6)} \qquad \text{s.t.} \quad 1)\ k_{t+1} = R_t k_t + y_t - c_t,$$

$$2)\ \lim_{t\to\infty} \left(\prod_{\ell=0}^{t-1} R_\ell \right)^{-1} k_t = 0 \text{ a.s.}$$

$$3)\ k_0 \text{ given}$$

where c_t = consumption, y_t = an endowment stream, k_t= nonhuman wealth at the beginning of period t (decisions at time t affect k_{t+1}, not k_t) and R_t is the ex-post real interest rate between time t and time t+1 (when stochastic, R_t is not known until the beginning of t+1). The second constraint rules out borrowing a dollar and rolling over the debt forever; it allows the period to period budget constraint given in (2.6) to be written in present value form

$$(2.7) \qquad k_0 = \sum_{t=0}^{\infty} \left(\prod_{\ell=1}^{t+1} R_\ell \right)^{-1} (c_t - y_t) \quad \text{a.s.}\ .$$

Let s^t denote the state of the economy at date t. For example, s^t can be a list of the current and past values of all relevant shocks. Then, the the consumer chooses a *consumption plan* $\{c_1(s^1),\ c_2(s^2),\ \ldots\}$ (the list extends over all dates and states) to maximize (2.5). The plan specifies how much to consume at each date t in each possible state s^t at that date.[4]

When finitely many states s_t can happen each period and the problem has a finite horizon T, the consumption plan has a finite number of elements (one for each date-state combination), and the budget constraint specifies a terminal condition for each of a finite number of states at the last date, so the consumer's dynamic, stochastic problem (2.5) - (2.6) is isomorphic to the static analysis of (2.1) - (2.4). Denote the optimal consumption plan

$$(2.8) \qquad \underline{c}^* = \{c_1^*(s^1),\ c_2^*(s^2),\ \ldots\ c_T^*(s^T)\}$$

where the list extends over all dates and states. Equations (2.3) - (2.5) apply directly, so (1) *deviations to an alternate plan*

$$(2.9) \qquad \underline{c}^+ = \{c_1^+(s^1),\ c_2^+(s^2),\ \ldots\ c_T^+(s^T)\}$$

have only second order effects on expected utility, and (2) *there is always an alternate plan for which the ratio of losses to the deviation is as small as one wishes*, where "small" is defined with the Euclidean norm.

To make the same statements in an infinite-period or continuous-state context, in which the consumption plan has an infinite number of elements, consider a deviation that satisfies the budget constraint. Let $\{c_t^*\}$ and $\{c_t^+\}$ denote the optimal and alternative plans, where $\{c_t^+\}$ satisfies the budget constraints in (2.6) - (2.7). Define the difference between the two plans $\Delta c_t = c_t^+ - c_t^*$, so the budget constraint (2.7) implies

$$(2.10) \qquad \sum_{t=0}^{\infty} \left(\prod_{\ell=0}^{t} R_\ell \right)^{-1} \Delta c_t = 0. \quad \text{a. s..}$$

Now consider suboptimal rules of the form $c_t^* + \alpha\Delta c_t$. If c_t^* is an optimum and u is differentiable, we must have

$$(2.11) \qquad \frac{d}{d\alpha} \left[E \sum_{t=0}^{\infty} \beta^t u(c_t^* + \alpha\Delta c_t) \right] \Bigg|_{\alpha=0} = E \sum_{t=0}^{\infty} \beta^t u'(c_t^*)\Delta c_t = 0 \quad .$$

The familiar statement of the Euler equation follows from particular choices for Δc_t. For example, $\Delta c_t = 0$, except Δc_t 1 at t in state s^t, and $\Delta c_{t+1} = R_t$ at time t+1 in states following s^t yields

$$(2.12) \qquad u'(c_t^*) = \beta\, E \left[R_t\, u'(c_{t+1}^*) \mid s^t \right] \quad .$$

(2.11) implies directly that (1) *first order deviations* $c^* + \alpha\Delta c_t$ *that respect the budget constraint have second order consequences*. Alternately, (2) *there are suboptimal consumption plans* $c_t^* + \alpha\Delta c_t$ *for which the ratio of losses to deviations is as small as one wishes*. Formally, the definition of a derivative in (2.11) states that for any $\epsilon > 0$ there is an $\alpha > 0$ such that the ratio of losses to the size of the deviation, measured by α, is smaller than the chosen ϵ,

$$\text{(2.13)} \qquad \frac{U(\{c_t^*\}) - U(\{c_t^* + \alpha\Delta c_t\})}{|\alpha|} < \epsilon .$$

The only real difference between this statement and the corresponding one for the finite date and state case is that the size of deviations is measured by α, instead of by the Euclidean norm of (2.4).

This formulation differs slightly from that in Akerlof and Yellen (1987). They consider a static maximizer whose objective was the one period maximization $f(x_t, a_t)$. They describe uncertainty by the evolution of a_t over time, and their central result is that "inertial behavior"-not changing x_t in response to a change in a_t-has second order effects. Here I consider an intertemporal maximizer, and the central proposition is that plans $\{x_t^+\}$ near $\{x_t^*\}$ have second order costs, which follows directly from the first order conditions.

3. Calculations of Utility Losses

"Second order" does not necessarily mean small: $100\epsilon^2$ is larger than $.01\epsilon$ for a range of ϵ. This section computes the actual utility costs of some economically interesting alternatives.

Utility costs depend on the consumer's environment (how much income he has, how variable that income is, and how rates of return vary over time), on the consumer's preferences (how he values deviations), and on the alternative decision rules we consider. The environments, preferences and alternatives in the empirical literature that tests the theory of the intertemporal allocation of consumption using aggregate data are similar, so there is some hope that the calculations in typical environments below are reasonable approximations to the utility loss of a wide variety of similar tests.

Many studies only test for misallocation of nondurable consumption (\$2,308 1982 dollars per capita in 1947, \$3,484 in 1985), but they use broad definitions of income, up to and including GNP (\$7,330 1982 dollars per capita in 1947, \$14,823 in 1985). If we specify the time series process for income and ask the consumer for the optimal level of consumption in a model like (2.5) - (2.6) we get a *total* consumption series, which averages about the same value as the income series. To produce a consumption series whose level is comparable to that of nondurable consumption, the calculations assume an income process whose average value is \$3,000 per year, and whose time series properties are the same as GNP (we can interpret this as a constant fraction of GNP devoted to nondurable consumption). Utility costs scale fairly well with income, so the costs in tests that use broader consumption aggregates are easy to extrapolate from calculations that use \$3000 per year.

Most tests specify either a quadratic or constant relative risk aversion utility function, and either specify or estimate a risk aversion coefficient between 1 and 10, and occasionally as high as 30. The calculations in this paper use those utility functions. Other forms for the utility function could raise (or lower) the costs of deviations.[5]

The alternatives in each case are motivated by the alternatives that typical tests have found in each environment. Sections 3.1-3.4 study economically interesting alternatives, including excess sensitivity and smoothness in the face of income shocks and slow reactions to changes in interest rates. Section 3.5 studies the costs of tolerating predictable Euler errors, which is typically the basis of statistical rejection.

