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IMPLICATIONS OF TECHNOLOGY FOR GROWTH, FACTOR SHARES AND EMPLOYMENT

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ABSTRACT

We examine the concerns that new technologies will render labor redundant in a framework in which tasks previously performed by labor can be automated and new versions of existing tasks, in which labor has a comparative advantage, can be created. In a static version where capital is fixed and technology is exogenous, automation reduces employment and the labor share, and may even reduce wages, while the creation of new tasks has the opposite effects. Our full model endogenizes capital accumulation and the direction of research towards automation and the creation of new tasks. If the long-run rental rate of capital relative to the wage is sufficiently low, the long-run equilibrium involves automation of all tasks. Otherwise, there exists a stable balanced growth path in which the two types of innovations go hand-in-hand. Stability is a consequence of the fact that automation reduces the cost of producing using labor, and thus discourages further automation and encourages the creation of new tasks. In an extension with heterogeneous skills, we show that inequality increases during transitions driven both by faster automation and introduction of new tasks, and characterize the conditions under which inequality is increasing or stable in the long run.

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1 Introduction

The accelerated automation of tasks performed by labor raises concerns that new technologies will make labor redundant (e.g., Brynjolfsson and McAfee, 2012, Akst, 2014, Autor, 2015). The recent declines in the labor share in national income and the employment to population ratio in the U.S. (e.g., Karabarbounis and Neiman, 2014, and Oberfield and Raval, 2014) are often interpreted as supporting evidence for the claims that, as digital technologies, robotics and artificial intelligence penetrate the economy, workers will find it increasingly difficult to compete against machines, and their compensation will experience a relative or even absolute decline. Yet, we lack a comprehensive framework incorporating such effects, as well as potential countervailing forces.

The need for such a framework stems not only from the importance of understanding how and when automation will transform the labor market, but also from the fact that similar claims have been made, but have not always come true, about previous waves of new technologies. Keynes famously foresaw the steady increase in per capita income during the 20th century from the introduction of new technologies, but incorrectly predicted that this would create widespread technological unemployment as machines replaced human labor (Keynes, 1930). In 1965, economic historian Robert Heilbroner confidently stated that "as machines continue to invade society, duplicating greater and greater numbers of social tasks, it is human labor itself—at least, as we now think of 'labor'—that is gradually rendered redundant"(quoted in Akst, 2014). Wassily Leontief was equally pessimistic about the implications of new machines. By drawing an analogy with the technologies of the early 20th century that made horses redundant, in an interview he speculated that "Labor will become less and less important... More and more workers will be replaced by machines. I do not see that new industries can employ everybody who wants a job"(The New York Times, 1983).

This paper is a first step in developing a conceptual framework to study how machines replace human labor and why this might (or might not) lead to lower employment and stagnant wages. Our main conceptual innovation is to propose a unified framework in which tasks previously performed by labor are automated, while at the same time other new technologies complement labor—specifically, in our model this takes the form of the introduction of new tasks in which labor has a comparative advantage. Herein lies our answer to Leontief's analogy: the difference between human labor and horses is that humans have a comparative advantage in new and more complex tasks. Horses did not. If this comparative advantage is sufficiently important and the creation of new tasks continues, employment and the labor share can remain stable in the long run even in the face of rapid automation.

The importance of these new tasks is well illustrated by the technological and organizational changes during the Second Industrial Revolution, which not only involved the replacement of the stagecoach by the railroad, sailboats by steamboats, and of manual dock workers by cranes, but also the creation of new labor-intensive tasks. These tasks generated jobs for a new class of engineers, machinists, repairmen, conductors, back-office workers and managers involved with the introduction and operation of new technologies (e.g., Landes, 1969, Chandler, 1977, and Mokyr, 1990).

Today, while industrial robots, digital technologies and computer-controlled machines replace labor, we are again witnessing the emergence of new tasks ranging from engineering and programming functions to those performed by audio-visual specialists, executive assistants, data administrators and analysts, meeting planners and computer support specialists. Indeed, during the last 30 years, new tasks and new job titles account for a large fraction of U.S. employment growth. To document this fact, we use data from Lin (2011) to measure the share of new job titles—in which workers perform newer tasks than those employed in more traditional jobs—within each occupation. In 2000, about 70% of computer software developers (an occupation employing one million people at the time) held new job titles. Similarly, in 1990 radiology technician and in 1980 management analyst were new job titles. Figure 1 shows that for each decade since 1980, employment growth has been greater in occupations with more new job titles. The regression line indicates that occupations with 10 percentage points more new job titles at the beginning of each decade grow 5.05% faster over the next 10 years (standard error=1.3%). From 1980 to 2007, total employment in the U.S. grew by 17.5%. About half (8.84%) of this growth is explained by the *additional* employment growth in occupations with new job titles relative to a benchmark category with no new job titles.[1]

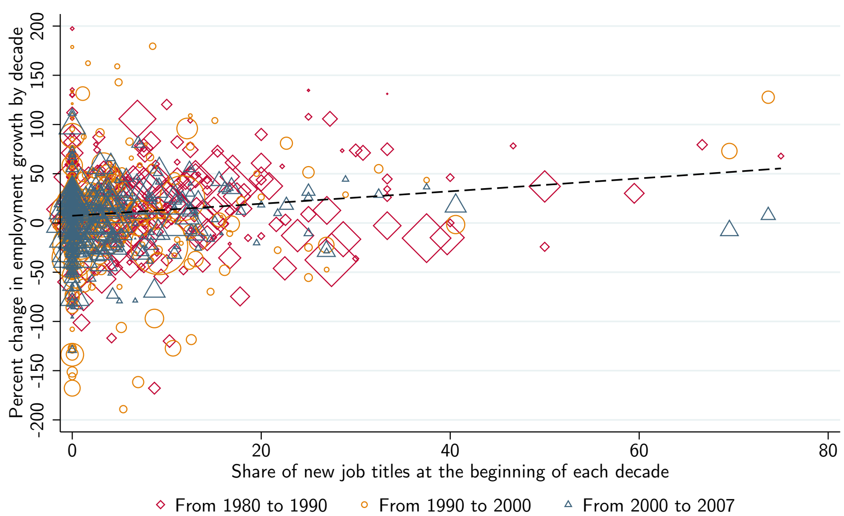

Figure 1: Employment growth by decade plotted against the share of new job titles at the beginning of each decade for 330 occupations. Data from 1980 to 1990 (in dark blue), 1990 to 2000 (in blue) and 2000 to 2007 (in light blue, re-scaled to a 10-year change). Data source: See Appendix B.

We start with a static model in which capital is fixed and technology is exogenous. There are two types of technological changes: the automation of existing tasks and the introduction of new tasks in which labor has a comparative advantage. Our static model provides a rich but tractable framework to study how automation and the creation of new tasks impact factor prices, factor

[1]The data for 1980, 1990 and 2000 are from the U.S. Census. The data for 2007 are from the American Community Survey. Additional information on the data and our sample is provided in Appendix B, where we also document in detail the robustness of the relationship depicted in Figure 1.

shares in national income and employment. Automation allows firms to substitute capital for tasks previously performed by labor, while the creation of new tasks enables the replacement of old tasks by new variants in which labor has a higher productivity. In contrast to the more commonly-used models featuring factor-augmenting technologies, here automation always reduces the labor share and employment, and may even reduce wages. Conversely, the creation of new tasks increases wages, employment and the labor share. These comparative statics follow because factor prices are determined by the range of tasks performed by capital and labor, and exogenous shifts in technology alter the range of tasks performed by each factor (see also Acemoglu and Autor, 2011).

We then embed this framework in a dynamic economy in which capital accumulation is endogenous, and we characterize restrictions under which the model delivers balanced growth with automation and creation of new tasks—which we take to be a good approximation to economic growth in the United States and the United Kingdom over the last two centuries. The key restrictions are that there is exponential productivity growth from the creation of new tasks and that the two types of technological changes—automation and the creation of new tasks—advance at equal rates. A critical difference from our static model is that capital accumulation responds to permanent shifts in technology in order to keep the interest rate and hence the rental rate of capital constant. As a result, the dynamic effects of technology on factor prices depend on the response of capital accumulation as well. The response of capital ensures that the productivity gains from both automation and the introduction of new tasks fully accrue to labor (the relatively inelastic factor). Although the real wage in the long run increases because of this productivity effect, automation always reduces the labor share.

Our full model endogenizes the rates of improvement of these two types of technologies by marrying our task-based framework with a directed technological change setup. This full version of the model remains tractable and allows a complete characterization of balanced growth paths. If the long-run rental rate of capital is very low relative to the wage, there will not be sufficient incentives to create new tasks, and the long-run equilibrium involves full automation—akin to Leontief's "horse equilibrium". Otherwise, however, the long-run equilibrium involves balanced growth based on equal advancement of the two types of technologies. Under natural assumptions, this (interior) balanced growth path is stable, so that when automation runs ahead of the creation of new tasks, market forces induce a slowdown in subsequent automation and more rapid countervailing advances in the creation of new tasks. This stability result highlights a crucial new force: a wave of automation pushes down the effective cost of producing with labor, discouraging further efforts to automate additional tasks and encouraging the creation of new tasks.

The stability of the balanced growth path implies that periods in which automation runs ahead of the creation of new tasks tend to trigger self-correcting forces, and as a result, the labor share and employment stabilize and may even return to their initial levels. Whether or not this is the case depends on the reason why automation paced ahead in the first place. If this is caused by the random arrival of a series of automation technologies, the long-run equilibrium takes us back to the same initial levels of employment and labor share. If, on the other hand, automation surges because of a change in the innovation possibilities frontier (making automation easier relative to the creation of new tasks), the economy will tend towards a new balanced growth path with lower levels of employment and labor share. In neither case does rapid automation necessarily bring about the demise of labor.[2]

We also consider three extensions of our model. First, we introduce heterogeneity in skills, and assume that skilled labor has a comparative advantage in new tasks, which we view as a natural assumption.[3] Because of this pattern of comparative advantage, automation directly takes jobs away unskilled labor who specialize in lower-indexed tasks and thus increases inequality, while new tasks directly benefit skilled workers and at first increase inequality as well. However, we also show that standardization of new tasks over time tends to help low-skill workers. This extension thus formalizes the intuitive idea that both automation and the creation of new tasks increase inequality in the short run, but also points out that standardization may limit the increase in inequality, and characterizes the conditions under which this force is sufficient to restore stable inequality in the long run. Our second extension modifies our baseline patent structure and reintroduces the creative destruction of the profits of previous innovators, which is absent in our main model, though it is often assumed in the endogenous growth literature. The results in this case are similar, but the conditions for uniqueness and stability of the balanced growth path are more demanding. Finally, we study the efficiency properties of the process of automation and creation of new technologies, and point to a new source of inefficiency leading to excessive automation: when the wage rate is above the opportunity cost of labor (due to labor market frictions), firms will choose automation to save on labor costs, while the social planner, taking into account the lower opportunity cost of labor, would have chosen less automation.

Our paper can be viewed as a combination of task-based models of the labor market with directed technological change models.[4] Task-based models have been developed both in the economic growth and labor literatures, dating back at least to Roy's (1955) seminal work. The first important recent contribution, Zeira (1998), proposed a model of economic growth based on capital-labor substitution. Zeira's model is a special case of our framework. Acemoglu and Zilibotti (2000) developed a simple task-based model with endogenous technology and applied it to the study of productivity differences across countries, illustrating the potential mismatch between new technologies and the skills of developing economies (see also Zeira, 2006, Acemoglu, 2010). Autor, Levy and Murnane (2003) suggested that the increase in inequality in the U.S. labor market reflects the automation and computerization of routine, labor-intensive tasks.[5] Our static model is most similar to Acemoglu and Autor (2011). Our full framework extends this model not only because of the dynamic

[2] Yet, it is also possible that some changes in parameters can shift us away from the region of stability to the full automation equilibrium.

[3] This assumption builds on Schultz (1965) (see also Greenwood and Yorukoglu, 1997, Caselli, 1999, Galor and Moav, 2000, Acemoglu, Gancia and Zilibotti, 2010, and Beaudry, Green and Sand, 2013).

[4] On directed technological change and related models, see Acemoglu (1998, 2002, 2003, 2007), Kiley (1999), Caselli and Coleman (2006), Gancia (2003), Thoenig and Verdier (2003) and Gancia and Zilibotti (2010).

[5] Acemoglu and Autor (2011), Autor and Dorn (2013), Jaimovich and Siu (2014), Foote and Ryan (2014), Burstein and Vogel (2012), and Burstein, Morales and Vogel (2014) provide various pieces of empirical evidence and quantitative evaluations on the importance of the endogenous allocation of tasks to factors in recent labor market dynamics.

equilibrium incorporating capital accumulation and directed technological change, but also because tasks are combined with a general elasticity of substitution, and because the equilibrium allocation of tasks critically depends both on factor prices and the state of technology.[6]

Three papers from the economic growth literature that are particularly related to our work are Acemoglu (2003), Jones (2005), and Hemous and Olson (2015). The first two papers develop growth models in which the aggregate production function is endogenous and, in the long run, adapts to make balanced growth possible. In Jones (2005), this occurs because of endogenous choices about different combinations of activities/technologies. In Acemoglu (2003), this long-run behavior is a consequence of directed technological change in a model of factor-augmenting technologies. Our task-based framework here is a significant departure from this model, especially since it enables us to address questions related to automation, its impact on factor prices and its endogenous evolution. In addition, our framework provides a more robust economic force ensuring the stability of the balanced growth path: while in models with factor-augmenting technologies stability requires an elasticity of substitution between capital and labor that is less than 1 (so that the more abundant factor commands a lower share of national income), we do not need such a condition in this framework.[7] Finally, Hemous and Olson (2015) develop a model of automation and horizontal innovation with endogenous technology, and use it to study consequences of different types of technologies on inequality. High wages (in their model for low-skill workers) encourage automation. But unlike in our model, the unbalanced dynamics that this generates are not countered by other types of innovations in the long run.[8]

The rest of the paper is organized as follows. Section 2 presents our task-based framework in the context of a static economy. Section 3 introduces capital accumulation and clarifies the conditions for balanced growth in this economy. Section 4 presents our full model with endogenous technology and establishes, under some plausible conditions, the existence, uniqueness and stability of a balanced growth path with two types of technologies advancing simultaneously. Section 5 considers the three extensions mentioned above. Section 6 concludes. Appendix A contains the proofs of our main results, while Appendix B, which is not for publication, contains the remaining proofs, some additional results and the details of the empirical analysis presented above.

[6] Acemoglu and Autor's model, like ours, is one in which a discrete number of labor types are allocated to a continuum of tasks. Costinot and Vogel (2010) develop a complementary model in which there is a continuum of skills and a continuum of tasks. See also the recent paper by Hawkins, Ryan and Oh (2015), which shows how a task-based model is more successful than standard models in matching the co-movement of investment and employment at the firm level.

[7] The role of technologies replacing tasks in this result can be seen by noting that with factor-augmenting technological changes, the impact on relative factor prices is ambiguous and the direction of innovation may be dominated by a strong market size effect (e.g., Acemoglu, 2002). Instead, in our model, the difference between factor prices regulates the future path of technological change and thus generates a powerful force that ensures stability.

[8] Kotlikoff and Sachs (2012) develop an overlapping generation model in which automation may have long-lasting effects, but this is for a very different reason—automation reduces the earnings of current workers, and via this channel, depresses their savings and capital accumulation.

2 Static Model

We start with a static version of our model with exogenous technology, which allows us to introduce our main setup in the simplest fashion and characterize the impact of different types of technological change on factor prices, employment and the labor share.

2.1 Environment

The economy produces a unique final good Y by combining a unit measure of tasks, $y(i)$, with an elasticity of substitution $\sigma \in (0, \infty)$:

$$Y = \widetilde{B} \left(\int_{N-1}^{N} y(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\widetilde{B} > 0$. All tasks and the final good are produced competitively. The fact that the limits of integration run between $N-1$ and N imposes that the measure of tasks used in production always remains at 1. A new (more complex) task replaces or upgrades the lowest-index task. Thus, an increase in N represents the upgrading of the quality (productivity) of the unit measure of tasks.[9]

Each task is produced by combining labor or capital with a task-specific intermediate $q(i)$, which embodies the technology used either for automation or for production with labor. To simplify the exposition, we start by assuming that these intermediates are supplied competitively, and that they can be produced using ψ units of the final good. Hence, they are also priced at ψ. In Section 4 we relax this assumption and allow intermediate producers to make profits so as generate endogenous incentives for innovation.

All tasks can be produced with labor. We model the technological constraints on automation by assuming that there exists $I \in [N-1, N]$ such that tasks $i \leq I$ are *technologically automated* in the sense that it is feasible to produce them with capital. Although tasks $i \leq I$ are technologically automated, whether they will be produced with capital or not depends on relative factor prices as we describe below. Conversely, tasks $i > I$ are not technologically automated, and must be produced with labor.

The production function for tasks $i > I$ takes the form

$$y(i) = \overline{B}(\zeta) \left[\eta^{\frac{1}{\zeta}} q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)^{\frac{1}{\zeta}} \left(\gamma(i) l(i) \right)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}, \tag{2}$$

where $\gamma(i)$ denotes the productivity of labor in task i, $\zeta \in (0, \infty)$ is the elasticity of substitution between intermediates and labor, $\eta \in (0,1)$ is the share parameter of this constant elasticity of substitution (CES) production function, and $\overline{B}(\zeta)$ is a constant included to simplify the algebra. In particular, we set $\overline{B}(\zeta) = \psi^{\eta}(1-\eta)^{\eta-1}\eta^{-\eta}$ when $\zeta = 1$ and $\overline{B}(\zeta) = 1$ otherwise.

[9] This formulation imposes that once a new task is created at N it will be immediately utilized and replace the lowest available task located at $N-1$. This is ensured by Assumption 3 imposed below, and avoids the need for additional notation at this point. We view newly-created tasks as higher productivity versions of existing tasks.

We are also assuming that task i is not compatible and will not be used together with tasks $i' < i - 1$ (see also footnote 19).

Tasks $i \leq I$ can be produced using labor or capital, and their production function is identical to (2) except for the presence of capital and labor as perfectly substitutable factors of production:[10]

$$y(i) = \overline{B}(\zeta)\left[\eta^{\frac{1}{\zeta}} q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)^{\frac{1}{\zeta}}\left(k(i)+\gamma(i)l(i)\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}. \tag{3}$$

Throughout, we impose the following assumption:

Assumption 1 $\gamma(i)$ *is strictly increasing*

Assumption 1 implies that labor has strict *comparative advantage* in tasks with a higher index, and will guarantee that, in equilibrium, lower-indexed tasks will be automated, while higher-indexed ones will be produced with labor.

We model the demand side of the economy using a representative household with preferences given by

$$u(C, L) = \frac{(Ce^{-\nu(L)})^{1-\theta} - 1}{1-\theta}, \tag{4}$$

where C is consumption, L denotes the labor supply of the representative household, and $\nu(L)$ designates the utility cost of labor supply, which we assume to be continuously differentiable, increasing and convex, and to satisfy $\nu''(L) + (\theta-1)(\nu'(L))^2/\theta > 0$ (which ensures that $u(C, L)$ is concave). The functional form in (4) ensures balanced growth (see King, Plosser and Rebelo, 1988; Boppart and Krusell, 2016). When we turn to the dynamic analysis in the next section, θ will be the inverse of the intertemporal elasticity of substitution.

Finally, in the static model, the capital stock, K, is taken as given (and will be endogenized via household saving decisions in Section 3).

2.2 Equilibrium in the Static Model

Given the set of technologies I and N, and the capital stock K, we now characterize the equilibrium in terms of Y, factor prices, employment and the threshold task I^*.

In the text, we simplify disposition by imposing:

Assumption 2 *One of the following two conditions holds: (i)* $\eta \to 0$*, or (ii)* $\zeta = 1$*.*

These two special cases ensure that the demand for labor and capital is homothetic. More generally, our qualitative results are identical as long as the degree of non-homotheticity is not too extreme, though in this case we no longer have closed-form expressions and this motivates our choice of presenting these more general results in Appendix A.[11]

[10] A simplifying feature of the technology described in equation (3) is that capital has the same productivity in all tasks. This assumption could be relaxed with no change to our results in the static model, but without other changes, it would not allow balanced growth in the next section. Another simplifying assumption is that non-automated tasks can be produced with just labor. Having these tasks combine labor and capital would have no impact on our main results as we show in Appendix B.

[11] The source of non-homotheticity in the general model is the substitution between factors (capital or labor) and intermediates (the $q(i)$'s). A strong substitution creates implausible features. For example, automation, which increases the productivity of capital, may end up raising the demand for labor more than the demand for capital. Assumption $2'$ in Appendix A imposes that $\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{2+2\sigma+\eta} > |\sigma-\zeta|$, and ensures that the degree of non-homotheticity is not too extreme and automation always reduces the relative demand for labor.

We proceed by characterizing the unit cost of producing each task as a function of factor prices and the automation possibilities represented by I. Because tasks are produced competitively, their price, $p(i)$, will be equal to the minimum unit cost of production:

$$p(i) = \begin{cases} \min\left\{R, \dfrac{W}{\gamma(i)}\right\}^{1-\eta} & \text{if } i \leq I, \\ \left(\dfrac{W}{\gamma(i)}\right)^{1-\eta} & \text{if } i > I, \end{cases} \tag{5}$$

where W denotes the wage rate and R denotes the rental rate of capital.

In equation (5), the unit cost of production for tasks $i > I$ is given by the effective cost of labor, $W/\gamma(i)$ (which takes into account that the productivity of labor in task i is $\gamma(i)$). The unit cost of production for tasks $i \leq I$, on the other hand, depends on $\min\left\{R, \frac{W}{\gamma(i)}\right\}$ reflecting the fact that capital and labor are perfect substitutes in the production of automated tasks. In these tasks, firms will choose whichever factor has a lower effective cost—R or $W/\gamma(i)$.

Because labor has a strict comparative advantage in tasks with a higher index, the expression for $p(i)$ implies that there is a (unique) threshold $\widetilde{I}$ such that

$$\frac{W}{R} = \gamma(\widetilde{I}). \tag{6}$$

This threshold represents the task for which the costs of producing with capital and labor are equal. For all tasks $i < \widetilde{I}$, we have $R < W/\gamma(i)$, and without any other constraints, these tasks will be produced with capital. However, if $\widetilde{I} > I$, firms cannot use capital all the way up to task $\widetilde{I}$ because of the constraint imposed by the available automation technology. This implies that there exists a unique equilibrium threshold task

$$I^* = \min\{I, \widetilde{I}\}$$

such that all tasks $i < I^*$ will be produced with capital, while all tasks $i > I^*$ will be produced with labor.[12]

Figure 2 depicts the resulting allocation of tasks to factors and also shows how, as already noted, the creation of new tasks replaces existing tasks from the bottom of the distribution.

As noted in footnote 9, we have simplified the exposition by imposing that new tasks created at N immediately replace tasks located at $N-1$, and it is therefore profitable to produce new tasks with labor (and hence we have not distinguished N, N^* and $\widetilde{N}$). In the static model, this will be the case when the capital stock is not too large, which is imposed in the next assumption.

Assumption 3 *Let $\underline{K}$ be such that $R = \frac{W}{\gamma(N)}$. We have $K < \underline{K}$.*

This assumption ensures that $R > \frac{W}{\gamma(N)}$ and consequently that new tasks will increase aggregate output and will be adopted immediately. Outside of this region, new tasks would not be utilized,

[12] Without loss of generality, we impose that firms use capital when they are indifferent between using capital or labor, which explains our convention of writing that all tasks $i \leq I^*$ (rather than $i < I^*$) are produced using capital.

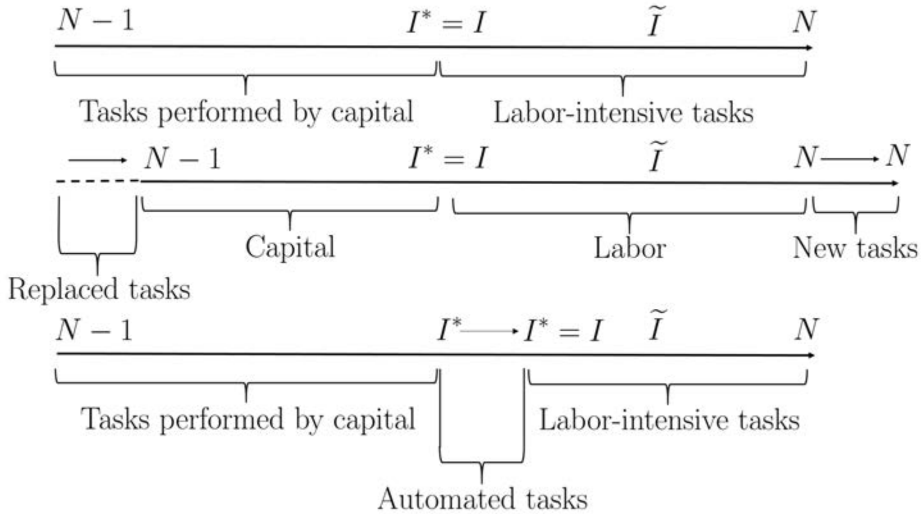

Figure 2: The task space and a representation of the effect of introducing new tasks (middle panel) and automating existing tasks (bottom panel).

which we view as the less interesting case. This assumption is relaxed in the next two sections where the capital stock is endogenous.

We next derive the demand for factors in terms of the (endogenous) threshold I^* and the technology parameter N. We choose the final good as the numeraire, which from equation (1) gives the demand for task i as

$$y(i) = \widetilde{B}^{\sigma-1} Y p(i)^{-\sigma}. \tag{7}$$

Let us define $\widehat{\sigma} = \sigma(1-\eta) + \zeta\eta$ and $B = \widetilde{B}^{\frac{\sigma-1}{\hat{\sigma}-1}}$. Then, under Assumption 2, equations (2) and (3) yield the demand for capital and labor in each task as

$$k(i) = \begin{cases} B^{\hat{\sigma}-1}(1-\eta) Y R^{-\hat{\sigma}} & \text{if } i \leq I^*, \\ 0 & \text{if } i > I^*. \end{cases} \quad \text{and } l(i) = \begin{cases} 0 & \text{if } i \leq I^*, \\ B^{\hat{\sigma}-1}(1-\eta) Y \dfrac{1}{\gamma(i)} \left(\dfrac{W}{\gamma(i)}\right)^{-\hat{\sigma}} & \text{if } i > I^*. \end{cases}$$

We can now define a *static equilibrium* as follows. Given a range of tasks $[N-1, N]$, automation technology $I \in (N-1, N]$, and a capital stock K, a static equilibrium is summarized by a set of factor prices, W and R, threshold tasks, $\widetilde{I}$ and I^*, and aggregate output, Y, such that:

- $\widetilde{I}$ is determined by equation (6) and $I^* = \min\{I, \widetilde{I}\}$;
- the capital and labor markets clear, so that

$$B^{\hat{\sigma}-1}(1-\eta) Y (I^* - N + 1) R^{-\hat{\sigma}} = K, \tag{8}$$

$$B^{\hat{\sigma}-1}(1-\eta) Y \int_{I^*}^{N} \frac{1}{\gamma(i)} \left(\frac{W}{\gamma(i)}\right)^{-\hat{\sigma}} di = L; \tag{9}$$

- factor prices satisfy the *ideal price index condition,*

$$(I^* - N + 1) R^{1-\hat{\sigma}} + \int_{I^*}^{N} \left(\frac{W}{\gamma(i)}\right)^{1-\hat{\sigma}} di = B^{1-\hat{\sigma}}; \tag{10}$$

- labor supply satisfies $\nu'(L) = W/C$. Since in equilibrium $C = RK + WL$, equilibrium labor supply is implicitly given by the increasing labor supply function:[13]

$$L = L^s\left(\frac{W}{RK}\right). \tag{11}$$

Proposition 1 (Equilibrium in the static model) *Suppose that Assumptions 1, 2 and 3 hold. Then a static equilibrium exists and is unique. In this static equilibrium, aggregate output is given by*

$$Y = \frac{B}{1-\eta}\left[(I^* - N + 1)^{\frac{1}{\hat{\sigma}}} K^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}} L^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}}. \tag{12}$$

Proof. See Appendix A. ■

Equation (12) shows that aggregate output is a CES aggregate of capital and labor, with the elasticity between capital and labor being $\hat{\sigma}$. The share parameters are endogenous and depend on the state of the two types of technologies in the economy. An increase in I^*—which corresponds to greater equilibrium automation—increases the share of capital and reduces the share of labor in this aggregate production function, while the creation of new tasks does the opposite.

Figure 3 illustrates the unique equilibrium described in Proposition 1. The equilibrium is given by the intersection of two curves in the (ω, I) space, where $\omega = \frac{W}{RK}$ is the wage level normalized by capital income; this ratio is a monotone transformation of the labor share and will play a central role in the rest of our analysis.[14] The upward-sloping curve represents the cost-minimizing allocation of capital and labor to tasks represented by equation (6), with the constraint that the equilibrium level of automation can never exceed I. The downward-sloping curve, $\omega(I^*, N, K)$, corresponds to the relative demand for labor, which can be obtained directly by differentiating equation (12) as

$$\ln\omega + \frac{1}{\hat{\sigma}}\ln L^s(\omega) = \left(\frac{1}{\hat{\sigma}} - 1\right)\ln K + \frac{1}{\hat{\sigma}}\ln\left(\frac{\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di}{I^* - N + 1}\right). \tag{13}$$

As we show in Appendix A, the relative demand curve always starts above the cost minimization condition and ends up below it, so that the two curves necessarily intersect, defining a unique equilibrium, as shown in Figure 3.

The figure also distinguishes between the two cases highlighted above. In the left panel, we have $I^* = I < \widetilde{I}$ and the allocation of factors is constrained by technology, while the right panel plots the case where $I^* = \widetilde{I} < I$ and firms choose the cost-minimizing allocation given factor prices.

A special case of Proposition 1 is also worth highlighting, because it leads to a Cobb-Douglas production function with an exponent depending on the degree of automation, which is particularly tractable in certain applications.

[13] This representation clarifies that the equilibrium implications of our setup are identical to one in which an upward-sloping quasi-labor supply determines the relationship between employment and wages (and does not necessarily equate marginal cost of labor supply to the wage). This follows readily by taking (11) to represent this quasi-labor supply relationship. Appendix B provides a non-competitive micro-foundation for such a quasi-labor supply relationship.

[14] The increasing labor supply relationship, (11), ensures that the labor share $s_L = \frac{WL}{RK+WL}$ is increasing in ω.

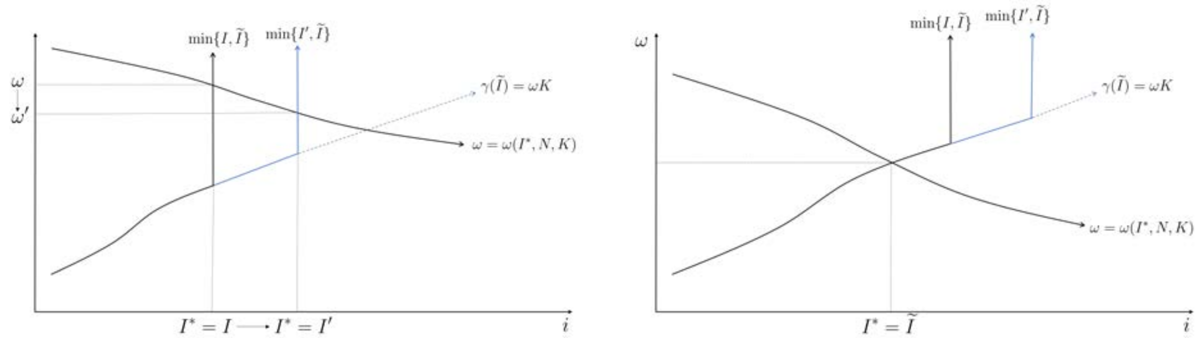

Figure 3: Static equilibrium. The left panel depicts the case in which $I^* = I < \widetilde{I}$ so that the allocation of factors is constrained by technology. The right panel depicts the case in which $I^* = \widetilde{I} < I$ so that the allocation of factors is not constrained by technology and is cost-minimizing. The blue curves show the shifts following an increase in I to I', which reduce ω in the left panel, but have no effect in the right panel.

