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DYNAMIC SAVINGS CHOICES WITH DISAGREEMENTS

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ABSTRACT

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We study a flexible dynamic savings game in continuous time, where decision makers rotate in and out of power. These agents value spending more highly while in power creating a time-inconsistency problem. We provide a sharp characterization of Markov equilibria. Our analysis proceeds by construction and isolates the importance of a local disagreement index, $\beta(c)$, defined as the ratio of marginal utility for those in and out of power. If disagreement is constant the model specializes to hyperbolic discounting. We also provide novel results for this case, offering a complete and simple characterization of equilibria. For the general model we show that dissaving occurs when disagreements are sufficiently high, while saving occurs when disagreements are sufficiently low. When disagreements vary sufficiently with spending, richer dynamics are possible. We provide conditions for continuous equilibria and also show that the model can be inverted for primitives that support any smooth consumption function. Our framework applies to individuals under a behavioral interpretation or to governments under a political-economy interpretation.

1 Introduction

Time-inconsistency problems that bias behavior towards the present may help explain a number of phenomena and have received ample attention from economists. However, the extent of these problems likely varies significantly according to the situation. In particular, there is no reason to expect the allure of the present over the future felt by the poor to be comparable to that experienced by the rich. Similarly, it has been noted that governments may suffer from a similar present bias for political economy reasons, yet the degree of this bias may be quite different for advanced countries than for developing countries. The general point is that the strength of time-inconsistency problems may depend on the

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level of wealth or spending. This possibility has received relatively little attention from the literature.

This paper introduces and studies an infinite-horizon continuous-time savings game that accommodates flexible forms of time inconsistency. Decision makers rotate in and out of power. An agent currently in power controls consumption and savings, choosing how much to spend subject to a borrowing constraint and a constant flow of income. Agents currently in power retain power for a stochastic interval of time and lose it at a Poisson rate to a successor. Once removed from power, an agent continues to care about the future spending path chosen by other agents. However, spending is enjoyed more while in power. This disagreement, represented by differences in the utility functions for those in and out of power, captures a form of present bias and leads to a time-inconsistency problem in savings choices. As a result, we approach the problem as a dynamic game and study Markov equilibria, a widely used refinement in this literature.

Our model admits both a behavioral and political-economy interpretation. For the behavioral one, following [Strotz \(1956\)](#), [Laibson \(1997\)](#) and many others, the model may describe the problem of a single consumer playing an intertemporal game against future 'selves' (a closely related literature, initiated by [Phelps and Pollak, 1968](#), studies paternalistic intergenerational growth models). The disagreement on the utility function that we allow generates a time inconsistency problem that is similar, but strictly generalizes, hyperbolic discounting. For the political economy interpretation, the model describes a situation where the ruling party controls the budget and obtains private benefits from spending while in power, due to pork spending or outright transfers to ruling party members. This relates our work to political economy models of government debt, such as [Alesina and Tabellini \(1990\)](#), [Amador \(2002\)](#), [Battaglini and Coate \(2008\)](#), [Azzimonti \(2011\)](#) and others.¹

With few exceptions, the existing time-inconsistency literature has focused on saving games that are effectively variants of the hyperbolic discounting setup. In our model this amounts to the assumption of a uniform disagreement, with utility out of power proportional to utility in power. Our analysis also applies to this special case and actually delivers new and sharp results.

Our first contribution, however, is to provide a framework to explore disagreements that vary with spending. To this end, we consider general differences in the utility functions for those in and out of power. This may give rise to a non-uniform time-inconsistency

¹A different strand of the literature models the endogenous transition of power by examining Markov equilibria in dynamic political economy games (e.g. [Timothy Besley, 1998](#); [Acemoglu and Robinson, 2001](#); [Bai and Lagunoff, 2011](#)), but abstracts from government debt or savings.

problem, where the incentives to save vary with wealth. We are especially interested in the long-run dynamics of wealth and how this depends on the form disagreements take.

Why would disagreements vary with spending? One straightforward answer is that there is no real reason to expect them to be constant and so the possibility that they are not must be contemplated. For example, in the behavioral context, it is plausible that present-biased impulses and behaviors decrease with spending. A deeper answer is offered by [Banerjee and Mullainathan \(2010\)](#), who provide a foundation for disagreements based on the notion that spending takes place over many goods, with disagreements on how to spend across these goods.² The overall disagreement on total spending then varies with the level of spending, except in special cases. This perspective explains a bias towards the present, but shifts the focus from intertemporal discounting to disagreements across different goods. For example, in a behavioral context, agents may feel drawn to consume certain tempting goods today—extreme examples may include unhealthy foods, alcohol or drugs—but do not value the consumption of these goods by future ‘selves’. If the marginal propensity to spend on such goods falls with greater spending this implies *decreasing* disagreements. A similar argument applies in political economy contexts. Indeed, the voting model in [Battaglini and Coate \(2008\)](#) implies *increasing* disagreements because the marginal propensity to spend on pork transfers is increasing in spending. One of the goals of this paper is to provide a framework that can encompass a wide class of assumptions on the form of disagreements, including increasing and decreasing disagreements.

Our second contribution is both technical and substantive, providing a sharp characterization of all Markov equilibria. As is well known, dynamic saving games in discrete time may be quite ill behaved.³ For example, [Harris and Laibson \(2001, 2002\)](#) point out that discontinuous equilibria are relatively pervasive in these standard settings. [Krusell and Smith \(2003\)](#) proved that the hyperbolic model has a continuum of local solutions with discontinuous policy functions. Recently [Chatterjee and Eyigungor \(2015\)](#) show that in discrete time *all* Markov equilibrium must be discontinuous (see also [Morris and Postlewaite, 1997](#) and [Morris, 2002](#) for the analogs in finite horizon settings). Properties

²In a behavioral context, [Banerjee and Mullainathan \(2010\)](#) focus on a finite-horizon model with many goods and additively separable utilities, with disagreements over which goods should be valued. In a political economy context, [Alesina and Tabellini \(1990\)](#) consider an infinite-horizon model with a relatively general form of disagreement in the composition of spending across different goods (see their equations 1). However, for their analysis they specialize to corner cases and a more extreme and uniform disagreement (see their equations 4 and 5).

³Linear equilibria exist in the absence of a borrowing constraint. Such equilibria were first studied by [Phelps and Pollak \(1968\)](#) and later put to good use by many others. However, linear equilibria do not exist in the presence of a binding borrowing constraint, a case that has been the primary focus of the literature.

such as these render these models relatively intractable and make it difficult to characterize equilibria.⁴ The literature has responded to these challenges in a number of ways. [Harris and Laibson \(2001\)](#) introduce income uncertainty to derive a Generalized Euler equation. [Harris and Laibson \(2013\)](#) propose a continuous-time model, focusing on a limit with ‘instant gratification’ and small noise in asset returns to apply the theory of viscosity solutions. [Chatterjee and Eyigungor \(2015\)](#) work in a discrete-time setting but introduce lotteries to smooth out the solution.

Despite these efforts from the hyperbolic literature, many fundamental questions remain open or have only received partial answers. For example, are there conditions that ensure that the equilibrium involves saving or dissaving? Can saving and dissaving equilibria coexist for some parameters? For a given equilibrium, can saving and dissaving coexist at different wealth levels? Do continuous equilibria exist, and, if so, under what conditions? Do these models display multiple equilibria?

Our model is cast in continuous time and this turns out to be crucial to our approach and results. We show that our continuous-time framework is relatively well behaved, without introducing uncertainty or lotteries. Our formulation builds on [Harris and Laibson \(2013\)](#), but extends it to allow for more general disagreements. In addition, we do not focus on the instant-gratification limit and our solution strategy is different.⁵ Our approach is to attack the differential equations for a Markov equilibrium head on. Due to the presence of singularities, no off-the-shelf results exist for such equations, but we provide a simple method to construct and characterize equilibria. Since wealth evolves continuously over time, we build up locally towards a global equilibrium. Since our proofs proceed by construction, we characterize equilibria sharply, delivering answers to the questions listed above, as well as providing a straightforward procedure for computation. In fact, our characterization is exhaustive, providing a complete description of the class of equilibria that are possible. Finally, we provide sufficient conditions for the existence and uniqueness of a continuous equilibrium.

⁴Despite these difficulties the literature has obtained various results on the existence of Markov equilibria, allowing for potentially discontinuous policy and value functions. [Bernheim et al. \(2015\)](#) shows the existence of Markov equilibrium in the standard model without uncertainty, assuming an interest rate that is strictly greater than the discount factor. [Harris and Laibson \(2001\)](#) provide existence results by adding i.i.d. uncertainty in income in the standard quasi-hyperbolic model ([Bernheim and Ray \(1989\)](#) provide a related result in an altruistic growth model with bounded transfers), while [Chatterjee and Eyigungor \(2015\)](#) does the same for a model with lotteries.

⁵The instantaneous gratification limit is tractable and provides a good approximation in some applications. Nevertheless, it is of obvious theoretical interest to obtain a more general characterization, away from this approximation. Moreover, in some applications, such as political economy settings, the approximation may be inadequate, since it would represent a situation of extremely high political turnover.

Although the entire analysis and results apply to our general framework, they are of interest even within the special hyperbolic case. For this case, we provide a novel graphical analysis that delivers a complete characterization for the entire set of equilibria in a particularly simple and visual manner. For example, we show that depending on parameters there can be either saving or dissaving, but never both; a continuous equilibrium may exist and there is at most one such equilibria; whenever the equilibrium involves savings then the equilibrium is unique. To the best of our knowledge, such results have no counterparts in the existing literature and settle various open questions mentioned above.

Our third contribution is to isolate the conditions for saving or dissaving and to characterize the resulting dynamics for wealth. A crucial innovation of our analysis here is our introduction a local disagreement index $\beta(c)$, defined as the ratio of marginal utilities from spending for agents in and out of power. The shape of this function summarizes how disagreements depend on spending (denoted by c). The hyperbolic case amounts to the special case where $\beta(c)$ is constant.

Our first set of results in this regard involve cases where the disagreement index does not vary too much and is either sufficiently high or low. Under these conditions, we show that an equilibrium exists that features either saving or dissaving at all wealth levels. Specifically, there exists a threshold $\hat{\beta}$ which depends on the interest rate and other parameters and show that when the disagreement index $\beta(c)$ lies above $\hat{\beta}$ then there is a unique equilibrium with positive savings and this equilibrium is continuous; when $\beta(c)$ lies below $\hat{\beta}$ all equilibria must feature dissaving and there is at most one continuous equilibrium. Although these results apply more generally, a special case of interest is the hyperbolic case with constant $\beta(c)$.

As a byproduct of our analysis of the hyperbolic case we touch on the issue of local indeterminacy. For the discrete-time quasi-hyperbolic model, [Krusell and Smith \(2003\)](#) constructed a continuum of local solutions to the equilibrium conditions. We provide an analogous result for our continuous-time setup: for *any* proposed steady state wealth a local solution exists, with wealth converging to this level. However, we show that these local constructs are *not* part of an equilibrium in our model.⁶ This conclusion actually follows as a corollary of our result showing that there exists a unique Markov equilibrium

⁶Indeed, the local constructs can only be interpreted as Markov equilibria for a modified saving game that adds ad hoc constraints, forcing the agent to choose wealth below a certain bound; this bound must be close enough to the proposed steady state. Such an ad hoc constraint is unnatural, however, and not standard in the literature. The local constructs do not characterize a part of any equilibrium of the original saving game, which imposes no upper bounds on wealth.

with strict savings.⁷

Our second set of results involve cases where the disagreement index $\beta(c)$ does not lie on one side of $\hat{\beta}$ but instead varies enough that it crosses this threshold. We find that rich dynamics then emerge, with saving and dissaving coexisting at different wealth levels. We focus on two polar opposite cases, when $\beta(c)$ is decreasing and when $\beta(c)$ is increasing.

In particular, if disagreements fall with spending then we show that a poverty trap may emerge, with dissavings below a threshold and savings above this same threshold. Intuitively, at low wealth levels the time-inconsistency problem is relatively severe because spending takes place in the range where disagreements are high. The incentive to consume is high because agents in power do not want to leave resources that may be spent when they are out of power. There is a feedback loop: the incentive to dissave is reinforced by the anticipation that successors overspend from the point of view of those in power. At high wealth levels the time-inconsistency problem is mitigated by the fact that spending takes place in regions with lower disagreement, so saving emerges. Again, a feedback loop reinforces these incentives: the incentive to save is enhanced if successors or future selves do not overspend too much. Poverty traps cannot arise without these feedback loops.

Banerjee and Mullainathan (2010) derive a related result. As mentioned earlier, their paper focuses on various implications of intraperiod disagreements on how to spend across goods. In the context of a two-period model with a single savings choice in the first period, they show there may be a downward jump in consumption as a function of wealth. This discontinuity may be interpreted as a poverty trap of sorts in their two-period setup. There are important differences, however. Obviously, a two-period framework does not permit the study of long-run wealth. In contrast, in our framework a poverty trap involves wealth getting trapped below a threshold forever. Moreover, a discontinuity like theirs cannot arise in a two-period adaptation of our model. Instead, in our setting poverty traps emerge from the strategic interactions across savings decisions over longer horizons. In this sense, the feedback loop described in the previous paragraph is crucial to our result, but is absent in their setup.

Our poverty trap result relies on non-uniform disagreements. Indeed, our results show that this is not only sufficient but also necessary: no Markov equilibrium exists featuring a poverty trap in the hyperbolic case. Poverty traps may emerge, however, in the hyperbolic case if one drops the Markov equilibrium requirement. Working in a

⁷The results presented here formally apply only to our continuous-time model and should be taken only as suggestive for the discrete-time model studied by **Krusell and Smith (2003)**.

discrete time setting, [Bernheim et al. \(2015\)](#) show that subgame perfect equilibria may feature poverty traps.⁸ As they argue, trigger strategies may be interpreted as self punishments that provide an internal and endogenous form of *self control*. Poverty traps may emerge in this context because this kind of self control is more effective for the rich, who are far from the asset limit. Our focus on Markov equilibria abstracts from these forms of self control and relies instead on the rich suffering less *disagreements* with future selves. Thus, [Bernheim et al. \(2015\)](#) focus on non-uniform self control with uniform disagreements; while we focus on non-uniform disagreements, without self control. Both are obviously compatible with one another and could act as complements. Interestingly, these two mechanisms have a few different implications: lower labor income or greater access to credit make poverty traps less likely in [Bernheim et al. \(2015\)](#), but more likely in our model. Such contrasting comparative statics serve to highlight the underlying differences in the mechanisms at work.

Turning to the opposite case, if disagreements rise with spending then the problem of time inconsistency becomes heightened at higher wealth levels. We show that an equilibrium exists where the wealthy dissave, while the poor save. Starting from any initial value, wealth converges to an interior steady state. Stability forces of this kind lie at the heart of the mean-reverting result in the political economy model of [Battaglini and Coate \(2008\)](#). Indeed, as we discuss in more detail in Section 5.2, their framework provides a foundation for disagreements that increase with spending. They set up and study a model of legislative voting, with this body determining the composition of spending. As we discuss, they prove a representation that fits our framework.

In addition to solving for an equilibrium, given primitives, using our differential approach, we show how our model can be fruitfully inverted to solve for primitives that support any given postulated equilibrium. We know of no parallel of this idea in the existing literature. In particular, we back out a disagreement function $\beta(c)$ for any given smooth consumption function. One advantage of this inverse perspective is that it is extremely tractable and insightful. Another is that it may also be more appropriate, at least conceptually, if the outside economist or econometrician observes behavior (the consumption function) but has no direct evidence on disagreements and how they vary.

Finally, we show that in our continuous-time framework there is at most one continuous equilibrium and provide sufficient conditions for its existence. The potential for a continuous equilibrium underscores, once again, that equilibria in our continuous-time framework can be extremely well-behaved compared to discrete-time counterparts, where no continuous equilibrium exists. We also provide conditions for the existence

⁸Their definition of a poverty trap is slightly different from ours, as we discuss in Section 5.1.

of discontinuous equilibria in our model. As a corollary, since both conditions turn out to be compatible in some cases, these results prove the possibility of multiple Markov equilibria.

2 A Dynamic Savings Game

This section introduces a continuous-time savings framework with a single consumption good that can accommodate relatively general forms of disagreement. We then offer one interpretation or motivation for our primitives, in a setting with many goods.

2.1 Preferences

Time is continuous with an infinite horizon, denoted by $t \in [0, \infty)$. The flow utility obtained from consumption by an agent in power is

$$U_1(c_t),$$

while utility for an agent out of power is

$$U_0(c_t).$$

The utility functions $U_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $U_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ are concave, increasing, continuous and differentiable. We also assume $U_1'(0) = \infty$ and $\lim_{c \rightarrow \infty} U_1'(c) = 0$.⁹

Agents in power are removed at a constant Poisson arrival rate $\lambda \geq 0$. To simplify the exposition we assume power cannot be regained, but show later that this is without loss of generality (see Section 3.3). The continuation lifetime utility at time t for an agent in power is then

$$V_t \equiv \mathbb{E}_t \left[\int_0^\tau e^{-\rho s} U_1(c_{t+s}) ds + e^{-\rho \tau} W_{t+\tau} \right] = \int_0^\infty e^{-(\rho+\lambda)s} (U_1(c_{t+s}) + \lambda W_{t+s}) ds \quad (1)$$

where $\rho > 0$ is the discount rate and τ the random time at which the agent currently in power is removed.¹⁰ Here W_t is the continuation lifetime utility for an agent out of power

$$W_t \equiv \int_0^\infty e^{-\rho s} U_0(c_{t+s}) ds. \quad (2)$$

⁹Concavity and differentiability of U_0 are not crucial for the analysis but simplify the exposition for most of the results. An earlier version of the paper focused on a case that had a convex kink in $U_0(c)$. Theorem 8 below also relaxes the concavity assumption and assumes a concave kink in U_0 .

¹⁰The only uncertainty present in the model is the timing for the alteration of power, τ . However, consumption is deterministic and does not depend on the realization of this uncertainty because of the symmetry of preferences (i.e. agents stepping up to power have identical preferences to those stepping down) and our focus on Markov strategies.

The difference between U_1 and U_0 is a form of disagreement that creates a time-inconsistency problem. The framework can be interpreted literally in a political economy setting as describing the alteration of power of different rulers or legislative majorities. Alternatively, a behavioral interpretation is that the different agents represent different selves or states of mind within an individual.

Crucial to our analysis is the introduction of a *local disagreement index*, which summarizes these differences, defined as the ratio of marginal utilities

$$\beta(c) \equiv \frac{U'_0(c)}{U'_1(c)}.$$

When $\beta(c) = 1$ for all c there is no disagreement and the utility functions coincide, up to a constant. As we shall show, the function $\beta(c)$ summarizes the relevant difference between the utility functions U_1 and U_0 . Throughout the paper we assume that the marginal utility from consumption is higher while in power.

Assumption 1 (Present Bias). *The utility functions U_1 and U_0 are such that, for some $\underline{\beta} > 0$*

$$\beta(c) \in [\underline{\beta}, 1] \quad \text{for all } c > 0.$$

When $\beta(c) < 1$ agents prefer to consume relatively more while in power, leading to a present-bias time-inconsistency problem. Those out of power want those in power to exercise restraint, to consume less and save more. Likewise, those currently in power would like to commit their successors somehow, but have no means to do so.

This simple and flexible framework allows us to capture different patterns of disagreement. In particular, for some applications it is natural to assume that disagreements are stronger at lower consumption levels, so that $\beta(c)$ is increasing. Yet in other cases it may be reasonable to suppose that disagreements grow with spending, so that $\beta(c)$ is decreasing.

Hyperbolic Discounting. An important special case occurs when disagreements are constant: $\beta(c) = \bar{\beta} < 1$ so that $U_0(c) = \bar{\beta}U_1(c)$. The model is then equivalent to the continuous-time hyperbolic discounting model introduced by [Harris and Laibson \(2013\)](#), which in turn builds on discrete-time quasi-hyperbolic counterparts in [Harris and Laibson \(2001\)](#), [Laibson \(1997\)](#) and [Phelps and Pollak \(1968\)](#). It is also common to adopt power utility functions: $U_1(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 0$.

[Harris and Laibson \(2013\)](#) focus on the limit as $\lambda \rightarrow \infty$, the so-called Instantaneous Gratification limit. They show that tractability is gained from the fact that then $V_t = W_t$ in the limit, so that a single continuation value function suffices.

As part of our analysis, we revisit this special hyperbolic case and provide some novel and sharp results. Indeed, we show that the equilibrium is unique in some cases, or

belongs to a simple class in others, and offer a tight characterization (see Section 4.3). As we explain below, we also leverage these results for more general cases where $\beta(c)$ is not constant. Throughout, we do not focus on the instant gratification limit, but instead allow for any finite λ .

Tail Assumptions. One of the goals of our framework is to allow for relatively general differences in utilities, extending disagreements past the hyperbolic discounting assumption. It is convenient to restrict these differences to an arbitrary bounded interval and assume that disagreements are constant outside this interval. This amounts to assuming hyperbolic discounting in the tail.¹¹

Assumption 2. *At high consumption levels disagreements are constant and utilities are power functions: $\beta(c) = \bar{\beta} \leq 1$ and $U_1(c) = \frac{1}{1-\bar{\sigma}}c^{1-\bar{\sigma}}$ with $\bar{\sigma} > 0$ for all $c \geq \bar{c}$ for some $\bar{c} > 0$. Furthermore, $(1 - \bar{\sigma})r < \rho$.*

For a few results, it is convenient to adopt the same hyperbolic assumption at the lower tail.

Assumption 3. *At low consumption levels disagreements are constant and utilities are power functions: $\beta(c) = \bar{\beta} \leq 1$ and $U_1(c) = \frac{1}{1-\bar{\sigma}}c^{1-\bar{\sigma}}$ with $\bar{\sigma} > 0$ for all $c \leq \bar{c}$ for some $\bar{c} > \underline{ra}$.*

Assumption 2 constrains preferences only above some arbitrarily high level of consumption and places no restrictions on the disagreement index or utility functions below this threshold. Assumption 3 constrains preferences only over an arbitrarily small interval. In this sense, both can be viewed as relatively weak constraints.

These mild assumptions are not required for all our results, but greatly simplify the analysis for others. We employ them in two related ways. First, when the equilibrium features positive savings, our constructive approach is aided by pinning things down above some asset level, and focusing on the remaining problem over a bounded interval of assets (when the equilibrium features dissaving, the boundary is automatically provided at the asset limit). As we shall see, Assumption 2 effectively provides us with a boundary condition at some high enough level of wealth \bar{a} . Secondly, for the hyperbolic model we prove uniqueness or show that the equilibrium belongs to a restricted class, offering a tight characterization. Assumptions 2 and 3 allow us to leverage these results outside the hyperbolic case.

¹¹The condition that $(1 - \bar{\sigma})r < \rho$ is a standard growth condition to ensure finite lifetime discounted utility.

2.2 Budget Constraints and Borrowing Limits

The agent in power chooses consumption and assets a_t subject to the budget constraint

$$\dot{a}_t = ra_t + y - c_t. \quad (3)$$

The interest rate $r > 0$ and income $y \geq 0$ are constant and given.

Wealth is subject to an asset or borrowing limit given by

$$\underline{a} \leq a_t. \quad (4)$$

The so-called natural borrowing constraint sets $\underline{a} = -\frac{y}{r} \leq 0$ and allows borrowing against the full present value of income. We mainly focus on tighter constraints, with $\underline{a} > -\frac{y}{r}$. Whenever $a_t = \underline{a}$ we require $c_t \leq y + r\underline{a}$ to ensure that $\dot{a} \geq 0$. Asset limits can also be interpreted as commitment devices. In the individual agent context, this may capture forced savings such as social security or illiquid assets. In the political-economy context, it may capture wealth funds with limits on discretionary spending from natural resources.

Without loss of generality we set income to zero, $y = 0$, and consider a positive asset limit, $\underline{a} > 0$. This is a normalization since, by a change of variables, one can transform a problem with positive income, $y > 0$, to one with zero income as follows. Defining $\tilde{a}_t = a_t + \frac{y}{r}$, then $\dot{\tilde{a}}_t = r\tilde{a}_t - c_t$ and $\tilde{a} \geq \underline{\tilde{a}} \equiv \underline{a} + \frac{y}{r}$. As a result of this transformation, the asset limit becomes a positive lower bound on assets, $\underline{\tilde{a}} > 0$, except in the natural borrowing limit case where $\underline{a} = 0$.

2.3 Many Goods, Engel Curves and Disagreements

One interesting motivation for time-inconsistency problems is to consider spending across various goods, with disagreement on how to spend across these goods. This notion is popular in political economy models on government spending and debt, appearing in [Persson and Svensson \(1989\)](#), [Alesina and Tabellini \(1990\)](#), [Amador \(2002\)](#) and [Azzimonti \(2011\)](#) among others. However, this literature typically adopts simple specifications that imply uniform disagreements. An important exception is [Battaglini and Coate \(2008\)](#) which instead derives specific non-uniform disagreements. [Banerjee and Mullainathan \(2010\)](#) work in a behavioral context and consider relatively rich and flexible disagreements across goods, providing a relationship between the shape of Engel curves and disagreements.

We apply these ideas to our formulation. Suppose consumption takes place over two goods, c_A and c_B , and normalize prices to unity. Utilities are additively separable, with $h(c_A)$ perceived by both those in and out of power, while $g(c_B)$ is perceived only by agents in power (less extreme assumptions work similarly). The static subproblem of spending

across c_A and c_B given total spending defines indirect utility functions

$$U_1(c) = \max_{c_A+c_B=c} \{h(c_A) + g(c_B)\} \quad \text{and} \quad U_0(c) = h(\hat{c}_A(c)), \quad (5)$$

where $(\hat{c}_A(c), \hat{c}_B(c))$ denotes the solution to the maximization. The next result shows that we can generate any desired U_1 and U_0 in this way, by appropriate choices of h and g .

Proposition 1. *Given U_1 and U_0 satisfying Assumption 1, there exists strictly concave functions h and g so that (5) holds.*

Proof. Appendix A.1. □

Note that $U_1'(c) = h'(\hat{c}_A(c))$ and $U_0'(c) = h'(\hat{c}_A(c))\hat{c}'_A(c)$, implying

$$\beta(c) = \hat{c}'_A(c) = 1 - \hat{c}'_B(c),$$

so that a high marginal propensity to spend on c_B increases disagreement, since those out of power do not value this good. When $\hat{c}_A(c)$ is concave, so that the marginal propensity to spend on c_A is decreasing, $\beta(c)$ is decreasing. In this way, the shape of the Engel curve dictates the shape of our disagreement index $\beta(c)$.¹²

3 Markov Equilibria

We focus on Markov equilibria with wealth as the state variable. The policy function for consumption maximizes utility for the agent in power (1), taking as given the value function for those out of power, $W(a)$, satisfying (2). We provide a more technical and detailed recursive definition of our equilibrium concept below. We also incorporate a standard refinement from the literature.

3.1 Regular Equilibria

Our Markov equilibrium concept is defined in terms of the functions $(\hat{c}(a), V(a), W(a))$ and imposes standard conditions. The following differential equations play a crucial role:

$$\rho V(a) = \max_c \{U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a))\}, \quad (6a)$$

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)), \quad (6b)$$

¹²For example, under hyperbolic discounting, $U_0(c) = \bar{\beta}U_1(c)$, the functions h, g in Proposition 1 become

$$h(c_A) = \bar{\beta}U_1\left(\frac{c_A}{\bar{\beta}}\right) \quad \text{and} \quad g(c_B) = (1 - \bar{\beta})U_1\left(\frac{c_B}{1 - \bar{\beta}}\right),$$

which implies that $\hat{c}_A(c) = \bar{\beta}c$ and $\hat{c}_B(c) = (1 - \bar{\beta})c$, i.e. constant marginal propensities to consume on both goods.

where $\hat{c}(a)$ denotes the solution to the maximization in (6a), which is equivalent to the first-order condition

$$U_1'(\hat{c}(a)) = V'(a), \quad (6c)$$

for $a > \underline{a}$. Equation (6a) is the Hamilton-Jacobi-Bellman equation providing a recursive representation of the problem of maximizing (1) taking the value function $W(a)$ as given. The last term takes into account the probability of transitioning out of power with probability λ , at which point the continuation value jumps from $V(a)$ to $W(a)$. Equation (6b) is a recursive representation of condition (2), which defines $W(a)$ given the policy function $\hat{c}(a)$. Finally, the implied wealth must satisfy

$$\dot{a}_t = ra_t - \hat{c}(a_t). \quad (7)$$

For any initial conditions, we require a solution path to this differential equation to exist and impose appropriate transversality conditions along this path (see below).

The elements of a Markov equilibrium spelled out above are familiar enough and all that is required when dealing with continuously differentiable V and W and continuous \hat{c} . However, we allow jumps in \hat{c} and W , as well as points of non-differentiability in V or W , since there is a priori reason to exclude such non smooth behavior. Indeed, in some cases jumps naturally arise or are even essential, as in our poverty trap equilibria. Accordingly, we adapt our definition of a Markov equilibrium as a triplet of functions $(\hat{c}(a), V(a), W(a))$ with the following properties: (a) V is piecewise continuously differentiable; (b) at all points of differentiability of V , equations (6a) and (6c) hold and equation (6b) holds if in addition $\hat{c}(a) \neq ra$; and (c) for any initial value $a_0 \geq \underline{a}$ the differential equation $\dot{a}_t = ra_t - \hat{c}(a_t)$ admits a path $\{a_t\}_{t \in [0, \infty)}$ that satisfies the asset limit $a_t \geq \underline{a}$ and implies that: $W(a_t)$ is continuous for all $t \geq 0$, the transversality conditions $\lim_{t \rightarrow \infty} e^{-(\rho+\lambda)t} V(a_t) = 0$ and $\lim_{t \rightarrow \infty} e^{-\rho t} W(a_t) = 0$ hold, and whenever $\hat{c}(a_0) = ra_0$ we also require that $W(a_0) = \int_0^\infty e^{-\rho t} U_0(\hat{c}(a_t)) ds$.

These conditions are relatively straightforward. The only subtle issue worth highlighting is the smoothness requirements for V and W . The function V must be everywhere continuous because it represents the value from a continuous-time optimal control problem with a controllable state (assets a) and payoffs that are continuous in the control (consumption c). Discontinuities in W do not induce discontinuities in V because c (hence \hat{a}) is unrestricted for all $a > \underline{a}$. Although V is continuous, by (6a) it inherits kinks at points where W is discontinuous; at these same points \hat{c} must be discontinuous.

In contrast, the function W may be discontinuous, because it is not the value from a smooth optimization. Jumps are severely limited, however. Condition (c) implies that $W(a_t) = \int_t^\infty e^{-\rho(s-t)} U_0(\hat{c}(a_s)) ds$ so that the value function W must be continuous and

differentiable along the equilibrium outcome path $\{a_t\}_{t \in [0, \infty)}$. Condition (b) then implies that (c, W) can only jump together to induce $\dot{a} = ra - \hat{c}(a)$ to cross zero. For example, we may jump from dissaving $\dot{a} < 0$ to saving $\dot{a} \geq 0$ (a jump that is important for our poverty trap equilibrium); or we may jump from dissaving $\dot{a} < 0$ to a steady state $\dot{a} = 0$ (this kind of jump occurs in discontinuous dissaving equilibria).¹³ In this latter type of jump we must have $V(a) = \bar{V}(a)$, so jump cannot occur at any point.

Finally, we impose a standard refinement from the literature. When utility is unbounded below it is difficult to rule out the possibility that future selves consume an infinitesimal amount which leads to ever lower continuation values. This in turn may induce the current self to save more and consume less. This feedback effect could potentially lead to an equilibrium with vanishing consumption and utility that is unbounded below. We have not been able to construct such an equilibrium, but it is useful to focus away from this possibility using the following refinement.

Definition 1 (Regular equilibria). A Markov equilibrium is *regular* if there exists $\nu > 0$ such that $\hat{c}(a) \geq \nu a$ for all $a \geq \underline{a}$.

Harris and Laibson (2013) impose precisely this refinement and call equilibria not satisfying it “pathological.” In the same spirit, **Bernheim et al. (2015)** add the constraint $c_t \geq \nu a_t$ to the decision maker problem.¹⁴

We are able to show that all Markov equilibria are regular for an important case of interest.

Lemma 1. *If U_1 is bounded below and satisfies Assumption 2 with $\bar{\sigma} < 1$ then all Markov equilibria are regular.*

From now on we limit ourselves to regular Markov equilibria and refer to such an equilibrium simply as ‘equilibrium’, for short.

3.2 Solution Approach

The system (6) can be thought of as a differential system in (V, W) . Equation (6a) can be solved for $V'(a)$ at any asset a given any pair of values $(V(a), W(a))$ satisfying $(\rho +$

¹³A symmetric case occurs with savings, jumping from a steady state $\dot{a} = 0$ to saving $\dot{a} > 0$. We can rule out, however, a jump from saving $\dot{a} > 0$ to dissaving $\dot{a} < 0$, since such a stable steady state requires continuity of W and hence \hat{c} .

¹⁴Adding a constraint in the decision maker problem is not exactly the same as focusing on equilibria satisfying these constraints. However, both approaches prevent situations where utility goes to $-\infty$.

$\lambda)V(a) - \lambda W(a) - U_1(ra) \geq 0$. Indeed, there are two solutions or roots, one root associated with saving $\hat{c}(a) \leq ra$ and one root associated with dissaving $\hat{c}(a) \geq ra$; the roots coincide if and only if $(\rho + \lambda)V(a) - \lambda W(a) - U_1(ra) = 0$ implying $\hat{c}(a) = ra$.

Define the values of constant wealth by

$$\bar{V}(a) = \frac{\rho}{\rho + \lambda} \frac{U_1(ra)}{\rho} + \frac{\lambda}{\rho + \lambda} \frac{U_0(ra)}{\rho} \quad \text{and} \quad \bar{W}(a) = \frac{U_0(ra)}{\rho}. \quad (8)$$

The value for those not in power, \bar{W} , is the present value of utility from consuming ra forever using U_0 . For those in power the value \bar{V} is a weighted average using U_1 and U_0 . These functions play an important role in our analysis, since the equilibrium value functions (V, W) must coincide with (\bar{V}, \bar{W}) at steady states. Indeed, note that $(\rho + \lambda)\bar{V}(a) - \lambda\bar{W}(a) - U_1(ra) = 0$ whenever $(V, W) = (\bar{V}, \bar{W})$, so that $\hat{c}(a) = ra$ is the unique root.

Our method for characterizing equilibria is to construct solutions to the differential system (6) by attacking these equations directly. Thus, we do not appeal to general existence or uniqueness results for the system (6). Indeed, we are unaware of any off-the-shelf results of this form for such equations for finite λ .^{15,16}

One technical challenge is that the differential system (6) features a singularity at steady states. As a result, we cannot apply standard existence theorems for regular ODEs.¹⁷ Thus, we must provide our own existence result. The following lemma proves the existence around singular points when $\beta < \hat{\beta}$, which will turn out to be the relevant case.

Lemma 2. *Suppose $\beta(ra_0) < \hat{\beta}$. Then the differential system (6) with initial condition*

$$(V(a_0), W(a_0)) = (\bar{V}(a_0), \bar{W}(a_0))$$

admits a solution over the interval $[a_0, a_0 + \omega]$ for some $\omega > 0$, with (i) $V(a) > \bar{V}(a)$ for $a > a_0$;

¹⁵This may seem surprising at first. After all, (6a) is a Hamilton-Jacobi-Bellman for V given W , and for which various existence results may apply (at least for a regular enough class of W functions). However, the main difficulty is not with solving (6a) for V given W . The problem lies in solving the system (6) jointly for both V and W (a fixed point). In particular, (6b) is reminiscent of a Hamilton-Jacobi-Bellman equation, but it is not since $\hat{c}(a)$ does not maximize the right hand side (6b), it instead maximizes (6a).

¹⁶Harris and Laibson (2013) apply general existence results, using viscosity theory, in the hyperbolic instantaneous-gratification limit as $\lambda \rightarrow \infty$. They show that, under some conditions, (6) is then equivalent to the condition for the value function of a time-consistent consumer with modified utility, implying existence and uniqueness.

¹⁷First, the differential system (6) seen as an ODE is not Lipschitz continuous around steady states. Second and more seriously, $W'(a)$ is not even determined at steady state points. When we rewrite system (6) as a differential algebraic equation (DAE), the steady states correspond to critical singular points. However, the DAE at this point does not satisfy the sufficient conditions provided in the literature, for example in Rabier and Rheinboldt (2002), for the existence and uniqueness of solutions around singular points of DAEs, except for the case $\lambda = 0$.

and (ii) $\hat{c}(a) > ra$ for $a > a_0$, $\lim_{a \downarrow a_0} \hat{c}(a) = ra_0$ and $\lim_{a \downarrow a_0} \hat{c}'(a) = \infty$.

Proof. Appendix C. □

Fortunately, with this lemma in hand, our constructive method is relatively straightforward and also provides an immediate characterization. We construct equilibria by solving the ODEs starting at the bottom and working up, when dissaving; or by starting at the top and working down, with saving; or by combining these procedures. In more detail, the construction involves the following: (i) imposing boundary conditions that serve as initial conditions; (ii) solving the ODEs with a saving or dissaving root over an interval of wealth; (iii) decide whether to engineer a jump in W . The great advantage of this approach is that (ii) is local in nature. Also, (iii) is aided by the fact that V must be continuous. Finally, the boundary conditions required for (i) are naturally supplied at the asset limit or at high enough level of wealth, appealing to Assumption 2.

3.3 Recovery of Power

We now justify our focus on a situation where agents do not return to power. Consider a situation where power can be recovered at Poisson rate $\lambda_r > 0$. The differential system (6) becomes

$$\begin{aligned}\rho V(a) &= \max_c \{U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a))\}, \\ \rho W(a) &= U_0(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)) + \lambda_r(V(a) - W(a)).\end{aligned}$$

The last term in the second equation captures the value from returning to power. Although this creates a difference with system 6, our next result establishes that the two settings are observationally equivalent.

Proposition 2. *Consider an economy with utilities and Poisson rates $(U_1, U_0, \lambda, \lambda_r)$ with positive recovery probability $\lambda_r > 0$. Equilibria for this economy coincide with equilibria for an economy with utilities and Poisson rates $(U_1, \tilde{U}_0, \tilde{\lambda}, 0)$ with no possible recovery of power, where $\tilde{\lambda} \equiv \lambda + \lambda_r$ and $\tilde{U}_0(c) \equiv \frac{\lambda}{\lambda + \lambda_r} U_0(c) + \frac{\lambda_r}{\lambda + \lambda_r} U_1(c)$.*

Proof. The pair (V, W) satisfies the differential system with power recovery above for $(U_1, U_0, \lambda, \lambda_r)$ if and only if the pair (V, \tilde{W}) with $\tilde{W} \equiv \frac{\lambda}{\lambda + \lambda_r} W + \frac{\lambda_r}{\lambda + \lambda_r} V$ satisfies the differential system without power recovery (6a)–(6b) for $(U_1, \tilde{U}_0, \tilde{\lambda}, 0)$ with $\tilde{\lambda} \equiv \lambda + \lambda_r$ and $\tilde{U}_0(c) \equiv \frac{\lambda}{\lambda + \lambda_r} U_0(c) + \frac{\lambda_r}{\lambda + \lambda_r} U_1(c)$. □

Intuitively, the possibility of recovering power makes an agent more invested in consumption after being ousted from power, which is similar to placing a higher value on

consumption while out of power. In a political economy setting, [Amador \(2002\)](#) and [Azzimonti \(2011\)](#) assume that there are no benefits from consuming out of power, $U_0 = 0$. Then with $\lambda_r = 0$ there is no time-inconsistency problem, only greater discounting, at rate $\rho + \lambda > \rho$. Thus, they assume that the agent returns to power with positive probability, $\lambda_r > 0$. Proposition 2 shows that this is equivalent to a model without recovery of power but with a positive utility for those out of power: $\tilde{U}_0 = \bar{\beta}U_1$ where $\bar{\beta} = \lambda/(\lambda + \lambda_r) \in (0, 1)$, i.e. a hyperbolic model.

3.4 Generalized Euler Equation

To economize on space we omit the calculations, but we have shown that

$$\begin{aligned} \frac{d}{dt}(U'_1(c_t)) &= (\rho - r)U'_1(c_t) + \lambda \int_0^{T^*} e^{-\int_0^s (\rho + \lambda - r + \hat{c}'(a_{t+z})) dz} U'_1(c_{t+s}) (1 - \beta(c_{t+s})) \hat{c}'(a_{t+s}) ds \\ &\quad + \lambda e^{-\int_0^{T^*} (\rho + \lambda - r + \hat{c}'(a_{t+z})) dz} U'_1(ra_{T^*}) (1 - \beta(ra_{T^*})), \end{aligned}$$

where T^* denotes the moment wealth reaches a steady state, i.e. $\hat{c}(a_{T^*}) = ra_{T^*}$; if $T^* = \infty$ the last term is zero. This is an adaptation of the Generalized Euler equation, derived by [Harris and Laibson \(2001\)](#) in the context of the discrete-time quasi-hyperbolic model, to our continuous-time model. When $\lambda = 0$ or $\beta(c) = 1$ this reduces to the standard Euler equation. However, when $\lambda > 0$, $\beta(c) < 1$ and $\hat{c}'(a) > 0$ the second and third terms on the right side are positive, acting in the same direction as higher ρ .

4 Dissaving and Saving

In this section we construct and characterize equilibria focusing on situations where the disagreement index $\beta(c)$ is relatively stable, implying global savings or dissavings.

When $r < \rho$ a time-consistent agent ($\lambda = 0$ or $\beta(c) = 1$) dissaves and time inconsistency (i.e. $\lambda > 0$ and $\beta(c) < 1$) only reinforces this conclusion. When $r > \rho$ a time consistent agent saves, but time inconsistency may overturn this conclusion. What turns out to be crucial is the value of our local disagreement index $\beta(c)$ relative to a cutoff defined by

$$\hat{\beta} \equiv \frac{\rho}{r} \left(1 - \frac{r - \rho}{\lambda} \right). \quad (9)$$

Note that $\hat{\beta} < 1$ if and only if $r > \rho$. Moreover, whenever $\hat{\beta} > 0$ then $\hat{\beta}$ is decreasing in r and increasing in ρ , and increasing (decreasing) in λ if $r > \rho$ (if $r < \rho$).

We will show that when $\beta(c) < \hat{\beta}$ the agent dissaves and when $\beta(c) > \hat{\beta}$ the agent saves. To anticipate this key role played by β relative to $\hat{\beta}$ it is useful to cover some special cases. We first state a very simple result showing that in the borderline case $\beta = \hat{\beta}$ an equilibrium exists with zero savings.

Theorem 1 (Zero Savings). *Assume that $\beta(c) = \hat{\beta}$ for all $c \geq r\underline{a}$. Then, $(V, W) = (\bar{V}, \bar{W})$ and $\hat{c}(a) = ra$ is an equilibrium.*

Proof. Appendix D. □

Next we present a result on linear equilibria for the hyperbolic case that extends the characterization offered already in Harris and Laibson (2013, Appendix F). The result requires no asset limit and power utility functions.