3.1 *A Simple Upper Bound*

Consider a small increase Δc_t in consumption at date t, balanced by future reductions in consumption. By taking Δc_t as the standard deviation of aggregate consumption, we will produce a cost per quarter of "mistakes" that would swamp the variation in aggregate consumption, and hence void any predictions the theory can make. This calculation can also be interpreted as an upper bound for the costs of following "reasonable" alternate rules, since alternatives cannot deviate from the optimum by much more than one standard deviation if they hope to be a plausible description of the data.

By the first order conditions for optimization, this perturbation has no first order effects. Its second order effects must be greater than the second order effects of changing c_t alone, which are

(3.1.1) $\Delta U \cong 1/2\ u''(c_t)\ (\Delta c_t)^2.$

Converting to dollars by dividing by the marginal utility of consumption,

$$(3.1.2) \quad \text{dollar loss} = \frac{\Delta U}{u'(c_t)} \cong \frac{1}{2} \frac{c_t\, u''(c_t)}{u'(c_t)} \frac{\Delta c_t}{c_t} \Delta c_t = \frac{1}{2} \gamma \frac{\Delta c_t}{c_t} \Delta c_t$$

where γ is the relative risk aversion coefficient.

Equation (3.1.2) is a lower bound for the effects of the perturbation, because it ignores the second order effects of the future changes in consumption needed to restore the budget constraint. We can derive upper bounds for the total effect of the perturbation by considering specific patterns of future consumption change. For example, if the consumer reestablishes the budget constraint at t+1 by $\Delta c_{t+1} = -R_t \Delta c_t$, the dollar value of the change in utility due to the change at t+1 is

$$(3.1.3) \quad \frac{\Delta U}{u'(c_t)} \cong \frac{1}{2} \frac{\beta\, u''(c_{t+1})\, (\Delta c_{t+1})^2}{u'(c_t)} \cong \frac{1}{2} \gamma \frac{\Delta c_t}{c_t} \Delta c_t$$

where the last approximation is for $R_t \cong 1/\beta$ and near 1, and $c_t \cong c_{t+1}$. The change in utility from the total perturbation is less than the sum of the second order effects due to the change at time t, (3.1.2), and the change at time t+1, (3.1.3):[6]

$$(3.1.4) \quad \frac{1}{2} \gamma \frac{\Delta c_t}{c_t} \Delta c_t \le \frac{\Delta U}{u'(c_t)} \lessapprox \gamma \frac{\Delta c_t}{c_t} \Delta c_t .$$

This equation captures much of the intuition of the calculations that follow: even if "mistakes" Δc are as large as the standard deviation of consumption, that standard deviation is on the order of \$10 per capita and $\Delta c/c$ is about 1%, so utility costs are less than 10¢ with risk aversion $\gamma = 1$ and less than \$1 with $\gamma = 10$.

Table 1 presents some evaluations of equation (3.1.4). There is a body of evidence that nondurable consumption is essentially a random walk (see Campbell and Deaton (1987) or Cochrane and Sbordone (1988)), so table 1 takes $\Delta c/c$ as the standard deviation of quarterly growth rates of nondurable per capita consumption, and c as its level in 1947 and 1985. The utility losses range from 4¢ to \$1.94 per quarter for values of the risk aversion coefficient γ between 1 and 30.

The essence of these calculations can also be found (in a completely different context) in Lucas (1987). Lucas calculated that the utility gain available from eliminating "cycles" in consumption was small compared to increases in the "trend", which implies that the utility costs of "misbehaving" over the cycle are similarly small.

3.2. *"Excess Sensitivity" and "Excess Smoothness" Tests of the Permanent Income Hypothesis*

Following Flavin (1981), consider an environment designed to represent detrended time series. Labor income is treated as an endowment, and is given exogenously by

$$(3.2.1) \qquad y_t = (1-\rho)\bar{y} + \rho y_{t-1} + \epsilon_t \qquad \epsilon_t \text{ i.i.d, } E(\epsilon_t) = 0, \ \text{var}(\epsilon_t) = \sigma_\epsilon^2$$

The consumer maximizes a quadratic utility function

$$(3.2.2) \qquad U = -1/2 \ E \sum_{t=0}^{\infty} \beta^t \ (c_t - \bar{c})^2 \ .$$

He can borrow and lend freely at a constant interest rate $R = (1+r)$ equal to the discount rate, $\beta = 1/(1+r)$, so the period to period budget constraint is

$$(3.2.3) \qquad k_{t+1} = (1+r) \ k_t + y_t - c_t; \qquad \lim_{t\to\infty} \beta^t k_{t+1} = 0 \text{ a.s.}$$

where k_t is accumulated capital or nonhuman wealth. The consumer's optimal decision rule is[7]

$$(3.2.4) \qquad c_t = rk_t + r\beta \sum_{j=0}^{\infty} \beta^j \ E_t(y_{t+j})$$

For the AR(1) income process (3.2.1), this decision rule becomes

$$(3.2.5) \qquad c_t = rk_t + \bar{y} + m^* \ (y_t - \bar{y}) \ ; \qquad m^* = \frac{r}{1+r-\rho} \ .$$

In summary, we can characterize the evolution of optimal consumption over time by the system

$$(3.2.6) \qquad y_t = (1-\rho)\bar{y} + \rho y_{t-1} + \epsilon_t$$

$$c_t^* = rk_t^* + \bar{y} + m^*(y_t - \bar{y})$$

$$k_{t+1}^* = (1+r) \ k_t^* + y_t - c_t^* = k_t^* + (1-m^*) \ (y_t - \bar{y}) \ .$$

(The asterisks on consumption and capital stock distinguish them from suboptimal versions that follow.)

Flavin and following authors aimed their tests at the alternative hypothesis that consumption is too sensitive to current income y_t. We can generate "excessively sensitive" consumption with decision rules with higher than optimum marginal propensities to consume

$$(3.2.7)\qquad \begin{aligned} y_t &= (1-\rho)\bar{y} + \rho y_{t-1} + \epsilon_t \\ c_t^+ &= rk_t^+ + \bar{y} + m^+(y_t - \bar{y}) \\ k_{t+1}^+ &= (1+r)k_t^+ + y_t - c_t^+ \quad = \quad k_t^+ + (1-m^+)(y_t - \bar{y}) \end{aligned}$$

where $m^+ \neq m^*$.