Corollary 1 *Suppose that $\sigma = \zeta = 1$ and $\gamma(i) = 1$ for all i. Then aggregate output is*

$$Y = \frac{B}{1-\eta} K^{1-N+I^*} L^{N-I^*}.$$

The next two propositions give a complete characterization of comparative statics.[15]

Proposition 2 (Comparative statics) *Suppose that Assumptions 1, 2 and 3 hold. Let $\varepsilon_L > 0$ denote the elasticity of the labor supply schedule $L^s(\omega)$; let $\varepsilon_\gamma = \frac{d \ln \gamma(I)}{dI} > 0$ denote the semi-elasticity of the comparative advantage schedule; and let*

$$\Lambda_I = \frac{\gamma(I^*)^{\hat{\sigma}-1}}{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di} + \frac{1}{I^* - N + 1} \qquad \text{and} \qquad \Lambda_N = \frac{\gamma(N)^{\hat{\sigma}-1}}{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di} + \frac{1}{I^* - N + 1}.$$

- *If $I^* = I < \widetilde{I}$—so that the allocation of tasks to factors is constrained by technology—then:*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d \ln(W/R)}{dI} = \frac{d \ln \omega}{dI} = -\frac{1}{\hat{\sigma} + \varepsilon_L} \Lambda_I < 0, \qquad \frac{d \ln(W/R)}{dN} = \frac{d \ln \omega}{dN} = \frac{1}{\hat{\sigma} + \varepsilon_L} \Lambda_N > 0;$$

  - *and the impact of capital on relative factor prices is given by*

$$\frac{d \ln(W/R)}{d \ln K} = \frac{d \ln \omega}{d \ln K} + 1 = \frac{1 + \varepsilon_L}{\hat{\sigma} + \varepsilon_L} > 0.$$

- *If $I^* = \widetilde{I} < I$—so that the allocation of tasks to factors is cost-minimizing—then:*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d \ln(W/R)}{dI} = \frac{d \ln \omega}{dI} = 0, \qquad \frac{d \ln(W/R)}{dN} = \frac{d \ln \omega}{dN} = \frac{1}{\sigma_{free} + \varepsilon_L} \Lambda_N > 0,$$

    where

$$\sigma_{free} = \hat{\sigma} + \frac{1}{\varepsilon_\gamma} \Lambda_I > \hat{\sigma};$$

[15] In this proposition, we do not explicitly treat the case in which $I^* = I = \widetilde{I}$ in order to save on space and notation, since in this case left and right derivatives with respect to I are different.

– *and the impact of capital on relative factor prices is given by*

$$\frac{d\ln(W/R)}{d\ln K} = \frac{d\ln\omega}{d\ln K} + 1 = \frac{1+\varepsilon_L}{\sigma_{free}+\varepsilon_L} > 0.$$

- *In all cases, the labor share and employment move in the same direction as ω. In particular, $\frac{dL}{dN} > 0$ and, when $I^* = I$, $\frac{dL}{dI} < 0$.*

Proof. The proof follows by straightforward differentiation of the relevant terms and is provided in Appendix B for completeness. ■

The main implication of Proposition 2 is that the two types of technological changes—automation and the creation of new tasks—have polar implications. An increase in I—an improvement in automation technology—reduces W/R, the labor share and employment (unless $I^* = \widetilde{I} < I$ and firms are not constrained by technology in their automation choice), while an increase in N—the creation of new tasks—raises W/R, the labor share and employment.[16]

Importantly, when $I^* = I < \widetilde{I}$, automation always reduces employment. Because automation raises aggregate output per worker more than it raises the wage (automation may even reduce the equilibrium wage as we will see next), the negative income effect on labor supply resulting from the greater aggregate output dominates any substitution effect that might follow from the higher wage. On the other hand, the creation of new tasks always increase employment—new tasks raise the wage more than aggregate output, increasing labor supply. Although these exact results rely on the balanced growth preferences in equation (4), similar forces operate in general and create a tendency for automation to reduce employment and for new tasks to increase it.

These comparative static results are illustrated in Figure 3 as well: automation moves us along the relative labor demand curve in the technology-constrained case shown in the left panel (and has no impact in the right panel), while the creation of new tasks shifts out the relative labor demand curve in both cases.

A final implication of Proposition 2 is that the "technology-constrained" elasticity of substitution between capital and labor, $\hat{\sigma}$, which applies when $I^* = I < \widetilde{I}$, differs from the "technology-free" elasticity, σ_{free} , which applies when the decision of which tasks to automate is not constrained by technology (i.e., when $I^* = \widetilde{I} < I$). This is because in the former case, as relative factor prices change, the set of tasks performed by each factor remains fixed. In the latter case, as relative factor prices change, firms reassign tasks to factors. This additional margin of adjustment implies that $\sigma_{\text{free}} > \hat{\sigma}$.

Proposition 3 (Impact of technology on productivity, wages, and factor prices) *Suppose that Assumptions 1, 2 and 3 hold, and denote the changes in productivity—the change in aggregate output holding capital and labor constant—by $d\ln Y\mid_{K,L}$.*

[16]Throughout, by "automation" or "automation technology" we refer to I, and use "equilibrium automation" to refer to I^*.

- *If* $I^* = I < \widetilde{I}$*—so that the allocation of tasks to factors is constrained by technology—then* $\frac{W}{\gamma(I^*)} > R > \frac{W}{\gamma(N)}$*, and*

$$d\ln Y\mid_{K,L} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I^*)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right)dI + \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.$$

That is, both technologies increase productivity.

Moreover, let s_L *denote the labor share. The impact of technology on factor prices in this case is given by:*

$$d\ln W = d\ln Y\mid_{K,L} + (1-s_L)\left(\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN - \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI\right),$$
$$d\ln R = d\ln Y\mid_{K,L} - s_L\left(\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN - \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI\right).$$

That is, a higher N *always increases the equilibrium wage but may reduce the rental rate, while a higher* I *always increases the rental rate but may reduce the equilibrium wage. In particular, there exists* $\overline{K} > \underline{K}$ *such that an increase in* I *increases the equilibrium wage when* $K < \overline{K}$ *and reduces it when* $K > \overline{K}$*.*

- *If* $I^* = \widetilde{I} < I$*—so that the allocation of tasks to factors is not constrained by technology—then* $\frac{W}{\gamma(I^*)} = R > \frac{W}{\gamma(N)}$*, and*

$$d\ln Y\mid_{K,L} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.$$

That is, new tasks increase productivity, but additional automation technologies do not.

Moreover, the impact of technology on factor prices in this case is given by:

$$d\ln W = d\ln Y\mid_{K,L} + (1-s_L)\frac{1}{\sigma_{free}+\varepsilon_L}\Lambda_N dN$$
$$d\ln R = d\ln Y\mid_{K,L} - s_L\frac{1}{\sigma_{free}+\varepsilon_L}\Lambda_N dN.$$

That is, an increase in N *(more new tasks) always increases the equilibrium wage but may reduce the rental rate, while an increase in* I *(greater technological automation) has no effect on factor prices.*

Proof. See Appendix B. ■

The most important result in Proposition 3 is that, when $I^* = I < \widetilde{I}$, automation—an increase in I—always increases aggregate output, but has an ambiguous effect on the equilibrium wage. On the one hand, there is a positive *productivity effect* captured by the term $d\ln Y\mid_{K,L}$: by substituting cheaper capital for expensive labor, automation raises productivity, and hence the demand for labor in the tasks that are not yet automated.[17] Countering this, there is a negative *displacement effect*

[17] This discussion also clarifies that our productivity effect is similar to the productivity effect in models of offshoring such as Grossman and Rossi-Hansberg (2008), Rodriguez-Clare (2010) and Acemoglu, Gancia and Zilibotti (2015). In these models, the productivity effect results from the substitution of cheap foreign labor for domestic labor in certain tasks.

captured by the term $\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I$. This negative effect occurs because automation contracts the set of tasks performed by labor. Because tasks are subject to diminishing returns in the aggregate production function, (1), bunching workers into fewer tasks puts downward pressure on the wage. As the equation for $d\ln Y\mid_{K,L}$ reveals, the productivity gains depend on the cost savings from automation, which are given by the difference between the effective wage at I^*, $\frac{W}{\gamma(I^*)}$, and the rental rate, R. When the productivity gains are small—which is guaranteed when $K > \overline{K}$—the gap between $\frac{W}{\gamma(I^*)}$ and R is small, and so is the productivity effect; in this case, the overall impact is necessarily negative.

Finally, Proposition 3 shows that an increase in N always increases productivity and the equilibrium wage (recall that Assumption 3 imposes that $R > \frac{W}{\gamma(N)}$), and when the productivity gains from the creation of new tasks are small, it can reduce the rental rate.

The fact that automation may increase productivity while simultaneously reducing wages is a key feature of the task-based framework developed here. With factor-augmenting technologies, capital-augmenting technological improvements always increase the equilibrium wage, but this is no longer the case when technological change alters the range of tasks performed by both factors (see also Acemoglu and Autor, 2011).[18] Furthermore, with factor-augmenting technologies, whether different types of technological improvements are biased towards one factor or the other, and hence their impact on factor shares, depends on the elasticity of substitution. But this too is different in our task-based framework; here automation is *always* capital-biased (that is, it reduces both W/R and the labor share), while the creation of new tasks is *always* labor-biased (that is, it increases both W/R and the labor share).

3 Dynamics and Balanced Growth

In this section, we extend our model to a dynamic economy in which the evolution of the capital stock is determined by the saving decisions of the representative household. We then investigate the conditions under which the economy admits a balanced growth path (BGP), where aggregate output, the capital stock and wages grow at a constant rate. We conclude by discussing the long-run effects of automation on wages, the labor share and employment.

3.1 Balanced Growth

We assume that the representative household's dynamic preferences are given by

$$\int_0^\infty e^{-\rho t} u(C(t), L(t)) dt, \tag{14}$$

where $u(C(t), L(t))$ is as defined in equation (4) and $\rho > 0$ is the discount rate.

[18] For instance, with a constant returns to scale production function and two factors, capital and labor are $q-$complements. Thus, capital-augmenting technologies always increases the marginal product of labor . To see this, let $F(A_K K, A_L L)$ be such a production function. Then $W = F_L$, and $\frac{dW}{dA_K} = KF_{LK} = -LF_{LL} > 0$ (because of constant returns to scale).

To ensure balanced growth, we impose more structure to the comparative advantage schedule. Because balanced growth is driven by technology, and in this model sustained technological change comes from the creation of new tasks, constant growth requires the productivity gains from new tasks to be exponential.[19] Thus, in what follows we strengthen Assumption 1 to:

Assumption 1′ $\gamma(i)$ *satisfies:*

$$\gamma(i) = e^{Ai} \text{ with } A > 0. \tag{15}$$

The path of technology, represented by $\{I(t), N(t)\}$, is exogenous, and we define

$$n(t) = N(t) - I(t)$$

as a summary measure of technology, and similarly let $n^*(t) = N(t) - I^*(t)$ be a summary measure of the state of technology used in equilibrium (since $I^*(t) \leq I(t)$, we have $n^*(t) \geq n(t)$). New automation technologies reduce $n(t)$, while the introduction of new tasks increases it.

From equation (12), aggregate output net of intermediates, or simply "net output," can be written as a function of technology represented by $n^*(t)$ and $\gamma(I^*(t)) = e^{AI^*(t)}$, the capital stock, $K(t)$, and the level of employment, $L(t)$, as

$$F\left(K(t), e^{AI(t)}L(t); n^*(t)\right) = B\left[(1-n^*(t))^{\frac{1}{\hat{\sigma}}}K(t)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1}di\right)^{\frac{1}{\hat{\sigma}}} e^{AI^*(t)}L(t))^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}}. \tag{16}$$

The resource constraint of the economy then takes the form

$$\dot{K}(t) = F\left(K(t), e^{AI(t)}L(t); n^*(t)\right) - C(t) - \delta K(t),$$

where δ is the depreciation rate of capital.

We characterize the equilibrium in terms of the employment level $L(t)$, and the normalized variables $k(t) = K(t)e^{-AI^*(t)}$, and $c(t) = C(t)e^{\frac{1-\theta}{\theta}\nu(L(t))-AI^*(t)}$. As in our static model, $R(t)$ denotes the rental rate, and $w(t) = W(t)e^{-AI^*(t)}$ is the normalized wage. These normalized variables determine factor prices as:

$$\begin{aligned} R(t) =& F_K[k(t), L(t); n^*(t)] \\ =& B(1-n^*(t))^{\frac{1}{\hat{\sigma}}}\left[(1-n^*(t))^{\frac{1}{\hat{\sigma}}} + \left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1}di\right)^{\frac{1}{\hat{\sigma}}}\left(\frac{L(t)}{k(t)}\right)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{1}{\hat{\sigma}-1}} \end{aligned}$$

[19] Notice also that in this dynamic economy, as in our static model, the productivity of capital is the same in all automated tasks. This does not, however, imply that any of the previously automated tasks can be used regardless of N. As N increases, as emphasized by equation (1) and in footnote 9, the set of feasible tasks shifts to the right, and only tasks above $N-1$ remain compatible with and can be combined with those currently in use. Just to cite a few motivating examples for this assumption: power looms of the 18th and 19th century are not compatible with modern textile technology; assembly lines based on the dedicated machinery are not compatible with numerically controlled machines and robots; first-generation calculators are not compatible with computers; and bookkeeping methods from the 19th and 20th centuries are not compatible with the modern, computerized office.

and

$$
\begin{aligned}
w(t) =&F_L[k(t),L(t);n^*(t)]\\
=&B\left(\int_0^{n^*(t)}\gamma(i)^{\hat{\sigma}-1}di\right)^{\frac{1}{\hat{\sigma}}}\left[(1-n(t))^{\frac{1}{\hat{\sigma}}}\left(\frac{k(t)}{L(t)}\right)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}+\left(\int_0^{n(t)}\gamma(i)^{\hat{\sigma}-1}di\right)^{\frac{1}{\hat{\sigma}}}\right]^{\frac{1}{\hat{\sigma}-1}}.
\end{aligned}
$$

The equilibrium interest rate is $R(t)-\delta$.

Given time paths for $g(t)$ (the growth rate of $e^{AI(t)}$) and $n(t)$, *a dynamic equilibrium* can now be defined as a path for the threshold task $n^*(t)$, (normalize) capital and consumption, and employment, $\{k(t),c(t),L(t)\}$, that satisfies

- $n^*(t)\geq n(t)$, with $n^*(t)=n(t)$ only if $w(t)>R(t)$, and $n^*(t)>n(t)$ only if $w(t)=R(t)$;
- the endogenous labor supply condition,

$$\nu'(L(t))e^{\frac{\theta-1}{\theta}\nu(L(t))}=\frac{F_L[k(t),L(t);n^*(t)]}{c(t)}; \tag{17}$$

- the Euler equation,

$$\frac{\dot{c}(t)}{c(t)}=\frac{1}{\theta}(F_K[k(t),L(t);n^*(t)]-\delta-\rho)-g(t); \tag{18}$$

- the representative household's transversality condition,

$$\lim_{t\to\infty}k(t)e^{-\int_0^t(F_K[k(s),L(s);n^*(s)]-\delta-g(s))ds}=0; \tag{19}$$

- and the resource constraint,

$$\dot{k}(t)=F\left(k(t),L(t);n^*(t)\right)-c(t)e^{-\frac{1-\theta}{\theta}\nu(L(t))}-(\delta+g(t))k(t). \tag{20}$$

We also define a *balanced growth path* (BGP) as a dynamic equilibrium in which the economy grows at a constant rate, factor shares are constant, and the rental rate of capital $R(t)$ is constant.

To characterize the growth dynamics implied by these equations, let us first consider a path for technology such that $g(t)\to g$ and $n(t)\to n$, consumption grows at the rate g and the Euler equation holds $R(t)=\rho+\delta+\theta g$. Suppose first that $n^*(t)=n(t)=0$, in which case F becomes linear and $R(t)=B$. Because the growth rate of consumption must converge to g as well, the Euler equation (18) is satisfied in this case only if ρ is equal to

$$\overline{\rho}=B-\delta-\theta g. \tag{21}$$

Lemma A2 in Appendix A shows that this critical value of the discount rate divides the parameter space into two regions as shown in Figure 4. To the left of $\overline{\rho}$, there exists a decreasing curve $\widetilde{n}(\rho)$ defined over $[\rho_{\min},\overline{\rho}]$ with $\widetilde{n}(\overline{\rho})=0$, and to the right of $\overline{\rho}$, there exists an increasing curve $\overline{n}(\rho)$ defined over $[\overline{\rho},\rho_{\max}]$ with $\overline{n}(\overline{\rho})=0$, such that:[20]

[20]The functions $w_N(n)$ and $w_I(n)$ depicted in this figure are introduced below.

- for $n > \widetilde{n}(\rho)$, we have $\frac{w(t)}{\gamma(N(t))} < R(t)$ and new tasks raise aggregate output and are immediately produced with labor;
- for $n > \overline{n}(\rho)$, we have $w(t) > R(t)$ and automated tasks raise aggregate output and are immediately produced with capital; and
- for $n < \overline{n}(\rho)$, we have $w(t) < R(t)$ and small changes in automation do not affect n^* and other equilibrium objects.

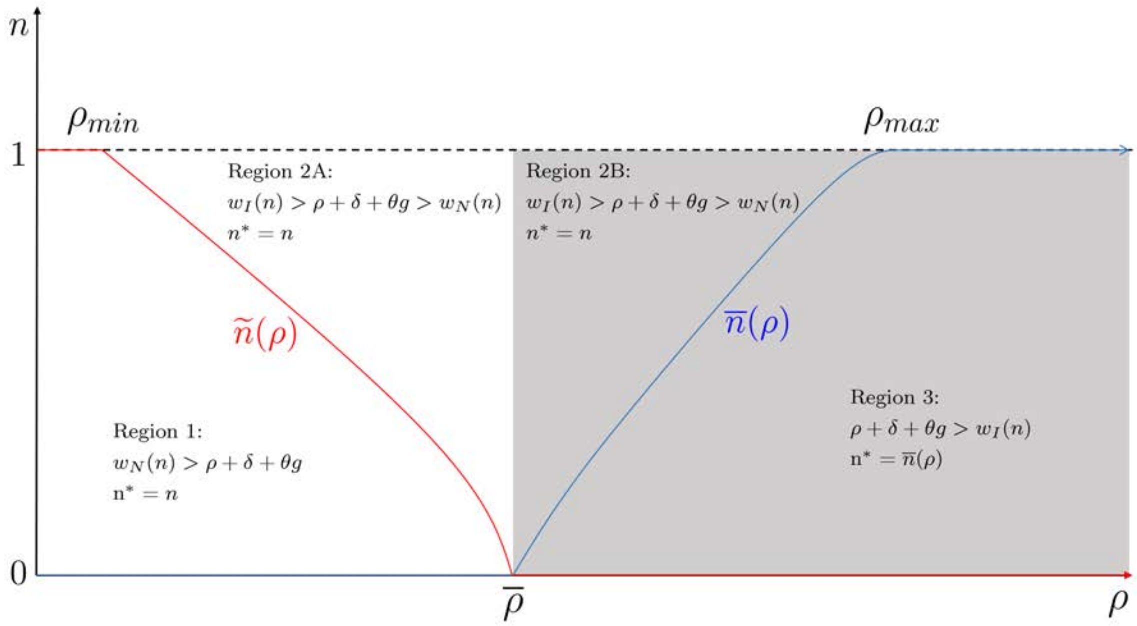

Figure 4: Behavior of factor prices in different parts of the parameter space.

The next proposition provides the conditions under which a BGP exists, and characterizes the BGP that results in each case. In what follows, we no longer impose Assumption 3, since depending on the value of ρ, the capital stock can become large and induce full automation.

Proposition 4 (Dynamic equilibrium with exogenous technological change) *Suppose that Assumptions 1′ and 2 hold. The economy admits a BGP if only if we are in one of the following cases:*

1. ***(Full automation)*** *if $\rho < \overline{\rho}$ and $N(t) = I(t)$ with $B > \delta + \rho > \frac{1-\theta}{\theta}(B - \delta - \rho) + \delta$, then there exists a unique and globally stable BGP. Moreover, in this BGP $n^*(t) = 0$, all tasks are produced with capital, and the labor share is zero;*
2. ***(Interior BGP with immediate automation)*** *if $\dot{N}(t) = \dot{I}(t) = \Delta$ with $\rho + (\theta - 1)A\Delta > 0$ and $n(t) = n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$, then there exists a unique and globally stable BGP. In this BGP $n^*(t) = n$ and $I^*(t) = I(t)$.*

3. ***(Interior BGP with eventual automation)*** *if* $\rho > \overline{\rho}$*,* $\dot{N}(t) = \Delta$ *and* $\dot{I}(t) \geq \Delta$ *with* $\rho + (\theta - 1)A\Delta > 0$*, and* $n(t) < \overline{n}(\rho)$*, then there exists a unique and globally stable BGP. In this BGP* $n^*(t) = \overline{n}(\rho)$ *and* $I^*(t) = \widetilde{I}(t) > I(t)$*.*

4. ***(No automation)*** *if* $\rho > \rho_{min}$ *and* $\dot{N}(t) = \Delta$ *with* $\rho + (\theta - 1)A\Delta > 0$*, then there exists a unique and globally stable BGP. In this BGP all tasks are produced with labor, and the capital share is zero.*

Proof. See Appendix A. ■

The first type of BGP in Proposition 4 involves the automation of all tasks, in which case aggregate output becomes linear in capital. This case was ruled out by Assumption 3 in our static analysis, but as the proposition shows, when the discount rate, ρ, is sufficiently small, it can emerge in the dynamic model. A BGP with no automation (case 4), where growth is driven entirely by the creation of new tasks, is also possible if the discount rate is sufficiently large.

More important for our focus are the two interior BGPs where automation and the introduction of new tasks go hand-in-hand, and as a result, $n^*(t)$ is constant at some value between 0 and 1; this implies that both capital and labor perform a fixed measure of tasks. In the more interesting case where automated tasks are immediately produced with capital (case 2), the proposition also highlights that this process needs to be "balanced" itself: the two types of technologies need to advance at the same rate so that $n(t) = n$.

Balanced growth with constant labor share emerges in this model because the net effect of automation and the creation of new technologies proceeding at the same rate is to augment labor while keeping constant the share of tasks performed by labor—as shown by equation (16). In this case, the gap between the two types of technologies, $n(t)$, regulates the share parameters in the resulting CES production function, while the levels of $N(t)$ and $I(t)$ determine the productivity of labor in the set of tasks that it performs. When $n(t) = n$, technology becomes purely labor augmenting on net because labor performs a fixed share of tasks, and labor becomes more productive over time in producing the newly-created tasks.[21]

To illustrate the main implication of the proposition, let us focus on part 2 with $\dot{I} = \dot{N} = \Delta$ and $n(t) = n \geq \overline{n}(\rho)$. Along such a path, $n^*(t) = n$ and $g(t) = A\Delta$. Figure 5 presents the phase diagram for the system of differential equations comprising the Euler equation (equation (18)) and the resource constraint (equation (20)). This system of differential equations determines the structure of the dynamic equilibrium and is identical to that of the neoclassical growth model with pure labor-augmenting technological change and endogenous labor supply (which makes the locus for $\dot{c} = 0$ downward-sloping because of the negative income effect on labor supply).

[21]This intuition connects Proposition 4 to Uzawa's Theorem, which implies that balanced growth requires a representation of the production function with purely labor-augmenting technological change (e.g., Acemoglu, 2009, or Grossman, Helpman and Oberfield, 2016) as the one that we obtain when $n(t) = n$.

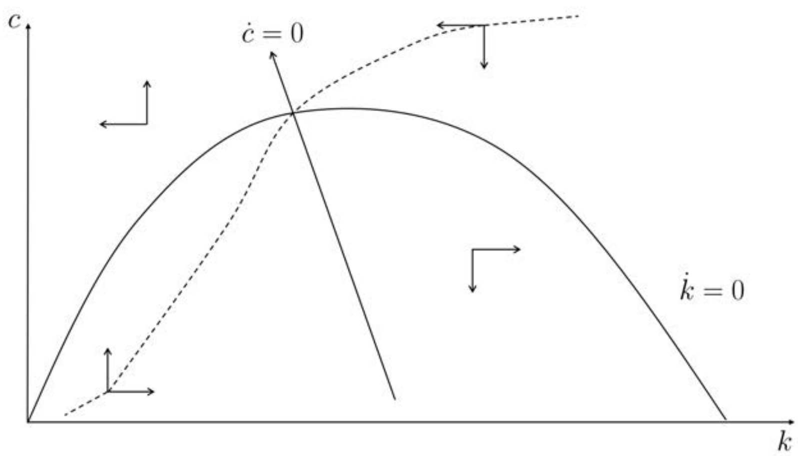

Figure 5: Dynamic equilibrium when technology is exogenous and satisfies $n(t) = n$ and $g(t) = A\Delta$.

3.2 Long-Run Comparative Statics

We next study the log-run implications of an unanticipated and permanent decline in $n(t)$, which corresponds to automation running ahead of the creation of new tasks. Because in the short run capital is fixed, the short-run implications of this change in technology are the same as in our static analysis in the previous section. But the fact that capital adjusts implies different long-run dynamics.

Consider an interior BGP in which $N(t) - I(t) = n \in (0, 1)$. Along this path, the equilibrium wage grows at the rate $A\Delta$. Define $w_I(n) = \lim_{t\to\infty} W(t)/\gamma(I^*(t))$ as the effective wage paid in the least complex task produced with labor and $w_N(n) = \lim_{t\to\infty} W(t)/\gamma(N(t))$ as the effective wage paid in the most complex task produced with labor. Both of these functions are defined well-defined and thus depend only on n. Figure 4 shows how these effective wages compare to the BGP value of the rental rate of capital, $\rho + \delta + \theta g$.

The next proposition characterizes the long-run impact of automation on factor prices, employment and the labor share in the interior BGPs.

Proposition 5 (Long-run comparative statics) *Suppose that Assumptions 1′ and 2 hold. Consider a path for technology in which $n(t) = n \in (0, 1)$ and $g(t) = g$ (so that we are in case 2 or 3 in Proposition 4). Then, in the unique BGP we have $R(t) = \rho + \delta + \theta g$, and*

- *for $n < \overline{n}(\rho)$, $w_I(n) = w_I(\overline{n}(\rho))$ and $w_N(n) = w_N(\overline{n}(\rho))$. Consequently, small changes in n do not affect the paths of effective wages, employment and the labor share.*
- *For $n > \overline{n}(\rho)$, $w_I(n)$ is increasing and $w_N(n)$ is decreasing in n. Moreover, the asymptotic values for employment and the labor share are increasing in n. Finally, if the increase in n is caused by an increase in I, the capital stock also increases.*

Proof. The proof is straightforward and is presented in Appendix B for completeness. ■

We discuss this proposition for $n > \overline{n}(\rho)$, so that we are in the most interesting region of the parameter space where $I^* = I$ and the level of automation is constrained by technology. The key implication in this case is that the long-run implications of automation are very different than its short-term impact. Automation reduces employment and the labor share in the long run, but always increases the long-run equilibrium wage. This result follows because $w_N(n)$ is decreasing in n, and thus a lower n implies a higher wage level;[22] but because $w_I(n)$ is increasing in n, this increase is less than proportional to $\gamma(I(t))$. Unlike the wage, the rental rate of capital reverts to its initial value of $\rho + \delta + \theta g$, which is pinned down by the Euler equation, (18), regardless of the extent of automation.

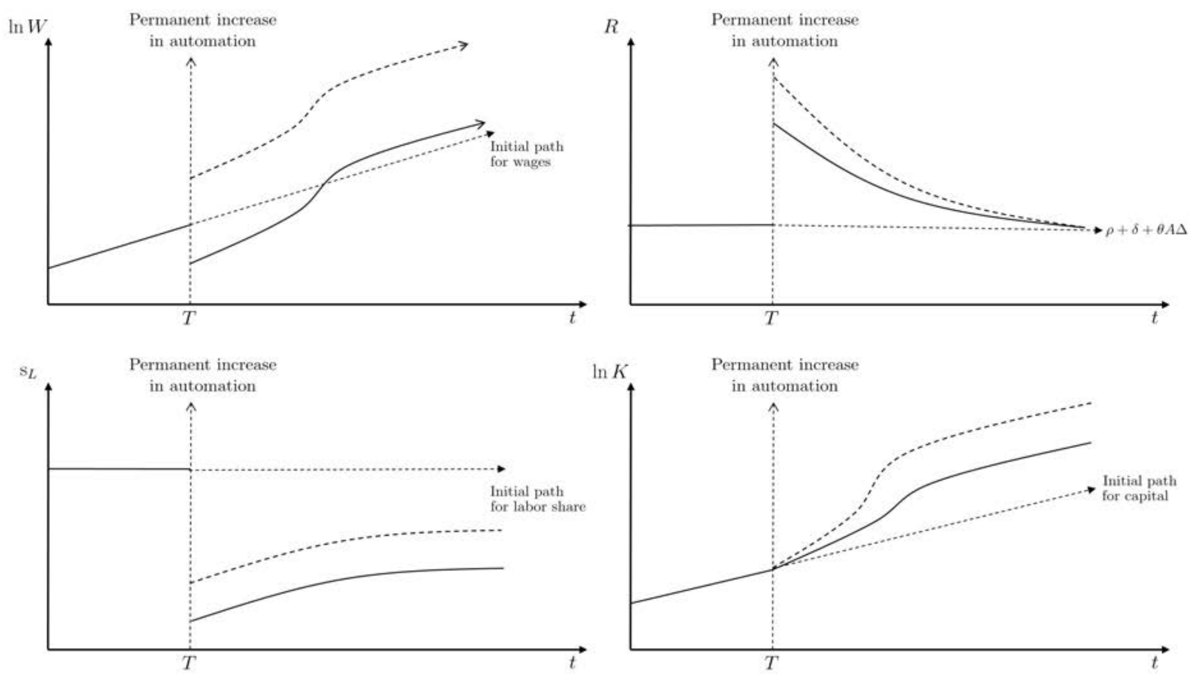

Figure 6: Dynamic behavior of wages ($\ln W$), the rental rate of capital (R), the labor share (s_L), and the capital stock following a permanent increase in automation.

Figure 6 illustrates the response of the economy to permanent changes in automation. It plots two potential paths for all variables. The dotted line depicts the case where $w_I(n)$ is large relative to R, so that there are significant productivity gains from automation. In this case, an increase in automation raises the wage immediately, followed by further increases in the long run. The solid line depicts the dynamics when $w_I(n) \approx R$, so that the productivity gains from automation are very small. In this case, an increase in automation reduces the wage in the short run and leaves

[22]This result can also be understood by noting that the ideal price index condition implies

$$d \ln Y \mid_{K,L} = s_L d \ln W + (1 - s_L) d \ln R.$$

In general, productivity gains from technological change accrue to both capital and labor. In the BGP, however, capital adjusts to keep the rental rate fixed at $R = \rho + \delta + \theta g$, and as a result, $d \ln W = \frac{1}{s_L} d \ln Y \mid_{K,L} > 0$, meaning that productivity gains accrue only to the inelastic factor—labor.

it approximately unchanged in the long run. In contrast to the concerns that highly productive automation technologies will reduce the wage and employment, our model thus shows that it is precisely when automation fails to raise productivity significantly that it has a more detrimental impact on wages and employment. In both cases, the duration of the period with stagnant or depressed wages depends on θ, which determines the speed of capital adjustment following an increase in the rental rate.

The remaining panels of the figure show that automation reduces employment and the labor share, as stated in Proposition 5. If $\hat{\sigma} < 1$, the resulting capital accumulation mitigates the short-run decline in the labor share but does not fully offset it (this is the case depicted in the figure). If $\hat{\sigma} > 1$, capital accumulation further depresses the labor share—even though it raises the wage.

The long-run impact of a permanent increase in $N(t)$ can also be obtained from the proposition. In this case, new tasks increase the wage (because $w_I(n)$ is increasing in n), aggregate output, employment, and the labor share, both in the short and the long run. Because the short-run impact of new tasks on the rental rate of capital is ambiguous, so is the response of capital accumulation.

In light of these results, the recent decline in the labor share and the employment to population ratio in the United States can be interpreted as a consequence of automation outpacing the creation of new labor-intensive tasks. Faster automation relative to the creation of new tasks might be driven by an acceleration in the rate at which $I(t)$ advances, in which case we would have stagnant or lower wages in the short run while capital adjusts to a new higher level. Alternatively, it might be driven by a deceleration in the rate at which $N(t)$ advances, in which case we would also have low growth of aggregate output and wages. We return to the productivity implications of automation once we introduce our full model with endogenous technological change in the next section.

4 Full Model: Tasks and Endogenous Technologies

The previous section established, under some conditions, the existence of an interior BGP with $\dot{N} = \dot{I} = \Delta$. This result raises a more fundamental question: why should these two types of technologies advance at the same rate? To answer this question we now develop our full model, which endogenizes the pace at which automation and the creation of new tasks proceeds.