Theorem 2 (Linear Equilibria). *Suppose*

$$\beta(c) = \bar{\beta} \leq 1, \quad U_1(c) = \frac{1}{1-\sigma} c^{1-\sigma} \quad \text{and} \quad \underline{a} = 0.$$

Then if $(1-\sigma)r < \rho$ there exists a unique linear equilibrium $\hat{c}(a) = \psi a$ with saving $\psi < r$ when $\bar{\beta} > \hat{\beta}$ and dissaving $\psi > r$ when $\bar{\beta} < \hat{\beta}$. When $\bar{\beta} > \hat{\beta}$ the result holds even if $\underline{a} > 0$.

Proof. Appendix D. □

These linear equilibria are similar to the well-known linear equilibria employed in discrete-time quasi-hyperbolic settings, such as Phelps and Pollak (1968), Laibson (1996) and others. An important difference is that Theorem 2 states that in continuous time linear equilibria are always unique, echoing Harris and Laibson (2013, Appendix F). Crucially, we show that the sign of the saving rate $\psi - r$ depends on β versus $\hat{\beta}$. When the interest rate is low enough or disagreements are high enough, $\beta < \hat{\beta}$, the agent dissaves. When the interest rate is high enough or disagreements are low enough, $\beta > \hat{\beta}$, the agent saves.

The linear equilibrium breaks down if $\beta(c)$ is not constant, if utility functions are not power functions, or in the presence of a non-trivial borrowing constraint. Although linear equilibria are special, the condition for savings and dissavings turn out to hinge on β versus $\hat{\beta}$ more generally.

4.1 Dissaving

Our first result constructs an equilibrium with dissaving, when $\beta(c) < \hat{\beta}$. Indeed, we show that all equilibria must have this property.

Theorem 3 (Dissaving). *Suppose Assumptions 1 and 2 hold and $\beta(c) < \hat{\beta}$ for all $c \geq ra$. Then there exists an equilibrium with dissavings, $\hat{c}(a) \geq ra$ for $a \geq \underline{a}$, and $V(a) > \bar{V}(a)$ in a neighborhood of a steady state. Moreover, all equilibria satisfy these properties. If in addition Assumption 3 holds then there is at most one continuous equilibrium and this equilibrium features strict dissavings, $\hat{c}(a) > ra$ for all $a > \underline{a}$.*

Proof. Appendix E. □

When $r < \rho$ dissaving is guaranteed, even in a time consistent situation. When $r \geq \rho$ dissaving occurs if the time-inconsistency problem created by disagreements is sufficiently strong, so that $\beta < \hat{\beta}$. Whenever $\hat{\beta} > 0$ then $\hat{\beta}$ is decreasing in r and increasing in ρ and λ (given $r > \rho$). Thus, lower r , higher ρ , higher λ and lower β all promote dissaving.

Our constructive proof solves the differential system (6) starting at \underline{a} with initial conditions $V(\underline{a}) = \bar{V}(\underline{a})$ and $W(\underline{a}) = \bar{W}(\underline{a})$, employing Lemma 2. We move up using the root for $V'(a)$ associated with dissaving. There are two possibilities. First, in some cases the solution may be continued indefinitely as $a \rightarrow \infty$, providing a continuous equilibrium; Assumption 3 then ensures that there is only one such solution. Alternatively, the solution to the differential system may reach a point where $\hat{c}(a) = ra$ and $V(a) < \bar{V}(a)$. Past this point there is no root associated with dissaving. An equilibrium must then involve a jump in W and \hat{c} at the asset point where V crosses \bar{V} from above. The system is then restarted at this point, by setting $(V, W) = (\bar{V}, \bar{W})$ again and continuing as before. Note that in this construction jumps only occur at steady states.

As this discussion makes clear, both continuous and discontinuous equilibria are possible and they may even coexist in some cases. We postpone a more detailed discussion regarding these possibilities until Section 7, where we provide sufficient conditions for the existence of continuous equilibria.

The existence portion of Theorem 3 only requires Assumption 1 and $\beta(c) < \hat{\beta}$. Assumption 2 is invoked to prove that all equilibria have the same property; this leverages our tight characterization of the hyperbolic case contained in Section 4.3. Assumption 3 is used to provide a unique local solution immediately above \underline{a} , which then implies that there is at most one continuous equilibrium.

Commitment Devices. Time inconsistency problems generally create a demand for commitment devices. In our setting, a simple form of commitment can be captured by raising the asset limit. This amounts to removing liquidity from the hands of those in power. Amador et al. (2006) argue that such minimum savings policies are optimal within a re-

lated class of models. Here we simply explore whether they would be adopted by those in power. The most extreme form of such a commitment device sets the new asset limit at the present asset level, effectively imposing a budget balance rule to hold assets constant. Such a commitment is desirable to those in power if $V(a) < \bar{V}(a)$.

Is such a commitment device desirable for the agents in power in our model? Theorem 3 shows that $V(a) \geq \bar{V}(a)$ near steady states. Moreover, the proof shows that there always exists an equilibrium (possibly discontinuous) with the property that $V(a) \geq \bar{V}(a)$ for all $a \geq \underline{a}$. Thus, when this inequality holds, the agent in power never willingly ties himself to the mast, so to speak, to adopt a budget balance commitment. This is not obvious, since by adopting a commitment device today the agent trades off lower consumption today with greater commitment in the future. Although our model has this somewhat surprising feature, that an equilibrium with this property always exists, in some cases a continuous equilibrium exists and features $V(a) < \bar{V}(a)$ for large enough assets. In such cases, the agent in power would like to commit to a balanced budget rule immediately to raise utility to $\bar{V}(a)$. We conclude that immediate commitment to a balanced budget may arise in some cases, but only when wealth is sufficiently high.

Even if $V(a) \geq \bar{V}(a)$, so that the current decision maker would not impose upon itself a budget balance rule, whenever $r \geq \rho$ one can show that $W(a) < \bar{W}(a)$ away from steady states. This implies that agents out of power would like to bind those in power to a budget balance rule to hold assets constant.¹⁸ Equivalently, the agent in power would like to commit its successors to such a rule. There are two ways the agent in power can achieve something similar. First, it may commit to a budget-balance rule that only comes into effect in the future, after a grace period. Second, it may commit immediately to an asset limit that is below the current asset level, so that this constraint binds in the future but not immediately.

4.2 Saving

We now consider the opposite case, where disagreement is low enough and $\beta > \hat{\beta}$. We show that this ensures positive savings. Indeed, we establish that there is a unique equilibrium.

Theorem 4 (Saving). *Suppose that $\beta(\bar{c}) > \hat{\beta}$ and that Assumption 1 and 2 hold. Then there exists a unique equilibrium for all $\underline{a} \geq \hat{a}$ for some $\hat{a} \in [0, \frac{\bar{c}}{r})$.*

Indeed, there exists $\bar{a} > \frac{\bar{c}}{r}$ and a unique consumption function $\hat{c}(a)$ for $a \geq \hat{a}$ with the property that \hat{c} is the unique equilibrium for any $\underline{a} \geq \hat{a}$ and $\hat{c}(a) = \psi a$ for $a \geq \bar{a}$, with ψ given by Theorem

¹⁸When $r < \rho$ then $W > \bar{W}$ is possible since some dissaving is desirable even for a time-consistent agent.

2. The equilibrium features strict savings $\hat{c}(a) < ra$ and $\hat{c}'(a) > 0$ for $a \geq \underline{a}$. Finally, if $\beta(c)$ is weakly increasing then either $\hat{a} = 0$ or $\beta(r\hat{a}) \leq \hat{\beta}$.

Proof. Appendix F. □

Just as with Theorem 3, the proof of this result is constructive and works by solving the differential system (6). We start the differential system at \bar{a} with initial conditions for $(V(\bar{a}), W(\bar{a}))$ provided by the linear solution described in Theorem 2. Assumption 2 ensures that this is a valid boundary condition. We then solve downwards using the higher root for $V'(a)$, which is associated with positive savings. Positive savings imply that \bar{a} will be reached from below, justifying its use as a boundary condition for the differential system. This construction comes to a stop when we reach zero or reach a point \hat{a} where $\hat{c}(\hat{a}) = r\hat{a}$; in this latter case we actually show that $(V(\hat{a}), W(\hat{a})) = (\bar{V}(\hat{a}), \bar{W}(\hat{a}))$. By leveraging the hyperbolic results in Section 4.3, we show that this construction is the unique equilibrium.

The last statement in the theorem implies that if $\beta(c)$ is increasing and always above $\hat{\beta}$ then $\hat{a} = 0$. This leaves open the possibility that if $\beta(c)$ is decreasing but always above $\hat{\beta}$ then $\hat{a} > 0$. We have been unable to develop such an example, but conjecture it may be possible. Intuitively, wealth accumulation may be discouraged if disagreements rise with spending, i.e. $\beta(c)$ decreasing.¹⁹

4.3 Hyperbolic Discounting with Power Utility

This subsection explores the hyperbolic discounting case $\beta(c) = \bar{\beta}$ with power utility:

$$U_1(c) = u(c) \quad \text{and} \quad U_0(c) = \bar{\beta}u(c),$$

$$u(c) = \frac{1}{1-\sigma}c^{1-\sigma}.$$

The differential system (6) can then be written as

$$\rho V(a) = u(\hat{c}(a)) + u'(\hat{c}(a))(ra - \hat{c}(a)) + \lambda(W(a) - V(a)), \quad (10a)$$

$$\rho W(a) = \bar{\beta}u(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)), \quad (10b)$$

$$V'(a) = u'(\hat{c}(a)). \quad (10c)$$

In accordance with our general approach, one can use equation (10a) to solve for $\hat{c}(a)$ at any asset a given any pair of values $(V(a), W(a))$ at this asset level satisfying $(\rho +$

¹⁹If $\hat{a} > 0$ and $\underline{a} < \hat{a}$ then equilibria can be constructed by restarting the system, analogously to the construction in Theorem 3, to continue the solution for $a < \hat{a}$. Theorem 8 actually involves a similar construction: solving for an interval where saving takes place to the left of a steady state under the condition that $\beta > \hat{\beta}$.

$\lambda)V(a) - \lambda W(a) - u(ra) \geq 0$. Equation (10a) has one root satisfying $\hat{c}(a) \leq ra$ and one root satisfying $\hat{c}(a) \geq ra$; the roots coincide if and only if $(\rho + \lambda)V(a) - \lambda W(a) - u(ra) = 0$. Selecting a root for $\hat{c}(a)$ and plugging it into equations (10b)–(10c) gives a differential system in V and W . One can solve for (V, W) , then back out the implied \hat{c} .

However, an equivalent way to proceed, which turns out to be particularly useful in this hyperbolic context, is to change variables and solve for (\hat{c}, W) , backing out V when needed. Indeed, differentiating the last equation and using the first equation gives

$$\hat{c}'(a) = \frac{1}{u''(\hat{c}(a))(ra - \hat{c}(a))} \left((\rho + \lambda - r)u'(\hat{c}(a)) + \lambda \frac{\bar{\beta}u(\hat{c}(a)) - \rho W(a)}{ra - \hat{c}(a)} \right), \quad (11a)$$

$$W'(a) = \frac{1}{ra - \hat{c}(a)} (\rho W(a) - \bar{\beta}u(\hat{c}(a))). \quad (11b)$$

Given a solution to (11) for \hat{c} and W one can back out V using equation (10a).

Next we exploit homogeneity to transform this non-autonomous differential system (11) into an autonomous one. Define the functions (c, w, v) by

$$\hat{c}(a) \equiv ac(x),$$

$$W(a) \equiv a^{1-\sigma}w(x)$$

$$V(a) \equiv a^{1-\sigma}v(x)$$

where $x = \log a$. Define the constants \bar{v}, \bar{w} analogously by $\bar{V}(a) = a^{1-\sigma}\bar{v}$ and $\bar{W}(a) = a^{1-\sigma}\bar{w}$. Here c represents the consumption rate and $r - c$ represents the savings rate. Homogeneity has been widely exploited in the literature to study linear equilibria, when c, w and v are constant. In contrast, we allow the functions (c, w, v) to be non constant, so our renormalization is simply a change of variables: any (\hat{c}, W, V) can be represented by (c, w, v) and vice versa.

Rewriting system (11) in terms of (c, w) one observes that assets a drops out and we obtain the autonomous system

$$c'(x) = \frac{\lambda c(x)}{\sigma(r - c(x))} \left(\frac{\rho w(x) - \bar{\beta}u(c(x))}{r - c(x)} \frac{1}{u'(c(x))} - \frac{r}{\rho} \hat{\beta} \right) - c(x), \quad (12a)$$

$$w'(x) = \frac{\rho w(x) - \bar{\beta}u(c(x))}{r - c(x)} - (1 - \sigma)w(x). \quad (12b)$$

Likewise condition (10a) gives v as a function of (c, w) ,

$$v(x) = \frac{1}{\rho + \lambda} (u(c(x)) + u'(c(x))(r - c(x)) - \lambda w(x)), \quad (12c)$$

The differential system (12) can be used to sharply characterize equilibria. Just as with the original system, there is a singularity at $c = r$ that requires some care. However, because of the reduced dimension of the system we are able to prove the uniqueness of a

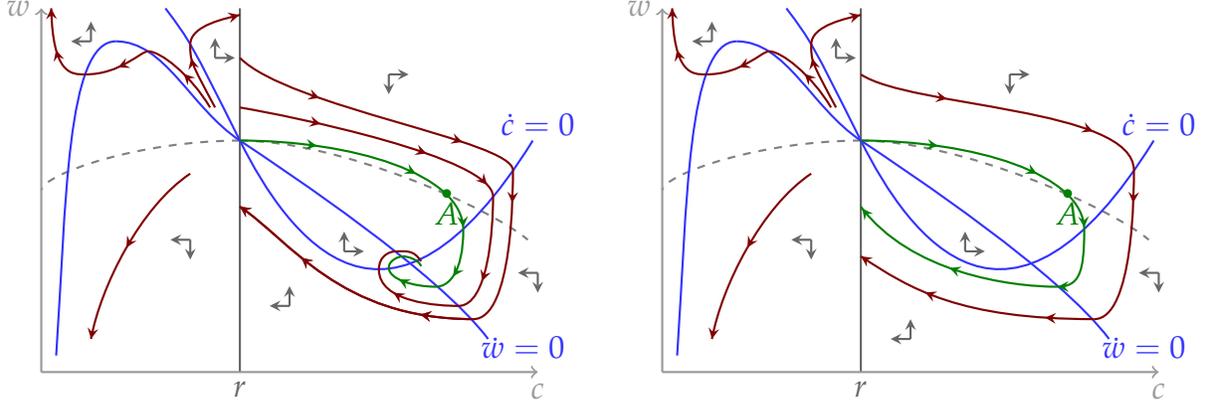


Figure 1: Phase diagrams for $\bar{\beta} < \hat{\beta}$ and $r > \rho$. The equilibrium path shown in green features dissavings. The left panel displays a case with a continuous equilibrium. In the right panel no continuous equilibrium exists. A discontinuous equilibrium exists in both cases, starting at $(c, w) = (r, \bar{w})$ and resetting when reaching point A .

solution when $\bar{\beta} < \hat{\beta}$ in the neighborhood of $c = r$ despite the singularity. Moreover, the autonomous system is more tractable and its dynamics can be grasped graphically, as we now demonstrate. Taken together our results will provide a complete characterization of equilibria in hyperbolic models. Current results provide existence of equilibria in some cases, but do not fully characterize the set of Markov equilibria.

The case with $\bar{\beta} < \hat{\beta}$. For this case we show that all regular equilibria involve dissaving. We are also able to provide a complete and simple characterization. Figure 1 depicts two phase diagrams (both for the case with $r > \rho$; the case with $r \leq \rho$ is similar). The loci where $\dot{c} = 0$ and $\dot{w} = 0$ are shown in solid blue; the dashed line shows the loci where $v = \bar{v}$, which peaks at $(c, w) = (r, \bar{w})$. This point turns out to be critical in our construction, serving as a boundary condition at $a = \underline{a}$. There is a steady state with $c > r$ and $w < \bar{w}$ corresponding to the linear equilibrium with no asset limit, $\underline{a} = 0$, from Theorem 2. There are no other steady states. The system is singular at $c = r$, so $(c, w) = (r, \bar{w})$ is not a steady state despite the apparent intersection of the loci for $\dot{c} = 0$ and $\dot{w} = 0$.

The green path in the left panel represents the unique continuous equilibrium. Starting from the asset limit $a = \underline{a}$ with initial conditions $(c, w) = (r, \bar{w})$ this path is the unique solution to the differential system and spans all $a \geq \underline{a}$. The right panel depicts a situation where no continuous equilibrium exists. The key difference is that the path leaving $(c, w) = (r, \bar{w})$ reaches $c = r$ with $w < \bar{w}$ at some finite asset level $a > \underline{a}$. This is incompatible with an equilibrium. A discontinuous equilibrium can be constructed by starting at $(c, w) = (r, \bar{w})$, following the path to point A where $v = \bar{v}$, and then restarting at (r, \bar{w}) ,

and repeating this indefinitely. Similar arguments imply that a discontinuous equilibria exists in the left panel, in addition to the continuous equilibrium.

In both panels, there are no other equilibrium paths. If $c < r$ the dynamical system either leads to $c \rightarrow 0$ with w diverging, which is incompatible with the regular equilibrium refinement, or else it reaches $c = r$ with $w > \bar{w}$ at some finite asset level, which is incompatible with a steady state, which requires $w = \bar{w}$. Since $c \geq r$ in the neighborhood of \underline{a} , an equilibrium requires $w = \bar{w}$, $v = \bar{v}$, and, thus, $c = r$. Despite the singularity, we prove that there is a unique solution starting from this point.²⁰

These results are summarized in the next theorem.

Theorem 5. *Consider the hyperbolic case, with $\beta(c) = \bar{\beta}$ and $U_1(c) = u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$. Suppose $\bar{\beta} < \hat{\beta}$ and assume $\underline{a} > 0$. Then equilibria exist and must involve dissaving. More specifically,*

(a) *there exists a unique solution (c, w) to the autonomous differential system (12) (with associated v) with initial condition $c(0) = r$, $w(0) = \bar{w}$. This solution can be defined over the interval $(0, \bar{x})$ with $c(x) > r$ for $x \in (0, \bar{x})$ and either $\bar{x} = \infty$ or $\lim_{x \rightarrow \bar{x}} c(x) = r$. Let $X \equiv \{x \in (0, \bar{x}) \mid v(x) = \bar{v}\}$, let $\underline{x} = \inf X$ with the convention that $\underline{x} = \infty$ if X is empty (which requires $\bar{x} = \infty$). Finally, $\underline{x} > 0$ and $v(x) > \bar{v}$ for all $x \in (0, \underline{x})$.*

(b) *the consumption function $\hat{c}(a)$ forms an equilibrium if and only if*

$$\hat{c}(a) = ac(\log a - \log a_n) \quad \forall a \in [a_n, a_{n+1})$$

with $a_0 = \underline{a}$ and for each $n = 1, 2, \dots$ (i) if $\bar{x} < \infty$ then $\log(\frac{a_n}{a_{n-1}}) \in X$, or (ii) if $\bar{x} = \infty$ then either $\log(\frac{a_n}{a_{n-1}}) \in X$ or $a_n = \infty$.

Proof. Appendix G. □

According to this theorem the set of regular equilibria is limited to paths generated by the unique solution to our autonomous differential system (12). Moreover, properties of this unique solution reveal important properties of the equilibria set. A few implications follow immediately. First, for an interval around the asset limit, given by $[\underline{a}, \underline{a}e^{\underline{x}})$, the consumption function must be continuous and coincide across all equilibria. Second, there is at most one continuous regular equilibrium, which exists if and only if $\bar{x} = \infty$. Third, discontinuous equilibria exists if and only if X is nonempty. Jumps occur at steady state asset levels and they restart the normalized consumption function.

²⁰To simplify the exposition in the text we ignored jumps in (c, w) except for those used in the right panel of Figure 1. Our equilibrium definition allows for jumps in $c(a)$ and $w(a)$ anywhere, as long as $v(a)$ remains continuous and $r - c(a)$ switching signs. The complete proof in Appendix G takes all possible jumps into account. However, they do not end up affecting the set of equilibria.

Equilibria in our continuous-time hyperbolic model are relatively well behaved, especially compared to their counterparts in the discrete-time quasi-hyperbolic model. Although, no complete characterization of equilibria has been offered in the literature, the equilibrium must be discontinuous, including jumps outside steady states (Chatterjee and Eyigungor, 2015), in addition to jumps at steady states. In contrast, in our continuous-time model, a continuous and differentiable equilibrium is possible here and equilibria are always piecewise differentiable. In this sense, jumps are minimized by formulating the model in continuous time. Section 7 provides sufficient conditions to ensure the existence of continuous equilibria in the more general model; providing conditions for the hyperbolic model as a special case.

We also note that the equilibrium may be unique. This may occur for a continuous equilibrium, if $\bar{x} = \infty$ and X is empty, which is necessarily true for $\bar{\beta}$ near 1, or for λ near 0. The unique equilibrium may also be discontinuous equilibrium, if $\bar{x} < \infty$ and X contains a single point. Multiple equilibria are also possible, if X has more than one element or if $\bar{x} = \infty$ and X is non-empty. When this is the case there exists a countable number of equilibria, determined by the choice in positioning jumps. Numerically, however, we find that a more typical configuration has X empty or X containing a single point.

This theorem requires a strictly positive asset limit, $\underline{a} > 0$, however similar constructions work when $\underline{a} = 0$. As is well known, when $\underline{a} = 0$ there exists a linear equilibrium with a constant dissaving rate. However, we argue that this is not necessarily the unique equilibrium. Indeed, whenever X is non-empty there exists a continuum of equilibria. For any $a_0 \in [1, e^{\bar{x}})$ use a two-sided periodic construction similar to that in Theorem 5, so that

$$\hat{c}(a) = ac(\log a - \log a_n) \quad \forall a \in [a_n, a_{n+1})$$

for $n = \dots, -2, 1, 0, 1, 2, \dots$ and $\log(\frac{a_{n+1}}{a_n}) = \bar{x}$. Since this equilibrium is indexed by the arbitrary steady state a_0 there is a continuum of such equilibria. These equilibria are discontinuous, but for some parameters there exists a continuum of continuous equilibria. These equilibria feature cyclical patterns for the consumption rate.²¹ For all these reasons, the degree of multiplicity of equilibria is greater when $\underline{a} = 0$ than when $\underline{a} > 0$. We conclude that imposing a positive asset limit, rather than opening the door to multiple

²¹This occurs when the dynamical system features a limit cycle around the interior steady state. In these cases the limit cycle and any other path inside the limit cycle serves as a possible equilibrium, starting at any asset limit a_0 and solving backwards and forwards along the path. Thus, each arc provides a continuum of solutions and there are a continuum of arcs.

A limit cycle occurs for a wide set of parameters. Indeed, by the Hopf Bifurcation Theorem a limit cycle is guaranteed to exist in a neighborhood of the parameter space where the interior steady state switches from being locally stable to locally unstable, with complex eigenvalues. Numerically we find that such cycles are supercritical, i.e. unstable in the direction of increasing assets.

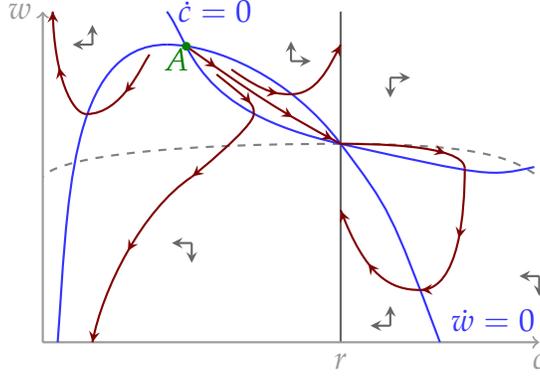


Figure 2: Phase diagrams for $\bar{\beta} \in (\hat{\beta}, \frac{r}{\rho}\hat{\beta})$. The unique equilibrium features savings at a constant savings rate, denoted by point A .

equilibria, contributes to restricting the set of equilibria.

The case with $\bar{\beta} > \hat{\beta}$. For this case we show that a single equilibrium exists. Figure 2 depicts the phase diagram. The unique equilibrium is the steady state labeled A , corresponding to the linear equilibrium with a constant consumption rate. As shown in the figure, there is a stable arm path from the neighborhood of this steady state to $c = r$ and $w = \bar{w}$; we discuss this path below. For now, note that starting from any other point with $c < r$ leads to either $c \rightarrow 0$ or hitting $c = r$ and $w > \bar{w}$ at some finite asset level. Both are incompatible with an equilibrium.

Turning to paths with $c > r$, note that the lowest asset level featuring dissavings requires $(c, w) = (r, \bar{w})$ to be compatible with a steady state. Unlike the case with $\beta < \hat{\beta}$, starting from this point either there are no solutions or there are a continuum of solutions.²² In the latter case all paths eventually hit $c = r$ with $w < \bar{w}$ at a finite asset level. These paths are incompatible with an equilibrium since they enter the $c < r$ region (outside the stable arm path).

Finally, we discuss the stable arm path with $c < r$ converging to $(c, w) = (r, \bar{w})$. Starting anywhere on this stable arm, the path reaches $c = r$ at some finite asset level. The path continues into the $c > r$ region, but all such paths reach $c = r$ and $w < \bar{w}$ at some finite asset level and, thus, are incompatible with an equilibrium.

We conclude that the interior steady state is the unique equilibrium. This characterization holds both for $\underline{a} > 0$ and $\underline{a} = 0$. These results are summarized in the next theorem.

²²When $\bar{\beta} > \frac{r}{\rho}\hat{\beta}$ the loci for $\dot{c} = 0$ slopes upward at $c = r$ and there are no solutions starting from $(c, w) = (r, \bar{w})$.

Theorem 6. Consider the hyperbolic case, with $\beta(c) = \bar{\beta}$ and $U_1(c) = u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$. Suppose $\bar{\beta} > \hat{\beta}$ and $\underline{a} \geq 0$. Then there exists a unique equilibrium. The consumption function is linear with strictly positive savings: $\hat{c}(a) = \psi a$ and $\psi < r$.

Proof. Appendix G. □

The existing literature has noted and used the existence of the linear equilibrium with positive savings. However, whether other equilibria exist has remained an open question. Our result settles this elusive question for the continuous-time hyperbolic setting. The analysis relies on the properties of our differential system (12), which has not been previously studied. In particular, the result rests on the instability of the steady state, which is not an obvious property. If, instead, the interior steady state had paths converging towards it, then all such paths would constitute a regular equilibrium when $\underline{a} > 0$. Indeed, when $\bar{\beta} < \hat{\beta}$ and $\underline{a} = 0$ the steady state with dissaving $c > r$ may be locally stable or unstable and a continuum of equilibria exist in some cases (see discussion above). By symmetry, the same would be true for $\bar{\beta} > \hat{\beta}$ for the dissaving steady state $c < r$ if we considered $\bar{\beta} > 1$. Thus, uniqueness subtly depends on $\bar{\beta} < 1$ and is not inherent to time-inconsistency models.

Finally, consider momentarily a variant of the saving game which incorporates an ad hoc finite asset limit: we add the constraint $a \in [\underline{a}, \bar{a}]$ on the decision maker. One can then show that the unique equilibrium lies on the stable path emanating from the interior steady state and leading to (r, \bar{w}) . The initial point $(c(\underline{a}), w(\underline{a}))$ is determined by the requirement that at the end point $(c(\bar{a}), w(\bar{a})) = (r, \bar{w})$. As the ad hoc bound \bar{a} is increased, then $(c(\underline{a}), w(\underline{a}))$ gets closer to the interior steady state and as $\bar{a} \rightarrow \infty$ then $(c(\underline{a}), w(\underline{a}))$ must converge to the steady state. Interestingly, these conclusions hold for Markov equilibria without the regularity refinement. We conclude that the game with an ad hoc upper asset limit converges to the unique equilibrium of the game without any asset limit. Indeed, taking a limit of games with $\bar{a} \rightarrow \infty$ provides an alternative to the introduction of the regularity refinement.

Local Indeterminacy. The paths going through (r, \bar{w}) in Figure 2 are worth further discussion, since they are the continuous-time analogs of the construction in [Krusell and Smith \(2003\)](#). For any arbitrary $a^* \geq \underline{a}$ we can set $(c(a^*), w(a^*)) = (r, \bar{w})$ and let $(c(a), w(a))$ be given locally by any of the paths solving the differential system. The asset level a^* is then a stable steady state. Since a^* was arbitrary this delivers a form of indeterminacy. Indeed, since there are many paths to the right side leaving (r, \bar{w}) there is a further dimension of indeterminacy. These constructions satisfies all the local requirements for an

equilibrium. However, we already know that, as a corollary to Theorem 6, they cannot be part of a global equilibrium. Indeed, the next proposition shows that these constructions cannot even be extended past some finite interval.²³

Proposition 3. *Consider the hyperbolic case, with $\beta(c) = \bar{\beta}$ and $U_1(c) = u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$. Suppose $\bar{\beta} \in (\hat{\beta}, \hat{\beta}_{\frac{r}{\rho}})$ and any $\underline{a} \geq 0$.*

For any $a^ > \underline{a}$ there exists a continuum of local solutions to the differential system (6); each solution (\hat{c}, V, W) is defined over a maximal interval $[\underline{a}, \bar{a}]$ that is bounded, $\bar{a} < \infty$, with the property that the dynamics for wealth are contained within this interval (i.e. it is self-referential). Moreover, for each solution, a^* is the unique and stable steady state within the interval $[\underline{a}, \bar{a}]$: $\hat{c}(a^*) = ra^*$, $\hat{c}(a) < ra$ for $a \in [\underline{a}, a^*)$ and $\hat{c}(a) > ra$ for $a \in (a^*, \bar{a}]$.*

These local constructions are not part of an equilibrium. Indeed, there is no equilibrium with an interior steady state.

Proof. Appendix G. □

The fact that local solutions that propose an interior and stable steady state cannot be part of an equilibrium, the very last statement in the proposition, is a direct corollary of Theorem 6, which rules out dissaving at any asset level. (We thus employ the term ‘local solution’ or ‘local construct’ to emphasize the distinction with an equilibrium.) The proposition shows that a continuum of local solutions can be defined, each over a bounded interval containing an interior and stable steady state. Moreover, this interval is self referential, in the sense that the dynamics for wealth pointing “inward”, towards the steady state. However, the proposition also states that none of these solutions can be extended in the same way to cover a wider interval. These results are not corollaries of Theorem 6.

²³In both the discrete- and continuous-time model there are two dimensions of indeterminacy. First, and most fundamentally, there are a continuum of steady states. Second, in discrete time, given a steady state level, [Krusell and Smith \(2003\)](#) provide a continuum of constructions in the neighborhood to the left of the steady state, but a single construction in the neighborhood to the right of the steady state. In continuous time our results actually indicate that there is no indeterminacy to the left. This can be easily reconciled with the aforementioned indeterminacy in discrete time. For a given steady state a^* , one can show that all the paths $\{a_t\}$ in [Krusell and Smith \(2003\)](#) solve the same difference equation $a_{t+1} = h(a_t)$, for a *unique and continuous* function h ; this implies $a_t = h^t(a_0)$ for $t = 0, 1, \dots$. Each path satisfying this difference equation is sustained by a policy function defined as a step function with jumps placed along the path $\{a_t\}$ (i.e. for any $x < x^*$ in the neighborhood of x^* , the policy function is defined as $g(a) \equiv h^t(x)$ for all $a \in [h^{t-1}(x), h^t(x))$ for $t = 0, 1, \dots$). Thus, although there are many policy functions, one for each initial condition, all of them are generated by the same function h . Moreover, the natural limit in continuous time of such a construction cannot be expected to feature jumps or indeterminacy.

For our continuous time model, we find a continuum of local solutions to the right that has not been reported in a discrete time setting. We conjecture, however, that this indeterminacy to the right of the steady state is also present in the discrete-time setting.

In greater detail, the extension fails as follows. To the left of the steady state a^* the path is unique and actually can be defined for all $a \in [\underline{a}, a^*]$; indeed, even if $\underline{a} = 0$. However, to the right of the steady state there exists a continuum of solutions starting from (r, \bar{w}) but all of these paths reach $c = r$ and $w < \bar{w}$ at some finite asset level $\bar{a} > a^*$ (the figure illustrates the outermost of these paths). Such solutions over the finite range $[\underline{a}, \bar{a}]$ can only be interpreted as an equilibrium if we now modify the saving game to incorporate a constraint on saving choices of agents, forcing them to keep wealth within the interval $[\underline{a}, \bar{a}]$. Formally, one must change the problem faced by the agent in power, adding this ad hoc constraint. Then, since the interval is self referential, these new constraints on the agent's optimization problem turn out not to bind. With the game modified this way, the local solution becomes an equilibrium. Needless to say, ad hoc constraints of this kind are unnatural and nonstandard in the literature, making this interpretation impractical.

As discussed already, any of these paths are incompatible with an equilibrium in the sense that if one attempted to extend them indefinitely, for all $a \geq \bar{a}$, then they would violate some equilibrium condition. However, what happens if we attempt to extend the local solution over a larger, but bounded, interval?

The answer is that none of these paths can be extended *anywhere* past \bar{a} . At this point one can show that the agent in power is actually indifferent between following the prescribed path with dissavings versus holding assets constant while in power (of course, doing so is not an equilibrium, since it would require $w = \bar{w}$ but we have $w < \bar{w}$). More intuitively, this signals that although dissaving by future selves may make the current self dissave, a strategic complementarity, it cannot do so indefinitely. At some high enough asset level the current self will prefer to hold wealth constant or actually save, even if future selves plan to dissave. The problem, it turns out, is that if the agent strictly prefers to save, the equilibrium will unravel.

More formally, past \bar{a} the path enters the $c < r$ region. If we attempt to define the solution using this path over some bounded interval $[\underline{a}, \bar{a}]$ with $\bar{a} > \bar{a}$ but close \bar{a} , then the solution will satisfy the differential system (6). However, this construct will no longer be self-referential: the dynamics for wealth now point "outward": at $a = \bar{a}$ the agent wishes to strictly save. The solution over such an interval then lacks any economic interpretation.^{24,25}

²⁴This is not to say that a self-referential solution does not exist, just that it cannot be the extension of the original construct. Indeed, we can show that for any bounded interval a solution with saving exists, with wealth converging to the upper bound and then remaining there. This solution has the interpretation of being an equilibrium with an ad hoc bound on assets that binds.

²⁵Krusell and Smith (2003) considers weakly concave saving technologies. The standard case in the hyperbolic discounting literature assumes that the decision maker faces a given market rate of interest, as we have assumed here, which is equivalent to a linear savings technology. Our discussion, showing that

5 Non-Uniform Disagreement

We now turn to situations where disagreement varies sufficiently so that $\beta(c)$ lies on both sides of $\hat{\beta}$, preventing the conditions for global saving or dissaving. We focus on two polar cases. In the first case, disagreements decrease with spending. We show how this creates conditions for a poverty trap: dissaving at lower wealth levels coexisting with positive savings at high wealth levels. In the second case, disagreements increase with spending. We show that this creates conditions for convergence of wealth to an interior steady state.

5.1 Decreasing Disagreement: Poverty Traps

We start with the case where $\beta(c)$ is increasing and lies on both sides of $\hat{\beta}$. Note that, given Assumption 1, this requires $r > \rho$.

Assumption 4. *The disagreement index $\beta(c)$ is nonconstant, weakly increasing and crosses $\hat{\beta}$.*

On the one hand, since $\beta(c) > \hat{\beta}$ for high c , Theorem 4 suggests an equilibrium with savings as long as the asset limit is high enough, so that $\underline{a} \geq \hat{a}$. On the other hand, since $\beta(c) < \hat{\beta}$ for all $c \leq r\hat{a}$, dissaving seems like a natural outcome for lower wealth levels, as in Theorem 3. This suggests the possibility of a poverty trap, with saving above a cutoff and dissaving below this same cutoff. When wealth is low, consumption is low, so disagreements are high; this leads to dissaving which perpetuates the time-inconsistency problem. When wealth is high, consumption is high, so disagreements are low. This leads to saving and the time-inconsistency problem is partly overcome. Indeed, an additional incentive to save is the anticipation of reaching wealth levels where disagreements are low and time-inconsistency problems are partly overcome.

The next result formalizes these ideas by combining the constructions underlying Theorems 3 and 4. The cutoff must be set at a point where the agent in power is indifferent between following the saving path versus the dissaving path.

Theorem 7 (Poverty Trap). *Suppose Assumptions 1, 2, and 4 hold. Then there exists a equilibrium with strict savings for $a > a^*$ and dissaving for $a < a^*$ for some cutoff $a^* < \infty$, i.e. $\hat{c}(a) < ra$ for $a > a^*$ and $\hat{c}(a) \geq ra$ for $a < a^*$. Moreover, all equilibria share these properties.*

Proof. Appendix H. □

the analog of their local construction cannot be extended globally, is limited to this linear case. We have verified numerically that global indeterminacy is possible with sufficiently concave savings technologies.

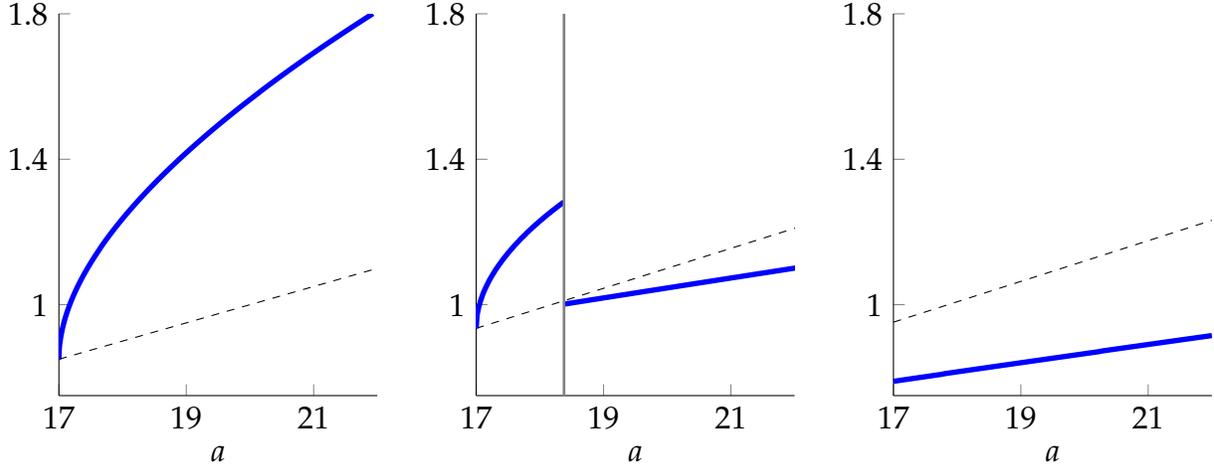


Figure 3: Equilibrium consumption functions $\hat{c}(ra)$ versus ra for three different interest rates. The left panel shows a low interest rate implying dissavings. The center panel shows an intermediate interest rate implying a poverty trap. The right panel shows a high interest rate leading to savings.

This result still leaves open the possibility that $a^* = \underline{a}$, so that there is no region with dissaving. Intuitively, this may occur if the incentives to save are very great, e.g. the region over which $\beta(c) < \hat{\beta}$ is small, $\hat{\beta} - \beta(c)$ is small, or both. The next result provides sufficient conditions for a^* to be interior.

Proposition 4. *Suppose Assumptions 1, 2, and 4 hold. Then there exists $\bar{r} > \underline{r} \geq \rho$ and $\underline{a} > 0$ such that for all $r \in (\underline{r}, \bar{r})$ the cutoff a^* defined in Theorem 7 is interior: $\underline{a} < a^* < \infty$.*

Proof. Appendix H. □

This result requires a low enough interest rate, but high enough to induce savings at the top, to make the benefit from saving relatively small and, thus, ensure dissavings at the bottom. Indeed, \underline{r} is set to that when $r = \underline{r}$ then $\beta(\bar{c}) = \bar{\beta} = \hat{\beta}$, which ensures that savings is an equilibrium at the top. Then \bar{r} is set close enough to \underline{r} to ensure that the utility value from saving is not too high, so that V is close to \bar{V} . This then implies that dissavings dominates savings at low wealth levels, since dissaving gives $V > \bar{V}$ for a near \underline{a} .

Example 1. We now illustrate the possibility of savings, dissavings and poverty traps. Let the utility for the agent in power be

$$U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$$

for $\bar{\sigma} > 0$ and let the disagreement be

$$\beta(c) = \begin{cases} \bar{\beta} \left(\alpha \left(\frac{c}{\bar{c}} \right)^\gamma + 1 - \alpha \right) & \text{if } c \leq \bar{c}, \\ \bar{\beta} & \text{if } c \geq \bar{c}. \end{cases}$$

with $\alpha, \bar{\beta} \leq 1$ and $\gamma > 0$. Under this specification $\beta(c)$ is increasing for $c < \bar{c}$ and constant for $c \geq \bar{c}$; the implied U_0 is concave as long as $\gamma\alpha \leq \bar{\sigma}$. We use the following parameters: $\rho = 0.05$, $\bar{\sigma} = \frac{1}{5}$, $\bar{\beta} = 1$, $\alpha = \frac{3}{4}$, $\gamma = \frac{\bar{\sigma}}{\alpha}$, $\lambda = 0.05$, $\bar{c} = 5$ and $\underline{a} = 17$.

Figure 3 depicts the consumption policy functions $\hat{c}(a)$ (solid line) against ra (dotted line) for three values for the interest rate. The left panel sets $r = 0.05$, the center panel sets $r = 0.55$ and the right panel $r = 0.56$. In the left panel $\beta(c) < \hat{\beta}$ so that Theorem 3 applies and $\hat{c}(a) > ra$ so the agent dissaves and wealth declines $\dot{a} < 0$, reaching \underline{a} in finite time. In the right panel $\beta(c) > \hat{\beta}$ so that Theorem 4 applies and $\hat{c}(a) < ra$ so the agent is saving and wealth rises without bound, $\dot{a} > 0$ and $a_t \rightarrow \infty$. For high enough wealth the consumption function coincides with the linear equilibrium provided by Theorem 2 providing the boundary condition for our construction. The center panel illustrates Theorem 7 and shows that a poverty trap emerges with $\hat{c}(a) \geq ra$ for $a < a^*$ and $\hat{c}(a) < ra$ for $a > a^*$, where $a^* \approx 18.37$.

Comparative Statics on Poverty Traps. We now discuss a few interesting comparative statics. First, suppose we lower the asset limit. This may be due to better access to credit which loosens the borrowing constraint (recall the connection in Section 2). We argue that this may worsen the incentives to save and prompts the agent to dissave over a greater range of asset levels. To see this, suppose initially that $\underline{a} \geq \hat{a}$ where \hat{a} is defined by Theorem 4. An equilibrium with positive savings then exists, where wealth and consumption rise over time starting from any initial wealth level $a \geq \underline{a}$. Next, suppose \underline{a} is lowered and that at its new level $\underline{a} < \hat{a}$, so that, according to Theorem 7, an equilibrium exists where the agent dissaves below a cutoff $a^* > \hat{a}$. Then for wealth in the intermediate region $[\hat{a}, a^*)$ the equilibrium switches from saving to dissaving when the asset limit is reduced. Thus, when the borrowing constraint is loosened it may prompt an agent that was previously saving to dissave.