These alternate decision rules respect the budget constraints. By iterating the capital accumulation rule in (3.2.7), capital accumulation follows

$$(3.2.8)\qquad k_{t+n}^+ = k_t^+ + (1-m^+)\sum_{j=0}^{n-1}(y_{t+j} - \bar{y}).$$

From (3.2.8) and the assumption that the present value of income is finite, it follows that $\lim_{n\to\infty} \beta^n k_{n+1}^+ = 0$ a.s.[8]

In this model it is possible to calculate the level of expected utility the consumer achieves by following any decision rule of the form (3.2.7). The calculation is presented in the appendix. The result is that the loss of time 0 expected utility (ΔU) suffered by a consumer who follows marginal propensity m^+ instead of the optimal m^* is

$$(3.2.9)\qquad \Delta U = \frac{(1+r)^2\ \sigma_\epsilon^2}{2\ r\ (1+r-\rho^2)}\,(m^+ - m^*)^2\ .$$

To convert this time 0 utility loss to dollars per quarter (the perpetuity of x dollars each quarter that would compensate the suboptimizing consumer), divide the utility loss by the marginal utility of a dollar at time 0, $u'(\bar{y}) = (\bar{c} - \bar{y})$, and multiply by r, since an additional dollar of income at every date is worth 1/r dollars of extra capital stock at time 0:

$$\text{(3.2.10)} \qquad \text{dollar/quarter loss} = \frac{r\,\Delta U}{u'(\bar{y})} = \frac{(1+r)^2\sigma_\epsilon^2}{2\,(1+r-\rho^2)}\,\frac{(m^+ - m^*)^2}{(\bar{c} - \bar{y})}\,.$$

Another measure of the utility loss is the dollar value of the time 0 utility loss as a fraction of the present value of the consumer's income stream, which is his total wealth when he has no initial capital stock. This present value is

$$\text{(3.2.11)} \qquad pv = E_0\,\beta\left[\textstyle\sum_{j=0}^{\infty}\beta^j\,y_{t+j}\right] = \bar{y}\,/\,r\,.$$

Hence,

$$\text{(3.2.12)} \quad \text{time 0 dollar loss / pv} = \frac{\Delta U}{u'(\bar{y})\cdot pv} = \frac{(1+r)^2\sigma_\epsilon^2}{2\,(1+r-\rho^2)}\,\frac{(m^+ - m^*)^2}{\bar{y}\,(\bar{c} - \bar{y})}\,.$$

The flow loss (3.2.10) divided by the expected value of the income flow $\bar{y}$ is equal to the ratio of total loss to wealth (3.2.12). The present value of income is \$60,750 in the calculations that follow.

Table 2 presents some evaluations of utility losses, (3.2.10) and (3.2.12). I used the following parameters, designed to evaluate a test using aggregate nondurable consumption data: 1) the real interest rate is 5% per year; 2) ρ=.95 from an OLS autoregression of detrended quarterly per capita real GNP; 3) $\bar{y}$ = \$3000/year, conformable to the level of nondurable consumption, as explained above; 4) σ_ϵ = \$120/4 x \$3000/\$14000 = \$6.43. σ_ϵ from the GNP autoregression was \$120. I divided this by 4 quarters/year so the units are quarterly consumption, and multiplied by nondurable consumption/GNP so the units are comparable to nondurable consumption. 5) $k_0 = 0$. Other k_0 simply increase both the optimal and alternative consumption by rk_0 in each period.

Table 2 presents utility costs for several values of the bliss point $\bar{c}$: \$937.50, \$1125, and \$1500, or 5/4, 3/2, and 2 times initial income and initial consumption of \$750. The choice of bliss point has no effect on the utility loss (3.2.9) because the utility function is quadratic, but it affects the dollar value of that loss by changing the marginal utility of a dollar.

The bliss point has not been a focus of empirical work as has the coefficient of risk aversion, so it is less clear what range of values is reasonable. Many studies do not report their estimated bliss point when it is identifiable, and the implied bliss points of many studies are negative or less than consumption (see Lewbel (1987)). Since quadratic utility is usually justified as a local approximation to a more reasonable utility function, we can assess how reasonable a bliss point is by calculating the local coefficient of relative risk aversion. This is

$$(3.2.13) \qquad \gamma(c,\bar{c}) = \frac{-cu''(c)}{u'(c)} = \frac{c}{(\bar{c}-c)} = \frac{1}{\bar{c}/c - 1},$$

so it is controlled by the ratio of the bliss point to consumption. This formula is also the coefficient of risk aversion to time 0 gambles, defined as $(k_o + \bar{y}/r)\ V''(k)/V'(k)$. This can be verified from the formula for the value function V(k) in the appendix. Table 2 includes a calculation of this quantity for each choice of bliss point. Table 2 stops at a bliss point of 1.25 times initial consumption and initial income, corresponding to a relative risk aversion coefficient of 4 at initial consumption. In simulations of the model with lower bliss points (say, 1.1 times initial consumption for $\gamma = 10$), consumption typically exceeded the bliss point within a few periods, suggesting that the linear quadratic model approximation is not useful in this range, because its results will depend too heavily on past bliss point behavior.

The costs in table 2 are less than 65¢ per quarter, or .09% of time zero wealth, and are mostly on the order of 1-10¢ per quarter or .01% of time 0 wealth. Figure 1 provides some intuition for the small size of the costs by contrasting a simulation of too sensitive consumption ($m^+ = 1$) with the optimal consumption path ($m^* = .2$). I included the origin of the vertical axis to emphasize that even with this extreme overreaction to current income, the *level* of consumption is not that affected. Since the consumer values deviations of the *level* of consumption from its optimal path, high frequency deviations cost very little.

For comparison, Flavin's point estimate of the excess marginal propensity was .355, so the corresponding costs are about those of the m =

$m^{*}+.355 \cong .6$ row of Table 2, or between 2¢ and 9¢ per quarter and less than .01% of time 0 income.[9]

Mankiw and Shapiro (1985), Campbell and Deaton (1987) and West (1988) criticized Flavin and her followers for using detrended data rather than assuming a process for income with a unit root. In the simplest case income follows a pure random walk,

$$(3.2.14) \qquad y_t = y_{t-1} + \epsilon_t ,$$

in which case the optimal consumption and capital stock evolve according to

$$(3.2.15) \qquad c^{*}_{t} = rk^{*}_{t} + y_t .$$
$$k^{*}_{t+1} = (1+r)k^{*}_{t} + y_t - c^{*}_{t} = k^{*}_{t}$$

Campbell and Deaton and West test models of this type and find that aggregate consumption is "too smooth."

We could capture "excess smoothness" by the same kind of alternate decision rules as in equation (3.2.7), with alternate marginal propensities $m^{+} < 1$. However, this choice produces an alternative decision rule with several undesirable properties when income follows a random walk. When income y_t follows a stationary process, y_t stays near its unconditional mean $\bar{y}$, so variation in m in the decision rule $c_t = rk_t + \bar{y} + m(y_t - \bar{y})$ has a bounded effect on consumption. When y_t is a random walk, however, $y_t - \bar{y}$ gets unboundedly large, so varying m has a big effect on consumption. Furthermore, since the spectral density of $(y_t - \bar{y})$ is concentrated at low frequencies, the excess smoothness that these decision rules capture is not the economically interesting high frequency or period to period failure to adjust, but a low frequency failure to adjust.

A way to capture excess smoothness that avoids these problems is to let the consumer respond to a long moving average of past income rather than to today's income alone:

$$(3.2.16) \qquad c_t^+ = rk_t^+ + \frac{1}{N+1} \sum_{j=0}^{N} y_{t-j} \quad .$$

Table 3 presents the utility loss from following this "too smooth" decision rule. Even when the consumer smooths the last ten years of income to determine current consumption, the utility loss is less than $1.28 per quarter. An explicit formula for the utility losses in this case is algebraically complicated. The calculation of utility losses is detailed in the appendix.