4.1 Endogenous and Directed Technological Change

To endogenize technological change, we deviate from our earlier assumption of a perfectly competitive market for intermediates, and assume that (intellectual) property rights to each intermediate, $q(i)$, are held by a technology monopolist which can produce it at the marginal cost $\mu\psi$ in terms of the final good, where $\mu \in (0, 1)$ and $\psi > 0$. We also assume that this technology can be copied by a fringe of competitive firms, which can replicate any available intermediate at a higher marginal cost of ψ, and that μ is such that the unconstrained monopoly price of an intermediate is greater than ψ. This ensures that the unique equilibrium price for all types of intermediates is a limit price of ψ, and yields a per unit profit of $(1-\mu)\psi > 0$ for technology monopolists. These profits generate incentives for creating new tasks and automation technologies.

In this section, we adopt a structure of intellectual property rights that abstracts from the creative destruction of profits. We assume that developing a new intermediate that automates or replaces an existing task is viewed as an infringement of the patent of the technology previously used to produce that task. Consequently, a firm must compensate the technology monopolist who owns the property rights over the production of the intermediate that it is replacing. We also assume that this compensation takes place with the new inventors making a take-it-or-leave-it offer to the holder of the existing patent.

Developing new intermediates that embody technology requires scientists.[23] There is a fixed supply of S scientists, which will be allocated to automation ($S_I(t) \geq 0$) or the creation of new tasks ($S_N(t) \geq 0$),

$$S_I(t) + S_N(t) \leq S.$$

When a scientist is employed in automation, she automates κ_I tasks per unit of time and receives a wage $W_I^S(t)$. When she is employed in the creation of new tasks, she creates κ_N new tasks per unit of time and receives a wage $W_N^S(t)$. We assume that automation and the creation of new tasks proceed in the order of the task index i. Thus, the allocation of scientists determines the evolution of both types of technology—summarized by the level of $I(t)$ and $N(t)$—as

$$\dot{I}(t) = \kappa_I S_I(t), \text{ and } \dot{N}(t) = \kappa_N S_N(t). \tag{22}$$

Because we want to analyze the properties of the equilibrium locally, we make a final assumption to ensure that the allocation of scientists varies smoothly when there is a small difference between $W_I^S(t)$ and $W_N^S(t)$ (rather than having discontinuous jumps). In particular, we assume that scientists differ in the cost of effort: when working in automation, scientist j incurs a cost of $\chi_I^j Y(t)$, and when working in the creation of new tasks, she incurs a cost of $\chi_N^j Y(t)$.[24] Consequently, scientist j will work in automation if $\frac{W_I^S(t) - W_N^S(t)}{Y(t)} > \chi_N^j - \chi_I^j$. We also assume that the distribution of $\chi_N^i - \chi_I^i$ among scientists is given by a smooth and increasing distribution function G over a support $[-\upsilon, \upsilon]$, where we take υ to be small enough that χ_N^j and χ_S^j are always less than $\max\left\{\frac{\kappa_N V_N(t)}{Y(t)}, \frac{\kappa_I V_I(t)}{Y(t)}\right\}$ and thus all scientists always work. For notational convenience, we also adopt the normalization $G(0) = \frac{\kappa_N}{\kappa_I + \kappa_N}$.

4.2 Equilibrium with Endogenous Technological Change

We first compute the present discounted value accruing to monopolists from automation and the creation of new tasks. Let $V_I(t)$ denote the value of automating task $i = I(t)$ (i.e., the highest-indexed task that has not yet been automated, or more formally $i = I(t) + \varepsilon$ for ε arbitrarily small and positive). Likewise, $V_N(t)$ is the value of a new technology creating a new task at $i = N(t)$.

[23]Focusing on an innovation possibilities frontier using just scientists, rather than variable factors such as in the lab-equipment specifications (see Acemoglu 2009), is convenient because it enables us to focus on the direction of technological change—and not on the overall amount of technological change.

[24]The cost of effort is multiplied by $Y(t)$ to capture the income effect on the costs of effort in a tractable manner.

To simplify the exposition, let us assume that in this equilibrium $n(t) > \max\{\bar{n}(\rho), \tilde{n}(\rho)\}(\rho)$, so that $I^*(t) = I(t)$ and newly-automated tasks start being produced with capital immediately. The flow profits that accrue to the technology monopolist that automated task i are:[25]

$$\pi_I(t,i) = bY(t)R(t)^{\zeta-\hat{\sigma}}, \tag{23}$$

where $b = (1-\mu)B^{\hat{\sigma}-1}\eta\psi^{1-\zeta}$. Likewise, the flow profits that accrue to the technology monopolist who created the labor-intensive task i are:

$$\pi_N(t,i) = bY(t)\left(\frac{W(t)}{\gamma(i)}\right)^{\zeta-\hat{\sigma}}. \tag{24}$$

The take-it-or-leave-it nature of offers implies that a firm that automates task I needs to compensate the existing technology monopolist by paying her the present discounted value of the profits that her inferior labor-intensive technology would generate if not replaced. This take-it-or-leave-it offer is given by:[26]

$$b\int_t^\infty e^{-\int_0^\tau (R(s)-\delta)ds}Y(\tau)\left(\frac{W(\tau)}{\gamma(I)}\right)^{\zeta-\hat{\sigma}}d\tau.$$

Likewise, a firm that creates task N needs to compensate the existing technology monopolist by paying her the present discounted value of the profits from the capital-intensive alternative technology. This take-it-or-leave-it offer is given by:

$$b\int_t^\infty e^{-\int_0^\tau (R(s)-\delta)ds}Y(\tau)R(\tau)^{\zeta-\hat{\sigma}}d\tau.$$

In both cases, the patent-holders will immediately accept theses offers and reject less generous ones.

We can then compute the values of innovating and becoming a technology monopolist as:

$$V_I(t) = bY(t)\int_t^\infty e^{-\int_t^\tau (R(s)-\delta-g_y(s))ds}\left(R(\tau)^{\zeta-\hat{\sigma}} - \left(w(\tau)e^{\int_t^\tau g(s)ds}\right)^{\zeta-\hat{\sigma}}\right)d\tau, \tag{25}$$

and

$$V_N(t) = bY(t)\int_t^\infty e^{-\int_t^\tau (R(s)-\delta-g_y(s))ds}\left(\left(\frac{w(\tau)}{\gamma(n(t))}e^{\int_t^\tau g(s)ds}\right)^{\zeta-\hat{\sigma}} - R(\tau)^{\zeta-\hat{\sigma}}\right)d\tau, \tag{26}$$

for automation and creation of new tasks, respectively, where $g_y(t)$ is the growth rate of aggregate output at time t and as noted above, $g(t)$ is the growth rate of $\gamma(N(t))$.

The expressions for the value functions, $V_I(t)$ and $V_N(t)$, share a common form: they subtract the lower cost of producing a task with the factor for which the new technology is designed from the higher cost of producing the same task with the older technology.[27] Because new entrants

[25]This follows because the demand for intermediates is $q(i) = B^{\hat{\sigma}-1}\eta\psi^{-\zeta}Y(t)R(t)^{\zeta-\hat{\sigma}}$, each intermediate is sold at a price ψ, and the technology monopolist makes a per unit profit of $1-\mu$.

[26]This expression is written by assuming that the patent-holder will also turn down subsequent less generous offers in the future. Writing it using dynamic programming and the one-step ahead deviation principle leads to the same conclusion.

[27]There is an important difference between the value functions in (25) and (26) and those in models of directed technological change building on factor-augmenting technologies (such as in Acemoglu, 1998, or 2002). In the latter case, the direction of technological change is determined by the interplay of a market size effect favoring the more

compensate the incumbent technology monopolists that they replace, our patent structure removes the creative destruction of profits, which is present in other models of quality improvements under the alternative assumption that new firms do not have to respect the intellectual property rights of the technology on which they are building (e.g., Aghion and Howitt, 1992; Grossman and Helpman, 1991). In Section 5, we explore how our main results change when the intellectual property rights regime allows for the creative destruction of profits.

To ensure that the value functions are well-behaved and non-negative, for the rest of the paper we impose the following assumption:

Assumption 4 $\hat{\sigma} > \zeta$.

This ensures that innovations are directed towards technologies that allow firms to produce tasks by using the cheaper (or more productive) factors, and consequently, that the present discounted values from innovation are positive. This assumption is intuitive and reasonable: since intermediates embody the technology that directly works with labor or capital, they should be highly complementary with the relevant factor of production in the production of tasks.[28]

An *equilibrium with endogenous technology* is given by paths $\{K(t), N(t), I(t)\}$ for capital and technology (starting from initial values $K(0), N(0), I(0)$), paths $\{R(t), W(t), W_I^S(t), W_N^S(t)\}$ for factor prices, paths $\{V_N(t), V_I(t)\}$ for the value functions of technology monopolists, and paths $\{S_N(t), S_I(t)\}$ for the allocation of scientists such that all markets clear, all firms and prospective technology monopolists maximize profits, the representative household maximizes its utility. Using the same normalizations as in the previous section, we can represent the equilibrium with endogenous technology by a path of the tuple $\{c(t), k(t), n(t), L(t), S_I(t), V_I(t), V_N(t)\}$ such that:

- consumption satisfies the Euler equation (18) and the labor supply satisfies equation (17);
- the transversality condition holds

$$\lim_{t\to\infty}(k(t) + \Pi(t))e^{-\int_0^t(\rho-(1-\theta)g(s))ds} = 0, \tag{27}$$

abundant factor and a price effect favoring the cheaper factor. The task-based framework here, combined with the assumption on the structure of patents, makes the benefits of new technologies only a function of the factor prices—in particular, the difference between the wage rate and the rental rate. This is because factor prices determine the profitability of producing with capital relative to labor. Without technological constraints, this would determine the set of tasks that the two factors perform. In the presence of technological constraints restricting which tasks can be produced with which factor, factor prices then determine the incentives for automation (to expand the set of tasks produced by capital) and the creation of new tasks (to expand the set of tasks produced by labor).

We should also note that despite this difference, the general results on absolute weak bias of technology in Acemoglu (2007) continue to hold here—in the sense that an increase in the abundance of a factor always makes technology more biased towards that factor.

[28]The profitability of introducing an intermediate that embodies a new technology depends on its demand. As a factor (labor or capital) becomes cheaper, there are two effects. First, the decline in costs allows firms to scale up their production, which increases the demand for the intermediate good. The extent of this positive scale effect is regulated by the elasticity of substitution $\hat{\sigma}$. Second, because the cheaper factor is substituted for the intermediate it is combined with, the demand for that intermediate good falls. This countervailing substitution effect is regulated by the elasticity of substitution ζ. The condition $\hat{\sigma} > \zeta$ guarantees that the former, positive effect dominates, so that prospective technology monopolists have an incentive to introduce technologies that allow firms to produce tasks more cheaply. When the opposite holds and $\zeta > \hat{\sigma}$, we have the paradoxical situation where technologies that work with more expensive factors are more profitable, and consequently, in this case the present discounted values from innovation is negative.

where in addition to the capital stock in equation (19), we now have the present value of corporate profits $\Pi(t) = I(t)V_I(t)/Y(t) + N(t)V_N(t)/Y(t)$ also added to the representative household's budget;

- capital satisfies the resource constraint

$$\dot{k}(t) = \left[1 + \frac{\eta}{1-\eta}(1-\mu)\right] F\left(k(t), L(t); n^*(t)\right) - c(t)e^{-\frac{1-\theta}{\theta}\nu(L(t))} - (\delta + g(t))k(t),$$

  where recall that $F\left(k(t), L(t); n^*(t)\right)$ is net output (aggregate output net of intermediates) and $\frac{\eta}{1-\eta}(1-\mu)F\left(k(t), L(t); n^*(t)\right)$ is profits of technology monopolists from intermediates;

- competition among prospective technology monopolists to hire scientists implies that $W_I^S(t) = \kappa_I V_I(t)$ and $W_N^S(t) = \kappa_N V_N(t)$. Thus,

$$S_I(t) = SG\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right), \quad S_N(t) = S\left[1 - G\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right)\right] S,$$

  and $n(t)$ according to:

$$\dot{n}(t) = \kappa_N S - (\kappa_N + \kappa_I)G\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right); \tag{28}$$

- and the value functions that determine the allocation of scientists, $V_I(t)$ and $V_N(t)$, are given by (25) and (26).

As before, a BGP is given by an equilibrium in which the normalized variables $c(t), k(t)$ and $L(t)$, and the rental rate $R(t)$ are constant, except that now $n(t)$ is determined endogenously. The definition of the equilibrium shows that the profits from automation and the creation of new tasks determine the evolution of $n(t)$: whenever one of the two types of innovation is more profitable, more scientists will be allocated to that activity.

Consider an allocation where $n(t) = n \in (0,1)$. Let us define the normalized value functions $v_I(n) = \lim_{t\to\infty} V_I(t)/Y(t)$ and $v_N(n) = \lim_{t\to\infty} V_N(t)/Y(t)$, which only depend on n. Equation (28) implies that $\dot{n}(t) > 0$ if and only if $\kappa_N V_N(t) > \kappa_I V_I(t)$, and $\dot{n}(t) < 0$ if and only if $\kappa_N V_N(t) < \kappa_I V_I(t)$. Thus if $\kappa_I v_I(n) \neq \kappa_N v_N(n)$, the economy converges to a corner with $n(t)$ equal to 0 or 1, and for an interior BGP with $n \in (0,1)$ we need

$$\kappa_I v_I(n) = \kappa_N v_N(n). \tag{29}$$

The next proposition gives the main result of the paper, and characterizes different types of BGPs with endogenous technology.

Proposition 6 (Equilibrium with endogenous technological change) *Suppose that Assumptions 1′, 2, and 4 hold. Then, there exists $\overline{S}$ such that, when $S < \overline{S}$, we have:*[29]

[29] The condition $S < \overline{S}$ ensures that the growth rate of the economy is not too high. If the growth rate is above the threshold implied by $\overline{S}$, the creation of new tasks is discouraged (even if current wages are low) because firms anticipate that the wage will grow rapidly, reducing the future profitability of creating new labor-intensive tasks. This condition also allows us to use Taylor approximations of the value functions in our analysis of local stability.

1 ***(Full automation)*** *For* $\rho < \overline{\rho}$*, there is a BGP in which* $n(t) = 0$ *and all tasks are produced with capital.*

For $\rho > \overline{\rho}$*, all BGPs feature* $n(t) = n > \overline{n}(\rho)$*. Moreover, there exist* $\overline{\kappa} \geq \underline{\kappa} > 0$ *such that:*

2 ***(Unique interior BGP)*** *if* $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$ *there exists a unique BGP. In this BGP we have* $n(t) = n \in (\overline{n}(\rho), 1)$ *and* $\kappa_N v_N(n) = \kappa_I v_I(n)$*. If, in addition,* $\theta = 0$*, then the equilibrium is unique everywhere and the BGP is globally (saddle-path) stable. If* $\theta > 0$*, then the equilibrium is unique in the neighborhood of the BGP and is asymptotically (saddle-path) stable;*

3 ***(Multiple BGPs)*** *if* $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$*, there are multiple BGPs;*

4 ***(No automation)*** *If* $\underline{\kappa} > \frac{\kappa_I}{\kappa_N}$*, there exists a unique BGP. In this BGP* $n(t) = 1$ *and all tasks are produced with labor.*

Proof. See Appendix A. ■

This proposition provides a complete characterization of different types of BGPs. Figure 7 shows visually how different BGPs arise in parts of the parameter space.

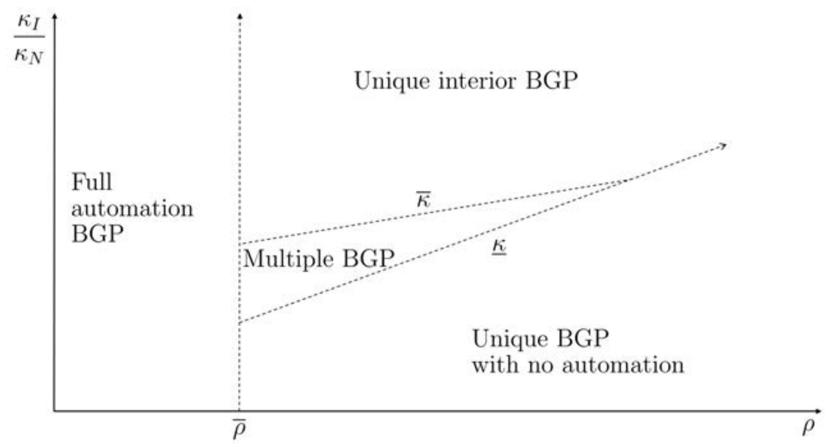

Figure 7: Varieties of BGPs.

Further intuition can be gained by studying the behavior $\kappa_I v_I(n)$ and $\kappa_N v_N(n)$, which we do in Figure 8. Lemma A3 shows that, for S small, the normalized value functions can be written as

$$v_I(n) = \frac{b\left((\rho+\delta+\theta g)^{\zeta-\hat{\sigma}} - w_I(n)^{\zeta-\hat{\sigma}}\right)}{\rho + (\theta-1)g} \qquad v_N(n) = \frac{b\left(w_N(n)^{\zeta-\hat{\sigma}} - (\rho+\delta+\theta g)^{\zeta-\hat{\sigma}}\right)}{\rho + (\theta-1)g}.$$

These expressions show that, asymptotically, the profitability of the two types of technologies depend on the effective wages, $w_I(n)$ and $w_N(n)$. As n increases, so does $w_I(n)$ and it becomes more expensive to produce the least complex tasks with labor, and this makes automation more profitable and triggers further improvements in the automation technology. As a result, $v_I(n)$ is increasing in n (recall that $\hat{\sigma} > \zeta$). However, $v_N(n)$ is also increasing in n because, as explained in the previous section, $w_N(n)$ is decreasing in n owing to the productivity effect and the fact that the rental rate is constant in the long run.

Panel A of Figure 8 illustrates the first part of Proposition 6 (which parallels the first part of Proposition 4): when $\rho < \overline{\rho}$, $\kappa_I v_I(0)$ is above $\kappa_N v_N(0)$ for $n < \widetilde{n}(\rho)$. In this region it is not optimal to create new tasks. Consequently, there exists a BGP with full automation, meaning that all tasks will be automated and produced with capital. Reminiscent of Leontief's "horse equilibrium," in this BGP labor becomes redundant. Intuitively, as also shown in Figure 4, when $\rho < \overline{\rho}$ and $n < \widetilde{n}(\rho)$, we have $w_N(n) > \rho + \delta + \theta g$, which implies that labor is too expensive relative to capital. This ensures that utilizing and thus creating new tasks is not profitable. Economic growth in this BGP is driven by capital accumulation (because when all tasks are automated, aggregate output is linear in capital).

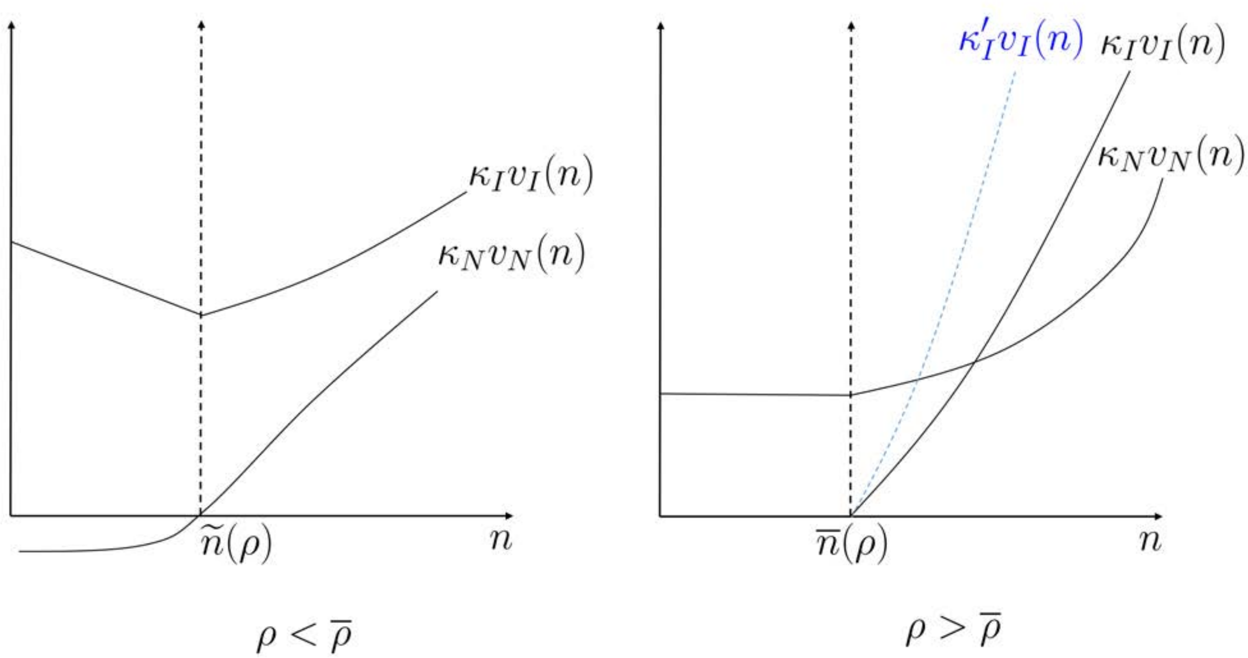

Figure 8: Asymptotic behavior of normalized values. In the left panel, $\kappa_I v_I(n)$ is everywhere above $\kappa_N v_N(n)$, and thus there is no effort devoted to creating new tasks in the BGP involves full automation. In the right panel, if κ_I/κ_N is sufficiently large, the two curves intersect and we have an interior BGP with both automation and creation of new tasks. The right panel also shows the effect of an increase in the productivity of scientists in automating tasks from κ_I to κ_I'.

Panel B of the figure illustrates the remaining three types of BGPs, which apply when $\rho > \overline{\rho}$. In this case, at $n = 0$ (or at any $n \leq \bar{n}(\rho)$), $\kappa_I v_I(n)$ is strictly below $\kappa_N v_N(n)$, and thus a full automation BGP is not possible, and the two curves can only intersect for $n \in (\bar{n}(\rho), 1]$, implying that in any BGP, utilizing and creating new labor-intensive tasks will be profitable. As explained above, both of these curves are increasing, but their relative slopes depend on $\frac{\kappa_I}{\kappa_N}$. When $\frac{\kappa_I}{\kappa_N} < \underline{\kappa}$, $\kappa_I v_I(n)$ is not sufficiently steep relative to $\kappa_N v_N(n)$, and the two never intersect. This means that even at $n = 1$, it is not profitable to create new automation technologies, and all tasks will be produced with labor. In this BGP, capital becomes redundant, and growth is driven by endogenous technological change increasing labor's productivity as in the standard quality ladder models such as Aghion and Howitt (1992) or Grossman and Helpman (1991).

Conversely, when $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, the curve $\kappa_I v_I(n)$ is sufficiently steep relative to $\kappa_N v_N(n)$ so that the two curves necessarily intersect and can only intersect once. Hence there exists a unique interior BGP (interior in the sense that now the BGP level of n is strictly between 0 and 1, and thus some tasks are produced with labor and some with capital).

Finally, when $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$, the two curves will intersect, but will do so multiple times, leading to multiple interior BGPs.

Proposition 6 also shows that, for $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, the unique interior BGP is globally stable provided that the intertemporal elasticity of substitution is infinite (i.e., $\theta = 0$), and locally stable otherwise (i.e., when $\theta > 0$). The critical economic force generating stability is that, as noted above, it is factor prices that guide the direction of technological change. This is the reason why $v_I(n)$ is increasing in n—implying that greater automation, i.e., lower n, reduces the profitability of additional automation. Countering this is the fact that, because of the productivity effect highlighted above, $v_N(n)$ is also increasing in n. The condition for a unique interior BGP in Proposition 6, $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, is sufficient to ensure that $\kappa_I v_I(n)$ is steeper than $\kappa_N v_N(n)$, guaranteeing asymptotic stability.

The asymptotic stability of the interior BGP implies that there are powerful self-correcting market forces pushing the economy towards balanced growth. An important consequence of this stability is that technological shocks that reduce n, for example the arrival of a series of new automation technologies, will set in motion self-correcting forces. Following such a change, there will be an adjustment process restoring the level of employment and the labor share back to their initial values.

This does not, however, imply that *all* shocks will leave the long-run prospects of labor unchanged. For one, this would not necessarily be the case in a situation with multiple steady states, and certain changes in the environment (for example, a large increase in B or a decline in ρ), can shift the economy from the region in which there is a unique interior BGP to the region with full automation, with disastrous consequences for labor. In addition, the next corollary shows that, if there is a change in the innovation possibilities frontier (in the κ's) that makes it permanently easier to develop new automation technologies, self-correcting forces still operate but will now only move the economy to a new BGP with lower employment and a lower labor share.

Corollary 2 *Suppose that $\rho > \overline{\rho}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. A one-time permanent increase in κ_I/κ_N leads to a BGP with lower n, employment and labor share.*

This corollary follows by noting that an increase in κ_I/κ_N shifts the intersection of the curves $\kappa_I v_I(n)$ and $\kappa_N v(n)$ to the left as shown by the blue dotted curve in Figure 8, leading to a lower value of n in the BGP. This triggers an adjustment process in which the labor share and employment decline over time, but ultimately settle to their new (interior) BGP values. The transition process will involve a slower rate of increase of N and a more rapid rate of increase of I than the

BGP. Interestingly, if new tasks generate larger productivity gains than automation, this transition process will also be associated with a slowdown in productivity growth because automation crowds out resources that could be used to develop new tasks.[30]

In summary, Proposition 6 characterizes the varieties of BGPs, and together with Corollary 2, it delineates the types of changes in technology that trigger self-correcting dynamics. Starting from the interior BGP, the effects of (small) increases in automation technology will reverse themselves over time, restoring employment and the labor share back to their initial values. Permanent changes in the ability of society to create new automation technologies trigger self-correcting dynamics as well, but these will take us towards a new BGP with lower employment and labor share.

5 Extensions

In this section we discuss three extensions. First we introduce heterogeneous skills, which allow us to analyze the impact of technological changes on inequality. Second, we study a different structure of intellectual property rights that introduces the creative destruction of profits. Finally, we discuss the welfare implications of our model.

5.1 Automation, New Tasks and Inequality

To study how automation and the creation of new tasks impact inequality, we now introduce heterogeneous skills. This extension is motivated by the observation that both automation and new tasks could increase inequality: new tasks favor high-skill workers who tend to have a comparative advantage in complex tasks, while automation substitutes capital for labor in lower-indexed tasks where low-skill workers have their comparative advantage.

The natural assumption that high-skill workers have a comparative advantage in new tasks receives support from the data. For instance, the left panel of Figure 9 shows that in each decade since 1980, occupations with more new job titles had higher skill requirements in terms of the average years of schooling among employees at the start of each decade (relative to the rest of the economy). However, the right panel of the same figure also shows a pattern of "mean reversion" whereby average years of schooling in these occupations decline in each subsequent decade, most likely, reflecting the fact that new job titles became more open to less skilled workers over time.

To incorporate these features, we assume that there are two types of workers: low-skill workers with time-varying productivity $\gamma_L(i,t)$ in task i, and high-skill workers with productivity $\gamma_H(i)$. We parametrize these productivities as follows:

[30]Forgone productivity gains from from slower creation of new tasks will exceed the gains from automation, causing a productivity slowdown during a transition to a higher level of automation, if $\rho > \rho_P$, where ρ_P is defined implicitly as the solution to the equation

$$\frac{1}{\sigma-1}(w_I(n)^{1-\sigma} - (\rho_P + \delta + \theta g)^{1-\sigma}) = \frac{1}{\sigma-1}((\rho_P + \delta + \theta g)^{1-\sigma} - w_N(n)^{1-\sigma}).$$

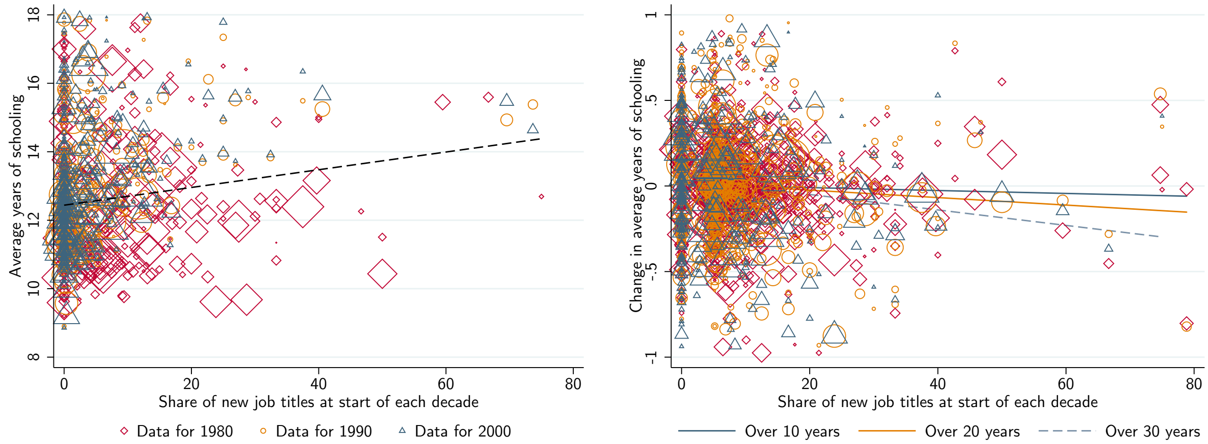

Figure 9: *Left panel:* Average years of schooling among employees against the share of new job titles at the beginning of each decade for 330 occupations. *Right panel:* Change in average years of schooling over the next 10 years (dark blue), next 20 years (blue) and next 30 years (in light blue) against the share of new job titles at the beginning of each decade. See Appendix B for data sources and detailed definitions.

Assumption 1$''$ *The productivities of high-skill and low-skill workers are given by*

$$\gamma_H(i) = e^{A_H i} \qquad\qquad \gamma_L(i,t) = e^{\xi A_H i}\Gamma(t - T(i)),$$

where Γ *is increasing with* $\lim_{x\to\infty}\Gamma(x) = 1$*,* $\xi \in (0,1]$*, and* $T(i)$ *denotes the time when task* i *was first introduced.*

Assumption 1$''$ is similar to but extends Assumption 1$'$ in several dimensions. First, because Γ is increasing, it implies that the productivity of low-skill workers in a task is greater when that task has been in existence for longer. This captures the idea that as new tasks become "standardized," they can be more productively performed by less skilled workers (e.g., Acemoglu, Gancia and Zilibotti, 2010), or that workers adapt to new technologies by acquiring human capital through training, on-the-job learning and schooling (e.g., Schultz, 1965, Nelson and Phelps, 1966, Greenwood and Yorukoglu, 1997, Caselli, 1999, Galor and Moav, 2000, Beaudry, Green and Sand, 2013, and Goldin and Katz, 2008). Second, because of standardization and the assumption that $\xi \in (0,1]$, the ratio $\frac{\gamma_H(i)}{\gamma_L(i,t)}$ is increasing in i. This implies that high-skill workers have a comparative advantage in higher-indexed tasks, as these tasks have had little time to be standardized. Finally, since the function Γ limits to 1 over time, the parameter ξ determines whether this standardization effect is complete or incomplete. When $\xi < 1$, as the time during which a task has existed tends to infinity, the productivity of low-skill workers relative to high-skill workers converges to $\gamma_L(i,t)/\gamma_H(i) = \gamma_H(i)^{\xi-1}$, and limits to zero as more and more advanced tasks are introduced. In contrast, when $\xi = 1$, the relative productivity of low-skill workers converges to 1 for tasks that have been around for a long time.

The structure of comparative advantage ensures that there exists a threshold task M such that high-skill labor performs tasks in $[M, N]$, low-skill labor performs tasks in (I^*, M), and capital performs tasks in $[N-1, I^*]$. In what follows, we denote the wages of high and low-skill labor by W_H and W_L, respectively, and to simplify the discussion, we focus on the economy with exogenous technology and assume that the supply of high-skill labor is fixed at H and the supply of low-skill labor is fixed at L.