Next, consider an increase in labor income y and suppose that, for the sake of the present discussion, the borrowing limit is proportional to income. Recall that, for convenience, we work with the change of variables outlined in Section 2 that allows us to set income to zero without loss of generality. According to this transformation, an increase in y amounts to an upward adjustment in the asset limit, together with a parallel upward shift in the initial transformed wealth level. As just argued, the asset limit and the cutoff

a^* may move in opposite directions. When this is the case an increase in income decreases the chances of being in a poverty trap, i.e. $a \leq a^*$. Indeed, a large enough increase in y ensures $\underline{a} \geq \hat{a}$ leaving global savings as the only equilibrium. We conclude that labor income may help prevent poverty traps in wealth. Conversely, being poor in terms of labor income makes accumulating wealth difficult.

Related Literature on Poverty Traps. [Banerjee and Mullainathan \(2010\)](#) study models with disagreements over many goods, which imply nonuniform disagreements over total spending, as discussed in our Section 2.3. They derive a result with decreasing disagreements that can be related to our poverty trap result. They work in a two-period model, with a single saving decision in the first period, and show the possibility of a downward discontinuity in the consumption function (i.e. an upward discontinuity in saving). Our poverty trap result also features a discontinuity of this kind at the threshold a^* .

There are some differences between their results and ours. First, since their model only has two periods, this prevents them from studying the long-run dynamics for wealth; indeed, it is unclear whether their upward discontinuity produces a switch in savings from negative to positive. In contrast, in our poverty trap equilibrium, we show that wealth declines and remains trapped forever below the threshold. Second, discontinuities in the consumption function are due to non-convexities in the optimization problem. However, such a discontinuity never arises in a two-period version of our model. Indeed, if we maximize $U_1(a_0 - a_1) + e^{-\rho}U_0(e^r a_1)$ with respect to a_1 then the solution is continuous, since we assume U_1 and U_0 to be strictly concave. In contrast, their result implicitly hinges on making U_0 sufficiently convex. We conclude that for our result it is crucial to include a longer horizon with future selves that also engage in savings choices. Unlike [Banerjee and Mullainathan \(2010\)](#), poverty traps arise from the strategic interactions across different selves over longer horizons.

Our construction of equilibria with a poverty trap relied on nonuniform disagreements. Indeed, we also proved a converse: if $\beta(c)$ is constant then poverty traps cannot arise as part of an equilibrium. However, poverty traps may emerge with uniform disagreements if one relaxes the equilibrium requirement from Markov to subgame perfection, as shown by [Bernheim et al. \(2015\)](#). Poverty traps are sustained by trigger strategies that punish deviations with overconsumption. Such punishments are not as effective near the asset limit, so savings can only be sustained for high enough wealth levels.

There are various differences between their result and ours. The first two are mostly of a technical nature. First, their main result provides conditions for the existence of two thresholds which need not coincide: below the first threshold there is dissaving and above

the second savings becomes possible. Second, they require an endogenous condition on the equilibrium set; in contrast, our results establish conditions on parameters for poverty trap equilibria with a single threshold. These theoretical differences notwithstanding, in practice, the numerical analysis in [Bernheim et al. \(2015\)](#) for a wide range of parameters reveals that the condition they require is met and their two thresholds coincide.²⁶ More substantively, poverty traps are driven by different and complementary mechanisms in the two frameworks. These differences are highlighted by differences in comparative statistics. [Bernheim et al. \(2015\)](#) show that loosening the borrowing constraint makes poverty traps less likely, since trigger punishments then become more severe. In addition, one can show that higher labor income, fixing wealth, raises the chances of being in a poverty trap in their setting. These predictions contrast with those of our model discussed above.

5.2 Increasing Disagreement: Convergence

We now consider the reverse situation, where disagreements are high for high spending and low for low spending. It is natural to expect the time-inconsistency problem to be aggravated at higher wealth levels, providing a force for dissaving at high wealth and saving at low wealth levels. These forces may imply the convergence of wealth to a unique interior steady state.

We now provide one such result. To simplify, we assume a downward jump in $\beta(c)$, but later confirm that similar conclusions are obtained with continuous disagreements.

Assumption 5. *There exists $c^* > 0$ such that $\beta(c) > \hat{\beta}$ for $c < c^*$ and $\beta(c) < \hat{\beta}$ for $c > c^*$. Furthermore, $\beta(c)$ is continuous except for a jump at c^* satisfying*

$$\lim_{c \uparrow c^*} \beta(c) > \hat{\beta} > \lim_{c \downarrow c^*} \beta(c).$$

This assumption requires $\beta(c)$ to single cross $\hat{\beta}$ from above with a jump, but does not require $\beta(c)$ to be decreasing.

²⁶In more detail, the main result in [Bernheim et al. \(2015\)](#) is as follows. Suppose parameters are such that the subgame perfect equilibrium set satisfies the following (“non-uniformity”): there exists some asset level with strict dissavings for *all* equilibria (i.e. self-control fails) and, in addition, there exists another asset level with strict savings for *some* equilibrium (i.e. self-control is possible). Then there exists a_1 and a_2 such that: for low wealth $a < a_1$ *all* equilibria feature strict dissaving with assets reaching the asset limit; for $a > a_2$ there is *at least one* equilibrium with strict saving with assets rising without bound. In other words, they show that if self control fails somewhere and if self control is possible somewhere, then the former occurs at the bottom (with assets reaching the limit) and the latter occurs at the top (with assets accumulating indefinitely).

[Bernheim et al. \(2015\)](#) report that it is not possible to provide conditions on primitives to ensure the non-uniformity condition required for their main result. However, they perform convincing numerical simulations showing that this condition holds for intermediate values of β ; they also provide sufficient conditions for a simplified model that only allows two savings choices and has assets lying on a fixed grid.

We now construct an equilibrium with a stable steady state at $a^* \equiv \frac{c^*}{r}$, with saving below a^* and dissaving above a^* .

Theorem 8 (Convergence). *Suppose Assumptions 1 and 5 hold. Then there exists an asset limit $\underline{a} \geq 0$ and an equilibrium with a stable steady state at $a^* = \frac{c^*}{r} > \underline{a}$, i.e. $\hat{c}(a^*) = ra^*$, $\hat{c}(a) < ra$ for $a < a^*$ and $\hat{c}(a) \geq ra$ $a \geq a^*$. If, in addition $\beta(c)$ is non-decreasing for $c < c^*$, then $\underline{a} = 0$.*

Proof. Appendix I. □

Intuitively, high wealth is associated with higher consumption, so lower $\beta(c)$ leads to dissaving. At low wealth levels consumption is low, so higher $\beta(c)$ leads to positive savings. This variation in disagreements provide a force for convergence, despite a constant interest rate.²⁷

Convergence in the Literature. Battaglini and Coate (2008) study a political economy model of debt and obtain a similar form of convergence.²⁸ They study a legislative voting game, where in each period, the legislature determines spending on two items, a valuable public goods and group-specific “pork” transfers. As they show, the outcome each period can be represented as a maximization over the two spending items and the continuation value from future debt. Spending on both item is only a function of total spending, just as in the many-good interpretation of our framework provided in Section 2.3.²⁹ In their model, pork spending hits corners when spending is low. As a result, in their model time-inconsistency is strongest when debt is low, wealth is high, because group-specific pork transfers are then strictly positive (the “business-as-usual” regime); conversely, there is no time-inconsistency when debt is high, wealth is low, and group-specific pork transfers are absent (the “responsible policymaking” regime). In terms of our notation, this maps into a specification where $\beta(c)$ takes on two values, $\bar{\beta} = 1$ above some \bar{c} and $\underline{\beta} < 1$ elsewhere.

Observational Non-Equivalence. Both the existence of equilibria with a poverty trap and the convergence to an interior steady state illustrate behavior that is patently *not* observationally equivalent to any time consistent consumer with additive utility and exponential discounting. A time-consistent agent would either save or dissave at all wealth

²⁷We conjecture that a similar result is obtained for continuous $\beta(c)$ as long as the function is sufficiently decreasing. Indeed, this is consistent with Example 2 below.

²⁸Their model includes shocks, but their main result is a form of mean reversion that renders the dynamics of debt ergodic, similar to the forces implying convergence of wealth in our deterministic setup.

²⁹See their equation (18), which shows that spending on each item is determined, once one conditions on current and future debt.

levels, depending on the sign of $r - \rho$. [Morris and Postlewaite \(1997\)](#) constructs an example where observational equivalence fails, based on discontinuities in the consumption function, since discontinuities are never optimal for time-consistent agents with concave utility. Our results provide counterexamples to observational equivalence without relying on discontinuities in the policy function. This is most obvious in the convergence case, since no discontinuity in the consumption policy function is involved.

6 Inverting Consumption Functions to Disagreements

We now explore the model from a different angle, inverting it to solve for the disagreement index $\beta(c)$ given an equilibrium policy function for consumption $\hat{c}(a)$. It turns out that, for sufficiently smooth consumption functions, this inverse mapping is very tractable, indeed providing a closed form. This allows us to reveal the model from a different angle.

Define the local curvature

$$\sigma(c) \equiv \frac{-U_1''(c)c}{U_1'(c)}, \quad (13)$$

which is strictly positive, since U_1 is strictly increasing and strictly concave. We then have the following result.

Theorem 9. *Suppose $\hat{c}(a)$ is an equilibrium consumption function that is strictly monotone (increasing or decreasing) and twice differentiable within an interval (a_1, a_2) . Let $\zeta(c)$ denote the local inverse of $\hat{c}(a)$ over the interval $(\hat{c}(a_1), \hat{c}(a_2))$. Then*

$$\beta(c) = \frac{1}{\lambda \zeta'} \left\{ \alpha_1 \zeta' + \alpha_2 (\zeta')^2 + \sigma c \zeta'' \left(\frac{r\zeta - c}{c} \right)^2 + \sigma \left(2 + (2\rho + \lambda - 3r)\zeta' \right) \frac{r\zeta - c}{c} + (\sigma^2 + \sigma - c\sigma') \left(\frac{r\zeta - c}{c} \right)^2 \right\} \quad (14)$$

where $\alpha_1 \equiv (\rho + \lambda - r)$ and $\alpha_2 \equiv (\rho - r)(\rho + \lambda - r)$.

Proof. Appendix J. □

We discuss two applications of this result.

Necessary Conditions. This result can be used to derive necessary conditions on the disagreement index $\beta(c)$ for certain properties to emerge in equilibrium. We next state a few examples.

First, suppose the equilibrium features dissaving for high enough wealth levels, so that $ra - \hat{c}(a) < 0$. Suppose further that $\hat{c}'(a)$ and $\hat{c}''(a)a$ converge as $a \rightarrow \infty$ with $\lim_{a \rightarrow \infty} \hat{c}'(a) > r$. In addition, suppose U_1 is such that σ and $\sigma'c$ converge as $c \rightarrow \infty$. Then it follows that $\beta(c) \rightarrow \bar{\beta} < \hat{\beta}$. This is a converse of sorts to Theorem 3. Similarly, suppose the equilibrium features saving for high enough wealth levels, then it must be that $\beta(c) \rightarrow \bar{\beta} > \hat{\beta}$.

Next, suppose that $r > \rho$ and the equilibrium features an interior steady state \tilde{a} with $\hat{c}(\tilde{a}) = r\tilde{a}$ that is locally stable, so that $\hat{c}'(\tilde{a}) > r$. Then at this steady state

$$\beta(r\tilde{a}) > \hat{\beta}.$$

If, in addition, $\hat{c}''(\tilde{a}) \leq 0$, i.e. the marginal propensity to consume falls with wealth, then

$$\beta'(r\tilde{a}) < 0,$$

This is consistent with our convergence result, Theorem 8, which featured decreasing $\beta(c)$, although that result assumed a discontinuous jump, whereas the present characterization features a smooth $\beta(c)$. Moreover, whether or not $\hat{c}''(\tilde{a}) \leq 0$, if \tilde{a} is globally stable with dissaving for all $a \geq \tilde{a}$ then $\beta(c) < \hat{\beta}$ as $c \rightarrow \infty$, as discussed above. Since at the steady state $\beta(r\tilde{a}) > \hat{\beta}$, we conclude that β must cross $\hat{\beta}$ from above at some point. In this sense, decreasing β that crosses $\hat{\beta}$ is necessary for convergence of wealth to an interior steady state.

Constructing Equilibria. This result can be used to construct smooth equilibria, by postulating any monotone and twice differentiable consumption function and backing out the required disagreement index $\beta(c)$. We only need to check that the disagreement index obtained from equation (14) satisfies $\beta(c) > 0$. The next example illustrates this procedure.

Example 2. Assume $U_1(c) \equiv \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ and $\rho + \lambda > r$. Consider a linear consumption function $\hat{c}(a) = r\tilde{a} + \Psi(a - \tilde{a})$ with $\Psi > r$ and $\tilde{a} > 0$. This implies $\dot{a} = -\Psi(a - \tilde{a})$ so that \tilde{a} is a stable steady state. Equation (14) then implies that β is quadratic in $\frac{1}{c}$,

$$\beta(c) = \bar{\beta}_0 + \bar{\beta}_1 \left(\frac{r\tilde{a}}{c} - 1 \right) + \bar{\beta}_2 \left(\frac{r\tilde{a}}{c} - 1 \right)^2,$$

with coefficients $\bar{\beta}_0 = \frac{(\rho-r+\Psi)(\rho+\lambda-r)}{\lambda\Psi}$, $\bar{\beta}_1 = \frac{\bar{\sigma}}{\lambda\Psi} (\lambda + 2\rho + 2\Psi - 3r) (\Psi - r)$ and $\bar{\beta}_2 = \bar{\sigma}(\bar{\sigma} + 1) \frac{1}{\lambda\Psi} (\Psi - r)^2$. One confirms that

$$\beta(r\tilde{a}) = \bar{\beta}_0 > \hat{\beta} \quad \text{and} \quad \beta'(r\tilde{a}) = -\bar{\beta}_1 \frac{1}{r\tilde{a}} < 0.$$

Figure 4 depicts an equilibrium with $\Psi = 0.10$ and $\tilde{a} = 10$ and the implied $\beta(c)$ using

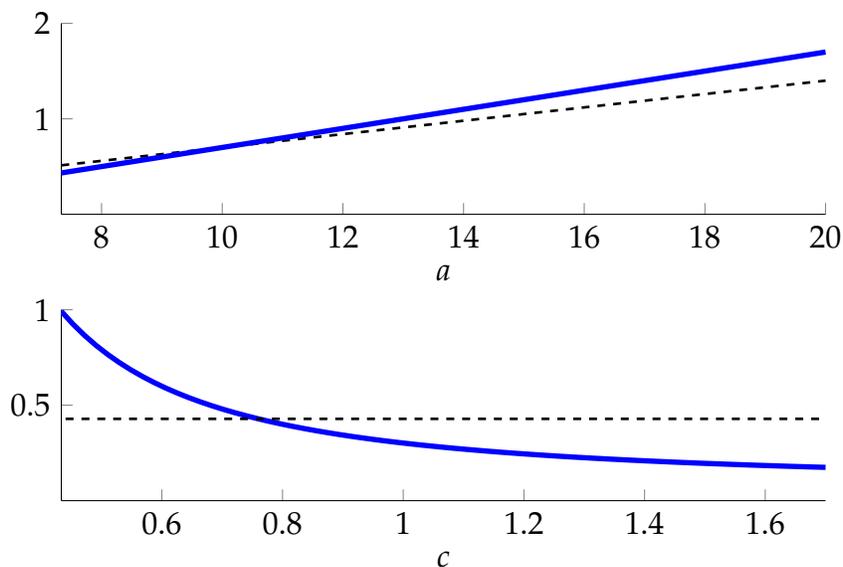


Figure 4: A linear consumption function $\hat{c}(a)$ with a stable interior steady state (top panel) and the implied disagreement index $\beta(c)$ (bottom panel).

parameters $\rho = 0.05$, $\bar{\sigma} = 0.8$, $\lambda = 0.05$ and $r = 0.07$. The upper panel shows the linear consumption function, while the lower panel shows the implied disagreement function $\beta(c)$ that sustains this linear policy function as an equilibrium. Equation (14) produces a $\beta(c)$ that is everywhere decreasing with $\lim_{c \rightarrow \infty} \beta(c) > 0$ so that the condition $\beta(c) > 0$ is met in this case.

7 Continuous Dissaving Equilibria

This last section takes a closer look at the dissaving case and provides sufficient conditions for the continuous equilibrium to exist (recall that, under mild conditions, there is at most one continuous equilibrium). As the Introduction explained, standard hyperbolic discrete-time models are somewhat ill behaved. Equilibria display a plethora of discontinuities, which can be challenging to work with. The literature has explored a few ways to overcome these issues and the objections they raise, especially by adding uncertainty.³⁰

Although there are obvious benefits to including uncertainty, doing so creates its own challenges and somewhat removes the focus from the time-inconsistency problem. Our

³⁰Harris and Laibson (2001) introduce uncertain income in a discrete time model and provide a generalized Euler equation and existence results, allowing for jumps; they also show that continuous equilibria exist when $r < \rho$ if the time-inconsistency problem is not too severe so that $\beta \approx 1$. Harris and Laibson (2013) introduced a continuous-time model allowing for uncertainty in asset returns and studied the instantaneous-gratification limit, $\lambda \rightarrow \infty$.

results indicate that casting the model in continuous time, following [Harris and Laibson \(2013\)](#), already affords substantial benefits even without uncertainty. In particular, equilibria are relatively well behaved. For instance, Section 4.3 provided a complete characterization for the hyperbolic case, showing that equilibria are either continuous or have limited jumps, only at interior steady states. Continuous time minimizes the discontinuities relative to discrete time.

We now investigate conditions for continuity of dissaving equilibria, allowing for nonuniform disagreements but assuming $\beta(c) < \hat{\beta}$ as in Section 4.1.³¹ Our first result provides a simple yet powerful result: the continuous equilibrium exists for interest rates below the discount rate.

Theorem 10 (Continuity I). *Suppose Assumptions 1 holds and that $r < \rho$ (implying $\beta < \hat{\beta}$). Then there exists a continuous equilibrium with dissaving, $\hat{c}(a) > ra$ for $a > \underline{a}$, and an increasing consumption function, with $\hat{c}'(a) > 0$.*

Proof. Appendix K. □

Recall that discontinuities are limited, with jumps only occurring at interior steady states. When $r < \rho$ the forces for strict dissaving are sufficiently strong, even without the time-inconsistency problem, to prevent such steady states.

When $r \geq \rho$ strict dissaving can be guaranteed if the time-inconsistency problem is strong enough. Our next result provides sufficient conditions for this to be the case. Define $\bar{r} \equiv \inf_{c \geq ra} \frac{\rho}{\beta(c)}$, so that $r \geq \bar{r}$ implies $\beta(c) > \hat{\beta}$ for some c for all large enough λ ; we require $r < \bar{r}$ and a condition on preferences.

Theorem 11 (Continuity II). *Suppose Assumption 2 holds, $r \in [\rho, \bar{r})$ and*

$$\inf_{c \geq ra} \frac{1 - \sigma(c)}{\beta(c)} > 1, \tag{15}$$

Then a continuous equilibrium exists for λ large enough or r close enough to ρ . This equilibrium features strict dissaving, $\hat{c}(a) > ra$ for $a > \underline{a}$, and an increasing consumption function, $\hat{c}'(a) > 0$. When $r = \rho$ these conclusions hold for any λ .

Proof. Appendix K. □

This result requires the curvature of the utility function to be low relative to disagreements, so that $1 - \sigma(c) > \beta(c)$. In particular, since $\beta(c) > 0$ this requires $\sigma(c) < 1$. The

³¹Equilibria with savings, $\beta > \hat{\beta}$, are naturally continuous, as we have already shown. Poverty traps may feature a discontinuity at the threshold separating dissaving from saving, but such a discontinuity is inherent to the economics of the situation. All other discontinuities we have encountered are associated with a dissaving equilibrium or a region with dissaving.

conditions of the proposition ensure that $\beta(c) < \hat{\beta}$, providing conditions for dissaving. Low $\sigma(c)$ then assures strict dissaving by heightening the strategic complementarity, so that substitution effects dominate income effects, and greater dissaving by future selves leads to greater dissaving by current selves.

These results on the existence of continuous equilibria can be contrasted with the discrete-time results in [Chatterjee and Eyigungor \(2015\)](#), Theorems 3 and 4). They show that in the hyperbolic case with $r \leq \rho$ and power utility functions, a continuous equilibrium does not exist. In contrast, we obtain continuity for $r < \rho$ for any $\beta(c)$ and utility functions; for $r \geq \rho$ we obtain continuity under some preference restrictions.^{32,33}

Finally, we provide sufficient conditions for the existence of discontinuous equilibria. The result relies on showing that eventually $V(a) < \bar{V}(a)$ for any solution to the differential system (6).

Theorem 12 (Discontinuity). *Suppose Assumptions 1–2 hold and $r \in (\rho, \bar{r})$. Then there exists an equilibrium with dissaving and a discontinuous policy function $\hat{c}(a)$ for all large enough λ .*

Proof. Appendix K. □

When $r > \rho$ the conditions in Theorems 11 and 12 are compatible. As a corollary, multiple equilibria are possible, with continuous and discontinuous equilibria coexisting. To the best of our knowledge, this is the first result on multiplicity in the context of time-inconsistent saving games.

When multiple equilibria of this sort exist, which one is more reasonable, the continuous or discontinuous one? Is there a reasonable selection criterion? One possibility is to introduce small amounts of uncertainty in income, returns or preferences. What happens when small noise is introduced in cases where continuous equilibria do not exist? Does the equilibrium become continuous or does it remain discontinuous? Or may an equilibrium even fail to exist? All these questions deserve further investigation, but are beyond the scope of the current paper.

³²To understand the difference between continuous and discrete time in this regard note that [Chatterjee and Eyigungor \(2015\)](#) rule out continuous equilibria by observing that there exists an interval of assets near the debt limit where the decision maker chooses to go to the debt limit (the policy function is flat). However, this is no longer true in continuous time because, since wealth moves continuously, it always takes a positive amount of time to reach the debt limit.

³³In the hyperbolic discounting case with power utility functions condition (15) is the opposite of the condition imposed by [Harris and Laibson \(2013\)](#) to guarantee the existence of an equilibrium in the instantaneous gratification limit $\lambda \rightarrow \infty$. Indeed, when $U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ and $U_0(c) = \bar{\beta}U_1(c)$, [Harris and Laibson \(2013\)](#) require $1 - \bar{\beta} < \bar{\sigma}$. Consistent with this observation, numerically we find that under condition (15) the continuous equilibrium identified by our theorem diverges with $\hat{c}(a) \rightarrow \infty$ as $\lambda \rightarrow \infty$ for any $a > \underline{a}$. Intuitively, strategic complementarities are strong and when λ increases they are amplified. This suggests that it may be reasonable to consider the limit of $\lambda \rightarrow \infty$ while simultaneously raising $\beta(c)$ to obtain a finite limit for consumption.

8 Conclusion

We put forth a continuous-time saving game that allows for flexible forms of disagreement. We provided a general method for constructing equilibria and use it to provide sharp characterizations.

We find that equilibria in our continuous-time framework are well behaved relative to standard discrete-time settings. For example, a unique continuous equilibrium exists under a wide range of conditions, even without the introduction of uncertainty. For the special hyperbolic case, we provided a simple graphical analysis and delivered results that address a number of open questions. Away from the hyperbolic case we found conditions for global saving or dissaving. We also showed that when disagreements vary sufficiently richer wealth dynamics are possible. When disagreements fall with spending poverty traps emerge; conversely, when disagreements rise with spending, the equilibrium may involve convergence to an interior wealth level.

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Appendix

A General Properties

A.1 Proof of Proposition 1

We show existence and uniqueness at the same time. By the Envelope Theorem

$$U_1'(c) = h'(\hat{c}_A(c)) = g'(\hat{c}_B(c)).$$

Moreover,

$$U_0'(c) = h'(\hat{c}_A(c))\hat{c}'_A(c).$$

Therefore, $\hat{c}'_A(c) = \frac{U_0'(c)}{U_1'(c)}$. Thus

$$\hat{c}_A(c) = \int_0^c \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})} d\tilde{c},$$

which is strictly increasing in c since $U_0', U_1' > 0$. Furthermore,

$$\hat{c}_B(c) = c - \hat{c}_A(c) = \int_0^c \left(1 - \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})}\right) d\tilde{c},$$

which is also increasing in c because $1 - \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})} \geq 0$ by Assumption 1. So $h(c_A)$ and $g(c_B)$ are uniquely determined (up to constants) by

$$h'(\hat{c}_A(c)) = U_1'(c)$$

and

$$g'(\hat{c}_B(c)) = U_1'(c).$$

h and g are increasing because $h', g' > 0$ and are concave because \hat{d} and \hat{e} are increasing in c and U_1' is decreasing in c .

A.2 Roots of Hamilton-Jacobi-Bellman Equations

Write the HJB equation (6a) as

$$(\rho + \lambda) V(a) - \lambda W(a) = H(V'(a), a), \quad (16)$$

where

$$H(p, a) \equiv \sup_c \{U_1(c) + p(ra - c)\}. \quad (17)$$

The next lemma characterizes the function H .

Lemma 3. For any a , the function $H(\cdot, a)$ defined by (17) is continuous, strictly convex and continuously differentiable for $p > 0$; has a unique interior minimum at $p = U'(ra)$; satisfies $\lim_{p \rightarrow \infty} H(p, a) = \infty$ and $H(0, a) = \lim_{p \rightarrow 0} H(p, a) = U_1(\infty)$.

Proof. For any $p > 0$ a maximum is attained on the right hand side of (17) uniquely by the first-order condition $U'_1(c) = p$. This implies that $H(a, \cdot)$ differentiable with derivative $H_p(p, a) = ra - (U'_1)^{-1}(p)$. This derivative is continuous and strictly increasing. Thus, $H(p, a)$ is strictly convex in p . Since $H_p(U'_1(ra), a) = 0$ then $p = U'_1(ra)$ is the unique minimum. Since $H(a, \cdot)$ is strictly convex it follows that $\lim_{p \rightarrow \infty} H(p, a) = \infty$.

Finally, by definition $H(0, a) \equiv \sup_c U_1(c) = \lim_{c \rightarrow \infty} U_1(c)$. This also coincides with $\lim_{p \rightarrow 0} H(p, a)$ since

$$\lim_{p \rightarrow 0} H(p, a) = \lim_{c \rightarrow \infty} (U_1(c) + U'_1(c)(ra - c)) \leq \lim_{c \rightarrow \infty} U_1(c),$$

$$\lim_{p \rightarrow 0} H(p, a) \geq \lim_{p \rightarrow 0} (U_1(p^{-\frac{1}{2}}) + p(ra - p^{-\frac{1}{2}})) = \lim_{c \rightarrow \infty} U_1(c). \quad \square$$

This has immediate implications for the possible solutions to equation (16).

Lemma 4. Consider solutions $p = V'(a)$ to equation (16), if

Case 1. $(\rho + \lambda)V(a) - \lambda W(a) < U_1(ra)$, then no solution exists;

Case 2. $(\rho + \lambda)V(a) - \lambda W(a) = U_1(ra)$, then the unique solution is given by $p = U'_1(ra)$;

Case 3. $U_1(ra) < (\rho + \lambda)V(a) - \lambda W(a) \leq U_1(\infty)$, then exactly two solutions p_1 and p_2 exist and $0 \leq p_1 < U'_1(ra) < p_2$;

Case 4. $U_1(\infty) < (\rho + \lambda)V(a) - \lambda W(a)$, then a unique solution exists and $U'_1(ra) < p$.

Given Lemma 4, we define the following subsets of \mathbb{R}^3 :

$$E \equiv \{(a, V, W) \mid a > 0 \text{ and } (\rho + \lambda)V - \lambda W > U_1(ra)\},$$

$$E_0 \equiv \{(a, V, W) \mid a > 0 \text{ and } U_1(\infty) > (\rho + \lambda)V - \lambda W > U_1(ra)\},$$

$$E_s \equiv \{(a, V, W) \mid a > 0 \text{ and } (\rho + \lambda)V - \lambda W = U_1(ra)\}$$

Lastly $\bar{E} = E \cup E_s$, and $\bar{E}_0 = E_0 \cup E_s$. Notice that E_s corresponds to the set of singular points of the differential (6) as an implicit ODE.

Using Lemma 4 we now rewrite system (6) as explicit ODEs. There are two systems to consider, depending on whether we consider the high or lower root.

Definition 2. Let $R_l(a, V, W)$ denote the lower root $p = V'(a)$ of equation (16). By Lemma 4, R_l is well-defined over \bar{E}_0 and is continuous in a, V, W . Let $S_l(a, V, W)$ denote the associated solution to W' in equation (6b), so that

$$S_l(a, V, W) = \frac{U_0(\hat{c}(a)) - \rho W}{\hat{c}(a) - ra}$$

with $\hat{c}(a) = (U'_1)^{-1}(V'(a)) = (U'_1)^{-1}(R_l(a, V, W)) > ra$, defined over E_0 . By the Implicit Function Theorem, R_l and S_l are continuously differentiable in a, V, W over E_0 .

Using R_l and S_l , system (6) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_l(a, V, W) \\ S_l(a, V, W) \end{pmatrix}. \quad (18)$$

This ODE is regular around $(a, V, W) \in E_0$. Around any regular point we can apply standard extension results (for example, Picard–Lindelöf theorem or Cauchy–Lipschitz theorem; see [Hartman \(2002\)](#) for a comprehensive exposition) to show that, the ODE (18) admits a unique solution (V, W) defined over a neighborhood of a , $(a - \epsilon, a + \epsilon)$, and is twice continuously differentiable (because R_l and S_l are continuously differentiable), such that $(V(a), W(a)) = (v, w)$.³⁴

The next definition is analogous, but using the higher root of equation (16).

Definition 3. Let $R_h(a, V, W)$ be the higher root for $p = V'(a)$ of equation (16). By Lemma 4, R_h is well-defined over \bar{E} and is continuous in a, V, W . Let $S_h(a, V, W)$ be the associated value W' in equation (6b), so that

$$S_h(a, V, W) = \frac{\rho W - U_0(\hat{c}(a))}{ra - \hat{c}(a)},$$

where $\hat{c}(a) = (U_1')^{-1}(V'(a)) = (U_1')^{-1}(R_h(a, V, W)) < ra$, defined over E . By the Implicit Function Theorem, R_h and S_h are continuously differentiable in (a, V, W) over E .

Using R_h and S_h , system (6) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_h(a, V, W) \\ S_h(a, V, W) \end{pmatrix}. \quad (19)$$

This ODE is regular around any $(a, V, W) \in E$. Just as with (18), standard extension results apply whenever (a, V, W) is regular.

A.3 Useful Observations

The following general properties of the solutions to system (6) is also important for their characterization.

Lemma 5. *Assume that V, W and \hat{c} constitutes a solution to the system (6). If V and W are continuously differentiable and V is twice differentiable at a , then*

$$(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a) (ra - \hat{c}(a)) \quad (20)$$

and if $\hat{c}(a) \neq ra$:

$$\hat{c}'(a) = \frac{V''(a)}{U_1''(\hat{c}(a))} = \frac{1}{U_1''(\hat{c}(a))} \frac{(\rho + \lambda - r) V'(a) - \lambda W'(a)}{ra - \hat{c}(a)} \quad (21)$$

Proof. Differentiating (6a) with respect to a , we obtain

$$(\rho + \lambda) V'(a) - \lambda W'(a) = U_1'(\hat{c}(a)) \hat{c}'(a) + V'(a)(r - \hat{c}'(a)) + V''(a) (ra - \hat{c}(a)).$$

Combining this with (6c) and rearranging yield (20). Now differentiating (6c) with respect to a ,

$$U_1''(\hat{c}(a)) \hat{c}'(a) = V''(a),$$

or equivalently $\hat{c}'(a) = \frac{V''(a)}{U_1''(\hat{c}(a))}$, which together with (20) yields (21). \square

³⁴For any solution $x(a)$ to an ODE $x'(a) = F(x(a))$. If F is continuously differentiable then x is twice continuously differentiable, and $x''(a) = \nabla F(x) \cdot x' = \nabla F(x) \cdot F(x)$.

Lastly, we will also use the follow result to connect the comparison between $\beta(\cdot)$ and $\hat{\beta}$ to the comparison between the slopes of \bar{V} and U'_1 .

Lemma 6. For $a > 0$, $\beta(ra) < \hat{\beta}$ if and only if

$$\bar{V}'(a) < U'_1(ra). \quad (22)$$

And $\beta(ra) = \hat{\beta}$ if and only if $\bar{V}'(a) = U'_1(ra)$.

Proof. Using the definition (8) for \bar{V} , we have

$$\bar{V}'(a) = \frac{r}{\rho + \lambda} U'_1(ra) + \frac{\lambda r}{(\rho + \lambda)\rho} U'_0(ra).$$

The condition that $\hat{\beta}(ra) < \hat{\beta}$ is equivalent to

$$\frac{r}{\rho + \lambda} U'_1(ra) + \frac{\lambda r}{(\rho + \lambda)\rho} U'_0(ra) < U'_1(ra).$$

The result then follows. Likewise, for the case with $\beta(ra) = \hat{\beta}$. \square

A.4 Proof of Lemma 1

Since $\sigma < 1$, the utility function U_1 is bounded from below and Assumption 2 is satisfied, we can find constants $\underline{u}, \bar{u}_0, \bar{u}_1$ such that

$$\underline{u} \leq U_0(c), U_1(c) \leq \bar{u}_0 \frac{c^{1-\sigma}}{1-\sigma} + \bar{u}_1.$$

This implies that the commitment solution is bounded above by

$$V_{sp}(a) \leq \bar{v}_0 \frac{a^{1-\sigma}}{1-\sigma} + \bar{v}_1$$

for some \bar{v}_0, \bar{v}_1 .³⁵

From the HJB for V , (6a), we have

$$V(a) = U_1(\hat{c}(a)) + U'_1(\hat{c}(a))(ra - \hat{c}(a)) + \lambda W(a) \leq (\rho + \lambda)V_{sp}(a).$$

In addition, $W(a) = \int_0^\infty e^{-\rho t} U_0(c_t) dt \geq \frac{\underline{u}}{\rho}$ and $U_1(\hat{c}(a)) > \underline{u}$.

Consequently, there exist $v_0^* > 0$ and v_1^* such that

$$U'_1(\hat{c}(a))(ra - \hat{c}(a)) < v_0^* \frac{a^{1-\sigma}}{1-\sigma} + v_1^* \quad (23)$$

for all $a \geq \underline{a}$.

We first choose \bar{a} such that

$$U'_1(\bar{c})(ra - \bar{c}) > v_0^* \frac{\bar{a}^{1-\sigma}}{1-\sigma} + v_1^*$$

and

$$v_0^* \frac{\bar{a}^{1-\sigma}}{1-\sigma} > v_1^*$$

³⁵ $V_{sp}(a)$ is defined by $\max_{\{c_t, a_t\}} \int_0^\infty e^{-\rho t} ((1 - e^{-\lambda t})U_1(c_t) + e^{-\lambda t}U_0(c_t))$ s.t. $\dot{a}_t = ra_t - c_t$ and $a_0 = a$.

for all $a \geq \bar{a}$.

We then choose v_1 sufficiently small so that

$$v_1^{-\sigma}(r - v_1) > 2\frac{v_0^*}{1 - \sigma}.$$

We now show by contradiction that $\hat{c}(a) > v_1 a$ for all $a \geq \bar{a}$. Indeed, if $\hat{c}(a) \leq \bar{c}$ then

$$U_1'(\hat{c}(a))(ra - \hat{c}(a)) \geq U_1'(\bar{c})(ra - \bar{c}) > v_0^* \frac{a^{1-\sigma}}{1 - \sigma} + v_1^*$$

which contradicts (23). Therefore $\hat{c}(a) \geq \bar{c}$ for all $a \geq \bar{a}$. In this case, $U_1'(\hat{c}(a)) = (\hat{c}(a))^{-\sigma}$. If $\hat{c}(a) \leq v_1 a$, then

$$U_1'(\hat{c}(a))(ra - \hat{c}(a)) > v_1^{-\sigma}(r - v_1)a^{1-\sigma} > 2v_0^* \frac{a^{1-\sigma}}{1 - \sigma} > v_0^* \frac{a^{1-\sigma}}{1 - \sigma} + v_1^*$$

which also contradicts (23). So

$$\hat{c}(a) > v_1 a \quad \forall a \geq \bar{a}$$

Because of the Inada condition on U_1 , there exists $\frac{ra}{2} > \epsilon > 0$ such that

$$U_1'(\epsilon) \frac{1}{2} ra > v_0^* \frac{\bar{a}^{1-\sigma}}{1 - \sigma} + v_1^*.$$

We show that $\hat{c}(a) > \epsilon$ for all $a \in [\underline{a}, \bar{a}]$. Assume by contradiction that $\hat{c}(a) \leq \epsilon$ for some $a \in [\underline{a}, \bar{a}]$:

$$U_1'(\hat{c}(a))(ra - \hat{c}(a)) > U_1'(\epsilon) \frac{1}{2} ra > v_0^* \frac{\bar{a}^{1-\sigma}}{1 - \sigma} + v_1^*,$$

which contradicts (23). So

$$\hat{c}(a) > \epsilon \quad \forall a \in [\underline{a}, \bar{a}]$$

From the two inequalities for \hat{c} and let $v = \min(v_1, \frac{\epsilon}{\bar{a}})$, it is immediate that $\hat{c}(a) > va$ for all $a \geq \underline{a}$. Q.E.D.

B A Single-Crossing Property

The following simple result on the comparison between two functions plays a crucial role in the characterization of the solutions to system (6). Although this result is very simple, we do not know of any reference, so include it here for completeness.³⁶

Lemma 7. *Let f and g be two continuously differentiable functions defined over $[\underline{a}, \bar{a}]$. Consider the subset satisfying the requirements that (1) $f(\underline{a}) \geq g(\underline{a})$; and (2) if $f(a) = g(a)$ for some $a \in [\underline{a}, \bar{a}]$ then $f'(a) > g'(a)$. Then $f(a) > g(a)$ for all $a \in (\underline{a}, \bar{a}]$.*

Proof. First, observe that, if $f(\underline{a}) = g(\underline{a})$, by property 2. $f'(\underline{a}) > g'(\underline{a})$, therefore $f(a) > g(a)$ in a neighborhood to the right of \underline{a} . If $f(\underline{a}) > g(\underline{a})$, we obtain the same result by continuity. Now, we prove the lemma by contradiction. Assume that, there exists $\tilde{a} \in [\underline{a}, \bar{a}]$ such that $f(\tilde{a}) \leq g(\tilde{a})$. By the Intermediate Value Theorem, we can assume that

³⁶See Cao (2014) for an earlier application of this result in the context of two-agent dynamic games.

$f(\tilde{a}) = g(\tilde{a})$, without loss of generality. Now let $a^* = \inf \{a \in [\underline{a}, \bar{a}] : f(a) = g(a)\}$. By continuity $f(a^*) = g(a^*)$. Moreover, $a^* > \underline{a}$ because $f(a) > g(a)$ in the right neighborhood of \underline{a} . By property 2), $f'(a^*) > g'(a^*)$. Together with $f(a^*) = g(a^*)$, this implies, $f(a) < g(a)$ in a neighborhood to the left of a^* . Therefore by the Intermediate Value Theorem, there exists $a^{**} \in (\underline{a}, a^*)$ such that $f(a^{**}) = g(a^{**})$. This contradicts the definition of a^* which is the infimum. \square

We also use a few variations of this lemma.

Variation 1. if 1) $f(\bar{a}) \geq g(\bar{a})$, and 2) if $f(a) = g(a)$, for some $a < \bar{a}$, then $f'(a) < g'(a)$, we have $f(a) > g(a)$ for all $a \in [\underline{a}, \bar{a}]$.

Variation 2. We can also relax condition 2, by the condition that if $f(a) = g(a)$ then $f'(\tilde{a}) > g'(\tilde{a})$ in a neighborhood to the left of a . Indeed, in the proof above, if $f(a^*) = g(a^*)$ and $f'(\tilde{a}) > g'(\tilde{a})$ in the left neighborhood of a^* , then for a in the left neighborhood of a^* ,

$$\begin{aligned} f(a) &= f(a^*) - \int_a^{a^*} f'(\tilde{a}) d\tilde{a} \\ &= g(a^*) - \int_a^{a^*} f'(\tilde{a}) d\tilde{a} \\ &< g(a^*) - \int_a^{a^*} g'(\tilde{a}) d\tilde{a} = g(a). \end{aligned}$$

We can then proceed as in the remaining of the proof. This variation is useful when f' or g' are not well-defined at some a .

C Local Existence

C.1 Proof of Lemma 2

For any $\epsilon > 0$ sufficiently small, indeed satisfying

$$\epsilon < \frac{\lim_{c \rightarrow +\infty} U_1(c) - U_1(ra_0)}{\lambda},$$

consider the solution (V_ϵ, W_ϵ) to the ODE (18) satisfying the initial condition

$$(V_\epsilon(a_0), W_\epsilon(a_0)) = (\bar{V}(a_0), \bar{W}(a_0) - \epsilon). \quad (24)$$

Given that (18) is regular around a_0 , we can apply standard ODE existence results to show that (V_ϵ, W_ϵ) exists and is unique over some interval $[a_0, a_0 + \omega_\epsilon]$ that depends on ϵ . We will use (V_ϵ, W_ϵ) , together with the supporting results, Lemmas 8-11 in Subsection C.2, to construct the equilibrium described in Lemma 2 as follows:

First, Lemma 8 shows that there exists an $\omega > 0$ and $\bar{\epsilon} > 0$ such that for $0 < \epsilon < \bar{\epsilon}$ such that (V_ϵ, W_ϵ) are defined over $[a_0, a_0 + \omega]$. Second, Lemma 10 shows that for $0 < \epsilon < \bar{\epsilon}$, the slopes of V_ϵ and W_ϵ are uniformly bounded over $[a_0, a_0 + \omega]$. Finally, using these two results and applying the Dominated Convergence Theorem, we show that (V_ϵ, W_ϵ) converges to (V, W) for a subsequence $\epsilon_N \rightarrow 0$ and (V, W) is a solution to system (6).

We now describe this last step in detail. Lemma 8 shows that there exist $\omega > 0$ and $\bar{\epsilon} > 0$ such that: for any $\epsilon < \bar{\epsilon}$ the solution $(V_\epsilon(a), W_\epsilon(a))$ are defined over $[a_0, a_0 + \omega]$ and that $V_\epsilon(a) > \bar{V}(a)$ for all $a \in (a_0, a_0 + \omega]$. Lemma 10 implies that for all $a \in [a_0, a_0 + \omega]$,

$$\begin{aligned} 0 &\leq W'_\epsilon(a) \leq U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra), \\ 0 &\leq V'_\epsilon(a) \leq U'_1(ra). \end{aligned}$$

Because the derivatives V'_ϵ and W'_ϵ are uniformly bounded, the families of functions $\{V_\epsilon\}$ and $\{W_\epsilon\}$ defined over $[a_0, a_0 + \omega]$ are uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, there exists a sequence ϵ_N such that $(V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ converges uniformly to continuous functions (V, W) . We now show that this candidate (V, W) is a solution to (6).