For comparison, Campbell and Deaton (table 6) report point estimates for the ratio of the actual to predicted innovation variance of $(\Delta c_t/y_{t-1})$ between .456 with a standard error of .20 and .747 with a standard error of .16, depending on which consumption variable they use and the number of included lags. Under the long moving average alternative, the innovation in Δc_t is $\Delta y_t/(N+1)$, so the inverse of the square root of Campbell and Deaton's ratios, between $1/\sqrt{.456} = 1.48$ and $1/\sqrt{.747} = 1.16$, is roughly comparable to (N+1). Hence, their finding of excess smoothness corresponds to a less than one period moving average of income, and carries utility costs of .7¢ to 2.7¢ per quarter.[10]

3.3 Euler equation tests and sensitivity to interest rate changes.

The second major category of tests of the intertemporal allocation of consumption are the Euler equation tests, following Hall (1979) and Hansen and Singleton (1983). The first-order conditions or Euler equations for maximization of the consumer's problem given in equations (2.5)-(2.6) are

$$(3.3.1) \qquad u'(c_t) = \beta\, E_t \left[R_t\, u'(c_{t+1}) \right].$$

Hence, if we define δ_{t+1} by

$$(3.3.2) \qquad \log u'(c_t) = \log \beta\, R_t + \log u'(c_{t+1}) + \delta_{t+1},$$

then $E(\delta_{t+1} | \text{time } t \text{ information}) = 0$, which is the basis of tests.

Euler equation tests are often used to test optimal responses to fluctuations in the conditional distribution of asset returns rather than optimal adjustment to income changes (in part because the models usually can't be solved for optimal adjustments to income). Hence, I examine the alternative to (3.3.1) that consumers fail to take optimal account of fluctuations in (real) rates of return.

To create a time-varying returns series, I generated quarterly real interest rates by an AR(1),

$$(3.3.3) \qquad R_t = \rho R_{t-1} + (1-\rho)\bar{R} + \epsilon_t.$$

I picked the mean interest rate $\bar{R} = 1 + .05/4$ and its standard deviation $\sigma_R = .05/4$ to give a generous variation over time in interest rates. This variance is roughly the variance in ex-ante returns that Poterba and Summers (1987) and Cochrane (1988b) argue is necessary to explain long horizon stock market data; it is also about the same as the variance of ex-post real interest rates. A lower variance of interest rates will give rise to less variance in both optimal and alternate consumption paths, and so lower utility costs.

I assume that consumers perfectly foresee the path of interest rates. This makes the calculations simpler; by making only part of the variation predictable we would again get less variance in optimal and alternate consumption and lower costs. Then, the optimal consumption path satisfies the Euler equation

$$(3.3.4) \qquad u'(c_t^*) = \beta R_t u'(c_{t+1}^*) .$$

With constant relative risk aversion utility $u = (c^{1-\gamma}-1)/(1-\gamma)$, the Euler equation is

$$(3.3.5) \qquad c_{t+1}^* / c_t^* = (\beta R_t)^{1/\gamma} .$$

For an alternate decision rule, suppose consumers react slowly to interest rate changes, by setting consumption growth proportional to a moving average of past interest rates. Define the alternative consumption rule c^+ by:

$$(3.3.6) \qquad c_{t+1}^+ / c_t^+ = (\beta R_t^{ma})^{1/\gamma}$$

where

$$(3.3.7) \qquad R_t^{ma} = \frac{1}{(N+1)} \sum_{j=0}^{N} R_{t-j} \quad .$$

Table 4 presents evaluations of the cost of following this alternative for various parameter values. I performed the calculations as follows: 1) I generated an interest rate path for 200 quarters using equation (3.3.3) and took a (1+N)-period moving average of the interest rate, as in equation (3.3.7); 2) starting with $c_0^* = c_0^+ = \$750$/quarter, I generated optimal and alternative consumption paths by (3.3.5) and (3.3.6); 3) I multiplied the alternative path by a constant, so that the present value of the optimal and alternate paths is the same; 4) I evaluated the achieved utility of the optimal and alternate consumption series by

$$(3.3.8) \qquad U^* = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{*(1-\gamma)} - 1}{1-\gamma} , \quad U^+ = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{+(1-\gamma)} - 1}{1-\gamma} \quad ;$$

5) I converted the utility losses to a dollar quarterly flow by dividing the utility loss, $\Delta U = U^* - U^+$, by the marginal utility of a time 0 dollar, $u'(c_0^*)$, and by the present value of a constant one dollar flow, $1 + \sum_{t=1}^{\infty} \prod_{j=0}^{t-1} R_j^{-1}$. To maintain comparability, I used the same interest rate path for each value of the parameters. The parameters are $\bar{R} = 1.012$ per quarter (corresponding to 1.05 per year), $\sigma_R = .012$ (.05 annual) , $c_0 = \$750$ per quarter (\$3000/year), $\beta = 1/\bar{R}$, and $\rho = .841$ (.5 annual).[11]

The costs in table 4 rise the longer the moving average of interest rates used to define c^+, and the costs are higher for more persistent interest rate movements. Both allow the alternate path to drift further away from the optimal path. Raising the coefficient of risk aversion γ *lowers* the costs of deviating from the optimal path. This occurs because less risk averse consumers adjust their consumption by greater amounts in response to given interest rate changes. Perfectly risk neutral consumers would set consumption to $+\infty$ every time $R_t < 1/\beta$ and vice versa. The greater difference between optimal and alternative consumption paths for less risk averse consumers more than offsets the lesser value placed on these differences.

3.4 Costs of ignoring information

In most studies, the strongest statistical evidence against the theory comes from predictability of Euler equation errors, rather than from a statistical rejection of the optimal decision rule in favor of a well-specified alternative as above. Evidence that $E(\delta_{t+1}|X_t)$ is not zero, where X_t is any variable observed at time t, is the basis for rejection of the model.

But consumers may rightly ignore information variables if the utility gained by using them to better adjust consumption does not outweigh the costs of obtaining and processing the information. If this is so,evidence of forecastability of Euler errors is not evidence against the basic theory of intertemporal optimization, and the variable X loses its status as an instrument. This section presents calculations of the utility costs of tolerating such predictable Euler errors.

Start with the upper bound derived in section 3.1, that the utility costs resulting from a perturbation Δc_t are

$$\text{(4.1.1)} \qquad \text{dollar loss} = \frac{\Delta U}{u'(c_t)} \lesssim \gamma \frac{\Delta c_t}{c_t} \Delta c_t .$$

Now, suppose that the Euler error, $\delta_{t+1} = \log(c_{t+1}/c_t)-\log(\beta R_t)$ in the CRRA case and $\delta_{t+1} = c_{t+1}-c_t$ in the linear-quadratic case, is predictable using a variable or vector of variables X_t. We can approximate the utility costs-how much utility the consumer loses by not readjusting consumption in response to the information variables X_t-by considering a perturbation from the optimum, Δc_t in (4.1.1), equal to the standard error of the predictable change in consumption.