Proposition 7 (Automation, new tasks and inequality) *Suppose Assumptions $1''$ and 2 hold. Suppose also that technology evolves exogenously with $\dot{N} = \dot{I} = \Delta$ and $n(t) = n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ (and $A_H(1-\theta)\Delta < \rho$). Then, there exists a unique BGP. Depending on the value of ξ this BGP takes one of the following forms:*

1. *If $\xi < 1$, in the unique BGP we have $\lim_{t\to\infty} W_H(t)/W_L(t) = \infty$, the share of tasks performed by low-skill workers converges to zero, and capital and high-skill workers perform a constant share of tasks.*

2. *If $\xi = 1$, in the unique BGP $W_H(t)$ and $W_L(t)$ grow at the same rate as the economy, the wage gap, $W_H(t)/W_L(t)$, remains constant, and capital, low-skill and high-skill workers perform constant shares of tasks. Moreover, $\lim_{t\to\infty} W_H(t)/W_L(t)$ is decreasing in n. Consequently, a permanent increase in N raises the wage gap $W_H(t)/W_L(t)$ in the short run, but reduces it in the long run, while a permanent increase in I raises the wage gap both in the short and the long run.*

Like all remaining proofs in the paper, the proof of this proposition is in Appendix B.

When $\xi < 1$, this extension confirms the pessimistic scenario about the implications of new technologies for wage inequality and the employment prospects of low-skill workers—both automation and the creation of new tasks increase inequality, the former because it displaces low-skill workers ahead of high-skill workers, and the latter because it directly benefits high-skill workers who have a comparative advantage in newer, more complex tasks relative to low-skill workers. As a result, low-skill workers are progressively squeezed into a smaller and smaller set of tasks, and wage inequality grows without bound.

However, our extended model also identifies a countervailing force, which becomes particularly potent when $\xi = 1$. Because new tasks become standardized, they can over time be as productively used by low-skill workers. In this case, automation and the creation of new tasks still reduce the relative earnings of low skill-workers in the short run, but their long-run implications are very different. In the long run, inequality is decreasing in n (because a higher n translates into a greater range of tasks for low-skill workers). Consequently, automation increases inequality both in the short and the long run, while the creation of new tasks that leads to a permanently lower level of n increases inequality in the short run, but reduces it in the long run. These observations suggest that inequality may be high following a period of adjustment in which the labor share first declines (due to increases in automation), and then recovers (due to the introduction and later standardization of new tasks).

5.2 Creative Destruction of Profits

In this subsection, we modify our baseline assumption on intellectual property rights and revert to the classical setup in the literature in which new technologies do not infringe the patents of the products that they replace (Aghion and Howitt, 1992, and Grossman and Helpman, 1991). This assumption introduces the creative destruction effects—the destruction of profits of previous inventors by new innovators. We will see that this alternative structure has similar implications, but necessitates more demanding conditions to guarantee its uniqueness and stability.

Let us first define $V_N(t,i)$ and $V_I(t,i)$ as the time t values for technology monopolist with, respectively, new task and automation technologies. These value functions satisfy the following Bellman equations:

$$r(t)V_N(t,i) - \dot{V}_N(t,i) = \pi_N(t,i) \qquad\qquad r(t)V_I(t,i) - \dot{V}_I(t,i) = \pi_I(t,i).$$

Here $\pi_I(t,i)$ and $\pi_N(t,i)$ denote the flow profits from automating and creating new tasks, respectively, which are given by the formulas in equations (23) and (24).

For a firm creating a new task i, let $T^N(i)$ denote the time at which it will be replaced by a technology allowing the automation of this task. Likewise, let $T^I(i)$ denote the time at which an automated task i will be replaced by a new task using labor. Since firms anticipate these deterministic replacement dates, their value functions also satisfy the boundary conditions $V_N(T^N(i),i) = 0$ and $V_I(T^I(i),i) = 0$. Together with these boundary conditions, the Bellman equations solve for

$$V_N^{CD}(t) = V_N(N(t),t) = b\int_t^{T^N(N(t))} e^{-\int_t^\tau (R(s)-\delta)ds} Y(\tau)\left(\frac{W(\tau)}{\gamma(N(t))}\right)^{\zeta-\hat{\sigma}} d\tau,$$
$$V_I^{CD}(t) = V_I(I(t),t) = b\int_t^{T^I(I(t))} e^{-\int_t^\tau (R(s)-\delta)ds} Y(\tau)\left(\min\left\{R(\tau), \frac{w(\tau)}{\gamma(I(t))}\right\}\right)^{\zeta-\hat{\sigma}} d\tau.$$

For reasons that will become evident, we modify the innovation possibilities frontier to

$$\dot{I}(t) = \kappa_I \iota(n(t)) S_I(t), \text{ and } \dot{N}(t) = \kappa_N S_N(t) \tag{30}$$

Here, the function $\iota(n(t))$ is included and assumed to be nondecreasing to capture the possibility that automating tasks closer to the frontier (defined as the highest indexed task available) may be more difficult.

Let us again define the normalized value functions as $v_I^{CD}(n) = \lim_{t\to\infty} \frac{V_I^{CD}(t)}{Y(t)}$ and $v_N^{CD}(n) = \lim_{t\to\infty} \frac{V_N^{CD}(t)}{Y(t)}$. In a BGP, the normalized value functions only depend on n because newly-created tasks are automated after a period of length $T^N(N(t)) - t = \frac{n}{\Delta}$, and newly-automated tasks are replaced by new ones after a period of length $T^I(I(t)) - t = \frac{1-n}{\Delta}$, where $\Delta = \frac{\kappa_I\kappa_N\iota(n)}{\kappa_I\iota(n)+\kappa_N}S$ is the endogenous rate at which N and I grow. The endogenous value of n in an interior BGP satisfies

$$\kappa_I\iota(n)v_I^{CD}(n) = \kappa_N v_N^{CD}(n).$$

The next proposition focuses on interior BGPs and shows that, because of creative destruction, we must impose additional assumptions on the function $\iota(n)$ to guarantee stability.

Proposition 8 (Equilibrium with creative destruction) *Suppose that $\rho > \overline{\rho}$, Assumptions $1'$, 2 and 4 hold and there is creative destruction of profits. Then:*

1. *There exist $\overline{\iota}$ and $\underline{\iota} < \overline{\iota}$ such that if $\iota(0) < \underline{\iota}$ and $\iota(1) > \overline{\iota}$, then there exists at least one locally stable interior BGP with $n(t) = n \in (\overline{n}(\rho), 1)$.*

2. *If $\iota(n)$ is constant, there is no stable interior BGP (with $n(t) = n \in (\overline{n}(\rho), 1)$). Any stable BGP involves $n(t) \to 0$ or $n(t) \to 1$.*

The first part of the proposition follows from an analogous argument to that in the proof of Proposition 6, with the only difference being that, because of the presence of the function $\iota(n)$ in equation (30), the key condition that pins down n becomes $\kappa_I \iota(n) v_I^{CD}(n) = \kappa_N v_N^{CD}(n)$.

The major difference with our previous analysis is that creative destruction introduces a new source of instability. Unlike the previous case with no creative destruction, we now have that $v_I^{CD}(n)$ is decreasing in n. As more tasks are automated, the rental rate remains unchanged and newly-automated tasks will be replaced less frequently (recall that newly-automated tasks are replaced after $(1-n)/\Delta$ units of time). As a result, automating tasks renders further automation more profitable. Moreover, $v_N^{CD}(n)$ continues to be increasing in n. This is for two reasons: first, as before, the productivity effect ensures that the effective wage in new tasks, $w_N(n)$, is decreasing in n; and second, because newly-created tasks are automated after $\frac{n}{\Delta}$ units of time, an increase in n increases the present discounted value of profits from new tasks. These observations imply that, if $\iota(n)$ were constant, the intersection between the curves $\kappa_N v_N^{CD}(n)$ and $\kappa_I \iota(n) v_I^{CD}(n)$ would correspond to an unstable BGP.

Economically, the instability is a consequence of the fact that, in contrast to our baseline model (and the socially planned economy that we will describe in the next subsection), here innovation incentives depend on the total revenue that a technology generates rather than its *incremental value* created by their innovation (the difference between these revenues and the revenues that the replaced technology generated). In our baseline model, the key force ensuring stability is that incentives to automate are shaped by the cost difference between producing a task with capital or with labor—by lowering the effective wage at the next tasks to be automated, current automation reduces the incremental value of additional automation. This force is absent when innovators destroy the profits of previous technology monopolists because they no longer care about the cost of production with the technology that they are replacing.

5.3 Welfare

We study welfare from two complementary perspectives. First, in Appendix B we discuss the socially optimal allocation in the presence of endogenous technology and characterize how this allocation can be decentralized. One of the main insights from Proposition 6 is that the expected path for factor prices determines the incentives to automate and create new tasks. We show that a planner would also allocate scientists according to the same principle—guided by the *cost savings* that each technology grants to firms. Although similar to the efficient allocation of scientists in this regard, the decentralized equilibrium is typically inefficient because the technology monopolists neither capture the full benefits from the new tasks they create nor internalize how their innovation affects other existing and future technology monopolists.

The second perspective is more novel and relevant to current debates about automation reducing employment and its policy implications. We examine whether an exogenous increase in automation could reduce welfare. Even though automation expands productivity—a force which always raises welfare—it also reduces employment. When the labor market is fully competitive as in our baseline model, this reduction in employment has no first-order welfare cost for the representative household (which was previously setting the marginal cost of labor supply equal to the wage). As a result, automation increases overall welfare. Next suppose that there are labor market frictions. In particular, suppose that there exists an upward-sloping quasi–labor supply schedule, $L_{qs}(\omega)$, which constrains the level of employment, so that $L \leq L_{qs}(\omega)$. This quasi-labor supply schedule then acts in a very similar fashion to the labor supply curve derived in (11) in Section 2, except that the marginal cost of labor supply is no longer equated to the wage. Crucially, the reduction in employment resulting from automation now has a negative impact on welfare, and this negative effect can exceed the positive impact following from the productivity gains, turning automation, on net, into a negative for welfare.

The next proposition provides the conditions under which automation can reduce welfare in the context of our static model with exogenous technology. Our focus on the static model is for transparency. The same forces are present in the dynamic model and also in the full model with endogenous technology.

Proposition 9 (Welfare implications of automation) *Consider the static economy and suppose that Assumptions 1, 2 and 3 hold, and that* $I^* = I > \widetilde{I}$. *Let* $\mathcal{W} = u(C, L)$ *denote the welfare of representative household.*

1. *Consider the baseline model without labor market frictions, so that the representative household chooses the amount of labor without constraints, and thus* $\frac{W}{C} = \nu'(L)$. *Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) > 0,$$

$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) > 0.$$

2. *Suppose that there are labor market frictions, so that employment is constrained by a quasi-labor supply curve* $L \leq L_{qs}(\omega)$. *Suppose also that the quasi-labor supply schedule* $L_{qs}(\omega)$ *is*

increasing in ω, has an elasticity $\widetilde{\varepsilon}_L > 0$, and is binding in the sense that $\frac{W}{C} > \nu'(L)$. Then:

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta}\left[\frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) - L\left(\frac{W}{C} - \nu'(L)\right)\frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_I\right] \lessgtr 0.$$

$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta}\left[\frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) + L\left(\frac{W}{C} - \nu'(L)\right)\frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_N\right] > 0.$$

The first part of the proposition shows that both types of technological improvements increase welfare when the labor market has no frictions. In this case, automation increases productivity by substituting cheaper capital for human labor, and this leads to less work for workers, but since they were previously choosing labor supply optimally, a small reduction in employment does not have a first-order impact on welfare, and overall welfare increases. The implications of the creation of new tasks are similar.

The situation is quite different in the presence of labor market frictions, however, as shown in the second part. Automation again increases productivity and reduces employment. But now, because workers are constrained in their labor supply choices, the lower employment that results from automation has a first-order negative effect on their welfare. Consequently, automation can reduce welfare if the productivity gains, captured by the first term, are not sufficiently large to compensate for the second, negative term. Interestingly, new tasks increase welfare even more than before, because they not only raise productivity but also expand employment, and by the same logic, the increase in labor supply has a welfare benefit for the workers (since they were previously constrained in their employment).

An important implication of this analysis emphasized further in Appendix B is that when labor market frictions are present and the direction of technological change is endogenized, there will be a force towards excessive automation. In particular, in this case, assuming that labor market frictions also constrain the social planner's choices, the decentralized equilibrium would have too much effort being devoted to improving automation relative to what she would like—because the social planner recognizes that additional automation has a negative effect through employment.

6 Conclusion

As automation, robotics and AI technologies are advancing rapidly, concerns that new technologies will render labor redundant have intensified. This paper develops a comprehensive framework in which these forces can be analyzed and contrasted. At the center of our model is a task-based framework. Automation is modeled as the (endogenous) expansion of the set of tasks that can be performed by capital, replacing labor in tasks that it previously produced. The main new feature of our framework is that, in addition to automation, there is another type of technological change complementing labor. In our model, this takes the form of the introduction of new, more complex versions of existing tasks, and it is assume that labor has a comparative advantage in these new tasks. We characterize the structure of equilibrium in such a model, showing how, given factor prices, the allocation of tasks between capital and labor is determined both by available technology and the endogenous choices of firms between producing with capital or labor.

One attractive feature of task-based models is that they highlight the link between factor prices and the range of tasks allocated to factors: when the equilibrium range of tasks allocated to capital increases (for example, as a result of automation), the wage relative to the rental rate and the labor share decline, and the equilibrium wage rate may also fall. Conversely, as the equilibrium range of tasks allocated to labor increases, the opposite result obtains. In our model, because the supply of labor is elastic, automation tends to reduce employment, while the creation of new tasks increases employment. These results highlight that, while both types of technological changes undergird economic growth, they have very different implications for the factor distribution of income and employment.

Our full model endogenizes the direction of research towards automation and the creation of new tasks. If in the long run capital is very cheap relative to labor, automation technologies will advance rapidly and labor will become redundant. However, when the long-run rental rate of capital is not so low relative to labor, our framework generates a BGP in which both types of innovation go hand-in-hand. Moreover in this case, under reasonable assumptions, the dynamic equilibrium is unique and converges to the BGP. Underpinning this stability result is the impact of relative factor prices on the direction of technological change. The task-based framework—differently from the standard models of directed technological change which are based on factor-augmenting technologies—implies that as a factor becomes cheaper, this not only influences the range of tasks allocated to it, but also generates incentives for the introduction of technologies that allow firms to utilize this factor more intensively. These economic incentives then imply that by reducing the effective cost of labor in the least complex tasks, automation discourages further automation and generates a self-correcting force towards stability.

We further show that, though market forces ensure the stability of the BGP, they do not necessarily generate the efficient composition of technology. If the elastic labor supply relationship results from rents (so that there is a wedge between the wage and the opportunity cost of labor), there is an important new distortion: because firms make automation decisions according to the wage rate, not the lower opportunity cost of labor, there will be a natural bias towards excessive automation.

In addition to claims about automation leading to the demise of labor, several commentators are concerned about the inequality implications of automation and related new technologies. We study this question by extending our model so that high-skill labor has a comparative advantage in new tasks relative to low-skill labor. In this case, both automation (which squeezes out tasks previously performed by low-skill labor) and the creation of new tasks (which directly benefits high-skill labor) increase inequality. Nevertheless, the long-term implications of the creation of new tasks could be very different, because new tasks are later standardized and used by low-skill labor. If this standardization effect is sufficiently powerful, there exists a BGP in which not only the factor distribution of income (between capital and labor) but also inequality between the two skill types stays constant.

We consider our paper to be a first step towards a systematic investigation of different types of technological changes that impact capital and labor differentially. Several areas of research appear fruitful based on this first step. First, our model imposes that it is always the tasks at the bottom that are automated; in reality, it may be those in the middle (e.g., Acemoglu and Autor, 2001). Incorporating the possibility of such "middling tasks" being automated is an important generalization, though ensuring a pattern of productivity growth consistent with balanced growth is more challenging. Second, there may be technological barriers to the automation of certain tasks and the creation of new tasks across industries (e.g., Polanyi, 1966, Autor, Levy and Murnane, 2003). An interesting step is to construct realistic models in which the sectoral composition of tasks performed by capital and labor as well as technology evolves endogenously and is subject to industry-level technological constraints (e.g., on the feasibility or speed of automation). Third, in this paper we have focused on the creation of new labor-intensive tasks as the type of technological change that complements labor and plays a countervailing role against automation. Another interesting area is to theoretically and empirically investigate different types of technologies that may complement labor. Finally, and perhaps most importantly, our model highlights the need for additional empirical evidence on how automation impacts employment and wages (which we investigate in Acemoglu and Restrepo, 2017) and how the incentives for automation and the creation of new tasks respond to policies, factor prices and supplies.

Appendix A: Proofs

General Model

The analysis in the text was carried out under Assumption 2, which imposed $\eta \to 0$ or $\zeta = 1$, and significantly simplified some of the key expressions. Throughout the Appendix, we relax Assumption 2 and replace it with:

Assumption 2′ *One of the following three conditions holds: (i) $\eta \to 0$, or (ii) $\zeta = 1$, or (iii) $\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{2+2\sigma+\eta} > |\sigma - \zeta|$.*

All of our qualitative results remain true and will be proved under this more general assumption. Intuitively, the conditions in Assumption 2 ensured homotheticity (see footnote 11). Assumption $2'$, on the other hand, requires that the departure from homotheticity is small relative to the inverse of the productivity gains from new tasks (where $\gamma(N)/\gamma(N-1)$ measures these productivity gains).

Task prices in this more general case are given by

$$p(i) = \begin{cases} c^u\left(\min\left\{R, \frac{W}{\gamma(i)}\right\}\right) = \left[\eta\psi^{1-\zeta} + (1-\eta)\min\left\{R, \frac{W}{\gamma(i)}\right\}^{1-\zeta}\right]^{\frac{1}{1-\zeta}} & \text{if } i \le I, \\ c^u\left(\frac{W}{\gamma(i)}\right) = \left[\eta\psi^{1-\zeta} + \left(\frac{W}{\gamma(i)}\right)^{1-\zeta}\right]^{\frac{1}{1-\zeta}} & \text{if } i > I. \end{cases} \quad \text{(A1)}$$

Here $c^u(\cdot)$ is the unit cost of production for task i, derived from the task production functions, (2) and (3). Naturally, this equation simplifies to (5) under Assumption 2.

From equations (5) and (7), equilibrium levels of task production are

$$y(i) = \begin{cases} B^{\hat{\sigma}-1} Y c^u \left(\min\left\{ R, \frac{W}{\gamma(i)} \right\} \right)^{-\sigma} & \text{if } i \leq I, \\ B^{\hat{\sigma}-1} Y c^u \left(\frac{W}{\gamma(i)} \right)^{-\sigma} & \text{if } i > I. \end{cases}$$

Combining this with equations (2) and (3), we obtain the task-level demand for capital and labor as

$$k(i) = \begin{cases} B^{\hat{\sigma}-1}(1-\eta) Y c^u (R)^{\zeta-\sigma} R^{-\zeta} & \text{if } i \leq I^*, \\ 0 & \text{if } i > I^*. \end{cases}$$

and

$$l(i) = \begin{cases} 0 & \text{if } i \leq I^*, \\ B^{\hat{\sigma}-1}(1-\eta) Y \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)} \right)^{\zeta-\sigma} W^{-\zeta} & \text{if } i > I^*. \end{cases}$$

Aggregating the preceding two equations across tasks, we obtain the following capital and labor market clearing equations,

$$B^{\hat{\sigma}-1}(1-\eta) Y (I^* - N + 1) c^u (R)^{\zeta-\sigma} R^{-\zeta} = K, \tag{A2}$$

and

$$B^{\hat{\sigma}-1}(1-\eta) Y \int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)} \right)^{\zeta-\sigma} W^{-\zeta} di = L^s \left(\frac{W}{RK} \right). \tag{A3}$$

Finally, from the choice of aggregate output as the numeraire, we obtain a generalized version of the ideal price condition,

$$(I^* - N + 1) c^u (R)^{1-\sigma} + \int_{I^*}^{N} c^u \left(\frac{W}{\gamma(i)} \right)^{1-\sigma} di = B^{1-\hat{\sigma}}, \tag{A4}$$

which again simplifies to the ideal price index condition in the text, (10), under Assumption 2.

Proofs from Section 2

Proof of Proposition 1: We prove Proposition 1 under the more general Assumption $2'$.

To prove the existence and uniqueness of the equilibrium, we proceed in three steps. First, we show that I^*, N and K, determine unique equilibrium values for R, W and Y, thus allowing us to define the function $\omega(I^*, N, K)$ representing the relative demand for labor, which was introduced in the text. Second, we prove a lemma which ensures that $\omega(I^*, N, K)$ is decreasing in I^* (and increasing in N). Third, we show that $\min\{I, \widetilde{I}\}$ is nondecreasing in ω and conclude that there is a unique pair $\{\omega^*, I^*\}$ such that $I^* = \min\{I, \widetilde{I}\}$ and $\omega^* = \omega(I^*, N, K)$. This pair uniquely determines the equilibrium relative factor prices and the range of tasks that get effectively automated.

Step 1: Consider I^*, N and K such that $I^* \in (N-1, N)$. Then, R, W and Y satisfy the system of equations given by capital and labor market clearing, equations (A2) and (A3), and the ideal price index, equation (A4).

Taking the ratio of (A2) and (A3), we obtain

$$\frac{\int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} di}{L^s \left(\frac{W}{RK}\right)(I^* - N + 1) c^u(R)^{\zeta-\sigma} R^{-\zeta}} = \frac{1}{K} \tag{A5}$$

In view of the fact that L^s is increasing and the function $c^u(x)^{\zeta-\sigma}x^{-\zeta}$ is decreasing in x (as it can be verified directly by differentiation), it follows that the left-hand side is decreasing in W and increasing in R . Therefore, (A5) defines an upward-sloping relationship between W and R, which we refer to as the relative demand curve.

On the other hand, inspection of equation (A4) readily shows that this equation gives a downward-sloping locus between R and W as shown in Figure A1, which we refer to as the ideal price curve.

Provided that an intersection exists and is unique, for a given I^*, N and K, the intersection point between the relative demand and the ideal price curves determines the equilibrium factor prices.

Because the relative demand curve is upward sloping and the ideal price index curve is downward sloping, there can be at most one intersection. To prove that there always exists an intersection, observe that $\lim_{x\to 0} c^u(x)^{\zeta-\sigma}x^{-\zeta} = \infty$, and that $\lim_{x\to\infty} c^u(x)^{\zeta-\sigma}x^{-\zeta} = 0$. These observations imply that as $W \to 0$, the numerator of (A5) limits to infinity, and hence, so must the denominator, i.e., $R \to 0$. This proves that the relative demand curve starts from the origin. Similarly, as $W \to \infty$, the numerator of (A5) limits to zero, and so must the denominator (i.e., $R \to \infty$). This then implies that the relative demand curve goes to infinity as $R \to \infty$. Thus, the upward-sloping relative demand curve necessarily starts below and ends above the ideal price curve, which ensures that there always exists an intersection between these curves. The unique intersection defines the equilibrium values of W and R, and therefore the function $\omega(I^*, N, K) = \frac{W}{RK}$.

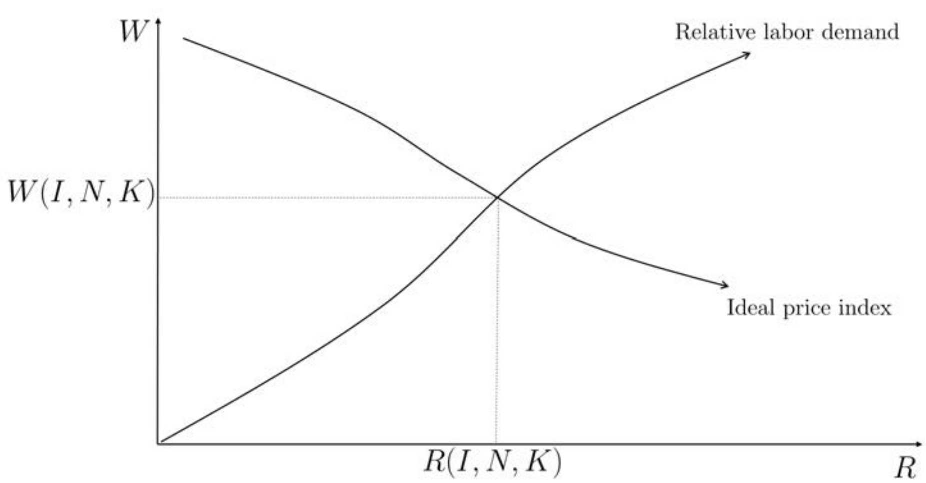

Figure A1: Construction of the function $\omega(I^*, N, K)$.

Step 2: This step follows directly from the following lemma, which we prove in Appendix B.

Lemma A1 *If Assumption $2'$, $\omega(I^*, N, K)$ is decreasing in I^* and is increasing in N.*

Although the general proof for this lemma is long (and thus relegated to Appendix B), the lemma is trivial under Assumption 2. In that case, equation (A5) yields:

$$\omega(I^*, N, K)^{\hat{\sigma}} L^s(\omega(I^*, N, K)) = \frac{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di}{I^* - N + 1} K^{1-\hat{\sigma}}.$$

Taking logs, we obtain equation (13) in the main text, which implies that $\omega(I^*, N, K)$ is increasing in N and decreasing in I^*.

Step 3: We now show that $I^* = \min\{I, \widetilde{I}\}$ is uniquely defined. Since $\gamma(\widetilde{I}) = \omega K$, $\widetilde{I}$ is increasing in ω, and thus $I^* = \min\{I, \widetilde{I}\}$ is nondecreasing in ω. Consider the pair of equations $\omega = \omega(I^*, N, K)$ and $I^* = \min\{I, \widetilde{I}\}$ plotted in Figure 3. Because $\omega = \omega(I^*, N, K)$ is decreasing in I^* and $I^* = \min\{I, \widetilde{I}\}$ is increasing in ω, there exists at most a single pair (ω, I^*) satisfying these two equations (or a single intersection in the figure).

To prove existence, we again verify the appropriate boundary conditions. Suppose that $I^* \to N - 1$. Then from (A2), $R \to 0$, while $W > 0$, and thus $\omega \to \infty$. This ensures that the curve $\omega(I^*, N, K)$ starts above $I^* = \min\{I, \widetilde{I}\}$ in Figure 3. Since $I^* = \min\{I, \widetilde{I}\}$, it is bounded above by I, and cannot be below $\omega = \omega(I^*, N, K)$ at $I^* = I$, ensuring that there must exist a unique intersection between the two curves over the interval $I^* \in (N - 1, I]$, which completes the proof of the existence and uniqueness of the equilibrium.

Finally, when Assumption 2 holds, the market clearing conditions, (8) and (9), imply

$$R = \left(B^{\hat{\sigma}-1}(1-\eta)(I^* - N + 1)\frac{Y}{K}\right)^{\frac{1}{\hat{\sigma}}} \qquad W = \left(B^{\hat{\sigma}-1}(1-\eta)\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di \frac{Y}{L}\right)^{\frac{1}{\hat{\sigma}}},$$

which combined with (10) yields (12), completing the proof of Proposition 1. ■

Proofs from Section 3

Lemma A2 (Derivation of Figure 4) *Suppose that Assumptions $1'$ and 2 hold. Consider a path of technology where $n(t) \to n$ and $g(t) \to g$, consumption grows at the rate g and the Euler equation* (18) *holds. Then, there exist $\rho_{min} < \overline{\rho} < \rho_{max}$ such that:*

1. *If $\rho \in [\rho_{min}, \overline{\rho}]$, there is a decreasing function $\widetilde{n}(\rho) : [\rho_{min}, \overline{\rho}] \to (0, 1]$ such that, for all $n > \widetilde{n}(\rho)$ we have $w_I(n) > \rho + \delta + \theta g > w_N(n)$, and $\rho + \delta + \theta g = w_N(\widetilde{n}(\rho))$. Moreover, $\widetilde{n}(\rho_{min}) = 1$ and $\widetilde{n}(\overline{\rho}) = 0$.*

2. *If $\rho \in [\overline{\rho}, \rho_{max}]$, there is an increasing function $\overline{n}(\rho) : [\overline{\rho}, \rho_{max}] \to (0, 1]$ such that, for all $n > \overline{n}(\rho)$ we have $w_I(n) > \rho + \delta + \theta g > w_N(n)$ and $\rho + \delta + \theta g = w_I(\overline{n}(\rho))$. Moreover, $\overline{n}(\rho_{max}) = 1$ and $\overline{n}(\overline{\rho}) = 0$.*

3. *If $\rho > \rho_{max}$, for all $n \in [0, 1]$ we have $\rho + \delta + \theta g > w_I(n) \geq w_N(n)$, which implies that automation is not profitable for any $n \in [0, 1]$.*

4. *If $\rho < \rho_{min}$, for all $n \in [0,1]$ we have $w_I(n) \geq w_N(n) > \rho + \delta + \theta g$, which implies that new tasks do not increase aggregate output and will not be adopted for any $n \in [0,1]$.*

Proof. Because consumption grows at the rate g and the Euler equation (18) holds, we have

$$R(t) = \rho + \delta + \theta g.$$

The effective wages $w_I(n)$ and $w_N(n)$ are then determined the *ideal price index condition*, (A4), as

$$\begin{aligned} B^{1-\hat{\sigma}} =& (1-n)c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^n c^u\left(\frac{w_I(n)}{\gamma(i)}\right)^{1-\sigma} di \\ =& (1-n)c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^n c^u(\gamma(i)w_N(n))^{1-\sigma} di. \end{aligned} \tag{A6}$$

Differentiating these expressions, we obtain

$$\begin{aligned} \frac{w_I'(n)}{w_I(n)} =& \frac{1}{1-\sigma}\left(c^u(\rho+\delta+\theta g)^{1-\sigma} - c^u(w_N(n))^{1-\sigma}\right)\frac{1}{\int_0^n c^{u\prime}(w_N(n)\gamma(i))c^u(w_N(n)\gamma(i))^{-\sigma}w_N(n)\gamma(i)di} \\ \frac{w_N'(n)}{w_N(n)} =& \frac{1}{1-\sigma}\left(c^u(\rho+\delta+\theta g)^{1-\sigma} - c^u(w_I(n))^{1-\sigma}\right)\frac{1}{\int_0^n c^{u\prime}(w_N(n)\gamma(i))c^u(w_N(n)\gamma(i))^{-\sigma}w_N(n)\gamma(i)di}. \end{aligned} \tag{A7}$$

To prove part 1, define ρ_{min} as

$$\rho_{min} + \delta + \theta g = w_N(1),$$

and define $\overline{\rho} > \rho_{min}$ as

$$c^u(\overline{\rho} + \delta + \theta g)^{1-\sigma} = B^{1-\hat{\sigma}}.$$

(When Assumption 2 holds we get $\overline{\rho} = B - \delta - \theta g$, as claimed in the main text).

To show that $\overline{\rho} > \rho_{min}$, note that

$$\begin{aligned} \frac{1}{1-\sigma}c^u(\rho_{min}+\delta+\theta g)^{1-\sigma} &= \frac{1}{1-\sigma}\int_0^1 c^u(\rho_{min}+\delta+\theta g)^{1-\sigma} di \\ &= \frac{1}{1-\sigma}\int_0^1 c^u(w_N(1))^{1-\sigma} di \\ &< \frac{1}{1-\sigma}\int_0^1 c^u(w_N(1)\gamma(i))^{1-\sigma} di \\ &= \frac{1}{1-\sigma}B^{1-\hat{\sigma}} \\ &= \frac{1}{1-\sigma}c^u(\overline{\rho}+\delta+\theta g)^{1-\sigma}. \end{aligned}$$

Because the function $\frac{1}{1-\sigma}c^u(x)^{1-\sigma}$ is increasing, we have $\overline{\rho} > \rho_{min}$.

Using the ideal price index condition, (A4), we define $\widetilde{n}(\rho)$ implicitly as

$$B^{1-\hat{\sigma}} = (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\widetilde{n}(\rho)} c^u(\gamma(i)(\rho+\delta+\theta g))^{1-\sigma} di.$$

Differentiating the above expression with respect to ρ shows that $\widetilde{n}(\rho)$ is decreasing. Moreover, $\widetilde{n}(\rho_{min}) = 1$ and $\widetilde{n}(\overline{\rho}) = 0$, so the function is well-defined for $\rho \in [\rho_{min}, \overline{\rho}]$.

For $n = \widetilde{n}(\rho)$, we have $w_I(\widetilde{n}(\rho)) > \rho + \delta + \theta g = w_N(\widetilde{n}(\rho))$. Thus, the formulas for $w_I'(n)$ and $w_N'(N)$ show that, for $\rho \in [\rho_{min}, \overline{\rho}]$ and starting at $\widetilde{n}(\rho)$, the curve $w_N(n)$ decreases in n and the curve $w_I(n)$ is increasing in n. Thus, for all $n > \widetilde{n}(\rho)$, we have

$$w_I(n) > w_I(\widetilde{n}(\rho)) > \rho + \delta + \theta g = w_N(\widetilde{n}(\rho)) > w_N(n),$$

as claimed. On the other hand, for all $n < \widetilde{n}(\rho)$, we have $w_N(n) > \rho + \delta + \theta g$.