Because (V_ϵ, W_ϵ) is a solution to the ODE (18), for any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$,

$$V_{\epsilon_N}(a_1) - V_{\epsilon_N}(a_2) = \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) da.$$

Since R_l is continuous

$$\lim_{N \rightarrow \infty} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = R_l(a, V(a), W(a)).$$

Moreover, by Lemma 10, $R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ is uniformly bounded over $[a_1, a_2]$: $0 \leq R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = V'_{\epsilon_N}(a) \leq U'_1(ra) < U'_1(ra_0)$. Therefore, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) da &= \int_{a_1}^{a_2} \lim_{N \rightarrow \infty} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) da \\ &= \int_{a_1}^{a_2} R_l(a, V(a), W(a)) da. \end{aligned}$$

Thus,

$$\begin{aligned} V(a_1) - V(a_2) &= \lim_{N \rightarrow \infty} (V_{\epsilon_N}(a_1) - V_{\epsilon_N}(a_2)) \\ &= \lim_{N \rightarrow \infty} \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) da \\ &= \int_{a_1}^{a_2} R_l(a, V(a), W(a)) da. \end{aligned} \tag{25}$$

Because R_l is continuous in a, V, W and V, W are continuous in a , the last equality implies that $V'(a) = R_l(a, V(a), W(a))$ for all $a \in [a_0, a_0 + \omega]$ (with V' standing for the right derivative of V at $a = a_0$).

Similarly, for any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$,

$$W_{\epsilon_N}(a_1) - W_{\epsilon_N}(a_2) = \int_{a_1}^{a_2} S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) da.$$

By choosing ω sufficiently small, the last property in Lemma 8 applies for each $a \in (a_0, a_0 + \omega]$. We show that, $(a, V(a), W(a)) \in E_0$ for each $a \in (a_0, a_0 + \omega]$ and

$$\lim_{N \rightarrow \infty} S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = S_l(a, V(a), W(a)). \quad (26)$$

Indeed, from the definition of $V_{\epsilon_N}, W_{\epsilon_N}$, $(\rho + \lambda)V_{\epsilon_N}(a) - \lambda W_{\epsilon_N}(a) > U_1(ra)$. Therefore, by pointwise convergence, $(\rho + \lambda)V(a) - \lambda W(a) \geq U_1(ra)$. We show by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. Assume to the contrary that $(\rho + \lambda)V(a) - \lambda W(a) = U_1(ra)$. From the last property of Lemma 8, $V_{\epsilon_N}(a) \geq \bar{V}(a) + \gamma_a$ for $\epsilon_N < \epsilon_a$. Therefore, by pointwise convergence, $V(a) \geq \bar{V}(a) + \gamma_a$. This, together with the contradiction assumption, implies that

$$W(a) < \bar{W}(a) - \frac{\rho + \lambda}{\lambda} \gamma_a.$$

In addition, by the continuity of R_l and by pointwise convergence,

$$\hat{c}_{\epsilon_N}(a) = (U_1')^{-1}(R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))) \rightarrow (U_1')^{-1}(R_l(a, V(a), W(a))) = ra$$

as $N \rightarrow \infty$. Consequently, there exists $\delta \in (0, 1)$ such that for N sufficiently high,

$$\begin{aligned} S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) &= \frac{\rho W_{\epsilon_N}(a) - U_0(\hat{c}_{\epsilon_N}(a))}{ra - \hat{c}_{\epsilon_N}(a)} > \frac{\rho \bar{W}(a) - \frac{\rho(\rho + \lambda)}{\lambda}(1 - \delta)\gamma_a - U_0(ra)}{ra - \hat{c}_{\epsilon_N}(a)} \\ &\rightarrow \frac{\rho \bar{W}(a) - \frac{\rho(\rho + \lambda)}{\lambda}(1 - \delta)\gamma_a - U_0(ra)}{ra - ra} = +\infty, \end{aligned}$$

as $N \rightarrow \infty$, which contradicts the boundedness of $S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ shown in Lemma 10. Therefore, we have shown by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. By the continuity of S_l in E_0 , we obtain the limit (26).

Since $0 < S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) < U_0'(ra) + \frac{\rho}{\lambda} U_1'(ra) < U_0'(ra_1) + \frac{\rho}{\lambda} U_1'(ra_1)$, by the Dominated Convergence Theorem, we can take the limit and conclude that

$$W(a_1) - W(a_2) = \int_{a_1}^{a_2} S_l(a, V(a), W(a)) da. \quad (27)$$

In addition, R_l, S_l are continuous over E_0 , therefore (25) and (27) imply that (V, W) is a solution to ODE (18) over $(a_0, a_0 + \omega_0]$; this immediately implies that (6) holds for all $a \in (a_0, a_0 + \omega_0]$.

Next we show that (6) holds at $a = a_0$. We showed that $V'(a_0) = R_l(a_0, V(a_0), W(a_0))$, so equation (6a) holds at $a = a_0$. Since $(V(a_0), W(a_0)) = \lim_{N \rightarrow \infty} (V_{\epsilon_N}(a_0), W_{\epsilon_N}(a_0)) = (\bar{V}(a_0), \bar{W}(a_0))$ this implies that $V'(a_0) = U_1'(ra_0)$. Since $V'(a_0) = U_1'(ra_0)$, this gives $\hat{c}(a_0) = ra_0$, and so equation (6b) holds.

Having established the existence of (V, W) , we turn to showing Properties 1) and 2).

Property 1: Notice that the right derivative of V at a_0 , $V'(a_0) = R_l(a_0, \bar{V}(a_0), \bar{W}(a_0)) =$

$U_1'(ra_0) > \bar{V}'(a_0)$, by Lemma 6. Together with $V(a_0) = \bar{V}(a_0)$, we have $V(a) > \bar{V}(a)$ in a neighborhood to the right of a_0 . Restricting ω so that $a_0 + \omega$ lies in this neighborhood, we obtain the first property in Lemma 2.

Property 2: Because $\hat{c}(a) = (U_1')^{-1}(V'(a))$ and $\lim_{a \downarrow a_0} V'(a) = \lim_{a \downarrow a_0} R_l(a, V(a), W(a)) = U_1'(ra_0)$, $\lim_{a \downarrow a_0} \hat{c}(a) = ra_0$.

By (21) in Lemma 5,

$$\hat{c}'(a) = \frac{1}{U_1''(\hat{c}(a))} \frac{(\rho + \lambda - r)V'(a) - \lambda W'(a)}{ra - \hat{c}(a)}.$$

From the derivation (30) in Lemma 10,

$$W'(a) = \lim_{\epsilon \rightarrow 0} W'_\epsilon(a) \leq \lim_{\epsilon \rightarrow 0} \left(U_0(ra) + \frac{\rho}{\lambda} U_1'(ra) - \frac{\rho}{\lambda} V'_\epsilon(a) \right),$$

and $V'(a) = \lim_{\epsilon \rightarrow 0} V'_\epsilon(a)$. Therefore

$$\hat{c}'(a) \geq \frac{1}{U_1''(\hat{c}(a))} \frac{\lim_{\epsilon \rightarrow 0} ((2\rho + \lambda - r)V'_\epsilon(a) - \lambda U_0(ra) - \rho U_1'(ra))}{ra - \hat{c}(a)}.$$

Because $\hat{c}(a) \rightarrow ra_0$ as $a \rightarrow a_0$, $\lim_{a \rightarrow a_0} (ra - \hat{c}(a)) = 0$. Moreover,

$$\lim_{(a, \epsilon) \rightarrow (a_0, 0)} ((2\rho + \lambda - r)V'_\epsilon(a) - \lambda U_0(ra) - \rho U_1'(ra)) = (\rho + \lambda - r)U_1'(ra_0) - \lambda U_0'(ra_0) > 0,$$

where the last inequality comes from the fact that

$$\frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) < \frac{\rho}{r} \left(\frac{\lambda + \rho - r}{\lambda} \right) \leq \frac{\lambda + \rho - r}{\lambda},$$

if $r \geq \rho$ and

$$\frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) \leq 1 < \frac{\rho + \lambda - r}{\lambda},$$

if $r < \rho$. As a result, $\lim_{a \rightarrow a_0} \hat{c}'(a) = +\infty$. We have established the second property in Lemma 2.

C.2 Supporting Results for Lemma 2

The proof of Lemma 2 given above draws on the following results.

The first lemma below shows that there exists $\omega > 0$ and $\bar{\epsilon}$ such that for each $\epsilon \in (0, \bar{\epsilon})$, the solution (V_ϵ, W_ϵ) to ODE (18) are defined over $[a_0, a_0 + \omega]$ and $V_\epsilon(a) > \bar{V}(a)$. The proof of this lemma uses Lemma 9 that follow.

Lemma 8. *There exist $\omega > 0$ and $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, $(V_\epsilon(a), W_\epsilon(a))$ constructed in the proof of Lemma 2 is defined on $[a_0, a_0 + \omega]$. Moreover, $V_\epsilon(a) > \bar{V}(a)$ for all $a \in (a_0, a_0 + \omega]$. Lastly, there exists $\omega_0 < \omega$ such that for each $a \in (a_0, a_0 + \omega_0]$, there exist $\epsilon_a, \gamma_a > 0$ such that $V_\epsilon(a) > \bar{V}(a) + \gamma_a$ for all $0 < \epsilon < \epsilon_a$.*

Proof. Let $\bar{\epsilon}_1 = \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0)) > 0$. For $0 < \epsilon < \bar{\epsilon}_1$, let $[a_0, \tilde{a}_\epsilon)$ denote the (right)

maximal interval of existence for (V_ϵ, W_ϵ) .³⁷ Lemma 9 shows that if $\tilde{a}_\epsilon < \infty$ then

$$(\tilde{a}_\epsilon, V_\epsilon(\tilde{a}_\epsilon), W_\epsilon(\tilde{a}_\epsilon)) \in E_s.$$

In addition, $V_\epsilon(\tilde{a}_\epsilon) \leq \bar{V}(\tilde{a}_\epsilon)$.

Because R_I is continuous,

$$\lim_{\epsilon \rightarrow 0} R_I(a_0, V_\epsilon(a_0), W_\epsilon(a_0)) = R_I(a_0, \bar{V}(a_0), \bar{W}(a_0)) = U_1'(ra_0) > \bar{V}'(a_0),$$

where the last inequality is an application of Lemma 6 at a_0 . Therefore, there exists $\bar{\epsilon}_2 > 0$, such that $V_\epsilon'(a_0) = R_I(a_0, V_\epsilon(a_0), W_\epsilon(a_0)) > \bar{V}'(a_0)$ for $0 < \epsilon < \bar{\epsilon}_2$. In this case, $V_\epsilon(a) > \bar{V}(a)$ in some neighborhood to the right of a_0 .

For $0 < \epsilon < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)$, let

$$\bar{a}_\epsilon = \sup \{a \in (a_0, \tilde{a}_\epsilon) : V_\epsilon(a') > \bar{V}(a') \text{ for all } a' \in (a_0, a)\}.$$

Because $V_\epsilon(a) > \bar{V}(a)$ in some neighborhood to the right of a_0 , as shown above, $\bar{a}_\epsilon > a_0$.

We show by contradiction that there exist $\omega > 0$ and $0 < \bar{\epsilon} < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)$, such that $\bar{a}_\epsilon > a_0 + \omega$ for all $\epsilon < \bar{\epsilon}$. Assume that this is not true, then there exists a sequence $\epsilon_N \rightarrow 0$ such that $\lim_{N \rightarrow \infty} \bar{a}_{\epsilon_N} = a_0$.

Because V_{ϵ_N} is continuous, $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) \geq \bar{V}(\bar{a}_{\epsilon_N})$ (otherwise, $V_{\epsilon_N}(a) < \bar{V}(a)$ in some neighborhood to the left of \bar{a}_{ϵ_N} , which contradicts the definition of \bar{a}_ϵ). If $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) > \bar{V}(\bar{a}_{\epsilon_N})$, then $\bar{a}_{\epsilon_N} < \tilde{a}_{\epsilon_N}$, because if $\tilde{a}_{\epsilon_N} < \infty$ then $V_{\epsilon_N}(\tilde{a}_{\epsilon_N}) \leq \bar{V}(\tilde{a}_{\epsilon_N})$ as shown in Lemma 9. This also contradicts the definition of \bar{a}_{ϵ_N} , because $V_{\epsilon_N}(a)$ is defined and is strictly greater than $\bar{V}(a)$ in a neighborhood of \bar{a}_{ϵ_N} . Therefore $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) = \bar{V}(\bar{a}_{\epsilon_N})$.

By the Mean Value Theorem, there exists $a_{\epsilon_N}^* \in [a_0, \bar{a}_{\epsilon_N}]$ such that

$$\frac{V_{\epsilon_N}(\bar{a}_{\epsilon_N}) - V_{\epsilon_N}(a_0)}{\bar{a}_{\epsilon_N} - a_0} = V_{\epsilon_N}'(a_{\epsilon_N}^*) = \frac{\bar{V}(\bar{a}_{\epsilon_N}) - \bar{V}(a_0)}{\bar{a}_{\epsilon_N} - a_0} \quad (28)$$

and by the definition of V_ϵ, W_ϵ :

$$V_{\epsilon_N}'(a_{\epsilon_N}^*) = R_I(a_{\epsilon_N}^*, V_{\epsilon_N}(a_{\epsilon_N}^*), W_{\epsilon_N}(a_{\epsilon_N}^*)).$$

By the monotonicity of V_ϵ and W_ϵ shown in Lemma 10,

$$\bar{V}(a_0) < V_{\epsilon_N}(a_{\epsilon_N}^*) < \bar{V}(\bar{a}_{\epsilon_N}),$$

and

$$W_{\epsilon_N}(a_0) = \bar{W}(a_0) - \epsilon_N < W_{\epsilon_N}(a_{\epsilon_N}^*).$$

Moreover, from the upper bound on W_ϵ' shown in Lemma 10 (using $V_{\epsilon_N}(a) \geq \bar{V}(a)$ for $a \in (a_0, \bar{a}_{\epsilon_N})$):

$$W_{\epsilon_N}(a_{\epsilon_N}^*) \leq W_{\epsilon_N}(a_0) + \left(U_0'(ra_0) + \frac{\rho}{\lambda} U_1'(ra_0) \right) (a_{\epsilon_N}^* - a_0).$$

Besides, by the contradiction assumption, $\lim_{N \rightarrow \infty} a_{\epsilon_N}^* = \lim_{N \rightarrow \infty} \bar{a}_{\epsilon_N} = a_0$. Therefore, by

³⁷The definition of the maximal interval of existence is standard in the ODE literature. See, for example, Hartman (2002).

the Squeeze Principle, using the four inequalities above, we obtain

$$\begin{aligned}\lim_{N \rightarrow \infty} V_{\epsilon_N}(a_{\epsilon_N}^*) &= \bar{V}(a_0) \\ \lim_{N \rightarrow \infty} W_{\epsilon_N}(a_{\epsilon_N}^*) &= \bar{W}(a_0).\end{aligned}$$

Thus, together with the continuity of R_l and (28), we obtain

$$\begin{aligned}\lim_{N \rightarrow \infty} R_l(a_{\epsilon}^*, V_{\epsilon}(a_{\epsilon}^*), W_{\epsilon}(a_{\epsilon}^*)) &= R_l(a_0, \bar{V}(a_0), \bar{W}(a_0)) = U_1'(ra_0) \\ &= \lim_{N \rightarrow \infty} \frac{\bar{V}(\bar{a}_{\epsilon_N}) - \bar{V}(a_0)}{\bar{a}_{\epsilon_N} - a_0} = \bar{V}'(a_0).\end{aligned}$$

This leads to the desired contradiction because Lemma 6 for $a = a_0$ implies that $\bar{V}'(a_0) < U_1'(ra_0)$.

Finally, we show the last property by contradiction. Assume that it does not hold. Then there exists a sequence $a_N \rightarrow a_0$ such that for each N , there exists a sequence $\epsilon_{N,M} \rightarrow 0$ such that $V_{\epsilon_{N,M}}(a_N) \rightarrow \bar{V}(a_N)$. By choosing M sufficiently high, we have $0 < \epsilon_{N,M} < \frac{1}{N}$ and

$$\left| \frac{V_{\epsilon_{N,M}}(a_N) - \bar{V}(a_N)}{a_N - a_0} \right| < \frac{1}{N}.$$

By the Mean Value Theorem, there exists $\tilde{a}_N \in [a_0, a_N]$ such that

$$\begin{aligned}\frac{V_{\epsilon_{N,M}}(a_N) - \bar{V}(a_N)}{a_N - a_0} &= \frac{V_{\epsilon_{N,M}}(a_N) - V_{\epsilon_{N,M}}(a_0) + \bar{V}_{\epsilon_{N,M}}(a_0) - \bar{V}(a_N)}{a_N - a_0} \\ &= V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) - \bar{V}'(\tilde{a}_{N,M}).\end{aligned}$$

Therefore

$$\left| V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) - \bar{V}'(\tilde{a}_{N,M}) \right| < \frac{1}{N}. \quad (29)$$

However,

$$V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) = R_l(\tilde{a}_{N,M}, V_{\epsilon_{N,M}}(\tilde{a}_{N,M}), W_{\epsilon_{N,M}}(\tilde{a}_{N,M})),$$

and by Lemma 10, as $N, M \rightarrow \infty$ $V_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \rightarrow \bar{V}(a_0)$ and $W_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \rightarrow \bar{W}(a_0)$. Therefore by the continuity of R_l ,

$$V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \rightarrow R_l(a_0, \bar{V}(a_0), \bar{W}(a_0)) = U_1'(ra_0).$$

Because $\tilde{a}_{N,M} \rightarrow a_0$

$$\bar{V}'(\tilde{a}_{N,M}) \rightarrow \bar{V}'(a_0).$$

Combining the last two limits with (29), we have $U_1'(ra_0) = \bar{V}'(a_0)$, which contradicts condition (22) for $a = a_0$ that $U_1'(ra_0) > \bar{V}'(a_0)$. Therefore by contradiction, the last property holds. \square

Lemma 9. Consider the (right) maximal interval of existence, $[a_0, \tilde{a})$ for the solution $(V_{\epsilon}, W_{\epsilon})$ to the ODE (18) with the initial condition (24) and $0 \leq \epsilon < \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0))$. If $\tilde{a} < \infty$, then $\lim_{a \uparrow \tilde{a}} V_{\epsilon}(a) = V(\tilde{a})$ and $\lim_{a \uparrow \tilde{a}} W_{\epsilon}(a) = W_{\epsilon}(\tilde{a})$ and $(\tilde{a}, V_{\epsilon}(\tilde{a}), W_{\epsilon}(\tilde{a})) \in E_s$. In addition, $V_{\epsilon}(\tilde{a}) \leq \bar{V}(\tilde{a})$.

Proof. By Lemma 10, $V'_{\epsilon}(a), W'_{\epsilon}(a) > 0$. Therefore, the limits $\lim_{a \uparrow \tilde{a}} V_{\epsilon}(a) = V_{\epsilon}(\tilde{a})$ and

$\lim_{a \uparrow \tilde{a}} W_\epsilon(a) = W_\epsilon(\tilde{a})$ exist. In addition, since $V'_\epsilon(a) = U'_1(\hat{c}_\epsilon(a)) < U'_1(ra)$, $V_\epsilon(\tilde{a}) < V_\epsilon(a_0) + \int_{a_0}^{\tilde{a}} U'_1(ra) da < \infty$ and $W_\epsilon(\tilde{a}) \leq \frac{(\rho+\lambda)V_\epsilon(\tilde{a}) - U_1(r\tilde{a})}{\lambda} < \infty$. By [Hartman \(2002, Theorem 3.1\)](#), $(\tilde{a}, V_\epsilon(\tilde{a}), W_\epsilon(\tilde{a}))$ must lie in the boundary of E_0 , i.e. Case 1: $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) = U_1(\infty)$ or Case 2: $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) = U_1(r\tilde{a})$. We first rule out Case 1 by showing that $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) < U_1(\infty)$.

If $U_1(\infty) = \infty$, this is obvious. Now if $U_1(\infty) < \infty$. Let $a(t)$ denote the solution to the ODE, $a(0) = \tilde{a}$ and $\frac{da(t)}{dt} = ra(t) - \hat{c}_\epsilon(a(t))$ where $\hat{c}_\epsilon(a) = (U'_1)^{-1}(R_I(a, V_\epsilon(a), W_\epsilon(a))) > ra$. Consider the derivative:

$$\begin{aligned} \frac{d\left(e^{-(\rho+\lambda)t} V_\epsilon(a(t))\right)}{dt} &= e^{-(\rho+\lambda)t} \left(-(\rho + \lambda)V_\epsilon(a(t)) + V'_\epsilon(a(t))(ra(t) - \hat{c}_\epsilon(a(t)))\right) \\ &= e^{-(\rho+\lambda)t} \left(U_1(\hat{c}_\epsilon(a(t))) + \lambda W_\epsilon(a(t))\right), \end{aligned}$$

where the second equality comes from the fact that V_ϵ is the solution of ODE (18). Let T denote the time at which $a(t)$ reaches a_0 (T can be $+\infty$), then

$$V_\epsilon(\tilde{a}) = \int_0^T e^{-(\rho+\lambda)t} (U_1(\hat{c}_\epsilon(a(t))) + \lambda W_\epsilon(a(t))) dt + e^{-(\rho+\lambda)T} \bar{V}(a_0).$$

Notice that, by Lemma 10, $W_\epsilon(a)$ is strictly increasing in a and a_t is strictly decreasing in t because $\hat{c}_\epsilon(a(t)) > ra(t)$. Therefore $W_\epsilon(a(t)) < W_\epsilon(a(0)) = W_\epsilon(\tilde{a})$. This implies

$$\begin{aligned} V_\epsilon(\tilde{a}) &< \int_0^T e^{-(\rho+\lambda)t} (U_1(\infty) + \lambda W_\epsilon(\tilde{a})) dt + e^{-(\rho+\lambda)T} \bar{V}(a_0) \\ &= (1 - e^{-(\rho+\lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho+\lambda)T}) W_\epsilon(\tilde{a}) + e^{-(\rho+\lambda)T} \bar{V}(a_0). \end{aligned}$$

By the definition of $\bar{V}(a)$,

$$\begin{aligned} \bar{V}(a_0) &= \frac{1}{\rho + \lambda} U_1(ra_0) + \frac{\lambda}{\rho + \lambda} \bar{W}(a_0) = \frac{1}{\rho + \lambda} U_1(ra_0) + \frac{\lambda}{\rho + \lambda} (W_\epsilon(a_0) + \epsilon) \\ &< \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_\epsilon(a_0), \end{aligned}$$

since $\epsilon < \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0))$. Thus

$$\begin{aligned} V_\epsilon(\tilde{a}) &< (1 - e^{-(\rho+\lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho+\lambda)T}) W_\epsilon(\tilde{a}) + e^{-(\rho+\lambda)T} \bar{V}(a_0) \\ &< (1 - e^{-(\rho+\lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho+\lambda)T}) W_\epsilon(\tilde{a}) \\ &\quad + e^{-(\rho+\lambda)T} \left(\frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_\epsilon(\tilde{a}) \right) = \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_\epsilon(\tilde{a}). \end{aligned}$$

Therefore $(\rho + \lambda) V_\epsilon(\tilde{a}) < U_1(\infty) + \lambda W_\epsilon(\tilde{a})$ which is equivalent to the desired inequality.

As we have ruled out Case 1, we must be in Case 2, i.e. $(\tilde{a}, V(\tilde{a}), W(\tilde{a})) \in E_s$.

Next, we show by contradiction that $V_\epsilon(\tilde{a}) \leq \bar{V}(\tilde{a})$. Assume to the contrary that $V_\epsilon(\tilde{a}) > \bar{V}(\tilde{a})$. Then $W_\epsilon(\tilde{a}) > \bar{W}(\tilde{a})$ because $(\rho + \lambda) \bar{V}(\tilde{a}) - \lambda \bar{W}(\tilde{a}) = U_1(r\tilde{a})$.

Since R_l is continuous over \bar{E}_0 , $\lim_{a \uparrow \bar{a}} V'_\epsilon(a) = U'_1(r\bar{a})$ and $\lim_{a \uparrow \bar{a}} \hat{c}_\epsilon(a) = r\bar{a}$. Therefore

$$\lim_{a \uparrow \bar{a}} W'_\epsilon(a) = \lim_{a \uparrow \bar{a}} \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = \frac{U_0(r\bar{a}) - \rho W_\epsilon(\bar{a})}{r\bar{a} - r\bar{a}} = -\infty,$$

which contradicts the property that $W'_\epsilon > 0$ established in Lemma 10. So by contradiction, $W_\epsilon(\bar{a}) \leq \bar{W}(\bar{a})$, and $V_\epsilon(\bar{a}) \leq \bar{V}(\bar{a})$. \square

The following lemma establishes the bounds on the derivative of V_ϵ and W_ϵ that are important to apply the Dominated Convergence Theorem in Lemma 2. To prove this result, we use Lemma 11.

Lemma 10. *Consider the solution (V_ϵ, W_ϵ) to ODE (18) with the initial condition (24) defined over some interval $[a_0, \bar{a}]$. We have $0 < V'_\epsilon(a) \leq U'_1(ra)$ and $0 < W'_\epsilon(a)$ for all $a \in [a_0, \bar{a}]$. Moreover, if $V_\epsilon(a) \geq \bar{V}(a)$, $W'_\epsilon(a) < U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra)$.*

Proof. Since $\hat{c}_\epsilon(a) > ra$ and $V'_\epsilon(a) = U'_1(\hat{c}_\epsilon(a))$, we have $0 < V'_\epsilon(a) \leq U'_1(ra)$ due to the concavity of U_1 . If $r \geq \rho$, from Lemma 11, $W_\epsilon(a) < \bar{W}(a)$. Therefore,

$$W'_\epsilon(a) = \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} > \frac{U_0(ra) - \rho \bar{W}(a)}{\hat{c}_\epsilon(a) - ra} = 0.$$

If $r < \rho$, Lemma 11 immediately implies that $W'_\epsilon(a) > 0$.

To show the upper bound on $W'_\epsilon(a)$ when $V_\epsilon(a) \geq \bar{V}(a)$, we use the facts that

$$(\rho + \lambda) V_\epsilon(a) - \lambda W_\epsilon(a) = U_1(\hat{c}_\epsilon(a)) + V'_\epsilon(a) (ra - \hat{c}_\epsilon(a))$$

and

$$(\rho + \lambda) \bar{V}(a) - \lambda \bar{W}(a) = U_1(ra).$$

By subtracting the two equalities side by side and rearranging,

$$\begin{aligned} \lambda (\bar{W}(a) - W_\epsilon(a)) &= -(\rho + \lambda) (V_\epsilon(a) - \bar{V}(a)) \\ &\quad + U_1(\hat{c}_\epsilon(a)) - U_1(ra) + V'_\epsilon(a) (ra - \hat{c}_\epsilon(a)) \\ &\leq U_1(\hat{c}_\epsilon(a)) - U_1(ra) + V'_\epsilon(a) (ra - \hat{c}_\epsilon(a)). \end{aligned}$$

where the last inequality comes from $V_\epsilon(a) \geq \bar{V}(a)$. It follows that

$$\begin{aligned} W'_\epsilon(a) &= \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra) + \rho (\bar{W}(a) - W_\epsilon(a))}{\hat{c}_\epsilon(a) - ra} \\ &\leq \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} (U_1(\hat{c}_\epsilon(a)) - U_1(ra) + V'_\epsilon(a) (ra - \hat{c}_\epsilon(a)))}{\hat{c}_\epsilon(a) - ra} \\ &= \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} (U_1(\hat{c}_\epsilon(a)) - U_1(ra))}{\hat{c}_\epsilon(a) - ra} - \frac{\rho}{\lambda} V'_\epsilon(a) \\ &< \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra)}{\hat{c}_\epsilon(a) - ra} + \frac{\rho}{\lambda} \frac{U_1(\hat{c}_\epsilon(a)) - U_1(ra)}{\hat{c}_\epsilon(a) - ra} < U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra), \quad (30) \end{aligned}$$

where the last inequality comes from the concavity of U_1 and U_0 and $\hat{c}_\epsilon(a) > ra$. \square

Lemma 11. *Consider the solution (V_ϵ, W_ϵ) to ODE (18) with the initial condition (24) defined over some interval $[a_0, \bar{a}]$. We have*

- 1) If $r \geq \rho$, $W_\epsilon(a) < \bar{W}(a) \forall a > a_0$.
 2) If $r < \rho$, $W_\epsilon(a) < U_0(\hat{c}_\epsilon(a)) \forall a > a_0$.

Proof. 1) $r \geq \rho$: We use Lemma 7 to show property 1). We just need to verify conditions 1) and 2) in Lemma 7. First by definition, $W_\epsilon(a_0) < \bar{W}(a_0)$, so condition 1) Lemma 7 is satisfied. For condition 2) in Lemma 7, we show that if $W_\epsilon(a) = \bar{W}(a)$, for some $\forall a > a_0$ then $W'_\epsilon(a) < \bar{W}'(a)$. Indeed,

$$\begin{aligned} W'_\epsilon(a) &= \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = \frac{U_0(\hat{c}_\epsilon(a)) - \rho \bar{W}(a)}{\hat{c}_\epsilon(a) - ra} \\ &= \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra)}{\hat{c}_\epsilon(a) - ra} < U'_0(ra), \end{aligned}$$

where the last inequality comes from the fact that U_0 is strictly concave and $\hat{c}_\epsilon(a) > ra$. On the other hand, we also have

$$\bar{W}'(a) = \frac{r}{\rho} U'_0(ra) \geq U'_0(ra),$$

because $r \geq \rho$. Therefore, $\bar{W}'(a) > W'_\epsilon(a)$.

- 2) $r < \rho$: We also use Lemma 7 to show property 2). By the definition of V_ϵ, W_ϵ :

$$W_\epsilon(a_0) = U_0(ra_0) - \epsilon < U_0(ra_0) < U_0(\hat{c}_\epsilon(a_0)).$$

So condition 1) in Lemma 7 is satisfied. Now we show that condition 2) in Lemma 7 is also satisfied, i.e. if at some $a > a_0$, $W_\epsilon(a) = U_0(\hat{c}_\epsilon(a))$, we show that $W'_\epsilon(a) < \frac{d}{da}(U'_0(\hat{c}_\epsilon(a)))$. Indeed,

$$W'_\epsilon(a) = \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = 0.$$

Moreover,

$$\frac{d}{da}(U_0(\hat{c}_\epsilon(a))) = U'_0(\hat{c}_\epsilon(a))\hat{c}'_\epsilon(a).$$

By (21), in addition to $W'_\epsilon(a) = 0$,

$$\begin{aligned} \hat{c}'_\epsilon(a) &= \frac{1}{U''_1(\hat{c}_\epsilon(a))} \frac{(\rho + \lambda - r) V'_\epsilon(a) - \lambda W'_\epsilon(a)}{ra - \hat{c}_\epsilon(a)} \\ &= \frac{1}{-U''_1(\hat{c}_\epsilon(a))} \frac{(\rho + \lambda - r) U'_1(\hat{c}_\epsilon(a))}{\hat{c}_\epsilon(a) - ra} > 0. \end{aligned}$$

Therefore $W'_\epsilon(a) = 0 < \frac{d}{da}(U'_0(\hat{c}_\epsilon(a)))$. □

D Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. Once we verify the differential system (6) all the equilibrium conditions in Subsection 3.1 are met. By Lemma 6, we have $\bar{V}'(a) = U'_1(ra)$, therefore $\hat{c}(a) = ra$ and equations (6) are satisfied by the definitions of \bar{V} and \bar{W} . □

Proof of Theorem 2. Assume $\sigma \neq 1$; the case with $\sigma = 1$ is similar. To proceed, we guess

$$V(a) = \bar{v} \frac{a^{1-\sigma}}{1-\sigma} \quad W(a) = \bar{w} \frac{a^{1-\sigma}}{1-\sigma},$$

and find \bar{v}, \bar{w} to verify that V, W satisfy (6).³⁸ To show the uniqueness of the linear equilibrium, notice that in any linear equilibrium, V, W must have the functional form above.

Given the conjectured functional form, the first-order condition (6a) implies

$$\hat{c}(a) = \psi a,$$

where $\psi = \bar{v}^{-\frac{1}{\sigma}}$. Plugging this back into (6a) gives

$$\begin{aligned} (\rho + \lambda) \bar{v} \frac{a^{1-\sigma}}{1-\sigma} &= \frac{1}{1-\sigma} \left((\bar{v} a^{-\sigma})^{-\frac{1}{\sigma}} \right)^{1-\sigma} + (\bar{v} a^{-\sigma}) \left(r a - (\bar{v} a^{-\sigma})^{-\frac{1}{\sigma}} \right) + \lambda \bar{w} \frac{a^{1-\sigma}}{1-\sigma} \\ &= \frac{\sigma}{1-\sigma} (\bar{v} a^{-\sigma})^{-\frac{1-\sigma}{\sigma}} + (\bar{v} a^{-\sigma}) r a + \lambda \bar{w} \frac{a^{1-\sigma}}{1-\sigma}. \end{aligned}$$

Canceling the $\frac{a^{1-\sigma}}{1-\sigma}$ terms and rearranging we obtain

$$(\lambda + \Delta) \bar{v} = \sigma \bar{v}^{1-\frac{1}{\sigma}} + \lambda \bar{w}, \quad (31)$$

where Δ is defined by

$$\Delta \equiv \rho - r(1 - \sigma) > 0, \quad (32)$$

where the inequality comes from the restriction that value functions V, W are finite.

From the second equation in system (6) we have

$$\rho \bar{w} \frac{a^{1-\sigma}}{1-\sigma} = \bar{\beta} \frac{1}{1-\sigma} \left((\bar{v} a^{-\sigma})^{-\frac{1}{\sigma}} \right)^{1-\sigma} + (\bar{w} a^{-\sigma}) \left(r a - (\bar{v} a^{-\sigma})^{-\frac{1}{\sigma}} \right).$$

Canceling the $\frac{a^{1-\sigma}}{1-\sigma}$ terms gives

$$\bar{w} = \frac{\bar{\beta} \bar{v}^{1-\frac{1}{\sigma}}}{\Delta + (1 - \sigma) \bar{v}^{-\frac{1}{\sigma}}}. \quad (33)$$

Combining equations (31) and (33), we obtain

$$\lambda + \Delta = \sigma \bar{v}^{-\frac{1}{\sigma}} + \lambda \bar{\beta} \frac{\bar{\beta} \bar{v}^{-\frac{1}{\sigma}}}{\Delta + (1 - \sigma) \bar{v}^{-\frac{1}{\sigma}}},$$

a single equation in \bar{v} . Define $\psi \equiv \bar{v}^{-\frac{1}{\sigma}}$. We then have a quadratic equation in ψ :

$$P(\psi) \equiv Q_2 \psi^2 + Q_1 \psi + Q_0 = 0, \quad (34)$$

³⁸If $\sigma = 1$

$$\begin{aligned} V(a) &= A_v + \frac{1}{\rho + \lambda} \left(1 + \lambda \frac{\bar{\beta}}{\rho} \right) \log a \\ W(a) &= A_w + \frac{\bar{\beta}}{\rho} \log a. \end{aligned}$$

with

$$\begin{aligned} Q_2 &\equiv (1 - \sigma) \sigma \\ Q_1 &\equiv (\sigma + \bar{\beta} - 1) \lambda + \Delta (2\sigma - 1) \\ Q_0 &\equiv -(\lambda + \Delta) \Delta. \end{aligned}$$

If $\sigma < 1$ then $Q_2 > 0$ and $Q_0 < 0$. This implies that there exists a unique strictly positive ψ that is the solution to (34). The implied consumption rule yields finite utility.

If $\sigma > 1$ we have that $Q_2 < 0$. This implies that

$$\begin{aligned} P\left(\frac{\Delta}{\sigma-1}\right) &= -\frac{\sigma}{\sigma-1}\Delta^2 + ((\sigma + \bar{\beta} - 1) \lambda + \Delta (2\sigma - 1)) \frac{\Delta}{\sigma-1} - (\lambda + \Delta) \Delta \\ &= \frac{\bar{\beta}}{\sigma-1} \lambda \frac{\Delta}{\sigma-1} > 0. \end{aligned}$$

Therefore, there exists two solutions $0 < \psi_1 < \frac{\Delta}{\sigma-1} < \psi_2$ such that $P(\psi) = 0$. To know which root corresponds to a solution to (6), we observe that $c_t = \psi a_t$ so $\dot{a}_t = (r - \psi)a_t$ or $a_t = e^{(r-\psi)t} a_0$. Thus $V \propto \int e^{-\rho t} e^{(1-\sigma)(r-\psi)t} dt = \int_0^\infty e^{(-\Delta + \psi(\sigma-1))t} dt$. For V to be finite, we require $\psi < \frac{\Delta}{\sigma-1}$. So only the smaller root to (34), ψ_1 , yields finite value functions, and corresponds to a solution to (6).

Lastly, the derivations above directly imply the uniqueness of the linear equilibrium.

Now, we turn to the second part of the theorem. Given that $\hat{c}(a) = \psi a$, $\hat{c}(a) < ra$ if and only if $\psi < r$. For $\sigma < 1$ we have $r > 0$ so that $0 < \psi < r$ if and only if $P(r) > P(\psi) = 0$. For $\sigma > 1$, because $\rho > 0$, $r < \frac{\rho - r(1-\sigma)}{\sigma-1} = \frac{\Delta}{\sigma-1}$. Given that $P(\frac{\Delta}{\sigma-1}) > 0$ and $P(\psi_1) = 0$, $r > \psi$ if and only if $P(r) > P(\psi) = 0$. Thus, we need to establish that $P(r) > 0$. This is equivalent to

$$\beta > \frac{\rho}{r} \left(1 - \frac{r - \rho}{\lambda}\right) = \hat{\beta}.$$

Similarly, $\hat{c}(a) > ra$, i.e. $\psi > r$ if and only if $\bar{\beta} < \hat{\beta}$. \square

E Proofs for Dissaving Equilibria

E.1 Proof of Theorem 3

Proof of Theorem 3 (Existence). We prove the existence of an equilibrium by construction.

Lemma 2 shows that starting from $a_0 = \underline{a}$, ODE (18) with the boundary condition

$$(V(a_0), W(a_0)) = (\bar{V}(a_0), \bar{W}(a_0))$$

admits a solution defined over $[\underline{a}, \underline{a} + \omega)$ for some $\omega > 0$. Let (V_0, W_0) denote this solution. In addition, let $[\underline{a}, a^*)$ be the right maximal interval of existence for this solution. It is immediate that $a^* \geq \underline{a} + \omega$. If $a^* = \infty$, we have found a (continuous) Markov equilibrium, with $(V, W) = (V_0, W_0)$.

If $a^* < \infty$, following the steps in the proof of Lemma 9, we can show that

$$\lim_{a \uparrow a^*} V_0(a) \leq \bar{V}(a^*).$$

Moreover, as shown in Lemma 2, $V_0(a) > \bar{V}(a)$ in a neighborhood to the right of \underline{a} . Thus, by the Intermediate Value Theorem, there exists $a_1 \in (\underline{a}, a^*]$ such that $V_0(a_1) = \bar{V}(a_1)$.

Starting from a_1 , we apply Lemma 2 again with a_1 standing for a_0 and construct the a solution (V_1, W_1) to ODE (18) with the boundary condition

$$(V_1(a_1), W_1(a_1)) = (\bar{V}(a_1), \bar{W}(a_1)).$$

Following this procedure, we obtain a sequence $a_0 = \underline{a} < a_1 < \dots$ with $\lim_{n \rightarrow \infty} a_n = +\infty$ and a sequence of value functions (V_n, W_n) defined over $[a_n, a_{n+1}]$ with the boundary condition

$$(V_n(a_n), W_n(a_n)) = (\bar{V}(a_n), \bar{W}(a_n)).$$

The divergence of $\{a_n\}$ is shown in Lemma 12 below.

We define the value and consumption functions (V, W, \hat{c}) over the whole interval $[\underline{a}, \infty)$ as

$$(V(a), W(a), \hat{c}) = (V_n(a), W_n(a), \hat{c}_n(a)) \text{ for } a \in [a_n, a_{n+1}).$$

We verify that this construction satisfies all the conditions in Subsection 3.1 for an equilibrium. Conditions (a)-(e) are satisfied by the construction of (V, W) . Condition (f) on the existence of a_t is satisfied because by construction $\hat{c}(a)$ is differentiable and $\hat{c}(a) > ra$ outside steady-states $\{a_n\}$. Indeed, if $a(0) = a_n$ then $a(t) \equiv a_n$ for all $t \geq 0$ satisfies ODE (7) for all $t \geq 0$. If $a(0) \in (a_n, a_{n+1})$, the solution $a(t)$ to ODE (7) with the initial condition $a(t) = a(0)$ exists and is unique over a small interval $[0, \epsilon]$ because $\hat{c}(a)$ is continuously differentiable over (a_n, a_{n+1}) . In addition, because $\hat{c}(a) > ra$, $a(t)$ is strictly decreasing in t . Let $[0, T)$ denote the right maximal interval of existence for $a(t)$ to ODE (7). If $T = \infty$, we obtain the existence of $a(t)$ to ODE (7) over the whole time interval $[0, \infty)$. If $T < \infty$ (Indeed, we can show that this is always the case in the equilibrium that we just constructed), by Hartman (2002, Theorem 3.1), $a(T) = a_n$. Defining $a(t) = a_n$ for all $t \geq T$, we also obtain the existence of $a(t)$ to ODE (7) over the whole time interval $[0, \infty)$. Finally, the limits $\lim_{t \rightarrow \infty} e^{-(\rho+\lambda)t} V(a_t)$ and $\lim_{t \rightarrow \infty} e^{-\rho t} W(a_t)$ are both equal to 0 because $\dot{a}_t \leq 0$, V and W are increasing over $[a_n, a_{n+1})$ and $a_t \geq a_n$. \square

Proof of Theorem 3 (Characterization). Under Assumption 2, we show that there cannot be strict saving at any asset level.

We use Lemma 14 to show this result. In order to apply the lemma, we need to rule out the possibility that for some \tilde{a} , $\hat{c}(a) < ra$ for all $a \geq \tilde{a}$. Assume by contradiction that this is the case. Let $\bar{a} = \max(\tilde{a}, \frac{\bar{c}}{v})$. Then for all $a \geq \bar{a}$, $\bar{c} < \hat{c}(a) < ra$. Therefore, the equilibrium consumption always stay in the homogenous portion of the utility function above \bar{a} . Therefore, $(V(a), W(a), \hat{c}(a))$ is an equilibrium for the homogenous system (11) over $[\bar{a}, \infty)$. However, Theorem 5 implies that $\hat{c}(a) \geq ra$ for all $a \geq \bar{a}$, which is a contradiction.