In what follows, I'll consider the case of constant interest rates, so that the standard deviation of consumption changes is equal to the standard deviations of the Euler error δ_{t+1}. (The standard deviation of forecastable returns is typically about the same or less than that of consumption, so this approximation is not misleading.)

Table 5 presents some evaluations of (4.1.1) for different values of the predictability of consumption changes or growth rates, where an R^2 of 1.00 correreposnds to the standard deviation of consumption changes (\$6.43) from table 1. The top four rows of table 5give four different and equivalent measures of the assumed predictability of returns for their column. The top row gives the ratio of the standard deviation of predictable consumption growth or change to total consumption growth or change. The next row gives the corresponding R^2 (the square of the top row). This is the R^2 of a regression of consumption growth or change on the information variable X_{t-1}. The third row gives the standard error of the predictable component in growth rate units, and the fourth row in changes or dollar units. The table entries are calculated by (4.1.1), with Δc_t = the standard error of predictable change in dollars (fourth row) (or $\Delta c_t/c_t$ = the standard deviation of predictable growth, third row), c_t = \$3000/4 and γ as given in the first column. The entries are thus dollars per quarter utility losses from tolerating the given predictability of consumption changes or growth rates. Comparing to Table 1, the perturbations Δc_t here are simply fractions of the perturbations Δc_t in table 1. Since utility losses are proportional to $(\Delta c_t)^2$, they are linear in the assumed R^2 of a regression, and are equal to the losses of table 1 at an R^2 of 1.00.

Typical values for R^2 of regressions that predict consumption growth or changes are below .1. I know of no study that claims an R^2 above .2. The column of table 5 with R^2 = .25 shows that tolerating this *overall* predictability carries utility costs less than 1¢ to 14¢ per quarter for risk aversion $\gamma \leq 10$, and 40¢ per quarter for the extreme of $\gamma = 30$. The predictability of consumption due to an *individual* variable is typically smaller; if consumers ignore that variable and hence invalidate its use as an instrument, their utility costs are determined by the R^2 of that varaible alone, and hence even lower than the 1¢ - 14¢ range.

4. Concluding Remarks

The calculations presented above suggest that in the majority of current tests of the intertemporal allocation of consumption on aggregate data, economically and statistically significant departures from the optimal decision rule have small utility costs, less than $1 per quarter or 30¢ per month. This suggests that the theory of the intertemporal allocation of consumption, applied to a representative consumer with certain typical preferences and used to explain aggregate phenomena in a period of mild consumption volatility such as the postwar U.S., does not generate predictions of behavior that are robust to small misspecifications by economists or small "mistakes" by consumers, in the sense that both economically and statistically extreme alternatives (for example, consumption proportional to income, or consumption growth that is predictable with an R^2 of .25) carry trivial utility costs.

In particular, the utility costs of deviations from an optimal path depend on the absolute deviation of the alternate path from the optimal path. Hence, high frequency deviations like lagged responses, temporary misuse of information, failure to adjust consumption immediately in response to the information content of typical observable macro variables, and so forth, have especially low utility costs. But it is the exact timing of the use of information and the exact timing of consumption changes that has been the focus of recent empirical work.

These observations are both good and bad news for macroeconomic applications of the theory. On one hand, they imply that the alternative behavior that typical tests search for and alternative behavior that can cause the tests to reject can be generated by small ($1 per quarter) costs of information acquisition or processing, transactions, etc., so finding those alternatives is not strong evidence against the basic theory that consumers intertemporally optimize. On the other hand, it implies that the theory as it stands provides few predictions about the relationship between aggregate consumption and asset price or aggregate quantity fluctuations that are robust to $1 "mistakes" or misspecifications.

These results are not a criticism of dynamic economic theory or its empirical application in general. Dynamic optimization by firms may be exempt because of firms' larger size and different structure. Studies of consumption in microeconomic data sets, in which income and investment opportunities show orders of magnitude greater variation over time and across individuals than in aggregate data, may well escape the criticism of this paper. Large utility costs could appear in studies that use aggregate data, if they include nonstandard utility functions with at least two orders of magnitude greater risk aversion, other frequencies (life-cycle allocation instead of cyclical allocation or period to period orthogonality), or data sets from other times or countries with orders of magnitude greater varaibility in consumption. If a theory departs from the representative consumer setup of most current empirical work to a disaggregated framework in which the cyclical variation in consumption is due to only a few people, the costs of misbehavior to those people may be high.

Nonetheless, the calculations and the existence of alternatives with arbitrarily small ratios of costs to deviations presented in this paper suggest that calculations such as these are a worthwhile robustness check in these other environments as well.

Table 1

Upper bound for utility loss from a perturbation Δc

Assumed c and Δc		Risk aversion coefficient γ			
$\Delta c/c$	Δc	1	5	10	30
.86%	$5.00	4.3¢	22¢	43¢	$1.29
.86%	$7.50	6.5¢	32¢	65¢	$1.94

Losses are computed as $\gamma\ \Delta c/c\ \Delta c$ (see equation (3.1.4)). The assumed values for c and Δc are motivated by the following (cnd = real nondurable consumption per capita):

.86% = standard deviation of quarterly percent growth of cnd

$5 = .86% x cnd per quarter in 1947 ($577)

$6.05 = standard deviation of $cnd_t - cnd_{t-1}$

$6.43 = .86% x $3000 per year / 4

$7.50 = .86% x cnd per quarter in 1986 ($871)

Table 2

Utility Loss from Excess sensitivity

	Bliss point and (risk aversion coefficient)					
	$937.50 (4)		$1125 (2)		$1500 (1)	
	$/q	$/pv	$/q	$/pv	$/q	$/pv
$m^+ = 0.0$	4.1¢	0.01%	2.1¢	0.00%	1.0¢	0.00%
$m^+ = 0.2$	0	0	0	0	0	0
$m^+ = 0.4$	4.1¢	0.01%	2.1¢	0.00%	1.0¢	0.00%
$m^+ = 0.6$	16.4¢	0.02%	8.2¢	0.01%	4.1¢	0.01%
$m^+ = 0.8$	37.0¢	0.05%	18.5¢	0.02%	9.2¢	0.01%
$m^+ = 1.0$	65.7¢	0.09%	32.9¢	0.04%	16.4¢	0.02%

The column marked "$/q" gives the dollar per quarter utility cost of following the indicated marginal propensity to consume, or $c_t^+ = rk_t + \bar{y} + m^+(y_t - \bar{y})$, calculated by equation (3.2.10). The column marked "$/pv" gives the time 0 dollar utility cost as a percent of the time 0 present value of income, equation (3.2.12). The local risk aversion coefficient corresponding to each bliss point is $1/(\bar{c}/c-1)$. The parameters are interest rate $r = .012$ (5 percent per year), AR(1) coefficient on income $\rho = .95$, standard error of income AR(1) $\sigma_{\epsilon} = \$6.43$, Initial = mean income $y_0 = \bar{y} = \$3000/4$. The optimal mpc is $m^* = \frac{r}{1+r-\rho} = .2$