To prove part 2, define $\rho_{max} > \overline{\rho}$ as

$$\rho_{max} + \delta + \theta g = w_I(1).$$

To show that $\overline{\rho} < \rho_{max}$, a similar argument establishes

$$\begin{aligned}
\frac{1}{1-\sigma} c^u (\rho_{max} + \delta + \theta g)^{1-\sigma} &= \frac{1}{1-\sigma} \int_0^1 c^u (\rho_{max} + \delta + \theta g)^{1-\sigma} di \\
&> \frac{1}{1-\sigma} \int_0^1 c^u (w_I(1)/\gamma(i))^{1-\sigma} di \\
&= \frac{1}{1-\sigma} c^u (\overline{\rho} + \delta + \theta g)^{1-\sigma}.
\end{aligned}$$

Because the function $\frac{1}{1-\sigma} c^u(x)^{1-\sigma}$ is increasing, we have $\overline{\rho} < \rho_{max}$.

Using (A6), we define the function $\overline{n}(\rho)$ implicitly as

$$B^{1-\hat{\sigma}} = (1 - \overline{n}(\rho)) c^u (\rho + \delta + \theta g)^{1-\sigma} + \int_0^{\overline{n}(\rho)} c^u ((\rho + \delta + \theta g)/\gamma(i))^{1-\sigma} di.$$

Differentiating this expression with respect to ρ shows that $\overline{n}(\rho)$ is increasing in ρ on $[\overline{\rho}, \rho_{max}]$. Moreover, $\overline{n}(\rho_{max}) = 1$ and $\overline{n}(\overline{\rho}) = 0$, so the function is well-defined for all $\rho \geq \overline{\rho}$.

For $n = \overline{n}(\rho)$, we have $w_I(\widetilde{n}(\rho)) = \rho + \delta + \theta g > w_N(\widetilde{n}(\rho))$. Thus, the formulas for $w_I'(n)$ and $w_N'(n)$ show that, for $\rho \in [\overline{\rho}, \rho_{max}]$ and starting at $\overline{n}(\rho)$, the curve $w_N(n)$ decreases in n and the curve $w_I(n)$ is increasing in n. Thus, for all $n > \overline{n}(\rho)$, we have

$$w_I(n) > w_I(\overline{n}(\rho)) = \rho + \delta + \theta g > w_N(\overline{n}(\rho)) > w_N(n),$$

as claimed. On the other hand, for all $n < \overline{n}(\rho)$, we have $w_I(n) < \rho + \delta + \theta g$. In this region we have $n^* = \overline{n}(\rho) > n$, and not all automated tasks are produced with capital.

To prove part 3, note that for $\rho > \rho_{max}$, we have

$$\rho + \delta + \theta g > w_I(1) > w_N(1).$$

The expressions for $w_I'(n)$ and $w_N'(n)$ show that, in this region, as n decreases, so does $w_I(n)$. Thus $\rho + \delta + \theta g > w_I(n) > w_N(n)$, and for all these values we have $n^* = 1$, and no task will be produced with capital.

To prove part 4 note that for $\rho < \rho_{min}$, we have

$$\rho + \delta + \theta g < w_N(1) < w_I(1).$$

The expressions for $w_I'(n)$ and $w_N'(N)$ show that, in this region, as n decreases, both $w_N(n)$ $w_N(n)$ increase. Thus $\rho + \delta + \theta g < w_N(n) < w_I(n)$ and for these values of ρ new tasks do not raise aggregate output. ■

Proof of Proposition 4: We prove this proposition under the more general Assumption $2'$.

We start by deriving necessary conditions on $N(t)$ and $I(t)$ such that the economy admits a BGP, and then show that these are also sufficient for establishing the existence of a unique and globally stable BGP.

The capital market clearing condition implies that:

$$c^u(R(t))^{\zeta-\sigma}R(t)^{-\zeta}\frac{K(t)}{Y(t)} = (1 - n^*(t)).$$

Because in a BGP the rental rate of capital, $R(t)$, and the capital to aggregate output ratio, $\frac{K(t)}{Y(t)}$, are constant, we must have $n^*(t) = n$, or in other words, labor and capital must perform a fixed measure of tasks.

Lemma A2 shows that we have four possibilities corresponding to the four cases in Proposition 4, each of which we now discuss in turn.

1. All tasks are automated: $n^*(t) = n = 0$. Because in this case capital performs all tasks, Lemma A2 implies that we must have $\rho < \overline{\rho}$ and $I(t) = N(t)$. In this part of the parameter space, net output is given by $A_K K$, and the economy grows at the rate $\frac{A_K - \delta - \rho}{\theta}$. The transversality condition, (19), is satisfied if and only if $A_K - \delta > \frac{A_K - \delta - \rho}{\theta}$—or $r > g$. Moreover, positive growth imposes $A_K > \delta + \rho$. The (generalized version of the) ideal price index condition, equation (A4), then implies that $R = c^{u-1}(B^{\frac{1-\hat{\sigma}}{1-\sigma}})$, and thus $A_K = c^{u-1}(B^{\frac{1-\hat{\sigma}}{1-\sigma}})$. Under Assumption 2, this last expression further simplifies to $A_K = B$ as claimed in the text.

We now show that The necessary conditions in this case are sufficient to generate balanced growth. Suppose $\rho < \overline{\rho}$ and $I(t) = N(t)$ so that $n^*(t) = 0$. Because all tasks all tasks are produced with capital, we also have $F_L = 0$, and thus the representative household supplies zero labor. Consequently, the dynamic equilibrium can be characterized as the solution to the system of differential equations

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(A_K - \delta - \rho)$$
$$\dot{K}(t) = (A_K - \delta)K(t) - C(t)e^{\nu(0)\frac{\theta-1}{\theta}},$$

together with the initial condition, $K(0) > 0$ and the trasversality condition, (19). We next show that there is a unique solution to the system, and this solution converges to the full automation BGP described in the proposition.

Define $\widetilde{c} = \frac{C}{K}$. The behavior of $\widetilde{c}$ is governed by the differential equation,

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \frac{1}{\theta}(A_K - \delta - \rho) - (A_K - \delta) + \widetilde{c}(t)e^{\nu(0)\frac{\theta-1}{\theta}}.$$

This differential equation has a stable rest point at zero and an unstable rest point at $c_B = \left(A_k - \delta - \frac{1}{\theta}(A_K - \delta - \rho)\right) e^{\nu(0)\frac{1-\theta}{\theta}} > 0$. There are therefore three possible equilibrium paths for $\widetilde{c}(t)$: (i) it immediately jumps to c_B and stays there; (ii) it starts at $[0, c_B)$ and converges to zero; (iii) it starts at (c_B, ∞) and diverges. The second and third possibilities violate, respectively, the transversality condition (19) (because the capital stock would grow at the rate $A_k - \delta$, implying $r = g$), and the resource constraint (when $\widetilde{c}(t) = \infty$). The first possibility, on the other hand, satisfies the transversality condition (since asymptotically it involves $r > g$), and yields an equilibrium path. In this path the economy converges to a unique BGP in which $C(t)$ and $K(t)$ grow at a constant rate $\frac{A_K - \delta - \rho}{\theta}$, thus also establishing uniqueness and global stability.

2. Interior equilibrium in which automated tasks are immediately produced with capital: $n^*(t) = n(t) = n \in (0, 1)$. Because capital performs all automated tasks, Lemma A2 implies that $n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ and $\dot{N}(t) = \dot{I}(t)$. Moreover, because in this candidate BGP $R(t)$ is constant, the general form of the ideal price index condition, (A4), implies that $W(t)/\gamma(I(t))$ must be constant too, and this is only possible if $\dot{I}(t) = \Delta$. Consequently, the growth rate of aggregate output is $A\Delta$. Finally, the transversality condition, (19), is satisfied given the condition $\rho + (\theta - 1)A\Delta > 0$ in this part of the proposition. Lemma A2 then verifies that $n^*(t) = n > \overline{n}(\rho)$. Substituting the market clearing conditions for capital and labor, (A2) and (A3), into (1), (2) and (3) and then subtracting the costs of intermediates, we obtain net output as $F(k, L; n)$. (When Assumption 2 holds, $F(k, L; n)$ is given by the CES aggregate in equation (16)). $F(k, L; n)$ exhibits constant returns to scale, and because factor markets are competitive, we also have $R(t) = F_K(k(t), L(t); n)$ and $w(t) = F_L(k(t), L(t); n)$.

To establish uniqueness, let w_B denote the BGP value of the wage rate, k_B the BGP value of the normalized capital stock, c_B the BGP value of normalized consumption, L_B the BGP value of employment, and R_B the BGP value of the rental rate of capital. These variables are, by definition, all constant. Then, the Euler equation, (18), implies

$R_B = \rho + \delta + \theta g$, and because $R_B = F_K(k_B, L_B; n)$, we must also have $\frac{k_B}{L_B} = \phi$, where ϕ is the unique solution to

$$F_K(\phi, 1; n) = \rho + \delta + \theta g.$$

Lemma B1 in Appendix B shows that, for $n \geq \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$, $F(\phi, 1; n)$ satisfies the following Inada conditions,

$$\lim_{\phi \to 0} F_K(\phi, 1; n) > \rho + \delta + \theta g \qquad \lim_{\phi \to \infty} F_K(\phi, 1; n) < \rho + \delta + \theta g,$$

which ensure that ϕ is well defined. Combining labor supply condition, (17), with the resource constraint, (20), we obtain $(F(\phi, 1; n) - (\delta + g)\phi)L_B = \frac{F_L(\phi,1)}{\nu'(L_B)}$. The left-hand side of this equation is linear and increasing in L (the concavity of F in k implies that $F(\phi, 1; n) > \phi F_K(\phi, 1; n) > (\delta + g)\phi)$, while the right-hand side is decreasing in L. This ensures that there exists a unique value $L_B > 0$ that satisfies this equation, and also pins down the value of the normalized capital stock as $k_B = \phi L_B$. Finally, c_B is uniquely determined from the resource constraint, (20), as

$$c_B = (F(\phi, 1; n) - (\delta + g)\phi) L_B e^{\nu(L_B)\frac{1-\theta}{\theta}}.$$

Note also that there cannot be any BGP with $L_B = 0$, since this would imply $c_B = 0$ from the resource constraint, (20). But then we would have $\nu'(0)e^{\frac{\theta-1}{\theta}\nu(0)} < \frac{F_L(\phi,1;n)}{c_B}$, which contradicts the labor supply optimality condition, (17). Hence, the only possible BGP is one in which $k(t) = k_B$, $c(t) = c_B$ and $L(t) = L_B > 0$. Moreover, in view of the fact that $\rho + (\theta - 1)A\Delta > 0$, this candidate BGP satisfies the transversality condition (19), and is indeed the unique BGP. The proof of the global stability of this unique BGP is similar to the analysis of global stability of the neoclassical growth model with endogenous labor supply, and for completeness, we provide the details in Appendix B.

Interior equilibrium in which automated tasks are eventually but not immediately produced with capital: $n^*(t) = \overline{n}(\rho) > n(t)$. Because capital does not immediately perform all automated tasks, Lemma A2 implies that $n(t) < \overline{n}(\rho)$ and $\rho > \overline{\rho}$. Moreover, because $R(t)$ is constant, the ideal price index condition, (A4), implies that $W(t)/\gamma(I^*(t))$ must be constant too. Thus, to generate constant growth of wages we must have $\dot{I}^*(t) = \Delta \leq I(t)$, so that the growth rate of the economy is given by $A\Delta$. Because $n^*(t) = \overline{n}(\rho)$, this also implies that $\dot{N}^*(t) = \Delta$. Finally, the transversality condition, (19), is satisfied in view of the fact that this part of the proposition imposes $\rho + (\theta - 1)A\Delta > 0$. The uniqueness and global stability of the BGP follow an identical arguments to part 2, with the only modification that $\overline{n}(\rho)$ plays the role of n in the preceding proof.

All tasks are always produced with labor: $n^*(t) = 1$. Because labor performs all tasks, Lemma A2 now implies $\rho > \rho_{min}$ and $n(t) \geq 1$, while the ideal price index condition, (A4), imposes that $W(t)/\gamma(N(t))$ must be constant. Thus, to generate constant wage, aggregate output and capital growth, we must have $\dot{N}(t) = \Delta$, with $\rho + (\theta - 1)\Delta > 0$ (where the last condition again ensures transversality). To show sufficiency of these conditions for balanced growth, let w_B denote the BGP value of the normalized wage, which is defined by

$$\int_0^1 c^u (w_B/\gamma(i))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

Consequently, net output is given by $F(k, L; n) = w_B \gamma(N(t) - 1)L(t)$, and thus depends linearly on labor and is independent of capital. This implies $K(t) = 0$ and $C(t) = w_B\gamma(N(t) - 1)L(t)$. The representative household's labor supply condition, (17), implies that in this BGP

$$\nu'(L(t)) = \frac{w_B\gamma(N(t) - 1)}{C(t)} = \frac{1}{L(t)},$$

which uniquely defines a BGP employment level L_B. Because this allocation also satisfies the transversality condition (in view of the fact that $\rho + (\theta - 1)A\Delta > 0$), it defines a unique BGP. Its global stability follows by noting that starting with any positive capital stock, $K(0)$, the representative household chooses zero investment and converges to this path. ■

Proofs from Section 4

All of the results in this section apply and will be proved, under Assumption $2'$.

Lemma A3 (Asymptotic behavior of the normalized value functions) *Suppose that Assumptions 1′, 2′ and 4. Let* $g = A\frac{\kappa_I \kappa_N}{\kappa_I+\kappa_N} S$ *denote the growth rate of the economy in a BGP. Then there exists a threshold* $\widetilde{S}$ *such that for* $S < \widetilde{S}$*, we have that* $\rho + (\theta - 1)g > 0$*, and*

- *if* $n \geq \max\{\overline{n}, \widetilde{n}\}$*, both* $v_N(n)$ *and* $v_I(n)$ *are positive and increasing in* n*;*
- *if* $n \leq \overline{n}(\rho)$ *(and* $\rho > \overline{\rho}$*), we have* $\kappa_N v_N(n) > \kappa_I v_I(n) = \mathcal{O}(g)$ *(meaning that it goes to zero as* $g \to 0$*);*
- *If* $n < \widetilde{n}(\rho)$ *(and* $\rho < \overline{\rho}$*), we have* $\kappa_I v_I(n) > 0 > \kappa_N v_N(n)$*. Moreover, in this region,* $v_I(n)$ *is decreasing and* $v_N(n)$ *is increasing in* n*.*

Proof. See Appendix B. ■

Proof of Proposition 6: We first show that all the scenarios described in the proposition are BGPs with endogenous technology. We then turn to analyzing the stability of interior BGPs.

Part 1: Characterization of the BGPs with endogenous technology.

Suppose that $S < \widetilde{S}$ so that Lemma A3 applies. We consider the two cases described in the proposition separately.

1. $\rho < \overline{\rho}$**:** Suppose that $n < \widetilde{n}(\rho)$. As depicted in the left panel of Figure 8 and shown in Lemma A3, in this region $v_I(n)$ is positive and decreasing in n, and $v_N(n)$ is negative and increasing in n. Thus, the only possible BGP in this region must be one with $n(t) = 0$. No interior BGP exists with $n \in (0, \widetilde{n}(\rho))$. Proposition 4 shows that for $\rho < \overline{\rho}$, a path for technology with $n(t) = 0$ yields balanced growth. Moreover, along this path all tasks are produced with capital, which implies that $V_I(t) = V_N(t) = 0$. Thus, a path for technology in which $n(t) = 0$ is consistent with the equilibrium allocation of scientists. The resulting BGP is an equilibrium with endogenous technology.

2. $\rho > \overline{\rho}$: Suppose $n(t) \leq \overline{n}(\rho)$. Then, we have $n^*(t) = \overline{n}(\rho)$ and therefore $v_N(n) = v_N(\overline{n}(\rho))$ and $v_I(n) = v_I(\overline{n}(\rho))$. Moreover, Lemma A3 implies that $\kappa_N v_N(\overline{n}(\rho)) > \kappa_I v_I(\overline{n}(\rho))$ and $v_I(\overline{n}(\rho)) = \mathcal{O}(g)$ with g small (again because $S < \widetilde{S}$). Therefore, in this region this region we always have that all scientists will be employed to create new tasks, and thus $\dot{n} > 0$ (and is uniformly bounded away from zero). But this contradicts $n(t) < \overline{n}(\rho)$. Suppose, instead, that $n(t) > \overline{n}(\rho)$. Then, Proposition 4 shows that the economy admits a BGP only if $n(t) = n$. Thus, a necessary and sufficient condition for an interior BGP is (29) in the text. Consequently, each interior BGP corresponds to a solution to this equation in $(\overline{n}(\rho), 1)$. Lemma A3 shows that at $\overline{n}$, $\kappa_N v_N(\overline{n})$ is above $\kappa_I v_I(\overline{n})$, and $\kappa_I v_I(\overline{n}) = O(g)$. Moreover, when $\frac{\kappa_I}{\kappa_N} = 0$, the entire curve $\kappa_N v_N(n)$ is above the curve $\kappa_I v_I(n)$. As this ratio increases, the curve $\kappa_I v_I(n)$ rotates up, and eventually crosses $\kappa_N v_N(n)$ at a point to the right of $\overline{n}(\rho)$. This defines the threshold $\underline{\kappa}$. Above this threshold, there exists another threshold $\overline{\kappa}$ such that if $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, there is a unique intersection of $\kappa_I v_I(n)$ and $\kappa_N v_N(n)$. (Note that one could have $\underline{\kappa} = \overline{\kappa}$). By continuity, there exists $\hat{S}$ such that, the thresholds $\underline{\kappa}$ and $\overline{\kappa}$ are defined for all $S < \hat{S}$ (recall that $g = A\frac{\kappa_I \kappa_N}{\kappa_I+\kappa_N} S$). It then follows that for $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, there exists a unique BGP, which is interior and satisfies $n(t) = n^*(t) = n_B \in (\overline{n}, 1)$. For $S < \min\{\widetilde{S}, \hat{S}\}$ and $\underline{\kappa} < \frac{\kappa_I}{\kappa_N} < \overline{\kappa}$ (provided that $\underline{\kappa} < \overline{\kappa}$), the economy admits multiple BGPs with endogenous technology. Finally, for $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} < \underline{\kappa}$, the only potential BGP in this case is the corner one with $n(t) = 1$, which we also described in Proposition 4. Because $\kappa_N v_N(1) > \kappa_I v_I(1)$, this path for technology is consistent with the equilibrium allocation of scientists and provides a BGP with endogenous technology.

Part 2: Stability analysis.

The stability analysis applies to the case in which $\rho > \overline{\rho}$, $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. In this case, the economy admits a unique BGP defined by $n_B \in (\overline{n}(\rho), 1)$. We denote by c_B, k_B and L_B the values of (normalized) consumption and capital, and employment in this BGP.

Proof of global stability when $\theta = 0$: Because $\theta = 0$, we also have $R = \rho + \delta$, and capital adjusts immediately and its equilibrium stock only depends on n, which becomes the unique state variable of the model.

Let $v = \kappa_I v_I - \kappa_N v_N$. Now starting from any $n(0)$, an equilibrium with endogenous technology is given by the path of (n, v) such that the evolution of the state variable is given by

$$\dot{n} = \kappa_N S - (\kappa_N + \kappa_I) G(v) S;$$

and the evolution of the difference of the normalized value functions, v, satisfies the forward looking equation:

$$\rho v - \dot{v} = b\kappa_I \left(c^u (\rho + \delta)^{\zeta - \sigma} - c^u (w_I)^{\zeta - \sigma} \right) - b\kappa_N \left(c^u (w_N)^{\zeta - \sigma} - c^u (\rho + \delta)^{\zeta - \sigma} \right) + \mathcal{O}(g);$$

and in addition the transversality condition (19) holds.

When $g = 0$, the locus for $\dot{v} = 0$ crosses zero from below at a unique point (recall that we are in the parameter region where there is a unique BGP). By continuity there exists a threshold $\breve{S}$ such that, for $S < \breve{S}$, the locus for $\dot{v} = 0$ crosses zero from below at a unique point n_B, which denotes the BGP value for $n(t)$ derived from (29).

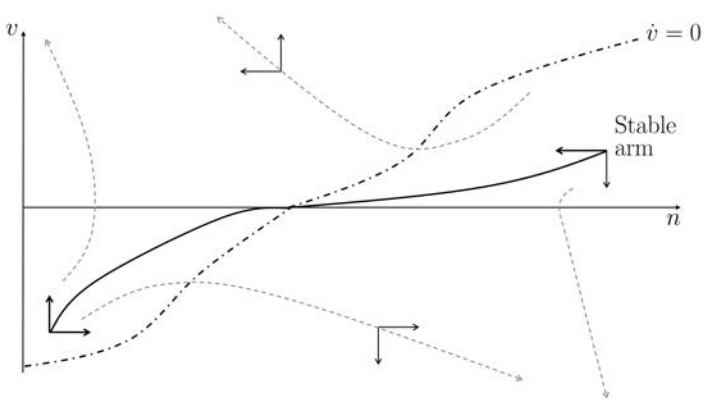

Figure A2: Phase diagram and global saddle path stability when $\theta = 0$. The figure plots the locus for $\dot{v} = 0$ and the locus for $\dot{n} = 0$. The unique BGP is located at their interception.

We now analyze the stability properties of the system and show that the BGP is globally saddle-path stable. Figure A2 presents the phase diagram of the system in (v, n). The locus for $\dot{v} = 0$

crosses $v = 0$ at n_B from below only once. This follows from the fact that $\kappa_I v_I(n)$ cuts $\kappa_N v_N(n)$ from below at n_B as shown in Figure 8. The laws of motion of the two variables, v and n, take the form shown in the phase diagram.[31] This implies the existence of the unique stable arm, and also establishes that there are no equilibrium paths that are not along this stable arm. In particular, all paths above the stable arm feature $\dot{v} > 0$ and eventually $n \to 0$ and $v \to \infty$, and since v_N is positive, $v_I \to \infty$. But this violates the transversality condition, (27). Similarly, all paths below the stable arm feature $\dot{v} < 0$ and eventually $n \to 1$ and $v \to -\infty$, and thus $v_N \to \infty$, once again violating the transversality condition.

Proof of local stability of the unique BGP when $\theta > 0$: Appendix B shows that there exists a threshold $\check{S}$ such that the BGP in this case is locally stable for $S < \check{S}$, and thus the conclusions of the proposition follow setting $\overline{S} = \min\{\widetilde{S}, \hat{S}, \breve{S}, \check{S}\}$. ■

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[31] This can also be verified locally from the fact that the behavior of n and v near the BGP can be approximated by the linear system $\dot{n} = -(\kappa_N + \kappa_I)G'(0)Sv$ and $\dot{v} = \rho v - Q$, where $Q > 0$ denotes the derivative of $-M\kappa_I c^u (w_I)^{\zeta-\sigma} + M\kappa_N c^u (w_N)^{\zeta-\sigma}$ with respect to n (this derivative is positive because $\kappa_I v_I(n)$ cuts $\kappa_N v_N(n)$ from below at n_B). Because the eigenvalues of the characteristic polynomial of this system add up to $\rho > 0$, and their product is $-Q(\kappa_N + \kappa_I)G'(0)S < 0$, there is one positive and one negative eigenvalue.

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Appendix B (Not-For-Publication): Omitted Proofs and Additional Results

Details of the Empirical Analysis

Here we provide information about the data used in constructing Figures 1 and 9. We also provide a regression analysis documenting the robustness of the patterns illustrated in these figures.

Data: We use data on the demographic characteristics of workers and employment counts in each of the 330 consistently defined occupations proposed by David Dorn (see `http://www.ddorn.net/data.htm`). Our sources of data are the U.S. Censuses for 1980, 1990 and 2000, and the American Community Survey for 2007. We focus on the set of workers between 16 and 64 years of age.

Our measure of new job titles is taken from Lin (2011), who computes the share of new job titles within each occupation for 1980, 1990 and 2000. Lin defines new job titles by comparing changes across waves of the Dictionary of Occupational Titles, and also by comparing the 1990 Census Index of Occupations with its 2000 counterpart. The data is available for 329 occupations in 1980, and 330 occupations for 1990 and 2000. The data are available from his website https://sites.google.com/site/jeffrlin/newwork.

Detailed Analysis for Figure 1: To document the role of new job titles in employment growth, we estimate the regression

$$\ln E_{it+10} - \ln E_{it} = \beta N_{it} + \delta_t + \Gamma_t X_{it} + \varepsilon_{it}. \tag{B1}$$

Here, the dependent variable is the percent change in employment from year t to $t+10$ in each occupation i. We stack the data for $t = 1980, 1990, 2000$. For $t = 2000$, we use the change from 2000 to 2007 as the dependent variable and re-scale it to a 10-year change. In all regressions we include a full set of decadal effects δ_t, and in some models we also control for differential decadal trends that vary depending on observable characteristics of each occupation, $\Gamma_t X_{it}$. These characteristics include the share of workers in different 5-year age brackets and from different races (Black, Hispanic), and the share of foreign and female workers. These covariates flexibly control for demographic changes that may affect the labor supply that is relevant for each occupation. Finally, ε_{it} is an error term. Throughout, all standard errors are robust against arbitrary heteroskedasticity and serial correlation of the error term within occupations.

The coefficient of interest is β, which represents the additional employment growth in occupations with a large share of new job titles, N_{it}.

Panel A in Table B1 presents estimates of equation (B1). Column 1 contains no additional covariates (the number of observations in this column is 989. We miss one observation because Lin's measure only covers 329 occupations in 1980). Our estimates indicate that occupations with 10 percentage points more new job titles at the beginning of each decade grew 5.05% faster over the decade (standard error= 1.29%). If occupations with more new job titles did not grow any faster than the benchmark category with no novel jobs, employment growth from 1980 to 2007 would have been, on average, 8.66% instead of the actual 17.5%, implying that approximately 8.84% of the 17.5% growth is accounted for by new job titles as reported at the bottom rows of the panel.

Table B1: Differential employment growth in occupations with more new job titles

	Dep. var: Percent change in employment growth by decade.					
					Weighted by size	
	(1)	(2)	(3)	(4)	(5)	(6)
	Panel A: Stacked differences over decades.					
Share of new job titles at the start of decade	0.505***	0.560***	0.479***	0.378***	0.140	0.351***
	(0.129)	(0.127)	(0.136)	(0.139)	(0.163)	(0.132)
log of employment at start of decade		-0.031**	-0.045***	-0.042***	0.002	0.005
		(0.012)	(0.014)	(0.013)	(0.011)	(0.011)
Average years of schooling at start of decade				9.602***	8.231***	8.179***
				(1.864)	(1.809)	(1.793)
R-squared	0.03	0.04	0.11	0.14	0.13	0.15
Observations	989	989	989	989	989	980
Occupations	330	330	330	330	330	327
Employment growth from 1980-2007 in p.p.	17.5	17.5	17.5	17.5	17.5	17.5
Contribution of novel tasks and jobs	8.84	9.8	8.38	6.62	2.45	6.14
	Panel B: Long differences from 1980-2007.					
Share of new job titles in 1980	1.247***	1.398***	1.458***	1.192***	0.028	0.450
	(0.392)	(0.348)	(0.353)	(0.333)	(0.532)	(0.483)
log of employment in 1980		-0.150***	-0.183***	-0.163***	-0.044	-0.037
		(0.031)	(0.035)	(0.034)	(0.032)	(0.032)
Average years of schooling in 1980				21.779***	15.978***	15.855***
				(4.162)	(4.204)	(4.226)
R-squared	0.02	0.08	0.17	0.24	0.18	0.18
Observations	329	329	329	329	329	326
Occupations	329	329	329	329	329	326
Employment growth from 1980-2007 in p.p.	17.5	17.5	17.5	17.5	17.5	17.5
Contribution of novel tasks and jobs	7.27	8.155	8.50	8.29	0.16	2.17
Covariates:						
Decade fixed effects	✓	✓	✓	✓	✓	✓
Demographics × decade effects			✓	✓	✓	✓

Notes: The table presents 10-years stacked-differences estimates (Panel A) and long-differences estimates (Panel B) of the share of new job titles in an occupation on subsequent employment growth. The bottom row in each panel reports the observed growth and the share explained by growth in occupations with more new job titles. The bottom rows indicate additional covariates included in each model. In column 5 we reweight the data using the baseline share of employment in each occupation in 1980, and in column 6 we exclude three large employment categories that are outliers in the model of column 5. These include office supervisors, office clerks, and production supervisors. Standard errors robust against heteroskedasticity and serial correlation within occupations are presented in parentheses.

In column 2 we control for the log of employment at the beginning of the decade (year t). The coefficient of interest increases slightly to 0.560 and continues to be precisely estimated. The log of employment at year t appears with a negative coefficient, which indicates that smaller occupations tend to grow more over time. The quantitative contribution of new tasks and job titles remains very similar to column 1, increasing slightly to 9.8%.

In column 3 we control for the trends that depend on the demographic covariates described above, which have little effect on the quantitative results. In column 4, we also control for the average years of schooling among workers in each occupation at the beginning of the decade. Although this covariate reduces the magnitude of the coefficient of the share of new job titles, our estimate for β remains highly significant. The contribution of new job titles is now estimated at 6.62% out of the 17.5% growth between 1980 and 2007.

Column 5 repeats the specification of column 4, but this time we reweight the data by the 1980 share of employment in each occupation. This weakens the relationship of interest, and the share of novel tasks and jobs is no longer statistically significant. However, this lack of significance is driven by a few large occupations that are outliers in the estimated relationship. (In contrast, there are no major outliers in the unweighted regressions reported in columns 1-4). These outliers include office supervisors, office clerks, and production supervisors; three occupations that had combined employment of about 4 million workers in 1980 and have been contracting since then. Though these occupations introduced a significant number of new job titles in 1980, they shed a large amount of workers in the subsequent years. In column 6, we exclude these three occupations from our analysis, and obtain a similar pattern to that of column 4.

Finally, in Panel B, we present a set of regressions, which are analogous to those in the top panel, but that focus on a long difference specification between 1980 and 2007. The overall patterns are very similar, and now the contribution of novel tasks and new job titles to the 17.5% growth in employment between 1980 and 2007 is between 7.27 and 8.5%.

Detailed Analysis for Figure 9: To document the presence of some amount of "standardization" of occupations with greater new job titles,

$$\Delta Y_{it} = \beta N_{it} + \delta_t + \Gamma_t X_{it} + \varepsilon_{it}. \tag{B2}$$

Here, the dependent variable is the change in the average years of schooling among workers employed in occupation i and measured over different time horizons (10 years, 20 years or 30 years). We stack the data for $t = 1980, 1990, 2000$, but our sample becomes smaller as we measure the change in average years of schooling over longer periods of time. We again use the ACS dayear 2007. The covariates and independent variable are the same ones that we used in equation (B1).

Panel A in Table B2 presents estimates of equation (B2). Columns 1 and 2 present models in which the dependent variable is the change in average years of education over a 10-year period (hence, we get 989 observations for 330 occupations). Columns 3 and 4 focus on the change in average years of education over a 20-year period. Columns 5 and 6 focus on the change in average years of education over a 30-year period. In the models presented in the even columns we include a full set of trends that are allowed to vary depending on the composition of employment in each occupational category.

Our estimates indicate that, although occupations with more new job titles tend to hire more skilled workers initially, this pattern slowly reverts over time. Figure 9 shows that, at the time of

Table B2: Reversal in skill content for occupations with more new job titles and in occupations that used to hire more educated workers.

	Dep. var: Change in average years of schooling.					
	Over 10 years		Over 20 years		Over 30 years	
	(1)	(2)	(3)	(4)	(5)	(6)
	Panel A: Change in occupations with new job titles.					
Share of new job titles at the start of decade	-0.085	-0.103	-0.219	-0.203	-0.452**	-0.411**
	(0.123)	(0.118)	(0.179)	(0.173)	(0.215)	(0.176)
R-squared	0.32	0.53	0.12	0.33	0.02	0.29
Observations	989	989	659	659	329	329
Occupations	330	330	330	330	329	329
	Panel B: Change in occupations with more educated workers.					
Average years of education at the start of decade	-0.030***	-0.077***	-0.028**	-0.102***	-0.083***	-0.149***
	(0.006)	(0.011)	(0.013)	(0.017)	(0.016)	(0.021)
R-squared	0.36	0.59	0.14	0.41	0.17	0.43
Observations	990	990	660	660	330	330
Occupations	330	330	330	330	330	330
Covariates:						
Decade fixed effects	✓	✓	✓	✓	✓	✓
Demographics × decade effects		✓		✓		✓

Notes: The table presents OLS estimates that explain the change in average years of schooling among workers employed in a given occupation. These changes are computed over 10 years (Columns 1 and 2), 20 years (Columns 3 and 4) and 30 years (Columns 5 and 6). In Panel A we explain the subsequent change in years of schooling as a function of share of new job titles in each occupation at the start of the decade. In Panel B we explain the subsequent change in years of schooling as a function of the years of schooling in each occupation at the start of the decade. The bottom rows indicate additional covariates included in each model. Standard errors robust against heteroskedasticity and serial correlation within occupations are presented in parentheses.

their introduction, occupations with 10 percentage points more new job titles hire workers with 0.35 more years of schooling). But our estimates in Column 6 of Table B2 show that this initial difference in the skill requirements of workers slowly vanishes over time. 30 years after their introduction, occupations with 10 percentage points more new job titles hire workers with 0.0411 fewer years of education than the workers hired initially (standard error= 0.0176).