Lemma 14 then implies that there cannot be strict saving at any asset level.

Now, under Assumption 3, we show that there is at most one continuous equilibrium with strict dissavings. Indeed, Lemma 21 for the homogenous system (12) (with $\sigma = \tilde{\sigma}$) shows that there exists a unique solution to (6) with dissavings determined over a neighborhood $[\underline{a}, \underline{a} + \epsilon]$ to the right of \underline{a} . Outside this neighborhood the solution to (6) with dissavings is uniquely determined since the ODE (18) is regular over $E = \{(V, W, a) : (\rho + \lambda)V - \lambda W - U_1(ra) > 0\}$. \square

E.2 Supporting Results for the Proof of Theorem 3

Lemma 12. *If the sequence $\{a_n\}$ constructed in Theorem 3 has an infinite number of elements then*

$$\lim_{n \rightarrow \infty} a_n = +\infty.$$

Proof. The result is shown by contradiction. Assume that the sequence is infinite and is bounded above. By construction $\{a_n\}_{n=0}^{\infty}$ is strictly increasing, thus the sequence converges to some a^∞ . We assume by contradiction that a^∞ is finite. By construction, $V_n(a_n) = \bar{V}(a_n)$, $W(a_n) = \bar{W}(a_n)$ and $V_n(a_{n+1}) = \bar{V}(a_{n+1})$ and $V_n(a) > \bar{V}(a)$ for $a \in (a_n, a_{n+1})$. We can then apply Lemma 10 to show that $0 \leq V'_n(a) \leq U'_1(ra)$ and $0 \leq W'_n(a) \leq U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra)$. By the Mean Value Theorem, there exists $a_n^* \in [a_n, a_{n+1}]$ such that

$$V'_n(a_n^*) = \frac{V_n(a_{n+1}) - V_n(a_n)}{a_{n+1} - a_n} = \frac{\bar{V}(a_{n+1}) - \bar{V}(a_n)}{a_{n+1} - a_n}.$$

Since $\{a_n\}$ converges to a^∞ ,

$$\lim_{n \rightarrow \infty} V'_n(a_n^*) = \bar{V}'(a^\infty). \quad (35)$$

On the other hand $V'_n(a_n^*) = R_l(a_n^*, V_n(a_n^*), W_n(a_n^*))$. Since $\bar{V}(a_n) \leq V_n(a_n^*) \leq \bar{V}(a_{n+1})$ and $\bar{W}(a_n) \leq W_n(a_n^*) \leq \bar{W}(a_n^*) + (U'_0(ra_n) + \frac{\rho}{\lambda} U'_1(ra_n))(a_{n+1} - a_n)$ and R_l is continuous, by the Squeeze Principle,

$$\begin{aligned} \lim_{n \rightarrow \infty} V'_n(a_n^*) &= \lim_{i \rightarrow \infty} R_l(a_n^*, V_n(a_n^*), W_n(a_n^*)) \\ &= \lim_{i \rightarrow \infty} R_l(a^\infty, \bar{V}(a^\infty), \bar{W}(a^\infty)) \\ &= U'_1(ra^\infty). \end{aligned} \quad (36)$$

The desired contradiction follows from the fact that (35) and (36) cannot happen at the same time given that $\bar{V}'(a^\infty) < U'_1(ra^\infty)$ by condition (22) at $a = a^\infty$. \square

Lemma 13. *If at $a = \tilde{a}$, $\hat{c}(a) \leq ra$ in some neighborhood to the left of \tilde{a} and $\hat{c}(\tilde{a}) \geq r\tilde{a}$ then $\hat{c}(\tilde{a}) = r\tilde{a}$ and $V(\tilde{a}) \leq \bar{V}(\tilde{a})$.*

Similarly, if $\hat{c}(a) \geq ra$ in some neighborhood to the right of \tilde{a} and $\hat{c}(\tilde{a}) \leq r\tilde{a}$ then $\hat{c}(\tilde{a}) = r\tilde{a}$ and $V(\tilde{a}) \leq \bar{V}(\tilde{a})$.

Proof. Case 1: $\hat{c}(a) \leq ra$ in some neighborhood to the left of \tilde{a} and $\hat{c}(\tilde{a}) \geq r\tilde{a}$.

By the definition of an equilibrium, there exists a solution $\{\hat{a}_t\}$ to the ODE: $\frac{d\hat{a}_t}{dt} = r\hat{a}_t - \hat{c}(\hat{a}_t)$ with the initial condition $\hat{a}_0 = \tilde{a}$. Since $\hat{c}(a) \leq ra$ in the left neighborhood of \tilde{a} , we have $\hat{a}_t \geq \tilde{a}$ for all $t \geq 0$. Therefore $r\tilde{a} - \hat{c}(\tilde{a}) = \lim_{t \rightarrow 0} \frac{\hat{a}_t - \tilde{a}}{t} \geq 0$. So $\hat{c}(\tilde{a}) = r\tilde{a}$.

Now we show that $V(\tilde{a}) \leq \bar{V}(\tilde{a})$. Indeed, if there exists a sequence $\{a_n\}$ such that $a_n > \tilde{a}$ and $a_n \downarrow \tilde{a}$ and $\hat{c}(a_n) > ra_n$, then $\hat{a}_t \leq a_n$ for all t, n so $\hat{a}_t = \tilde{a}$ for all $t \geq 0$. Therefore, by the equilibrium definition in Subsection 3.1, $W(\tilde{a}) = \bar{W}(\tilde{a})$ and hence, $V(\tilde{a}) = \bar{V}(\tilde{a})$, which is $\leq \bar{V}(\tilde{a})$ as desired.

Otherwise, $\hat{c}(a) \leq ra$ in some neighborhood to the right of \tilde{a} , therefore \hat{a}_t is non-decreasing.

In this case, if $\hat{a}_t = \tilde{a}$ for all $t \geq 0$, then $V(\tilde{a}) = \bar{V}(\tilde{a})$.

Otherwise, there exists T such that $\hat{a}_T > \tilde{a}$. Therefore, (V, W, \hat{c}) are continuous over $[\tilde{a}, \hat{a}_T]$. Assume by contradiction that $V(\tilde{a}) > \bar{V}(\tilde{a})$, then $W(\tilde{a}) > \bar{W}(\tilde{a})$. As $a \downarrow \tilde{a}$, we have $W'(a) = \frac{\rho W(a) - U'(\hat{c}(a))}{ra - \hat{c}(a)} \rightarrow +\infty$ since $\hat{c}(a) - ra \rightarrow 0$ when $a \downarrow \tilde{a}$.

Now

$$\begin{aligned} & \frac{d}{da} ((\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)) \\ &= (\rho + \lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\ &= (\rho + \lambda)U_1'(\hat{c}(a)) - \lambda W'(a) - rU_1'(ra) \end{aligned}$$

As $a \rightarrow \tilde{a}$, $\hat{c}(a) \rightarrow r\tilde{a}$, therefore at $a = \tilde{a}$

$$\frac{d}{da} ((\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)) = -\infty,$$

which contradicts the property that $(\rho + \lambda)V(a) - \lambda W(a) - U_1(ra) \geq 0$ for $a \geq \tilde{a}$ and $(\rho + \lambda)V(\tilde{a}) - \lambda W(\tilde{a}) - U_1(r\tilde{a}) = 0$.

Case 2: $\hat{c}(a) \geq ra$ in some neighborhood to the right of \tilde{a} and $\hat{c}(\tilde{a}) \leq r\tilde{a}$. Similar to the proof of Case 1, we have $\hat{c}(\tilde{a}) = r\tilde{a}$ and $V(\tilde{a}) \leq \bar{V}(\tilde{a})$. \square

Lemma 14. Assume that $\beta(c) \leq \hat{\beta}$ for all c . There cannot be strict saving at any asset level if for any a , there exists $a' > a$ such that $\hat{c}(a') \geq ra'$.

Proof. Suppose by contradiction that, at $a = \check{a}$, $\hat{c}(\check{a}) < r\check{a}$. Let $\tilde{a} = \sup\{a > \check{a} : \hat{c}(b) < rb \text{ for } b \in [\check{a}, a]\}$. Let $\{a_t\}_{t=0}^{\infty}$ denote the solution of the ODE $\dot{a}_t = ra_t - \hat{c}(a_t)$ with the initial condition $a_0 = \check{a}$ (a_t exists by the equilibrium definition in Subsection 3.1). Since $\frac{da(0)}{dt} = r\check{a} - \hat{c}(\check{a}) > 0$, $a_t > \check{a}$ for $t > 0$. Also by the equilibrium definition (W, \hat{c}) is continuous on path a_t , thus in a neighborhood to the right of \check{a} . Therefore, $\hat{c}(a) > ra$ in the neighborhood, which implies $\tilde{a} > \check{a}$. Because there exists $a' > \check{a}$ such that $\hat{c}(a') \geq ra'$, $\tilde{a} < \infty$.

Now, going back to the main proof. At \tilde{a} , $\hat{c}(\tilde{a}) \geq r\tilde{a}$, otherwise the solution to the ODE $\dot{a}_t = ra_t - \hat{c}(a_t)$ with the initial condition $a_0 = \tilde{a}$ admits a solution that extends to a neighborhood the right of \tilde{a} , and by continuity, in this neighborhood $\hat{c}(a) < ra$ which contradicts the definition of \tilde{a} . Consequently, by Lemma 13, we must have $\hat{c}(\tilde{a}) = r\tilde{a}$. There are two cases:

Case 1: $\hat{c}(a_n) > ra_n$ for some sequence $a_n \downarrow \tilde{a}$ in a neighborhood to the right of \tilde{a} then $V(\tilde{a}) = \bar{V}(\tilde{a})$, $W(\tilde{a}) = \bar{W}(\tilde{a})$. Case 2: $\hat{c}(a) \leq ra$ in a neighborhood to the right of \tilde{a} , then by Lemma 13, $V(\tilde{a}) \leq \bar{V}(\tilde{a})$. In either case, we have $V(\tilde{a}) \leq \bar{V}(\tilde{a})$.

For $a < \tilde{a}$,

$$V'(a) = U_1'(\hat{c}(a)) \geq U_1'(ra) \geq \bar{V}'(a),$$

since $\beta < \hat{\beta}$. Therefore, $V(a) \leq \bar{V}(a)$ for all $a < \tilde{a}$. Consequently $W(a) < \bar{W}(a)$ for $a < \tilde{a}$, since $\hat{c}(a) < ra$. Therefore, for $a < \tilde{a}$,

$$W'(a) = \frac{\rho W(a) - U_0(\hat{c}(a))}{ra - \hat{c}(a)} < \frac{U_0(ra) - U_0(\hat{c}(a))}{ra - \hat{c}(a)} < U_0'(ra).$$

Now

$$\begin{aligned} & \frac{d}{da} ((\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)) \\ &= (\rho + \lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\ &> (\rho + \lambda)U_1'(\hat{c}(a)) - \lambda U_0'(ra) - rU_1'(ra) \end{aligned}$$

Now as $a \rightarrow \tilde{a}$, $\hat{c}(a) \rightarrow r\tilde{a}$, therefore at $a = \tilde{a}$

$$\frac{d}{da} ((\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)) \geq (\rho + \lambda - r)U_1'(r\tilde{a}) - \lambda U_0'(r\tilde{a}),$$

If $r > \rho$, since $\beta \leq \hat{\beta} = \frac{\rho}{r} \frac{\rho + \lambda - r}{\lambda} < \frac{\rho + \lambda - r}{\lambda}$, we have

$$(\rho + \lambda - r)U_1'(r\tilde{a}) - \lambda U_0'(r\tilde{a}) > 0.$$

If $r \leq \rho$,

$$(\rho + \lambda - r)U_1'(r\tilde{a}) - \lambda U_0'(r\tilde{a}) \geq \lambda(U_1'(r\tilde{a}) - U_0'(r\tilde{a})) > 0.$$

Either way, we have

$$\left. \frac{d}{da} ((\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)) \right|_{a=\tilde{a}} > 0.$$

This is a contradiction since $(\rho + \lambda)V(a) - \lambda W(a) - U_1(ra) > 0$ for $a > \tilde{a}$ and $(\rho + \lambda)V(\tilde{a}) - \lambda W(\tilde{a}) - U_1(r\tilde{a}) = 0$. \square

F Proofs for Saving Equilibria

F.1 Proof of Theorem 4

Proof of Theorem 4 (Existence). We prove the existence of equilibrium by construction.

We define an wealth level a_u , and the value functions (V_u, W_u) over $[a_u, \infty)$ satisfying the differential equations (6) as following. In Theorem 2, we show that for $a \geq a_u$, $(V_u(a), W_u(a)) = \left(\bar{v} \frac{a^{1-\sigma}}{1-\sigma}, \bar{w} \frac{a^{1-\sigma}}{1-\sigma} \right)$ and $\hat{c}_u(a) = \psi a$ satisfy the differential equations (6) over $[a_u, \infty)$ where $a_u = \frac{\bar{c}}{\psi}$. It is immediate that $\hat{c}'_u(a) = \psi > 0$. Moreover, because $\bar{\beta} > \hat{\beta}$, $\psi < r$, so $\hat{c}_u(a) < ra$.

Having determined the value and policy functions at and beyond a_u , we construct the value and policy functions below a_u . Noticing that the initial values $(a_u, V_u(a_u), W_u(a_u)) \in E$, a solution (V_d, W_d) to the ODE (19) with the initial condition

$$(V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))$$

exists and is unique locally over an interval $(a_u - \epsilon, a_u + \epsilon)$.³⁹ Let $(\hat{a}, a_u + \epsilon)$ denote the (left) maximal interval of existence for this solution. We will show that $\hat{a} < \frac{\bar{c}}{r}$.

If $\hat{a} = 0$, this is immediate. If $\hat{a} > 0$, by Lemma 15, $V'_d(a), W'_d(a) > 0$ for all $a > \hat{a}$. So the limits $\lim_{a \downarrow \hat{a}} V_d(a) = V_d(\hat{a})$ and $\lim_{a \downarrow \hat{a}} W_d(a) = W_d(\hat{a})$ exist.⁴⁰ By Hartman (2002,

³⁹Because of the uniqueness of the solution, $(V_d(a), W_d(a)) = (V_u(a), W_u(a))$ for all $a \in [a_u, a_u + \epsilon)$.

⁴⁰We also show that $\lim_{a \downarrow \hat{a}} W_d(a) > -\infty$ and $\lim_{a \downarrow \hat{a}} V_d(a) > -\infty$. This is immediate if U_1 is bounded

Theorem 3.1), $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$. Therefore $\hat{c}_d(\hat{a}) = r\hat{a}$. As shown in Lemma 15, $V_d''(a) < 0$. This implies $\hat{c}'_d(a) = \frac{V_d''(a)}{U_1''(\hat{c}_d(a))} > 0$. Thus,

$$r\hat{a} = \hat{c}_d(\hat{a}) < \hat{c}_d(a_u) = \bar{c}.$$

So $\hat{a} < \frac{\bar{c}}{r}$.

Given the value and policy functions (V_u, W_u, \hat{c}_u) and (V_d, W_d, \hat{c}_d) , for $\underline{a} > \hat{a}$, we define the value and policy functions (V, W, \hat{c}) over $[\underline{a}, \infty)$ as follows

$$(V, W, \hat{c}) = \begin{cases} (V_d, W_d, \hat{c}_d) & \text{if } \underline{a} \leq a < a_u \\ (V_u, W_u, \hat{c}_u) & \text{if } a \geq a_u. \end{cases}$$

As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for an equilibrium. In addition, $\hat{c}(a) < ra$ and $\hat{c}'(a) > 0$.

If $\hat{a} > 0$, Lemma 30 below shows that $(V(\hat{a}), W(\hat{a})) = (\bar{V}(\hat{a}), \bar{W}(\hat{a}))$ and $\hat{c}(\hat{a}) = r\hat{a}$. In addition, when $\beta(c)$ is increasing, Lemma 31 shows that $V(a) > \bar{V}(a)$ for all $a > \hat{a}$. Therefore, $V'(\hat{a}) = U_1'(r\hat{a}) \geq \bar{V}'(\hat{a})$. By Lemma 6, $\beta(r\hat{a}) \leq \hat{\beta}$. \square

Proof of Theorem 4 (Uniqueness). As shown in Lemma 18, $\hat{c}(a) \leq ra$ for all $a \geq \underline{a}$. Theorem 6 shows that the linear equilibrium is the unique equilibrium above $\frac{\bar{c}}{v}$. Let $a^* = \inf\{a : \hat{c}(b) < rb \forall b \geq a\}$. As we show in the existence proof, $V(a) > \bar{V}(a)$ for all $a \geq a^*$. If $a^* > \underline{a}$, then there exists a sequence $\{a_n\}$ converges to a^* to the left such that $\hat{c}(a_n) = ra_n$. Therefore, by Lemma 13, $V(a_n) \leq \bar{V}(a_n)$. So $V(a^*) = \lim V(a_n) \leq \bar{V}(a^*)$, which contradicts the earlier result that $V(a^*) > \bar{V}(a^*)$. Therefore $\hat{c}(a) < ra$ for all $a \geq \underline{a}$, the solution then is uniquely determined as a global solution to the ODE (19) with the initial condition at a_u given by the linear solution. \square

F.2 Supporting Results for Proof of Theorem 4

Lemma 15. Assume $\rho < r$. Consider a solution (V, W) to ODE (19) defined over $(\hat{a}, a_u]$ with the initial condition $(V(a_u), W(a_u)) = (V_u(a_u), W_u(a_u))$ with a_u, V_u, W_u defined in Subsection F.1. Then for all $a < a_u$

- 1) $(\rho + \lambda - r)V'(a) - \lambda W'(a) < 0$ and $W'(a) > 0$
- 2) $V''(a) < 0$
- 3) $V'(a) > W'(a)$.

from below, and consequently U_0 is bounded from below by some \underline{u} , because $\beta(c) \leq 1$. Because $W_d'(a) > 0$ and $\hat{c}_d(a) < ra$ for $a > \hat{a}$, (6b) implies that $W_d(a) > \frac{1}{\rho}\underline{u}$ for $a > \hat{a}$. If U_1 is unbounded from below, we make the additional technical assumption that $\underline{\sigma} = \inf_c \sigma(U_1, c) > 1 - \frac{\rho}{r}$. We have

$$\begin{aligned} \hat{c}'_d(a) &= \frac{(\rho + \lambda - r)V_d'(a) - \lambda W_d'(a)}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} < \frac{(\rho - r)V_d'(a)}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} \\ &= \frac{(\rho - r)U_1'(\hat{c}_d(a))}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} < \frac{r - \rho}{\underline{\sigma}} \frac{\hat{c}_d(a)}{ra - \hat{c}_d(a)}, \end{aligned}$$

because $W_d'(a) < V_d'(a)$ as shown in Lemma 15. Therefore $\hat{c}_d(a) > c^*(a) > 0$ where $c^*(a)$ is the solution to the ODE $c^{*'}(a) = \frac{r - \rho}{\underline{\sigma}} \frac{c^*(a)}{ra - c^*(a)}$ and $c^*(a_u) = \hat{c}_d(a_u) < ra_u$ ($c^*(a)$ can be solved in closed form). Therefore,

$\lim_{a \downarrow \hat{a}} W_d(a) > \frac{1}{\rho} U_0(c^*(\hat{a})) > -\infty$ for all $a \geq \hat{a}$. Finally, $\lim_{a \downarrow \hat{a}} V_d(a) > \frac{\frac{1}{\rho} U_0(c^*(\hat{a})) + U_1(\hat{a})}{\rho + \lambda} > -\infty$.

Proof. We prove this lemma in two steps. Step 1: If properties 1),2) and 3) hold in a neighborhood to the left of a_u then they hold for all $a < \hat{a}$. Step 2: Verify that properties 1),2) and 3) hold in a neighborhood to the left of a_u separately under Assumption 2 with $\bar{\beta} = 1$ or $\bar{\beta} < 1$.

Step 1: Assume that properties 1),2) and 3) hold in a neighborhood to the left of a_u , We show that 1),2), and 3) hold for all $a < a_u$.

We prove 1) separately for two cases: Case 1: $\rho + \lambda - r > 0$ and Case 2: $\rho + \lambda - r \leq 0$.

Case 1: By construction, $V'(a) = U_1'(\hat{c}(a)) > 0$. Therefore if $(\rho + \lambda - r) V'(a) - \lambda W'(a) < 0$, $W'(a) > 0$. We just need to show the first inequality.

We prove this inequality using Lemma 7 (Variation 1). Condition 1) of Lemma 7 (at a_u) is satisfied by assumption. We just need to verify Condition 2) of Lemma 7, i.e. if there exists $\check{a} < a_u$ such that

$$(\rho + \lambda - r) V'(\check{a}) - \lambda W'(\check{a}) = 0. \quad (37)$$

then

$$(\rho + \lambda - r) V''(\check{a}) > \lambda W''(\check{a}).$$

By Lemma 5,

$$(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a) (ra - \hat{c}(a)). \quad (38)$$

At $a = \check{a}$, because of (37), and $r\check{a} > \hat{c}(\check{a})$, $V''(\check{a}) = 0$.

Differentiating the second equation, (6b), in system (6), and using (21), we have

$$\begin{aligned} \rho W'(a) &= \frac{U_0'(\hat{c}(a))}{U_1''(\hat{c}(a))} V''(a) \\ &+ W''(a) (ra - \hat{c}(a)) + W'(a) \left(r - \frac{1}{U_1''(\hat{c}(a))} V''(a) \right). \end{aligned} \quad (39)$$

At $a = \check{a}$, using the previous result that $V''(\check{a}) = 0$, and rearranging, we arrive at

$$W''(\check{a}) (r\check{a} - \hat{c}(\check{a})) = (\rho - r) W'(\check{a}).$$

Because, $W'(\check{a}) = \frac{(\rho + \lambda - r) V'(\check{a})}{\lambda} > 0$ and $\rho - r < 0$, the right hand side is strictly negative. Moreover $ra - \hat{c}(a) > 0$, therefore $W''(\check{a}) < 0$. Thus,

$$W''(\check{a}) < 0 = V''(\check{a}),$$

i.e. we have verified the second condition in Lemma 7. Given that the two conditions of Lemma 7 are satisfied, this lemma implies the first property.

Case 2: Because $\rho + \lambda \leq r$ and $V'(a) > 0$, $(\rho + \lambda - r) V'(a) - \lambda W'(a) < 0$ if $W'(a) > 0$. Therefore we just need to show the last inequality. Again we prove this inequality using Lemma 7 (Variation 1). Condition 1) of Lemma 7 (at a_u) is shown in the proof of Theorem 4. We now verify Condition 2). If there exists $\check{a} < a_u$ such that $W'(\check{a}) = 0$ we show that $W''(\check{a}) < 0$. From equation (38) at \check{a} , $(\rho + \lambda - r) V'(\check{a}) = V''(\hat{a}) (r\check{a} - \hat{c}(\check{a}))$. This implies $V''(\check{a}) < 0$. From (39), since $W'(\check{a}) = 0$,

$$0 = \frac{U_0'(\hat{c}(\check{a}))}{U_1''(\hat{c}(\check{a}))} V''(\check{a}) + W''(\check{a}) (r\check{a} - \hat{c}(\check{a})).$$

Therefore $W''(\check{a}) < 0$. Given that the two conditions of Lemma 7 are satisfied, this lemma implies $W'(a) > 0$ for all a .

The second property immediately follows using (38) and $ra - \hat{c}(a) > 0$.

We also prove the third property similarly by using Lemma 7. Condition 1) in Lemma 7 is satisfied. We now verify that condition 2) is also satisfied. Indeed, if there exists $\check{a} < a_u$ such that $W'(\check{a}) = V'(\check{a})$. By (38), at $a = \check{a}$,

$$V''(\check{a}) = \frac{(\rho - r)V'(\check{a})}{r\check{a} - \hat{c}(\check{a})} < 0.$$

Again by equation (39),

$$\begin{aligned} (r\check{a} - \hat{c}(\check{a})) W''(\check{a}) &= (\rho - r) W'(\check{a}) - \underbrace{(U'_0(\hat{c}(\check{a})) - W'(\check{a})) \frac{1}{U'_1(\hat{c}(\check{a}))}}_{< 0} V''(\check{a}) \\ &> (\rho - r) W'(\check{a}) = (\rho - r) V'(\check{a}), \end{aligned}$$

The second line comes from the assumption that $W'(\check{a}) = V'(\check{a}) = U'_1(\hat{c}(\check{a})) > U'_0(\hat{c}(\check{a}))$ (by properties 1) and 2), $\hat{c}'(a) > 0$ therefore $\hat{c}(\check{a}) < \hat{c}(a_u) = \bar{c}$ and that $U'_1(c) > U'_0(c)$ for all $c < \bar{c}$ by Assumption 1). So

$$W''(\check{a}) > \frac{(\rho - r) V'(\check{a})}{r\check{a} - \hat{c}(\check{a})} = V''(\check{a}).$$

So by Lemma 7, we obtain the third property.

Step 2: We show that properties 1), 2) and 3) hold in a neighborhood to the left of a_u . We treat the two cases associated with Assumption 2 with $\bar{\beta} < 1$ or $\bar{\beta} = 1$ separately.

Under Assumption 2 with $\bar{\beta} < 1$, given the closed form solution given in Appendix D, we show that properties 1), 2), 3) are satisfied at a_u in Lemma 16 below. By continuity, properties 1), 2), 3) hold in a neighborhood to the left of a_u .

Under Assumption 2 with $\bar{\beta} = 1$. It is easy to verify that properties 1) and 2) are also satisfied from the close form solution. By continuity, properties 1) and 2) hold in a neighborhood to the left of a_u . Property 3) is not satisfied at a_u because $V'_u(a_u) = W'_u(a_u)$ so $V'(a_u) = W'(a_u)$. WOLG, by defining \bar{c} and a_u such that $\beta(c) < 1$, $\beta(c) = 1$ for all $c \geq \bar{c} = \hat{c}(a_u)$ (above a_u , V and W differ only by a constant and both are the value functions of a time-consistent consumer with utility U_1 up to some constant; properties 1) and 2) also hold because of this property). We show that $V'(a) < W'(a)$ in some neighborhood to the left of a_u .

Indeed, consider the solution (V_ϵ, W_ϵ) to ODE (19) with the initial condition

$$(V_\epsilon(a_u), W_\epsilon(a_u)) = \left(V_u(a_u) - \frac{\epsilon}{\rho + \lambda}, W_u(a_u) - \frac{\epsilon}{\lambda} \right).$$

Because $(a_u, V_u(a_u), W_u(a_u)) \in E$, by Hartman (2002, Theorem 3.2), there exists $\omega > 0$ such that (V_ϵ, W_ϵ) are defined over, $[a_u - \omega, a_u]$ and $(V_\epsilon, W_\epsilon) \rightarrow (V, W)$ uniformly over $[a_u - \omega, a_u]$ as $\epsilon \rightarrow 0$.

It is easy to verify that $V'_\epsilon(a_u) > W'_\epsilon(a_u)$. Indeed, from the initial conditions, we have

$\hat{c}_\epsilon(a_u) = \hat{c}_u(a_u)$, and $V'_u(a_u) = U'_1(\hat{c}_u(a_u)) = U'_1(\hat{c}_\epsilon(a_u)) = V'_\epsilon(a_u)$,

$$\begin{aligned} W'_\epsilon(a_u) &= \frac{\rho(W_u(a_u) - \frac{\epsilon}{\lambda}) - U_0(\hat{c}_\epsilon(a_u))}{ra_u - \hat{c}_\epsilon(a_u)} < \frac{\rho W_u(a_u) - U_0(\hat{c}_u(a_u))}{ra_u - \hat{c}_u(a_u)} \\ &= W'_u(a_u) = V'_u(a_u) = V'_\epsilon(a_u). \end{aligned}$$

In addition, when ϵ sufficiently small, we also have properties 1) and 2) holds for V_ϵ, W_ϵ at a_u . Therefore, following the proofs in Step 1 above, we can show that properties 1),2),3) hold for all $a \in [a_u - \omega, a_u]$ for (V_ϵ, W_ϵ) . In particular, $V'_\epsilon(a) > W'_\epsilon(a)$ for all $a \in [a_u - \omega, a_u]$.

Now as $\epsilon \rightarrow 0$, $(V_\epsilon, W_\epsilon) \rightarrow (V, W)$. So $V'(a) \geq W'(a)$ for all $a \in [a_u - \omega, a_u]$. We show by contradiction that $V'(a) > W'(a)$ for all $a \in (a_u - \omega, a_u)$. Assume to the contrary that $V'(\check{a}) = W'(\check{a})$ for some $\check{a} < a_u$. As in Step 1, this implies that $V''(\check{a}) < W''(\check{a})$ strictly, because $\beta(\hat{c}(\check{a})) < 1$ given that $\hat{c}(\check{a}) < \hat{c}(a_u) = \bar{c}$. Therefore in the right neighborhood of \check{a} , $V'(a) < W'(a)$, which contradicts the earlier result that $V'(a) \geq W'(a)$. Thus, $V'(a) > W'(a)$ for all $a \in (a_u - \omega, a_u)$. \square

Lemma 16. *The linear equilibria in Theorem 2 with $\hat{\beta} < \bar{\beta} < 1$ satisfies, for all $a > 0$*

- 1) $(\lambda + \rho - r)V'(a) < \lambda W'(a)$
- 2) $V''(a) > 0$
- 3) $W'(a) < V'(a)$

Proof. As shown in Theorem 2, because $\bar{\beta} > \hat{\beta}$, $\hat{c}(a) < ra$. By Lemma 5,

$$(\rho + \lambda - r)V'(a) - \lambda W'(a) = V''(a)(ra - \hat{c}(a)).$$

Because $V''(a) = -\bar{\nu}\sigma a^{-\sigma-1} < 0$, and $\hat{c}(a) < ra$,

$$(\lambda + \rho - r)V'(a) - \lambda W'(a) < 0.$$

The second inequality 2) is immediate because $V''(a) = -\bar{\nu}\sigma a^{-\sigma-1} < 0$.

The last inequality 3) is equivalent to $\frac{\bar{\beta}\psi}{\Delta + (1-\sigma)\psi} < 1$. After algebra manipulation, this inequality holds if and only if $(\sigma + \bar{\beta} - 1)\psi < \Delta$. This obviously holds if $\sigma + \bar{\beta} \leq 1$, because $\Delta, \psi > 0$. If $\sigma + \bar{\beta} > 1$, we show that $\psi < \frac{\Delta}{\sigma + \bar{\beta} - 1}$. Indeed, $P\left(\frac{\Delta}{\sigma + \bar{\beta} - 1}\right) = \frac{\bar{\beta}(1-\bar{\beta})}{(\sigma + \bar{\beta} - 1)^2} > 0$. Therefore, $\psi < \frac{\Delta}{\sigma + \bar{\beta} - 1}$. \square

Lemma 17. *Assume Assumption 2 and that $\beta(c) > \hat{\beta}$ for all $c > 0$ and β is non-decreasing. In any equilibrium, for any $a > \underline{a}$, there exists $a' > a$ such that $\hat{c}(a') \leq ra'$.*

Proof. Assume that there exists a^* such that $\hat{c}(a) > ra$ for all $a \geq a^*$.

Let $\tilde{a} = \inf\{a^* > a > \underline{a} : \hat{c}(b) > rb \text{ for } b \in [a, a^*]\}$. Follow the argument in the beginning of Lemma 14, we have $\tilde{a} < a^*$.

If $\tilde{a} = \underline{a}$, then by the limit on assets, $\hat{c}(\tilde{a}) = r\tilde{a}$. If $\tilde{a} > \underline{a}$, by Lemma 13, $\hat{c}(\tilde{a}) = r\tilde{a}$ (similar to the argument in Lemma 14, we must have $\hat{c}(\tilde{a}) \leq r\tilde{a}$). Either way we have $\hat{c}(\tilde{a}) = r\tilde{a}$. Also by Lemma 13, $V(\tilde{a}) \leq \bar{V}(\tilde{a})$. By the definition of \tilde{a} , $\hat{c}(a) > ra$ for all $a > \tilde{a}$. Since $V'(a) = U'_1(\hat{c}(a)) < U'_1(ra) < \bar{V}'(a)$, we have $V(a) < \bar{V}(a)$ for all $a > \tilde{a}$.

Now for $\check{a} = \max(\tilde{a}, \frac{\bar{c}}{r})$, by choosing $\epsilon > 0$ sufficiently small, we have $V(\check{a}) < \bar{V}(\check{a}) - \epsilon(\check{a})^{1-\sigma}$ and $V'(a) < \bar{V}'(a) - \epsilon(\check{a})^{-\sigma}$ for all $a \geq \check{a}$. Therefore, for all $a > \check{a}$

$$\frac{V(a)}{a^{1-\sigma}} < \bar{v} - \epsilon.$$

As in the proof of Lemma 28, using the phase diagram, any solution to the differential system (12), with the initial condition $c(x^*) > r$ and $w_0(c(x^*)) - \frac{\rho+\lambda}{\lambda}\epsilon > w(c(x^*))$, reaches a point where $c(x^{**}) = r$, at some $x^{**} > x^*$. This contradicts the assumption that $\hat{c}(a) > ra$ for all $a \geq a^*$. \square

Lemma 18. *Assume Assumption 2 and that $\beta(c) > \hat{\beta}$ for all $c > 0$ and β is non-decreasing. In any equilibrium, there cannot be strict dissaving at any asset level.*

Proof. Suppose by contradiction that there is strict dissaving at $a = \check{a} > \underline{a}$. Let $\tilde{a} = \inf\{\check{a} > a > \underline{a} : \hat{c}(b) > rb \text{ for } b \in [a, \check{a}]\}$. If $\tilde{a} = \underline{a}$, then by borrowing constraint, $\hat{c}(\tilde{a}) = r\tilde{a}$. If $\tilde{a} > \underline{a}$, following the arguments in the proof of Lemma 14, we have $\tilde{a} < \check{a}$ and $\hat{c}(\tilde{a}) \leq r\tilde{a}$. By Lemma 13, $V(\tilde{a}) \leq \bar{V}(\tilde{a})$.

Let $\tilde{\tilde{a}} = \sup\{a > \check{a} : \hat{c}(b) > rb \text{ for } b \in [\check{a}, a]\}$. By Lemma 17, $\tilde{\tilde{a}} < \infty$. Since for $a \in (\tilde{a}, \tilde{\tilde{a}})$,

$$V'(a) = U_1'(\hat{c}(a)) < U_1'(ra) < \bar{V}'(a)$$

we have $V(a) < \bar{V}(a)$ for all $\tilde{a} < a < \tilde{\tilde{a}}$.

Also by Lemma 17, there exists $a^* > \tilde{\tilde{a}}$ and $a^* > \frac{\bar{c}}{v}$ such that $\hat{c}(a^*) \leq ra^*$. Above a^* , $\hat{c}(a) > \bar{c}$, so the equilibrium stays in homogenous portion of the utility function starting above a^* in addition starting from any asset level $a \geq a^*$, asset never goes below a^* since $\hat{c}(a^*) \leq ra^*$. Therefore, by Theorem 6, $\hat{c}(a) = \psi a$ for all $a \geq a^*$.

Let $a^0 = \inf\{a \in [\underline{a}, a^*] : \hat{c}(b) < rb \text{ for all } b \in [a, a^*]\}$. By definition, $a^0 \geq \tilde{\tilde{a}}$. As shown in the existence proof of Theorem 4, $V(a) > \bar{V}(a)$ for all $a \geq a^0$.

We show by contradiction that $a^0 = \tilde{\tilde{a}}$. Assuming that this is not true, i.e. $a^0 > \tilde{\tilde{a}}$. There are two cases:

Case 1: There exists a sequence $\{a_n\}_{n=1}^\infty \in (\tilde{\tilde{a}}, a^0)$ such that $a_n \uparrow a^0$ and $\hat{c}(a_n) > ra_n$. Applying the arguments for $a_n = \check{a}$ above, we must have $V(a_n) < \bar{V}(a_n)$. Therefore $V(a^0) = \lim_{n \rightarrow \infty} V(a_n) \leq \bar{V}(a^0)$ which contradicts the property that $V(a^0) > \bar{V}(a^0)$ shown earlier.

Case 2: There exists $\epsilon > 0$ such that $\hat{c}(a) \leq ra$ for all $a \in (a^0 - \epsilon, a^0)$. Then $\hat{c}(a^0) \leq ra^0$. Since $V(a^0) > \bar{V}(a^0)$, $\hat{c}(a^0) < ra^0$. Now, by the definition of a^0 , there exists a sequence $\{a_n\}_{n=1}^\infty \in (a^0 - \epsilon, a^0)$ such that $a_n \rightarrow a^0$ and $\hat{c}(a_n) = ra_n$. By Lemma 13, $V(a_n) \leq \bar{V}(a_n)$. Therefore $V(a^0) = \lim_{n \rightarrow \infty} V(a_n) \leq \lim_{n \rightarrow \infty} \bar{V}(a_n) = \bar{V}(a^0)$. This contradicts $V(a^0) > \bar{V}(a^0)$.

So either in Case 1 or Case 2, we have a contradiction. Therefore $a^0 = \tilde{\tilde{a}}$. But as we show above, $V(a) < \bar{V}(a)$ for $a < \tilde{\tilde{a}} = a^0$ and $V(a^0) > \bar{V}(a^0)$. This contradicts the continuity of V .

Thus we have shown by contradiction that there cannot be strict dissaving at any asset level.

Following the arguments above, we can show that there cannot be an asset level without strict saving. Therefore $\hat{c}(a) < ra$ for all $a \geq \underline{a}$. By Theorem 6, $\hat{c}(a) = \psi a$ for all $a \geq \max\{\underline{a}, \frac{\bar{c}}{v}\}$. Since the equilibrium features strict dissaving for all $\underline{a} \leq a \leq \max\{\underline{a}, \frac{\bar{c}}{v}\}$,

the equilibrium is uniquely determined given the boundary condition provided by the linear equilibrium above $\max\{\underline{a}, \frac{\bar{c}}{v}\}$. \square

G Proofs for Subsection 4.3

G.1 Phase Diagram and Loci

The proofs for the results in Subsection 4.3 rely on the phase diagram. In this subsection, we analyze the preliminary properties of the phase diagram, in particular, the behaviors of the loci $c' = 0$, $w' = 0$, and $v = \bar{v}$.

Loci $c'(x) = 0$ The loci of points where $c'(x) = 0$ is given by

$$\lambda\rho w = (\rho + \lambda)u'(c)(r - c) + \lambda\beta u(c) - (1 - \sigma)(r - c)(u(c) + u'(c)(r - c))$$

or

$$w = w_1(c) = \frac{(\rho + \lambda)u'(c)(r - c) + \lambda\beta u(c) - (1 - \sigma)(r - c)(u(c) + u'(c)(r - c))}{\lambda\rho}$$

Lemma 19. *In the relevant regions, $w \geq 0$ for $\sigma < 1$ and $w < 0$ for $\sigma > 1$, w_1 has U-shape.*

Proof. The derivative of $w_1(c)$ is equal to

$$u''(c) \left(A_0 + A_1c + A_2c^2 \right)$$

where

$$\begin{aligned} A_0 &= (\rho - (1 - \sigma)r + \lambda)r \\ A_1 &= \frac{(\rho - (1 - \sigma)r + \lambda)(1 - \sigma) - \lambda\beta + r(1 - \sigma)\sigma}{\sigma} \\ A_2 &= -(2 - \sigma) \end{aligned}$$

After simplifying

$$w_1(c) = \frac{1}{\lambda\rho} u'(c) \left((\rho - (1 - \sigma)r + \lambda)(r - c) + \lambda\beta \frac{c}{1 - \sigma} - r\sigma c + \sigma c^2 \right)$$

when $c \rightarrow 0$, the right hand side goes to ∞ and when $c \rightarrow \infty$, the right-hand side goes to $+\infty$ if $2 < \sigma$ and goes to 0 if $2 < \sigma$ and to $\frac{\sigma}{\lambda\rho}$ if $\sigma = 2$.

Local optima of $w_1(c)$: It is easy to see that $A_0 > 0$.

If $\sigma < 2$ then $A_2 < 0$. The roots of $w'(c) = 0$ are

$$\frac{-A_1 + \sqrt{A_1^2 - 4A_2A_0}}{2A_2}, \frac{-A_1 - \sqrt{A_1^2 - 4A_2A_0}}{2A_2}$$

with one root negative and the other one positive. Therefore since $w_1(c) \rightarrow +\infty$ as $c \rightarrow 0$ and $w_1(c) \rightarrow +\infty$ as $c \rightarrow +\infty$, the positive root is a global minimum.

If $\sigma > 2$, then $A_2 > 0$. Since $w_1(c) \rightarrow +\infty$ as $c \rightarrow 0$ and $w_1(c) \rightarrow 0$ as $c \rightarrow +\infty$, and $w_1(c^*) < 0$, where c^* is the interior steady-state. At local minimum of $w_1''(c)$ exists.

Therefore

$$A_1^2 - 4A_0A_2 > 0$$

The two local optima of w_1 are then

$$c_1 = \frac{-A_1 + \sqrt{A_1^2 - 4A_2A_0}}{2A_2}, c_2 = \frac{-A_1 - \sqrt{A_1^2 - 4A_2A_0}}{2A_2}$$

Consider the second derivative at these local optima:

$$\begin{aligned} w_1''(c) &= w_1'''(c) (A_0 + A_1c + A_2c^2) + u''(c)(A_1 + 2A_2c) \\ &= u''(c)(A_1 + 2A_2c) \end{aligned}$$

At the lower root, $w_1''(c) > 0$ and at the higher root $w_1''(c) < 0$. Therefore c_1 is a local minimum and c_2 is a local optimum. In addition $\lim_{c \rightarrow \infty} w_1(c) = 0$, therefore $w_1(c_2) > 0$ and $w_1(c) > 0$ for all $c > c_2$.

Now since in equilibrium $w < 0$, we can ignore the part of the phase diagram in which $w_1 > 0$, i.e. $c \geq c_2$. So restricting attention to $(0, c_2)$, w_1 has U-shape with the local minimum at c_1 . \square

Loci $w'(x) = 0$ Loci of points where $w'(x) = 0$ is determined by

$$w = w_2(c) = \frac{\beta u(c)}{\rho - (1 - \sigma)(r - c)}$$

Lemma 20. w_2 has inverted U-shape.

Proof. If $\sigma < 1$, since $c > 0$, and $\rho - (1 - \sigma)r > 0$, $\rho - (1 - \sigma)(r - c) > 0$, so $w_2(c) > 0$ for all $c > 0$.

If $\sigma > 1$, $\rho - (\sigma - 1)(c - r) < 0$ if $c > \frac{\rho - (1 - \sigma)r}{\sigma - 1}$. Therefore, $w_2(c) < 0$ if $c < \frac{\rho - (1 - \sigma)r}{\sigma - 1}$ and $w_2(c) > 0$ if $c > \frac{\rho - (1 - \sigma)r}{\sigma - 1}$.

In any Markov equilibrium, $w \leq 0$, therefore the positive part of $w_2(c)$ is not relevant for the study the Markov equilibria. In particular,

$$\dot{w} = (\rho - (1 - \sigma)(r - c)) w - \beta u(c),$$

therefore, $\dot{w} > 0$ for all $c > \frac{\rho - (1 - \sigma)r}{\sigma - 1}$ and $w \geq 0$.

