

LOW INTEREST RATES, MARKET POWER, AND
PRODUCTIVITY GROWTH

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ABSTRACT

This study provides a new theoretical result that a decline in the long-term interest rate can trigger a stronger investment response by market leaders relative to market followers, thereby leading to more concentrated markets, higher profits, and lower aggregate productivity growth. This strategic effect of lower interest rates on market concentration implies that aggregate productivity growth declines as the interest rate approaches zero. The framework is relevant for anti-trust policy in a low interest rate environment, and it provides a unified explanation for rising market concentration and falling productivity growth as interest rates in the economy have fallen to extremely low levels.

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Interest rates have fallen to extreme lows across advanced economies, and they are projected to stay low. At the same time, market concentration, business profits, and markups have been rising steadily. The rise in concentration has been associated with a substantial decline in productivity growth; furthermore, the productivity gap between leaders and followers within the same industry has risen. This study investigates the effect of a decline in interest rates on investments in productivity enhancement when firms engage in dynamic strategic competition. The results suggest that these broad secular trends—declining interest rates, rising market concentration, and falling productivity growth—are closely linked.

In traditional models, lower interest rates boost the present value of future cash flows associated with higher productivity, and therefore lower interest rates encourage firms to invest in productivity enhancement. This study highlights a second strategic force that reduces aggregate investment in productivity growth at very low interest rates. When firms engage in strategic behavior, market leaders have a stronger investment response to lower interest rates relative to followers, and this stronger investment response leads to more market concentration and eventually lower productivity growth.

The model is rooted in the dynamic competition literature (e.g. [Aghion et al. \(2001\)](#)). Two firms compete in an industry both intra-temporally, through price competition, and inter-temporally, by investing in productivity-enhancing technology. Investment increases the probability that a firm improves its productivity position relative to its competitor. The decision to invest is a function of the current productivity gap between the leader and the follower, which is the state variable in the industry. A larger productivity gap gives the leader a larger share of industry profits, thereby making the industry more concentrated. The model includes a continuum of industries, all of which feature the dynamic game between a leader and follower. Investment decisions within each industry induce a steady state stationary distribution of productivity gaps across markets and hence overall industry concentration and productivity growth.

The theoretical analysis is focused on the following question: What happens to aggregate investment in productivity enhancement when the interest rate used to discount profits falls? The model's solution includes the "traditional effect" through which a decline in the interest rate leads to more investment by market leaders and market followers. However, the solution to the model also reveals a "strategic effect" through which market leaders invest more aggressively relative to market followers when interest rates fall. The central theoretical result of the analysis shows that the strategic effect dominates the traditional effect at a sufficiently low interest rate; as the interest rate approaches zero, it is guaranteed that economy-wide measures of market concentration will rise and aggregate

productivity growth will fall.

The intuition behind the strategic effect can be seen through careful consideration of the investment responses of market leaders and market followers when the interest rate falls. When the interest rate is low, the present value of a persistent market leader becomes extremely high. The attraction of becoming a persistent leader generates fierce and costly competition especially if the two firms are close to one another in the productivity space. When making optimal investment decisions, both market leaders and market followers realize that their opponent will fight hard when their distance closes. However, they respond asymmetrically to this realization when deciding how much to invest. Market leaders invest more aggressively in an attempt to ensure they avoid neck-and-neck competition. Market followers, understanding that the market leader will fight harder when they get closer, become discouraged and therefore invest less aggressively. The realization that competition will become more vicious and costly if the leader and follower become closer in the productivity space discourages the follower while encouraging the leader. The main proposition shows that this strategic effect dominates as the interest rate approaches zero.

The dominance of the strategic effect at low interest rates is a robust theoretical result. This result is shown first in a simple example that captures the basic insight, and then in a richer model that includes a large state space and hence richer strategic considerations by firms. The existence of this strategic effect and its dominance as interest rates approach zero rests on one key realistic assumption, that technological catch-up by market followers is gradual. That is, market followers cannot “leapfrog” the market leader in the productivity space and instead have to catch up one step at a time. This feature provides an incentive for market leaders to invest not only to reach for higher profits but also to endogenously accumulate a strategic advantage and consolidate their leads. This incentive is consistent with the observations that real-world market leaders may conduct defensive R&D, erect entry barriers, or engage in predatory acquisition as in [Cunningham et al. \(2019\)](#). This assumption is also supported by the fact that gradual technological advancement is the norm in most industries, especially in recent years (e.g., [Bloom et al. \(2020\)](#)).

The exploration of the supply side of the economy is embedded into a general equilibrium framework to explore whether the mechanism is able to quantitatively account for the decline in productivity growth. The general equilibrium analysis follows the literature in assuming that the long-run decline in interest rates is generated by factors on the demand side of the economy, which is modeled as a reduction in the discount rate of households. We conduct a simple calibration of the model and show that the model

generates a quantitatively meaningful rise in the profit share and decline in productivity growth following the decline in the interest rate from 1984 to 2016 in the United States.

The insights from the model have implications for anti-trust policy. Policies that tax leader profits or subsidize follower investments are less effective than one that dynamically facilitates technological advancements of followers. Furthermore, more aggressive anti-trust policy is needed during times of low interest rates. The baseline model abstracts from financial frictions by assuming that market leaders and market followers face the same interest rate. We believe that the introduction of financial frictions that generate a gap between the interest rates faced by market leaders and market followers would lead to even stronger leader dominance when interest rates fall.¹ For example, using data on interest rates and imputed debt capacity, we show that the decline in long-term rates has disproportionately favored industry leaders relative to industry followers.

The model developed here is rooted in dynamic patent race models (e.g. [Budd et al. \(1993\)](#)). These models are notoriously difficult to analyze; earlier work relies on numerical methods (e.g. [Budd et al. \(1993\)](#), [Acemoglu and Akcigit \(2012\)](#)) or imposes significant restrictions on the state space to keep the analysis tractable (e.g. [Aghion et al. \(2001\)](#) and [Aghion et al. \(2005\)](#)). We bring a new methodology to this literature by analytically solving for the recursive value functions when the discount rate is small. This new technique enables us to provide sharp, analytical characterizations of the asymptotic equilibrium as discounting tends to zero, even as the ergodic state space becomes infinitely large. The technique should be applicable to other stochastic games of strategic interactions with a large state space and low discounting.

This study also contributes to the large literature on endogenous growth.² The key difference between the model here and other studies in the literature, e.g. [Aghion et al. \(2001\)](#) and [Peters \(forthcoming\)](#), is the assumption that followers have to catch up to the leader gradually and step-by-step instead of being able to close all gaps at once. This assumption provides an incentive for market leaders to accumulate a strategic advantage, which is a key strategic decision that is relevant in the real world. We show this key “no-leapfrog” feature overturns the traditional intuition that low interest rates always promote investment, R&D, and growth; instead, when interest rates are sufficiently low, this strategic effect always dominates the traditional effect, and aggregate investment and productivity growth will fall.

¹For related work on financial constraints and productivity growth, see [Caballero et al. \(2008\)](#), [Gopinath et al. \(2017\)](#) and [Aghion et al. \(2019a\)](#).

²Recent contributions to this literature include [Acemoglu and Akcigit \(2012\)](#), [Akcigit et al. \(2015\)](#), [Akcigit and Kerr \(2018\)](#), [Cabral \(2018\)](#), [Garcia-Macia et al. \(2018\)](#), [Acemoglu et al. \(forthcoming\)](#), [Aghion et al. \(forthcoming\)](#), and [Atkeson and Burstein \(forthcoming\)](#), among others.

In contemporaneous work, [Akcigit and Ates \(2019\)](#) and [Aghion et al. \(2019b\)](#) respectively argue that a decline in technology diffusion from leaders to followers and the advancement in information and communication technology—which enables more efficient firms to expand—could have contributed to the rise in firm inequality and low growth. While we do not explicitly study these factors in the model here, the economic forces highlighted by our theory suggest that low interest rates could magnify market leaders’ incentives to take advantage of these changes in the economic environment. More broadly, the theoretical result of this study suggests that the literature exploring the various reasons behind rising market concentration and declining productivity growth should consider the role of low interest rates in contributing to these patterns.

This paper is also related to the broader discussion surrounding “secular stagnation” in the aftermath of the Great Recession. Some explanations, e.g., [Summers \(2014\)](#), focus primarily on the demand side and highlight frictions such as the zero lower bound and nominal rigidities.³ Others such as [Barro \(2016\)](#) have focused more on the supply-side, arguing that the fall in productivity growth is an important factor in explaining the slow recovery. This study suggests that these two views might be complementary. For example, the decline in long-term interest rates might initially be driven by a weakness on the demand side. But a decline in interest rates can then have a contractionary effect on the supply-side by increasing market concentration and reducing productivity growth. An additional advantage of this framework is that one does not need to rely on financial frictions, liquidity traps, nominal rigidities, or a zero lower bound to explain the persistent growth slowdown such as the one we have witnessed since the Great Recession.

1 Motivating Evidence

Existing research points to four secular trends in advanced economies that motivate the model. First, there has been a secular decline in interest rates across almost all advanced economies. [Rachel and Smith \(2015\)](#) show a decline in real interest rates across advanced economies of 450 basis points from 1985 to 2015. The nominal 10-year Treasury rate has declined further from 2.7% in January 2019 to 0.6% in July 2020. This motivates the consideration of extremely low interest rates on firm incentives to invest in productivity enhancement.

Second, measures of market concentration and market power have risen substantially over this same time frame. Rising market power can be seen in rising markups (e.g., [Hall](#)

³See e.g., [Krugman \(1998\)](#), [Eggertsson and Krugman \(2012\)](#), [Guerrieri and Lorenzoni \(2017\)](#), [Benigno and Fornaro \(forthcoming\)](#), and [Eggertsson et al. \(2019\)](#).

(2018), De Loecker et al. (2020), Autor et al. (2020)), higher profits (e.g., Barkai (forthcoming), De Loecker et al. (2020)), and higher concentration in product markets (e.g., Grullon et al. (2019), Autor et al. (2020)). Diez et al. (2019) from the International Monetary Fund put together a firm-level cross-country dataset from 2000 onward to show a series of robust facts across advanced economies. Measures of markups, profitability, and concentration have all risen. They also show that the rise in markups has been concentrated in the top 10% of firms in the overall markup distribution, which are firms that have over 80% of market share in terms of revenue.

Third, productivity growth has stalled across advanced economies (e.g., Cette et al. (2016), Byrne et al. (2016)). It is important to note that this slowdown in productivity began before the Great Recession, as shown convincingly in Cette et al. (2016). The slowdown in productivity growth has been widespread, and was not initiated by the Great Recession.

Fourth, the decline in productivity growth has been associated with a widening productivity gap between leaders and followers and reduced dynamism in who becomes a leader. Andrews et al. (2016b) show that the slowdown in global productivity growth is associated with an expanding productivity gap between “frontier” and “laggard” firms. In addition, the study shows that industries in which the productivity gap between the leader and the follower is rising the most are the same industries where sector-aggregate productivity is falling the most.

Berlingieri et al. (2017) use firm level productivity data from OECD countries to estimate productivity separately for “leaders,” defined as firms in the 90th percentile of the labor productivity distribution for a given 2-digit industry, and “followers,” defined as firms in the 10th percentile of the distribution. The study shows that the gap between leaders and followers increased steadily from 2000 to 2014. Both the Andrews et al. (2016b) and Berlingieri et al. (2017) studies point to the importance of the interaction between market leaders and market followers in understanding why productivity growth has fallen over time. Andrews et al. (2016a) show that the tendency for leaders, which they call frontier firms, to remain market leaders has increased substantially from the 2001 to 2003 period to the 2011 to 2013 period. They conclude that it has become harder for market followers to successfully replace market leaders over time.

As shown below, these facts are consistent with the model’s prediction of what happens when interest rates fall to low levels. Furthermore, even the timing of these patterns is consistent with the results of the model. As shown below, the model predicts that market power increases as the interest rate declines, but productivity growth has an inverted-U relationship and only declines when the interest rate becomes sufficiently low. In the real-world, the decline in real interest rates began in the 1980s, and measures of

market concentration began rising in the late 1990s. The trends in productivity growth, in contrast, began later. Most studies place the beginning of the period of a decline in productivity growth between 2000 and 2005, and the rising productivity gap between market leaders and followers also emerged at this time.

2 A Stylized Example

Declining interest rates in advanced economies have been associated with a rise in market power, a widening of productivity and markups between market leaders and followers, and a decline in productivity growth. This section begins the theoretical analysis of the effect of a decline in interest rates on market concentration and productivity growth. More specifically, we begin by presenting a stylized example to illustrate the key force in the model: low interest rates boost the incentive to invest for industry leaders more than for industry followers. Section 3 below presents the full model.

Consider two firms competing in an industry. Time is continuous and as in the dynamic patent race literature, there is a technological ladder such that firms that are further ahead on the ladder are more productive and earn higher profits. The distance between two firms on the technological ladder represents the state variable for an industry. To keep the analysis as simple as possible, we assume that an industry has only three states: firms can compete neck-and-neck (state=0) with flow profit $\pi_0 = 1/2$ each, they can be one step apart earning flow profits $\pi_1 = 1$ for the leader and $\pi_{-1} = 0$ for the follower, or they can be two steps apart. If firms are two steps apart, that state becomes permanent, with leader and follower earning $\pi_2 = 1$ and $\pi_{-2} = 0$ perpetually.

Firms compete by investing at the rate η in technology in order to out-run the other firm on the technological ladder. The firm pays a flow investment cost $c(\eta) = -\eta^2/2$ and advances one step ahead on the technological ladder with Poisson rate η . Starting with a technological gap of one step, if the current follower succeeds before the leader, their technological gap closes to zero. The two firms then compete neck-and-neck, both earning flow profit $1/2$ and continue to invest in order to move ahead on the technological ladder. Ultimately each firm is trying to get two steps ahead of the other firm in order to enjoy permanent profit of $\pi_2 = 1$.

Given the model structure, we can solve for equilibrium investment levels. At an interest rate r , the value of a permanent leader is $v_2 \equiv 1/r$ and the value of a permanent follower is $v_{-2} \equiv 0$. Firms that are zero or one-step apart choose investment levels to maximize their firm values, taking the other firm's investment level as given. The equilibrium

firm value functions satisfy the following HJB equations:

$$rv_1 = \max_{\eta} \pi_1 - \eta^2/2 + \eta(v_2 - v_1) + \eta_{-1}(v_0 - v_1) \quad (1)$$

$$rv_0 = \max_{\eta} \pi_0 - \eta^2/2 + \eta(v_1 - v_0) + \eta_0(v_{-1} - v_0) \quad (2)$$

$$rv_{-1} = \max_{\eta} \pi_{-1} - \eta^2/2 + \eta(v_0 - v_{-1}) + \eta_1(v_{-2} - v_{-1}) \quad (3)$$

where $\{\eta_{-1}, \eta_0, \eta_1\}$ denote the investment choices in equilibrium.

The intuition behind the HJB equations can be understood using equation (1) that relates the flow value rv_1 for a one-step-ahead leader to its three components: flow profits minus investment costs ($\pi_1 - \eta^2/2$), a gain in firm value of $(v_2 - v_1)$ with Poisson rate η if the firm successfully innovates, and a loss in firm value of $(v_0 - v_{-1})$ with Poisson rate η_{-1} if the firm's competitor successfully innovates.

Both firms compete dynamically for future profits and try to escape competition in order to enjoy high profits π_2 indefinitely. Suppose the industry is in state 1. Then the investment intensity for the leader and the follower are given by the first order conditions from HJB equations, $\eta_1 = v_2 - v_1$ and $\eta_{-1} = v_0 - v_{-1}$, respectively. Intuitively, the magnitude of investment effort depends on the slope of the value function for the leader and the follower. The follower gains value from reaching state=0 so it has a chance to become the leader in the future; the leader gains value from reaching to state=2 not because of higher flow profits (note $\pi_1 = \pi_2 = 1$) but, importantly, by turning its temporary leadership into a permanent one.

The key question is, what happens to equilibrium investment efforts in state 1 if there is a fall in interest rate r ? The answer, summarized in proposition 1, is that the leader's investment η_1 rises by more than the follower's investment η_{-1} as r falls. In fact, as $r \rightarrow 0$, the difference between leader's and follower's investment diverges to infinity.

Proposition 1. *A fall in the interest rate r raises the market leader's investment more than it raises the follower's, and their investment gap goes to infinity as r goes to zero. Formally, $d\eta_1/dr < d\eta_{-1}/dr$ with $\lim_{r \rightarrow 0} (\eta_1 - \eta_{-1}) = \infty$.*

All proofs are in the appendix. The intuition for $(\eta_1 - \eta_{-1}) \rightarrow \infty$ is as follows. Since $\eta_1 = v_2 - v_1$, a fall in r increases investment for the leader as the present value of its monopoly profits ($v_2 \equiv 1/r$) is higher were it to successfully innovate. However, for the follower $\eta_{-1} = v_0 - v_{-1}$, and the gain from a fall in r is not as high due to the *endogenous* response of its competitor in state=0 were the follower to successfully innovate. In particular, a fall in r also makes firms compete *more fiercely* in the neck-and-neck state

zero. A fall in r thus increases the expectation of a tougher fight were the follower to successfully catch up to $s = 0$. While the expectation of a more fierce competition in future state-zero disincentivizes the follower from catching up, the possibility of “escaping” the fierce competition through investment raises the incentive for the leader. This strategic asymmetry continues to amplify as $r \rightarrow 0$, giving us the result.

The core intuition in this example does not depend on simplifying assumptions such as exogenous flow profits, quadratic investment cost, state independent investment cost, or limiting ourselves to three states. For example, the full model that follows allows for an infinite number of possible states, microfounded flow profits with Bertrand competition, investment cost advantage for the follower, and other extensions. Also note that we do not impose any financing disadvantage for the follower vis-a-vis the leader as they both face the same cost of capital r . Any additional cost of financing for the follower, as is typically the case in practice, is likely to further strengthen our core result.

The key assumption for the core result is that follower cannot “leapfrog” the leader. As we explained, the key intuition relies on the expectation that the follower will have to “duke it out” in an intermediate state (state zero in our example) before it can get ahead of the leader. This expectation creates the key strategic asymmetry between the response by the leader and the follower to a lower interest rate. The follower is discouraged by the fierce competition in the future if it were to successfully close the technological gap between itself and the leader. The same is not true for the leader. In fact for the leader in state $s = 1$, the expectation of more severe competition in state $s = 0$ makes the leader want to escape competition with even greater intent. All of this gives the leader a larger reward for investment relative to the follower as r falls. We discuss the plausibility and applicability of the no-leapfrogging assumption in more detail below, and we show that the idea of incremental innovation applies to a wide range of settings in the real world.

The next section moves to a more general setting with a potentially infinite number of states. This breaks the rather artificial restriction of the simple example that leadership becomes perpetual in state 2. In the general set up, the leader can continue to create distance between itself and the follower by investing, but it cannot guarantee permanent leadership. Adding this more realistic dimension to the framework brings out additional important insights: not only does a fall in r increase the investment gap between the leader and the follower, but for r low enough, the average follower stops investing all together thereby killing competition in the industry. Therefore, while the example imposed permanent leadership exogenously, the full model shows that leadership endogenously becomes permanent. And as in the example, the expectation of permanent leadership makes the temporary leader invest more aggressively in a low interest rate environment.

As a result, a fall in the interest rate to a very low level raises market concentration and profits, and ultimately reduces productivity growth.

3 Model

The model has a continuum of markets with each market having two firms that compete with each other for market leadership. Firms compete along a technological ladder where each step of the ladder represents productivity enhancement. The number of steps, or states, is no longer bounded, so firms can move apart indefinitely. Firms' transition along the productivity ladder is characterized by a Poisson process determined by the level of investment made by each firm.

We aggregate across all markets and define a stationary distribution of market structures and the aggregate productivity growth rate. Section 4 characterizes the equilibrium and analyzes how market dynamism, aggregate investment, and productivity growth evolve as the interest rate declines toward zero. This section and Section 4 evaluate the model in partial equilibrium taking the interest rate and the income of the consumer as exogenously given. Section 5 then endogenizes these objects by embedding the model into general equilibrium.