Table 3

Utility Loss from "too smooth" consumption when income is a random walk

	Bliss point and (risk aversion coefficient)					
	$937.50 (4)		$1125 (2)		$1500 (1)	
	$/q	$/pv	$/q	$/pv	$/q	$/pv
N+1 = 2	2.8¢.	.004%	1.4¢	.002%	0.7¢	.001%
N+1 = 4	9.7¢	.013%	4.9¢	.006%	2.4¢	.003%
N+1 = 8	24.0¢	.032%	12.0¢	.016%	6.0¢	.008%
N+1 = 20	65.4¢	.087%	32.7¢	.044%	16.3¢	.022%
N+1 = 40	$1.28	.170%	64.0¢	.085%	32.0¢	.043%

The entries in the table are the dollar per quarter utility loss and the total utility loss / time 0 value of income. Income y_t is a random walk; the optimal marginal propensity is 1, and the alternative is that consumption responds to a long moving average of past income

$$c_t^+ = rk_t^+ + \frac{1}{(N+1)} \sum_{j=0}^{N} y_{t-j} \quad .$$

The calculations are detailed in the Appendix.

Table 4

Dollar loss / quarter from smoothing interest rates

Risk aversion γ and autocorrelation ρ	Moving average of past interest rates 1 year	5 years	10 years
$\gamma = 2$ $\rho = .841$	5¢	76¢	$1.45
$\gamma = 5$ $\rho = .841$	2¢	30¢	56¢
$\gamma = 10$ $\rho = .841$	1¢	15¢	28¢
$\gamma = 5$, $\rho = 0$	0	8¢	26¢

The entries are the dollar cost per quarter of using a moving average of past interest rates in the place of the one period rate. To calculate the entries I 1) generated an interest rate path for 200 quarters and took an N+1 period moving average of the interest rate; 2) generated optimal and alternative consumption paths; 3) multiplied the alternative path by a constant, so that the present value of the optimal and alternate paths is the same; 4) evaluated the achieved utility of each consumption series; 5) converted the utility losses to a dollar quarterly flow. I used the same interest rate path for each entry. The parameters are $\bar{R} = 1.012$ per quarter (1.05 per year), $\sigma_R = .012$ (.05 annual), $c_0 =$ \$750/quarter (\$3000/year), $\beta = 1/(1.012)$ or 1/(1.05) annual. $\rho = .841$ (corresponds to $\rho = .5$ annual).

Table 5.

Utility costs of tolerating predictable Euler errors.

		Assumed predictability						
$\Delta c^{predictable}/\Delta c^{total}$		0.01	0.05	0.10	0.25	0.50	0.90	1.00
$R^2 = (\Delta c^{pred.}/\Delta c^{tot.})^2$		0.00	0.00	0.01	0.06	0.25	0.81	1.00
$\Delta c^{pred.}/c$ (%)		0.01	0.04	0.09	0.21	0.43	0.77	0.86
$\Delta c^{pred.}$ ($)		0.06	0.32	0.64	1.61	3.21	5.79	6.43
γ = coeff. of relative risk aversion	1	0.00¢	0.01¢	0.06¢	0.34¢	1.38¢	4.46¢	5.51¢
	2	0.00¢	0.03¢	0.11¢	0.69¢	2.76¢	8.93¢	11.0¢
	5	0.00¢	0.07¢	0.28¢	1.72¢	6.89¢	22.3¢	27.6¢
	10	0.01¢	0.14¢	0.55¢	3.44¢	13.8¢	44.6¢	55.1¢
	30	0.02¢	0.41¢	1.65¢	10.3¢	41.3¢	$1.33	$1.65

Utility costs are in dollars per quarter. The top four rows give four measures of the assumed predictability of consumption growth or changes, which equal the Euler error with constant interest rates. The table entries are calculated as $\gamma\,(\Delta c^{predictable})^2/c$.

Appendix

I. Near-Rationality near an optimum.

If the objective is twice differentiable, first order deviations have second order effects even if we don't start precisely at an optimum. The problem is

$$\max_{\{x\}} f(x) \quad .$$

Expand f about a point x^0 near x^*. Then

$$\Delta f \cong f'(x^0)\ \Delta x + \frac{1}{2} f''(x^0)\ \Delta x^2 \qquad .$$

We can expand the derivatives around the optimum x^* as well:

$$f'(x^0) \cong f'(x^*) + f''(x^*)(x^0 - x^*)$$

$$f''(x^0) \cong f''(x^*) + f'''(x^*)(x^0 - x^*)$$

so, keeping only second order terms,

$$\Delta f \cong f''(x^*)\ (\Delta x\ (x^0 - x^*) + \Delta x^2) \quad .$$

For fixed x^0, deviations Δx have first order losses, but the ratio of losses to deviations can be made arbitrarily small by choosing small enough regions for x^0 as well as small enough Δx.

$$\forall\ \epsilon > 0\ \ \exists\ \delta,\ \nu \text{ s.t. } |x^0 - x^*| < \delta \text{ and } \Delta x < \nu \Rightarrow \Delta f/\Delta x < \epsilon .$$

This point carries over to the consumer's problem.

II. Attained Expected Utility for Linear-Quadratic Problems.

A. General Problem.

The general problem can be stated as: find

(A.1) $$U(X_t) = E_t \sum_{j=0}^{\infty} \beta^j X'_{t+j} R X_{t+j}$$

where X_t evolves according to

(A.2) $$X_t = AX_{t-1} + \xi_t \ . \ E_t(\xi_{t+1}) = 0 \ ; \ E_t(\xi_t \xi'_t) = \Sigma \ .$$

X_t is a vector of state variables; the decision rule relating consumption to state variables has been substituted in to derive (A.1) and (A.2). Either substituting (A.2) in (A.1), or guessing a quadratic form and verifying it, we have

(A.3) $$U(X_t) = X'_t P X_t + 1/r \text{ Trace}(P\Sigma)$$

where

(A.4) $$P = \sum_{j=0}^{\infty} \beta^j A^{j\prime} R A^j \quad ;$$

or

(A.5) $$P = R + \beta A'PA \ .$$

(See Sargent (1987).) Different decision rules will yield different values for A, and hence different achieved utilities $U(X_t)$.