Relatedly, Panel B of the same table shows a similar pattern when we look at occupations that start with greater average years of schooling at the beginning of a decade. In particular, for each decade since 1980, employment growth has been faster in occupations with greater skill requirements—as measured by the average years of education among employees at the start of each decade (see Table B1). But estimating a version of (B2) with average years of schooling at the beginning of the decade on the right-hand side, we find significant mean reversion. For example, Column 6 shows that occupations that used to hire workers with one additional year of schooling workers reduce their average years of schooling by 0.149 years relative to baseline after 30 years (standard error=0.021).

Remaining Proofs from Section 2

We start by providing the proof of Lemma A1.

Proof of Lemma A1. Denote by $W(I^*,N,K)$ and $R(I^*,N,K)$ the wage and interest rate for task thresholds I^*,N and a supply of capital K. $W(I^*,N,K)$ and $R(I^*,N,K)$ are given by the intercept of the ideal price index condition in equation (A4) and the equation

$$L^s\left(\frac{W}{RK}\right)(I^*-N+1)c^u(R)^{\zeta-\sigma}R^{-\zeta}-K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}di=0,$$

which can be derived from the market clearing conditions for capital and labor in equations (A2) and (A3).

Taking total derivatives of this equation with respect to I^*,N, we obtain

$$\begin{aligned}
&dN\left(L^sc^u(r)^{\zeta-\sigma}r^{-\zeta}+K\gamma(N)^{\zeta-1}c^u\left(\frac{W}{\gamma(N)}\right)^{\zeta-\sigma}W^{-\zeta}\right)-dI^*\left(L^sc^u(r)^{\zeta-\sigma}r^{-\zeta}+K\gamma(I^*)^{\zeta-1}c^u\left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma}W^{-\zeta}\right)\\
&=(d\ln W-d\ln r)\left(K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\left[\hat{s}(i)(\sigma-\zeta)+\zeta\right]di+\varepsilon_L L^s(I^*-N+1)c^u(r)^{\zeta-\sigma}r^{-\zeta}\right)\\
&+d\ln r\left(K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\left[(\hat{s}(i)-s_k)(\sigma-\zeta)\right]di\right).
\end{aligned}$$

Here, $\hat{s}(i)$ is the share of labor in the production of task i and $\hat{s}_k$ is the share of capital in tasks produced with capital. The last term in the above equation captures the non-homotheticity introduced by the presence of intermediate goods. When $(\hat{s}(i)-\hat{s}_k)(\sigma-\zeta)=0$, the demand system is homothetic and the results outlined in Lemma A1 follow easily. These cases include the limits when $\zeta\to 1$ (and $\hat{s}(i)=\hat{s}_k$ are constant) or $\eta\to 0$ (and $\hat{s}(i)=\hat{s}_k=1$ are constant).

When $(\hat{s}(i)-\hat{s}_k)(\sigma-\zeta)\neq 0$, we also need to take into account the movements in the ideal price curve in equation (A4) to determine the behavior of relative factor prices.

Differentiation of (A4) gives:

$$\begin{aligned}
&dN\frac{1}{1-\sigma}\left(c^u(r)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)+dI\frac{1}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma}-c^u(r)^{1-\sigma}\right)\\
=&(d\ln W-d\ln r)\left(\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right)+d\ln r\left((I^*-N+1)s_kc^u(r)^{1-\sigma}+\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right),
\end{aligned}$$

which implies that we can solve for $d\ln W-d\ln R$ as

$$(d\ln W-d\ln R)\Theta=dNP_N-d_IP_I,$$

where:

$$\begin{aligned}
\Theta=&\left(K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\left[\hat{s}(i)(\sigma-\zeta)+\zeta\right]di+\varepsilon_L L^s(I^*-N+1)c^u(r)^{\zeta-\sigma}r^{-\zeta}\right)\\
&\times\left((I^*-N+1)s_kc^u(r)^{1-\sigma}+\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right)\\
&-\left(\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right)\times\left(K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\left[(\hat{s}(i)-s_k)(\sigma-\zeta)\right]di\right),
\end{aligned}$$

$$\begin{aligned}
P_N=&\left(L^sc^u(r)^{\zeta-\sigma}r^{-\zeta}+K\gamma(N)^{\zeta-1}c^u\left(\frac{W}{\gamma(N)}\right)^{\zeta-\sigma}W^{-\zeta}\right)\\
&\times\left((I^*-N+1)s_kc^u(r)^{1-\sigma}+\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right)\\
&-\frac{1}{1-\sigma}\left(c^u(r)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\times\left(K\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\left[(\hat{s}(i)-s_k)(\sigma-\zeta)\right]di\right),
\end{aligned}$$

$$
\begin{aligned}
P_I =& \left(L^s c^u(r)^{\zeta-\sigma} r^{-\zeta} + K\gamma(I^*)^{\zeta-1} c^u \left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma} W^{-\zeta} \right) \\
&\times \left((I^* - N + 1) s_k c^u(r)^{1-\sigma} + \int_{I^*}^{N} \hat{s}(i) c^u \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di \right) \\
&+ \frac{1}{1-\sigma} \left(c^u \left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u(r)^{1-\sigma} \right) \times \left(K \int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} \left[(\hat{s}(i) - s_k)(\sigma - \zeta) \right] di \right).
\end{aligned}
$$

We now show that, under the conditions of Lemma A1, we have $\Theta, P_N, P_I > 0$.

A sufficient condition for $\Theta > 0$ is that:

$$
\begin{aligned}
&\left(K \int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} \left[\hat{s}(i)(\sigma - \zeta) + \zeta \right] di \right) \times \left(\int_{I^*}^{N} \hat{s}(i) c^u \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di \right) \\
\geq& \left(\int_{I^*}^{N} \hat{s}(i) c^u \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di \right) \times \left(K \int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} \left[(\hat{s}(i) - s_k)(\sigma - \zeta) \right] di \right).
\end{aligned}
$$

After canceling common terms on both sides of this inequality, it boils down to:

$$
\int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} \left[\hat{s}(i)(\sigma - \zeta) + \zeta \right] di \geq \int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} \left[(\hat{s}(i) - \hat{s}_k)(\sigma - \zeta) \right] di.
$$

This can be rewritten as:

$$
\int_{I^*}^{N} \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} \left[\hat{s}(i)(\sigma - \zeta) + \zeta - (\hat{s}(i) - \hat{s}_k)(\sigma - \zeta) \right] di \geq 0.
$$

The last inequality always holds because $\hat{s}(i)(\sigma - \zeta) + \zeta - (\hat{s}(i) - \hat{s}_k)(\sigma - \zeta) = \sigma \hat{s}_k + \zeta(1 - \hat{s}_k) > 0$.

To determine the signs of P_N and P_I, we regroup terms as follows. First, we group the terms that are multiplied by $\hat{s}_k$ in the expression for P_I. To guarantee that these terms add up to a positive number, a sufficient condition is given by:

$$
\begin{aligned}
&\left(L^s c^u(r)^{\zeta-\sigma} r^{-\zeta} \right) \times (I^* - N + 1) s_k c^u(r)^{1-\sigma} \\
\geq& \left| \frac{\sigma - \zeta}{1-\sigma} \left(c^u \left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u(r)^{1-\sigma} \right) \right| \times \left(K \int_{I^*}^{N} s_k \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} di \right).
\end{aligned}
$$

We can use equations (A2) and (A3) to cancel terms on both sides of the above inequality and obtain the sufficient condition:

$$
c^u(R)^{1-\sigma} \geq \left| \frac{\sigma - \zeta}{1-\sigma} \left(c^u \left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u(R)^{1-\sigma} \right) \right|. \tag{B3}
$$

Second, we group the terms that are multiplied by $\hat{s}(i)$ in the expression for P_I. To guarantee that these terms add up to a positive number, a sufficient condition is given by:

$$
\begin{aligned}
&\left(K\gamma(I^*)^{\zeta-1} c^u \left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma} W^{-\zeta} \right) \times \left(\int_{I^*}^{N} \hat{s}(i) c^u \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di \right) \\
\geq& \left| \frac{\sigma - \zeta}{1-\sigma} \left(c^u \left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u(r)^{1-\sigma} \right) \right| \times \left(K \int_{I^*}^{N} \hat{s}(i) \gamma(i)^{\zeta-1} c^u \left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma} W^{-\zeta} di \right).
\end{aligned}
$$

After removing common terms and re-grouping, this condition becomes

$$\int_{I^*}^{N} \hat{s}(i)W^{-\zeta}\left(\gamma(I^*)^{\zeta-1}c^u\left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} - \gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|\right) \geq 0. \tag{B4}$$

We now show that, for this condition to hold, it suffices that

$$\left(\frac{\gamma(I^*)}{\gamma(i)}\right)^{\max\{\sigma,\zeta\}-1} c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} \geq \left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|. \tag{B5}$$

We first show this in the case in which $\sigma > \zeta$. In this case, we have

$$c^u\left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma} \geq c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}\left(\frac{\gamma(i)}{\gamma(I^*)}\right)^{\zeta-\sigma}.$$

This follows from the fact that

$$\frac{c^u\left(\frac{W}{\gamma(i)}\right)}{c^u\left(\frac{W}{\gamma(I^*)}\right)} \geq \frac{\gamma(I^*)}{\gamma(i)},$$

an inequality which can be proven by straightforward differentiation of the function $c^u(x)/x$, which is decreasing.

Plugging this in the inequality in equation (B4), we obtain the sufficient condition

$$\int_{I^*}^{N} \hat{s}(i)W^{-\zeta}\gamma(i)^{\zeta-\sigma}\left(\gamma(I^*)^{\sigma-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} - \gamma(i)^{\sigma-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|\right),$$

which holds provided that

$$\left(\frac{\gamma(I^*)}{\gamma(i)}\right)^{\sigma-1} c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} \geq \left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|.$$

This inequality coincides with the condition in equation (B5), completing this step of the proof.

We now turn to the case $\zeta \geq \sigma$. Because the cost function is increasing we have:

$$c^u\left(\frac{W}{\gamma(I^*)}\right)^{\zeta-\sigma} \geq c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}$$

Plugging this in the sufficient condition in equation (B4) yields:

$$\int_{I^*}^{N} \hat{s}(i)W^{-\zeta}\left(\gamma(I^*)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} - \gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|\right),$$

which holds provided that

$$\left(\frac{\gamma(I^*)}{\gamma(i)}\right)^{\zeta-1} c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} \geq \left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(I^*)}\right)^{1-\sigma} - c^u\left(R\right)^{1-\sigma}\right)\right|,$$

once again establishing the sufficiency of (B5).

Third, we group the terms that are multiplied by $\hat{s}(i)$ in the expression for P_N. To guarantee that these terms add up to a positive number, a sufficient condition is given by:

$$\left(K\gamma(N)^{\zeta-1}c^u\left(\frac{W}{\gamma(N)}\right)^{\zeta-\sigma}W^{-\zeta}\right)\times\left(\int_{I^*}^{N}\hat{s}(i)c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di\right)$$
$$\geq\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(r\right)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\right|\times\left(K\int_{I^*}^{N}\hat{s}(i)\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}di\right).$$

This condition can be rewritten as:

$$\int_{I^*}^{N}\hat{s}(i)W^{-\zeta}\left[c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}\gamma(N)^{\zeta-1}c^u\left(\frac{W}{\gamma(N)}\right)^{\zeta-\sigma}-\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(R\right)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\right|\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}\right]di\geq 0.$$

In Step 1 of the proof of this proposition, we showed that $x^{-\zeta}c^u(x)^{\sigma-\zeta}$ is decreasing in x. This implies that $\gamma(N)^{\zeta}c^u\left(\frac{W}{\gamma(N)}\right)^{\zeta-\sigma}>\gamma(i)^{\zeta}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}$.

Therefore, a sufficient condition for the terms that are multiplied by $\hat{s}(i)$ in the expression for P_N to add up to a positive number is

$$\int_{I^*}^{N}\hat{s}(i)\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}\times\left[\frac{\gamma(I)}{\gamma(N)}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}-\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(R\right)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\right|\right]di\geq 0.$$

This expression implies that, to guarantee that these terms are positive, the following relationship would be sufficient

$$\frac{\gamma(I)}{\gamma(N)}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}\geq\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(R\right)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\right|. \tag{B6}$$

Fourth, and lastly, we group the terms that are multiplied by s_K in the expression for P_N. To guarantee that these terms add up to a positive number, a sufficient condition is given by:

$$L^s c^u(r)^{\zeta-\sigma}r^{-\zeta}\times(I^*-N+1)s_k c^u(r)^{1-\sigma}$$
$$\geq\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(r\right)^{1-\sigma}-c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}\right)\right|\times\left(K\int_{I^*}^{N}s_k\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}di\right).$$

Using the market clearing conditions for capital and labor in equations (A2) and (A3), we can rewrite the sufficient condition as:

$$c^u(R)^{1-\sigma}\geq\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(\frac{W}{\gamma(N)}\right)^{1-\sigma}-c^u\left(R\right)^{1-\sigma}\right)\right|. \tag{B7}$$

To finalize the proof of the lemma, we use the fact that for any effective factor prices p_1,p_2 (the price per effective unit of labor or capital at any given task), we have

$$\left|\frac{\sigma-\zeta}{1-\sigma}\left(c^u\left(p_1\right)^{1-\sigma}-c^u\left(p_2\right)^{1-\sigma}\right)\right|\leq|\sigma-\zeta|c^u\left(\frac{W}{\gamma(N-1)}\right)c^u\left(\frac{W}{\gamma(N)}\right)^{-\sigma}. \tag{B8}$$

This inequality follows because $f(x)=\frac{1}{1-\sigma}x^{1-\sigma}$ is a concave function and the effective factor prices satisfy $p_1,p_2\in\left[\frac{W}{\gamma(N)},\frac{W}{\gamma(N-1)}\right]$ (recall that because $K>\overline{K}$, we have that $\frac{W}{\gamma(N)}<R$. Likewise, the definition of I^* imposes that $\frac{W}{\gamma(N-1)}>R$).

Inequality (B8) then implies that to guarantee (B3), (B5), (B6) and (B7), the following would suffice

$$c^u(R)^{1-\sigma} \geq |\sigma-\zeta| c^u\left(\frac{W}{\gamma(N-1)}\right) c^u\left(\frac{W}{\gamma(N)}\right)^{-\sigma}.$$

$$\left(\frac{\gamma(I^*)}{\gamma(i)}\right)^{\max\{\sigma,\zeta\}-1} c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} \geq |\sigma-\zeta| c^u\left(\frac{W}{\gamma(N-1)}\right) c^u\left(\frac{W}{\gamma(N)}\right)^{-\sigma}.$$

$$\frac{\gamma(I)}{\gamma(N)} c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma} \geq |\sigma-\zeta| c^u\left(\frac{W}{\gamma(N-1)}\right) c^u\left(\frac{W}{\gamma(N)}\right)^{-\sigma}.$$

These inequalities can be, in turn, rewritten as

$$\left(\frac{c^u(R)}{c^u\left(\frac{W}{\gamma(N-1)}\right)}\right)\left(\frac{c^u\left(\frac{W}{\gamma(N)}\right)}{c^u(R)}\right)^{\sigma} \geq |\sigma-\zeta|.$$

$$\left(\frac{\gamma(I^*)}{\gamma(i)}\right)^{\max\{\sigma,\zeta\}-1}\left(\frac{c^u\left(\frac{W}{\gamma(i)}\right)}{c^u\left(\frac{W}{\gamma(N-1)}\right)}\right)\left(\frac{c^u\left(\frac{W}{\gamma(N)}\right)}{c^u\left(\frac{W}{\gamma(i)}\right)}\right)^{\sigma} \geq |\sigma-\zeta|.$$

$$\frac{\gamma(I)}{\gamma(N)}\left(\frac{c^u\left(\frac{W}{\gamma(i)}\right)}{c^u\left(\frac{W}{\gamma(N-1)}\right)}\right)\left(\frac{c^u\left(\frac{W}{\gamma(N)}\right)}{c^u\left(\frac{W}{\gamma(i)}\right)}\right)^{\sigma} \geq |\sigma-\zeta|.$$

From the properties of unit cost functions, it follows that for all $p \in \left[\frac{W}{\gamma(N)}, \frac{W}{\gamma(N-1)}\right]$ we have $c^u(p) \geq c^u\left(\frac{W}{\gamma(N-1)}\right)\frac{\gamma(N-1)}{\gamma(N)}$ and $c^u\left(\frac{W}{\gamma(N)}\right) \geq c^u(p)\frac{\gamma(N-1)}{\gamma(N)}$. Using these properties, it follows that the sufficient conditions above hold whenever

$$\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{1+\sigma} \geq |\sigma-\zeta|$$

$$\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{\max\{\sigma,\zeta\}+\sigma} \geq |\sigma-\zeta|$$

$$\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{2+\sigma} \geq |\sigma-\zeta|.$$

The first inequality above ensures the sufficient conditions (B3) and (B7) simultaneously; the second inequality above ensures the sufficient condition (B5); and the last inequality above ensures the sufficient condition (B6)

Thus a sufficient condition for all three inequalities to hold is

$$\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{2+2\sigma+\zeta} \geq |\sigma-\zeta|,$$

which completes the proof of the Lemma. ■

We now turn to Proposition 2. We first provide a similar Proposition which applies in general. We then specialize to Assumption 2 to derive a sharp characterization of the comparative statics. In what follows, we let $\frac{\partial\omega}{\partial I^*}$, $\frac{\partial\omega}{\partial N}$ and $\frac{\partial\omega}{\partial K}$ denote the partial derivatives of the function $\omega(I^*, N, K)$ with respect to its arguments.

Proposition B1 (Comparative statics in the general model) *Suppose that Assumptions 1, 2′ and 3 hold. Let $\varepsilon_L > 0$ denote the elasticity of the labor supply schedule $L^s(\omega)$ with respect to ω; let $\varepsilon_\gamma = \frac{d \ln \gamma(I)}{dI} > 0$ denote the semi-elasticity of the comparative advantage schedule.*

- *If $I^* = I < \widetilde{I}$—so that the allocation of tasks to factors is constrained by technology—then:*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d \ln(W/R)}{dI} = \frac{d \ln \omega}{dI} = \frac{1}{\omega}\frac{\partial \omega}{\partial I^*} < 0, \frac{d \ln(W/R)}{dN} = \frac{d \ln \omega}{dN} = \frac{1}{\omega}\frac{\partial \omega}{\partial N} > 0$$

  - *the impact of capital on relative factor prices is given by*

$$\frac{d \ln(W/R)}{d \ln K} = \frac{d \ln \omega}{d \ln K} + 1 = \frac{1 + \varepsilon_L}{\sigma_{cons} + \varepsilon_L} > 0,$$

  where $\sigma_{cons} \in (0, \infty)$ is the elasticity of substitution between labor and capital that applies when technology constraints the allocation of factors to tasks. This elasticity is given by a weighted average of σ and ζ.

- *If $I^* = \widetilde{I} < I$—so that tasks are allocated to factors in the unconstrained cost minimizing fashion—then*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d \ln(W/R)}{dI} = \frac{d \ln \omega}{dI} = 0, \quad \frac{d \ln(W/R)}{dN} = \frac{d \ln \omega}{dN} = \frac{\sigma_{cons} + \varepsilon_L}{\sigma_{free} + \varepsilon_L}\frac{1}{\omega}\frac{\partial \omega}{\partial N} > 0 \text{ and}$$

  - *the impact of capital on relative factor prices is given by*

$$\frac{d \ln(W/R)}{d \ln K} = \frac{d \ln \omega}{d \ln K} + 1 = \left(\frac{1 + \varepsilon_L}{\sigma_{free} + \varepsilon_L}\right) > 0,$$

  where

$$\sigma_{free} = (\sigma_{cons} + \varepsilon_L)\left(1 - \frac{1}{\omega}\frac{\partial \omega}{\partial I^*}\right) - \varepsilon_L > \hat{\sigma};$$

- *In both parts of the proposition, the labor share and employment move in the same direction as ω.*

- *Finally, when Assumption 2 holds, both parts of the proposition apply as stated, with the additional difference that now we can explicitly compute the derivatives as*

$$\frac{1}{\omega}\frac{\partial \omega}{\partial I^*} = -\frac{1}{\hat{\sigma} + \varepsilon_L}\Lambda_I \qquad \frac{1}{\omega}\frac{\partial \omega}{\partial N} = -\frac{1}{\hat{\sigma} + \varepsilon_L}\Lambda_N,$$

  and the elasticities of substitution as:

$$\sigma_{cons} = \hat{\sigma} \qquad \sigma_{free} = \hat{\sigma} + \frac{1}{\varepsilon_\gamma}\Lambda_I.$$

Note: In this proposition, we do not explicitly treat the case in which $I^* = I = \widetilde{I}$ in order to save on space and notation, since in this case left and right derivatives with respect to I are different.

Proof. We first establish the comparative statics of ω with respect to I, N and K when both $I^* = I < \widetilde{I}$ and $I^* = \widetilde{I} < I$.

Comparative statics for K: The curve $I^* = \min\{I, \widetilde{I}\}$ does not depend on K, all comparative statics are entirely determined by the effect of capital on $\omega(I^*, N, K)$. An increase in K shifts up the relative demand locus in Figure A1 (this does not affect the ideal price index condition, which simplifies the analysis in this case), and thus increases W and reduces R . The impact on $\omega = \frac{W}{RK}$ depends on whether the initial effect on W/R has elasticity greater than one (since K is in the denominator).

Notice that the function $\omega(I^*, N, K)$ *already* incorporates the equilibrium labor supply response. To distinguish this supply response from the elasticity of substitution determined by factor demands, we define $\omega^L(I^*, N, K, L)$ as the static equilibrium for a fixed level of the labor supply L.

The definition of σ_{cons} implies that $\frac{\partial \omega^L}{\partial K}\frac{K}{\omega^L} = \frac{1}{\sigma_{\text{cons}}} - 1$ and $-\frac{\partial \omega^L}{\partial L}\frac{L}{\omega^L} = \frac{1}{\sigma_{\text{cons}}}$. Thus, when $I^* = I < \widetilde{I}$, we have

$$d\ln(W/R) = d\ln\omega + 1 = \left(\frac{1}{\sigma_{\text{cons}}} - 1\right) d\ln K - \frac{1}{\sigma_{\text{cons}}}\varepsilon_L d\ln\omega + d\ln K = \frac{1+\varepsilon_L}{\sigma_{\text{cons}} + \varepsilon_L} d\ln K,$$

where we have used the fact that $\omega(I^*, N, K) = \omega^L(I^*, N, K, L^s(\omega))$. This establishes the claims about the comparative statics with respect to K when $I^* = I < \widetilde{I}$.

For the case where $I^* = \widetilde{I} < I$, we have that the change in K also changes the threshold task $I^* = \widetilde{I}$. In particular, $dI^* = \frac{1}{\varepsilon_\gamma} d\ln\omega$. Thus,

$$d\ln(W/R) = \frac{1+\varepsilon_L}{\sigma_{cons} + \varepsilon_L} d\ln K + \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma} d\ln(W/R) = \frac{1+\varepsilon_L}{\sigma_{cons} + \varepsilon_L}\frac{1}{1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I}\frac{1}{\varepsilon_\gamma}} d\ln K = \frac{1+\varepsilon_L}{\sigma_{\text{free}} + \varepsilon_L} d\ln K,$$

where we define σ_{free} as in the proposition.

Comparative statics with respect to I**:** The relative demand locus $\omega = \omega(I^*, N, K)$ does not directly depend on I . Thus, the comparative statics are entirely determined by the effect of changes in I on the $I^* = \min\{I, \widetilde{I}\}$ schedule depicted in Figure 3. When $I^* = \widetilde{I} < I$, small changes in I have no effect as claimed in the proposition. Suppose next that $I^* = I < \widetilde{I}$. In this case, an increase in I shifts the curve $I^* = \min\{I, \widetilde{I}\}$ to the right in Figure 3. Lemma A1 implies that $\omega(I^*, N, K)$ is decreasing in I^*. Thus, the shift in I increases I^* and reduces ω—as stated in the proposition. Moreover, because $I^* = I$, we have

$$\frac{d\ln(W/R)}{dI} = \frac{d\ln\omega}{dI^*} = \frac{1}{\omega}\frac{\partial\omega}{\partial I^*} < 0,$$

where $\frac{\partial\omega}{\partial I^*}$ denotes the partial derivative of $\omega(I^*, N, K)$ with respect to I^*.

Comparative statics for N: From Lemma A1, changes in N only shift the relative demand curve up in Figure 3. Hence, when $I^* = I < \widetilde{I}$, we have

$$\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN} = \frac{1}{\omega}\frac{\partial\omega}{\partial N} > 0,$$

where $\frac{\partial\omega}{\partial N}$ denotes the partial derivative of $\omega(I^*, N, K)$ with respect to N.

Turning next to the case where $I^* = \widetilde{I} < I$, note that the threshold task is given by $\gamma(I^*) = \omega K$. Therefore, $dI^* = \frac{1}{\varepsilon_\gamma} d\ln\omega$ (where recall that ε_γ is the semi-elasticity of the γ function as defined in the proposition). Therefore, $\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN}$, and we can compute this total derivative as claimed in proposition:

$$\frac{d\ln\omega}{dN} = \frac{1}{\omega}\frac{\partial\omega}{\partial N} + \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}\frac{d\ln\omega}{dN} = \frac{\frac{1}{\omega}\frac{\partial\omega}{\partial N}}{1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}} = \frac{\sigma_{\text{ cons}} + \varepsilon_L}{\sigma_{\text{free}} + \varepsilon_L}\frac{1}{\omega}\frac{\partial\omega}{\partial N}.$$

To conclude the proposition, we specialize to the case in which Assumption 2 holds. The explicit expressions for the partial derivative $\frac{\partial\omega}{\partial I^*}$, $\frac{\partial\omega}{\partial N}$ and $\hat{\sigma}$ presented in the proposition follow directly from differentiating equation (13) in the main text. Finally, the definition of σ_{free} in the proposition implies that, in this case:

$$\sigma_{\text{free}} = (\hat{\sigma} + \varepsilon_L)\left(1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\right) - \varepsilon_L = \hat{\sigma} + \frac{1}{\varepsilon_\gamma}\Lambda_I,$$

which proofs the claims in Proposition 2 in the main text. ■

Proof of Proposition 3: The formulas provided for $d\ln Y\mid_{K,L}$ in this proposition only apply when Assumption 2 holds. Thus, we only prove this proposition for the particular model analyzed in the main text.

We start by deriving the formulas for $d\ln Y\mid_{K,L}$ in the case in which technology binds and $I^* = I < \widetilde{I}$. To do so, we first consider a change in dN and totally differentiate equation (12) in the main text. We get:

$$\begin{aligned}
d\ln Y\mid_{K,L} =& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\gamma(N)^{\hat{\sigma}-1}\left(\frac{\int_{I^*}^N \gamma(i)^{\hat{\sigma}-1}di}{L}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \left(\frac{I^*-N+1}{K}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dN \\
=& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\gamma(N)^{\hat{\sigma}-1}\left(\frac{B^{1-\hat{\sigma}}W^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \left(\frac{B^{1-\hat{\sigma}}R^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dN \\
=& B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.
\end{aligned}$$

Likewise, following a change in dI^* we get:

$$d\ln Y\mid_{K,L} =\frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\left(\frac{I^*-N+1}{K}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}-\gamma(I)^{\hat{\sigma}-1}\left(\frac{\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di}{L}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dI$$

$$=\frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\left(\frac{B^{1-\hat{\sigma}}R^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}-\gamma(I)^{\hat{\sigma}-1}\left(\frac{B^{1-\hat{\sigma}}W^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dI$$

$$=B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}}-R^{1-\hat{\sigma}}\right)dI.$$

We now derive the formulas for the impact of technology on factor prices. Let s_L denote the labor share in net output. Because $WL+RK=(1-\eta)Y$, we obtain:

$$s_L d\ln W+(1-s_L)d\ln R=d\ln Y\mid_{K,L}. \tag{B9}$$

Moreover, Proposition 2 implies that

$$d\ln W-d\ln R=\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN-\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI. \tag{B10}$$

Solving the system of equations given by (B9) and (B10), we obtain the formulas for $d\ln W$ and $d\ln R$ in the proposition.

Now, consider the case in which $I^*=\widetilde{I}<I$. In this case we have:

$$d\ln Y\mid_{K,L}=B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}}-\left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN+B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(\widetilde{I})}\right)^{1-\hat{\sigma}}-R^{1-\hat{\sigma}}\right)dI^*$$

$$=B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}}-\left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.$$

Thus, changes in I^* do not affect aggregate output because the marginal firm at $\widetilde{I}$ is indifferent between producing with capital or producing with labor. On the other hand, because I is not binding, changes in I do not affect aggregate output.

We derive the formulas for the impact of technology on factor prices as before, with the difference that now equation (B10) becomes

$$d\ln W-d\ln R=\frac{1}{\sigma_{\text{free}}+\varepsilon_L}\Lambda_N dN.$$

■

Remaining Proofs from Section 3

We start by providing an additional lemma showing that, for a path of technology in which $g(t)=g$ and $n>\max\{\overline{n},\widetilde{n}(\rho)\}$, the resulting production function $F(k,L;n)$ satisfies the Inada conditions required in a BGP.

Lemma B1 (Inada conditions) *Suppose that Assumptions 1′ and 2 hold. Consider a path of technology in which $n(t) \to n$ and $g(t) \to g$. Let $F(k, L; n)$ denote net output introduced in the proof of Proposition 4. If $n \geq \max \overline{n}(\rho), \widetilde{n}(\rho)$ we have F satisfies the Inada conditions:*

$$\lim_{\phi \to 0} F_K(\phi, 1; n) > \rho + \delta + \theta g \qquad \lim_{\phi \to \infty} F_K(\phi, 1; n) < \rho + \delta + \theta g.$$

Proof. Let $\phi = \frac{k}{L}$. Let $MPK(\phi) = F_K(\phi, 1; n)$ and $w(\phi) = F_L(\phi, 1; n)$ denote the rental rate of capital and the wage at this ratio, respectively.

When $n \geq \max \overline{n}(\rho), \widetilde{n}(\rho)$, these factor prices satisfy the system of equations given by the ratio of the market clearing conditions (A2) and (A3),

$$\phi = \frac{(1-n)c^u(MPK(\phi))^{\zeta-\sigma} MPK(\phi)^{-\zeta}}{\int_0^n \gamma(i)^{\zeta-1} c^u(w(\phi)/\gamma(i))^{\zeta-\sigma} w(\phi)^{-\zeta}},$$

coupled with the ideal price index condition in equation (A4), which we can rewrite succinctly as:

$$B^{1-\hat{\sigma}} = (1-n)c^u(MPK(\phi))^{1-\sigma} + \int_0^n c^u(w(\phi)/\gamma(i))^{1-\sigma} di. \tag{B11}$$

We start by considering the limit case in which $\phi = 0$. The factor-demand equation requires that either (i) $MPK(\phi) = \infty$, or (ii) $w(\phi) = 0$. In the first case, we have $MPK(\phi) > \rho + \delta + \theta g$ as claimed. In the second case we have:

$$c^u(0) = \begin{cases} 0 & \text{if } \zeta \geq 1 \\ c_0^u & \text{if } \zeta < 1. \end{cases}$$

We show that in both cases $MPK(0) > \rho + \delta + \theta g$:

1. Suppose that $\zeta \geq 1$. For the ideal price index condition in (B11) to hold, we require $1 > \sigma$ (otherwise the right-hand side diverges). Moreover, the ideal price index condition in (B11) implies that $MPK(0)$ is implicitly given by:

   $$(1-n)c^u(MPK(0))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

   First, suppose that $\rho \leq \overline{\rho}$. We have that

   $$c^u(MPK(0))^{1-\sigma} > (1-n)c^u(MPK(0))^{1-\sigma} = B^{1-\hat{\sigma}} = c^u(\overline{\rho} + \delta + \theta g)^{1-\sigma}.$$

   Here we have used the fact that $n \geq 0$ and the definition of $\overline{\rho}$ introduced in Lemma A2. Because $1 > \sigma$, the above inequality implies $MPK(0) > \overline{\rho} + \delta + \theta g \geq \rho + \delta + \theta g$ as claimed.