3.1 Consumer Preferences

Time is continuous. At each instance t , a representative consumer decides how to allocate one unit of income across a continuum of duopoly markets indexed by ν , maximizing

$$\max_{\{y_1(t;\nu), y_2(t;\nu)\}} \exp \left\{ \int_0^1 \ln \left[y_1(t;\nu)^{\frac{\sigma-1}{\sigma}} + y_2(t;\nu)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} d\nu \right\} \quad (4)$$

$$\text{s.t.} \quad \int_0^1 p_1(t;\nu) y_1(t;\nu) + p_2(t;\nu) y_2(t;\nu) d\nu = 1,$$

where $y_i(t;\nu)$ is the quantity produced by firm i of market ν and $p_i(t;\nu)$ its price. The consumer preferences in (4) is a Cobb-Douglas aggregator across markets ν , nesting a CES aggregator with elasticity of substitution $\sigma > 1$ across the two varieties within each market.

Let $P(t) \equiv \exp \left(\int_0^1 \ln [p_1(t;\nu)^{1-\sigma} + p_2(t;\nu)^{1-\sigma}]^{\frac{1}{1-\sigma}} \right)$ be the consumer price index. Cobb-Douglas preferences imply that total revenue of each market is always one, i.e. $p_1(t;\nu) y_1(t;\nu) + p_2(t;\nu) y_2(t;\nu) = 1$. Hence firm-level sales only depend on the

relative prices within each market and are independent of prices in other markets, i.e. $\frac{y_1(t;\nu)}{y_2(t;\nu)} = \left(\frac{p_1(t;\nu)}{p_2(t;\nu)}\right)^{-\sigma}$. This implies that all strategic considerations on the firm side take place within a market and are invariant to prices outside a given market.

3.2 Firms: Pricing and Investment Decisions

The two firms in a market are indexed by $i \in \{1, 2\}$ and we drop the market index ν to avoid notational clutter. Each firm has productivity z_i with unit cost of production equal to λ^{-z_i} for $\lambda > 1$. Given consumer demand described earlier, each firm engages in Bertrand competition to solve,

$$\max_{p_i} (p_i - \lambda^{-z_i}) y_i \quad \text{s.t. } p_1 y_1 + p_2 y_2 = 1 \text{ and } y_1/y_2 = (p_1/p_2)^{-\sigma}. \quad (5)$$

The solution to this problem can be written in terms of state variable $s = |z_1 - z_2| \in \mathbb{Z}_{\geq 0}$ that captures the productivity gap between the two firms. When $s = 0$, two firms are said to be neck-and-neck; when $s > 0$, one firm is a temporary leader while the other is a follower. Let π_s denote leader's profit in a market with productivity gap s , and likewise let π_{-s} be the follower's profit of the follower in the market. Conditioning on the state variable s , firm profits π_s and π_{-s} no longer depend on the time index or individual productivities and have the following properties.

Lemma 1. *Given productivity gap s , the solution to Bertrand competition leads to flow profits*

$$\pi_s = \frac{\rho_s^{1-\sigma}}{\sigma + \rho_s^{1-\sigma}}, \quad \pi_{-s} = \frac{1}{\sigma \rho_s^{1-\sigma} + 1},$$

where ρ_s defined implicitly by $\rho_s^\sigma = \lambda^{-s} \frac{(\sigma \rho_s^{\sigma-1} + 1)}{\sigma + \rho_s^{\sigma-1}}$ is the relative price between the leader and the follower. Equilibrium markups are $m_s = \frac{\sigma + \rho_s^{1-\sigma}}{\sigma - 1}$ and $m_{-s} = \frac{\sigma \rho_s^{1-\sigma} + 1}{(\sigma - 1) \rho_s^{1-\sigma}}$.

Lemma 2. *Under Bertrand competition, follower's flow profit π_{-s} is weakly-decreasing and convex in s ; leader's and joint profits, π_s and $(\pi_s + \pi_{-s})$, are bounded, weakly-increasing, and eventually concave in s .⁴ Moreover, $\lim_{s \rightarrow \infty} \pi_s > \pi_0 \geq \lim_{s \rightarrow \infty} \pi_{-s}$.*

Lemma 2 states that a higher productivity gap is associated with higher profits for the leader and for the market as a whole. We therefore interpret markets in a lower state to be more competitive than markets in a higher state. Markups are also (weakly) increasing in the state s .

⁴A sequence $\{a_s\}$ is eventually concave iff there exists \bar{s} such that a_s is concave in s for all $s \geq \bar{s}$.

Our main theoretical results hold under any sequence of flow profits $\{\pi_s\}_{s=-\infty}^{\infty}$ that satisfy the properties in Lemma 2, as our proofs show. Such a profit sequence could be generated by alternative forms of competition (e.g., Cournot) or anti-trust policies (e.g., constraints on markups or taxes on profits). For clarity, even though $\lim_{s \rightarrow \infty} \pi_s = 1$ under Bertrand, we let $\pi_{\infty} \equiv \lim_{s \rightarrow \infty} \pi_s$ denote the limiting profit of an infinitely-ahead leader, and we derive our theory using the notation π_{∞} .

As an example under Bertrand competition, when duopolists produce perfect substitutes ($\sigma \rightarrow \infty$), profits are $\pi_s = 1 - e^{-\lambda s}$ for leaders and $\pi_{-s} = 0$ for followers and neck-and-neck firms. As another example outside of the Bertrand microfoundation, our main results hold for the following sequence of profits: $\pi_s = 0$ if $s < 1$ and $\pi_s = \pi_{\infty} > 0$ if $s \geq 1$, i.e. all leaders receive the identical flow profits whereas followers and neck-and-neck firms have zero profit.

Investment Choice

The most important choice in the model is the investment decision of firms competing for market leadership. A firm that is currently in the leadership position incurs investment cost $c(\eta_s)$ in exchange for Poisson rate η_s to improve its productivity by one step and lower the unit cost of production by a factor of $1/\lambda$. The corresponding follower firm chooses its own investment η_{-s} and state s transitions over time interval Δ according to,

$$s(t + \Delta) = \begin{cases} s(t) + 1 & \text{with probability } \Delta \cdot \eta_s, \\ s(t) - 1 & \text{with probability } \Delta \cdot (\kappa + \eta_{-s}), \\ s(t) & \text{otherwise.} \end{cases}$$

where parameter $\kappa \geq 0$ is the exogenous catch-up rate for the follower. There is a natural catch-up advantage that the follower enjoys due to technological diffusion from the leader to the follower; this guarantees the existence of a non-degenerate steady-state and is a standard feature in patent-race-based growth models (e.g., [Aghion et al. \(2001\)](#), and [Acemoglu and Akcigit \(2012\)](#)).

Firms discount future payoffs at interest rate r which is taken to be exogenous from the perspective of firm decision-making.⁵ Firm value $v_s(t)$ equals the expected present-

⁵We illustrate in Section 5.1 how r is endogenously determined in general equilibrium and can be viewed as coming from the household discount rate.

discount-value of future profits net of investment costs:

$$v_s(t) = \mathbb{E} \left[\int_0^\infty e^{-r\tau} \{ \pi(t+\tau) - c(t+\tau) \} d\tau \mid s \right]. \quad (6)$$

Value function (6) illustrates the various incentives that collectively determine how a firm invests. The basic problem is not only inter-temporal, but most importantly, *strategic*. A firm bears the investment cost today but obtains the likelihood of enhancing its market position by one-step which earns it higher profits in the future. However, there is also an important strategic dimension embedded in (6), as a firm's expected gain from investment today is also implicitly a function of how its competitor is expected to behave in the future. For instance, in the example of Section 2, intensified competition in the neck-and-neck state has a discouragement effect on the follower's investment and a motivating effect on the leader's.

We impose regularity conditions on the cost function $c(\cdot)$ so that firm's investment problem is well-defined and does not induce degenerate solutions. Specifically, we assume $c(\cdot)$ is twice continuously differentiable and weakly convex over a compact investment space: $c'(\eta_s) \geq 0$, $c''(\eta_s) \geq 0$ for $\eta_s \in [0, \eta]$. We assume the investment space is sufficiently large, $c(\eta) > \pi_\infty$ and $\eta > \kappa$ —so that firms can compete intensely if they choose to—and $c'(0)$ is not prohibitively high relative to the gains from becoming a leader ($c'(0) \kappa < \pi_\infty - \pi_0$)—otherwise no firm has any incentive to ever invest.

We look for a stationary Markov-perfect equilibrium such that the value functions and investment decisions are time invariant and depend only on the state. The HJB equations for firms in state $s \geq 1$ are

$$rv_s = \pi_s + (\kappa + \eta_{-s})(v_{s-1} - v_s) + \max_{\eta_s \in [0, \eta]} [\eta_s(v_{s+1} - v_s) - c(\eta_s)] \quad (7)$$

$$\begin{aligned} rv_{-s} = & \pi_{-s} + \eta_s(v_{-(s+1)} - v_{-s}) + \kappa(v_{-(s-1)} - v_{-s}) \\ & + \max_{\eta_{-s} \in [0, \eta]} [\eta_{-s}(v_{-(s-1)} - v_{-s}) - c(\eta_{-s})]. \end{aligned} \quad (8)$$

In state zero, the HJB equation for either market participant is

$$rv_0 = \pi_0 + \eta_0(v_{-1} - v_0) + \max_{\eta_0 \in [0, \eta]} [\eta_0(v_1 - v_0) - c(\eta_0)]. \quad (9)$$

These HJB equations have the same intuition as those in equations (1) through (3) in our earlier example. The flow value in state s is composed of current profit net of investment

cost, capital gain from successfully advancing on the technological ladder, and capital loss if the firm is pushed back on the ladder.

Definition 1. (Equilibrium) Given interest rate r , a symmetric Markov-perfect equilibrium is an infinite collection of value functions and investments $\{v_s, v_{-s}, \eta_s, \eta_{-s}\}_{s=0}^{\infty}$ that satisfy equations (7) through (9). The collection of flow profits $\{\pi_s, \pi_{-s}\}_{s=0}^{\infty}$ is generated by Bertrand competition as in Lemma 1.

3.3 Aggregation Across Markets: Steady State and Productivity Growth

The state variable in each market follows an endogenous Markov process with transition rates governed by investment decisions $\{\eta_s, \eta_{-s}\}_{s=0}^{\infty}$ of market participants. We define a steady-state equilibrium as one in which the distribution of productivity gaps in the entire economy, $\{\mu_s\}_{s=0}^{\infty}$, is time invariant. The steady-state distribution of productivity gaps must satisfy the property that, over each time instance, the density of markets leaving and entering each state must be equal.

Definition 2. (Steady-State) Given equilibrium investment $\{\eta_s, \eta_{-s}\}_{s=0}^{\infty}$, a steady-state is the distribution $\{\mu_s\}_{s=0}^{\infty}$ ($\sum \mu_s = 1$) over the state space that satisfies:

$$\underbrace{2\mu_0\eta_0}_{\substack{\text{density of markets} \\ \text{going from state 0 to 1}}} = \underbrace{(\eta_{-1} + \kappa)\mu_1}_{\substack{\text{density of markets} \\ \text{going from state 1 to 0}}}, \quad (10)$$

$$\underbrace{\mu_s\eta_s}_{\substack{\text{density of markets} \\ \text{going from state } s \text{ to } s+1}} = \underbrace{(\eta_{-(s+1)} + \kappa)\mu_{s+1}}_{\substack{\text{density of markets} \\ \text{going from state } s+1 \text{ to } s}} \quad \text{for all } s > 0. \quad (11)$$

where the number “2” in equation (10) reflects the fact that a market leaves state zero if either firm’s productivity improves.

We define aggregate productivity $Z(t)$ as the inverse of the total production cost per unit of the consumption aggregator:

$$\lambda^{Z(t)} \equiv \frac{\exp\left(\int_0^1 \ln\left[y_1(t; \nu)^{\frac{\sigma-1}{\sigma}} + y_2(t; \nu)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\right)}{\int_0^1 \lambda^{-z_1(t; \nu)} y_1(t; \nu) + \lambda^{-z_2(t; \nu)} y_2(t; \nu) d\nu}, \quad (12)$$

where recall λ is the step size of productivity increments. Note that $\lambda^{-Z(t)}$ is also the ideal cost index for the nested CES demand system in (4).

The next lemma characterizes the steady-state productivity growth rate as a function of the steady-state distribution in productivity gaps $\{\mu_s\}_{s=0}^{\infty}$ and firm-level investments $\{\eta_s, \eta_{-s}\}_{s=0}^{\infty}$.

Lemma 3. *In a steady state, the aggregate productivity growth rate $g \equiv \frac{d \ln \lambda^{Z(t)}}{dt}$ is*

$$\begin{aligned} g &= \ln \lambda \cdot \left(\sum_{s=0}^{\infty} \mu_s \eta_s + \mu_0 \eta_0 \right) \\ &= \ln \lambda \cdot \sum_{s=1}^{\infty} \mu_s (\eta_{-s} + \kappa). \end{aligned}$$

The productivity gap distribution is stationary in a steady state and, on average, the productivity growth rate at the frontier—leaders and neck-and-neck firms—is the same as that of market followers. Consequently, Lemma 3 states that aggregate productivity growth g is equal to the average rate of productivity improvements for leaders and neck-and-neck firms, weighted by the fraction of markets in each state (first equality), and that g can be equivalently written as the average rate of productivity improvements for market followers (second equality).

4 Analytical Solution

4.1 Linear Cost Function

The dynamic game between the two firms is complex and has rich strategic interactions, with potentially infinite state-contingent investment levels by each player to keep track of. To achieve analytical tractability, throughout this section we assume the cost function is linear in investment intensity: $c(\eta_s) = c \cdot \eta_s$ for $\eta_s \in [0, \eta]$. The model with a convex cost function is solved numerically in Section 5, where we show that the core results carry through. Because of linearity, firms generically invest at either the upper or lower bound in any state; hence, investment effectively becomes a binary decision, and any interior investments can be interpreted as firms playing mixed strategies. For expositional ease, we focus on pure-strategy equilibria in which $\eta_s \in \{0, \eta\}$, but all formal statements apply to mixed-strategy equilibria as well.

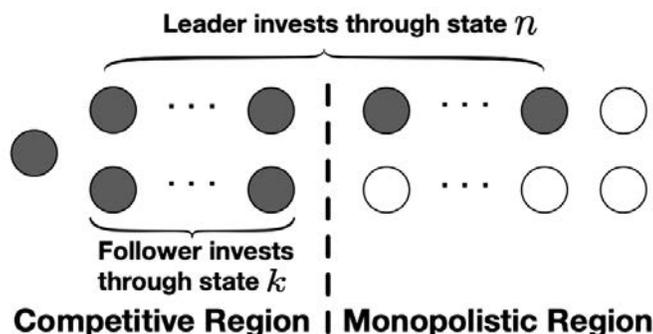
4.2 Market Equilibrium

The rest of this section solves for the equilibrium investment decisions by the leader and the follower in a market, dropping the market index ν for brevity. When the interest rate is prohibitively high, there may be a trivial equilibrium in which even the neck-and-neck firms in state zero do not invest, and aggregate investment and productivity growth are both zero in the steady state. Because the main result evaluates the effect of low interest rates r , for expositional simplicity we restrict analysis to equilibria with positive investment in the neck-and-neck state, and we present results that hold across all non-trivial equilibria.

Let $n + 1 \equiv \min \{s | \eta_s < \eta\}$ be the first state in which the market leader does not strictly prefer to invest, and likewise, let $k + 1 \equiv \min \{s | \eta_{-s} < \eta\}$ be the first state in which the market follower does not strictly prefer to invest.

Lemma 4. *In any non-trivial equilibrium, the leader invests in more states than the follower, $n \geq k$. Moreover, the follower does not invest ($\eta_{-s} = 0$) in states $s = k + 2, \dots, n + 1$.*

Figure 1: Illustration of Equilibrium Structure



Lemma 4 establishes that the leader must maintain investment in (weakly) more states than the follower does. The structure of an equilibrium can thus be represented by Figure 1. States are represented by circles, going from state 0 on the left to state $n + 1$ on the right. The coloring of a circle represents investment decisions: states in which the firm invests are represented by dark circles, whereas white ones represent those in which the firm does not invest. The top row represents leaders' investment decisions while the bottom row represents followers'. The corresponding steady-state features positive mass of markets in states $\{0, 1, \dots, n + 1\}$, and we can partition the set of non-neck-and-neck states into two regions: one in which the follower invests ($\{1, \dots, k\}$) and the other in which the follower does not ($\{k + 1, \dots, n + 1\}$). In the first region, the productivity gap widens

with Poisson rate η and narrows with rate $(\eta + \kappa)$. In expectation, the state s tends to decrease in this region, and the market structure tends to become more competitive. For this reason, we refer to this as the *competitive* region. Note this label does not reflect competitive market conduct or low flow profits—leaders’ profits can still be high in this region—instead, the label reflects the fact that joint profits tend to decrease over time. In the second region, the downward state transition occurs at a lower rate (κ), and the market structure tends to stay or become more monopolistic and concentrated. We refer to this as the *monopolistic* region.

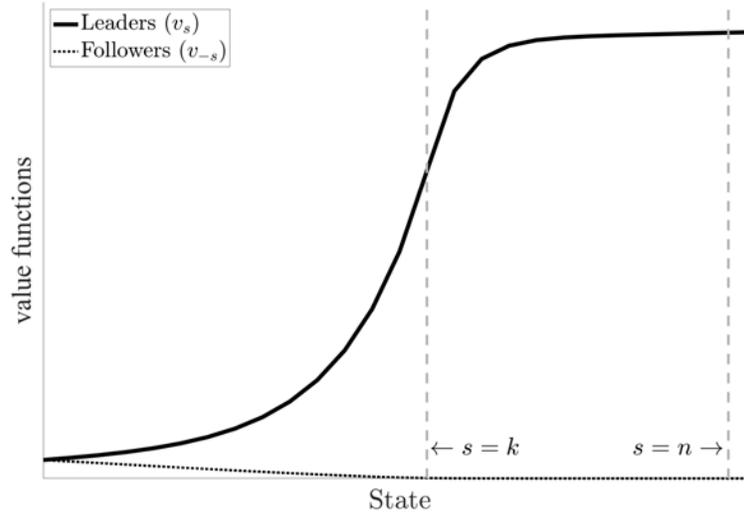
The formal proof of Lemma 4 is in the appendix; the intuition behind the $n \geq k$ proof is as follows. Suppose the leader stops investing before the follower does, $n < k$. In this case, the high flow payoff π_{n+1} is transient for the leader and the market leadership of being $n + 1$ steps ahead is fleeting, because the follower invests in state $n + 1$ and the rate of downward state transition is high $(\eta + \kappa)$. This implies a relatively low upper bound on the value for the leader in state $n + 1$. However, because firms are forward-looking and their value functions depend on future payoffs, the low value in state $n + 1$ “trickles down” to value functions in all states, meaning the incentive for the follower to invest—motivated by the future prospect of eventually becoming the leader in state $n + 1$ —is low. This generates a contradiction to the presumption that follower invests in more states than the leader does.

Figure 2 shows the value functions for both the leader and the follower, which help explain their investment decisions. The solid black curve represents the value function of the leader, whereas the dotted black curve represents the value function of the follower. The two dashed and gray vertical lines respectively represent k and n , the last states in which the follower and the leader invest, respectively.

The firm value in any state is a weighted average of the flow payoff in that state and the firm value in neighboring states, with weights being functions of the Poisson rate of state transitions.⁶ Figure 2 shows that $v_0 - v_{-k}$ is substantially lower than $v_k - v_0$; in fact, the joint value of both firms is strictly increasing in the state: $2v_0 < v_1 + v_{-1} < \dots < v_{n+1} + v_{-(n+1)}$. This is due to three complementary forces. First, joint profits $(\pi_s + \pi_{-s})$ are increasing in the state (Lemma 2). Second, as both firms invest in the competitive region, their investment cost further lowers the flow payoffs in the competitive region relative to the later, monopolistic region, i.e., states $k + 1$ through $n + 1$. Third, again because both players invest in the competitive region, a firm close to state 0 expects having to incur investment costs for a substantial amount of time before it will be able to escape

⁶For instance, for s in the competitive region ($0 < s < k$), $v_s = \frac{\pi_s - c\eta_s + \eta_s v_{s+1} + (\eta_{-s} + \kappa)v_{s-1}}{r + \eta_s + \eta_{-s} + \kappa}$, as implied by equation (7).

Figure 2: Value functions



the region and move beyond state $k + 1$.