B. Utility losses from excess sensitivity

For the model in equations (3.2.1) - (3.2.3) with decision rules of the form (3.2.7), the attained level of utility is

(A.6) $$U = -1/2 \ E_t \sum_{j=0}^{\infty} \beta^j \ (c_{t+j} - \bar{c})^2 .$$

Define the vector of state variables

(A.7) $X_t = [1\ k_t\ z_t]'$

where $z_t = y_t - \bar{y}$. Then, consumption is

(A.8) $c_t - \bar{c} = rk_t + \bar{y} - \bar{c} + m^+ z_t = [(\bar{y}-\bar{c})\ r\ m^+]\ X_t = F'X_t$.

so we may write the objective (A.6) in the form (A.1) with

$$\text{(A.9)} \qquad R = -1/2\ FF' = -1/2 \begin{bmatrix} (\bar{y}-\bar{c}) & r(\bar{y}-\bar{c}) & m^+(\bar{y}-\bar{c}) \\ r(\bar{y}-\bar{c}) & r^2 & m^+ r \\ m^+(\bar{y}-\bar{c}) & m^+ r & r^2 \end{bmatrix} .$$

X_t evolves as follows: using the laws of motion for income and capital,

(A.10) $z_t = \rho z_{t-1} + \epsilon_t$

(A.11) $k_{t+1} = (1+r)\ k_t + y_t - c_t = k_t + (1-m^+) z_t$

we can write the law of motion for X_t in the form (A.2) with

$$\text{(A.12)} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & (1-m) \\ 0 & 0 & \rho \end{bmatrix} ,\ \xi_t = \begin{bmatrix} 0 \\ 0 \\ \epsilon_t \end{bmatrix} .$$

For this model, I will derive (3.2.14) as an analytic solution to (A.5). The idea of the analytic solution is the following: from (A.5), form

(A.13) $\mathrm{Vec}(P) = \mathrm{Vec}(R) + \beta\ \mathrm{Vec}(A'PA)$

where Vec(.) creates a vector by stacking the rows of a matrix. Using

(A.14) $\mathrm{Vec}(AB) = (I \otimes A)\mathrm{Vec}(B) = (B' \otimes I)\mathrm{Vec}(A)$,

we have

(A.15) $\mathrm{Vec}(P) = \mathrm{Vec}(R) + \beta\ (A' \otimes A')\mathrm{Vec}(P)$.

We can't quite collect terms in Vec P and invert because P is symmetric, so only the diagonal and one off diagonal side can be chosen independently. To remedy this problem, let M be a matrix that deletes redundant rows of Vec(P), and let N be a matrix that takes M Vec(P) and restores the redundant rows, so that Vec(P) = N (M Vec(P)). Then, from (A.15),

$$\text{(A.16)} \qquad M\,Vec(P) = M\,Vec(R) + \beta\, M\,(A' \otimes A')\,(N\,M\,VecP)$$

so,

$$\text{(A.17)} \qquad (M\,Vec\,P) = (I - \beta\, M\,(A' \otimes A')\,N\,)^{-1}\, M\,Vec(R).$$

Equation (A.17) can be used to calculate P and hence $U = X'PX + 1/r\ \mathrm{Trace}(P\Sigma)$ for a given A R and Σ.

For the consumer's problem, denote the elements of P by

$$\text{(A.18)} \qquad P = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Then, (A.17) becomes

$$\text{(A.19)} \qquad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 1/2 \begin{bmatrix} -(\bar{y}-\bar{c})^2 \\ r(\bar{c}-\bar{y}) \\ m(\bar{c}-\bar{y}) \\ -r^2 \\ -rm \\ -m^2 \end{bmatrix} + \beta \begin{bmatrix} a \\ b \\ (1-m)b + \rho c \\ d \\ (1-m)d + \rho e \\ (1-m)^2 d + 2\rho(1-m)e + \rho^2 f \end{bmatrix}.$$

Solving,

$$\text{(A.20)} \qquad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -(\bar{c}-\bar{y})^2/2(1-\beta) \\ r(\bar{c}-\bar{y})/(2(1-\beta)) \\ (\bar{c}-\bar{y})/2(1-\beta\rho) \\ -r^2/2(1-\beta) \\ -r/(1-\beta\rho)^2 \\ -1/(2(1-\beta\rho^2)(m^2+r(1-m)^2+2\rho m^*(1-m)) \end{bmatrix}.$$

And using these, we can evaluate achieved utility $X'PX + 1/r\ \mathrm{Trace}(P\Sigma)$.

Since m only enters in f in (A.20), utility losses from following a different m evaluated at $y_0 = \bar{y}$ or $z_0 = 0$ depend only on $1/r$ Trace $P\Sigma$. In turn,

(A.21) $$1/r\ \mathrm{Trace}\ P\Sigma = \sigma_{\epsilon}^{2} f/r =$$

$$= -\frac{\sigma_{\epsilon}^{2}}{2r(1-\beta\rho^{2})}\left[m^{2} + r(1-m)^{2} + 2\rho m^{*}(1-m) \right]$$

$$= -\frac{\sigma_{\epsilon}^{2}(1+r)}{2r(1-\beta\rho^{2})}\left[(m-m^{*})^{2} - m^{*2} + \frac{1+r+\rho}{1+r} m^{*} \right] .$$

Hence, the utility loss of using m rather than m^{*} is

(A.22) $$\Delta U = \frac{\sigma_{\epsilon}^{2}(1+r)}{4(1-\beta\rho^{2})} \qquad (m^{*}-m^{2}) = \frac{(1+r)^{2}\sigma_{\epsilon}^{2}}{2r(1+r-\rho^{2})} (m-m^{*})^{2} .$$

which is equation (3.2.9) in the text.

C. Utility losses from consumption equal to a long moving average of income

Income follows

(A.23) $$y_t = y_{t-1} + \epsilon_t ;$$

optimal and alternate consumption are

(A.24) $$c_t^{*} = rk_t^{*} + y_t$$
$$k_t^{*} = (1+r)\ k_{t-1}^{*} + y_{t-1} - c_{t-1}^{*} = k_{t-1}^{*}$$

$$c_t^+ = rk_t^+ + \frac{1}{N+1} \sum_{j=0}^{N} y_{t-j}$$

$$k_t^+ = (1+r)k_{t-1}^+ + y_{t-1} - c_{t-1} = k_{t-1}^+ + \frac{N}{N+1} y_{t-1} - \frac{1}{N+1} \sum_{j=1}^{N} y_{t-j} \quad .$$

Thus, we can take the state vector as

(A.25) $\quad X_t = [1 \; k_t \; (y_t-\bar{y}) \;\; (y_{t-1}-\bar{y}) \; \ldots . \; (y_{t-N}-\bar{y}) \;]' \quad ;$

the matrices A, F and Σ are

(A.26)
$$A = \begin{vmatrix} 1 & 0 & 1 & 1 & & 1 \\ 0 & 1 & N/(N+1) & -1/(N+1) & \ldots & -1/(N+1) \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \ldots & 0 \end{vmatrix}$$

(A.27)
$$F = \left[\; (\bar{y}-\bar{c}) \;\; r \;\; 1/(N+1) \;\; 1/(N+1) \;\; \ldots \;\; (1/N+1) \; \right]'$$

(A.28)
$$\Sigma = \begin{vmatrix} 0 & 0 & 0 & \ldots & \\ 0 & 0 & 0 & & \\ 0 & 0 & \sigma_\epsilon^2 & 0 & 0 \\ & \vdots & & & \end{vmatrix} \quad .$$

To calculate the entries of Table 4, I calculated P using (A.4) and then Trace $P\Sigma$. By using a doubling algorithm, the entries in Table 4 include 2^{13} elements of the sum.

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Footnotes

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[1]For example, Cochrane (1988a) presents an asset pricing model derived from firm's first order conditions.