Differentiate $w_2(c)$, we arrive at

$$w_2'(c) = \frac{\beta u'(c) \sigma \left(\frac{\rho - (1 - \sigma)r}{\sigma} - c \right)}{(\rho - (1 - \sigma)(r - c))^2}$$

Therefore $w_2(c)$ has single peak at $c = \frac{\rho - (1 - \sigma)r}{\sigma} = r + \frac{\rho - r}{\sigma}$.

Lastly,

$$w_2(c) = \frac{\beta}{\frac{\rho - (1 - \sigma)r}{(1 - \sigma)c} + 1} \frac{u(c)}{(1 - \sigma)c} \rightarrow_{c \rightarrow \infty} 0$$

and if $\sigma > 1$

$$w_2(c) = \frac{\beta}{1 - \frac{\rho - (1-\sigma)r}{(\sigma-1)c}} \frac{u(c)}{(1-\sigma)c} \xrightarrow{c \rightarrow \frac{\rho - (1-\sigma)r}{\sigma-1}} -\infty$$

On $\left(\frac{\rho - (1-\sigma)r}{\sigma-1}, +\infty\right)$, $w(c) > 0$.

As $c \rightarrow 0$, the limit of $w_2(c)$ depends on whether $\sigma < 1$ or $\sigma \geq 1$:

$$w_2(c) = \frac{\beta u(c)}{\rho - (1-\sigma)(r-c)} \propto_{c \rightarrow 0} \frac{\beta u(c)}{\rho - (1-\sigma)r}.$$

□

The relationship between $c' = 0$ and $w' = 0$ At $c = r$, the loci of $c' = 0$ and $w' = 0$, are both equal to \bar{w} . Here we compare the slopes of the two loci at this point.

The slope of the loci of $c' = 0$ is given by

$$w'_1(r) = \frac{-(\rho + \lambda)u'(r) + \lambda\beta u'(r) + ru'(r)}{\lambda\rho}$$

and the slope of the loci of $w' = 0$ is given by

$$w'_2(r) = \frac{\beta u'(r)(\rho - r)}{\rho^2}$$

Therefore the comparison between the two slopes is equivalent to comparing

$$\frac{\lambda\beta - (\rho + \lambda - r)}{\lambda} \leq \frac{\beta(\rho - r)}{\rho}$$

which is also equivalent to

$$\beta \leq \hat{\beta}.$$

This also implies that depending on β versus $\hat{\beta}$, $c' = 0$ and $w' = 0$ loci intersects at a unique steady state to the left ($\beta > \hat{\beta}$) or to the right ($\beta < \hat{\beta}$) of $c = r$.

Now we turn to the slopes of the loci at the interior steady-states. Assume that $\beta > \hat{\beta}$, and consequently $r > \rho$. Therefore the loci $w' = 0$ has single peak at $\check{c} = \frac{\rho - (1-\sigma)r}{\sigma} = r + \frac{\rho - r}{\sigma} < r$.

We show that $w_1(\check{c}) > w_2(\check{c})$. Indeed, this is equivalent to

$$\frac{\beta u(\check{c})}{\rho - (1-\sigma)(r - \check{c})} < \frac{(\rho + \lambda)u'(\check{c})(r - \check{c}) + \lambda\beta u(\check{c}) - (1-\sigma)(r - c)(u(\check{c}) + u'(\check{c})(r - c))}{\lambda\rho}$$

after algebra manipulations, this is equivalent to

$$\rho < (\rho + \lambda(1 - \beta))$$

which is obviously true since $\beta < 1$.

Therefore, w_1 and w_2 intersects at c^* where w_2 is strictly decreasing. Since, $w_1(c) < w_2(c)$ in the left neighborhood of $c = r$ and the interior steady-state is unique, we must have $w'_1(c^*) < 0$ (otherwise, $w_1(c) \geq w_1(c^*) = w_2(c^*) > w_2(c)$ in the right neighborhood of c^* , which implies that there is another steady-state between c^* and r). Now by Lemma 20, w_1 is U-shaped. Therefore, w_1 is decreasing from 0 until $c^{**} > c^*$ and increasing from

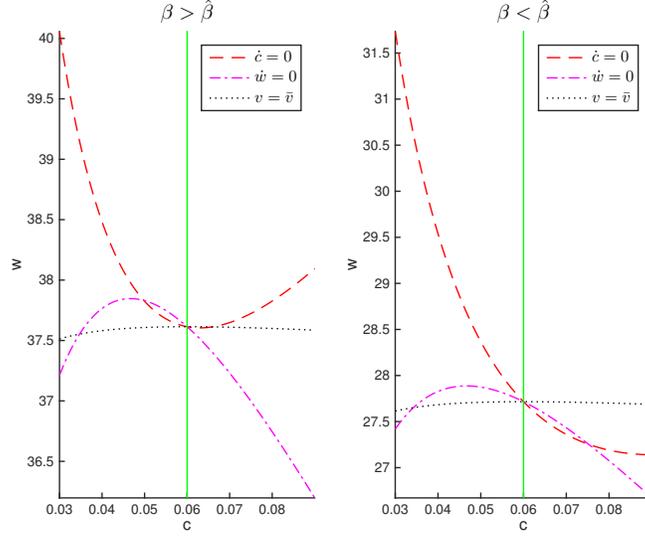


Figure 5: Loci for $\dot{c} = 0$ and $\dot{w} = 0$

c^{**} on (as long as w_1 stays in the relevant region). We can show that $c^{**} > r$ if and only if $\frac{\rho - r + \lambda}{\lambda} > \beta$.

$V = \bar{V}$ curve Another important set of points is the set of points at which $V(a) = \bar{V}(a)$. This is equivalent to

$$v = \bar{v} = \frac{\bar{V}(a)}{a^{1-\sigma}}$$

Since

$$(\rho + \lambda)v - \lambda w = u(c) + u'(c)(r - c),$$

this leads to

$$w = w_0(c) = \frac{(\rho + \lambda)v - u(c) - u'(c)(r - c)}{\lambda}.$$

It is immediate that w is single-peaked with the peak exactly at $c = r$.

Figure 5 presents two examples of the loci, for $\beta > \hat{\beta}$ and $\beta < \hat{\beta}$.

G.2 Local Dynamics

Before tackling the global dynamics of system (12), the following results establish the local properties of the system, including existence and uniqueness.

Lemma 21. *Assume that $\beta < \hat{\beta}$, the differential system (12) admits a unique solution which converges to the steady state $(r, \bar{w} = \frac{\beta u(r)}{\rho})$ with the property that $c > r$ and $c' < 0$ near the steady state.*

Proof. The local existence is a corollary of Lemma 2. Now we show uniqueness.

Indeed, since $\dot{c} < 0$, c decreases as the solution converges to the steady state (r, \bar{w}) . Therefore, we can find a function \hat{w} such that

$$w(x) = \hat{w}(z(x)) + \bar{w}$$

where $c = r + z(x)$. We show that the function \hat{w} is uniquely determined.

Indeed, from (12), we obtain the following differential equation for \hat{w} :

$$\hat{w}'(z) = \frac{-u''(r+z)z(\rho\hat{w} - \beta z\tilde{u}(z) + (1-\sigma)z(\bar{w} + \hat{w}))}{-(\rho + \lambda)u'(r+z)z + \lambda\beta z\tilde{u}(z) + (1-\sigma)z(u(r+z) - u'(r+z)z) - \lambda\rho\hat{w}}, \quad (40)$$

where $\tilde{u}(z) = \frac{u(r+z)-u(r)}{z}$. We simplify this equation as:

$$\hat{w}'(z) = F(z) + \frac{G(z)}{K(z) + \lambda\rho\hat{w}}$$

where

$$F(z) = u''(r+z)z \frac{\rho - (1-\sigma)z}{\lambda\rho}$$

$$G(z) = u''(r+z)z^2 * \left\{ -\beta\tilde{u}(z) + (1-\sigma)\bar{w} - \frac{(\rho + \lambda)u'(r+z) - \lambda\beta\tilde{u}(z) - (1-\sigma)(u(r+z) - u'(r+z)z)}{\lambda\rho} (\rho - (1-\sigma)z) \right\}$$

$$K(z) = (\rho + \lambda)u'(r+z)z - \lambda\beta z\tilde{u}(z) - (1-\sigma)z(u(r+z) - u'(r+z)z)$$

First, we observe that, as $z \rightarrow 0$, $\frac{G(z)}{u''(r+z)z^2} \rightarrow \left(-\beta + \beta\frac{r}{\rho} - \frac{(\rho+\lambda-r)-\lambda\beta}{\lambda}\right) u'(r) < 0$, since $\beta < \hat{\beta}$. Therefore $G(z) > 0$ in a neighborhood to the right of $z = 0$, i.e. $G(z) > 0$ for all $z \in (0, z^*)$ for some $z^* > 0$.

We assume by contradiction that the ODE (40) has to distinct solutions \hat{w}_1 and \hat{w}_2 . From the phase diagram, the two solutions cannot intersect when $z > 0$. Without loss of generality, we assume that $\hat{w}_1 < \hat{w}_2$ for all $0 < z < z^*$. Then

$$\frac{d}{dz}(\hat{w}_2 - \hat{w}_1) = G(z) \frac{\lambda\rho(\hat{w}_1(z) - \hat{w}_2(z))}{(K(z) + \lambda\rho\hat{w}_2(z))(K(z) + \lambda\rho\hat{w}_1(z))} < 0,$$

since, $K(z) + \lambda\rho\hat{w}_2(z) > 0$ and $K(z) + \lambda\rho\hat{w}_1(z) > 0$ because $\dot{c} < 0$ in the two solutions. Therefore

$$\lim_{z \rightarrow 0}(\hat{w}_2(z) - \hat{w}_1(z)) > \hat{w}_2(z^*) - \hat{w}_1(z^*) > 0.$$

This is a contradiction since, $\lim_{z \rightarrow 0} \hat{w}_2(z) = \lim_{z \rightarrow 0} \hat{w}_1(z) = 0$. □

Lemma 22. Assume that $\beta > \hat{\beta}$, the differential system (12) admits a unique solution which converges to the steady state $(r, \bar{w} = \frac{\beta u(r)}{\rho})$ with the property that $c < r$ and $c'(x) > 0$ near the steady state.

Proof. Similar to the proof of the last result. The proof of Theorem 8 below shows the existence of a solution, with the properties that $c(x) \geq \nu$ and $w(x) \geq \bar{w}$ for all t

First, we observe that, as $z \rightarrow 0$, $\frac{G(z)}{u''(r+z)z^2} \rightarrow \left(-\beta + \beta\frac{r}{\rho} - \frac{(\rho+\lambda-r)-\lambda\beta}{\lambda}\right) u'(r) > 0$, since

$\beta > \hat{\beta}$. Therefore $G(z) < 0$ in a neighborhood to the right of $z = 0$, i.e. $G(z) < 0$ for all $z \in (0, z^*)$ for some $z^* > 0$.

We assume by contradiction that the ODE (40) has two distinct solutions \hat{w}_1 and \hat{w}_2 . From the phase diagram, the two solutions cannot intersect when $z < 0$. Without loss of generality, we assume that $\hat{w}_1 < \hat{w}_2$ for all $0 > z > z^*$. Then

$$\frac{d}{dz} (\hat{w}_2 - \hat{w}_1) = G(z) \frac{\lambda \rho (\hat{w}_1(z) - \hat{w}_2(z))}{(K(z) + \lambda \rho \hat{w}_2(z)) (K(z) + \lambda \rho \hat{w}_1(z))} > 0,$$

since, $K(z) + \lambda \rho \hat{w}_2(z) > 0$ and $K(z) + \lambda \rho \hat{w}_1(z) > 0$ because $\dot{c} > 0$ in the two solutions. Therefore

$$\lim_{z \rightarrow 0} (\hat{w}_2(z) - \hat{w}_1(z)) > \hat{w}_2(z^*) - \hat{w}_1(z^*) > 0.$$

This is a contradiction since, $\lim_{z \rightarrow 0} \hat{w}_2(z) = \lim_{z \rightarrow 0} \hat{w}_1(z) = 0$. \square

Lemma 23. Consider the ODE (12) and a singular point (r, w^*) at $x = x^*$

If $w^* > \bar{w}$, locally there exists a unique solution to (12) defined in finite intervals $(x^* - \epsilon, x^*)$ and $(x^*, x^* + \epsilon)$ continuous at the boundaries of the interval and with the property that $c'(x) > 0$.

If $w^* < \bar{w}$, locally there exists a unique solution to (12) defined in finite intervals $(x^* - \epsilon, x^*)$ and $(x^*, x^* + \epsilon)$ continuous at the boundaries of the interval and with the property that $c'(x) < 0$.

Proof. We use the tools from the DAE literature, in particular **Rabier and Rheinboldt (2002)**, to show this result. First, we use a change of variable from x to t such that $\frac{dx}{dt} = r - c(x_t)$. Then the system (12) becomes

$$w'(t) = (\rho w(t) - \bar{\beta} u(c(t))) - (1 - \sigma)w(t) (r - c(t)) \quad (41a)$$

$$c'(t) = \frac{1}{u''(c(t))} \left((\rho + \lambda - r)u'(c(t)) - \frac{\lambda}{r - c(t)} (\rho w(t) - \bar{\beta} u(c(t))) \right) - c(t) (r - c(t)) \quad (41b)$$

To use the DAE tools, let $y = \begin{bmatrix} c \\ w \end{bmatrix}$ and we write this ODE system as

$$A(y)\dot{y} = G(y),$$

where

$$A(y) = \begin{bmatrix} u''(c)(r - c) & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$G(y) = \begin{bmatrix} (\rho + \lambda)u'(c)(r - c) + \lambda\beta u(c) - (1 - \sigma)(r - c)(u(c) + u'(c)(r - c)) - \lambda\rho w \\ \rho w - \beta u(c) - (1 - \sigma)(r - c)w \end{bmatrix}$$

We have

$$\det(A(y))\dot{y} = \text{adj}(A(y))G(y),$$

where

$$\text{adj}(A(y)) = \begin{bmatrix} 1 & 0 \\ 0 & u''(c)(r - c) \end{bmatrix}$$

Let

$$\gamma_0(y) = \det(A(y)) = u''(c)(r - c)$$

and

$$\tilde{G}(y) = \begin{bmatrix} (\rho + \lambda)u'(c)(r - c) + \lambda\beta u(c) - (1 - \sigma)(r - c)(u(c) + u'(c)(r - c)) - \lambda\rho w \\ (\rho w - \beta u(c) - (1 - \sigma)(r - c)w)u''(c)(r - c) \end{bmatrix}$$

In addition

$$\begin{aligned} \gamma_1(y) &= D\gamma_0(y)\tilde{G}(y) \\ D\gamma_0(y) &= [u'''(c)(r - c) - u''(c) \quad 0] \end{aligned}$$

Therefore at $y^* = (r, w^*)$

$$\gamma_1(y^*) = -u''(r)(\lambda\beta u(r) - \lambda\rho w^*).$$

By **Rabier and Rheinboldt (2002, Theorem 39.1)**, if $w^* > \bar{w}$, $\gamma_1(y^*) < 0$, then y^* is attractive and there are two solutions y^1 and y^2 defined over $[-T, 0]$, such that (41) is satisfied over $[-T, 0)$ and $\lim_{t \rightarrow 0} y^i(t) = y^*$. It is easy to see that one solution has $r > c_t$ and the other one has $r < c_t$. Since y_t^i is continuous, c_t^i also is. Let $x_t = x^* + \int_t^0 (r - c_s) ds$, we can verify that $(c^i(x_t) = c_t^i, w^i(x_t) = w_t^i)$, $i = 1, 2$ are the desired solution to (12) over $[x_T + x^*, x^*]$.

Similarly if $w^* < \bar{w}$, $\gamma_1(y^*) < 0$ so y^* is repulsive, we obtain the desired solutions similarly. \square

G.3 Proofs for Theorem 5

G.3.1 Proof of Theorem 5

Part a. is a direct application of Lemma 27.

Now we prove the complete characterization in Part b. Consider an equilibrium. By Lemma 24, we have $\hat{c}(a) \geq ra$ for all $a \geq \underline{a}$. We first show that if $\underline{a} < \tilde{a} < \bar{a}$ such that $\hat{c}(\tilde{a}) = r\tilde{a}$ and $\hat{c}(\bar{a}) = r\bar{a}$ then $\frac{\bar{a}}{\tilde{a}} > \delta > 1$ where δ is defined in Lemma 24.

Indeed, if this is not the case then

$$(\tilde{V}(a), \tilde{W}(a), \tilde{c}(a)) = \left(\left(\frac{\tilde{a}}{\underline{a}} \right)^{1-\sigma} V\left(a \frac{\underline{a}}{\tilde{a}}\right), \left(\frac{\tilde{a}}{\underline{a}} \right)^{1-\sigma} W\left(a \frac{\underline{a}}{\tilde{a}}\right), \frac{\tilde{a}}{\underline{a}} \hat{c}\left(a \frac{\underline{a}}{\tilde{a}}\right) \right)$$

with $a \geq \tilde{a}$ is an equilibrium defined over $[\tilde{a}, \infty)$ with $\tilde{V}(\tilde{a}) = \bar{V}(\tilde{a})$, which contradicts the result in Lemma 24 with \tilde{a} stands for \underline{a} .

Given this result, we can define the sequence $\{a_n\}$ with $a_0 = \underline{a}$ and

$$a_{n+1} = \inf \{a > a_n : \hat{c}(a) = ra\}.$$

We have $\hat{c}(a_{n+1}) = ra_{n+1}$ and $\frac{a_{n+1}}{a_n} > \delta$. In addition $\hat{c}(a) > ra$ for all $a \in (a_n, a_{n+1})$. Now for $a \in [a_n, a_{n+1})$, let $c(x) = \frac{\hat{c}(a)}{a}$ where $x = \log a - \log a_n$ and $w(x) = \frac{W(a)}{a^{1-\sigma}}$. We have $\lim_{x \rightarrow \log a_{n+1} - \log a_n} v(x) = \bar{v}$ and $(c(x), w(x))$ is a solution to 12, with $c(x) > r$. By Lemma 27, $c(x)$ corresponds to the solution defined in that lemma. In addition, $\log a_{n+1} - \log a_n \in X$.

G.3.2 Supporting Results for Theorem 5

Lemma 24. *All equilibria must feature dissaving at all assets. In addition, there exists $\delta = \delta(\rho, r, \beta, \sigma) > 1$ such that for any $\underline{a} > 0$, all equilibria must have the same consumption and value functions over $[\underline{a}, \underline{a}\delta]$ and $V(a) > \bar{V}(a)$ for all $a \in (\underline{a}, \underline{a}\delta]$. In addition, if a continuous (regular) equilibrium exists, it is unique.*

Proof. Consider a regular equilibrium. By Lemma 25, $\hat{c}(a) \geq ra$ for all $a \geq \underline{a}$. Therefore at the debt limit, $\hat{c}(\underline{a}) = r\underline{a}$.

Given $\gamma > r$ define in Lemma 26 (depending on whether $r \geq \rho$ or $r < \rho$). Let δ be defined by

$$\delta = \frac{\gamma}{2\Delta},$$

where

$$\Delta = \frac{\gamma - r}{2(-u''(\frac{\gamma}{2})) \left((\rho + \lambda - r)u'(\frac{\gamma}{2}) + \frac{2\lambda}{\gamma - r}(\rho\hat{w} - \bar{\beta}u(\frac{\gamma}{2})) \right)}.$$

We first show by contradiction that $\frac{\hat{c}(a)}{a} < \gamma$ for all $a \leq \underline{a}\delta$. If there exists $\tilde{a} \in (\underline{a}, \underline{a}\delta]$ such that $\frac{\hat{c}(\tilde{a})}{\tilde{a}} > \gamma$. Let $c(x) = \frac{\hat{c}(e^x)}{e^x}$ and $\tilde{x} = \log \tilde{a}$ and $\underline{x} = \inf\{x \geq \log \underline{a} : c(z) > r \forall z \in (x, \tilde{x})\}$. So $c(x)$ is continuous over $[\underline{x}, \tilde{x}]$ and $c(\underline{x}) = r$.

We have $c(\tilde{x}) > \gamma$ and $c(\underline{x}) = r < \gamma$. Let $x^* = \inf\{x \in [\underline{x}, \tilde{x}] : c(z) \geq \frac{\gamma+r}{2} \forall z \in [x, \tilde{x}]\}$. By the continuity of $c(\cdot)$, $c(x^*) = \frac{\gamma+r}{2}$ and $c(x) \geq \frac{\gamma+r}{2}$ for all $x \in [x^*, \tilde{x}]$. Therefore,

$$\begin{aligned} c'(x) &= \frac{1}{u''(c(x))(r - c(x))} \left((\rho + \lambda - r)u'(c(x)) - \frac{\lambda}{r - c(x)}(\rho w(x) - \bar{\beta}u(c(x))) \right) - c(x) \\ &< \frac{1}{(-u''(\frac{\gamma}{2})) \frac{\gamma-r}{2}} \left((\rho + \lambda - r)u'(\frac{\gamma}{2}) + \frac{2\lambda}{\gamma - r}(\rho\hat{w} - \bar{\beta}u(\frac{\gamma}{2})) \right) - \frac{\gamma}{2} \\ &< \Delta. \end{aligned}$$

Consequently,

$$\Delta(\tilde{x} - x^*) > c(\tilde{x}) - c(x^*) > \frac{\gamma}{2}$$

So

$$\frac{\gamma}{2\Delta} < \tilde{x} - x^* < \delta,$$

which contradicts the definition of δ . Therefore $\frac{\hat{c}(a)}{a} < \gamma$ for all $\underline{a} \leq a \leq \underline{a}\delta$.

Consider two equilibria with consumption and policy functions $(V_1(a), W_1(a), \hat{c}_1(a))$ and $(V_2(a), W_2(a), \hat{c}_2(a))$. Let $(c_1(x), w_1(x))$ and $(c_2(x), w_2(x))$ be the associated homogenized functions, defined over $[\log \underline{a}, \log \underline{a} + \log \delta]$. By choosing one of the two equilibria, say V_2, W_2, \hat{c}_2 as the one constructed in Lemma 2, lowering δ if necessary, we can assume that $V_2(a) > \bar{V}(a)$ for all $\underline{a} < a < \delta\underline{a}$. There are two cases:

Case 1: $\hat{c}_1(\underline{a}\delta) \geq \hat{c}_2(\underline{a}\delta) > r$. There exists $\tilde{x} \in (\log \underline{a}, \log \underline{a} + \log \delta]$ such that $c_1(\tilde{x}) = c_2(\log \underline{a} + \log \delta)$ (c_1 is continuous from some $\tilde{\tilde{x}} \geq \log \underline{a}$ up until $\underline{a}\delta$, where $c_1(\tilde{\tilde{x}}) = r$). By Lemma 26, we have $w_1(\tilde{x}) = w_2(\log \underline{a} + \log \delta)$. Therefore, since system (12) is autonomous: $(c_1(x), w_1(x)) = (c_2(x + \log \underline{a} + \log \delta - \tilde{x}), w_2(x + \log \underline{a} + \log \delta - \tilde{x}))$ for all

$x \in [\log \underline{a}, \tilde{x}]$. If $\tilde{x} < \log \underline{a} + \log \delta$, then

$$(c_2(\log \underline{a} + \log \underline{a} + \log \delta - \tilde{x}), w_2(\log \underline{a} + \log \underline{a} + \log \delta - \tilde{x})) = (r, \bar{w})$$

and therefore $V_2(\frac{\underline{a}\delta}{\exp(\tilde{x})}) = \bar{V}(\frac{\underline{a}\delta}{\exp(\tilde{x})})$, which contradicts the property that $V_2 > \bar{V}$ over $(\underline{a}, \underline{a}\delta)$. Therefore $\tilde{x} = \log \underline{a} + \log \delta$, so $(c_1, w_1) \equiv (c_2, w_2)$ over $[\log \underline{a}, \log \underline{a} + \log \delta]$.

Case 2: $\hat{c}_1(\underline{a}\delta) < \hat{c}_2(\underline{a}\delta)$. There exists $\tilde{x} \in (\log \underline{a}, \log \underline{a} + \log \delta)$ such that $c_2(\tilde{x}) = c_1(\log \underline{a} + \log \delta)$. Similar to Case 1, we can show that at $x^* = \frac{\underline{a}\delta}{\exp(\tilde{x})}$, $V_1(x^*) = \bar{V}(x^*)$ and $c_1(x^*) = r$. Now there must exist a sequence $\log \underline{a} < x^{**} < x^*$ such that $c_1(x^{**}) > r$. Let (x_1, x_2) denote the maximum interval containing x^{**} such that $c(x) > r$ inside the interval. Then $x_1 \geq \underline{a}$ and $x_2 \leq x^*$. In addition $V_1(x_1) = \bar{V}(x_1)$ and $V_1(x_2) = \bar{V}(x_2)$. Lemma 26 then implies that $c_1(x) = c_2(x - x_1 + \log \underline{a})$ for all $x \in (x_1, x_2)$. But this means $V_2(\underline{a} \exp(x_2 - x_1)) = \bar{V}(\underline{a} \exp(x_2 - x_1))$, which contradicts the assumption $V_2 > \bar{V}$ over $(\underline{a}, \underline{a}\delta)$.

So the only possibility is in Case 1, which implies $(c_1, w_1) \equiv (c_2, w_2)$, which in turn is equivalent to $(V_1, W_1, \hat{c}_1) \equiv (V_2, W_2, \hat{c}_2)$.

Now we show by contradiction that if a continuous equilibrium exists, it must be unique. Assume that this is not the case, i.e. there exist two distinct continuous equilibria with the consumption and policy functions $(V_1(a), W_1(a), \hat{c}_1(a))$ and $(V_2(a), W_2(a), \hat{c}_2(a))$. The associated homogenized functions $(c_1(x), w_1(x))$ and $(c_2(x), w_2(x))$ are defined over $[\log \underline{a}, +\infty)$.

Let $x^* = \inf\{x \geq \log \underline{a} : c_1 \neq c_2\}$. By the contradiction assumption $x^* < +\infty$. By continuity $c_1(x) = c_2(x)$ for all $x \in [\log \underline{a}, x^*]$. If $c_1(x^*) = c_2(x^*) > r$, then $(c_1(x^*), w_1(x^*))$ is not a singular point of the system (12). Therefore, $(c_1, w_1) \equiv (c_2, w_2)$ in a neighborhood to the right of x^* , which contradicts the definition of x^* . Therefore, $c_1(x^*) = r$.

Now applying the result in the first part of this theorem for $\underline{a} = \exp(x^*)$, there exists $\delta > 1$ such that $c_1(x) = c_2(x)$ over $(x^*, x^* + \log \delta)$, which also contradicts the definition of x^* .

In all cases, we obtain a contradiction. Therefore, a continuous equilibrium is unique if it exists. \square

Lemma 25. *In any equilibrium, $\hat{c}(a) \geq ra$ for all $a \geq \underline{a}$.*

Proof. We show this result by contradiction. Assume that there exists \tilde{a} such that $\hat{c}(\tilde{a}) < r\tilde{a}$.

Consider the unique solution to the differential system (12), $\{c(x), w(x)\}$ with the initial condition $(\tilde{c}, \tilde{w}) = \left(\frac{\hat{c}(\tilde{a})}{\tilde{a}}, \frac{W(\tilde{a})}{(\tilde{a})^{1-\sigma}}\right)$ at $x = \log \tilde{a}$. Let $(\tilde{x}, \tilde{\hat{x}})$ denote the right maximal interval of existence of this solution, as defined in Hartman (2002). There are three cases:

Case 1: $\tilde{w} \geq w_1(\tilde{c})$. From the phase diagram, there exists ϵ such that $c'(x) > 0$ and $w(x) > 0$ for all $x \in [\tilde{x}, \tilde{x} + \epsilon]$. We show that $c'(x) > 0$ for all $x > \tilde{x}$. If this is not the case, let \hat{x} denote the first time $c'(x)$ reaches 0 after $\tilde{x} + \epsilon$ i.e. $w(\hat{x}) = w_1(c(\hat{x}))$ and $w(x) > w_1(c(x))$ for all $\tilde{x} < x < \hat{x}$. Since $w(x) > w_1(c(x)) > w_2(c(x))$, $w'(x) > 0$. However, since $w'_1 < 0$ for all $c < r$, we have $w(\hat{x}) > \tilde{w} > w_1(c(\tilde{x})) > w_1(c(\tilde{x}))$, which contradicts $w(\hat{x}) = w_1(c(\hat{x}))$. Therefore $c'(x) > 0$ and $w'(x) > 0$ for all $x \geq \tilde{x}$. As shown in Case 1-i) of Lemma 29, $\tilde{\hat{x}}$ is finite and $\hat{c}(\tilde{\hat{x}}) = r$. We also have $\tilde{\hat{w}} > \tilde{w} > w_1(\tilde{w}) > w_1(r) =$

\bar{w} . Therefore, at $\tilde{a} = \exp(\tilde{x})$, we have $W(\tilde{a}) = \tilde{w}(\tilde{a})^{1-\sigma} > \bar{W}(\tilde{a})$. However $\hat{c}(\tilde{a}) = r\tilde{a}$ and $\hat{c}(a) < ra$ for $a < \tilde{a}$, therefore by Lemma 13, $W(\tilde{a}) \leq \bar{W}(\tilde{a})$, which contradicts the preceding inequality.

Case 2: $w_2(\tilde{c}) > \tilde{w}$. If $\tilde{c} > \frac{\rho-(1-\sigma)r}{\sigma}$, then by Lemma 20 $w_2(c)$ is decreasing in c . So as long as $c(x) > \frac{\rho-(1-\sigma)r}{\sigma}$, $w'(x) < 0$, and consequently, $c'(x) < 0$, since $w(x) < \tilde{w} < w_2(\tilde{c}) < w_2(c(x))$. Since $W_t \geq \int u(v)e^{-\rho t} a_t^{(1-\sigma)t} dt > a_0^{(1-\sigma)} \underline{w}$ for some \underline{w} , we have $w(x) \geq \underline{w}$ for all $x \geq \tilde{x}$. Let $c^* = \inf_{x \geq \tilde{x}} \{c(x) : c(x) > \frac{\rho-(1-\sigma)r}{\sigma}\}$. Therefore $c^* \geq \frac{\rho-(1-\sigma)r}{\sigma}$. If $c^* > \frac{\rho-(1-\sigma)r}{\sigma}$, $c(x) \downarrow c^*$ and $w(x) \downarrow w^*$. Since (c^*, w^*) is regular, c^*, w^* is reached at some $x^* > \tilde{x}$. However, $c'(x^*) < 0$, so there exists $x > x^*$ such that $\frac{\rho-(1-\sigma)r}{\sigma} < c(x) < c(x^*) = c^*$, which contradicts the definition of c^* .

Therefore $c(x)$ goes below $\frac{\rho-(1-\sigma)r}{\sigma}$ for some x . By letting this x standing for \tilde{x} , we can assume that $\tilde{c} \leq \frac{\rho-(1-\sigma)r}{\sigma}$. If $w'(x) < 0$ for all $x \geq \tilde{x}$. Since $w(x) \geq \underline{w}$. Let $w^* = \inf_{x > \tilde{x}} w(x) \geq \underline{w}$. Since $w_2(c) < w_1(c)$, $c'(x) < 0$, let $c^* = \inf_{x > \tilde{x}} c(x) \geq v$. (c^*, w^*) is regular and must be reached at some x^* . So $c'(x^*) < 0$, which implies $c(x) < c(x^*)$ for some $x > x^*$. This contradicts the definition of c^* .

Consequently, $w'(x) = 0$ for some $x \geq \tilde{x}$, or $w(x) = w_2(c(x))$. By letting this x stand for \tilde{x} , we are in Case 3.

Case 3: $w_1(\tilde{c}) > \tilde{w} \geq w_2(\tilde{c})$. If $\tilde{c} \leq \frac{\rho-(1-\sigma)r}{\sigma}$, then $c'(x) < 0$ and $w'(x) > 0$ for all $x > \tilde{x}$. Since $w(x) < \hat{w} < \infty$ defined in (42), let $w^* = \sup_{x > \tilde{x}} w(x)$ and $c^* = \inf_{x > \tilde{x}} c(x) \geq v$. (c^*, w^*) must be reached at some x^* . So $c'(x^*) < 0$, which implies $c(x) < c(x^*)$ for some $x > x^*$. This contradicts the definition of c^* .

We end up with the case $\tilde{c} > \frac{\rho-(1-\sigma)r}{\sigma}$. If $w'(x) = 0$ at some x , we are back to Case 2, which leads to a contradiction (after \tilde{c} goes below $\frac{\rho-(1-\sigma)r}{\sigma}$). If $w'(x) > 0$ for all $x > \tilde{x}$, we can show by contradiction that $c'(x) < 0$ for all $x > \tilde{x}$ and goes below $\frac{\rho-(1-\sigma)r}{\sigma}$. We then get a contradiction as in the first part of this case.

In all cases, we get a contradiction. Therefore, in any regular equilibrium, $\hat{c}(a) \geq ra$ for all $a \geq \underline{a}$. \square

Lemma 26. Consider the system (12) over $c(x) \geq r$ for all $x \geq 0$. There exists $\gamma = \gamma(\rho, r, \beta, \sigma) > r$ such that: for any $\tilde{c} \in (r, \gamma]$ there exists a unique \tilde{w} such that the solution $(c(x), w(x))$ to system (12) with the initial condition (\tilde{c}, \tilde{w}) at some \tilde{x} and the left maximal interval of existence (\tilde{x}, \tilde{x}) satisfies $r < c(x) < \gamma$ for all $\tilde{x} < x < \tilde{x}$ and $w(\tilde{x}) = \tilde{w}$ if $c(\tilde{x}) = r$.

Proof. First of all, the existence result in Lemma 1 in the main paper shows that, there exists a solution (c^e, w^e) to 12 defined over $[x_1 = \log \underline{a}, x_2]$ such that $c^e(x_1) = r$ and $c^e(x_2) = \gamma^e$ with $r < c^e \leq \gamma^e$ for all $x \in (x_1, x_2)$.

We show the result in this lemma for two cases separately, $r \geq \rho$ and $r < \rho$.

Case 1: If $r > \rho$. Let $\phi = \arg \min_{c > r} w_1(c)$ and $\gamma = \min \left\{ \psi, \phi, \frac{\rho-r(1-\sigma)}{\sigma}, \gamma^e \right\} > r$.

Fixing $\tilde{c} \in (r, \gamma]$:

i) If $\tilde{w} \geq w_2(\tilde{c})$, then following the phase diagrams of (12) as in the proof of Lemma 28, we have $c'(x) > 0$ and $w'(x) < 0$. Therefore by Lemma 23, the solutions hits (r, w^*) for some $w^* > \tilde{w}$ at some finite $\tilde{x} < \tilde{x}$.

ii) If $\tilde{w} \leq w_1(\tilde{c})$, then following the phase diagrams of (12) as in the proof of Lemma 28, we have $c(x) > \gamma$ for some $x < \tilde{x}$.

iii) If $w_1(\tilde{c}) < \tilde{w} < w_2(\tilde{c})$. If $w'(x) = 0$ at some $x < \tilde{x}$, then we are in case i), this cannot lead to a solution. If $c'(x) = 0$ at some $x < \tilde{x}$, then we are in case ii), this cannot lead to a solution either. Therefore, a solution must have $c'(x) < 0$ and $w'(x) < 0$ for all $x < \tilde{x}$.

Given the solution (c^e, w^e) , there exists $x_0 \in (x_1, x_2)$ such that $c^e(x_0) = \tilde{c}$. It is immediate that $(c^e(x + x_0 - \tilde{x}), w^e(x + x_0 - \tilde{x}))$ is a solution satisfies the desired properties. By Lemma 21, this solution is unique. Therefore, $\tilde{w} = w^e(x_0)$ is the unique \tilde{w} which gives rise to the solution with the desired properties stated in the lemma.

Case 2: If $r < \rho$. The proof is exactly the same, exact for $\gamma = \min\{\psi, \phi\}$ and now the solutions converging to (r, \bar{w}) has $\dot{c} < 0$ and $\dot{w} > 0$. \square

Lemma 27. *There exists a unique solution (c, w) to the autonomous differential system (12) (with associated v) defined over $[0, \bar{x})$ with initial condition $c(0) = r, w(0) = \bar{w}$. This solution features $c(x) > r$ for $x \in (0, \bar{x})$ and either $\bar{x} = \infty$ or $\lim_{x \rightarrow \bar{x}} c(x) = r$. Let $X \equiv \{x \in (0, \bar{x}) \mid v(x) = \bar{v}\}$, let $\underline{x} = \inf X$ with the convention that $\underline{x} = \infty$ if X is empty. Then $\underline{x} > 0$ and $v(x) > \bar{v}$ for all $x \in (0, \underline{x})$.*

Proof. Similar to the proof of Lemma 25, a solution to the autonomous differential system (12) must have $c(x) \geq r$ for all $x \geq 0$. Lemma 26 implies that (12) has a unique solution defined over $[0, \epsilon]$ for some $\epsilon > 0$ such that $c(x) > r$ for all $x \in (0, \epsilon)$. Let $[0, \bar{x})$ denote the right maximal interval of existence for this solution. If $\bar{x} = \infty$ then we immediately obtain the desired result. If $\bar{x} < \infty$. Then by Hartman (2002, Theorem 3.1) and Lemma 23, we have $\lim_{x \rightarrow \bar{x}} c(x) = r$.

Now, we turn to the characterization of X . First of all by the existence result in Lemma 1 of the main paper, we have $v(x) > \bar{v}$ for all $x \in (0, \epsilon_1)$ for some $\epsilon_1 > 0$. Therefore $\underline{x} \geq \epsilon_1 > 0$. By definition of \underline{x} if $x \in (0, \underline{x})$, we have $v(x) \neq \bar{v}$. If $v(x) < \bar{v}$, then by the Intermediate Value Theorem, $v(\tilde{x}) = \bar{v}$ for some $\tilde{x} \in (0, x)$ which again contradicts the definition of \underline{x} . Therefore, $v(x) > \bar{v}$ for all $x \in (0, \underline{x})$. \square

G.4 Proofs for Theorem 6

G.4.1 Proof of Theorem 6

First we rule out the possibility that $\hat{c}(a) > ra$ at some asset level a .

Assume by contradiction that there exists a^* such that $\hat{c}(a) > ra$ for all $a \geq a^*$. Let $[\tilde{a}, a^*]$ denote the maximal interval containing a^* such that $\hat{c}(a) > ra$ in this interval: $\tilde{a} = \inf\{a^* > a > \underline{a} : \hat{c}(b) > rb \text{ for } b \in [a, a^*]\}$ and $\check{a} = \sup\{a > a^* : \hat{c}(b) > rb \text{ for } b \in [a^*, a]\}$. Because of the asset limit $\tilde{a} \geq \underline{a}$ and by Lemma 28, there exists $a' > \tilde{a}$ such that $\hat{c}(a') \leq ra'$, so $\check{a} < +\infty$.

If $\tilde{a} = \underline{a}$, then by borrowing constraint, $\hat{c}(\tilde{a}) = r\tilde{a}$. If $\tilde{a} > \underline{a}$, by the definition of Markov equilibrium we also have $\hat{c}(\tilde{a}) = r\tilde{a}$ (otherwise the ODE for asset $\dot{a}_t = ra_t - \hat{c}(a_t)$ would not have a solution). Either way we have $\hat{c}(\tilde{a}) = r\tilde{a}$. Therefore $V(\tilde{a}) = \bar{V}(\tilde{a})$ and $W(\tilde{a}) = \bar{W}(\tilde{a})$. For $a \in (\tilde{a}, \check{a})$, since $V'(a) = U'_1(\hat{c}(a)) < U'_1(ra) < \bar{V}'(a)$, we have $V(a) < \bar{V}(a)$ for all $\check{a} \geq a > \tilde{a}$. One direct implication of this is that $\hat{c}(\check{a}) \neq r\check{a}$. By

the definition of \check{a} , there exists a sequence $a_n \downarrow \check{a}$ such that $\hat{c}(a_n) \leq ra_n$. By Lemma 29, $V(a_n) \geq \bar{V}(a_n)$. Therefore $\lim_{n \rightarrow \infty} V(a_n) = V(\check{a}) \geq \bar{V}(\check{a})$. This is a contradiction.

Therefore, $\hat{c}(a) \leq ra$ for all $a \geq \underline{a}$. $\hat{c}(a) \equiv ra$ is obviously not an equilibrium. Now if $\hat{c}(\check{a}) < r\check{a}$ and $\hat{c}(\check{a}) \neq \psi\check{a}$ for some \check{a} . Lemma 29 shows that there exists $a^* > \check{a}$ such that $(V(a^*), W(a^*), \hat{c}(a^*)) = (\bar{V}(a^*), \bar{V}(a^*), ra^*)$ and $\hat{c}(a) < ra$ for all $a \in (\check{a}, a^*)$.

If there exists $a^{**} > a^*$ such that $\hat{c}(a^{**}) = ra^{**}$. There exists $\hat{a} \in (a^*, a^{**})$ such that $\hat{c}(\hat{a}) < r\hat{a}$ (otherwise $\hat{c}(a) \equiv ra$ for all $a \in (a^*, a^{**})$ which is clearly not an equilibrium). Let $\hat{a}_1 = \sup\{a > \hat{a} : \hat{c}(b) < rb \forall b \in [\hat{a}, a]\}$. By definition $\hat{a}_1 \leq a^{**}$ and $\hat{c}(\hat{a}_1) = r\hat{a}_1$. Let $\hat{a}_2 = \sup\{a < \hat{a} : \hat{c}(b) < rb \forall b \in [a, \hat{a}]\}$. By definition, $\hat{a}_2 \geq a^*$. In addition, there exists a sequence $\{a_n\}$, $a_n \uparrow \hat{a}_2$ such that $\hat{c}(a_n) = ra_n$. Therefore $\lim V(a_n) = \lim \bar{V}(a_n) = \bar{V}(\hat{a}_2) = \bar{V}(\hat{a}_2)$. Lemma 29 shows that $V(\hat{a}_2) > \bar{V}(\hat{a}_2)$ which contradicts the earlier result that $V(\hat{a}_2) = \bar{V}(\hat{a}_2)$.

If $\hat{c}(a) > ra$ for all $a > a^*$, Lemma 29 shows that $\hat{c}(a) = \psi a$ for all $a > a^*$. However, this implies $\lim_{a \downarrow a^*} V(a) > \bar{V}(a^*) = V(a^*)$ which contradicts the continuity of V .

In all cases, we obtain a contradiction. Therefore $\hat{c}(a) \equiv \psi a$ for all $a \geq \underline{a}$.

G.4.2 Proof of Proposition 3

Let $x^* = \log a^*$.

To the left of x^* , Lemma 22 shows that existence and uniqueness of a solution to the system (12) over $[x^* - \epsilon, x^*]$ with the initial condition $(c, w) = (r, \bar{w})$ at $x = x^*$ and $c(x) < r$ for all $x < x^*$. The proof of Theorem 8 below also implies that the solution can be extended over $[\log \underline{a}, x^*]$.