The inequalities $2v_0 < v_s + v_{-s} < v_n + v_{-n}$ hold for any $s < n$ and imply that the leader's incentive to invest and move from state 0 to s is always higher than the follower's incentive to move from state $-s$ to 0 (as $v_s - v_0 > v_0 - v_{-s}$). Likewise, the leader's incentive to move from state s to n is always higher than follower's incentive to move from state $-n$ to $-s$. The valuation difference $v_0 - v_{-s}$ is low precisely because both firms compete intensely in states 0 through k , and their investment costs dissipate future rents. This is the sense in which strategic competition serves as a deterrent to the follower. The fact that competition serves as an endogenous motivator to the leader for racing ahead manifests itself through the convexity of the value function of a leader in the competitive region. As the leader approaches the end of the competitive region ($s = k$), its value function increases sharply, as maintaining its leadership would become substantially easier once the leader escapes the competitive region and gets to the monopolistic region. Conversely, falling back is especially costly to a leader within reach of the monopolistic region, precisely because of the intensified competition when $s < k$.

Why does the leader continue to invest in states $k + 1$ through n , even though the follower does not invest in those states? It does so to consolidate its strategic advantage. Because of technological diffusion κ , leadership is never guaranteed to be permanent, and a leader always has the possibility of falling back. As the value of being a far-ahead leader is substantially higher than being in the competitive region—due to intense competition in states 0 through k —it is worthwhile for the leader to create a “buffer” between its current state and the competitive region. The further ahead is the leader, the longer it expects to

stay in the monopolistic region before falling back to state k .

For sufficiently large s , both firms cease to invest. This happens to the follower because it is too far behind—its firm value is low, and the marginal value of catching up by one step is not worth the investment cost. This is known as the “discouragement effect” in the dynamic contest literature (Konrad (2012)). The leader eventually ceases investment as well, due to a “lazy monopolist” effect: the “buffer” has diminishing value, and once the lead ($n - k$) is sufficiently large, an additional step of security is no longer worth the investment costs.

4.3 Steady State

The steady-state of an equilibrium can be characterized by the investment cutoff states, n and k .⁷ The aggregate productivity growth rate in the steady-state is a weighted average of the productivity growth rate in each market; hence, aggregate growth depends on both the investment decisions in each state as well as the stationary distribution over states, which in turn is a function of the investment decisions. Given the investment cutoffs (n, k) , equations (10) and (11) enable us to solve for the stationary distribution $\{\mu_s\}$ in closed form. The following result builds on Lemma 3 and shows that the aggregate growth rate can be succinctly summarized by the fraction of markets in the competitive and monopolistic regions.

Lemma 5. *In a steady-state induced by equilibrium investment cutoffs (n, k) , the aggregate productivity growth rate is*

$$g = \ln \lambda \left(\mu^C \cdot (\eta + \kappa) + \mu^M \cdot \kappa \right),$$

where $\mu^C \equiv \sum_{s=1}^k \mu_s$ is the fraction of markets in the competitive region and $\mu^M \equiv \sum_{s=k+1}^{n+1} \mu_s$ is the fraction of markets in the monopolistic region. The fraction of markets in each region

⁷Technically, because we do not assume leader profits $\{\pi_s\}$ are always concave, the leader may resume investment after state $n + 1$. However, because market leaders do not invest in state $n + 1$, the investment decisions beyond state $n + 1$ are irrelevant for characterizing the steady-state because there are no markets in those states. Moreover, because $\{\pi_s\}$ is eventually concave in s (c.f. Lemma 2), all equilibria follow a monotone structure when interest rate r is small.

satisfies

$$\begin{aligned} \mu_0 + \mu^C + \mu^M &= 1, & \mu_0 &\propto (\kappa/\eta)^{n-k+1}(1 + \kappa/\eta)^k/2, \\ \mu^C &\propto (\kappa/\eta)^{n-k}((1 + \kappa/\eta)^k - 1), & \mu^M &\propto \frac{1 - (\kappa/\eta)^{n-k+1}}{1 - \kappa/\eta}. \end{aligned}$$

Lemma 5 follows from the fact that aggregate productivity growth is equal to the average rate of productivity improvements for market followers (Lemma 3). It shows the fractions of markets in the competitive and monopolistic regions are sufficient statistics for steady-state growth, and that markets in the competitive region contribute more to aggregate growth than those in the monopolistic region. Intuitively, both firms invest in the competitive region, and, consequently, productivity improvements are rapid, the state transition rate is high, dynamic competition is fierce, leadership is contentious, and market power tends to decrease over time. On the other hand, the follower ceases to invest in the monopolistic region, and, once markets are in this region, they tend to become more monopolistic over time. The monopolistic region also includes state $n + 1$, where even the leader stops investing. On average, this region features a low rate of state transition and low productivity growth.

Equilibrium investment cutoffs (n, k) affect aggregate growth through their impact on the fraction of markets in each region. Lemma 5 implies that holding n constant, a higher k always draws more markets into the competitive region, thereby raising the steady-state productivity growth rate. On the other hand, holding $k \geq 1$ constant—that followers invest at all—a higher n reduces productivity growth by expanding the monopolistic region and reducing the fraction of markets in the competitive region. We formalize this discussion into a Corollary, and we further provide lower bounds for the steady-state investment and growth rate when $k \geq 1$.

Corollary 1. *Consider an equilibrium with investment cutoffs (n, k) . The steady-state growth rate g is always increasing in k , and g is decreasing in n if and only if $k \geq 1$.*

Lemma 6. *Consider an equilibrium with investment cutoffs (n, k) . If $k \geq 1$, then in a steady-state, the aggregate investment is bounded below by $c \cdot \kappa$, and the productivity growth rate is bounded below by $\ln \lambda \cdot \kappa$.*

4.4 Comparative Steady-State: Declining Interest Rates

The key theoretical results of the model concern the limiting behavior of aggregate steady-state variables as the interest rate declines toward zero. Conventional intuition suggests

that, when firms discount future profits at a lower rate, the incentive to invest should increase because the cost of investment declines relative to future benefits. This intuition holds in our model, and we formalize it into the following lemma.

Lemma 7. $\lim_{r \rightarrow 0} k = \lim_{r \rightarrow 0} (n - k) = \infty$.

The result suggests that, as the interest rate declines, firms in all states tend to raise investment. In the limit as $r \rightarrow 0$, firms sustain investment even when arbitrarily far behind or ahead: followers are less easily discouraged, and leaders are less lazy.

However, the fact that firms raise investment in all states does not translate into high aggregate investment and growth. These aggregate variables are averages of the investment and productivity growth rates in each market, weighted by the steady-state distribution. A decline in the interest rate not only affects the investment decisions in each state but also shifts the steady-state distribution. As Lemma 5 shows, a decline in the interest rate can boost aggregate productivity growth if and only if it expands the fraction of markets in the competitive region; conversely, if more markets are in the monopolistic region—for instance if n increases at a “faster” rate than k —aggregate productivity growth rate could slow down, as Corollary 1 suggests.

Our main result establishes that, as $r \rightarrow 0$, a slow down in aggregate productivity growth is inevitable and is accompanied by a decline in investment and a rise in market power.

Theorem 1. *As $r \rightarrow 0$, aggregate productivity growth slows down:*

$$\lim_{r \rightarrow 0} g = \ln \lambda \cdot \kappa.$$

In addition,

1. *No markets are in the competitive region, and all markets are in the monopolistic region:*

$$\lim_{r \rightarrow 0} \mu^C = 0; \quad \lim_{r \rightarrow 0} \mu^M = 1.$$

2. *The productivity gap between leaders and followers diverges:*

$$\lim_{r \rightarrow 0} \sum_{s=0}^{\infty} \mu_s s = \infty.$$

3. *Aggregate investment to output ratio declines:*

$$\lim_{r \rightarrow 0} c \cdot \sum_{s=0}^{\infty} \mu_s (\eta_s + \eta_{-s}) = c\kappa.$$

4. *Leaders take over the entire market, with high profit shares and markups:*

$$\lim_{r \rightarrow 0} \sum_{s=0}^{\infty} \mu_s \pi_s = \pi_{\infty}.$$

Under Bertrand competition, the average sales of market leaders converges to 1 and that of followers converges to zero; aggregate labor share in production converges to zero.

5. *Market dynamism declines, and leadership becomes permanently persistent:*

$$\lim_{r \rightarrow 0} \sum_{s=0}^{\infty} M_s \mu_s = \infty,$$

where M_s is the expected time before a leader in state s reaches state zero.

6. *Relative market valuation of leaders and followers diverges:*

$$\lim_{r \rightarrow 0} \frac{\sum_{s=0}^{\infty} \mu_s v_s}{\sum_{s=0}^{\infty} \mu_s v_{-s}} = \infty.$$

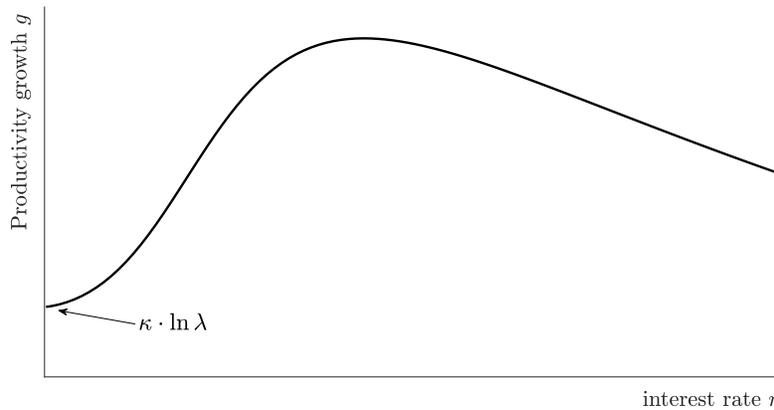
The Theorem states that, as $r \rightarrow 0$, all markets in a steady-state are in the monopolistic region, and leaders almost surely stay permanently as leaders. Followers cease to invest completely, and leaders invest only to counteract technology diffusion κ . As a result, aggregate investment and productivity growth decline and converge to their respective lower bounds governed by the parameter κ .

In the model, a low interest rate affects steady-state growth through two competing forces. As in traditional models, a lower rate is expansionary, as firms in all states tend to invest more (Lemma 7). On the other hand, a low rate is also anti-competitive, as the leader's investment response to a decline in r is stronger than follower's response. This anti-competitive force changes the distribution of market structure toward greater market power, thereby reducing aggregate investment and productivity growth. Theorem 1 shows that the second force always dominates when the level of the interest rate r is sufficiently low.

In fact, the limiting rate of productivity growth, $\kappa \cdot \ln \lambda$, is independent of the limiting profit $\lim_{s \rightarrow \infty} \pi_s$ and the investment cost c . Theorem 1 therefore has precise implications for anti-trust policies. As we elaborate in Section 6, policies that raise κ can promote growth, whereas policies that reduce leader profits or reduce the follower’s investment costs are ineffective when r is low.

Because $\kappa \ln \lambda$ is the lower bound on productivity growth (c.f. Lemma 6), Theorem 1 implies an inverted-U relationship between steady-state growth and the interest rate, as depicted in Figure 3. In a high- r steady-state, few firms invest in any markets, and aggregate productivity growth is low. A marginally lower r raises all firms’ investments, and the expansionary effect dominates. When the interest rate is too low, however, most markets are in the monopolistic region, in which followers cease to invest, and aggregate productivity growth is again low. The anti-competitive effect of a low interest rate also generates other implications: the leader-follower productivity gap widens, the relative leader-follower market valuation diverges, the profit share and markups rise, and business dynamism declines.

Figure 3: Steady-state growth and the interest rate: inverted-U



Lemma 7 shows that, as $r \rightarrow 0$, the *number* of states in both the competitive and monopolistic regions grow to infinity, but n and k may grow at different rates. Theorem 1 shows that the *fraction* of markets in the monopolistic region μ^M converges to one, which can happen only if the monopolistic region expands at a “faster rate” than the competitive region, i.e., the leader raises investment “faster” in response to a low r than the follower does. In the Appendix, we provide a sharp characterization on the exact rate of divergence for k and $(n - k)$ and the rate of convergence for $\mu^M \rightarrow 1$ (Lemma A.4).

To understand the leader’s stronger investment response, we again turn to Figure 2. The shape of the value functions in the figure holds for any r . As the figure demonstrates,

the leader's value close to state 0 in the competitive region is small relative to its value in state $n + 1$, and the leader would experience a sharp decline in value if it slips back from the monopolistic region into the competitive region. When the interest rate is low, a patient leader invests even far into the monopolistic region (i.e., $n - k$ grows indefinitely as r declines) in order to avoid the future prospect of falling back, and the leader stops investing only when it expects to stay in the monopolistic region for a sufficiently long time. As $r \rightarrow 0$, a leader behaves as if it is infinitely patient. Even the distant threat of losing market power is perceived to be imminent; consequently, leaders scale back investment only if they expect to never leave the monopolistic region, causing market leadership to become endogenously permanent.

Why does a symmetric argument not apply to the follower? Consider the follower in state $k + 1$. As $r \rightarrow 0$, $k + 1$ grows indefinitely (Lemma 7), and the follower in this marginal state is further and further behind. Because both firms invest in all states 0 through k , the follower in state $k + 1$ expects to fight a longer and longer war before it can reach state 0 and has a chance to become the leader. As the fight for leadership involves intense competition and large investment costs for a long time, the follower is eventually discouraged from the fight—when it is more than k steps behind—despite low r . Once again, the intense competition in states 0 through k dissipates future rents and serves as an endogenous deterrent to the follower in state $k + 1$. Low interest rates motivate investment only if future leadership is attainable. As $r \rightarrow 0$ and as k grows, it becomes infinitely costly to overcome the competition in states 0 through k , and the prospect of becoming a future leader is perceived to be too low even for a patient follower in state $k + 1$.

Theorem 1 is an aggregate result that builds on sharp analytical characterizations of the dynamic game between duopolists in each market. The duopolist game is rooted in models of dynamic patent races and is notoriously difficult to analyze: the state variable follows an endogenous stochastic process, and firms' value functions are recursively defined and therefore depend on flow payoffs and investment decisions in every state of the ergodic steady-state distribution $\{\mu_s\}_{s=0}^{n+1}$. Even seminal papers in the literature rely on numerical methods (e.g. Budd et al. (1993), Acemoglu and Akcigit (2012)) or restrictive simplifications to make the analysis tractable.⁸ Relative to the literature, our analysis of an economy in a low-rate environment is further complicated by the fact that, as r declines, the ergodic state space $\{0, 1, \dots, n + 1\}$ becomes infinitely large.

In order to obtain Theorem 1, we fully characterize the asymptotic equilibrium as $r \rightarrow$

⁸For instance, Aghion et al. (2001) and Aghion et al. (2005) assume leaders do not invest in all $s \geq 1$, effectively restricting the ergodic state space as $\{0, 1\}$.

0. We analytically solve for the recursive value functions as a first-order approximation in r around $r = 0$, and we analytically characterize the rate at which equilibrium objects—value functions, investment cutoffs, the stationary distribution of productivity gaps—grow as $r \rightarrow 0$. Theorem 1 is a distillation of the full characterization, and we relegate the formal proof to the appendix. In what follows, we provide a sketch of the proof, in four steps. Each step aims to explain a specific feature in the shape of value functions shown in Figure 2. Note because total revenue in a market is always equal to 1, $rv_s \leq 1$ for all s .

Step 1: The leader’s value in state $n + 1$ is asymptotically large.

Formally, Lemma A.1 shows $\lim_{r \rightarrow 0} rv_{n+1} \rightarrow \pi_\infty - c\kappa > 0$. To see this, note the leader stops investing in state $n + 1$ if and only if the marginal investment cost is higher than the change in value function, implying

$$c \geq v_{n+2} - v_{n+1} \geq \frac{\pi_{n+2} - rv_{n+1}}{r + \kappa}, \quad (13)$$

where the last inequality follows from rearranging the HJB equation (7) for state $n + 2$. This in turn generates a lower bound for rv_{n+1} :

$$rv_{n+1} \geq \pi_{n+2} - c(r + \kappa) \xrightarrow{\text{(Lemma 7)}} \pi_\infty - c\kappa.$$

Step 2: The follower’s value in state $k + 1$ is asymptotically small.

Formally, Lemma A.2 shows $rv_{-(k+1)} \rightarrow 0$. To understand this, note the follower stops investing in state $k + 1$ only if the marginal change in value function is lower than the investment cost ($v_{-k} - v_{-(k+1)} < c$). As $r \rightarrow 0$, the leader continues to invest in infinitely many states beyond k , and the follower stops investing in state $k + 1$ despite knowing that once it gives up, the market structure tends to move in the leader’s favor indefinitely, and that investing instead could delay or prevent falling back indefinitely. Lemma A.2 shows that follower not investing in state $k + 1$ must imply follower’s value in that state is asymptotically small.

Step 3: The value of a neck-and-neck firm is asymptotically small.

Formally, $rv_0 \rightarrow 0$. This is because as $k \rightarrow \infty$ (Lemma 7), firms in state zero expect to spend an indefinitely long time in the competitive region (states $s = 1, \dots, k$), in which both firms invest at the upperbound, with a negative joint flow payoff due to intense competition. In fact, k must grow at a rate exactly consistent with an asymptotically small

v_0 ; this is because v_0 can be asymptotically large only if k grows slowly, but a large v_0 in turn implies that v_{-k} must be large—as the follower in state $-k$ is forward looking—which contradicts the earlier statement, that $rv_{-k} \rightarrow 0$. Conversely, the fact that v_0 must be non-negative—firms can always guarantee at least zero payoff—imposes an upper bound on the rate at which k diverges.

Step 4: A leader experiences an asymptotically large decline in value as it falls from the monopolistic region into the competitive region.

Formally, $\lim_{r \rightarrow 0} r(v_{k+1} - v_k) > 0$. This follows from the fact that v_{n+1} is asymptotically large (step 1) and v_0 is asymptotically small (step 3).

Step 4 implies that falling back into the competitive region is costly for the leader. Hence, starting from state $k + 1$, the leader continues to invest in additional states in order to consolidate market power and reduce the future prospect of falling back. Its firm value increases as the productivity gap widens, and the leader stops only when the value function is sufficiently high, as characterized by inequalities (13). As a leader becomes infinitely patient, he must invest in sufficiently many states beyond k until the prospect of falling back into the competitive region vanishes, thereby endogenously perpetuating market leadership and causing the monopolistic region to become absorbing.

5 General Equilibrium and Quantitative Analysis

5.1 General Equilibrium

Up to this point, the analysis has taken the interest rate as exogenous, and we exogenously specify that the representative consumer has unit expenditure at each time t . We now embed the model into a general equilibrium framework by endowing the consumer with intertemporal preferences and endogenous income.

We limit our attention to the steady-state equilibrium, i.e., a balanced growth path, with aggregate productivity and consumption both growing at a constant rate g . Let \hat{r} be the interest rate faced by the consumer. All of the formal statements in earlier sections continue to hold along the balanced growth path if we re-define $r \equiv \hat{r} - g$. In other words, what we have been calling “the interest rate” in earlier sections is the growth-adjusted interest rate in the context of general equilibrium, which, as we show, is also equal to the discount rate of the representative consumer.

Formally, the consumer has the following intertemporal preferences:

$$\begin{aligned} & \max_{\{y_1(t;\nu), y_2(t;\nu), L(t)\}} \int_0^\infty e^{-\rho t} (\ln C(t) - L(t)) dt & (14) \\ \text{s.t. } & C(t) = \exp \left(\int_0^1 \ln \left[y_1(t;\nu)^{\frac{\sigma-1}{\sigma}} + y_2(t;\nu)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} d\nu \right), \\ & \int_0^1 p_1(t;\nu) y_1(t;\nu) + p_2(t;\nu) y_2(t;\nu) d\nu = w(t) L(t) + \Pi(t), \end{aligned}$$

where ρ is the discount rate and $\Pi(t)$ is the total profit net of the investment cost accrued to producers in the economy.