   Finally, suppose that $\rho > \overline{\rho}$. Because $n \geq \overline{n}(\rho)$, we have:

   $$\begin{aligned}(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} &\geq (1-n)c^u(MPK(0))^{1-\sigma} \\ &= B^{1-\hat{\sigma}} \\ &= (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\overline{n}(\rho)} c^u((\rho+\delta+\theta g)/\gamma(i))^{1-\sigma} di \\ &> (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma}.\end{aligned}$$

Here we have also used the definition of $\overline{n}(\rho)$ introduced in Lemma A2. Because $1 > \sigma$, the above inequality implies $MPK(0) > \rho + \delta + \theta g$ as claimed.

2. Suppose that $\zeta < 1$. We have that $0 < c_0^u \leq c^u(x) \forall x$. The ideal price index condition in (B11) implies that $MPK(0)$ is implicitly given by:

$$(1-n)c^u(MPK(0))^{1-\sigma} + nc_0^{u\,1-\sigma} = B^{1-\hat{\sigma}}.$$

When $1 > \sigma$, we have the following series of inequalities:

$$\begin{aligned}(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma} &\geq (1-n)c^u(MPK(0))^{1-\sigma} + nc_0^{u\,1-\sigma} \\ &= B^{1-\hat{\sigma}} \\ &= (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} \\ &+ \int_0^{\overline{n}(\rho)} c^u((\rho+\delta+\theta g)/\gamma(i))^{1-\sigma} di \\ &> (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma}.\end{aligned}$$

Here, we have used the fact that $n \geq \overline{n}(\rho)$ and $0 < c_0^u \leq c^u(x)\forall x$, and the definition of $\overline{n}(\rho)$ introduced in Lemma A2. Because $1 > \sigma$, the above inequality implies $MPK(0) > \rho + \delta + \theta g$ as claimed.

When $1 < \sigma$, the previous inequalities are reversed, and we get:

$$(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma} < (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma}.$$

Because $1 < \sigma$, the above inequality implies $MPK(0) > \rho + \delta + \theta g$ as claimed.

We next consider the limit case in which $\phi = \infty$. With a slight abuse of notation, we define $MPK(\infty) = \lim_{\phi\to\infty} MPK(\phi)$ and $w(\infty) = \lim_{\phi\to\infty} w(\phi)$. The factor-demand equation requires that either (i) $MPK(\infty) = 0$, or (ii) $w(\infty) = \infty$. In the first case, $MPK(\infty) < \rho + \delta + \theta g$. In the second case we have:

$$c^u(0) = \begin{cases} \infty & \text{if } \zeta \geq 1 \\ c_\infty^u & \text{if } \zeta > 1. \end{cases}$$

We show that in both cases $MPK(\infty) < \rho + \delta + \theta g$.

1. Suppose that $\zeta \leq 1$. For the ideal price index condition in (B11) to hold, we require $\sigma > 1$ (otherwise the right-hand side diverges). Moreover, the ideal price index condition in (B11) implies that $MPK(\infty)$ is implicitly given by:

$$(1-n)c^u(MPK(\infty))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

First, suppose that $\rho \geq \overline{\rho}$. We have that

$$c^u(MPK(\infty))^{1-\sigma} > (1-n)c^u(MPK(\infty))^{1-\sigma} = B^{1-\hat{\sigma}} = c^u(\overline{\rho}+\delta+\theta g)^{1-\sigma}.$$

Here we have used the fact that $n \geq 0$ and the definition of $\overline{\rho}$ introduced in Lemma A2. Because $\sigma > 1$, the above inequality implies $MPK(\infty) < \overline{\rho} + \delta + \theta g \leq \rho + \delta + \theta g$as claimed.

Finally, suppose that $\rho < \overline{\rho}$. Because $n \geq \widetilde{n}(\rho)$, we have:

$$
\begin{aligned}
(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} &\geq (1-n)c^u(MPK(\infty))^{1-\sigma} \\
&= B^{1-\hat{\sigma}} \\
&= (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\widetilde{n}(\rho)} c^u((\rho+\delta+\theta g)\gamma(i))^{1-\sigma} di \\
&> (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma}.
\end{aligned}
$$

Here we have also used the definition of $\widetilde{n}(\rho)$ introduced in Lemma A2. Because $\sigma > 1$, the above inequality implies $MPK(\infty) < \rho + \delta + \theta g$ as claimed.

2. Suppose that $\zeta > 1$. We have that $0 < c^u(x) \leq c^u_\infty \forall x$. The ideal price index condition in (B11) implies that $MPK(\infty)$ is implicitly given by:

$$(1-n)c^u(MPK(\infty))^{1-\sigma} + n{c^u_\infty}^{1-\sigma} = B^{1-\hat{\sigma}}.$$

When $1 > \sigma$, we have the following series of inequalities:

$$
\begin{aligned}
(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma} &\leq (1-n)c^u(MPK(\infty))^{1-\sigma} + n{c^u_\infty}^{1-\sigma} \\
&= B^{1-\hat{\sigma}} \\
&= (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} \\
&+ \int_0^{\widetilde{n}(\rho)} c^u((\rho+\delta+\theta g)\gamma(i))^{1-\sigma} di \\
&< (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \overline{n}(\rho){c^u_\infty}^{1-\sigma}.
\end{aligned}
$$

Here, we have used the fact that $n \geq \widetilde{n}(\rho)$ and $0 < c^u(x) \leq c^u_\infty \forall x$, and the definition of $\widetilde{n}(\rho)$ introduced in Lemma A2. Because $1 > \sigma$, the above inequality implies $MPK(\infty) < \rho+\delta+\theta g$ as claimed.

When $1 < \sigma$, the previous inequalities are reversed, and we get:

$$(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma} > (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma}.$$

Because $1 < \sigma$, the above inequality implies $MPK(\infty) < \rho + \delta + \theta g$ as claimed.

■

Proof of Global Stability for Part 2 of Proposition 4: Here we provide the details of global stability of the interior equilibrium where all automated tasks are immediately produced with capital (part 2 of Proposition 4). In particular, we show that the BGP given by $k(t) = k_B$, $c(t) = c_B$ and $L(t) = L_B$ is globally stable.

For a given level of capital and consumption, we can define the equilibrium labor supply schedule, $L^E(k,c)$, implicitly as the solution to the first-order condition:

$$\nu'^E(k,c))e^{\nu(L^E(k,c))\frac{\theta-1}{\theta}} = \frac{F_L(k, L^E(k,c))}{c}.$$

The left-hand side of this equation is increasing in L^E. Thus, the optimal labor supply $L^E(k,c)$ is increasing in k (because of the substitution effect) and is decreasing in c (because of the income effect). In addition, because F_L is homogeneous of degree zero, one can verify that $\frac{L}{k} > L_k^E > 0$, so that labor responds less than one-to-one to an increase in capital.

Any dynamic equilibrium must solve the system of differential equations:

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} =& \frac{1}{\theta}(F_K(k(t), L^E(k(t), c(t)); n) - \delta - \rho) - g \\ \dot{k}(t) =& F(k(t), L^E(k(t), c(t)); n) - (\delta + g)k(t) - c(t)e^{\nu(L^E(k(t),c(t)))\frac{\theta-1}{\theta}},\end{aligned}$$

together with the transversality condition in equation (19).

We analyze this system in the (c,k) space. We always have one of the two cases portrayed in Figure B1; either $\lim_{c\to 0} L^E(k,c) = \overline{L}$ or $\lim_{c\to 0} L^E(k,c) = \infty$.

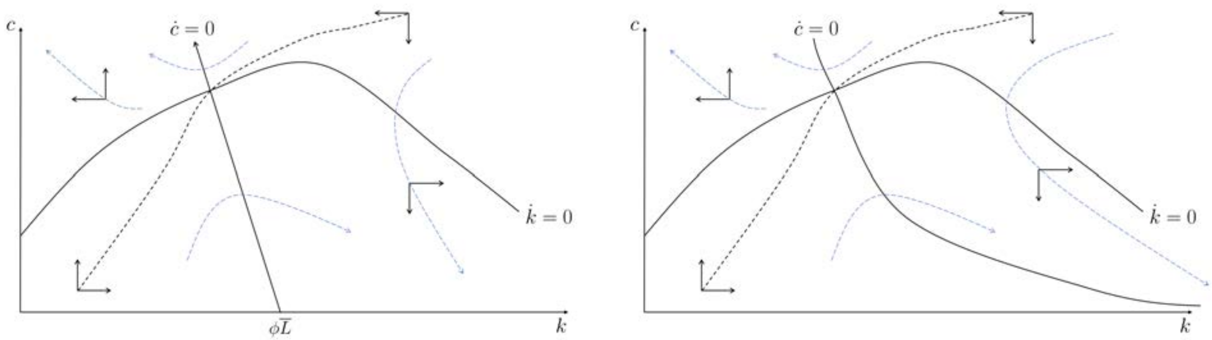

Figure B1: The left panel shows the phase diagram of the equilibrium system when $\lim_{c\to 0} L^E(k,c) = \overline{L}$. The right panel shows the phase diagram of the equilibrium system when $\lim_{c\to 0} L^E(k,c) = \infty$.

The locus for $\dot{k} = 0$ yields a curve that defines the maximum level of consumption that can be sustained at each level of capital. This level is determined implicitly by:

$$F(k, L^E(k,c); n) - (\delta + g)k = ce^{\nu(L^E(k,c))\frac{\theta-1}{\theta}}.$$

The locus for $\dot{c} = 0$ is given by $k = \phi L^E(k,c)$, which defines a decreasing curve between c and k. Depending on whether $\nu'(L)$ has a vertical asymptote or not, as $c \to 0$, this locus converges to $k = \phi\overline{L}$ (left panel in figure B1) or $k = \infty$ (right panel in figure B1).

Importantly, we always have that, as $c \to 0$, the locus for $\dot{k} = 0$ is above the locus for $\dot{c} = 0$. This is clearly the case when $\lim_{c\to 0} L^E(k,c) = \overline{L}$. To show this when $\lim_{c\to 0} L^E(k,c) = \infty$, consider a point (c_0, k_0) in the locus for $\dot{k} = 0$. We have that

$$F_K\left(1, \frac{L^E(k_0,c_0)}{k_0}\right) < F\left(1, \frac{L^E(k_0,c_0)}{k_0}\right) = \delta + g + \mathcal{O}(c_0)$$

Thus, for $c_0 \to 0$, the condition $\rho + (\theta - 1)g > 0$ implies that

$$F_K\left(1, \frac{L^E(k_0, c_0)}{k_0}\right) < \rho + \delta + \theta g.$$

This inequality implies that $\frac{L^E(k_0,c_0)}{k_0} < \phi$, which is equivalent to the point (c_0, k_0) being in the northeast region of the locus for $\dot{c} = 0$.

As shown in Appendix A, both when $\lim_{c\to 0} L^E(k,c) = \overline{L}$ or $\lim_{c\to 0} L^E(k,c) = \infty$, we have a unique interior equilibrium at (c_B, k_B). Moreover, because as $c \to 0$ the locus for $\dot{c} = 0$ is below the locus for $\dot{k} = 0$, we must have that the locus for $\dot{c} = 0$ always cuts the locus for $\dot{k} = 0$ from above at (c_B, k_B). Thus, as shown in the phase diagrams in Figure B1 the unique interior equilibrium at (c_B, k_B) is saddle-path stable.

One could also establish the saddle-path stability locally as follows. Around the interior BGP, the system of differential equations that determines the equilibrium can be linearized as (suppressing the arguments of the derivatives of the production function)

$$\begin{aligned}\dot{k}(t) &= \left(\rho + (\theta-1)g + \frac{1}{\theta}F_L L_k^E\right)(k(t) - k_B) + \left(-e^{\nu(L_B)\frac{\theta-1}{\theta}} + \frac{1}{\theta}F_L L_c^E\right)(c(t) - c_B)\\ \dot{c}(t) &= \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)(k(t) - k_B) + \frac{c_B}{\theta}F_{KL}L_c^E(c(t) - c_B).\end{aligned}$$

The characteristic matrix of the system is therefore given by

$$M_{\text{exog}} = \begin{pmatrix} \rho + (\theta-1)g + \frac{1}{\theta}F_L L_k^E & -e^{\nu(L_B)\frac{\theta-1}{\theta}} + \frac{1}{\theta}F_L L_c^E \\ \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E) & \frac{c_B}{\theta}F_{KL}L_c^E \end{pmatrix}.$$

To analyze the properties of this matrix, we will use two facts: that $F_L L_K^E + c_B F_{KL} L_c^E = 0$ and $F_{KK} + F_{KL}L_K^E < 0$. The first fact follows by implicitly differentiating the optimality condition for labor, which yields:

$$L_k^E = \frac{\frac{1}{c}F_{Lk}}{e^{\nu(L)\frac{\theta-1}{\theta}}\left(\nu''(L) + \frac{\theta-1}{\theta}\right) - \frac{1}{c}F_{LL}} \qquad L_c^E = -\frac{\frac{1}{c^2}F_L}{e^{\nu(L)\frac{\theta-1}{\theta}}\left(\nu''(L) + \frac{\theta-1}{\theta}\right) - \frac{1}{c}F_{LL}}.$$

The second fact follows by noting that, because $L_K^E < \frac{L^E}{k}$, we have

$$F_{KK} + F_{KL}L_K^E < F_{KK} + F_{KL}\frac{L^E}{k} = 0.$$

Using these facts, we can compute the trace of M_{exog}:

$$\text{Tr}(M_{\text{exog}}) = \rho + (\theta-1)g + \frac{1}{\theta}F_L L_k^E + \frac{c_B}{\theta}F_{KL}L_c^E = \rho + (\theta-1)g > 0.$$

In addition, the determinant of M_{exog} is given by:

$$\text{Det}(M_{\text{exog}}) = \frac{c_B}{\theta}F_{KL}L_c^E\left(\rho + (\theta-1)g + \frac{1}{\theta}F_L L_k^E\right) - \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L_B)\frac{\theta-1}{\theta}}\right) < 0.$$

The inequality follows by noting that $F_{KL}L_c^E < 0$ $\rho + (\theta-1)g + \frac{1}{\theta}F_L L_k^E > 0$ $F_{KK} + F_{KL}L_k^E < 0$ and $\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} < 0$.

(The negative determinant is equivalent to the fact established above that the curve for $\dot{c} = 0$ cuts the curve for $\dot{k} = 0$ from above. Moreover, the algebra here shows that, at the interception point (c_B, k_B), the locus for $\dot{k}$ is increasing).

The sign of the trace and the determinant imply that the matrix has one positive and real eigenvalue and one negative and real eigenvalue. Theorem 7.19 in Acemoglu (2009) shows that, locally, the economy with exogenous technology is saddle-path stable as wanted.

To show the global stability of the unique BGP (c_B, k_B), all we need is to discard two types of paths: the candidate paths that converge to zero capital, which we will show are not feasible, and the candidate paths that converge to zero consumption, which we will show are not optimal.

To rule out the paths that converge to zero capital, note that all of them converge to an allocation with $k(t) = 0$ and $c(t) > \underline{c}$. Here $\underline{c} \geq 0$ is the maximum level of consumption that can be sustained when $k = 0$, which is given by:

$$F(0, L^E(0, \underline{c})) = \underline{c} e^{\nu(L^E(0,\underline{c}))\frac{\theta-1}{\theta}}.$$

To rule out the paths that converge to zero consumption, we show that they violate the transversality condition in equation (19). In all these paths we have $c(t) \to 0$. There are two possible paths for capital. Either capital converges to $\overline{k}$—even at zero consumption the economy only sustains a finite amount of capital—, or capital grows with no bound. In the first case, note that:

$$F_K\left(1, \frac{L^E(k,c)}{k}\right) \leq F\left(1, \frac{L^E(k,c)}{k}\right) = \delta + g.$$

Thus, the transversality condition in (19) does not hold. In the second case, we have that capital grows at an asymptotic rate of $F\left(1, \frac{L^E(k,c)}{k}\right) - \delta - g$. This is greater than or equal to the discount rate used in the transversality condition in equation (19), which is $F_K\left(1, \frac{L^E(k,c)}{k}\right) - \delta - g$. Thus, the transversality condition does not hold in this case either. ■

Proof of Proposition 5: We prove the proposition in the more general case in which Assumption 2' holds.

Proposition 4 shows that for this path of technology the economy admits a unique BGP.

If $n < \overline{n}(\rho)$, we have that in the BGP $n^*(t) = \overline{n}(\rho) > n$. Thus, small changes in n do not affect the BGP equations; n does not affect effective wages, employment, or the labor share.

If $n > \overline{n}(\rho)$, we have that in the BGP $n^*(t) = n$. In this case, the behavior of the effective wages follows from the formulas for $w_I'(n)$ and $w_N'(N)$ in equation (A7), whose signs can be determined given the ordering between $w_I(n)$, $\rho + \delta + \theta g$, and $w_N(n)$ derived in Lemma A2.

To characterize the behavior of employment, note that we can rewrite the first order condition for the BGP level of employment in equation (17) as

$$\frac{1}{L\nu'(L)} = \frac{c}{wL} = \frac{1}{s_L}\frac{\rho + (\theta - 1)g}{\rho + \delta + \theta g} + \frac{\delta + g}{\rho + \delta + \theta g}.$$

It follows that, asymptotically, there is a positive relationship between employment and the labor share. Thus, we can denote the BGP level of employment by an increasing function $L^{LR}(\omega)$, whose elasticity we denote by ε_L^{LR}.

To characterize the behavior of the labor share we use Lemma A1. This lemma was derived for the static model when the labor supply was given by $L^s(\omega)$, but we can use it here to describe the asymptotic behavior of the economy when the supply of labor is given by $L^{LR}(\omega)$. In what follows we use the same terminology employed in Proposition 2.

We consider two cases. First, suppose that $\sigma_{\text{const}} \leq 1$. Let $k_I(n)$ denote the BGP value for $K(t)/\gamma(I(t))$. The definition of $\omega(I^*, N, K)$ implies that

$$\omega(0, n, k_I(n)) = \frac{w_I(n)}{(\rho + \delta + \theta g) k_I(n)}.$$

Differentiating this expression, we obtain

$$k_I'(n) = \frac{w_I'(n)\frac{1}{Rk} - \frac{\partial \omega}{\partial N}}{\frac{\omega}{k}\frac{1+\varepsilon_L^{LR}}{\sigma_{\text{const}}+\varepsilon_L^{LR}}}.$$

Using this expression for $k_I'(n)$, it follows that the total effect of technology on ω is given by

$$\begin{aligned}\frac{d\omega}{dn} =& \frac{\partial \omega}{\partial N} + \frac{\partial \omega}{\partial K} k_I'(n) \\ =& \frac{\partial \omega}{\partial N}\left(\frac{\sigma_{\text{const}} + \varepsilon_L^{LR}}{1+\varepsilon_L^{LR}}\right) + \frac{w_I'(n)}{Rk}\left(\frac{1-\sigma_{\text{const}}}{1+\varepsilon_L^{LR}}\right).\end{aligned}$$

Because $\frac{\partial \omega}{\partial N} > 0$ and $w_I'(n) > 0$, we have that, whenever $\sigma_{\text{const}} \leq 1$, ω is increasing in n. Moreover, because the BGP level of employment is given by the increasing function $L^{LR}(\omega)$, n raises employment too.

Next suppose that $\sigma_{\text{const}} > 1$. Let $k_N(n)$ denote the BGP value for $K(t)/\gamma(N(t))$. The definition of $\omega(I^*, N, K)$ implies that

$$\omega(-n, 0, k_N(n)) = \frac{w_N(n)}{(\rho + \delta + \theta g) k_N(n)}.$$

Differentiating this expression, we have

$$k_N'(n) = \frac{w_N'(n)\frac{1}{Rk} + \frac{\partial \omega}{\partial I^*}}{\frac{\omega}{k}\frac{1+\varepsilon_L^{LR}}{\sigma_{\text{const}}+\varepsilon_L^{LR}}} < 0.$$

Using this expression for $k_N'(n)$, it follows that the total effect of technology on ω is given by

$$\begin{aligned}\frac{d\omega}{dn} =& -\frac{\partial \omega}{\partial I^*} + \frac{\partial \omega}{\partial K} k_N'(n) \\ =& -\frac{\partial \omega}{\partial I^*}\left(\frac{\sigma_{\text{const}} + \varepsilon_L^{LR}}{1+\varepsilon_L^{LR}}\right) + \frac{w_N'(n)}{Rk}\left(\frac{1-\sigma_{\text{const}}}{1+\varepsilon_L^{LR}}\right).\end{aligned}$$

Because $\frac{\partial \omega}{\partial I^*} < 0$ and $w_N'(n) < 0$, we have that, whenever $\sigma_{\text{const}} \geq 1$, ω is increasing in n. Moreover, because the BGP level of employment is given by the increasing function $L^{LR}(\omega)$, n raises employment too.

The previous observations show that automation reduces the labor share in the long run. In addition, we have shown that $k_N'(n) < 0$, which implies that in response to automation, capital increases above its trend. The induced capital accumulation implies that the impact of automation on the labor share worsens over time if $\sigma_{\text{const}} > 1$ and eases if $\sigma_{\text{const}} < 1$.

(When Assumption 2 holds the capital share is given by $s_K(t) = (1-n)\left(\frac{R(t)}{B}\right)^{1-\hat{\sigma}}$. In this case, it follows trivially that n reduces the capital share and thus increases the labor share. This expression also shows that, by bringing back the rental rate of capital to its BGP level, the induced capital accumulation may worsen the decline in the labor share if $\hat{\sigma} > 1$ or partially offset it if $\hat{\sigma} < 1$.) ■

Remaining Proofs from Section 4

Proof of Lemma A3: In a BGP we have that the economy grows at the rate $g = A\frac{\kappa_I \kappa_N}{\kappa_I + \kappa_N}S$.

Suppose that $n \geq \max\{\overline{n}, \widetilde{n}\}$, we can write the value functions in the BGP as:

$$v_N(n) = b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(w_N(n)e^{g\tau}\right)^{\zeta-\sigma} - c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}\right]d\tau,$$
$$v_I(n) = b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left(w_I(n)e^{g\tau}\right)^{\zeta-\sigma}\right]d\tau.$$

Thus, the value functions only depend on the unit cost of labor $w_N(n)$ and $w_I(n)$, and on the rental rate, which is equal to $\rho + \delta + \theta g$ in the BGP.

Now consider Taylor expansions of both of these expressions (which are continuously differentiable) around $S = 0$—so that the growth rate of the economy is small:

$$v_N(n) = \frac{b}{\rho}\left[c^u\left(w_N(n)\right)^{\zeta-\sigma} - c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}\right] + \mathcal{O}(g), \tag{B12}$$
$$v_I(n) = \frac{b}{\rho}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left(w_I(n)\right)^{\zeta-\sigma}\right] + \mathcal{O}(g).$$

Because $\mathcal{O}(g) \to 0$ as $S \to 0$, we can approximate the above integrals when S is small with the explicit expressions evaluated at $g = 0$.

Differentiating the value functions in (B12) establishes that they are both strictly increasing in n. This follows from the result established in Proposition 5 that, in this region, $w_I(n)$ increases in n and $w_N(n)$ decreases in n. Moreover, as $S \to 0$ we find that both $v_N(n)$ and $v_I(n)$ are positive. Thus, there exists $\widetilde{S}_1$ such that for $S < \widetilde{S}_1$, both $v_N(n)$ and $v_I(n)$ are positive ans strictly increasing in n.

Now suppose that $n \leq \overline{n}(\rho)$ (this case requires that $\rho > \overline{\rho}$). In this region we have $n^*(t) = \overline{n}$ and therefore $v_N(n) = v_N(\overline{n})$ and $v_I(n) = v_I(\overline{n})$. The value of automating a task is given by:

$$\begin{aligned} v_I(n) &= v_I(\overline{n}) \\ &= b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left(\min\{w_I(\overline{n}(\rho))\gamma(n-\overline{n}(\rho))e^{g\tau}, \rho+\delta+\theta g\}\right)^{\zeta-\sigma}\right]d\tau. \\ &= b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left((\rho+\delta+\theta g)\min\{\gamma(n-\overline{n}(\rho))e^{g\tau}, 1\}\right)^{\zeta-\sigma}\right]d\tau. \end{aligned}$$

The min operator $\min\{w_I(\overline{n}(\rho))\gamma(n-\overline{n}(\rho))e^{g\tau}, \rho+\delta+\theta g\}$ shows that a task that is automated at time t will only generate profits in the future starting at a time $\tau > t$ such that $I^*(\tau) = I(t)$. At this point in time, $w_I(n)\gamma(n-\overline{n}(\rho))e^{g\tau} = \rho+\delta+\theta g$, and it becomes profitable to use capital to produce the automated task.

Because $\lim_{g\to 0} v_I(n) = 0$, we have $v_I(\overline{n}) = \mathcal{O}(g)$. On the other hand, we have $v_N(n) = v_N(\overline{n})$ and this value function remains bounded away from zero as $S \to 0$. Thus, there exists $\widetilde{S}_2 > 0$ such that for $S < \widetilde{S}_2$, $\kappa_N v_N(\overline{n}) > \kappa_I v_I(\overline{n}) > 0$ as claimed, and $v_I(\overline{n}) = \mathcal{O}(g)$.

Finally, consider the case in which $n < \widetilde{n}(\rho)$ (this case requires that $\rho < \overline{\rho}$). Because $w_N(n)e^{g\tau} > \rho+\delta+\theta g$ and $w_I(n)e^{g\tau} > \rho+\delta+\theta g$ for all $\tau \geq 0$, it follows that, in this region, $v_I(n) > 0 > v_N(n)$ as claimed. Moreover, the derivatives for $w_I(n)$ and $w_N(n)$ in equation (A7) imply that, in this region, both $w_I(n)$ and $w_N(n)$ are decreasing in n. Thus, in this region, $v_I(n)$ is decreasing and $v_N(n)$ is increasing in n.

To finalize the proof of the Lemma we simply take $\widetilde{S} = \min\{\widetilde{S}_1, \widetilde{S}_2\}$ if $\theta \geq 1$, and $\widetilde{S} = \min\{\widetilde{S}_1, \widetilde{S}_2, \frac{\rho(\kappa_I+\kappa_N)}{(1-\theta)A\kappa_I\kappa_N}\}$ if $\theta < 1$. This choice also ensures that $\rho + (\theta-1)g > 0$ as required in the Lemma. ■

Proof of local stability for the unique BGP when $\theta > 0$: The local stability analysis applies to the case in which $\rho > \overline{\rho}$, $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. In this case, the economy admits a unique BGP.

An equilibrium is given by a solution to the following system of differential equations: Starting from any $n(0), k(0)$ the equilibrium with endogenous technology can be summarized by paths for $\{c(t), k(t), n(t), v(t), S_I(t)\}$ such that:

1. The normalized consumption satisfies the Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(F_K(k,L;n) - \delta - \rho) + \mathcal{O}(g).$$

2. The endogenous labor supply is given by $L^E(k,c;n)$, and is defined implicitly by:

$$c\nu'^{\nu(L)\frac{\theta-1}{\theta}} \geq F_L(k,L;n),$$

   with equality if $L^E(k,c;n) > 0$.

3. Capital satisfies the resource constraint:

$$\dot{k} = F(k,L;n) + X(k,L;n) - \delta k - ce^{\nu(L)\frac{\theta-1}{\theta}} + \mathcal{O}(g).$$

   Here, $X(k,L;n) = b(1-n^*)yc^u(F_K)^{\zeta-\sigma} + by\int_0^{n^*} c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di$ are the profits from the intermediate sales.

4. Households face the transversality condition:

$$\lim_{t\to\infty} (k(t) + \pi(t))e^{-\int_0^t F_K(k(s),L(s);n(s))-\delta-\mathcal{O}(g))ds} = 0,$$

   where $\pi(t) = I(t)v_I(t) + N(t)v_N(t)$ are (the normalized) corporate profits.

5. Technology evolves endogenously according to:

$$\dot{n} = \kappa_N S - (\kappa_I + \kappa_N) G(v) S.$$

6. The value function, $v = \kappa_I v_I - \kappa_N v_N$, satisfies

$$(F_K - \delta - g)v - \dot{v} = b\kappa_I \pi_I(k, L; n) - b\kappa_N \pi_N(k, L; n) + \mathcal{O}(g).$$

Around $g = 0$, the above system of differential equations is Lipschitz continuous (on their right-hand side, the equations for $\dot{c}$, $\dot{k}$, $\dot{n}$ and $\dot{v}$ have bounded derivatives around the BGP $\{c_B, k_B, n_B, v_B\}$. This can be seen from the matrix containing these derivatives M_{endog}, which we present below). Thus, from the theorem of the continuous dependence of trajectories of a dynamical system on parameters (e.g., Walter, 1998, page 146, Theorem VI), there exists a neighborhood of $g = 0$ and a threshold $\overline{S}_1$ such that, for $S < \overline{S}_1$, the trajectories that solve the above system have the same direction as the trajectories of the system evaluated at $g = 0$. In particular, for $S < \overline{S}_1$, the BGP is locally saddle-path stable if and only if it is also locally saddle path stable in the limit in which $g = 0$.

The previous argument shows that, to analyze the local stability of the BGP when $S < \overline{S}_1$, it is enough to analyze the limit case in which $g = 0$. Thus, in what follows we focus on this limit.

As in the proof of Proposition 4, the Euler equation and the resource constraint can be linearized around the BGP (denoted with the subscript B) as follows:

$$\begin{aligned}
\dot{c} =& \frac{c_B}{\theta}(F_{Kn} + F_{KL}L_n^E)(n - n_B) + \frac{c_B}{\theta}F_{KL}L_c^E(c - c_B) + \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)(k - k_B) \\
\dot{k} =& \left(F_n + \frac{1}{\theta}F_L L_n^E + X_n + X_L L_n^E\right)(n - n_B) + \left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)(c - c_B) \\
&+ \left(F_K + \frac{1}{\theta}F_L L_k^E - \delta + X_k + X_L L_k^E\right)(k - k_B).
\end{aligned}$$

Here, X_K, X_L, X_n are the partial derivatives of $X(k, L; n)$ with respect to each of its arguments. Moreover, we have $X_K, X_L > 0$—that is, the demand for intermediates increases with K and L. We show that this property holds under Assumption 4, which imposes that $\sigma > \zeta$, and ensures that capital and labor are complements to intermediates. We first show this for L. We can rewrite X as

$$X = \frac{b}{B^{\hat{\sigma}-1}(1-\eta)} k R^{\zeta} + by \int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di.$$

Thus:

$$\begin{aligned}
X_L =& \frac{b}{B^{\hat{\sigma}-1}(1-\eta)} k\zeta R^{\zeta-1} F_{KL} \\
&+ M y_L \int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di + (\zeta - \sigma) by \int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} \widetilde{s}_L(i) \frac{F_{LL}}{F_L} di > 0.
\end{aligned}$$

Here, $\widetilde{s}_L(i)$ is the share of labor in the production of task i. The above inequality follows from the fact that $y_L > 0$, $F_{KL} > 0$, and $F_{LL} < 0$ (recall that Assumption 4 requires $\sigma > \zeta$).

To show that $X_K > 0$ we first differentiate the labor market clearing condition:

$$\begin{aligned}
\frac{y_k}{y} =&(\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}\widetilde{s}_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}di}+\zeta\frac{F_{Lk}}{F_L}\\
>&(\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}\widetilde{s}_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}di}\\
>&(\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\widetilde{s}_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}di}. \qquad \text{(B13)}
\end{aligned}$$

The last line in inequality (B13) follows from Chebyshev's sum inequality, which applies here because both $\gamma(i)^{\zeta-1}$ and $\widetilde{s}_L(i)$ are increasing in i (when $\zeta > 1$) or both are decreasing in i (when $\zeta < 1$).

Now, we have

$$\begin{aligned}
X_K =&b(1-n)y_k c^u(F_K)^{\zeta-\sigma}+(\zeta-\sigma)b(1-n^*)yc^u(F_K)^{\zeta-\sigma}\widetilde{s}_k\frac{F_{KK}}{F_K}\\
&+by_k\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}di+by(\zeta-\sigma)\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\widetilde{s}_L(i)\frac{F_{Lk}}{F_L}di>0.
\end{aligned}$$

Here $\widetilde{s}_k$ is the share of capital in the production of automated tasks. The inequality follows from the fact that $y_K > 0$, $F_{KK} < 0$ (recall that $\sigma > \zeta$ under Assumption 4), and the inequality in equation (B13) derived above.