To the right of x^* , using the limiting arguments similar to the ones in the existence proofs for Lemma 2 and Theorem 8, we can show the existence of a solution to the system (12) over $[x^*, x^* + \epsilon]$ with the initial condition $(c, w) = (r, \bar{w})$ at $x = x^*$ and $c(x) > r$ for all $x > x^*$. Let (x^*, \tilde{x}) denote the right maximal interval of existence for this solution. Lemma 28 shows that $\tilde{x} < \infty$ and $c(\tilde{x}) = r, w(\tilde{x}) < \bar{w}$.

Let $\bar{a} = \exp(\tilde{x})$ and $(V(a), W(a), \hat{c}(a)) = (v(\log a), w(\log a), c(\log a))$, we obtain an equilibrium defined over $[\underline{a}, \bar{a}]$. Since (x^*, \tilde{x}) is a maximal interval of existence, this equilibrium cannot be extended past \bar{a} .

The phase diagram in Figure 2 also implies that there exists a continuum of solution (\tilde{c}, \tilde{w}) to the system (12) over $[x^*, x^* + \epsilon]$ with the initial condition $(\tilde{c}, \tilde{w}) = (r, \bar{w})$ at $x = x^*$ and $\tilde{c}(x) > r$ for all $x > x^*$ and the trajectories of (\tilde{c}, \tilde{w}) lie below and are bounded by the one generated by (c, w) .

G.4.3 Supporting Results for the Proofs of Theorem 6 and Proposition 3

Let \hat{w} be the maximal value for an agent with utility $\beta u(c)$:

$$\hat{w} = \max \int_0^\infty e^{-\rho t} \beta u(c_t) dt \quad (42)$$

subject to $a_0 = 1$ and $\dot{a}_t = ra_t - c_t$.

Lemma 28. *In any equilibrium, at any asset level, a , there exists $a' > a$ such that $\hat{c}(a') \leq ra'$.*

Proof. We prove this result by contradiction. Assume that there exists a^* such that $\hat{c}(a) > ra$ for all $a \geq a^*$.

Let $\tilde{a} = \inf\{a^* > a > \underline{a} : \hat{c}(b) > rb \text{ for } b \in [a, a^*]\}$. If $\tilde{a} = \underline{a}$, then by borrowing constraint, $\hat{c}(\tilde{a}) = r\tilde{a}$. If $\tilde{a} > \underline{a}$, by the definition of Markov equilibrium we also have $\hat{c}(\tilde{a}) = r\tilde{a}$, otherwise we can extend to left of \tilde{a} . Either way we have $\hat{c}(\tilde{a}) = r\tilde{a}$. Therefore $V(\tilde{a}) = \bar{V}(\tilde{a})$ and $W(\tilde{a}) = \bar{W}(\tilde{a})$. By the definition of \tilde{a} , $\hat{c}(a) > ra$ for all $a > \tilde{a}$. Since $V'(a) = U'_1(\hat{c}(a)) < U'_1(ra) < \bar{V}'(a)$, we have $V(a) < \bar{V}(a)$ for all $a \geq \tilde{a}$.

Let a^* be any asset level such that $a^* > \tilde{a}$. Let $w^* = \frac{W(a^*)}{a^*}$ and $c^* = \frac{\hat{c}(a^*)}{a^*} > r$. By choosing $\epsilon > 0$ sufficiently small, we have $V(\tilde{a}) < \bar{V}(\tilde{a}) - \epsilon(\tilde{a})^{1-\sigma}$ and $V'(a) < \bar{V}'(a) - \epsilon(\tilde{a})^{-\sigma}$ for all $a \geq \tilde{a}$. Therefore, for all $a > a^*$

$$\frac{V(a)}{a^{1-\sigma}} < \bar{v} - \epsilon.$$

which implies

$$w(\log a) < \bar{w} - \frac{\rho + \lambda}{\lambda} \epsilon$$

for all $a > a^*$.

We consider two cases separately:

Case 1: $\beta \geq \frac{\rho + \lambda - r}{\lambda}$. In this case, by Lemma 19 and Lemma 20, w_1 is strictly increasing in c when $c > r$ and $w_0(c), w_2(c)$ are strictly decreasing in c when $c > r$.

Since $V(a) < \bar{V}(a)$, we have $w^* < w_0(c^*) < w_1(c^*)$. Consider the unique solution to the differential system (12), $\{c(x), w(x)\}$ with the initial condition (c^*, w^*) at $x = \log a^*$. Let (x^*, \tilde{x}) denote the right maximal interval of existence of this solution, as defined in Hartman (2002).

Since $w(x) \leq w_0(c) - \epsilon < w_1(c)$ for all $c > r$, $c'(x) < 0$ for all $x > x^*$. Now we have, $r < c(x) < c(x^*) = c^*$ for all $x > x^*$. First we show that, $w(x) > w_2(c^*)$ for all $x \in (x^*, \tilde{x})$. Indeed, if $w(x) \geq w_2(c(x))$ then $w(x) \geq w_2(c(x)) > w_2(c(x^*))$ since w_2 is decreasing in c and $c(x) < c(x^*)$. If $w(x) < w_2(c(x))$, then $w'(x) > 0$, therefore $w(x) > w(x - \epsilon) < w_2(c^*)$. We have shown that $w_2(c^*) < w(x) < w_0(c^*) - \frac{\rho + \lambda}{\lambda} \epsilon < \bar{w} - \frac{\rho + \lambda}{\lambda} \epsilon$. By Hartman (2002, Theorem 3.1), there exists $\{x_n\} \uparrow \tilde{x}$ such that $\{c(x_n), w(x_n)\}$ converges to some point (\tilde{c}, \tilde{w}) and (\tilde{c}, \tilde{w}) must lie in the boundary of the region where the differential system (12) is well-defined and Lipschitz continuous. Since $c^* > c(x) > r$ and $w_2(c^*) < w(x) < \bar{w} - \frac{\rho + \lambda}{\lambda} \epsilon$, we must have $(\tilde{c}, \tilde{w}) = (r, \tilde{w})$. However, Lemma 23 shows that \tilde{x} is finite, so $c(\tilde{x}) = r$, which contradicts the property that $\hat{c}(a) > ra$ for all $a > a^*$.

Case 2: $\beta < \frac{\rho + \lambda - r}{\lambda}$. In this case, by Lemma 20, w_1 is decreasing in c from r until \tilde{c} and increasing for all $c \geq \tilde{c}$. As before w_2 is decreasing for all $c > r$ and $w_1(c) > w_2(c)$ for all $c > r$.

Again, consider the unique solution to the system (12), $\{c(x), w(x)\}$ with the initial condition (c^*, w^*) at $x = \log(a^*)$. We show that there exists $\hat{x} > 0$, such that for all $x > \hat{x}$, $c'(x) < 0$ i.e. $w(x) < w_1(c(x))$. To show this, we consider three possibilities:

i) $w^* > w_1(c^*)$: so $w'(x^*) < 0$ and $c'(x^*) > 0$. From the phase diagram, there exists $\epsilon > 0$ such that $w'(x) < 0$ and $c'(x) > 0$ for all $x \in (x^*, x^* + \epsilon]$. Let $\hat{x} = \sup\{x \geq x^* + \epsilon : \dot{w}_s < 0, \dot{c}_s > 0 \forall s \in [x^*, x]\}$. Since c is increasing and w is decreasing,

and $w_1(c(x)) < w(x) < w_0(c(x))$. It is obvious from the phase diagram that $w(c(\hat{x})) = w_2(c(\hat{x}))$. By letting \hat{x} standing for x^* , we arrive as Case ii)

ii) $w^* \leq w_1(c^*)$. From the phase diagram, there exists $\epsilon > 0$ such that $c'(x) < 0$ for all $x \in (x^*, x^* + \epsilon]$. We show by contradiction that $c'(x) < 0$ for all $x > x^* + \epsilon$. If $c'(x) \geq 0$ for some $x = x^{**} > x^* + \epsilon$. Let $\hat{x} = \sup \{x \geq x^* + \epsilon : c'(s) < 0 \forall s \in [x^*, x]\}$. By definition $\hat{x} < x^{**}$. As in Case 1 we can show that $w_2(c^*) < w(x) < \bar{w} - \frac{\rho+\lambda}{\lambda}\epsilon$ for all $x < \hat{x}$. Therefore $c(\hat{x})$ and $w(\hat{x})$ are well-defined. By the definition of \hat{x} , we must have $c'(\hat{x}) = 0$ or equivalently, $w(\hat{x}) = w_1(\hat{c}(\hat{x}))$.

Now, consider the function \tilde{w} defined over $(c(\hat{x}), c(x^*))$ such that $\tilde{w}(c(x)) = w(x)$. Then $\tilde{w}(c) < w_1(c)$ and $\tilde{w}(c(\hat{x})) = w_1(c(\hat{x}))$. Therefore $\tilde{w}'(c(\hat{x})^+) \leq w_1(c(\hat{x})^+)$. However

$$\tilde{w}'(c(\hat{x})^+) = \lim_{x \rightarrow \hat{x}} \frac{w'(x)}{c'(x)} = +\infty,$$

since $c'(\hat{x}) = 0$ and $w'(\hat{x}) < 0$ since $w(x) = w_1(x) > w_2(x)$ which is a contradiction.

Therefore, $c'(x) < 0$ for all $x > x^*$. Following the steps in Case 1. we obtain a contradiction.

In all cases, a contradiction arises. Therefore, we obtain the result in this lemma. \square

Lemma 29. *In any equilibrium, if at an asset level \tilde{a} , $\hat{c}(\tilde{a}) < r\tilde{a}$, then $V(\tilde{a}) > \bar{V}(\tilde{a})$. In addition, either:*

1) *There exists $a^* > a$ such that $(V(a^*), W(a^*), \hat{c}(a^*)) = (\bar{V}(a^*), \bar{W}(a^*), ra^*)$ and $\hat{c}(a) < ra$ for all $a \in (\tilde{a}, a^*)$.*

2) $\hat{c}(a) = \psi a$ for all $a \geq \tilde{a}$.

Proof. Let $\hat{a} = \sup\{a > \tilde{a} : \hat{c}(b) < rb \forall b \in [\tilde{a}, a]\}$. By definition, $\hat{a} > \tilde{a}$. There are two cases, $\hat{a} = \infty$ or $\hat{a} < \infty$.

Case 1: $\hat{a} = \infty$. We show that $\hat{c}(a) = \psi a$ for all $a \geq \tilde{a}$, where ψ is the linear equilibrium. Assume by contradiction that this is not the case.

Consider the unique solution to the differential system (12), $\{c(x), w(x)\}$ with the initial condition $(\tilde{c}, \tilde{w}) = \left(\frac{\hat{c}(\tilde{a})}{\tilde{a}}, \frac{W(\tilde{a})}{(\tilde{a})^{1-\sigma}}\right)$ at $x = \log \tilde{a}$. Let $(\tilde{x}, \tilde{\tilde{x}})$ denote the right maximal interval of existence of this solution, as defined in Hartman (2002).

From the analysis of the phase diagram and the loci above, we know that $w_1(\psi) = w_2(\psi)$, $w_1(c) > w_2(c)$ for all $c < \psi$ and $w_1(c) < w_2(c)$ for all $c > \psi$. In addition, we also know that $w_2'(c) < 0$ for all $c \leq \psi$ and $w_1'(c) < 0$ for all $c \geq \psi$. We obtain contradiction in different sub-cases.

1-i) If $\tilde{c} > \psi$ and $w_0 \geq w_2(\tilde{c})$: the phase diagram, there exists $\epsilon > 0$ such that $w'(x) > 0$ and $c'(x) > 0$ for all $x \in (\tilde{x}, \tilde{x} + \epsilon]$. We show that $w'(x) > 0$ and $c'(x) > 0$ for all $x > \tilde{x}$. Indeed, $w(x) > w(\tilde{x}) \geq w_2(\tilde{c}) > w_2(c(x))$ since $c(x) > c(\tilde{x})$ and w_2 is strictly decreasing over $[\psi, r]$. Therefore $w'(x) > 0$. Similarly, we have $c'(x) > 0$. In addition $c(x)$ is bounded above by r and $w(x) \leq \hat{w}$ where \hat{w} is defined in (42). So $(c(x), w(x)) \rightarrow (c^*, w^*)$. By Hartman (2002, Theorem 3.1), (c^*, w^*) must lie at the boundary, i.e. $c^* = r$. In addition, $w^* \geq w(\tilde{x}) > w_1(\tilde{c}) > w_1(r) = \bar{w}$. Lemma 23 also implies that (c^*, w^*) is reached in finite asset. i.e $\tilde{\tilde{x}} < \infty$. Therefore, at $\tilde{a} = \exp(\tilde{\tilde{x}})$, we have $W(\tilde{a}) = \tilde{w}(\tilde{a})^{1-\sigma} > \bar{W}(\tilde{a})$. However $\hat{c}(\tilde{a}) = r\tilde{a}$ and $c(a) < ra$ for $a < \tilde{a}$, therefore by Lemma 13, $W(\tilde{a}) \leq \bar{W}(\tilde{a})$, which contradicts the preceding inequality.

1-ii) If $\tilde{c} > \psi$ and $w_1(\tilde{c}) < \tilde{w} < w_2(\tilde{c})$. In this case, there are three possibilities. First, $w(x) = w_2(c(x))$ for some $x > \tilde{x}$, by letting x standing for \tilde{x} , we are back to case i) which implies a contradiction. Second $w(x) = w_1(c(x))$ for some x , we arrive at Case 1-iii) below which also leads to a contradiction. Lastly, $w_1(c(x)) < w(x) < w_2(c(x))$ for all $x \geq \tilde{x}$. In this case, we have $w'(x) < 0$ and $c'(x) > 0$ for all $x > \tilde{x}$. Lemma 22 shows that there exists a unique solution. This solution is defined over some $[x_1, x_2]$ with $(c(x_2), w(x_2)) = (r, \tilde{w})$. This contradicts the property that $\hat{a} = +\infty$.

1-iii) If $\tilde{c} > \psi$ and $w_1(\tilde{c}) \geq \tilde{w}$. From the phase diagram, there exists $\epsilon > 0$ such that $w'(x) < 0$ and $c'(x) < 0$ for all $x \in (\tilde{x}, \tilde{x} + \epsilon]$. If $c(x) < \psi$ for all $x > \tilde{x}$, then $c'(x) < 0$ and $w'(x) < 0$ for all $x \geq \tilde{x}$. Since $W_t \geq \int u(v)e^{-\rho t} a_t^{(1-\sigma)t} dt > a_0^{(1-\sigma)} \underline{w}$ for some \underline{w} , we have $w(x) \geq \underline{w}$ for all $x \geq \tilde{x}$. Let $c^* = \inf_{x \geq \tilde{x}} c(x) \geq \psi$ and $w^* = \inf_{x \geq \tilde{x}} w(x)$. We must reach (c^*, w^*) in finite asset \hat{x} . Therefore $c(\hat{x}) < c^*$ for all $x > \hat{x}$, which contradicts the definition of c^* . So $c(x) = \psi$ at some x and $w(x) < \tilde{w} \leq w_2(\tilde{w}) \leq w_2(\psi)$. By letting x standing for \tilde{x} , we arrive at Case 1-iv) below.

1-iv) If $\tilde{c} \leq \psi$ and $\tilde{w} < w_2(\tilde{c})$. If $c'(x) < 0$ for all $x \geq \tilde{x}$, since $\underline{w} \leq w < \hat{w}$, this implies that $c(x) = v$ for some x , which violates the regularity condition. Therefore, $c'(x) = 0$ at some $\hat{x} > \tilde{x}$. At \hat{x} , $w(\hat{x}) = w_1(c(\hat{x}))$ and $c(\hat{x}) < \psi$. Let \hat{x} standing for \tilde{x} , we show that $w'(x) > 0$ and $c'(x) > 0$ for all $x > \tilde{x}$. Similar to Case 1-i). we obtain a contradiction.

In all sub-cases i)-iv), we obtain a contradiction, therefore it must be that $c(x) = \psi$ and $w(x) = w_1(\psi) = w_2(\psi)$ for all t . In particular $V(\tilde{a}) > \bar{V}(\tilde{a})$.

Case 2: $\hat{a} < \infty$. By the definition of Markov equilibrium, we must have $\hat{c}(\hat{a}) = r\hat{a}$, and $V(\hat{a}) = \bar{V}(\hat{a})$ and $W(\hat{a}) = \bar{W}(\hat{a})$. Again consider the unique solution to the differential system (12), $\{c(x), w(x)\}$ with the initial condition $(\tilde{c}, \tilde{w}) = \left(\frac{\hat{c}(\tilde{a})}{\tilde{a}}, \frac{W(\tilde{a})}{(\tilde{a})^{1-\sigma}}\right)$ at $x = \log \tilde{a}$. Going through the sub-cases as in Case 1, we can show that $\{c(x), w(x)\}$ must coincide with the solution in Lemma 22. Therefore, $V(\tilde{a}) > \bar{V}(\tilde{a})$. \square

H Proofs for Poverty Trap Equilibria

H.1 Proof of Theorem 7

Proof of Theorem 7 (Existence). We prove the existence by construction.

As shown in the proof of Theorem 4, there exists a unique solution (V_d, W_d) to ODE (19) that satisfies $(V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))$ defined over a maximal interval of existence $(\hat{a}, a_u]$, where $\hat{a} < \frac{\underline{c}}{r}$. We also show in the proof of Theorem 4 that V_d, W_d is defined and is continuous at \hat{a} , i.e. the limits $\lim_{a \downarrow \hat{a}} V_d(a) = V_d(\hat{a})$ and $\lim_{a \downarrow \hat{a}} W_d(a) = W_d(\hat{a})$ exist, and $\hat{c}_d(\hat{a}) = r\hat{a}$.

If $\hat{a} \leq \underline{a}$, let

$$(V, W, \hat{c}) = \begin{cases} (V_d, W_d, \hat{c}_d) & \text{if } \underline{a} \leq a < \hat{a} \\ (V_u, W_u, \hat{c}_u) & \text{if } a \geq \hat{a}. \end{cases}$$

If $\underline{a} < \hat{a}$, Lemma 30 below shows that $V_d(\hat{a}) = \bar{V}(\hat{a})$. In addition, Lemma 31 shows that

$V_d(a) > \bar{V}(a)$ for all $a > \hat{a}$. Therefore,

$$V'_d(\hat{a}) = U'_1(r\hat{a}) \geq \bar{V}'(\hat{a}).$$

By Lemma 6, $\beta(r\hat{a}) \leq \hat{\beta}$.

If $\beta(r\hat{a}) = \hat{\beta}$. Let $\tilde{a} = \min \{a \geq \underline{a} : \beta(ra) = \hat{\beta}\}$. Because β is weakly increasing, $\beta(ra) = \hat{\beta}$ for all $a \in [\tilde{a}, \hat{a}]$ (Assumption 4). We define V_h, W_h over $[\tilde{a}, \infty)$ such that

$$(V_h(a), W_h(a), \hat{c}_h) = \begin{cases} (V_u(a), W_u(a), \hat{c}_u(a)) & \text{if } a_u \leq a \\ (V_d(a), W_d(a), \hat{c}_d(a)) & \text{if } \hat{a} \leq a < a_u \\ (\bar{V}(a), \bar{W}(a), ra) & \text{if } \tilde{a} \leq a < \hat{a}. \end{cases}$$

By Theorems 1 and 4, V_u, W_u satisfy (6) over $[\tilde{a}, \infty)$. Replacing \hat{a} by \tilde{a} if $\beta(r\hat{a}) = \hat{\beta}$, without loss of generality, we can assume that $\beta(ra) < \hat{\beta}$ for $a < \hat{a}$.

Iteratively, we construct a sequence $\{a_i\}$ starting with $a_0 = \underline{a}$, and for each $i \geq 0$, $a_i < \hat{a}$ and $\beta(ra_i) < \hat{\beta}$ and the value and policy functions (V_i, W_i, \hat{c}_i) are determined as following:

Iteration i : Starting from $a_i < \hat{a}$, because $\beta(ra_i) < \hat{\beta}$, using Lemma 2, we show that ODE (18) admits a solution (V_i, W_i) , with the initial condition $(V_i(a_i), W_i(a_i)) = (\bar{V}(a_i), \bar{W}(a_i))$, defined over a (right) maximal interval of existence $[a_i, a_i^*]$. Moreover $V_i(a) > \bar{V}(a)$ in a neighborhood to right of a_i . There are three possibilities:

$i-1$) $a_i^* < \hat{a}$. Then following the steps in Lemma 9, we can show that $V_i(a_i^*) \leq \bar{V}(a_i^*)$. By the Intermediate Value Theorem, there exists $a_i < a_{i+1} < \hat{a}$, such that $V_i(a_{i+1}) = \bar{V}(a_{i+1})$ and $V_i(a) > \bar{V}(a)$ for $a \in (a_i, a_{i+1})$. Because $a_{i+1} < \hat{a}$, $\beta(ra_{i+1}) < \hat{\beta}$. Go to iteration $i + 1$ with a_{i+1} standing for a_i .

$i-2$) $a_i^* \geq \hat{a}$ and $V_i(a) \leq \bar{V}(a)$ for some $a < \hat{a}$. By the Intermediate Value Theorem, there exists $a_i \leq a_{i+1} < \hat{a}$, such that $V_i(a_{i+1}) = \bar{V}(a_{i+1})$ and $V_i(a) > \bar{V}(a)$ for $a \in (a_i, a_{i+1})$. Go to iteration $i + 1$ with a_{i+1} standing for a_i .

$i-3$) $a_i^* \geq \hat{a}$ and $V_i(a) > \bar{V}(a)$ for all $a < \hat{a}$. We stop the construction.

Following this procedure, we produce a strictly increasing sequence $\{a_i\}$ such that for each $i \geq 0$, $a_i < \hat{a}$ and $\beta(ra_i) < \hat{\beta}$ and the value functions (V_i, W_i) satisfies $(V_i(a_i), W_i(a_i)) = (\bar{V}(a_i), \bar{W}(a_i))$ and $V_i(a_{i+1}) = \bar{V}(a_{i+1})$ and $V_i(a) > \bar{V}(a)$ for all $a \in (a_i, a_{i+1})$. Let

$$(V_l(a), W_l(a), \hat{c}_l) = (V_i(a), W_i(a), \hat{c}_i(a)) \text{ for } a \in [a_i, a_{i+1}),$$

with $a_{n+1} = a_n^*$ if possibility $n-3$) is reached at some iteration n .

There are two possible cases:

Case 1: The sequence $\{a_i\}$ is finite, i.e. possibility $n-3$) is reached at some iteration n : We obtain a sequence $a_0 < a_1 < \dots < a_n < \hat{a}$.

If $a_n^* = \infty$, we define the value and consumption functions (V, W, \hat{c}) over the whole interval $[a, \infty)$ as

$$(V(a), W(a), \hat{c}(a)) = (V_l(a), W_l(a), \hat{c}_l(a)).$$

If $a_n^* < \infty$, following the steps in the proof of Lemma 9, we can show that $V_n(a_n^*) \leq \bar{V}(a_n^*)$.

Therefore, both V_n and V_h are defined over $[\hat{a}, a_n^*]$ and

$$\begin{aligned} V_h(\hat{a}) &= \bar{V}(\hat{a}) \leq V_n(\hat{a}) \\ V_h(a_n^*) &\geq \bar{V}(a_{n+1}^*) \geq V_n(a_{n+1}^*). \end{aligned}$$

By the Intermediate Value Theorem, there exists $a^* \in [\hat{a}, a_n^*]$ such that

$$V_n(a^*) = V_h(a^*).$$

We define (V, W, \hat{c}) as

$$(V, W, \hat{c}) = \begin{cases} (V_l, W_l, \hat{c}_l) & \text{if } a < a^* \\ (V_h, W_h, \hat{c}_h) & \text{if } a \geq a^*. \end{cases}$$

Case 2: The sequence $\{a_i\}$ is infinite (possibility *i-3* is never reached). Then $\lim_{i \rightarrow \infty} a_i = a_\infty \leq \hat{a}$ and $\beta(ra_\infty) = \hat{\beta}$. Because $\beta(ra) < \hat{\beta}$ for $a < \hat{a}$. We have $a_\infty = \hat{a}$. In this case

$$(V, W, \hat{c}) = \begin{cases} (V_l, W_l, \hat{c}_l) & \text{if } a < a_\infty = \hat{a} \\ (V_h, W_h, \hat{c}_h) & \text{if } a \geq \hat{a}. \end{cases}$$

In all cases we can construct the value and policy functions (V, W, \hat{c}) over $[\underline{a}, \infty)$. As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. In addition, $\hat{c}(a) < ra$ for all $a \geq a^*$ and $\hat{c}(a) \geq ra$ for all $a < a^*$.

It remains to show that $a^* < \infty$. Above a_u , $\hat{c}_l(a) > ra > \psi a = \hat{c}(a) > \bar{c}$, so the solution (V_l, W_l, \hat{c}_l) stays in the homogenous region of the utility function defined by Assumption 2. As shown in the proof of Theorem 5 using the phase diagram (to the right of $c = r$), this solution cannot be extended forever and there exists $a^{**} < \infty$ such that $V_l(a^{**}) < \bar{V}(a^{**}) < V_u(a^{**})$. Therefore $a^* < a^{**} < \infty$. \square

Proof of Theorem 7 (Characterization). We show the characterization of poverty trap under Assumption 2 and that β is non-decreasing and $\bar{\beta} > \hat{\beta}$. In any equilibrium, there exists a^* such that $\hat{c}(a) \geq ra$ for all $a < a^*$ and $\hat{c}(a) < ra$ for all $a > a^*$.

Indeed, let $a^* = \inf\{a \geq \underline{a} : \hat{c}(a) < ra\}$. If $a^* = \infty$ then $\hat{c}(a) \geq ra$ for all $a \geq \underline{a}$, the result is immediate. We just need to show the result for $a^* < +\infty$. By the definition of a^* , $\hat{c}(a) \geq ra$ for all $a < a^*$.

We show by contradiction that $\hat{c}(a) \leq ra$ for all $a \geq a^*$. Assume otherwise, then there exists $\tilde{a} > a^*$, such that $\hat{c}(\tilde{a}) > r\tilde{a}$. Let $a^{**} = \inf\{a > a^* : \hat{c}(a) > ra\}$.

There are two possibilities:

Case 1: $a^{**} > a^*$: Then $\hat{c}(a) \leq ra$ for all $a \in (a^*, a^{**})$. Therefore by Lemma 13, $\hat{c}(a^{**}) = ra^{**}$.

i) $\beta(ra^{**}) \leq \hat{\beta}$. Then $\beta(ra) \leq \hat{\beta}$ for all $a \leq a^{**}$. Following the proof of Lemma 14, there cannot be strict saving for $a < a^{**}$, which contradicts the definition of a^* .

ii) $\beta(ra^{**}) > \hat{\beta}$: Then $\beta(ra) > \hat{\beta}$ for all $a \geq a^{**}$. Following the proof of Lemma 18, there cannot be strict dissaving for $a > a^{**}$, which contradicts the definition of a^* .

Case 2: $a^{**} = a^*$: Then $\hat{c}(a^*) \geq ra^*$, otherwise $\hat{c}(a) < ra$ in a neighborhood to the right of a^* , which contradicts the definition of $a^{**} = a^*$.

By the definition of a^* and a^{**} , there exist two sequences $\{a_n^s\}$, $\{a_n^d\}$ both decrease towards a^* and $\hat{c}(a_n^s) < ra_n^s$ and $\hat{c}(a_n^d) > ra_n^d$.

If $\beta(ra^*) < \hat{\beta}$, there exists ϵ such that $\beta(ra) < \hat{\beta}$ for all $a < a^* + \epsilon$. Following the proof of Lemma 14, there cannot be strict saving for $a < a^* + \epsilon$, which contradicts the existence of the strict saving sequence a_n^s .

If $\beta(ra) > \hat{\beta}$, for $a > a^*$. Consider a dissaving point $a^* < a_n^{ds}$. There exists a saving point $a_m^s \in (a^*, a_n^{ds})$. Following the proof of Lemma 18, there cannot not be strict dissaving above a_n^s which contradicts the definition of a_n^{ds} .

If $\beta(ra) = \hat{\beta}$, for some interval $[a^*, a^* + \epsilon]$. Following the proof of Lemma 14, we can show that there cannot be strict dissaving for any $a \leq a^* + \epsilon$. This contradicts the existence of the strict saving sequence a_n^s .

In all cases we obtain contradiction. Therefore $\hat{c}(a) \leq ra$ for all $a \geq a^*$.

Now we show that $\hat{c}(a) < ra$ for all $a > a^*$. Let $a^{**} = \inf\{a \geq a^* : \hat{c}(b) < rb \forall b \geq a\}$. We just need to consider the case $a^{**} < +\infty$ ($a^* < +\infty$ and $a^{**} = +\infty$ is a violation of Theorem 6). Now if $a^{**} > a^*$, then there exists a sequence $\{a_n\}$ converging to a^{**} from the left, such that $\hat{c}(a_n) = ra_n$. By Lemma 13, $V(a_n) \leq \bar{V}(a_n)$. Therefore, $V(ra^{**}) \leq \bar{V}(ra^{**})$, thus $\beta(ra^{**}) \leq \hat{\beta}$. By Lemma 14, there cannot be strict saving for $a \leq a^{**}$. Therefore, $\hat{c}(a) = ra$ for all $a \in (a^*, a^{**})$. This contradicts the definition of a^* . \square

H.2 Supporting Results for Theorem 7

Lemma 30. *Given the definition of V_d and \hat{a} in the of Theorem 7, if $\hat{a} > 0$ then $(V_d(\hat{a}), W_d(\hat{a})) = (\bar{V}(\hat{a}), \bar{W}(\hat{a}))$.*

Proof. As shown in the proof of Theorem 4, if $\hat{a} > 0$, $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$. Therefore $\hat{c}_d(\hat{a}) = r\hat{a}$.

First, we show that $W_d(\hat{a}) \leq \bar{W}(\hat{a})$. Assume by contradiction that, $W_d(\hat{a}) > \bar{W}(\hat{a})$, then

$$W'_d(a) = \frac{\rho W_d(a) - U_0(\hat{c}_d(a))}{ra - \hat{c}_d(a)} \longrightarrow \frac{\rho W_d(\hat{a}) - \rho U_0(r\hat{a})}{r\hat{a} - r\hat{a}} = +\infty$$

as a approaches \hat{a} from the right because $\hat{c}_d(a) \rightarrow r\hat{a}$. Moreover, by the continuity of R_l , $\lim_{a \downarrow \hat{a}} V'_d(a) = U'_1(r\hat{a})$. This contradicts the property 3) in Lemma 15 that $W'_d(a) < V'_d(a)$ for all $a_u > a > \hat{a}$. Therefore, $W_d(\hat{a}) \leq \bar{W}(\hat{a})$.

We also show that $W_d(\hat{a}) \geq \bar{W}(\hat{a})$. Assume by contradiction that, $W_d(\hat{a}) < \bar{W}(\hat{a})$, then, similarly to the previous case,

$$W'_d(a) = \frac{\rho W_d(a) - U_0(\hat{c}_d(a))}{ra - \hat{c}_d(a)} \longrightarrow \frac{\rho W_d(\hat{a}) - \rho U_0(r\hat{a})}{r\hat{a} - r\hat{a}} = -\infty$$

as a approaches \hat{a} from the right. This contradicts the property 1) in Lemma 15 that $W'_d(a) > 0$ for all $a > \hat{a}$. Therefore, $W_d(\hat{a}) \geq \bar{W}(\hat{a})$.

The two results imply that $W_d(\hat{a}) = \bar{W}(\hat{a})$. Combining this equality with the fact that $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$ yields $V_d(\hat{a}) = \bar{V}(\hat{a})$. \square

Lemma 31. *Given the definition of V_d and \hat{a} in the proof of Theorem 7, $V_d(a) > \bar{V}(a)$ for all $a > \hat{a}$.*

Proof. Let $\tilde{U}(c) \equiv U_1(c) + \frac{\lambda}{\rho} U_0(c)$. By the concavity of U_1 and U_0 , \tilde{U} is also strictly concave. We first show that $ra > \hat{c}_d(a) > c^*(a)$ where $c^*(a)$ is defined by

$$\tilde{U}'(c^*(a)) = V'(a) + \frac{\lambda}{\rho} W'(a).$$

Indeed, because \tilde{U} is strictly concave, this is equivalent to $\tilde{U}'(c^*(a)) > \tilde{U}'(\hat{c}_d(a))$ or

$$V'_d(a) + \frac{\lambda}{\rho} W'_d(a) > U'_1(\hat{c}_d(a)) + \frac{\lambda}{\rho} U'_0(\hat{c}_d(a)).$$

Because $V'_d(a) = U'_1(\hat{c}_d(a))$ and $W'_d(a) > U'_0(\hat{c}_d(a))$ by Lemma 32, we obtain the desired inequality.

Now using system (6), substituting W_d by the right hand side of the second equation into the first equation, we obtain

$$(\rho + \lambda)V_d(a) = \tilde{U}(\hat{c}(a)) + (V'_d(a) + \frac{\lambda}{\rho} W'_d(a))(ra - \hat{c}(a)).$$

Let

$$F(a, c) \equiv \tilde{U}(c) + (V'_d(a) + \frac{\lambda}{\rho} W'_d(a))(ra - c).$$

Because \tilde{U} is strictly concave, F is strictly concave in c . By the definition of \tilde{U} and $c^*(a)$, $\frac{\partial F(a, c)}{\partial c} = 0$ at $c = c^*(a)$ and $\frac{\partial F(a, c)}{\partial c} < 0$ for $c > c^*(a)$. Therefore

$$F(a, c^*(a)) > F(a, \hat{c}_d(a)) > F(a, ra) = (\rho + \lambda)\bar{V}(a).$$

Moreover, $F(a, \hat{c}_d(a)) = (\rho + \lambda)V_d(a)$, so $V_d(a) > \bar{V}(a)$. \square

Lemma 32. *Given the definition of W_d and \hat{a} in the proof of Theorem 7, $W'_d(a) > U'_0(\hat{c}(a))$ for all $a \in (\hat{a}, a_u)$.*

Proof. Assumption 4 is equivalent to

$$\frac{-U''_0(c)}{U'_0(c)} \leq \frac{-U''_1(c)}{U'_1(c)} \quad (43)$$

for all $c \leq \bar{c}$. We use Lemma 7 to prove this lemma. Indeed, we first show that condition 2) in Lemma 7 is satisfied, i.e. if $W'_d(a) = U'_0(\hat{c}_d(a))$ then

$$\frac{d}{da}(W'_d(a)) < \frac{d}{da}(U'_0(\hat{c}_d(a))). \quad (44)$$

Indeed, differentiating equation (6b) with respect to a implies

$$\rho W'_d(a) = U'_0(\hat{c}_d(a))\hat{c}'_d(a) + W''_d(a)(ra - \hat{c}_d(a)) + W'_d(a)(r - \hat{c}'_d(a)).$$

Because $W'_d(a) = U'_0(\hat{c}_d(a))$, this equation simplifies to

$$W''_d(a) = \frac{(\rho - r) W'_d(a)}{ra - \hat{c}_d(a)} = \frac{(\rho - r) U'_0(\hat{c}_d(a))}{ra - \hat{c}_d(a)}.$$

On the other hand,

$$\begin{aligned}
\frac{d}{da}(U'_0(\hat{c}_d(a))) &= U''_0(\hat{c}_d(a))\hat{c}'_d(a) = U''_0(\hat{c}_d(a))\frac{V'_d(a)}{U'_1(\hat{c}_d(a))} \\
&= U''_0(\hat{c}_d(a))\frac{(\rho-r)U'_1(\hat{c}_d(a)) + \lambda(U'_1(\hat{c}_d(a)) - U'_0(\hat{c}_d(a)))}{U'_1(\hat{c}_d(a))(ra - \hat{c}_d(a))} \\
&\geq U''_0(\hat{c}_d(a))\frac{(\rho-r)U'_1(\hat{c}_d(a))}{U'_1(\hat{c}_d(a))(ra - \hat{c}_d(a))}, \tag{45}
\end{aligned}$$

where the last inequality comes from $U'_1(\hat{c}_d(a)) \geq U'_0(\hat{c}_d(a))$. Combining this with $\rho < r$ and condition (43), we have (44), but with weak inequality. Now we show that it holds with strict inequality. For $a < a_u$, because $\hat{c}'_d > 0$, $\hat{c}_d(a) < \hat{c}_d(a_u) = \bar{c}$, therefore $U'_1(\hat{c}_d(a)) > U'_0(\hat{c}_d(a))$ (this also holds for $a = a_u$ under Assumption 2 with $\bar{\beta} < 1$). Thus (45) holds with strict inequality. If $a = a_u$ and under Assumption 2 with $\bar{\beta} = 1$, $\hat{c}_d(a) = \bar{c}$, (43) holds with strict inequality (as argued in the existence proof of Theorem 4, we can assume that $U'_1(c) > U'_0(c)$ for $c < \bar{c}$ WOLG). Hence, in either case, (44) holds with strict inequality.

Now, we show that condition 1) in Lemma 7 is also satisfied. Under Assumption 2 with $\bar{\beta} < 1$, it is shown in Lemma 33 that at a_u that $W'_d(a_u) > U'_0(\hat{c}_d(a_u))$. Under Assumption 2 with $\bar{\beta} = 1$, $U'_1(c) = U'_0(c)$ for $c \geq \bar{c}$, given how a_u is defined in the existence proof of Theorem 4, we have $W'_d(a_u) = V'_d(a_u) = U'_1(\bar{c}) = U'_0(\bar{c})$, so $W'_d(a) = U'_0(\hat{c}_d(a))$ at $a = a_u$. Therefore, by (44), $W'_d(a) > U'_0(\hat{c}_d(a))$ in the left neighborhood of a_u (assuming that $U'_1(c) > U'_0(c)$ for $c < \bar{c} = \hat{c}_d(a_u)$ WOLG).⁴¹

Given that both conditions in Lemma 7 are satisfied, it implies that $W'(a) > U'_0(\hat{c}_d(a))$ for all $a \in (\hat{a}, a_u)$. \square

Lemma 33. *The linear equilibria described in Theorem 2 with $\bar{\beta} < 1$ satisfies $W'(a) > U'_0(\hat{c}_d(a))$ for all $a > 0$.*

Proof. From Theorem 2, $W'(a) > U'_0(\hat{c}_d(a))$ is equivalent to

$$\frac{\bar{\beta}\bar{v}^{1-\frac{1}{\sigma}}}{\Delta + (1-\sigma)\bar{v}^{-\frac{1}{\sigma}}} > \bar{\beta}\bar{v},$$

or equivalently $\psi > \frac{\Delta}{\sigma}$. This inequality holds because $P\left(\frac{\Delta}{\sigma}\right) = (\bar{\beta} - 1)\frac{\Delta}{\sigma} < 0$. \square

H.3 Proof of Proposition 4

Let \underline{r} be defined such that $\hat{\beta} = \bar{\beta}$, where $\hat{\beta}$ is defined in (9). Since $\bar{\beta} \leq 1$, $\underline{r} \geq \rho$.

⁴¹Another way to show this is to proceed as in the proof of Theorem 3 by considering the solution (V_ϵ, W_ϵ) to the ODE (19) with the initial condition $(V_\epsilon(a_u), W_\epsilon(a_u)) = \left(V_u(a_u) + \frac{\epsilon}{\rho+\lambda}, W_u(a_u) + \frac{\epsilon}{\lambda}\right)$. It is easy to verify that $W'_\epsilon(a_u) > U'_0(\hat{c}_\epsilon(a_u))$ because $\hat{c}_\epsilon(a_u) = \hat{c}(a_u)$. Therefore by Lemma 32, $V'_\epsilon(a) > W'_\epsilon(a)$ for all $a < a_u$. As $\epsilon \rightarrow 0$, $(V_\epsilon, W_\epsilon) \rightarrow (V_d, W_d)$. As a result, $W'_d(a) \geq U'_0(\hat{c}_d(a))$ for all $a < a_u$. We can then apply Lemma 32 to show that $W'_d(a) > U'_0(\hat{c}_d(a))$ for all $a < a_u$ because $\beta(c) < 1$ for all $c < \bar{c}$, which implies that (45), and consequently (44), holds with strict inequality.

Consider the construction in Theorem 7. Fixing $\tilde{a} \in (0, \frac{\bar{c}}{r})$, such that β is not constant over $[\underline{r}\tilde{a}, \bar{c}]$. First we show that there exists \bar{r} such that for $r \in (\underline{r}, \bar{r})$

$$V_d(a_u) < \int_{\tilde{a}}^{a_u} U_1'(ra) da + \bar{V}(\tilde{a}), \quad (46)$$

where $a_u = \frac{\bar{c}}{\psi}$ and ψ is defined in Theorem 2 (which depends on r).

Indeed, when $r \rightarrow \underline{r}$, $\psi \rightarrow \underline{\psi}$ and $V_d(a_u) \rightarrow \frac{1}{\rho+\lambda} \left(U_1(\bar{c}) + \frac{\lambda}{\rho} U_0(\bar{c}) \right) = \bar{V}(a_u(\underline{r}))$. Therefore,

$$V_d(a_u) - \bar{V}(\tilde{a}) \rightarrow \bar{V}(a_u(\underline{r})) - \bar{V}(\tilde{a}) = \int_{\tilde{a}}^{a_u(\underline{r})} \bar{V}'(a) da.$$

Since β is not constant over $[\underline{r}\tilde{a}, \bar{c}]$, $\beta(c) < \bar{\beta} = \hat{\beta}(\underline{r})$ for a non-zero measure subset of $[\underline{r}\tilde{a}, \bar{c}]$ (and $\beta(c) = \bar{\beta}$ outside the subset). Hence, $\bar{V}'(a; \underline{r}) < U_1'(\underline{r}a)$ for a non-zero measure subset of $(\tilde{a}, a_u(\underline{r}))$ and therefore,

$$\bar{V}(a_u(\underline{r})) - \bar{V}(\tilde{a}) < \int_{\tilde{a}}^{a_u(\underline{r})} U_1'(\underline{r}a) da.$$

From the limits and inequalities above, we obtain (46) for r sufficiently close to \underline{r} .

Under (46), we show by contradiction that $\hat{a} > \tilde{a}$. Assume $\tilde{a} \geq \hat{a}$. By Lemma 31, $V_d(\tilde{a}) \geq \bar{V}(\tilde{a})$. Because $\hat{c}(a) < ra$ for $\tilde{a} < a < a_u$, $V_d'(a) > U_1'(ra)$. So

$$V_d(a_u) - V_d(\tilde{a}) > \int_{\tilde{a}}^{a_u} U_1'(ra) da,$$

which contradicts (46). So $\hat{a} > \tilde{a} > 0$. Now pick any \underline{a} such that $0 < \underline{a} < \tilde{a}$. We have $\hat{a} > \underline{a}$.

The construction in Theorem 7 implies that $a^* > \underline{a}$. The proof for existence also shows that $a^* < \infty$, therefore a^* is strictly interior.