We normalize the wage rate $w(t) \equiv 1$ for all t , and specify that production and the investment cost are both paid in labor. The labor market clearing condition is

$$L(t) = \int_0^1 [y_1(t;\nu) \lambda^{-z_1(t;\nu)} + y_2(t;\nu) \lambda^{-z_2(t;\nu)}] d\nu + \left(\sum_{s=1}^{\infty} \mu_s(t) (c(\eta_s) + c(\eta_{-s})) + 2\mu_0(t) c(\eta_0) \right).$$

The consumption aggregator $C(t)$ in (14) is once again CES across varieties within each market and Cobb-Douglas across markets. Given our normalization, the consumer's intratemporal problem implies total expenditure on all consumption goods is equal to one, thereby inducing instantaneous demand functions that coincide with the preferences in (4) of Section 3. In addition, the intertemporal preferences in (14) imply an Euler equation:

$$g(t) \equiv \frac{d \ln C(t)}{dt} = \hat{r}(t) - \rho \quad (15)$$

where $\hat{r}(t)$ is the general equilibrium interest rate and $g(t)$ is the growth rate of aggregate consumption.

On a balanced growth path, the consumption price index $P(t)$ takes the same form as defined in Section 3.1, and it declines at a constant rate g relative to the numeraire; hence, the value function of a firm currently in state s is

$$\begin{aligned} v_s(t) &= \mathbb{E} \left[\int_0^\infty e^{-\hat{r}\tau} \left\{ \frac{\pi(t+\tau) - c(t+\tau)}{P(t+\tau)/P(t)} \right\} d\tau \middle| s \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-(\hat{r}-g)\tau} \{ \pi(t+\tau) - c(t+\tau) \} d\tau \middle| s \right] \end{aligned}$$

The model presented in earlier Sections 3 and 4 represents the production-side of this

economy, and the earlier analysis demonstrates an inverted-U relationship between g and $\hat{r} - g$. That production-side relationship and the consumer-side Euler equation (15) together pin down the interest rate \hat{r} and aggregate productivity growth g on a balanced growth path.

The Euler equation implies that in any equilibrium, $\hat{r} - g$ must be equal to the consumer discount rate ρ . Consequently, the general equilibrium version of the main result of Theorem 1 states that, if the consumer discount rate ρ declines towards zero, then in the limit, $\hat{r} - g \rightarrow 0$, and aggregate productivity growth rate g must decline and converge to $\kappa \cdot \ln \lambda$ (which implies that $\hat{r} \rightarrow \kappa \cdot \ln \lambda$ as well). Aggregate investment must decline, along with market dynamism and the aggregate labor share; markets become more concentrated, with high levels of markups, profits, and firm inequality.

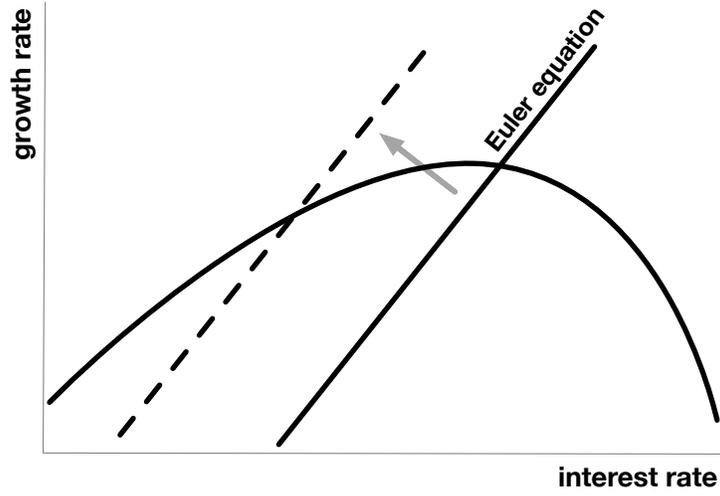
What could be the sources of a decline in ρ ? We follow Krugman (1998) and note that a decline in ρ can be seen as a catch-all shock that stands in place for any secular changes on the consumer side that pushes consumers towards saving more and consuming less, including a change in preferences, tightened borrowing constraints (e.g., Eggertsson and Krugman (2012)), or structural shifts such as an aging population (e.g., Eggertsson et al. (2019)) and rising inequality (e.g., Summers (2014), Mian et al. (2020)). Hence, the model presents an alternative view of the reasons behind “secular stagnation.” As in traditional secular stagnation explanations, an initial inward shift in the consumer-side curve can lower equilibrium interest rates to very low levels. However, “stagnation” is not due to monetary constraints such as the zero lower bound or nominal rigidities. Instead, a large fall in interest rates can make the economy more monopolistic for reasons laid out above, thereby lowering investment and productivity growth. This is depicted in Figure 4.

5.2 Quantitative Analysis

This section explores the quantitative properties of the model, with two goals. The first goal is validation—the model is numerically solved with a convex investment cost function, and we show that the limiting properties of the steady-state in Theorem 1, as well as other qualitative features of the equilibrium discussed in Section 4, continue to hold as we dispense with the linear-investment-cost assumption. Second, we show that, despite its parsimony, the model has some quantitative bite in explaining long-run trends in productivity growth and the profit shares. The quantitative model also has the added benefit that it illustrates some of the main mechanisms of the model.

According to Theorem 1, the steady-state distance between leaders and followers diverges as $r \rightarrow 0$. Hence, under either Bertrand or Cournot competition, the steady-state

Figure 4: Growth and the interest rate in general equilibrium



profit share converges to one. For quantitative relevance, we continue to assume Bertrand competition but modify the microfoundation for flow profits $\{\pi_s\}$ as follows. We specify that the production cost of the follower is $\lambda^{\max\{s, \bar{s}\}}$ times the cost of the leader for some parameter \bar{s} ; hence, while a greater distance always implies a bigger strategic advantage for the leader—it takes the follower more steps to catch up with the leader—a greater s only translates into an additional production cost advantage up to $s \leq \bar{s}$. For simplicity, we set $\bar{s} = 1$, so that flow profits for both firms are constant for all $s \geq 1$.

The calibration is purposefully simple with only four other parameters. The cost function is specified to be quadratic, $c(\eta_s) \equiv (c \cdot \eta_s)^2$, where c is a cost-shifter, and the investment space is assumed to be sufficiently large so that η_s is always interior. The other three parameters are κ , the rate of technological diffusion; λ , the step-size of productivity gains; σ , the elasticity of substitution between two firms in the same market. The parameters σ and λ jointly determine flow profits $\{\pi_s\}$, which, along with c and κ , determine the equilibrium growth rate.

The calibration is done using the general equilibrium version of the model, with $r \equiv \hat{r} - g$, i.e., a firm's discount rate r is indeed the real interest rate \hat{r} minus the productivity growth rate g . The calibration of these parameters $\{c, \kappa, \sigma, \lambda\}$ targets four moments: TFP growth rate and profit shares in high- and low-interest rate steady-states. The high-interest rate steady-state represents the U.S. economy during the years 1984–2000, and the low-interest rate steady-state for the years 2001–2016. For TFP growth—1.10% in the high- \hat{r} period and 0.76% in the low- \hat{r} period—we use the unadjusted total factor productivity for the business sector from the Federal Reserve Bank of San Francisco's database

(Fernald (2015)).

For the profit share, we target 0.14 in the high- \hat{r} period and 0.17 in the low- \hat{r} period, and we compute it from our model as average profits net of investment cost relative to revenue across all firms. These profit shares translate into markups of 16% and 20%, respectively; they capture the rise in markups in the United States and correspond roughly to the midpoint of recent estimates in the literature.⁹ Finally, for the real interest rate, the U.S. AA corporate bond rate net of current inflation is used, which is 4.69% for the high- \hat{r} period and 1.09% for low- \hat{r} period (Farhi and Gourio (2019)). We use the AA corporate bond rate instead of the 10-year treasury rate—3.94% and 1.06% in the two periods—because the former is more relevant as the firms’ discount rate, but the quantification is not sensitive to this choice. Table 1 shows the parameter values for the model’s fit.

Table 1: Calibration: Parameters and Model Fit

Definition	Parameter	Value	Moment	Target	Model
Elasticity of substitution	σ	12	TFP growth, high- \hat{r}	1.10%	1.09%
Productivity step size	λ	1.21	TFP growth, low- \hat{r}	0.76%	0.76%
Technology diffusion rate	κ	3.93	Profit share, high- \hat{r}	0.14	0.14
Investment cost shifter	c	33.4	Profit share, low- \hat{r}	0.17	0.17

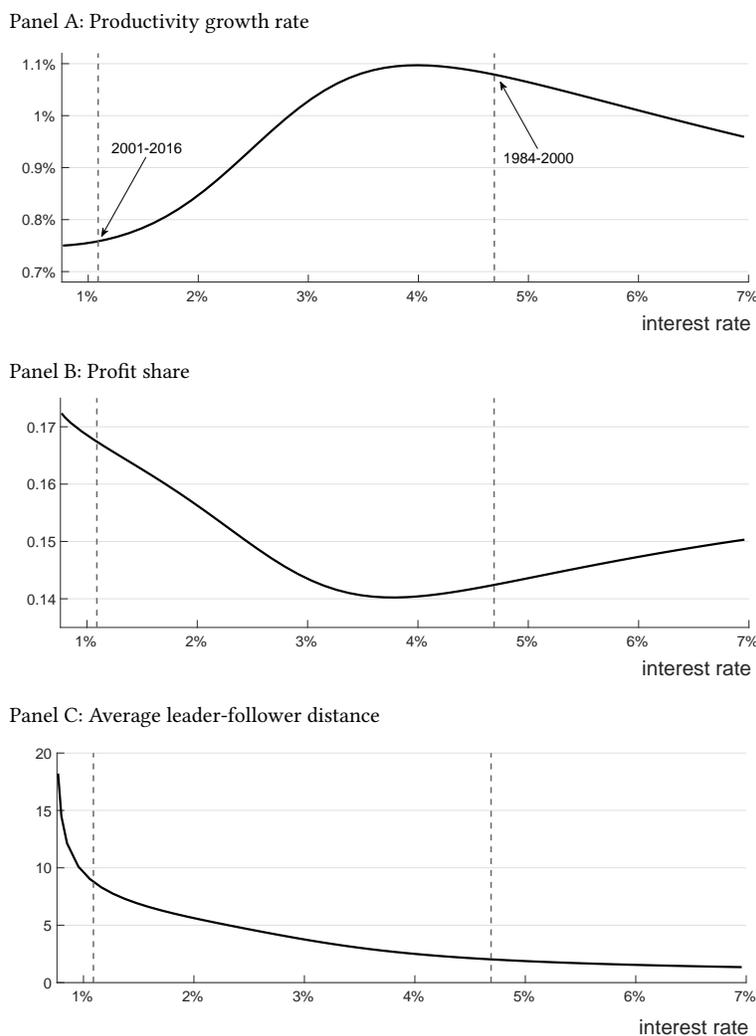
Figure 5 shows aggregate variables as steady states are compared for a decline in interest rates under the calibration. Panel A plots the productivity growth rate g against the interest rate \hat{r} .¹⁰ There are three noteworthy features. First, as the theory predicts, g has an inverted-U relationship with \hat{r} . Moving from right to left, as \hat{r} declines, g first increases as in traditional models. But eventually g declines. Second, in the limit as $\hat{r} - g \rightarrow 0$, g converges to $\kappa \cdot \ln \lambda$. This is a sharp prediction that is shown above analytically in Theorem 1 under the linear-investment-cost assumption. The numerical solution here shows that the prediction continues to hold under a convex investment cost. Note that this is not an artifact of the calibration, as we find g converges to $\kappa \ln \lambda$ under any calibration of the model. Third, growth is maximized at $g = 1.1\%$ when the real interest rate is around $\hat{r} = 4\%$. The productivity growth rate therefore starts to decline well above the limit, implying that the mechanism is empirically relevant.

Panel B of Figure 5 shows an U-shaped relationship between the net profit share and the interest rate. As \hat{r} declines, competition always intensifies in any given state, but the

⁹See Gutiérrez and Philippon (2016, 2017); Hall (2018); Barkai (forthcoming); Edmond et al. (2019); De Loecker et al. (2020), among many others.

¹⁰Recall that in general equilibrium, $r \equiv \hat{r} - g$ is the discount rates used by firms when making investment decisions.

Figure 5: Comparative steady states: low interest rates on productivity growth, profit share, and average leader-follower distance

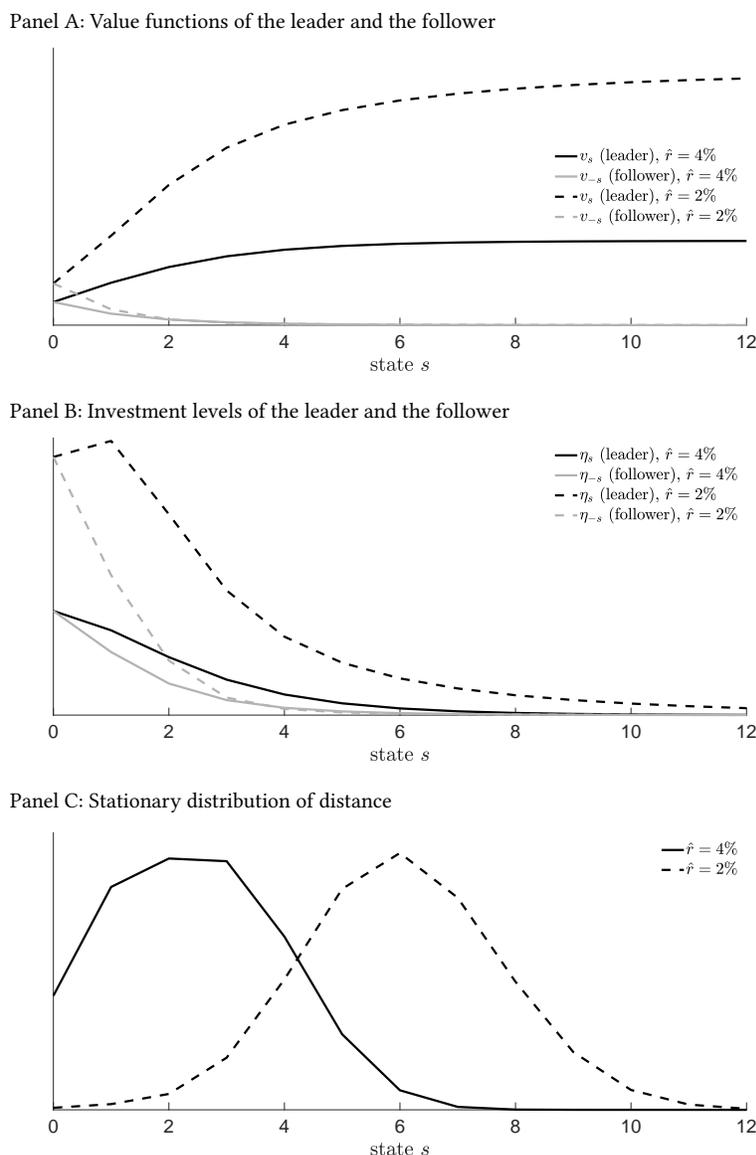


leader-follower distance tends to widen. Intensified competition raises investment costs, whereas widening leader-follower-distance raises gross profits. As \hat{r} declines (right to left in the figure), initially the first force dominates and the profit share declines; eventually for \hat{r} sufficiently low, the second force dominates, steady-state competition and investment decline, and the profit share increases.

Panel C of Figure 5 shows the average leader-follower distance monotonically increases as \hat{r} declines and tends to infinity as $\hat{r} - g \rightarrow 0$. This is consistent with Theorem 1, which establishes the divergence in leader-follower distance analytically under a linear investment cost.

Figure 6 shows additional comparative steady state results by illustrating how state-

Figure 6: Comparative steady states: state-by-state value functions, investment, and stationary distribution for $\hat{r} = 4\%$ (solid line) and $\hat{r} = 2\%$ (dashed line)



by-state value functions (Panel A), investment levels (Panel B), and the stationary distribution of leader-follower distance (Panel C) vary at two levels of interest rates, $\hat{r} = 4\%$ (solid line) and $\hat{r} = 2\%$ (dashed line). These figures confirm that the qualitative properties established analytically in Section 4 continue to hold under a convex investment cost. Panel A shows that the gain in the leader's value (black line) from state 0 to being far ahead (e.g. in state 12) is greater than the loss in the follower's value (grey line), and the asymmetry is greater under a lower interest rate. As explained above, this feature of

the equilibrium value functions is due to the intensified competition between firms when their distance s is small, as shown in Panel B. Strategic competition therefore serves as an endogenous motivator for the leader and a deterrent for the follower, resulting in the leader investing more than the follower in every state along the intensive margin. Panel B further demonstrates that the leader-follower investment gap widens in every state as r declines. Finally, Panel C plots the stationary distribution of firm distance $\{\mu_s\}$ and shows the distribution undergoes a first-order-stochastic-dominant shift to the right as r declines.

6 Additional Discussion

This section presents a number of extensions. The policy implications of the framework are discussed in Section 6.1. Section 6.2 discusses the implication of introducing real-world financial frictions, and Section 6.3 discusses the key “no-leapfrog” feature of the model and its relevance in the real world. Section 6.4 discusses transitional dynamics and the model’s asset-pricing implications.

6.1 Policy Implications

The main result in Theorem 1, that $\lim_{r \rightarrow 0} g = \kappa \cdot \ln \lambda$, has interesting implications for anti-trust policies in a low interest rate environment. As with traditional endogenous growth models, it is the incentive to gain market power that drives investment and growth in this framework. The additional insight in the model studied here is that investment by market leaders responds more aggressively to lower interest rates than the investment by market followers. Correspondingly, a low interest rate environment creates an expectation that market leaders will fight much more fiercely if market followers were to try to close in on the leader. This expectation of tougher competition, and the associated higher cost, discourages challengers from investing to unseat market leaders.

The expectation of tougher resistance by market leaders in a low interest rate environment reduces competition and growth. In these situations, regulation that reduces the expectation of tougher competition from market leaders can help raise investment and productivity growth. The model therefore shows why anti-trust regulation may become *more* important in a low interest rate environment.

But which anti-trust policies are most effective in this model? Broadly speaking, there are two types of potential policies that are relevant. The first type aims at helping market followers in terms of flow payoffs—such as taxing the leader’s flow profits or subsidizing

the follower’s flow investment costs. The second type facilitates technological transfers from leaders to followers—such as policies that directly raise κ by restricting defensive R&D and removing barriers for followers to compete.

In principle, policies focused directly on flow payoffs may promote investment by discouraging leaders’ investment and encouraging followers’. However, as Theorem 1 suggests, these policies are ineffective at promoting investment and growth in a low interest rate environment. As $r \rightarrow 0$, the leader-follower strategic asymmetry continues to prevail, and the growth rate slows down to the same limit ($\kappa \cdot \ln \lambda$) whether these policies are in place or not. Because the strategic asymmetry is so strong, it is ineffective to merely *encourage* the followers; policies must target technological transfers directly by raising κ , thereby helping followers even as they become endogenously discouraged.

The calibrated model from section 5.2 can be used to demonstrate these intuitions. Two policies focused on flow payoffs are considered: one which taxes leader profits by 10% and the other which reduces the follower’s investment cost by 10%. We also consider a policy that raises the rate of technological diffusion κ by 10%. Figure 7 shows the effects of these interventions; Panel A plots the relationship between the growth rate and the interest rate, and Panel B plots the relationship between the profit share and the interest rate. The baseline calibration is the solid line in black; the counterfactuals are represented in grey with various markers.