[2]An economic theory of the contracting problems that prevent the emergence of markets in the ownership of people or other institutions that could remove the small surpluses of cyclical mis-allocation is beyond the scope of this paper. Beyond the obvious agency questions and the unobservability of utility (as compared to earnings), the fact that cyclical allocations ($7.50 more this quarter, $7.50 less the next) are so small compared to other elements of individual consumption decisions is probably part of the reason that we do not observe such institutions.

[3]Jones and Stock (1987) also consider this extension.

[4]In a more formal presentation, we specify an underlying probability space (Ω,F,P), and a nondecreasing sequence of sub σ-algebras of F, s^t, that comprise the consumer's information set. s^t can be generated by current and past observations of a collection of observed random variables x(t). Then the object that the consumer chooses is a sequence of random variables $c_t:\Omega \to R$ where each c_t is measurable with respect to s^t.

[5]Non-state-separable preferences (see Epstein and Zin (1987) or Kocherlakota (1988)) are typically adopted to plausibly use a value of the coefficient of risk aversion or intertemporal substitution around 30. Utility costs are roughly linear in the coefficient of risk aversion, so one needs risk aversion coefficients of 100 or so to get an order of magnitude increase in utility costs, or $10 per quarter. On the other hand, utility functions that include durability ("nondurable consumption" includes clothes) allow greater variation in consumption for the same utility costs.

[6]The consumer's problem can often be written as a dynamic program,

$$V(k_t, \lambda_t) = \max_{\{c_t\}} u(c_t) + \beta E_t V(R_t k_t + y_t - c_t, \lambda_{t+1}) \quad .$$

Where λ_t is a listing of shocks. Then, the second order effects are

$$\Delta U \cong \frac{1}{2} [u''(c_t^*) + \beta E_t V_{kk}(k_{t+1}^*, \lambda_{t+1})] \Delta c_t^2 \; .$$

Since $u''(c_{t+1}^*) < V_{kk}(k_{t+1}^*, \lambda_{t+1}) < 0$ the same approximate bounds follow. Since the functional form of V is often not known, the upper bound in (3.1.4) is easier to use for order-of-magnitude calculations.

[7] To derive the consumer's optimal decision rule, express the budget constraint in present value form:

$$k_t + \beta \sum_{j=0}^{\infty} \beta^j \, y_{t+j} = \beta \sum_{j=0}^{\infty} \beta^j \, c_{t+j} \quad \text{a.s.} \; .$$

The consumer's first order conditions (2.2) are $c_t = E_t (c_{t+j})$. Then, take the expected value of the budget constraint, plug in the first order conditions to obtain the decision rule:

$$k_t + \beta \sum_{j=0}^{\infty} \beta^j \, E_t(y_{t+j}) = c_t \, \beta \sum_{j=0}^{\infty} \beta^j = c_t \, / \, r \; .$$

(Hansen (1987) and Eichenbaum Christiano and Marshall (1988) derive similar decision rules in more general versions of this model.)

[8] We can also vary the coefficient on the k_t term in the consumption decision rule, to r^+ instead of r. As long as $|1+r-r^+| < 1/\beta$ the budget constraint will be satisfied.

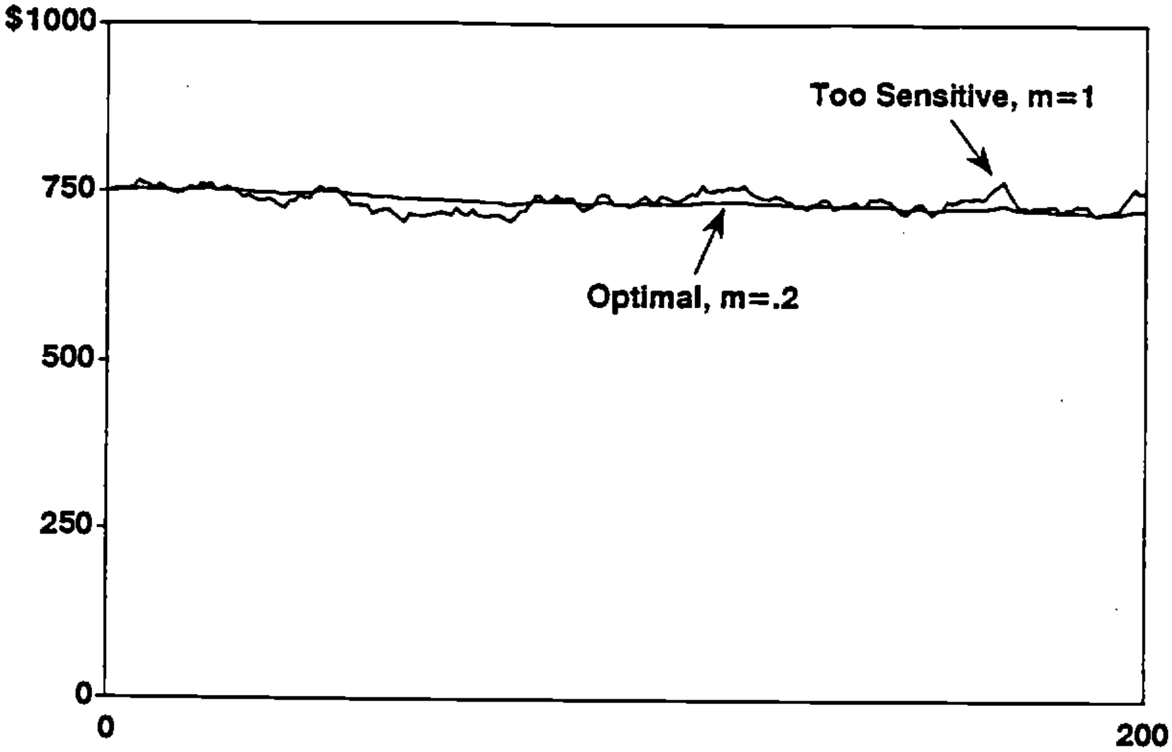
[9] Assessing whether this alternative could generate Flavin's statistical rejection is a little more subtle. Though Flavin's estimate of $m-m^*$ was nearly 2 standard errors away from 0, the weight of Flavin's statistical evidence came from combined excess sensitivity to eight lags of income rather than from contemporaneous income alone, and from the predictive power of all eight lags of income for consumption changes. However, the excess smoothness alternative considered here generates about the same predictability of consumption changes (R^2) as is found in a replication of Flavin's regression of consumption changes on eight lags of income, which is a more precise indication that this alternative can account for the statistical rejection.

[10]The weight of Campbell and Deaton's statistical evidence also came from predictability of Euler equation errors rather than rejection of the optimal innovation variance of $\Delta c/y$ in favor of these alternatives. The excess smoothness alternative generates predictable consumption changes (R^2) larger than those found by Campbell and Deaton, so it can also account for the statistical rejection.

[11]In more complex environments, for example those that include stochastic interest rates, we can find utility costs as in the linear quadratic case, by solving a Bellman-like equation

$$V(k_t, \text{shocks}_t) = u(c_t^+) + \beta E_t V(k_{t+1}, \text{shocks}_{t+1}).$$

after we specify the alternative decision rule relating c_t^+ to k_t etc.


$1000
750
500
250
0
0
200
Too Sensitive, m=1
Optimal, m=.2