(When Assumption 2 holds, we have $X = \frac{\eta}{1-\eta}F(k, L; n)$. In this case, it is clear that $X_K, X_L > 0$).

In addition, let $Q_n, Q_k, Q_c > 0$ denote the change in the incentives to automate (vs the incentives to introduce new tasks) as n, k, c increase. All of these are positive.

The matrix that determines the local stability of the system is given by $M_{\text{endog}} =$

$$\begin{pmatrix}
0 & -(\kappa_I+\kappa_N)G'(0)S & 0 & 0\\
-Q_n & \rho & -Q_c & -Q_k\\
\frac{c^*}{\theta}(F_{Kn}+F_{KL}L_n^E) & 0 & \frac{c^*}{\theta}F_{KL}L_c^E & \frac{c^*}{\theta}(F_{KK}+F_{KL}L_k^E)\\
F_n+\frac{1}{\theta}F_L L_n^E+X_n+X_L L_n^E & 0 & \frac{1}{\theta}F_L L_c^E-e^{\nu(L)\frac{\theta-1}{\theta}}+X_L L_c^E & \rho+\frac{1}{\theta}F_L L_k^E+X_k+X_L L_k^E
\end{pmatrix}.$$

The eigenvalues of this matrix $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ satisfy the following properties:

- The trace satisfies:

$$\text{Tr}(M_{\text{endog}}) = \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2\rho+X_k+X_L L_k^E > 0.$$

  The last inequality follows from the fact that $X_L, L_k^E > 0$

- The determinant satisfies:

$$\begin{aligned}\text{Det}(M_{\text{endog}}) =&\lambda(\kappa_I+\kappa_N)G'(0)S\times\\ &\left[\frac{c_B}{\theta}\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)\left(Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E)\right)\right.\\ &+\frac{c_B}{\theta}\left(\rho+\frac{1}{\theta}F_L L_k^E + X_k + X_L L_k^E\right)\left(Q_c(F_{Kn}+F_{KL}L_n^E) - Q_n F_{KL}L_c^E\right)\\ &+\frac{c_B}{\theta}\left(F_n+\frac{1}{\theta}F_L L_n^E + X_n + X_L L_n^E\right)\left(Q_k F_{KL}L_c^E - Q_c(F_{KK}+F_{KL}L_k^E)\right).\end{aligned}$$

The expression for the determinant can be further simplified by noting that $Q_c(F_{KK} + F_{KL}L_k^E) = Q_k F_{KL} L_c^E$. To show this, note that the impact of k, c on Q—the relative incentives for automation—depends on the ratio $k/L^E(k, c; n)$. For a given value of n, this ratio determines factor prices and hence Q. Let $\phi = \frac{k}{L^E(k,c;n)}$. Then:

$$Q_k = Q_\phi\left(\frac{1}{L} - \frac{k}{L^2}L_k^E\right) \qquad\qquad Q_c = -Q_\phi\frac{k}{L^2}L_c^E.$$

These equations imply:

$$Q_c = -Q_k\frac{kL_c^E}{L-kL_K^E} = Q_k F_{KL}L_c^E\frac{1}{\frac{kL_k^E - L}{k}F_{KL}} = Q_k F_{KL} L_c^E \frac{1}{F_{KK}+F_{KL}L_k^E},$$

which gives the desired identity.

Replacing this expression for Q_c in the determinant, we get

$$\begin{aligned}\text{Det}(M_{\text{endog}}) =&\lambda(\kappa_I+\kappa_N)G'(0)S\times\\ &\left[\frac{c_B}{\theta}\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)\left(Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E)\right)\right.\\ &+\frac{c_B}{\theta}F_{KL}L_c^E\left(\rho+\frac{1}{\theta}F_L L_k^E + X_k + X_L L_k^E\right)\left(Q_k\frac{F_{Kn}+F_{KL}L_n^E}{F_{KK}+F_{KL}L_k^E} - Q_n\right).\end{aligned}$$

Because we are focusing on a BGP in which, asymptotically, $\kappa_I v_I(n)$ intercepts $\kappa_N v_N(n)$ from below, we have

$$Q_n - Q_k\frac{F_{Kn}+F_{KL}L_n^E}{F_{KK}+F_{KL}L_k^E} > 0.$$

(Note that this is equivalent to the derivative of the profit function Q with respect to n when the capital adjusts to keep the interest rate constant. This derivative is positive precisely when $\kappa_I v_I(n)$ intercepts $\kappa_N v_N(n)$ from below). Because $F_{KK} + F_{KL}L_k^E < 0$ (as shown in the proof of Proposition 4), we also have that

$$Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E) < 0.$$

Thus, both terms in the determinant are positive and $\text{Det}(M_{\text{endog}}) > 0$ (recall that $L_c^E < 0$ and $L_k^E > 0$, and $X_L, X_K > 0$).

Let $Z(M_{\text{endog}}) = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2$. We also have that

$$Z(M_{\text{endog}}) = \rho \, \text{Det}(M_{\text{exog}}) + \mathcal{O}(S).$$

Because $\text{Det}(M_{\text{exog}}) < 0$ and this determinant does not depend on S, we have that there exists a $\overline{S}_2$ such that, for $S < \overline{S}_2$ we have that $Z(M_{\text{endog}}) < 0$.

We now show that the matrix has exactly two eigenvalues with negative real parts for $S < \overline{S}_2$. Descarte's rule of signs shows that the matrix has 0 or 2 positive real eigenvalues.

Suppose that the matrix has two positive real eigenvalues, $p_1, p_2 > 0$. Suppose to obtain a contradiction that the remaining eigenvalues are complex and have a real part. Let z and its conjugate $\overline{z}$ denote the eigenvalues. We then have:

$$S(M_{\text{endog}}) = 2\Re(z)p_1p_2 + |z|^2(p_1 + p_2) > 0,$$

a contradiction. Thus, in this case, exactly two eigenvalues have a negative real part.

Now suppose that the matrix has no positive real eigenvalues. Then we have two possible cases. First, we could have that the eigenvalues are all complex and given by λ_1, λ_2 and their conjugates $\overline{\lambda_1}, \overline{\lambda_2}$. We then have:

$$Z(M_{\text{endog}}) = 2\Re(\lambda_1)|\lambda_2| + 2\Re(\lambda_2)|\lambda_1| < 0.$$

Thus, at least one of λ_1 or λ_2 has a negative real part. Moreover, exactly one of them has a negative real part because

$$\text{Tr}(M_{\text{endog}}) = 2\Re(\lambda_1) + 2\Re(\lambda_2) > 0.$$

Second, we could have that the matrix has two negative real eigenvalues and a complex pair, λ and $\overline{\lambda}$. In this case, because the trace is positive, the real part of λ must be positive as well.

These observations show that when $S < \overline{S}_2$, M_{endog} has exactly two eigenvalues with negative real parts. Thus, Theorem 7.19 in Acemoglu (2009) shows that, in the limit case in which $g \to 0$ the economy with endogenous technology is saddle-path stable. Let $\check{S} = \min\{\overline{S}_1, \overline{S}_2\}$. Thus, for $S < \check{S}$, the unique BGP is locally stable. ■

Proofs from Section 5

Proof of Proposition 7: We prove this proposition under Assumption 2 only. Consider an exogenous path for technology in which $\dot{N} = \dot{I} = \Delta$ (with $\rho + (\theta - 1)A_H\Delta > 0$) and suppose that $n(t) > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$. This implies that in any candidate BGP $I^*(t) = I(t)$ and $n^*(t) = n(t)$.

Define $M \in [I, N]$ as in the main text. Two equations determine M. First, because firms are indifferent between producing task M with low-skill or high-skill workers, we have:

$$\frac{W_H(t)}{W_L(t)} = \frac{\gamma_H(M(t))}{\gamma_L(M(t), t)} = \frac{\gamma_H(M(t))^{1-\xi}}{\Gamma(t - T(M(t)))}.$$

In addition, the relative demand for high-skill and low-skill labor yields:

$$\frac{L}{H} \frac{\int_{M(t)}^{N(t)} \gamma_H(i)^{\hat{\sigma}-1} di}{\int_{I(t)}^{M(t)} \gamma_L(i,t)^{\hat{\sigma}-1} di} = \left(\frac{W_H(t)}{W_L(t)}\right)^{\hat{\sigma}}.$$

These two equations can be combined into the equilibrium condition:

$$\frac{L}{H}\frac{\int_{M(t)}^{N(t)}\gamma_H(i)^{\hat{\sigma}-1}di}{\int_{I(t)}^{M(t)}\gamma_L(i,t)^{\hat{\sigma}-1}di}=\left(\frac{\gamma_H(M(t))^{1-\xi}}{\Gamma(t-T(M(t)))}\right)^{\hat{\sigma}}.$$

Let $m(t)=M(t)-I(t)$ and $n=N(t)-I(t)$. Using the formula for $\gamma_L(i,t)$ and the change of variables $i=N-i'$ to rewrite the integrals in the previous equations we get that $m(t)$ is uniquely pinned down by:

$$\frac{L}{H}\frac{\int_0^{n-m(t)}\gamma_H(i)^{1-\hat{\sigma}}di}{\int_{n-m(t)}^{n}\gamma_H(i)^{\xi(1-\hat{\sigma})}\Gamma\left(\frac{i}{\Delta}\right)^{\hat{\sigma}-1}di}=\frac{\gamma_H(N(t))^{1-\xi}}{\gamma_H(n-m(t))^{\sigma(1-\xi)}\Gamma\left(\frac{n-m(t)}{\Delta}\right)^{\hat{\sigma}}}. \tag{B14}$$

This expression also uses the fact that, because both $\dot{N}=\dot{I}=\Delta$, we have $t-T(i)=\frac{N(t)-i}{\Delta}$. The left-hand side of equation (B14)—the relative demand curve—is decreasing in $m(t)$, converges to zero as $m(t)\to n$, and converges to infinity as $m(t)\to 0$. Moreover, the right-hand side—the comparative advantage schedule—is increasing in $m(t)$. Thus, this equation uniquely determines $m(t)$ as a function of $N(t)$ and n.

To prove the first part of the proposition, consider the case in which $\xi<1$. Taking the limit as $t\to\infty$ we have that the right-hand side of equation (B14) converges to infinity. To maintain the equality, we must have $m(t)\to 0$, which implies that asymptotically $M(t)=I(t)$ and low-skill workers produce no task. Moreover, we have that inequality explodes, since

$$\frac{W_H(t)}{W_L(t)}\to\frac{\gamma_H(N(t))^{1-\xi}}{\gamma(n)^{1-\xi}\Gamma\left(\frac{n}{\Delta}\right)}\to\infty.$$

To prove the second part of the proposition, consider the case in which $\xi=1$. We now show that there is a BGP in which $m(t)=m$ and $\frac{W_H(t)}{W_L(t)}$ is constant. Equation (B14) shows that, in this case, m only depends on n as claimed. Moreover, the wage gap is also constant over time and given by

$$\frac{W_H(t)}{W_L(t)}=\frac{1}{\Gamma\left(\frac{n-m}{\Delta}\right)}.$$

Now, consider an increase in n, and let $s=n-m$ denote the measure of tasks performed by high-skill workers. Holding s constant, the left-hand side of equation (B14) is decreasing in n. Because the left-hand side of equation (B14) is increasing in s and the right-hand side of equation (B14) is decreasing in s, we must have that s increases as a result. This implies that, as stated in the proposition, the wage gap, which is a decreasing function of s, declines with n. ■

Proof of Proposition 8:

We prove this proposition in the more general case in which Assumption 2′ holds.

From the Bellman equations provided in the main text, it follows that along a BGP we have

$$v_N(n)=b\int_0^{\frac{n}{\Delta}}e^{-(\rho-(1-\theta)g)\tau}c^u\left(w_N(n)e^{g\tau}\right)^{\zeta-\sigma}d\tau,$$
$$v_I(n)=b\int_0^{\frac{1-n}{\Delta}}e^{-(\rho-(1-\theta)g)\tau}c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}d\tau.$$

Here $\Delta = \frac{\kappa_I \kappa_N \iota(n^D)}{\kappa_I \iota(n^D) + \kappa_N} S$ is the endogenous rate at which both technologies grow in a BGP and $g = A\Delta$.

As before, a BGP requires that n satisfies

$$\kappa_I \iota(n) v_I(n) = \kappa_N v_N(n).$$

Using these formulas, the proof of the proposition follows from the properties of the effective wages derived in Proposition 5. Following the same steps involved in the proof of Proposition 6, we also obtain that the equilibrium in this case is locally stable whenever $\kappa_I \iota(n) v_I(n)$ cuts $\kappa_N v_N(n)$ from below. ■

We now turn to Proposition 9. We prove a similar statement in the more general case in which Assumption 2′ holds. In particular, we show that:

Proposition B2 (Welfare implications of automation in the general model) *Consider the static economy and suppose that Assumptions 1, 2′ and 3 hold, and that $I^* = I > \widetilde{I}$. Let $\mathcal{W} = u(C, L)$ denote the welfare of households and let $F(K, L; I, N)$ denote the resulting replace when the amount of labor supplied is L and capital is K.*

1. *Consider the baseline model without labor market frictions, so that the representative household chooses the amount of labor without constraints, and thus $\frac{W}{C} = \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_I}{F} > 0,$$
$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_N}{F} > 0.$$

2. *Suppose that there are labor market frictions, so that employment is constrained by a quasi-labor supply curve $L \leq L_{qs}(\omega)$. Suppose also that the quasi-labor supply schedule $L_{qs}(\omega)$ is increasing in ω, has an elasticity $\widetilde{\varepsilon}_L > 0$, and is binding in the sense that $\frac{W}{C} > \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{F_I}{F} - L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\omega} \frac{\partial \omega}{\partial I^*}\right] \lesseqgtr 0.$$
$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{F_N}{F} + L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\omega} \frac{\partial \omega}{\partial N}\right] > 0.$$

Proof. The unconstrained allocation of employment maximizes:

$$\mathcal{W} = \max_L u(F(K, L; I, N), L).$$

Thus, the envelope theorem implies that:

$$\mathcal{W}_I = u_C F_I = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_I}{F} > 0.$$

(recall that $F_I > 0$ because we assumed $I^* = I$).

Likewise, the envelope theorem implies that:

$$\mathcal{W}_N = u_C F_N = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_N}{F} > 0.$$

(recall that $F_N > 0$ because we imposed Assumption 3).

Now suppose that $L \leq L_{qs}(\omega)$. The allocation of employment now solves:

$$\mathcal{W} = \max_L u(F(K, L; I, N), L) + \lambda(L_{qs}(\omega) - L),$$

where $\lambda = u_c F_L + u_L = cu_c\left(\frac{F_L}{c} - \nu'(L)\right) > 0$ is the multiplier on the employment constraint (by assumption this constraint is binding). The envelope theorem now implies:

$$\mathcal{W}_I = u_C F_I + \lambda L'_{qs}(\omega)\frac{\partial \omega}{\partial I^*} = \left(Ce^{-\nu(L)}\right)^{1-\theta}\left[\frac{F_I}{F} - L\left(\frac{W}{C} - \nu'(L)\right)\frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial I^*}\right] \lessgtr 0.$$

Likewise,

$$\mathcal{W}_N = u_C F_N + \lambda L'_{qs}(\omega)\frac{\partial \omega}{\partial N} = \left(Ce^{-\nu(L)}\right)^{1-\theta}\left[\frac{F_I}{F} - L\left(\frac{W}{C} - \nu'(L)\right)\frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial N}\right] > 0.$$

The explicit formulas presented in Proposition 9 follow from the formulas presented here, but using the fact that, when Assumption 2 holds, we have:

$$\begin{aligned}
\frac{F_I}{F} &= \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) & \qquad \frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial I} &= \frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_I \\
\frac{F_N}{F} &= \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) & \qquad \frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial N} &= \frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_N.
\end{aligned}$$

■

Properties of the constraint efficient allocation:

We now derive the constrained efficient allocation both when the labor market is frictionless and when there is a friction as the one introduced in Proposition 9. We focus on the case in which Assumption 2 holds, although similar insights apply in general.

First the planner removes markups. This implies that net output is given by

$$\begin{aligned}
F^p(K, L; I^*, N) =& \mu^{\frac{\eta}{\eta-1}} B\left[(I^* - N + 1)^{\frac{1}{\hat{\sigma}}} K^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}} L^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}}, \\
=& \mu^{\frac{\eta}{\eta-1}} F(K, L; I^*, N).
\end{aligned}$$

Using this expression, we can write the planner's problem as:

$$\max_{C(t), L(t), S(t)} \int_0^\infty e^{-\rho t}\frac{[C(t)e^{-\nu(L(t))}]^{1-\theta} - 1}{1-\theta} dt$$

Subject to

$$\dot{K}(t) = \mu^{\frac{\eta}{\eta-1}} F(K, L; I^*, N) - \delta K(t) - C(t).$$

Let $\mu_N(t)$ denote the marginal value of new tasks (increasing N) in terms of the final good. Let $\mu_I(t)$ denote the marginal value of automation (increasing I) in terms of the final good. These marginal values are the social counterparts to $V_N(t)$ and $V_I(t)$ in the decentralized economy. Assuming that the planner operates in the region where $I^*(t) = I(t)$, we can write these marginal values as

$$\mu_N(t) =(1-\eta)\mu^{\eta(1-\sigma)} \int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds} \frac{\sigma}{1-\sigma} Y(\tau) \left(R(\tau)^{1-\widetilde{\sigma}} - \gamma(n(\tau))^{\sigma-1} w(\tau)^{1-\widetilde{\sigma}} \right) d\tau,$$

$$\mu_I(t) =(1-\eta)\mu^{\eta(1-\sigma)} \int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds} \frac{\sigma}{1-\sigma} Y(\tau) \left(w(\tau)^{1-\widetilde{\sigma}} - R(\tau)^{1-\widetilde{\sigma}} \right) d\tau.$$

With some abuse of notation and to maximize the parallel with the decentralized expressions for V_N and V_I, we are using $R(t)$ to denote the marginal product of capital $\mu^{\frac{\eta}{\eta-1}} F_K$ and $w(t)$ to denote the (normalized) marginal product of labor $\mu^{\frac{\eta}{\eta-1}} F_L e^{-AI^*(t)}$.

These observations show that the efficient allocation satisfies similar conditions to the decentralized economy in our main model in Section 4. The only difference is that now, the allocation of scientists is guided by $\mu_N(t)$ and $\mu_I(t)$ and satisfies:

$$S_I(t) = SG\left(\frac{\kappa_I \mu_I(t) - \kappa_N \mu_N(t)}{Y(t)}\right), \quad S_N(t) = S\left[1 - G\left(\frac{\kappa_I \mu_I(t) - \kappa_N \mu_N(t)}{Y(t)}\right)\right],$$

so that in the efficient allocation, $n(t)$ changes endogenously according to:

$$\dot{n}(t) = \kappa_N S - (\kappa_N + \kappa_I) G\left(\frac{\kappa_I \mu_I(t) - \kappa_N \mu_N(t)}{Y(t)}\right) S.$$

One of the key insights from Proposition 6 is that the expected path for factor prices determines the incentives to automate and create new tasks. The equations for μ_N and μ_I show that a planner would also allocate scientists to developing both types of technologies following a similar principle; guided by the cost savings that each technology grants to firms. However, the fact that $\mu_N \neq V_N$ and $\mu_I \neq V_I$ shows that the decentralized allocation is not necessarily efficient. The inefficiency arises because technology monopolists do not earn the full gains that their technology generates, nor internalize how their innovations affect other existing and future technology monopolists.

We now show that labor market frictions change the planner's incentives to allocate scientists. By contrast, conditional on the wage level, such frictions do not change the market incentives to automate or create new tasks.

Without frictions, the efficient level of labor satisfies

$$\left(\mu^{\frac{\eta}{\eta-1}} F_L - c\nu'(L)\right) \mu_K \leq 0,$$

with equality if $L > 0$.

Now suppose that there is an exogenous constraint on labor that requires $L \leq L_{qs}(\omega)$. Let μ_L be the multiplier of this constraint. We have that:

$$\mu_L = \begin{cases} \left(\mu^{\frac{\eta}{\eta-1}} F_L - c\nu'(L)\right) \mu_K > 0 & \text{if } L = L_{qs}(\omega) \\ 0 & \text{if } L < L_{qs}(\omega) \end{cases}$$

Because the planner takes into account the first-order effects from changes in the employment level, the values for μ_N and μ_I change to:

$$\mu_N(t) = (1-\eta)\mu^{\eta(1-\sigma)} \int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds} \left[\frac{\sigma}{1-\sigma} Y(\tau) \left(R(\tau)^{1-\widetilde{\sigma}} - \gamma(n(\tau))^{\sigma-1} w(\tau)^{1-\widetilde{\sigma}} \right) \right.$$
$$\left. + \left(\mu^{\frac{\eta}{\eta-1}} F_L - c\nu'(L) \right) L \frac{\widetilde{\varepsilon}_L}{\hat{\sigma} + \widetilde{\varepsilon}_L} \Lambda_N \right] d\tau,$$
$$\mu_I(t) = (1-\eta)\mu^{\eta(1-\sigma)} \int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds} \left[\frac{\sigma}{1-\sigma} Y(\tau) \left(w(\tau)^{1-\widetilde{\sigma}} - R(\tau)^{1-\widetilde{\sigma}} \right) \right.$$
$$\left. - \left(\mu^{\frac{\eta}{\eta-1}} F_L - c\nu'(L) \right) L \frac{\widetilde{\varepsilon}_L}{\hat{\sigma} + \widetilde{\varepsilon}_L} \Lambda_I \right] d\tau.$$

Thus, when the level of employment is below its unconstrained optimum, the planner values the introduction of new tasks more because they raise the marginal product of labor and ease the constraint on total employment. Likewise, the planner values automation less because she recognizes that by reducing employment automation has a first-order cost on workers. Importantly, the market does not recognize the first-order costs from automation or the first-order benefits from introducing new tasks. As the expressions for V_N and V_I show, only factor prices—not the extent of frictions in the labor market— determine the incentives to introduce these technologies.

When New Tasks Also Use Capital

In our baseline model, new tasks use only labor. This simplifying assumption facilitated our analysis, but is not crucial or even important for our results. Here we outline a version of the model where new tasks also use capital and show that all of our results continue to hold in this case. Suppose, in particular, that the production function for non-automated tasks is

$$y(i) = \left[\eta q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta) \left(B_\nu (\gamma(i) l(i))^\nu k(i)^{1-\nu} \right)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}, \quad \text{(B15)}$$

where $k(i)$ is the capital used in the production of the task, $\nu \in (0,1)$, and $B_\nu = \nu^{-\nu}(1-\nu)^{-(1-\nu)}$ is a constant that is re-scaled to simplify the algebra.

Automated tasks $i \leq I$ can be produced using labor or capital, and their production function takes the form

$$y(i) = \left[\eta q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta) \left(k(i) + B_\nu (\gamma(i) l(i))^\nu k(i)^{1-\nu} \right)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}. \quad \text{(B16)}$$

Comparing these production functions to those in our baseline model (2) and (3), we readily see that the only difference is the requirement that labor has to be combined with capital in all tasks (while automated tasks continue not to use any labor). Note also that when $\nu \to 1$, we recover the model in the main text as a special case. It can be shown using a very similar analysis to that in our main model that most of the results continue to hold with minimal modifications. For example, there will exist a threshold $\widetilde{I}$ such that tasks below $I^* = \min\{I, \widetilde{I}\}$ will be produced using capital and the remaining more complex tasks will be produced using labor. Specifically, whenever $R \in \arg\min \left\{ R, R^{1-\nu} \left(\frac{W}{\gamma(i)} \right)^{\nu} \right\}$ and $i \le I$, the relevant task is produced using capital, and otherwise it is produced using labor. Since $\gamma(i)$ is strictly increasing, this implies that there exists a threshold $\widetilde{I}$ at which, if technologically feasible, firms would be indifferent between using capital and labor. Namely, at task $\widetilde{I}$, we have $R = W/\gamma(\widetilde{I})$, or

$$\frac{W}{R} = \gamma(\widetilde{I}).$$

This threshold represents the index up to which using capital to produce a task yields the cost-minimizing allocation of factors. However, if $\widetilde{I} > I$, firms will not be able to use capital all the way up to task $\widetilde{I}$ because of the constraint imposed by the available automation technology. For this reason, the equilibrium threshold below which tasks are produced using capital is given by

$$I^* = \min\{I, \widetilde{I}\},$$

meaning that $I^* = \widetilde{I} < I$ when it is technologically feasible to produce task $\widetilde{I}$ with capital, and $I^* = I < \widetilde{I}$ otherwise.

The demand curves for capital and labor are similar, with the only modification that the demand for capital also comes from non-automated tasks. Following the same steps as in the text, we can then establish analogous results. This requires the more demanding Assumption 2″, which guarantees that the demand for factors is homothetic:

Assumption 2″: One of the following three conditions holds:

1. $\sigma - \zeta \to 0$, or
2. $\zeta \to 1$, or
3. $\eta \to 0$.

The following proposition is very similar to Proposition 1, with the only difference being in the ideal price condition.

Proposition B3 (Equilibrium in the static model when $\epsilon \in (0, 1)$) *Suppose that Assumption 1″ holds. Then, for any range of tasks $[N-1, N]$, automation technology $I \in (N-1, N]$, and capital stock K, there exists a unique equilibrium characterized by factor prices, W and R, and threshold tasks, $\widetilde{I}$ and I^*, such that: (i) $\widetilde{I}$ is determined by equation (6) and $I^* = \min\{I, \widetilde{I}\}$; (ii) all tasks $i \le I^*$ are produced using capital and all tasks $i > I^*$ are produced using labor; (iii) the capital and labor market clearing conditions, equations (8) and (9), are satisfied; and (iv) factor prices satisfy the ideal price index condition:*

$$(I^* - N + 1)c^u(R)^{1-\sigma} + \int_{I^*}^{N} c^u \left(R^{1-\nu} \left(\frac{W}{\gamma(i)} \right)^{\nu} \right)^{1-\sigma} di = 1. \tag{B17}$$

Proof. The proof follows the same steps as Proposition 1. ■

Comparative statics in this case are also identical to those in the baseline model (as summarized in Proposition 2) and we omit them to avoid repetition. The dynamic extension of this more general model is also very similar, and in fact, Proposition 4 applies identically, and is also omitted. One can also define $\overline{\rho}$, $\overline{n}(\rho)$ and $\widetilde{n}(\rho)$ in an analogous nway as we did in the proof of Lemma A2. To highlight the parallels, we just present the equivalent of Proposition 6.

Proposition B4 (Equilibrium with endogenous technology when $\nu \in (0,1)$) *Suppose that Assumptions 1′, 2′, and 4 hold. Then, there exists $\overline{S}$ such that, when $S < \overline{S}$, we have:*

1 ***(Full automation)*** *For $\rho < \overline{\rho}$, there is a BGP in which $n(t) = 0$ and all tasks are produced with capital.*

For $\rho > \overline{\rho}$, all BGPs feature $n(t) = n > \overline{n}(\rho)$. Moreover, there exist $\overline{\kappa} > \underline{\kappa} > 0$ such that:

2 ***(Unique interior BGP)*** *if $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$ there exists a unique BGP. In this BGP we have $n(t) = n \in (\overline{n}(\rho), 1)$ and $\kappa_N v_N(n) = \kappa_I v_I(n)$. If, in addition, $\theta = 0$, then the equilibrium is unique everywhere and the BGP is globally (saddle-path) stable. If $\theta > 0$, then the equilibrium is unique in the neighborhood of the BGP and is asymptotically (saddle-path) stable;*

3 ***(Multiple BGPs)*** *if $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$, there are multiple BGPs;*

4 ***(No automation)*** *If $\underline{\kappa} > \frac{\kappa_I}{\kappa_N}$, there exists a unique BGP. In this BGP $n(t) = 1$ and all tasks are produced with labor.*

Proof. The proof of this result closely follows that of Proposition 6, especially exploiting the fact that the behavior of profits of automation and the creation of new tasks behave identically to those in the baseline model, and thus the value functions behave identically also. ■

Microfoundations for the Quasi-Labor Supply Function

We provide various micro-foundations for the quasi-labor supply expression used in the main text, $L^s\left(\frac{W}{RK}\right)$.

Efficiency wages: Our first micro-foundation relies on an efficiency wage story. Suppose that, when a firm hires a worker to perform a task, the worker could shirk and, instead of working, use her time and effort to divert resources away from the firm.

Each firm monitors its employees, but it is only able to detect those who shirk at the flow rate q . If the worker is caught shirking, the firm does not pay wages and retains its resources. Otherwise, the worker earns her wage and a fraction of the resources that she diverted away from the firm.

In particular, assume that each firm holds a sum RK of liquid assets that the worker could divert, and that if uncaught, a worker who shirks earns a fraction $u(i)$ of this income. We assume that the sum of money that the worker may be able to divert is RK to simplify the algebra. In general, we obtain a similar quasi-supply curve for labor so long as these funds are proportional to total income $Y = R\,K + WL$.

In this formulation, $u(i)$ measures how untrustworthy worker i is, and we assume that this information is observed by firms. $u(i)$ is distributed with support $[0, \infty)$ and has a cumulative density function G. Moreover, we assume there is a mass L of workers. A worker of type $u(i)$ does not shirk if and only if:

$$W \geq (1-q)[W + u(i)RK] \rightarrow \frac{W}{RK}\frac{q}{1-q} \geq u(i).$$

Thus, when the market wage is W, firms can only afford to hire workers who are sufficiently trustworthy. The employment level is therefore given by:

$$L^s = G\left(\frac{W}{RK}\frac{q}{1-q}\right)L.$$

When $q = 1$—so that there is no monitoring problem—, we have $G\left(\frac{W}{RK}\frac{q}{1-q}\right) = 1$, and the supply of labor is fixed at L for all wages $W \geq 0$. However, when $q < 1$—so that there is a monitoring problem—, we have $L^s < L$. Even though all workers would rather work than stay unemployed, the monitoring problem implies that not all of them can be hired at the market wage. Notice that, though it is privately too costly to hire workers with $u(i) > \frac{W}{RK}\frac{q}{1-q}$, these workers strictly prefer employment to unemployment.

Alternatively, one could also have a case in which firms do not observe $u(i)$, which is private information. This also requires that firms do not learn about workers. To achieve that, we assume that workers draw a new value of $u(i)$ at each point in time.

When the marginal value of labor is W, firms are willing to hire workers so long as the market wage $\widetilde{W}$ satisfies:

$$(W - \widetilde{W})G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) - (1-q)\left(\widetilde{W} + RK\int_{\frac{\widetilde{W}}{RK}\frac{q}{1-q}}^{\infty} udG(u)\right) \geq 0.$$

This condition guarantees that the firm makes positive profits from hiring an additional worker, whose type is not known.

Competition among firms implies that the equilibrium wage at each point in time satisfies:

$$(W - \widetilde{W})G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) - (1-q)\left(\widetilde{W} + RK\int_{\frac{\widetilde{W}}{RK}\frac{q}{1-q}}^{\infty} udG(u)\right) = 0.$$

This curve yields an increasing mapping from $\frac{W}{RK}$ to $\frac{\widetilde{W}}{RK}$, which we denote by

$$\frac{\widetilde{W}}{RK} = h\left(\frac{W}{RK}\right).$$

Therefore, the effective labor supply in this economy, or the quasi-supply of labor, is given by

$$L^s = G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) = G\left(h\left(\frac{W}{RK}\right)\frac{q}{1-q}\right)L.$$

As in the previous model, even though the opportunity cost of labor is zero, the economy only manages to use a fraction of its total labor.

Minimum wages: Following Acemoglu (2003), another way in which we could obtain a quasi-labor supply curve is if there is a binding minimum wage. Suppose that the government imposes a (binding) minimum wage $\widetilde{W}$ and indexes it to the income level (or equivalently the level of consumption):

$$\widetilde{W} = \varrho \cdot (RK + WL),$$

with $\varrho > 0$. Here, $RK + WL$ represents the total income in the economy (net of intermediate goods' costs).

Suppose that the minimum wage binds. Then:

$$L = \frac{1}{\varrho} s_L,$$

which defines the quasi-labor supply in this economy as an increasing function of the labor share.

Additional References

Acemoglu, Daron (2009) *Introduction to Modern Economic Growth,* Princeton University Press.

Walter, Wolfgang (1998) *Ordinary Differential Equations,* Springer; Graduate Texts in Mathematics.