Figure 7: Counterfactual productivity growth and profit share: 10% tax on leader profits and 10% higher κ

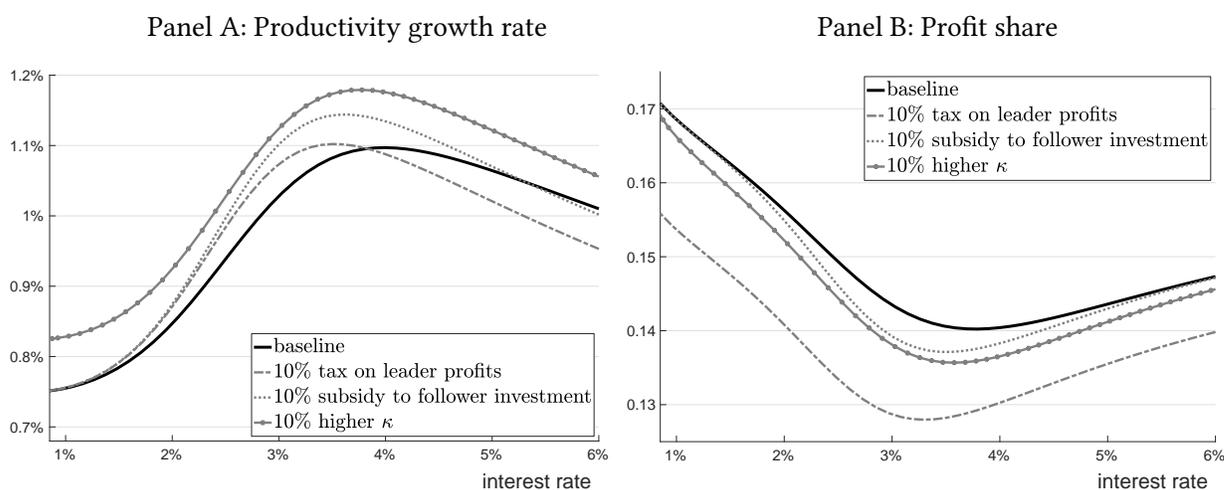


Figure 7 shows that taxing leader profits, while effective in reducing market power (Panel B), are not effective in stimulating investment and growth when the interest rate is sufficiently low (Panel A). Intuitively, the value to erecting barriers for a strategic ad-

vantage as r approaches zero is so large that even with 10% taxes, the value of being a permanent leader goes to infinity; market leaders therefore still have the incentive to completely discourage the followers. The intuition from Theorem 1 continues to hold under policies that constrain markups and profits, and, as $r \rightarrow 0$, growth will decline to the very same limit, $\kappa \cdot \ln \lambda$, regardless of whether the profit tax is in place or not. Taxing leader profits also has the undesirable effect that, for \hat{r} sufficiently high (e.g. at the rates that prevail in 1984–2000) the policy also reduces growth by discouraging the incentive to become a leader. For similar reasons, subsidizing the follower’s investment is also ineffective in promoting growth when the interest rate is low.

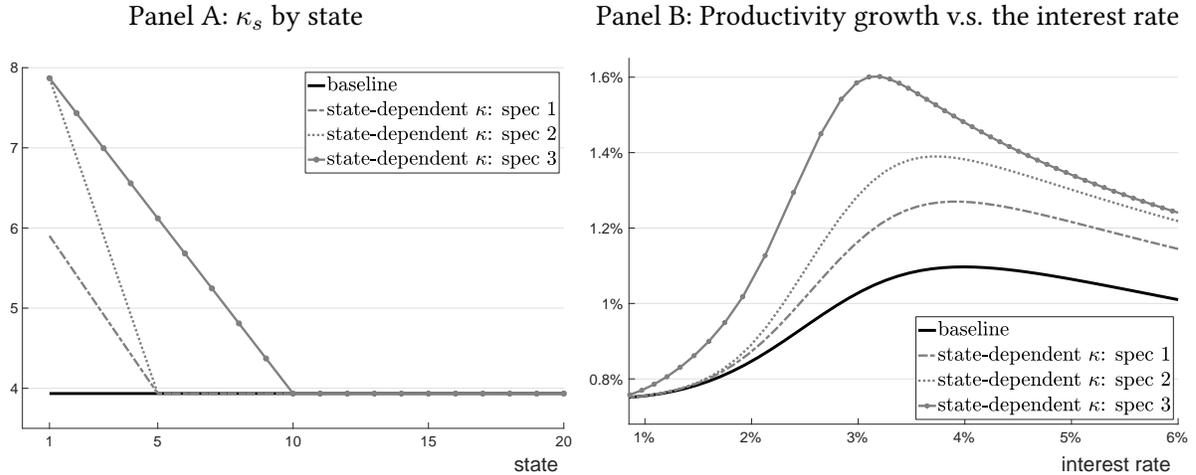
Antitrust policies focused on technological transfers on the other hand are effective in increasing investment and growth in our model. Theorem 1 implies that raising κ stimulates growth by directly raising the limiting growth rate of the economy. Figure 7 further shows that a higher κ raises the growth rate even when the interest rate is significantly above its lower limit. This is because a greater κ facilitates technological diffusion from leaders to followers, helping the followers even as they become endogenously discouraged. The steady-state therefore features more markets in states with stronger competition and greater investment, which leads to a higher aggregate growth rate. The fact that a higher κ implies more competitive markets is also evident in Panel B, which shows that the aggregate profit share is lower than the baseline for all levels of the interest rate, despite κ not directly affecting the flow profits in any given market.

Finally, we note that it is important for policy to raise κ in all states. If the rate of technology diffusion were state-dependent $\{\kappa_s\}_{s=1}^{\infty}$ and always finite—for instance, if policy facilitates technology transfer only if followers were not too far behind—then it is the limiting rate $\lim_{s \rightarrow \infty} \kappa_s$ that matters in for aggregate growth in a low interest rate environment: $\lim_{r \rightarrow 0} g = (\lim_{s \rightarrow \infty} \kappa_s) \cdot \ln \lambda$. Intuitively, because the leader-follower distance tends to diverge, bounded variations in κ_s for finite distance does not affect the steady-state as $r \rightarrow 0$.

Figure 8 demonstrates this result. We consider three alternative policies that facilitate technology transfer but only for finite states. The state-dependent κ_s that these policies represent are shown in Panel A. Specification 1 sets κ_1 to be 50% higher than κ in the baseline calibration, and κ_s decays linearly towards the baseline over five states. Specification 2 sets κ_1 to be 100% higher than the baseline and κ_s again decays linearly over five states. Specification 3 sets κ_1 to be 100% higher than baseline and decays over 10 states. Panel B shows how steady-state growth rate varies with the interest rate under these policies. Evident from the figure, all three policies raise productivity growth when $r > 0$; however, the effectiveness declines as $r \rightarrow 0$, and, in the limit, the growth rate always converges to

$\kappa \cdot \ln \lambda$.

Figure 8: Counterfactual productivity growth and profit share: state-dependent κ



6.2 Introducing Financial Frictions

The model assumes that firms face no financial frictions. In particular, both the market leader and the follower use the same interest rate r to discount cash flows and neither faces an external financing premium. Financial frictions are intentionally assumed away to highlight that even when firms are not handicapped by an asymmetric financing constraint, the strategic incentive of market leaders becomes stronger as $r \rightarrow 0$, and such an incentive causes market power to increase in low-rate environments. We conjecture that introducing financial frictions would strengthen the core results of the model. Related points are made in the literature such as [Caballero et al. \(2008\)](#) and [Gopinath et al. \(2017\)](#).

Empirical evidence on financial frictions further suggests that financial frictions hurt market followers more than market leaders, especially in a low interest rate environment. In particular, a declining long-term interest rate is associated with a larger financing gap between industry leaders and followers. This fact is shown by constructing the interest rate faced by industry leaders (the top 5% of firms in any industry) versus industry followers in Compustat data. A firm's interest rate is calculated by dividing annual interest expense by total debt. Then the median imputed interest rate for industry leaders and followers is plotted in the left panel of [Figure 9](#) over time, along with bootstrapped standard errors.

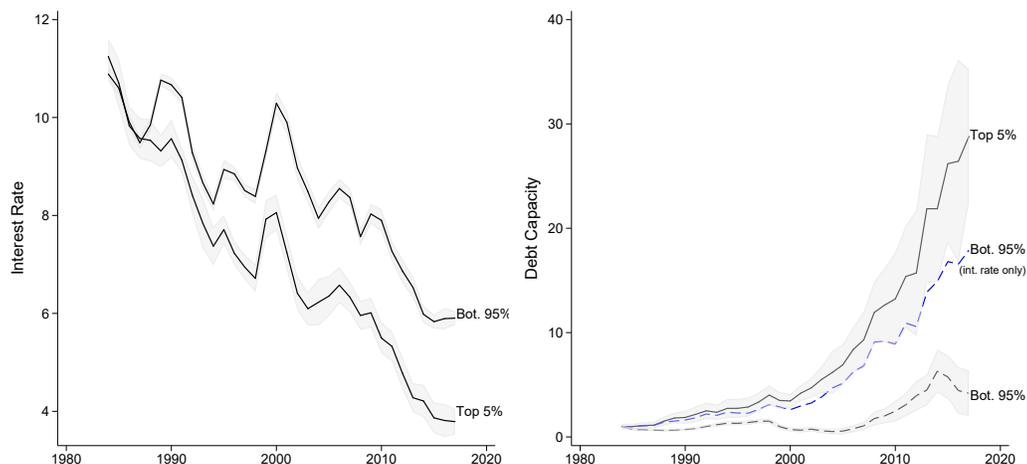
The figure shows that as the risk-free rate falls over time, the interest rate paid by the industry leaders and followers also falls. However, the spread in basis points increases as r

declines. This shows that financing costs fall less than one-for-one for industry followers relative to industry leaders. The decline in the relative cost of borrowing for industry leaders when the interest rate is low gives an additional reason to build a strategic advantage over followers.

This financing advantage that leaders enjoy can further be seen in the right panel of Figure 9. This figure plots the relative debt capacities for the median industry leader and follower in the Compustat data. Let π be EBIT for a firm, i the firm's interest rate, and D its maximum debt capacity. The maximum debt capacity D can be calculated using a minimum interest coverage ratio, k_{min} that lenders require, along with the formula $k_{min} = \frac{\pi}{i * D}$. A recent note from the Federal Reserve suggests that $k_{min} = 2$ (Palomino et al. (2019)).

The right panel of Figure 9 plots imputed debt capacity for the median industry leader and follower using the firm's interest rate and EBIT. Debt capacities are normalized to one at the beginning of sample. We can see that over time, as the long-term interest rate has fallen, the debt-capacity gap between industry leaders and followers has expanded considerably. The middle blue line shows how much of the gap is coming from the median leader and follower facing different interest rates alone. That is, it plots the follower's debt capacity assuming the follower continues to earn the leader's EBIT throughout the sample period. Figure 9 makes it clear that low interest rates have given industry leaders a large financing advantage over followers. They can use this advantage to, for example, threaten potential entrants with price wars or predatory acquisitions. All of this is assumed away in the model, but would likely strengthen the results if considered explicitly.

Figure 9: Interest rate and debt capacity for industry leaders and followers



6.3 Discussion of Model Assumptions

The key feature of the model that delivers the main result is that technological progress is incremental and follows a step-by-step process. In other words, the follower cannot “leapfrog” the leader in a single step. As explained earlier, it is the expectation of tougher competition for the follower when rates are low that discourages the follower relative to the leader as interest rates fall. For this expectation to remain relevant, investment today should bring the follower closer to leader, but it cannot allow the follower to leapfrog the leader regardless of how far back the follower is.

The condition of incremental innovation is plausible and relevant in a wide variety of contexts. Most of the innovation that happens is gradual and incremental, with each patent or scientific paper making an incremental contribution without creating a whole new paradigm. Recent empirical work by [Bloom et al. \(2020\)](#) suggests that if anything, innovation may be becoming more incremental and gradual in recent years. Moreover, low interest rates in a leapfrogging world would raise investment levels which is counter-factual.

The “no leapfrogging” condition is also realistic in that it helps to understand the real-world phenomena of market leaders conducting defensive R&D, erecting entry barriers, and engaging in predatory acquisitions. In the model, market leaders invest not only for higher flow profits but, importantly, also to acquire a strategic advantage and to prolong leadership—the main theorem holds even if a leader’s flow profit does not increase with distance, e.g. when $\pi_s = \pi_\infty > 0 \forall s \geq 1$. The model’s insight also helps to explain the ever expanding “kill zone” around industry giants’ area of influence that makes it difficult for young startups to thrive ([Cunningham et al. \(2019\)](#)). As the Economist headlined in its report on June 2, 2018, “American tech giants are making life tough for startups”. One can show that in our model, if market followers always leapfrog the leader with one successful investment, the leader no longer has the incentive to create such an empirically realistic strategic advantage. Instead, the leader invests only to acquire higher flow profits.

Nonetheless, the “no leapfrogging” condition cleanly identifies the scope and limit of the theoretical result. An economy can break-out of the low investment and low productivity equilibrium in a low interest rate environment if there appears on the horizon the possibility of investing in paradigm-shifting technology that will enable followers to leapfrog leaders (e.g., [Cabral \(2018\)](#)). However, if such paradigm-shifting opportunities are rare, or only apply to a small set of industries, the insight from the framework will continue to hold.

Finally, it is important to note that the key results are insensitive to other auxiliary fea-

tures of the model. Figure 7 shows that productivity growth converges to the same limit $\kappa \ln \lambda$ under a convex cost function, alternative profit levels, and follower cost advantage (i.e., state-dependent cost function). Figure 8 further shows that, when the rate of technology diffusion is state-dependent, growth converges to $\lim_{s \rightarrow \infty} \kappa_s \cdot \ln \lambda$. Intuitively, Theorem 1 characterizes the asymptotic equilibrium as $r \rightarrow 0$; consequently, bounded variations of κ_s in finite states do not affect firms' decisions in the limit.

6.4 Transitional Dynamics and Asset-Pricing Test

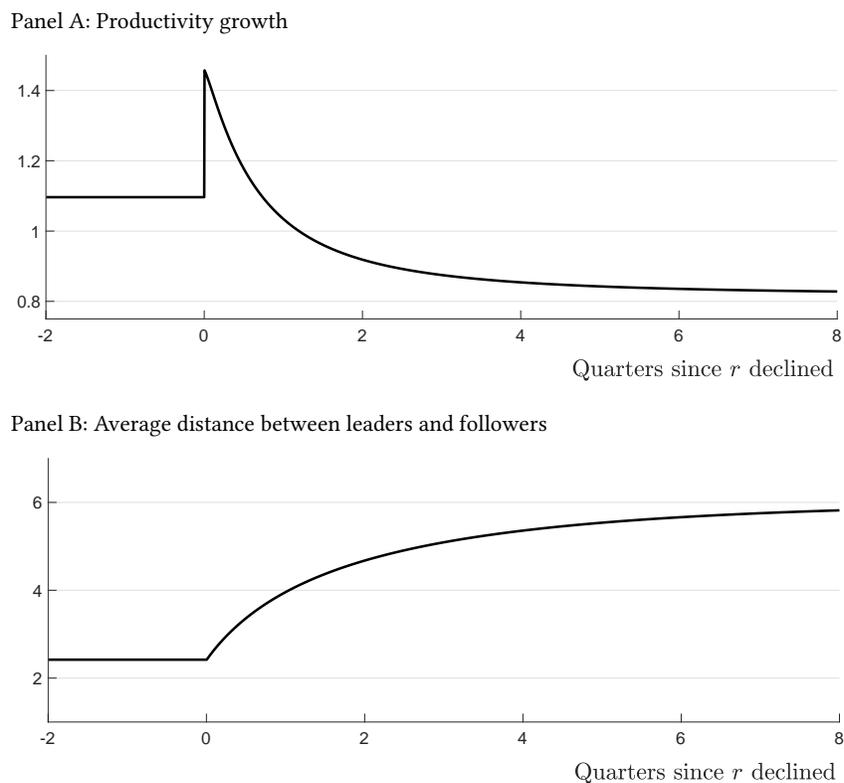
This study focuses mainly on the analysis of steady-states of the model. How long does it take for the economy to transition from one steady-state into another, following an unexpected and permanent interest rate shock? Figure 10 answers this question by showing the impulse response of a decline in the interest rate from 4% to 2%. Panel A shows the time path of productivity growth and Panel B is for the average productivity gap between leaders and followers. Starting from a steady-state, a permanent decline in the interest rate immediately moves market participants to a new equilibrium, featuring higher investments and productivity growth given any productivity gap (Panel A). The average productivity gap starts to rise, although it moves slowly (Panel B). Over time, as the distribution of the state variable converges to the new steady-state and as the average productivity gap increases, the equilibrium growth rate and investment eventually decline to the new steady-state level.

Figure 10 shows the convergence is rapid. Productivity growth is 1.1% in the initial steady-state and 0.82% in the new-steady-state; Panel A shows that it takes about 1.5 quarters for the growth rate to decline to 0.96%, closing about half of the steady-state difference. The initial boost in productivity growth lasts only 0.75 quarters, after which the growth rate declines below 1.1%.

A companion paper (Liu et al. (2020)) examines the transitional dynamics of the model for an unexpected shock to the interest rate. It shows that, starting from a steady-state with a low interest rate, an unexpected but permanent decline in the interest rate benefits industry leaders more than industry followers, and this asymmetric effect becomes stronger at lower levels of the initial interest rate. In the language of asset pricing, the model predicts that, when interest rates are low, market leaders have higher “duration”—log-sensitivity of firm valuation to the interest rate—and also higher “convexity”—the second derivative of log-valuation with respect to the interest rate.

The companion study tests this hypothesis using CRSP-Compustat merged data from 1962 onward. It constructs a “leader portfolio” that is long industry leaders and is short

Figure 10: Impulse response: reduction of interest rate from 4% to 2%



industry followers, and it examines the portfolio's performance in response to quarterly changes in interest rates. As the analysis there shows, the leader portfolio exhibits higher returns in response to a decline in interest rates for interest rates below a threshold, and this response becomes stronger at lower levels of the initial interest rate. Therefore, the model's asset pricing implications of a decline in the interest rate are supported in the data.

7 Conclusion

This study highlights a new strategic force for the determination of firm investment in productivity enhancement. This strategic force leads to an asymmetric investment response of market leaders to market followers when interest rates fall to low levels. Market leaders aggressively invest to escape competition when interest rates are low, whereas market followers become discouraged by the fierce competition that would be necessary to gain market leadership.

This strategic force delivers a unified explanation for the presence across advanced

economies of low interest rates, high market concentration, high profits, large productivity gaps between market leaders and followers, and low productivity growth. The slowdown in productivity growth has been pervasive across almost all advanced economies. The slowdown started well before the Great Recession, suggesting that cyclical forces related to the crisis are unlikely to be the trigger. Furthermore, the slowdown in productivity is highly persistent, lasting well over a decade. The long-run pattern suggests that explanations relying on price stickiness or the zero lower bound on nominal interest rates are less likely to be the complete explanation. This paper introduces the possibility of low interest rates as the common global factor that can potentially explain the slowdown in productivity growth.

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A Appendix: Proofs

A.1 Proof of claims in Sections 2 and 3

Proof of Proposition 1 The solution to HJB equations (1) through (3) imply that equilibrium investment and value functions must satisfy $\eta_s = v_{s+1} - v_s$ for $s \in \{-1, 0, 1\}$. The HJB equations can thus be re-written as

$$(r + \eta_s/2 + \eta_{-s}) v_s = \pi_s + \eta_s v_{s+1}/2 + \eta_{-s} v_{s-1} \quad \text{for } s \in \{-1, 0, 1\}. \quad (\text{A.1})$$

Substitute using $v_2 = \pi_2/r$, $v_{-2} = \pi_{-2}/r$, $v_1 = v_2 - \eta_1$, $v_0 = v_2 - \eta_1 - \eta_0$, and $v_{-1} = v_2 - \eta_1 - \eta_0 - \eta_{-1}$, the HJB equations become a system of three quadratic equations involving three endogenous variables $\{\eta_{-1}, \eta_0, \eta_1\}$ with exogenous parameters $\{\pi_s\}$ and r . That $d\eta_s/dr < 0$ follows from totally differentiating the system of equations and applying the implicit function theorem.

We prove a generalized version of the limiting result that as $r \rightarrow 0$, $\eta_1 \rightarrow \infty$, $\eta_{-1} \rightarrow \infty$, and $(\eta_1 - \eta_{-1}) \rightarrow \infty$, under a quadratic cost function with a leader disadvantage. Specifically, we define $c_s = 1$ if $s < 1$ and $c_s = c$ if $s = 1$, and we write the HJB equation for state $s \in \{-1, 0, 1\}$ as

$$r v_s = \max_{\eta} \pi_s - c_s \eta^2/2 + \eta (v_{s+1} - v_s) + \eta_{-s} (v_{s-1} - v_s).$$

The parameter c is a cost shifter for the leader. The example in Section 2 has $c = 1$. When $c > 1$, leader holds a cost disadvantage relative to the follower. We now prove the limiting result for a generic c . Note optimal investment satisfies $\eta_{-1} = v_0 - v_{-1}$, $\eta_0 = v_1 - v_0$, and $c\eta_1 = v_2 - v_1$. After substituting these expressions into the HJB equation and then taking the limit $r \rightarrow 0$, we obtain

$$v_1 \sim \frac{\eta_1 v_2 + 2\eta_{-1} v_0}{\eta_1 + 2\eta_{-1}}, \quad v_0 \sim \frac{v_1 + 2v_{-1}}{3}, \quad v_{-1} \sim \frac{\eta_{-1} v_0 + 2\eta_1 v_{-2}}{\eta_{-1} + 2\eta_1},$$

where we use $x \sim y$ to denote $\lim_{r \rightarrow 0} (x - y) = 0$. Using optimal investment decisions to substitute out v_{-1} , v_0 and v_1 , we obtain

$$c\eta_1 \sim \frac{8\eta_{-1} (v_2 - v_{-2})}{6\eta_1 + 9\eta_{-1}}, \quad \eta_{-1} \sim \frac{2\eta_1 (v_2 - v_{-2})}{6\eta_1 + 9\eta_{-1}},$$

thereby implying $c\eta_1^2 \sim 4\eta_{-1}^2$. As $r \rightarrow 0$, $v_2 - v_{-2} \rightarrow \infty$, implying that $\eta_1 \rightarrow \infty$, $\eta_{-1} \rightarrow \infty$, and $(\eta_1 - \eta_{-1}) \rightarrow \infty$ if and only if $c < 4$. In particular, when the leader does not have a cost disadvantage ($c = 1$), the difference between leader and follower investment diverges.

Proof of Lemmas 1 and 2 The CES demand within each market implies that the market share of firm i is $\delta_i \equiv \frac{p_i y_i}{p_1 y_1 + p_2 y_2} = \frac{p_i^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$. Under Bertrand competition, the price of a firm with productivity z_i must solve $p_i = \frac{\sigma(1-\delta_i) + \delta_i}{(\sigma-1)(1-\delta_i)} \lambda^{-z_i}$, with markup $m_i \equiv \frac{p_i}{\lambda^{-z_i}} = \frac{\sigma(1-\delta_i) + \delta_i}{(\sigma-1)(1-\delta_i)}$ and profits $\pi_i = \delta_i \left(\frac{p_i - \lambda^{-z_i}}{p_i} \right)$. Now define ρ_s as the relative price between leader and follower in a market with productivity gap s . Taking ratios of the prices and re-arrange, we derive that ρ_s must solve $\rho_s^\sigma = \lambda^{-s} \frac{(\sigma \rho_s^{\sigma-1} + 1)}{\sigma + \rho_s^{\sigma-1}}$. Market share is therefore $\delta_s = \frac{\rho_s^{1-\sigma}}{\rho_s^{1-\sigma} + 1}$ for the leader and $\delta_{-s} = \frac{1}{\rho_s^{1-\sigma} + 1}$ for the follower and profits are $\pi_s = \frac{1}{\sigma \rho_s^{\sigma-1} + 1}$ and $\pi_{-s} = \frac{\rho_s^{\sigma-1}}{\sigma + \rho_s^{\sigma-1}}$, respectively. Leader's markup is $m_s = \frac{\sigma + \rho_s^{1-\sigma}}{\sigma-1}$ and follower's markup is $m_{-s} = \frac{\sigma \rho_s^{1-\sigma} + 1}{(\sigma-1)\rho_s^{1-\sigma}}$.

The fact that follower's flow profits are convex in s follows from algebra. Moreover, $\lim_{s \rightarrow \infty} \rho_s^\sigma \lambda^s = 1/\sigma$; hence, for large s , $\pi_s \approx \frac{1}{\sigma \frac{1}{\lambda} \lambda^{-\frac{\sigma-1}{\sigma} s + 1}}$ and $\pi_{-s} \approx \frac{1}{\sigma \frac{2\sigma-1}{\sigma} \lambda^{\frac{\sigma-1}{\sigma} s + 1}}$. The eventual concavity of π_s and $(\pi_s + \pi_{-s})$ as $s \rightarrow \infty$ is immediate. Also note that, as $s \rightarrow \infty$, $\pi_s \rightarrow 1$, $\pi_{-s} \rightarrow 0$, $m_s \rightarrow \infty$, $m_{-s} \rightarrow 0$.

Proof of Lemma 3 The expression $g = \ln \lambda \left(\sum_{s=0}^{\infty} \mu_s \eta_s + \mu_0 \eta_0 \right)$ shows that aggregate growth is equal to $\ln \lambda$ times the weighted-average investment rate of firms at the frontier—leaders and neck-and-neck firms. In a steady-state, the growth rate of the productivity frontier must be the same as the growth rate of followers; hence, aggregate growth rate g can also be written as $g = \ln \lambda \left(\sum_{s=1}^{\infty} \mu_s (\eta_{-s} + \kappa) \right)$.

To prove the expression formally, we proceed in two steps. First, we express aggregate productivity growth as a weighted average of productivity growth in each market. We then use the fact that, given homothetic within-market demand, if a follower in state s improves productivity by one step (i.e. by a factor λ) and a leader in state $s-1$ improves also by one step, the net effect is equivalent to one step improvement in the overall productivity of a single market.

Let $p(\nu) \equiv \left[p_1(\nu)^{1-\sigma} + p_2(\nu)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$ be the price index of a single market ν . We can equivalently index for markets not using ν but instead using (s, z^F) , the productivity gap and the productivity of the follower. The growth rate g of aggregate productivity defined

in (12) is equal to $-\frac{d \ln P}{dt}$, where P is the ideal consumer price index, and can be written as:

$$g \equiv \frac{d \ln \lambda^Z}{dt} = -\frac{d \ln P}{dt} = -\frac{d \int_0^1 \ln p(\nu) d\nu}{dt} = -\sum_{s=0}^{\infty} \mu_s \times \frac{d \left[\int_{z^F} \ln p(s, z^F) dF(z^F) \right]}{dt}.$$

Now recognize that productivity growth rate in each market, $-\frac{d \ln p(s, z^F)}{d \ln t}$, is a function of only the productivity gap s and is invariant to the productivity of follower, z^F . Specifically, suppose the follower in market (s, z^F) experiences an innovation, the market price index becomes $p(s-1, z^F+1)$. If instead the leader experiences an innovation, the price index becomes $p(s+1, z^F)$. The corresponding log-changes in price indices are respectively

$$\begin{aligned} a_s^F &\equiv \ln p(s-1, z^F+1) - \ln p(s, z^F) = -\ln \lambda + \ln [\rho_{s-1}^{1-\sigma} + 1]^{\frac{1}{1-\sigma}} - \ln [\rho_s^{1-\sigma} + 1]^{\frac{1}{1-\sigma}}, \\ a_s^L &\equiv \ln p(s+1, z^F) - \ln p(s, z^F) = \ln [\rho_{s+1}^{1-\sigma} + 1]^{\frac{1}{1-\sigma}} - \ln [\rho_s^{1-\sigma} + 1]^{\frac{1}{1-\sigma}}, \end{aligned}$$

where ρ_s is the implicit function defined in the proof for Lemma 1. The log-change in price index is independent of z^F in either case. Hence, over time interval $[t, t + \Delta]$, the change in price index for markets with state variable s at time t follows

$$\Delta \ln p(s, z^F) = \begin{cases} a_s^L & \text{with probability } \eta_s \Delta, \\ a_s^F & \text{with probability } (\eta_{-s} + \kappa \cdot \mathbf{1}(s \neq 0)) \Delta. \end{cases}$$

The aggregate productivity growth can therefore be written as

$$g = -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times (\eta_s a_s^L + (\eta_{-s} + \kappa) a_s^F),$$

where $a_0 \equiv a_0^F = a_0^L$. Finally, note that if both leader and follower in a market experiences productivity improvements, regardless of the order in which these events happen, the price index in the market changes by a factor of λ^{-1} : $a_s^F + a_{s-1}^L = a_s^L + a_{s+1}^F = -\ln \lambda$ for

all $s \geq 1$. Hence,

$$\begin{aligned}
g &= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times (\eta_s a_s^L + (\eta_{-s} + \kappa) a_s^F) \\
&= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times (\eta_s a_s^L + (\eta_{-s} + \kappa) (-\ln \lambda - a_{s-1}^L)) \\
&= \ln \lambda \cdot \sum_{s=1}^{\infty} \mu_s (\eta_{-s} + \kappa) - \left(\sum_{s=1}^{\infty} \mu_s \times (\eta_s a_s^L - a_{s-1}^L (\eta_{-s} + \kappa)) + \mu_0 2\eta_0 a_0 \right).
\end{aligned}$$

Given that steady-state distribution $\{\mu_s\}$ must follow equations (10) and (11), we know

$$\sum_{s=1}^{\infty} \mu_s \times (\eta_s a_s^L - a_{s-1}^L (\eta_{-s} + \kappa)) + \mu_0 2\eta_0 a_0 = \sum_{s=1}^{\infty} \mu_s \eta_s a_s^L + \mu_0 2\eta_0 a_0 - \left(\sum_{s=1}^{\infty} \mu_s a_{s-1}^L (\eta_{-s} + \kappa) \right) = 0.$$

Hence aggregate growth rate simplifies to $g = \ln \lambda \cdot \sum_{s=1}^{\infty} \mu_s (\eta_{-s} + \kappa)$, which traces the growth rate of productivity laggards. We can also apply equations (10) and (11) again to express productivity growth as a weighted average of frontier growth: $g = \ln \lambda \cdot (\sum_{s=1}^{\infty} \mu_s \eta_s + 2\mu_0 \eta_0)$.

A.2 Proof of claims in Sections 4.2 and 4.3

Section 4 maintains the assumption that investment cost is linear, $c(\eta_s) = c \cdot \eta_s$ for $\eta_s \in [0, \eta]$. As discussed in Section 3.2, we assume the investment space is sufficiently large— $c\eta > \pi_\infty$ and $\eta > \kappa$ —so that firms can compete intensely if they choose to—and c is not prohibitively high relative to the gains from becoming a leader ($c\kappa < \pi_\infty - \pi_0$)—otherwise no firm has any incentive to ever invest.

Proof of Lemma 4 Recall $n + 1$ is the first state in which market leaders choose not to invest, and $k + 1$ is the first state in which followers choose not to invest: $n + 1 \equiv \min \{s | s \geq 0, \eta_s < \eta\}$ and $k + 1 \equiv \min \{s | s \leq 0, \eta_s < \eta\}$. Suppose $n < k$, i.e. leader invests in states 1 through n whereas follower invests in states 1 through at least $n + 1$. We first show that, if these investment decisions were optimal, the value functions of both leader and follower in state $n + 1$ must be supported by certain lower bounds. We then reach for a contradiction, showing that, if $n < k$, then market power is too transient to support these lower bounds on value functions.

The HJB equation for the leader in state $n + 2$ implies

$$\begin{aligned} rv_{n+2} &= \max_{\eta_{n+2} \in [0, \eta]} \pi_{n+2} + \eta_{n+2} (v_{n+3} - v_{n+2} - c) + (\eta_{-(n+2)} + \kappa) (v_{n+1} - v_{n+2}) \\ &\geq \pi_{n+2} + (\eta + \kappa) (v_{n+1} - v_{n+2}). \end{aligned} \quad (\text{A.2})$$

That the leader does not invest in state $n + 1$ implies $c \geq v_{n+2} - v_{n+1}$; combining with (A.2) to obtain

$$rv_{n+1} \geq \pi_{n+2} - c(\eta + \kappa + r).$$

The HJB equation for the follower in state $n + 1$ implies

$$\begin{aligned} rv_{-(n+1)} &= \max_{\eta_{-(n+1)} \in [0, \eta]} \pi_{-(n+1)} + (\eta_{-(n+1)} + \kappa) (v_{-n} - v_{-(n+1)}) - c\eta_{-(n+1)} \\ &\geq \pi_{-(n+1)} + \kappa (v_{-n} - v_{-(n+1)}). \end{aligned} \quad (\text{A.3})$$

That the follower invests in state $n + 1$ implies $c \leq v_{-n} - v_{-(n+1)}$; combining with (A.3) to obtain

$$rv_{-(n+1)} \geq \pi_{-(n+1)} + c\kappa. \quad (\text{A.4})$$

Combining this with the earlier inequality involving rv_{n+1} , we obtain an inequality on the joint value $w_{n+1} \equiv v_{n+1} + v_{-(n+1)}$:

$$rw_{n+1} \geq \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r) \quad (\text{A.5})$$

We now show that inequalities (A.4) and (A.5) cannot both be true. To do so, we construct alternative economic environments with value functions $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$ such that $\hat{w}_1^{(0)} \geq w_{n+1}$ and $\hat{v}_{-1}^{(0)} \geq v_{-(n+1)}$; we then show that even these dominating value functions $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$ cannot satisfy both inequalities.

First, fix n and fix investment strategies (leader invests until state $n + 1$ and follower invests at least through $n + 1$); suppose for all states $1 \leq s \leq n + 1$, follower's profits are equal to $\pi_{-(n+1)}$ and leader's profits are equal to π_{n+2} ; two firms each earn $\frac{\pi_{-(n+1)} + \pi_{n+2}}{2}$ in state zero. The joint profits in this modified economic environment are independent of the state by construction; moreover, the joint flow profits always weakly dominate those in the original environment and strictly dominate in state zero ($\pi_{n+2} + \pi_{-(n+1)} \geq \pi_1 + \pi_{-1} > 2\pi_0$). Let \hat{w}_s denote the value function in the modified environment; $\hat{w}_s > w_s$ for all $s \leq n + 1$.

Consider the joint value in this modified environment but under alternative investment strategies. Let \bar{n} index for investment strategies: leader invests in states 1 through \bar{n} whereas the follower invests at least through $\bar{n} + 1$. Let $\hat{w}_s^{(\bar{n})}$ denote the joint value in state s under investments indexed by \bar{n} . We argue that $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in \bar{n} . To see this, note the joint flow payoffs in all states 0 through \bar{n} is constant by construction and is equal to $x \equiv (\pi_{n+2} + \pi_{-(n+1)} - 2c\eta)$ —total profits net of investment costs—and the joint flow payoff in state $\bar{n} + 1$ is $(\pi_{n+2} + \pi_{-(n+1)} - c\eta) = x + c\eta$. $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is equal to a weighted average of x/r and $(x + c\eta)/r$, and the weight on $(x + c\eta)/r$ is higher when \bar{n} is smaller. Hence, $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in \bar{n} , and that $\hat{w}_1^{(0)} \geq \hat{w}_{n+1}^{(n)} > w_{n+1}$. The same logic also implies $\hat{v}_0^{(0)} = \frac{1}{2}\hat{w}_0^{(0)} > \frac{1}{2}w_0 = v_0$.

Consider follower's value $\hat{v}_{-1}^{(0)}$ in the alternative environment, when investment strategies are indexed by zero, i.e. firms invest in states 0 and -1 only. We know $\hat{v}_{-1}^{(0)}$ must be higher than $v_{-(n+1)}$ because

$$\hat{v}_{-1}^{(0)} = \frac{\pi_{-(n+1)} - c\eta + \kappa\hat{v}_0^{(0)}}{r + \kappa + \eta} > \frac{\pi_{-(n+1)} - c\eta + \kappa v_0}{r + \kappa + \eta} \geq \frac{\pi_{-(n+1)} - c\eta + \kappa v_{-n}}{r + \kappa + \eta} = v_{-(n+1)}.$$

We now show that the inequalities $r\hat{v}_{-1}^{(0)} \geq \pi_{-(n+1)} + c\kappa$ and $r\hat{w}_1^{(0)} \geq \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r)$ cannot both hold. We can explicitly solve for the value functions from the HJB equations:

$$\begin{aligned} \hat{w}_0^{(0)} &= \frac{\pi_{n+2} + \pi_{-(n+1)} - 2c\eta + 2\eta\hat{w}_1^{(0)}}{r + 2\eta} \\ \hat{w}_1^{(0)} &= \frac{\pi_{n+2} + \pi_{-(n+1)} - c\eta + (\eta + \kappa)\hat{w}_0^{(0)}}{r + \eta + \kappa} \\ \hat{v}_{-1}^{(0)} &= \frac{\pi_{-(n+1)} - c\eta + (\eta + \kappa)\hat{w}_0^{(0)}/2}{r + \eta + \kappa} \end{aligned}$$

Solving for $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$, we obtain

$$\begin{aligned} r\hat{w}_1^{(0)} &= \pi_{n+2} + \pi_{-(n+1)} - c\eta \left(1 + \frac{\eta + \kappa}{r + 3\eta + \kappa} \right) \\ (r + \eta + \kappa)r\hat{v}_{-1}^{(0)} &= r \left(\pi_{-(n+1)} - c\eta \right) + (\eta + \kappa) \left(\frac{\pi_{n+2} + \pi_{-(n+1)}}{2} - c\eta \frac{r + 2\eta + \kappa}{r + 3\eta + \kappa} \right) \end{aligned}$$

That $r\hat{v}_{-1}^{(0)} \geq \pi_{-(n+1)} + c\kappa$ implies

$$\begin{aligned} (r + \eta + \kappa) r\hat{v}_{-1}^{(0)} &= r (\pi_{-(n+1)} - c\eta) + (\eta + \kappa) \left(\frac{\pi_{n+2} + \pi_{-(n+1)}}{2} - c\eta \frac{r + 2\eta + \kappa}{r + 3\eta + \kappa} \right) \geq (r + \eta + \kappa) (\pi_{-(n+1)}) \\ &\implies (\eta + \kappa) \left(\frac{\pi_{n+2} - \pi_{-(n+1)}}{2} - c\eta \frac{r + 2\eta + \kappa}{r + 3\eta + \kappa} \right) \geq (r + \eta + \kappa) c\kappa + c\eta r \end{aligned}$$

Since $\frac{\pi_{n+2} - \pi_{-(n+1)}}{2} \leq \frac{\pi_{n+2}}{2} < c\eta$, it must be the case that

$$(\eta + \kappa) c\eta > (r + \eta + \kappa) c\kappa + c\eta r + (\eta + \kappa) c\eta \frac{r + 2\eta + \kappa}{r + 3\eta + \kappa}.$$

On the other hand, that $r\hat{w}_1^{(0)} \geq \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r)$ implies $r \geq \eta \frac{\eta + \kappa}{r + 3\eta + \kappa}$; hence the previous inequality implies

$$\begin{aligned} (\eta + \kappa) c\eta &> (r + \eta + \kappa) c\kappa + (\eta + \kappa) c\eta \frac{\eta}{r + 3\eta + \kappa} + (\eta + \kappa) c\eta \frac{r + 2\eta + \kappa}{r + 3\eta + \kappa} \\ &= (r + \eta + \kappa) c\kappa + (\eta + \kappa) c\eta, \end{aligned}$$

which is impossible; hence $n \geq k$.

We now show that the follower does not invest in states $s \in \{k + 1, \dots, n + 1\}$. First, note

$$\begin{aligned} (r + \eta + \kappa) (v_{-s} - v_{-s-1}) &= \pi_{-s} - \pi_{-s-1} + \kappa (v_{-s+1} - v_{-s}) + \eta (v_{-s-1} - v_{-s-2}) \\ &\quad + \max \{ \eta (v_{-s+1} - v_{-s} - c), 0 \} - \max \{ \eta (v_{-s} - v_{-s-1} - c), 0 \}. \end{aligned}$$

Suppose $v_{-s+1} - v_{-s} \geq (v_{-s} - v_{-s-1})$, then

$$\begin{aligned} (r + \eta + \kappa) (v_{-s} - v_{-s-1}) &\geq \pi_{-s} - \pi_{-s-1} + \kappa (v_{-s+1} - v_{-s}) + \eta (v_{-s-1} - v_{-s-2}) \\ &\implies (r + \eta) (v_{-s} - v_{-s-1}) \geq \pi_{-s} - \pi_{-s-1} + \eta (v_{-s-1} - v_{-s-2}). \end{aligned}$$

If $v_{-s+1} - v_{-s} < (v_{-s} - v_{-s-1})$, then

$$\begin{aligned} (r + \eta) (v_{-s} - v_{-s-1}) &< \pi_{-s} - \pi_{-s-1} + \eta (v_{-s-1} - v_{-s-2}) \\ &\quad + \max \{ \eta (v_{-s+1} - v_{-s} - c), 0 \} - \max \{ \eta (v_{-s} - v_{-s-1} - c), 0 \} \\ &\leq \pi_{-s} - \pi_{-s-1} + \eta (v_{-s-1} - v_{-s-2}). \end{aligned}$$

To summarize, for all s ,

$$v_{-s+1} - v_{-s} \geq (v_{-s} - v_{-s-1}) \iff (r + \eta)(v_{-s} - v_{-s-1}) \geq \pi_{-s} - \pi_{-s-1} + \eta(v_{-s-1} - v_{-s-2}) \quad (\text{A.6})$$

Now suppose $\eta_{-k-1} = 0$ but $\eta_{-s'} = \eta$ for some $s' \in \{k+2, \dots, n+1\}$. This implies

$$v_{-(k-1)} - v_{-k} \geq c > v_{-k} - v_{-k-1} < v_{-s'+1} - v_{-s'},$$

implying there must be at least one $s \in \{k+2, \dots, n+1\}$ such that $v_{-s+1} - v_{-s} \geq v_{-s} - v_{-s-1} < v_{-s-1} - v_{-s-2}$. Applying (A.6),

$$(r + \eta)(v_{-s} - v_{-s-1}) \geq \pi_{-s} - \pi_{-s-1} + \eta(v_{-s-1} - v_{-s-2}) \quad (\text{A.7})$$

$$(r + \eta)(v_{-s-1} - v_{-s-2}) < \pi_{-s-1} - \pi_{-s-2} + \eta(v_{-s-2} - v_{-s-3}) \quad (\text{A.8})$$

Inequality (A.7) and $v_{-s} - v_{-s-1} < v_{-s-1} - v_{-s-2}$ implies $r(v_{-s} - v_{-s-1}) > \pi_{-s} - \pi_{-s-1}$; convexity in follower's profit functions further implies $r(v_{-s} - v_{-s-1}) > \pi_{-s-1} - \pi_{-s-2}$. Substitute into inequality (A.8), and using the fact $v_{-s} - v_{-s-1} < v_{-s-1} - v_{-s-2}$, we deduce it must be the case that $(v_{-s-2} - v_{-s-3}) > (v_{-s-1} - v_{-s-2})$. Applying (A.6) again,

$$(r + \eta)(v_{-s-2} - v_{-s-3}) < \pi_{-s-2} - \pi_{-s-3} + \eta(v_{-s-3} - v_{-s-4}).$$

That $r(v_{-s-2} - v_{-s-3}) > \pi_{-s-2} - \pi_{-s-3}$ further implies $(v_{-s-3} - v_{-s-4}) > (v_{-s-2} - v_{-s-3})$. By induction, we can show $v_{s-1} - v_{s-2} < v_{s-2} - v_{s-3} < \dots < v_{-n} - v_{-(n+1)}$. But

$$(r + \eta + \kappa)(v_{-n} - v_{-(n+1)}) \leq \pi_{-n} - \pi_{-(n+1)} + \kappa(v_{-n+1} - v_{-n}) + \eta(v_{-n+1} - v_{-n+1})$$

$$\implies (r + \eta)(v_{-n} - v_{-(n+1)}) \leq \pi_{-n} - \pi_{-(n+1)}$$

which is a contradiction, given convexity of the profit functions. Hence, we have shown $v_{-k} - v_{-(k+1)} \geq v_{-s} - v_{-s-1}$ for all $s \in \{k+1, \dots, n+1\}$, establishing that follower cannot invest in these states.

Proof of Lemma 5 Given the cutoffs (n, k) , aggregate productivity growth is (from Lemma 3) $g = \ln \lambda \cdot (\sum_{s=1}^n \mu_s \eta + 2\mu_0 \eta)$. The steady-state distribution must follow

$$\mu_s \eta = \begin{cases} \mu_1 (\eta + \kappa) / 2 & \text{if } s = 0 \\ \mu_{s+1} (\eta + \kappa) & \text{if } 1 \leq s \leq k - 1 \\ \mu_{s+1} \kappa & \text{if } k \leq s \leq n + 1 \\ 0 & \text{if } s > n + 1 \end{cases} \quad (\text{A.9})$$

Hence we can rewrite the aggregate growth rate as

$$\begin{aligned} g &= \ln \lambda \cdot \left(2\mu_0 \eta + \sum_{s=1}^{k-1} \mu_s \eta + \sum_{s=k-1}^n \mu_s \eta \right) \\ &= \ln \lambda \cdot \left(\mu_1 (\eta + \kappa) + \sum_{s=2}^k \mu_s (\eta + \kappa) + \sum_{s=k}^{n+1} \mu_s \kappa \right) \\ &= \ln \lambda \cdot (\mu^C (\eta + \kappa) + \mu^M \kappa), \end{aligned}$$

as desired. To solve for μ_0 , μ^C , and μ^M as functions of n and k , we use (A.9) to write μ_s as a function of μ_{n+1} for all s . Let $\alpha \equiv \kappa/\eta$, then

$$\mu_s = \begin{cases} \mu_{n+1} \alpha^{n+1-s} & \text{if } n+1 \geq s \geq k \\ \mu_{n+1} \alpha^{n+1-k} (1 + \alpha)^{k-s} & \text{if } k-1 \geq s \geq 1 \\ \mu_{n+1} \alpha^{n+1-k} (1 + \alpha)^k / 2 & \text{if } s = 0 \end{cases}$$

Hence $\mu_0 = \mu_{n+1} \alpha^{n+1-k} (1 + \alpha)^k / 2$. The fraction of markets in the competitive and monopolistic regions can be written, respectively, as

$$\begin{aligned} \mu^M &= \mu_{n+1} \sum_{s=k+1}^{n+1} \alpha^{n+1-s} = \mu_{n+1} \frac{1 - \alpha^{n-k+1}}{1 - \alpha}, \\ \mu^C &= \mu_{n+1} \alpha^{n+1-k} \sum_{s=1}^k (1 + \alpha)^{k-s} = \mu_{n+1} \alpha^{n-k} \left((1 + \alpha)^k - 1 \right). \end{aligned}$$

Proof of Lemma 6 Given $k \geq 1$, the fraction of markets in the competitive region can be written as

$$\mu^C = \sum_{s=1}^k \mu_s = \mu_1 + \underbrace{\mu_1 (1 + \alpha)^{-1}}_{=\mu_2} + \cdots + \underbrace{\mu_1 (1 + \alpha)^{-(k-1)}}_{=\mu_k} = \underbrace{\mu_0 \frac{\kappa + \eta}{2\eta}}_{=\mu_1} \frac{1 - (1 + \alpha)^{-k}}{1 - (1 + \alpha)^{-1}} \geq \mu_0 \frac{\kappa + \eta}{2\eta}$$

Aggregate growth rate can be re-written as

$$g = \ln \lambda \cdot [(1 - \mu_0) \kappa + \mu^C \eta] \geq \ln \lambda \cdot \left[(1 - \mu_0) \kappa + \mu_0 \frac{\kappa + \eta}{2} \right] \geq \ln \lambda \cdot \kappa.$$

Aggregate investment is $I = 2\eta (\mu^C + \mu_0) + \eta (\mu^M - \mu_{n+1})$. The definition of a steady-state implies $2\eta\mu_0 + \eta (\mu^M - \mu_{n+1}) = (\eta + \kappa) \mu^C + \kappa\mu^M$, thus $I = 2\eta\mu^C + \kappa (1 - \mu_0) \geq \kappa$, as desired.

A.3 Proof of claims in Section 4.4

Consider the following recursive equations of value functions $\{u_s\}_{s=-\infty}^{\infty}$:

$$ru_{s+1} = \lambda_{s+1} + p_{s+1} (u_s - u_{s+1}) + q (u_{s+2} - u_{s+1}) \quad (\text{A.10})$$

where λ_{s+1} is the flow payoff, p_{s+1} and q are respectively the Poisson rate of transition from state $s+1$ into state s and state $s+2$. Given u_s and $\Delta u_s \equiv u_{s+1} - u_s$, we can solve for all u_{s+t} , $t > 0$ as recursive functions of u_s and Δu_s . The recursive formulation generically does not have a closed-form representation. However, as $r \rightarrow 0$, the value functions do admit asymptotic closed form expressions, as Proposition A.1 shows. In what follows, let \sim denote asymptotic equivalence as $r \rightarrow 0$, i.e. $x \sim y$ iff $\lim_{r \rightarrow 0} (x - y) = 0$.

Proposition A.1. Consider value functions $\{u_s\}_{s=-\infty}^{\infty}$ satisfying (A.10). Fix state s and integer $t > 0$.

Suppose $\lambda_{s'} \equiv \lambda$ and $p_{s'} \equiv p$ for all states $s \leq s' \leq t$. Let $\delta \equiv \frac{ru_s - \lambda}{q}$, $a \equiv \frac{p}{q}$, $b \equiv \frac{r}{q}$, then for all $t > 0$,

$$\begin{aligned} u_{s+t} - u_s &\sim (\Delta u_s) \frac{1 - a^t}{1 - a} + \delta \frac{t - \frac{a - a^t}{1 - a}}{1 - a} + \Delta u_s \cdot b \frac{(t - 1) (1 + a^t) (1 - a) - (2 - a) (a^t - a)}{(1 - a)^3} \\ &\quad + \delta b \frac{1}{(1 - a)^3} \left(\frac{(t - 2) (t - 1)}{2} (1 - a) - (t - 3) a^t - a (2 - a) (t - 1) + 2a (1 - a) \right) \end{aligned}$$

$$\begin{aligned}
u_{s+t} - u_{s+t-1} \sim & \Delta u_s a^{t-1} + \delta \frac{1 - a^{t-1}}{1 - a} + \Delta u_s b \frac{((t-1)(1+a^t) - (t-2)(1+a^{t-1}))}{(1-a)^2} \\
& - \Delta u_s b \frac{((2-a)(a^t - a^{t-1}))}{(1-a)^3} + \frac{\delta b}{(1-a)^2} \frac{(t-2)(t-1) - (t-2)(t-3)}{2} \\
& - \frac{\delta b}{(1-a)^3} (t-3)a^t + (t-4)a^{t-1} - a(2-a) \Big). \tag{A.12}
\end{aligned}$$

If $t \rightarrow \infty$ as $r \rightarrow 0$, then the formulas can be simplified as follows:

1. If $a < 1$, then $u_{s+t} - u_{s+t-1} \sim \Delta u_s a^{t-1} + \frac{\delta}{1-a} + \frac{b\Delta u_s}{(1-a)^2}$; further,

(a) if $r\Delta u_s \rightarrow 0$, then $u_{s+t} - u_s \sim \Delta u_s \frac{1}{1-a} + \frac{t\delta}{1-a}$;

(b) if $r\Delta u_s \not\rightarrow 0$, then $r(u_{s+t} - u_s) \sim \frac{r\Delta u_s}{1-a}$.

2. Suppose $a > 1$ and $r\Delta u_s \rightarrow 0$.

(a) If $\Delta u_s + \frac{\delta}{a-1} \not\sim 0$, then $r(u_{s+t} - u_s) \sim \left(\Delta u_s + \frac{\delta}{a-1}\right) \frac{ra^t}{a-1}$ and
 $r(u_{s+t} - u_{s+t-1}) \sim \left(\Delta u_s + \frac{\delta}{a-1}\right) ra^{t-1}$.

(b) If $\Delta u_s + \frac{\delta}{a-1} \sim 0$, then $u_{s+t} - u_s \sim -\frac{b\delta}{(1-a)^4} \cdot a^{t+1}$.

Suppose $\lambda_{s'}$ and $p_{s'}$ are state-dependent. Let $\lambda \geq \lambda_{s'}$ and $p \leq p_{s'}$ for all $s \leq s' \leq t$. The formulas in (A.11) and (A.12) provide asymptotic lower bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$. Conversely, if $\lambda \leq \lambda_{s'}$ and $p \geq p_{s'}$ for all $s \leq s' \leq t$, then the formulas provide asymptotic upper bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$.

Remark. Proposition A.1. expresses u_{s+t} and Δu_{s+t} as functions of u_s and Δu_s . One can also apply the Proposition write u_s and Δu_s as functions of Δu_{s+t} and u_{s+t} . Proposition A.1. thus enables us to solve for value functions asymptotically, and we apply it repeated throughout the rest of this appendix.

Proof of Proposition A.1. First suppose $\lambda_{s'} \equiv \lambda$ and $p_s \equiv p$ are constant for all states $s \leq s' \leq t$. Given u_s and Δu_s , we can solve for value functions u_{s+t} as

$$\begin{aligned}
u_{s+1} - u_s &= \Delta u_s \\
\begin{cases} u_{s+2} - u_{s+1} = a\Delta u_s + b\Delta u_s + \delta \\ u_{s+2} - u_s = (1+a)\Delta u_s + b\Delta u_s + \delta \end{cases} & \tag{A.13}
\end{aligned}$$

$$\begin{cases} u_{s+3} - u_{s+2} = a^2 \Delta u_s + (1 + 2a) b \Delta u_s + (1 + a) \delta + o(r) \\ u_{s+3} - u_s = (1 + a + a^2) \Delta u_s + (1 + 1 + 2a) b \Delta u_s + (1 + 1 + a) \delta + b \delta + o(r) \end{cases}$$

where $o(r)$ captures terms that vanishes as $r \rightarrow 0$. Applying the formula iteratively, one can show that

$$u_{s+t+1} - u_{s+t} = a^t \Delta u_s + \delta \sum_{z=0}^{t-1} a^z + b \Delta u_s \sum_{z=1}^t z a^{z-1} + b \delta \sum_{z=1}^{t-1} \sum_{m=1}^z m a^{m-1} + o(r)$$

$$u_{s+t+1} - u_s = \Delta u_s \sum_{z=0}^t a^z + \delta \sum_{z=0}^t \sum_{m=0}^{z-1} a^m + b \Delta u_s \sum_{z=1}^t \sum_{m=1}^z m a^{m-1} + b \delta \sum_{x=1}^{t-1} \sum_{z=1}^x \sum_{m=1}^z m a^{m-1} + o(r)$$

One obtains the proposition by applying the following formulas for power series summations:

1. $\sum_{z=0}^t a^z = \frac{1-a^{t+1}}{1-a}$;
2. $\sum_{z=0}^t \sum_{m=0}^{z-1} a^m = \frac{t+1-\frac{a-a^{t+1}}{1-a}}{1-a}$;
3. $\sum_{z=1}^t \sum_{m=1}^z m a^{m-1} = \frac{t(1+a^{t+1})(1-a) - (2-a)(a^{t+1}-a)}{(1-a)^3}$;
4. $\sum_{x=1}^{t-1} \sum_{z=1}^x \sum_{m=1}^z m a^{m-1} = \frac{1}{(1-a)^3} \left(\frac{t(t-1)}{2} (1-a) - (t-2) a^{t+1} - a(2-a)t + 2a(1-a) \right)$.

The third and fourth summations formulas follow because

$$\begin{aligned} \sum_{m=1}^z m a^{m-1} &= (1 + 2a + 3a^2 + \dots + z a^{z-1}) = (1 - a^z + a(1 - a^{z-1}) + \dots + a^{z-1}(1 - a)) / (1 - a) \\ &= (1 + a + \dots + a^{z-1} - z a^z) / (1 - a) = (1 - a^z - (1 - a) z a^z) / (1 - a)^2 \end{aligned}$$

$$\begin{aligned} \sum_{z=1}^s \sum_{m=1}^z m a^{m-1} &= \sum_{z=1}^s (1 - a^z - (1 - a) z a^z) / (1 - a)^2 = \left(s - (a - a^{s+1}) - a(1 - a) \sum_{z=1}^s z a^{z-1} \right) / (1 - a)^2 \\ &= (s - (a - a^{s+1}) - a((1 - a^s) / (1 - a) - s a^s)) / (1 - a)^2 \\ &= (s(1 - a) - (a(1 - a) - (1 - a) a^{s+1}) - (a - a^{s+1}) + s a^{s+1} (1 - a)) / (1 - a)^3 \\ &= (s(1 + a^{s+1})(1 - a) - (2 - a)(a^{s+1} - a)) / (1 - a)^3 \end{aligned}$$

$$\begin{aligned}
& \sum_{x=1}^{s-1} \sum_{z=1}^x \sum_{m=1}^z ma^{m-1} = \sum_{x=1}^{s-1} (x(1+a^{x+1})(1-a) - (2-a)(a^{x+1}-a)) / (1-a)^3 \\
&= \left(\sum_{x=1}^{s-1} x(1-a) + xa^{x+1}(1-a) - (2-a)(a^{x+1}-a) \right) / (1-a)^3 \\
&= \left(s(s-1)(1-a)/2 + a^2(1-a) \sum_{x=1}^{s-1} xa^{x-1} - a(2-a)(s-1) - (2-a)a^2 \frac{1-a^{s-1}}{1-a} \right) / (1-a)^3 \\
&= \left(\frac{s(s-1)}{2}(1-a) - (s-2)a^{s+1} - a(2-a)s + 2a(1-a) \right) / (1-a)^3.
\end{aligned}$$

Now suppose λ_s and p_s are state-dependent, and $\lambda \geq \lambda_{s'}$, $p \leq p_{s'}$ for all $s \leq s' \leq t$. Let $\delta_s \equiv \frac{ru_s - \lambda_s}{q}$, $a_s \equiv \frac{p_s}{q}$ and note $\delta_s > \delta \equiv \frac{ru_s - \lambda}{q}$, $a_s > a$. By re-writing equations in this proof as inequalities (e.g. rewrite (A.13) as $u_{s+2} - u_{s+1} > a\Delta u_s + b\Delta u_s + \delta$ and $u_{s+2} - u_s > (1+a)\Delta u_s + b\Delta u_s + \delta$), the formulas in the Proposition provide asymptotic lower bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$ as functions of u_s and Δu_s . Conversely, if $\lambda \leq \lambda_{s'}$ and $p \geq p_{s'}$ for all $s \leq s' \leq t$, then the formulas provide asymptotic upper bounds for $u_{s+t} - u_{s+t-1}$ and $u_{s+t} - u_s$. QED.

Proof of Lemma 7 Recall n and k are the last states in which the leader and the follower, respectively, chooses to invest in an equilibrium. Both n and k are functions of the interest rate r . Also recall that we use $w_s \equiv v_s + v_{-s}$ to denote the total firm value of a market in state s .

We first prove $\lim_{r \rightarrow 0} n = \infty$. Consider the sequence of value functions \hat{v}_s generated by an alternative sequence investment decisions: leader follows equilibrium strategies and invests in n states whereas follower does not invest in any state. Under these alternative investments, flow payoff is higher in every state, hence the joint value of both firms is higher in every state—including state 0—thus $\hat{v}_0 \geq v_0$. One can further show by induction that the alternative value functions dominate the equilibrium value functions ($\hat{v}_s \geq v_s$) for all $s \geq 0$; intuitively, leader's value is higher in any state because it expects to spend more time in higher payoff states, since the follower does not invest. Also by induction one can show $\Delta v_s \geq \Delta \hat{v}_s$ for all $s \geq 0$; intuitively, when the follower does not invest, leader has less of an incentive to invest as well.

Now suppose n is bounded, and we look for a contradiction. Let N be the smallest integer such that (1) $N > n$ for all r , and (2) $\pi_N - \pi_0 > c\kappa$. Note $rv_N = r \cdot \frac{\pi_N + \kappa v_{N-1}}{r + \kappa} \rightarrow rv_{N-1}$ as $r \rightarrow 0$; hence $rv_N \sim rv_{N-1}$. By induction, because N is finite, $rv_s \sim rv_t \sim rv_{-s}$ for any $s, t \leq N$. Likewise, $r\hat{v}_s \sim r\hat{v}_t$ for any $s, t \leq N$. The fact that leader does not

invest in state $N - 1$ implies $\lim_{r \rightarrow 0} (v_N - v_{N-1}) < c \implies \lim_{r \rightarrow 0} r v_{N-1} > \pi_N - c\kappa$, which further implies $\lim_{r \rightarrow 0} r \hat{v}_0 \geq \lim_{r \rightarrow 0} r v_0 = \lim_{r \rightarrow 0} r v_{N-1} > \pi_N - c\kappa$. Also note that $\Delta \hat{v}_0 > \Delta \hat{w}_0 = \frac{r \hat{w}_1 - (2\pi_0 - 2c\eta)}{r + 2\eta} \rightarrow \frac{r \hat{w}_0 - (2\pi_0 - 2c\eta)}{2\eta} = \frac{r \hat{v}_0 - (\pi_0 - c\eta)}{\eta}$. We now put these pieces together and apply Proposition A.1 to compute a lower bound for $\Delta \hat{v}_n$ as a function of \hat{v}_0 and $\Delta \hat{v}_0$ (substitute $u_s = \hat{v}_0$, $u_{s+t} = \hat{v}_N$, $a = \kappa/\eta$, $b = r/\eta$, $\delta = \frac{r \hat{v}_0 - (\pi_N - c\eta)}{\eta}$):

$$\begin{aligned}
\lim_{r \rightarrow 0} \Delta \hat{v}_N &\geq \lim_{r \rightarrow 0} \left(\Delta \hat{v}_0 (\kappa/\eta)^{N-1} + \frac{r \hat{v}_0 - (\pi_N - c\eta)}{\eta} \frac{1 - (\kappa/\eta)^{N-1}}{1 - \kappa/\eta} \right) \\
&> \lim_{r \rightarrow 0} \frac{r \hat{v}_0 - (\pi_0 - c\eta)}{\eta} (\kappa/\eta)^{N-1} + \frac{r \hat{v}_0 - (\pi_N - c\eta)}{\eta} \frac{1 - (\kappa/\eta)^{N-1}}{1 - \kappa/\eta} \\
&> \lim_{r \rightarrow 0} \frac{\pi_N - c\kappa - (\pi_0 - c\eta)}{\eta} (\kappa/\eta)^{N-1} + \frac{\pi_N - c\kappa - (\pi_N - c\eta)}{\eta} \frac{1 - (\kappa/\eta)^{N-1}}{1 - \kappa/\eta} \\
&> \lim_{r \rightarrow 0} c (\kappa/\eta)^{N-1} + \frac{c(\eta - \kappa)}{\eta} \frac{1 - (\kappa/\eta)^{N-1}}{1 - \kappa/\eta} \\
&= c,
\end{aligned}$$

where the last inequality follows the fact that $\pi_N - \pi_0 > c\kappa$. Thus $\lim_{r \rightarrow 0} \Delta v_N \geq \lim_{r \rightarrow 0} \Delta \hat{v}_N > c$ and the leader must invest in state N , a contradiction.

Next, suppose $\lim_{r \rightarrow 0} k = \infty$ but $(n - k)$ remain bounded. Let $\epsilon \equiv 2c\eta - \pi_\infty > 0$. The joint flow payoff $\pi_s + \pi_{-s} - 2c\eta$ is negative and bounded above by $-\epsilon$ in all states $s \leq k$. As $k \rightarrow \infty$, if $n - k$ remain bounded, then there are arbitrarily many states in which the total flow payoffs for both firms is negative and only finitely many states in which the flow payoffs may be positive. The firm value in state 0 is therefore negative. Since firms can always ensure non-negative payoffs by not taking any investment, this cannot be an equilibrium, reaching a contradiction. Hence $\lim_{r \rightarrow 0} (n - k) = \infty$.

To show $\lim_{r \rightarrow 0} k = \infty$, we first establish a few additional asymptotic properties of the model.

Lemma A.1. (1) $r v_n \sim \pi_\infty - c\kappa$; (2) $v_{n+1} - v_n \sim c$; (3) $r(n - k) \sim 0$; (4) $r k \sim 0$.

Proof. (1) The fact that leader invests in state n but not in state $n + 1$ implies

$$\frac{\pi_{n+2} - r v_{n+1}}{r + \kappa} = v_{n+2} - v_{n+1} \leq c \leq v_{n+1} - v_n = \frac{\pi_{n+1} - r v_n}{r + \kappa}$$

$$\implies \pi_\infty - c\kappa = \lim_{r \rightarrow 0} (\pi_{n+2} - c\kappa) \geq \lim_{r \rightarrow 0} r v_n \geq \lim_{r \rightarrow 0} (\pi_{n+1} - c\kappa) = \pi_\infty - c\kappa, \text{ Q.E.D.}$$

$$(2) \text{ The claim follows from the previous one: } v_{n+1} - v_n = \frac{\pi_{n+1} - r v_n}{r + \kappa} \sim \frac{\pi_\infty - r v_n}{\kappa} \sim c.$$

(3) The previous claims show $rv_n \sim \pi_\infty - c\kappa$ and $\Delta v_n \sim c$. We apply Proposition A.1 to iterate backwards and obtain a lower bound for $(v_k - v_n)$:

$$\lim_{r \rightarrow \infty} r(v_k - v_n) \geq \lim_{r \rightarrow \infty} -\frac{r^2 rv_n - (\pi_\infty - c\eta)}{\kappa^2 (1 - \eta/\kappa)^4} (\eta/\kappa)^{n-k+1} \sim -\frac{r^2 c(\eta - \kappa)}{\kappa^2 (1 - \eta/\kappa)^4} (\eta/\kappa)^{n-k+1}$$

Since $|\lim_{r \rightarrow 0} r(v_k - v_n)| \leq \pi_\infty$, $\lim_{r \rightarrow 0} r^2 (\eta/\kappa)^{n-k+1}$ must remain bounded, implying $r(n-k) \sim 0$.

(4) We apply Proposition A.1 to find a lower bound for $w_k - w_0$ (where $a \equiv \eta/\kappa > 1$):

$$\lim_{r \rightarrow 0} r(w_k - w_0) \geq \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{rw_0 - (\pi_\infty - 2c\eta)}{a-1} \right) \frac{ra^k}{a-1} \geq \lim_{r \rightarrow 0} \left(\frac{2c\eta - \pi_\infty}{a-1} \right) \frac{ra^k}{a-1}.$$

Since $r(w_k - w_0)$ is bounded, it must be that ra^k is bounded; therefore $rk \sim 0$. QED.

Lemma A.2. $rv_{-k} \sim r\Delta v_{-k} \sim rv_{-n} \sim \Delta v_{-n} \sim 0$.

Proof. First, note that follower not investing in state $k+1$ implies $c \geq \Delta v_{-(k+1)}$. We apply Proposition A.1 to find an upper bound for $(v_{-n} - v_{-k})$ as a function of rv_{-k} and $\Delta v_{-(k+1)}$: $v_{-n} - v_{-k} \leq \lim_{r \rightarrow 0} \left(-\Delta v_{-(k+1)} \frac{\eta}{\eta - \kappa} + (n-k) \frac{rv_{-k}}{\eta - \kappa} \right)$, which implies $r(v_{-n} - v_{-k}) \sim 0$. Let $m \equiv \text{floor}(\frac{n+k}{2})$. That the follower does not invest in state m implies $c \geq \Delta v_{-m}$. Proposition A.1. provides a lower bound for $v_{-(n+1)} - v_{-n}$ as a function of rv_{-m} and Δv_{-m-1} : $\lim_{r \rightarrow 0} (v_{-(n+1)} - v_{-n}) \geq \lim_{r \rightarrow 0} -\Delta v_{-(m+1)} (\kappa/\eta)^{n-m} + \frac{rv_{-m} - \pi_{-m}}{\eta - \kappa} = \lim_{r \rightarrow 0} \frac{rv_{-m}}{\eta - \kappa}$, where the equality follows from $\lim_{r \rightarrow 0} (\kappa/\eta)^{n-m} = 0$ and $\lim_{r \rightarrow 0} \pi_{-m} \rightarrow 0$. Since the LHS is non-positive, it must be the case that $\lim_{r \rightarrow 0} \Delta v_{-n} = \lim_{r \rightarrow 0} rv_{-m} = 0$. But since $rv_{-n} \leq rv_{-m}$, it must be that $rv_{-n} \sim 0$, which, together with $rv_{-n} \sim rv_{-k}$, further implies $rv_{-k} \sim 0$. That $r\Delta v_{-k} \sim 0$ follows directly from the HJB equation for state k . QED.

We now prove $\lim_{r \rightarrow 0} k = \infty$. We show k bounded $\implies rv_k \sim r\Delta w_k \sim 0$, and we look for a contradiction. First, we use the fact that $0 \leq \pi_{-s}$ for all $0 \leq s \leq k$ and apply Proposition A.1 (simplification 1a, substituting $u_s \equiv v_{-k+1}$, $u_{s+t} = v_0$, $t = k+1$, $\Delta u_s = \Delta v_{-k}$, $a = \frac{\eta}{\eta + \kappa}$, $b = \frac{r}{\eta + \kappa}$, $\delta = \frac{rv_{-(k+1)} - (-c\eta)}{\eta + \kappa}$) to find an asymptotic upper bound for rv_0 :

$$\lim_{r \rightarrow 0} rv_0 = \lim_{r \rightarrow 0} r(v_0 - v_{-(k+1)}) \leq \lim_{r \rightarrow 0} \frac{r}{1 - \kappa/\eta} \left(\Delta v_{-(k+1)} + k \frac{rv_{-(k+1)} + c\eta}{\eta} \right)$$

By Lemma A.1 part (4) and Lemma A.2, the RHS converges to 0, implying that $rw_0 \sim rw_0 \sim 0$. Further, using the HJB equation for state 0, we find that $\Delta w_0 \equiv w_1 - w_0 = \frac{rw_0 + 2c\eta - 2\pi_0}{2\eta} \sim c - \pi_0/\eta$.

Lower and upper bounds for rw_k and $r\Delta w_k$ can be found, as functions of Δw_0 and rw_0 , using Proposition A.1 (simplification 2(a), substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta + \kappa}{\eta}$, $b = \frac{r}{\eta}$, and $\delta = \frac{rw_0 - (-2c\eta)}{\eta}$ for the upper bound, $\delta = \frac{rw_0 - (\pi_\infty - 2c\eta)}{\eta}$ for the lower bound):

$$\begin{aligned} \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{rw_0 + 2c\eta - \pi_\infty}{\kappa} \right) \frac{\eta}{\kappa} r \left(\frac{\eta + \kappa}{\eta} \right)^k &\leq \lim_{r \rightarrow 0} (rw_k - rw_0) & (A.14) \\ &\leq \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{rw_0 + 2c\eta}{\kappa} \right) \frac{\eta}{\kappa} r \left(\frac{\eta + \kappa}{\eta} \right)^k \end{aligned}$$

$$\begin{aligned} \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{rw_0 + 2c\eta - \pi_\infty}{\kappa} \right) r \left(\frac{\eta + \kappa}{\eta} \right)^{k-1} &\leq \lim_{r \rightarrow 0} (r\Delta w_k) & (A.15) \\ &\leq \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{rw_0 + 2c\eta}{\kappa} \right) r \left(\frac{\eta + \kappa}{\eta} \right)^{k-1}. \end{aligned}$$

If k is bounded, these inequalities imply $rw_k \sim r\Delta w_k \sim 0$.

Now suppose $rw_k \sim r\Delta w_k \sim 0$ and we look for a contradiction. Let $\hat{k} \equiv \max\{k, N\}$ where N is the smallest integer such that $\pi_N - \pi_0 > c\kappa$. That $|N - k|$ is finite and $rw_k \sim r\Delta w_k \sim 0$ jointly imply $rw_N \sim r\Delta w_N \sim 0$. Note that $\pi_{\hat{k}}$ is a lower bound for π_s for all $n \geq s \geq \hat{k}$; we apply Proposition A.1 (simplification 1, substituting $u_s \equiv w_{\hat{k}}$, $u_{s+t} = w_{n+1}$, $t = n + 1 - \hat{k}$, $\Delta u_s = \Delta w_{\hat{k}}$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_{\hat{k}} - (\pi_{\hat{k}} - c\eta)}{\eta}$) and obtain $\frac{rw_{\hat{k}} - (\pi_{\hat{k}} - c\eta)}{\eta - \kappa}$ as an asymptotic upper bound for $w_{n+1} - w_n$. Lemma A.1 part 2 further implies that

$$\lim_{r \rightarrow 0} \frac{rw_{\hat{k}} - (\pi_{\hat{k}} - c\eta)}{\eta - \kappa} \geq c \iff \lim_{r \rightarrow 0} rw_{\hat{k}} \geq \pi_{\hat{k}} - c\kappa > 0. \quad (A.16)$$

This contradicts the presumption that $rw_{\hat{k}} \sim 0$. QED.

Note that (A.14), (A.15), and the contradiction above jointly imply $\lim_{r \rightarrow 0} rw_k > 0$ and $\lim_{r \rightarrow 0} r\Delta w_k > 0$, and that $r \left(\frac{\eta + \kappa}{\eta} \right)^k$ converges to a positive constant. We summarize these findings into a Lemma.

Lemma A.3. $\lim_{r \rightarrow 0} r \Delta w_k > 0$, and $r \left(\frac{\eta + \kappa}{\eta} \right)^k$ converges to a positive constant as $r \rightarrow 0$.

Proof of Theorem 1. We show $\lim_{r \rightarrow 0} (\kappa/\eta)^{n-k} (1 + \kappa/\eta)^k = 0$, which, based on Lemma 3, is a sufficient condition for $\mu^M \rightarrow 1$, $\mu^C \rightarrow 0$, and $g \rightarrow \kappa \cdot \ln \lambda$.

To proceed, we first find a lower bound for Δw_k by applying simplification 2 of Proposition A.1 (substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta + \kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{r w_0 - (\pi_\infty - 2c\eta)}{\eta}$):

$$\lim_{r \rightarrow 0} r \Delta w_k \geq \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{r w_0 - (\pi_\infty - 2c\eta)}{\kappa} \right) r \left(\frac{\eta + \kappa}{\eta} \right)^k. \quad (\text{A.17})$$

Simplification 1 of Proposition A.1 provides asymptotic bounds for Δw_n (substituting $u_s = w_k$, $u_{s+t} = w_n$, $t = n - k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$; the upper bound is obtained using $\delta = \frac{r w_k - (\pi_k - c\eta)}{\eta}$ and the lower bound is obtained using $\delta = \frac{r w_k - (\pi_\infty - c\eta)}{\eta}$):

$$\lim_{r \rightarrow 0} \left[\Delta w_k \left(\left(\frac{\kappa}{\eta} \right)^{n-k} + \frac{r\eta}{(\eta - \kappa)^2} \right) + \frac{r w_k + c\eta - \pi_k}{\eta - \kappa} \right] \geq \lim_{r \rightarrow 0} \Delta w_n$$

$$\lim_{r \rightarrow 0} \Delta w_n \geq \lim_{r \rightarrow 0} \left[\Delta w_k \left(\left(\frac{\kappa}{\eta} \right)^{n-k} + \frac{r\eta}{(\eta - \kappa)^2} \right) + \frac{r w_k + c\eta - \pi_\infty}{\eta - \kappa} \right].$$

Since $\lim_{r \rightarrow 0} \pi_k = \pi_\infty$, the lower and upper bounds coincide asymptotically. Furthermore, Lemma A.1 shows $\Delta w_n \sim c$; hence,

$$c \sim \Delta w_k \left(\left(\frac{\kappa}{\eta} \right)^{n-k} + \frac{r\eta}{(\eta - \kappa)^2} \right) + \frac{r w_k + c\eta - \pi_\infty}{\eta - \kappa}. \quad (\text{A.18})$$

Next, we apply simplification 1(b) of Proposition A.1 to obtain (substituting $u_s \equiv w_k$, $u_{s+t} = w_n$, $t = n - k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$; the simplification applies because $\lim_{r \rightarrow 0} r \Delta w_k > 0$, as stated in Lemma A.3): $r(w_n - w_k) \sim \frac{r \Delta w_k}{(\eta - \kappa)/\eta}$. Part 1 of Lemma A.1 further implies

$$\pi_\infty - c\kappa - r w_k \sim \frac{r \Delta w_k}{(\eta - \kappa)/\eta}. \quad (\text{A.19})$$

Substituting the asymptotic equivalence (A.19) into (A.18), we obtain

$$c \sim c + \Delta w_k \left(\left(\frac{\kappa}{\eta} \right)^{n-k} + \frac{r\eta}{(\eta - \kappa)^2} \right) - \frac{r\eta \Delta w_k}{(\eta - \kappa)^2}$$

$$\implies 0 \sim \Delta w_k (\kappa/\eta)^{n-k}.$$

Further substitute into inequality (A.17),

$$0 \geq \lim_{r \rightarrow 0} \left(\Delta w_0 + \frac{r w_0 - (\pi_\infty - 2c\eta)}{\kappa} \right) \left(\frac{\eta + \kappa}{\eta} \right)^k (\kappa/\eta)^{n-k}$$

Given $\Delta w_0 \geq 0$, $r w_0 \geq 0$, and $2c\eta - \pi_\infty > 0$, the inequality can hold if and only if $\lim_{r \rightarrow 0} \left(\frac{\eta + \kappa}{\eta} \right)^k (\kappa/\eta)^{n-k} = 0$, as desired. All other claims in Theorem 1 follows directly. QED.

Finally, the next result characterizes the relative rate of divergence between $(n - k)$ and k , as well as the rate of convergence of μ^M .

Lemma A.4. 1) $\lim_{r \rightarrow 0} \frac{n-k}{k} = \frac{2 \ln(1+\kappa/\eta)}{\ln \eta/\kappa}$; 2) $\lim_{r \rightarrow 0} \frac{1-\mu^M}{r}$ converges to a positive constant.

Proof of Lemma A.4. We first prove $\frac{n+k}{k} \sim \frac{2 \ln(1+\alpha)}{-\ln \alpha}$. Note Lemmas A.1 and A.2 jointly imply $\frac{r w_{n+1} - (\pi_\infty - c\eta)}{\eta - \kappa} \sim c \sim \Delta w_n$. We apply Proposition A.1 simplification 2(b) to find $\lim_{r \rightarrow 0} r w_k$. We substitute $u_s = w_{n+1}$, $u_{s+t} = w_k$, $\Delta u_s = w_n - w_{n+1} = -\Delta w_n$, $a = \frac{\eta}{\kappa}$, $b = \frac{r}{\kappa}$; the upper bound is obtained using $\delta = \frac{r w_{n+1} - (\pi_k - c\eta)}{\kappa}$ and the lower bound is obtained using $\delta = \frac{r w_{n+1} - (\pi_\infty - c\eta)}{\kappa}$, and that the lower and upper bounds coincide as $r \rightarrow 0$. Simplification 2(b) applies because $\Delta u_s + \frac{\delta}{a-1} \sim -c + \frac{r w_{n+1} - (\pi_\infty - c\eta)}{\kappa(\eta/\kappa - 1)} \sim 0$. Proposition A.1 implies

$$\begin{aligned} w_k - w_{n+1} &\sim -\frac{r}{\kappa(\eta/\kappa - 1)^4} \frac{c(\eta - \kappa)}{\kappa} (\eta/\kappa)^{n+1-k} \\ \implies r(w_{n+1} - w_k) &\sim \frac{c(\eta - \kappa)}{\kappa^2(\eta/\kappa - 1)^4} r^2 (\eta/\kappa)^{n+1-k} \end{aligned}$$

substitute into (A.19) $\implies r \Delta w_k \sim \varphi_1 \cdot r^2 (\eta/\kappa)^{n-k}$ for some constant $\varphi_1 > 0$.

We denote $a = \Phi(f(r))$ if $a/f(r)$ converges to a positive constant as $r \rightarrow 0$. By Lemma A.3, $\lim_{r \rightarrow 0} r \Delta w_k > 0$, hence $(\kappa/\eta)^{n-k} = \Phi(r^2)$. Lemma A.3 also states that $(1 + \kappa/\eta)^{-k} = \Phi(r)$; hence $(\eta/\kappa)^{n-k} \sim \varphi_2 (1 + \kappa/\eta)^{2k}$ for some constant $\varphi_2 > 0$, im-

plying

$$(n - k) \ln(\eta/\kappa) \sim \ln \varphi_2 + 2k \ln \left(\frac{\eta + \kappa}{\eta} \right)$$

$$\implies \frac{n - k}{2k} \sim \frac{2 \ln(1 + \kappa/\eta)}{\ln \eta/\kappa}, \quad \text{as desired.}$$

We now prove $1 - \mu^M = \Phi(r)$. By Lemma 3 and denoting $\alpha \equiv \kappa/\eta$,

$$1 - \mu^M = \frac{\alpha^{n-k} \left((1 + \alpha)^k - 1 \right) + \alpha^{n-k+1} (1 + \alpha)^k / 2}{\frac{1 - \alpha^{n-k+1}}{1 - \alpha} + \alpha^{n-k} \left((1 + \alpha)^k - 1 \right) + \alpha^{n-k+1} (1 + \alpha)^k / 2}.$$

Hence $(1 - \mu^M) \sim (\kappa/\eta)^{n-k} (1 + \kappa/\eta)^k$. But we have established above that $(\kappa/\eta)^{n-k} = \Phi(r^2)$ and $(1 + \kappa/\eta)^{-k} = \Phi(r)$; jointly, these asymptotic relationships imply $1 - \mu^M = \Phi(r)$, as desired.