I Proofs for Convergence Equilibria

Proof of Theorem 8. First of all let $\alpha_1 = \lim_{c \uparrow c^*} \beta(c)$ and $\alpha_2 = \lim_{c \downarrow c^*} \beta(c)$. Assumption 5 implies that

$$\underline{\beta} \leq \alpha_2 < \alpha_1 \leq 1.$$

Let $\underline{r} = \rho \frac{\rho+\lambda}{\rho+\lambda(1-\alpha_2)}$ and $\bar{r} = \rho \frac{\rho+\lambda}{\rho+\lambda(1-\alpha_1)}$. Also by Assumption 5, we have

$$\rho < \underline{r} < r < \bar{r}.$$

Consider the initial condition at $a^* = \frac{c^*}{r}$, $(V(a^*), W(a^*)) = (\bar{V}(a^*), \bar{W}(a^*))$. We show that the ODE (18) admits a solution over $[a^*, \infty)$ with the initial condition at a^* . Similarly, we show that the ODE (19) admits a solution over $[\underline{a}, a^*]$, with the initial condition at a^* . Combining the two solutions, we obtain an equilibrium defined over $[\underline{a}, \infty)$.

Indeed, starting at $a^* = \frac{c^*}{r}$, and the initial condition $(\bar{V}(a^*), \bar{W}(a^*))$, because $r < \rho \frac{\rho+\lambda}{\rho+\lambda(1-\alpha_2)}$, $\beta(ra^*) < \hat{\beta}$, we can use Lemma 2, to show the existence of a solution (V, W) to ODE (18), given the initial condition. The solution has a (right) maximal interval of existence $[a^*, \tilde{a})$. If $1 - \alpha_2 > \sigma(U_1, c)$ for all $c \geq c^*$, Theorem 11 shows that $\tilde{a} = +\infty$, for

λ sufficiently high. Otherwise, we follow the steps in the proof of Theorem 3 to restart the procedure each time V crosses \bar{V} (we can always do so since $\beta(c) < \hat{\beta}$ for all $c > c^*$ by Assumption 5). We then obtain an equilibrium over $[a^*, \infty)$ with $\hat{c}(a) > ra$ except for a countable set of steady states at which $\hat{c}(a) = ra$.

Starting at $a^* = \frac{c^*}{r}$, and the initial condition $(\bar{V}(a^*), \bar{W}(a^*))$, we also show that the ODE (19) admits a solution defined over $(\tilde{a}, a^*]$, where $\tilde{a} < a^*$. First consider the case $\alpha_1 < 1$ (the case $\alpha_1 = 1$ will be considered below). The proof follows closely the steps of Lemma 2, i.e. we start with the initial condition

$$(\bar{V}(a^*) + \epsilon, \bar{W}(a^*) + \delta(\epsilon)\epsilon), \quad (47)$$

where $\delta(\epsilon) \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$ is chosen appropriately. In Lemma 34 below, we show that there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$, $\delta(\epsilon)$ can be chosen such that

$$\max \{(\rho + \lambda - r) V'_\epsilon(a^*), \lambda U'_0(\hat{c}_\epsilon(a^*))\} < \lambda W'_\epsilon(a^*) < \lambda V'_\epsilon(a^*).$$

Therefore, following the steps in Lemma 15, we can show that

$$\max \{(\rho + \lambda - r) V'_\epsilon(a), 0\} < \lambda W'_\epsilon(a) < \lambda V'_\epsilon(a), \quad (48)$$

for all a in the (left) maximal interval of existence for V_ϵ, W_ϵ .

As in the proof of Lemma 2, we show that there exists $\bar{\epsilon} > 0$ and $\omega > 0$ such that the ODE (19) with the initial condition (47) admits a unique solution (V_ϵ, W_ϵ) defined over $[a^* - \omega, a^*]$. Moreover, since $\beta(ra_-) > \hat{\beta}$, $V_\epsilon(a) > \bar{V}(a)$ for $a < a^*$.⁴² Therefore, we follow the steps in Lemma 15 to show that $V''_\epsilon(a) < 0$, for all $0 < \epsilon < \bar{\epsilon}$ and $a^* - \omega \leq a \leq a^*$.

Now let $\underline{a} = a^* - \frac{\omega}{2}$, we have

$$\bar{V}(a^*) + \bar{\epsilon} - \bar{V}(a^* - \omega) \geq V_\epsilon(a^*) - V_\epsilon(a^* - \omega) \geq V_\epsilon(\underline{a}) - V_\epsilon(a^* - \omega) > \frac{\omega}{2} V'_\epsilon(\underline{a}),$$

where the last inequality comes from the concavity of V_ϵ . So $V'_\epsilon(\underline{a}) < \frac{2}{\omega} (\bar{V}(a^*) + \bar{\epsilon} - \bar{V}(a^* - \omega))$. Also by the concavity of V_ϵ

$$V'_\epsilon(a) \leq V'_\epsilon(\underline{a}) < \frac{2}{\omega} (\bar{V}(a^*) + \bar{\epsilon} - \bar{V}(a^* - \omega)),$$

for all $a \in [\underline{a}, a^*]$.

Together with (48), we have

$$0 < V'_\epsilon(a), W'_\epsilon(a) < \frac{2}{\omega} (\bar{V}(a^*) + \bar{\epsilon} - \bar{V}(a^* - \omega))$$

for all $a \in [\underline{a}, a^*]$ and $\epsilon \in (0, \bar{\epsilon})$. Therefore, as in Lemma 2, we can apply Dominated Convergence Theorem to show that $(V_\epsilon, W_\epsilon) \rightarrow (V, W)$ over $[\underline{a}, a^*]$ for some subsequence of ϵ and (V, W) is a solution to the ODE (19) over $[\underline{a}, a^*]$. Furthermore, for all $a \in (\underline{a}, a^*)$, $(a, V(a), W(a))$ is a regular point.

⁴²We also prove by contradiction: if $V_{\epsilon_N}(a_N) = \bar{V}(a_N)$ and $a_N \rightarrow a^*$ as $N \rightarrow \infty$,

$$\begin{aligned} \frac{V_{\epsilon_N}(a^*) - \bar{V}(a_N)}{a^* - a_N} &> \frac{\bar{V}(a^*) - \bar{V}(a_N)}{a^* - a_N} \\ &= R_h(\tilde{a}_N, V_{\epsilon_N}(\tilde{a}_N), W_{\epsilon_N}(\tilde{a}_N)) \end{aligned}$$

which at the limit contradicts the condition that $\bar{V}'(a^*) > U'_1(ra^*)$ since $\beta(ra_-) > \hat{\beta}$.

When $\beta(c) = 1 - \alpha_1$ for $c \leq c^*$. Consider left maximal interval of existence, $(\hat{a}, a^*]$ of (V_ϵ, W_ϵ) as a solution to the ODE (19) from a^* with the initial conditions $(V_\epsilon(a^*), W_\epsilon(a^*)) = (\bar{V}(a^*) + \epsilon, \bar{W}(a^*) + \epsilon)$. As shown above, for ϵ sufficiently small, at a^* ,

$$\begin{aligned} \max \left\{ \frac{\rho + \lambda - r}{\lambda} V'_\epsilon(a^*), 0 \right\} &< W'_\epsilon(a^*) \leq V'_\epsilon(a^*) \\ V''_\epsilon(a^*) &< 0 \\ W'_\epsilon(a^*) &\geq U'_0(\hat{c}_\epsilon(a^*)) \\ V_\epsilon(a^*) &> \bar{V}(a^*). \end{aligned}$$

By Lemmas 15, 31, and 32 (when $\lim_{c \uparrow c^*} \beta(c) = 1$, $W'_\epsilon(a) = V'_\epsilon(a) = U'_1(\hat{c}_\epsilon(a)) = U'_0(\hat{c}_\epsilon(a))$ for $a < a^*$, and $W'_\epsilon(a) > U'_0(\hat{c}_\epsilon(a))$ for $a < a^*$), these properties hold for all $a \in (\hat{a}, a^*)$. We show by contradiction that $\hat{a} = 0$. Assume by contradiction that this is not the case, i.e. $\hat{a} > 0$. Then the ODE (19) reaches a singular point at \hat{a} . By Lemma 30, $V_\epsilon(\hat{a}) = \bar{V}(\hat{a})$. Because $V_\epsilon(a) > \bar{V}(a)$ for all $a > \hat{a}$, $V'_\epsilon(\hat{a}) \geq \bar{V}'(\hat{a})$. In addition, because \hat{a} is a singular point, $V'_\epsilon(\hat{a}) = U'_1(r\hat{a})$. So $U'_1(r\hat{a}) \geq \bar{V}'(\hat{a})$, or equivalently $\beta(r\hat{a}) = 1 - \alpha_1 \leq \hat{\beta}$, by Lemma 6. This contradicts the assumption that $r < \bar{r}$. We obtain the desired contradiction. Thus, (V_ϵ, W_ϵ) is defined over $(0, a^*]$.

Given any $\underline{a} > 0$, let $\omega \in (0, \underline{a})$. We have

$$\bar{V}(a^*) + \epsilon - \bar{V}\left(\frac{\omega}{2}\right) \geq V_\epsilon(a^*) - V_\epsilon\left(\frac{\omega}{2}\right) \geq V_\epsilon(\omega) - V_\epsilon\left(\frac{\omega}{2}\right) > \frac{\omega}{2} V'_\epsilon(\omega),$$

where the last inequality comes from the concavity of V_ϵ . So $V'_\epsilon(\omega) < \frac{2}{\omega} (\bar{V}(a^*) + \epsilon - \bar{V}(\frac{\omega}{2}))$. Also by the concavity of V_ϵ

$$0 < V'_\epsilon(a) \leq V'_\epsilon(\omega) < \frac{2}{\omega} \left(\bar{V}(a^*) + \bar{\epsilon} - \bar{V}\left(\frac{\omega}{2}\right) \right),$$

for all $a \in [\omega, a^*]$. W'_ϵ are bounded by the same bounds. Similar to the proof of Lemma 2, there exists a subsequence $\epsilon \rightarrow 0$ such that (V_ϵ, W_ϵ) converges to some limit (V, W) which is a solution to ODE (19) over $[\omega, a^*]$, thus over $[\underline{a}, a^*]$, with the initial condition $(a^*, \bar{V}(a^*), \bar{W}(a^*))$. \square

Lemma 34. Assume that $\alpha_1 < 1$. There exists $\bar{\epsilon} > 0$ such that for $\epsilon \in (0, \bar{\epsilon})$, there exists $\delta(\epsilon) \in \left[1, \frac{\rho + \lambda}{\lambda}\right)$, such that

$$\max \{ (\rho + \lambda - r) V'_\epsilon(a^*), \lambda U'_0(\hat{c}_\epsilon(a^*)) \} < \lambda W'_\epsilon(a^*) < \lambda V'_\epsilon(a^*),$$

where $(V'_\epsilon(a^*), W'_\epsilon(a^*)) = (R_h, S_h) (a^*, \bar{V}(a^*) + \epsilon, \bar{W}(a^*) + \delta(\epsilon)\epsilon)$ and

$$\hat{c}_\epsilon(a^*) = (U'_1)^{-1} (V'_\epsilon(a^*)) < ra^*.$$

Proof. Because

$$r > \underline{r} = \rho \frac{\rho + \lambda}{\rho + \lambda(1 - \alpha_2)} > \rho,$$

we have $(\rho + \lambda - r) < \lambda$.

First, we show that there exists, $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\delta \in \left[1, \frac{\rho + \lambda}{\lambda}\right)$,

$$U'_0(\hat{c}_{\epsilon, \delta}(a^*)) < V'_{\epsilon, \delta}(a^*) \tag{49}$$

where $(V'_{\epsilon,\delta}(a^*), W'_{\epsilon,\delta}(a^*)) = (R_h, S_h)(a^*, \bar{V}(a^*) + \epsilon, \bar{W}(a^*) + \delta\epsilon)$ and

$$\hat{c}_{\epsilon,\delta}(a^*) = (U'_1)^{-1}(V'_{\epsilon,\delta}(a^*)) < ra^*.$$

Indeed, because R_h is continuous, for each $\delta \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$,

$$\begin{aligned}\lim_{\epsilon \rightarrow 0} V'_{\epsilon,\delta}(a^*) &= U'_1(ra^*) \\ \lim_{\epsilon \rightarrow 0} \hat{c}_{\epsilon,\delta}(a^*) &= ra^*.\end{aligned}$$

In addition, $\beta(ra^*_-) = \alpha_1 < 1$ which is equivalent to $U'_0(ra^*_-) < U'_1(ra^*)$. Therefore there exists, $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\delta \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$ such that (49) holds. Because $\frac{\rho+\lambda-r}{\lambda} < 1$, this also implies

$$\max \left\{ \frac{\rho + \lambda - r}{\lambda} V'_{\epsilon,\delta}(a^*), U'_0(\hat{c}_{\epsilon,\delta}(a^*)) \right\} < V'_{\epsilon,\delta}(a^*).$$

By Lemma 35, we can choose $\bar{\epsilon}$ such that for all $\epsilon \in (0, \bar{\epsilon})$, $W'_{\epsilon,1}(a^*) < V'_{\epsilon,1}(a^*)$. It is easy to see that

$$\lim_{\delta \uparrow \frac{\rho+\lambda}{\lambda}} W'_{\epsilon,\delta}(a^*) = +\infty > \max \left\{ \frac{\rho + \lambda - r}{\lambda} V'_{\epsilon,\delta}(a^*), U'_0(\hat{c}_{\epsilon,\delta}(a^*)) \right\}.$$

So by the Intermediate Value Theorem, there exists $\delta(\epsilon)$ such that

$$\max \{(\rho + \lambda - r) V'_\epsilon(a^*), \lambda U'_0(\hat{c}_\epsilon(a^*))\} < \lambda W'_\epsilon(a^*) < \lambda V'_\epsilon(a^*).$$

□

Lemma 35. For $\epsilon > 0$, let $V_\epsilon(a^*) = \bar{V}(a^*) + \epsilon$ and $W_\epsilon(a^*) = \bar{W}(a^*) + \epsilon$. We have

$$\lim_{\epsilon \rightarrow 0} S_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*)) = U'_0(ra^*_-).$$

Proof. From the definition of R_h, S_h ,

$$\begin{aligned}V'_\epsilon(a^*) &= R_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*)) \\ W'_\epsilon(a^*) &= S_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*)).\end{aligned}$$

Also by the definition of $V_\epsilon(a^*), W_\epsilon(a^*)$, $(\lambda + \rho)V_\epsilon(a^*) - \lambda W_\epsilon(a^*) = U_1(ra^*) + \rho\epsilon$. Using the Taylor expansion for $H(p, a)$ around $p^* = U'_1(ra^*)$, we obtain

$$\begin{aligned}\lambda\epsilon + U_1(ra^*) &= H(V'_\epsilon(a^*), a^*) \\ &= H(p^*, a^*) + \frac{\partial H(p^*, a^*)}{\partial p}(V'_\epsilon(a^*) - p^*) \\ &\quad + \frac{1}{2} \frac{\partial^2 H(p^*, a^*)}{\partial p^2}(V'_\epsilon(a^*) - p^*)^2 + o((V'_\epsilon(a^*) - p^*)^2).\end{aligned}$$

From the proof of Lemma 4, $H(p^*, a^*) = U_1(ra^*)$ and $\frac{\partial H(p^*, a^*)}{\partial p} = 0$. In addition,

$$\frac{\partial^2 H(p^*, a^*)}{\partial p^2} = -\frac{1}{U''_1((U'_1)^{-1}(p^*))} = -\frac{1}{U''_1(ra^*)} > 0.$$

Therefore

$$\rho\epsilon = -\frac{1}{2U_1''(ra^*)}(V_\epsilon'(a^*) - U_1'(ra^*))^2 + o((V_\epsilon'(a^*) - p^*)^2).$$

Consequently

$$V_\epsilon'(a^*) - U_1'(ra^*) = \sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon}).$$

By the definition of \hat{c}_ϵ ,

$$\begin{aligned}\hat{c}_\epsilon(a^*) - ra^* &= (U_1')^{-1}(V_\epsilon'(a^*)) - (U_1')^{-1}(U_1'(ra^*)) \\ &= \frac{1}{U_1''(ra^*)}(V_\epsilon'(a^*) - U_1'(ra^*)) + o(V_\epsilon'(a^*) - U_1'(ra^*)) \\ &= \frac{1}{U_1''(ra^*)}\sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon}).\end{aligned}$$

Therefore,

$$\begin{aligned}W_\epsilon'(ra) &= \frac{\rho W_\epsilon(a^*) - U_0(\hat{c}_\epsilon(a^*))}{ra^* - \hat{c}_\epsilon(a)} = \frac{U_0(ra^*) - U_0(\hat{c}_\epsilon(a^*)) - \rho\epsilon}{ra^* - \hat{c}_\epsilon(a^*)} \\ &= \frac{U_0(ra^*) - U_0(\hat{c}_\epsilon(a^*))}{ra^* - \hat{c}_\epsilon(a^*)} - \frac{\rho\epsilon}{\frac{1}{-U_1''(ra^*)}\sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon})} \\ &\longrightarrow U_0'(ra_-),\end{aligned}$$

as $\epsilon \rightarrow 0$. □

J Derivations for Inverting Results in Section 6

Proof of Theorem 9. Differentiating (6a), and noticing that $V'(a) = U_1'(\hat{c}(a))$, we obtain

$$(\rho + \lambda - r)U_1'(\hat{c}(a)) = U_1''(\hat{c}(a))\hat{c}'(a)(ra - \hat{c}(a)) + \lambda W'(a).$$

From this equation, we can solve for $W'(a)$ as a function of $\hat{c}(a), \hat{c}'(a), a$:

$$\lambda W'(a) = (\rho + \lambda - r)U_1'(\hat{c}(a)) - U_1''(\hat{c}(a))\hat{c}'(a)(ra - \hat{c}(a)).$$

Differentiating the last equation, we can also write $W''(a)$ as a function of $\hat{c}, \hat{c}', \hat{c}'', a$

$$\begin{aligned}\lambda W''(a) &= (\rho + \lambda - r)U_1''(\hat{c}(a))\hat{c}'(a) - U_1'''(\hat{c}(a))(\hat{c}'(a))^2(ra - \hat{c}(a)) \\ &\quad - U_1''(\hat{c}(a))\hat{c}''(a)(ra - \hat{c}(a)) - U_1''(\hat{c}(a))\hat{c}'(a)(r - \hat{c}'(a)) \\ &= U_1''(\hat{c}(a))\left((\rho + \lambda - 2r)\hat{c}'(a) + (\hat{c}'(a))^2 - \hat{c}''(a)(ra - \hat{c}(a))\right) \\ &\quad - U_1'''(\hat{c}(a))(\hat{c}'(a))^2(ra - \hat{c}(a)).\end{aligned}$$

Now differentiating (6b) and rearranging, we obtain

$$(\rho - r + \hat{c}'(a))W'(a) = U_0'(\hat{c}(a))\hat{c}'(a) + W''(a)(ra - \hat{c}(a)).$$

Substituting in the expressions of W' and W'' above, and using the fact that $U'_0(c) = \beta(c)U'_1(c)$, we arrive at

$$\begin{aligned} & (\rho - r + \hat{c}'(a)) \left((\rho + \lambda - r)U'_1(\hat{c}(a)) - U''_1(\hat{c}(a))\hat{c}'(a) (ra - \hat{c}(a)) \right) \\ &= \lambda\beta(\hat{c}(a))U'_1(\hat{c}(a))\hat{c}'(a) \\ &+ U''_1(\hat{c}(a)) \left((\rho + \lambda - 2r)\hat{c}'(a) + (\hat{c}'(a))^2 - \hat{c}''(a) (ra - \hat{c}(a)) \right) (ra - \hat{c}(a)) \\ &- U'''_1(\hat{c}(a)) (\hat{c}'(a))^2 (ra - \hat{c}(a))^2. \end{aligned}$$

Finally, dividing both sides by $U'_1(\hat{c}(a))$ and simplifying, we get

$$\begin{aligned} & (\rho - r + \hat{c}'(a)) (\rho + \lambda - r) \\ &= \lambda\beta(\hat{c}(a))\hat{c}'(a) - \frac{U'''_1(\hat{c}(a))\hat{c}(a)^2}{U'_1(\hat{c}(a))} (\hat{c}'(a))^2 \left(\frac{ra - \hat{c}(a)}{\hat{c}(a)} \right)^2 \\ &- \sigma(U_1, \hat{c}(a)) \left((2\rho + \lambda - 3r)\hat{c}'(a) + 2(\hat{c}'(a))^2 - \hat{c}''(a) (ra - \hat{c}(a)) \right) \frac{(ra - \hat{c}(a))}{\hat{c}(a)}. \end{aligned}$$

Since ζ is the inverse of \hat{c} , $a = \zeta(\hat{c}(a))$. Therefore, $\zeta' = \frac{1}{\hat{c}'}$ and $\zeta'' = \frac{-\hat{c}''}{\hat{c}'^2}$. In addition, from the definition of σ , (13), $\sigma' = -\frac{U'''_1}{U'_1}c + \frac{U''^2}{U'^2}c - \frac{U''}{U'} = \frac{1}{c} \left(-\frac{U'''_1}{U'_1}c^2 + \sigma^2 + \sigma \right)$, which implies $-\frac{U'''_1}{U'_1}c^2 = c\sigma' - \sigma^2 - \sigma$. Plugging these identities into the last equation, we arrive at (14). \square

Now, we apply (14) to the parametric Example 2. Noticing that $\sigma \equiv \bar{\sigma}$, then $\sigma' \equiv 0$, and $\zeta(c) \equiv \frac{c-r\tilde{a}}{\Psi} + \tilde{a}$, $\zeta' \equiv \frac{1}{\Psi}$, (14) becomes

$$\begin{aligned} \lambda\beta\frac{1}{\Psi} &= \alpha_1\frac{1}{\Psi} + \alpha_2 \left(\frac{1}{\Psi} \right)^2 \\ &+ \bar{\sigma} \left(2 + (2\rho + \lambda - 3r)\frac{1}{\Psi} \right) \left(1 - \frac{r}{\Psi} \right) \left(\frac{r\tilde{a}}{c} - 1 \right) + (\bar{\sigma}^2 + \bar{\sigma}) \left(\left(1 - \frac{r}{\Psi} \right) \left(\frac{r\tilde{a}}{c} - 1 \right) \right)^2, \end{aligned}$$

since $\frac{r\zeta-c}{c} = \frac{r(\frac{c-r\tilde{a}}{\Psi} + \tilde{a})-c}{c} = \left(1 - \frac{r}{\Psi} \right) \left(\frac{r\tilde{a}}{c} - 1 \right)$. Dividing both sides by $\lambda\frac{1}{\Psi}$ and simplifying, we obtain the expression for $\beta(c)$ given in Example 2.

K Proofs for Further Characterizations

Proof of Theorem 10. Consider the construction of equilibrium in Theorem 3 and let $(V, W) = (V_0, W_0)$ and $[\underline{a}, a^*)$ denote its maximal interval of existence. First of all, we show that there exists $\epsilon \in (0, 1)$ such that $\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r)V'(a)$, for all $a \in [\underline{a}, a^*)$. Then we show that $a^* = \infty$.

Because $r < \rho$, there exists $\epsilon \in (0, 1)$ (sufficiently small) such that

$$\frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} > 1$$

and

$$(\rho - r) \frac{1}{\lambda} \frac{1 - \epsilon}{\epsilon} > \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda}.$$

In the proof of Lemma 2, we show that

$$\lim_{a \downarrow \underline{a}} W'(a) \leq U'_0(r\underline{a}) \leq U'_1(r\underline{a}) = \lim_{a \downarrow \underline{a}} V'(a).$$

Therefore, in the right neighborhood of \underline{a} ,

$$\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r)V'(a),$$

because $(1 - \epsilon)(\lambda + \rho - r) > \lambda$.

We use Lemma 7 (Variation 2) to show that $\lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r)V'(a)$ for all $a^* > a > \underline{a}$. We just showed that this is true in the right neighborhood of \underline{a} , so the first condition in Lemma 7 is satisfied. Now, we show that the second (relaxed) condition in Lemma 7 is also satisfied, i.e. if there exists $\tilde{a} > \underline{a}$ such that $\lambda W'(\tilde{a}) = (1 - \epsilon)(\rho + \lambda - r)V'(\tilde{a})$, then $\lambda W''(a) < (1 - \epsilon)(\rho + \lambda - r)V''(a)$ in the left neighborhood of \tilde{a} .⁴³

Indeed, in the left neighborhood of \tilde{a} , $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a)$, therefore

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r)V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon(\rho + \lambda - r)V'(a)}{\hat{c}(a) - ra} < 0,$$

Differentiating equation (6b), we obtain

$$\begin{aligned} W''(a) &= \frac{(U'_0(\hat{c}(a))\hat{c}'(a) - \rho W'(a))(\hat{c}(a) - ra) - (U_0(\hat{c}(a)) - \rho W(a))(\hat{c}'(a) - r)}{(\hat{c}(a) - ra)^2} \\ &= \frac{(U'_0(\hat{c}(a))\hat{c}'(a) - \rho W'(a))}{\hat{c}(a) - ra} - \frac{W'(a)(\hat{c}'(a) - r)}{\hat{c}(a) - ra}. \end{aligned}$$

Therefore,

$$\begin{aligned} W''(a) &= \frac{(U'_0(\hat{c}(a)) - W'(a))\hat{c}'(a)}{\hat{c}(a) - ra} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra} \\ &= \frac{(U'_0(\hat{c}(a)) - W'(a))}{\hat{c}(a) - ra} \frac{V''(a)}{U'_1(\hat{c}(a))} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra} \\ &= \frac{(W'(a) - U'_0(\hat{c}(a)))}{(\hat{c}(a) - ra)(-U''_1(\hat{c}(a)))} V''(a) + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}. \end{aligned}$$

When a close to \tilde{a} , we also have:

$$\begin{aligned} V''(a) &= \frac{\lambda W'(a) - (\lambda + \rho - r)V'(a)}{\hat{c}(a) - ra} \\ &\approx \frac{-\lambda \frac{\epsilon}{1 - \epsilon} W'(a)}{\hat{c}(a) - ra}, \end{aligned}$$

because, by continuity, when a close to \tilde{a} , $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a)$. Therefore, $W'(a) \approx -\frac{1}{\lambda} \frac{1 - \epsilon}{\epsilon} V''(a)(\hat{c}(a) - ra)$. Plugging this back to the expression for W'' above, we

⁴³We use Variation 2 of Lemma 7 because if $\tilde{a} = a^*$, W' and W'' might not exist at a^* .

have

$$\begin{aligned}
W''(a) &\approx \frac{\left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda}U_1'(\hat{c}(a)) - U_0'(\hat{c}(a))\right)}{(\hat{c}(a) - ra)(-U_1''(\hat{c}(a)))}V''(a) - \frac{(r-\rho)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}V''(a)(\hat{c}(a) - ra)}{\hat{c}(a) - ra} \\
&= \left(\left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))}\right)\frac{1}{\sigma(U_1, \hat{c}(a))}\frac{\hat{c}(a)}{\hat{c}(a) - ra} + (\rho-r)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}\right)V''(a) \\
&< \frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda}V''(a),
\end{aligned}$$

where the last inequality comes from

$$\begin{aligned}
&\left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))}\right)\frac{1}{\sigma(U_1, \hat{c}(a))}\frac{\hat{c}(a)}{\hat{c}(a) - ra} \\
&\geq \left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} - 1\right)\frac{1}{\sigma(U_1, \hat{c}(a))}\frac{\hat{c}(a)}{\hat{c}(a) - ra} > 0,
\end{aligned}$$

and

$$(\rho-r)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon} > \frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda}$$

and $V''(a) < 0$. Therefore both conditions 1) and 2) in Lemma 7 are satisfied, and by that lemma, $\lambda W'(a) \leq (1-\epsilon)(\rho+\lambda-r)V'(a)$ for all $a < a^*$.

We prove by contradiction that a^* is infinite. Assume by contradiction that a^* is finite. Let $F(a) = (\rho+\lambda)V(a) - \lambda W(a) - U_1(ra)$. At $a = a^*$, $F(a) = 0$ and

$$\begin{aligned}
F'(a) &= (\rho+\lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\
&> (\rho+\lambda)V'(a) - (1-\epsilon)(\rho+\lambda-r)V'(a) - rU_1'(ra) \\
&= (\rho+\lambda - (1-\epsilon)(\rho+\lambda-r) - r)U_1'(ra) \\
&= \epsilon(\rho+\lambda-r)U_1'(ra) > 0.
\end{aligned}$$

So $F(a) < 0$ in the left neighborhood of a^* . This is a contradiction. Thus $a^* = +\infty$, i.e. (V, W) is defined over $[a, \infty)$.

By Lemma 5,

$$\begin{aligned}
\hat{c}'(a) &= \frac{V''(a)}{U_1''(\hat{c}(a))} = \frac{(\lambda+\rho-r)V'(a) - \lambda W'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} \\
&> \frac{\epsilon(\lambda+\rho-r)V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > \frac{\epsilon\lambda V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > 0,
\end{aligned}$$

where the last inequality comes from $r < \rho$. □

Proof of Theorem 11. As in the proof of Theorem 10 (using the same definition of V, W and a^*), first, we show that there exists $\epsilon \in (0, 1)$ such that $\lambda W'(a) < (1-\epsilon)(\lambda+\rho-r)V'(a)$, for all $a \in [a, a^*)$. Then we show that $a^* = \infty$.

Condition (15) implies that $\sup_{c > ra} \beta(c) < 1$. Therefore, there exists $\epsilon \in (0, 1)$ such that

$$\beta(c) < 1 - \epsilon$$

and

$$(1 - \epsilon) - \beta(c) > (1 - \epsilon)\sigma(c)$$

for all $c > r\underline{a}$.⁴⁴ Therefore, given $\rho \leq r$, there exists $\bar{\lambda} \geq 0$ such that for all $\lambda > \bar{\lambda}$, we have $\beta(r\underline{a}) < \hat{\beta}(r, \rho, \lambda)$ (since $\beta(r\underline{a}) < \frac{\rho}{r}$) and

$$\lambda U'_0(r\underline{a}) < (1 - \epsilon)(\lambda + \rho - r) U'_1(r\underline{a}) \quad (50)$$

and for all $c > r\underline{a}$,

$$\frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U'_0(c)}{U'_1(c)} > \left(\frac{r - \rho}{\lambda} \frac{1 - \epsilon}{\epsilon} + \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} \right) \sigma(c). \quad (51)$$

Moreover $\bar{\lambda}$ can be chosen to be increasing in r and $\bar{\lambda}(\rho) = 0$.^{45,46}

By (50),

$$\rho + \lambda - r > 0. \quad (52)$$

Since $\beta(r\underline{a}) < \hat{\beta}$ we can apply Lemma 2. Besides, in the proof of Lemma 2, we show that

$$\begin{aligned} \lim_{a \downarrow \underline{a}} W'(a) &\leq U'_0(r\underline{a}), \\ \lim_{a \downarrow \underline{a}} V'(a) &= U'_1(r\underline{a}). \end{aligned}$$

Therefore, by (50),

$$\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r) V'(a)$$

in the right neighborhood of \underline{a} .

Given these three conditions, as in the proof of Theorem 10, we use Lemma 7 (Variation 2) to show that $\lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r) V'(a)$ for all $a > \underline{a}$. As shown above, this is true in the right neighborhood of \underline{a} so the first condition in Lemma 7 is satisfied. Now we show that the second (relaxed) condition in Lemma 7 is also satisfied, i.e. if there exists $\tilde{a} > \underline{a}$ such that $\lambda W'(\tilde{a}) = (1 - \epsilon)(\rho + \lambda - r) V'(\tilde{a})$, we show that $\lambda W''(a) < (1 - \epsilon)(\rho + \lambda - r) V''(a)$ in the left neighborhood of \tilde{a} . Indeed, in the left neighborhood of \tilde{a} , $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r) V'(a)$, therefore

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon(\rho + \lambda - r) V'(a)}{\hat{c}(a) - ra} < 0,$$

Differentiating equation (6b) and simplifying as done in the proof of Theorem 10:

$$W''(a) = \frac{(W'(a) - U'_0(\hat{c}(a)))}{(\hat{c}(a) - ra)(-U''_1(\hat{c}(a)))} V''(a) + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}.$$

When a close to \tilde{a} , we also have:

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx \frac{-\lambda \frac{\epsilon}{1 - \epsilon} W'(a)}{\hat{c}(a) - ra},$$

⁴⁴This is equivalent to $1 - \sigma(c) > \frac{1}{1 - \epsilon} \beta(c)$, for some $\epsilon \in (0, 1)$ which is true given (15).

⁴⁵When $r = \rho$, (51) becomes $(1 - \epsilon) - \beta(c) > (1 - \epsilon)\sigma(U_1, c)$.

⁴⁶Similarly, given $\lambda > 0$ there exists $r_1 > \rho$ such that for $r \in [\rho, r_1]$, (50) and (51) hold. The proof for existence of continuous Markov equilibrium then proceeds in exactly the same way.

because, by continuity, when a close to \tilde{a} , $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a)$. Therefore, $W'(a) \approx -\frac{1}{\lambda} \frac{1-\epsilon}{\epsilon} V''(a)(\hat{c}(a) - ra)$. Plugging this back to the expression for W'' above, we have

$$\begin{aligned} W''(a) &\approx \frac{\left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} U_1'(\hat{c}(a)) - U_0'(\hat{c}(a))\right)}{(\hat{c}(a) - ra) (-U_1''(\hat{c}(a)))} V''(a) - \frac{(r - \rho) \frac{1}{\lambda} \frac{1-\epsilon}{\epsilon} V''(a)(\hat{c}(a) - ra)}{\hat{c}(a) - ra} \\ &= \left(\left(\frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))} \right) \frac{1}{\sigma(U_1, \hat{c}(a))} \frac{\hat{c}(a)}{\hat{c}(a) - ra} - (r - \rho) \frac{1}{\lambda} \frac{1 - \epsilon}{\epsilon} \right) V''(a) \\ &< \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} V''(a), \end{aligned}$$

where the last inequality comes from (51) and $V''(a) < 0$. Therefore both conditions 1) and 2) in Lemma 7 are satisfied, and by that lemma, $\lambda W'(a) \leq (1 - \epsilon)(\rho + \lambda - r)V'(a)$ for all $a \leq a^*$.

As in the proof of Theorem 10, we prove by contradiction that a^* is infinite. Assume by contradiction that a^* is finite. Let $F(a) = (\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)$. At $a = a^*$, $F(a) = 0$ and

$$\begin{aligned} F'(a) &= (\rho + \lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\ &> (\rho + \lambda)V'(a) - (1 - \epsilon)(\rho + \lambda - r)V'(a) - rU_1'(ra) \\ &= (\rho + \lambda - (1 - \epsilon)(\rho + \lambda - r) - r) U_1'(ra) \\ &= \epsilon(\rho + \lambda - r) U_1'(ra) > 0, \end{aligned}$$

where the last inequality comes from (52). So $F(a) < 0$ in the left neighborhood of a^* . This is a contradiction. Thus $a^* = +\infty$.

Similar to the proof of Theorem 10,

$$\hat{c}'(a) = \frac{(\lambda + \rho - r) V'(a) - \lambda W'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > \frac{\epsilon(\lambda + \rho - r) V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > 0,$$

where the last inequality also comes from (52). \square

Proof of Theorem 12. We use the notation $\bar{V}_\lambda, \bar{W}_\lambda$ for the functions defined in (8). Notice that

$$\bar{W}_\lambda(a) = \frac{1}{\rho} U_0(ra),$$

independent of λ , so we can drop the subscript λ .

First, we notice that since $\sup_{c \geq ra} \beta(c) < \frac{\rho}{r}$, there exists $\lambda^* > 0$ such that $\beta(c) < \hat{\beta}$ for all $\lambda \geq \lambda^*$. Therefore, we can apply Lemma 2 and Theorem 3 to construct Markov equilibria with dissaving. Let V_λ, W_λ denote the equilibrium value functions constructed in the proof of Theorem 3. We show that there exists $\bar{\lambda} > \lambda^*$, such that for all $\lambda \geq \bar{\lambda}$, V_λ crosses \bar{V}_λ at some $a_1(\lambda) > \underline{a}$ and $\lim_{\lambda \rightarrow \infty} a_1(\lambda) = \underline{a}$. In addition, $W_\lambda(a_1(\lambda)) < \bar{W}_\lambda(a_1(\lambda))$. This result immediate implies the existence of a Markov equilibrium with dissaving and discontinuous policy function because starting from $a_1(\lambda)$ we can apply Theorem 3 to obtain a Markov equilibrium defined over $[a_1(\lambda), \infty)$ with $(V, W) = (\bar{V}_\lambda, \bar{W}_\lambda)$ at $a_1(\lambda)$. Combining this equilibrium with (V_λ, W_λ) defined over $[\underline{a}, a_1(\lambda))$, we obtain a discontinuous equilibrium over $[\underline{a}, \infty)$.

We prove the result by contradiction. Assume that the result does not hold, then there exists $\bar{a} > \underline{a}$ and a sequence of $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$, such that

$$V_{\lambda_n}(a) > \bar{V}_{\lambda_n}(a)$$

for all $a \in (\underline{a}, \bar{a})$.⁴⁷ Because $\lim_{a \downarrow \underline{a}} \frac{U_0(ra) - U_0(\underline{a})}{a - \underline{a}} = rU'_0(ra)$, and $r > \rho$, there exists $a_1 \in (\underline{a}, \bar{a})$ and $0 < \gamma$ such that

$$\frac{1}{\rho} \frac{U_0(ra_1) - U_0(\underline{a})}{a_1 - \underline{a}} > (\gamma + 1) U'_0(ra).$$

First, using Lemma 36 below, we show that

$$\lim_{n \rightarrow \infty} W_{\lambda_n}(a_1) = \bar{W}(a_1).$$

Indeed, by Lemma 11, $W_{\lambda_n}(a_1) - \bar{W}(a_1) \leq 0$. Therefore

$$\limsup_{n \rightarrow \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) \leq 0. \quad (53)$$

Now,

$$\begin{aligned} W_{\lambda_n}(a_1) - \bar{W}(a_1) &= W_{\lambda_n}(a_1) - V_{\lambda_n}(a_1) + V_{\lambda_n}(a_1) - \bar{V}_{\lambda_n}(a_1) \\ &\quad + \bar{V}_{\lambda_n}(a_1) - \bar{W}_{\lambda_n}(a_1). \end{aligned}$$

By Lemma 36,

$$\lim_{n \rightarrow \infty} (V_{\lambda_n}(a_1) - W_{\lambda_n}(a_1)) = 0.$$

By the definition of $\bar{V}_{\lambda}, \bar{W}_{\lambda}$ in (8)

$$\lim_{n \rightarrow \infty} (\bar{V}_{\lambda_n}(a_1) - \bar{W}_{\lambda_n}(a_1)) = 0,$$

and by the contradiction assumption

$$V_{\lambda_n}(a_1) - \bar{V}_{\lambda_n}(a_1) \geq 0.$$

Thus

$$\liminf_{n \rightarrow \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) \geq 0. \quad (54)$$

Therefore by (53) and (54)

$$\lim_{n \rightarrow \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) = 0.$$

Given this limit, for $\epsilon > 0$, sufficiently small, there exists N such that $W_{\lambda_n}(a_1) - \bar{W}(a_1) > -\epsilon$ for all $n \geq N$. Now,

$$\frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(\underline{a})}{a_1 - \underline{a}} > \frac{\bar{W}(a_1) - \bar{W}(\underline{a}) - \epsilon}{a_1 - \underline{a}} > (\gamma + 1) U'_0(ra) - \frac{\epsilon}{a_1 - \underline{a}}. \quad (55)$$

⁴⁷By Lemma 11, $W_{\lambda_n}(a) < \bar{W}_{\lambda}(a)$. Therefore $(\rho + \lambda_n) V_{\lambda_n}(a) - \lambda_n W_{\lambda_n}(a) > (\rho + \lambda_n) \bar{V}_{\lambda_n}(a) - \lambda_n \bar{W}_{\lambda_n}(a) = U_1(ra)$. Thus $V_{\lambda_n}, W_{\lambda_n}$ are defined and continuous over $[\underline{a}, \bar{a}]$.

By the Mean Value Theorem, there exists $a_n \in (\underline{a}, a_1)$ such that,

$$\begin{aligned} W'_{\lambda_n}(a_n) &= \frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(\underline{a})}{a_1 - \underline{a}} \\ &\leq U'_0(ra_n) + \frac{\rho}{\lambda_n} U'_1(ra_n) \\ &\leq U'_0(r\underline{a}) + \frac{\rho}{\lambda_n} U'_1(r\underline{a}), \end{aligned} \quad (56)$$

where the first inequality comes from the proof of Lemma 10 (especially inequality (30)). By choosing ϵ sufficiently small and n sufficiently large such that

$$\frac{\epsilon}{a_1 - \underline{a}} + \frac{\rho}{\lambda_n} U'_1(r\underline{a}) < \gamma U'_0(r\underline{a}),$$

which contradicts (55) and (56). We obtain the desired contradiction. \square

Lemma 36. *Assume that there exists $\bar{a} > \underline{a}$ and a diverging sequence $\{\lambda_n\}$ such that $V_{\lambda_n}(a) > \bar{V}_{\lambda_n}(a)$ for all $a \in (\underline{a}, \bar{a})$. Then*

$$\lim_{n \rightarrow \infty} (V_{\lambda_n}(a) - W_{\lambda_n}(a)) = 0, \quad (57)$$

for all $a \in (\underline{a}, \bar{a})$.

Proof. By Lemma 11, $W_{\lambda_n} \leq \bar{W}$ therefore

$$V_{\lambda_n}(a) - W_{\lambda_n}(a) \geq \bar{V}_{\lambda_n}(a) - \bar{W}_{\lambda_n}(a)$$

for all $a \in (\underline{a}, \bar{a})$.

To find an upper bound on $V_{\lambda_n} - W_{\lambda_n}$. We rewrite equation (6a) as

$$\lambda (V_\lambda(a) - W_\lambda(a)) = U_1(\hat{c}_\lambda(a)) + V'_\lambda(a)(ra - \hat{c}_\lambda(a)) - \rho V_\lambda(a).$$

Therefore

$$\lambda (V_\lambda(a) - W_\lambda(a)) \leq U_1(\hat{c}_\lambda(a)) - \rho \bar{V}_\lambda(\underline{a}), \quad (58)$$

because $V_\lambda(a) > V_\lambda(\underline{a}) = \bar{V}_\lambda(\underline{a})$, and $V'_\lambda \geq 0$, and $ra - \hat{c}_\lambda(a) < 0$.

Now if U_1 is bounded above

$$\lambda (V_\lambda(a) - W_\lambda(a)) \leq \sup_c U_1(c) - \rho \bar{V}_\lambda(\underline{a}).$$

Thus $\lambda |V_\lambda(a) - W_\lambda(a)|$ is bounded when $\lambda \rightarrow \infty$. Therefore (57) holds.

If U_1 is not bounded, by Assumption 2, for some $\underline{\sigma} > 0$, $\sigma(c) > \underline{\sigma}$ for all $c \geq \bar{c}$. We show, using Lemma 7, that there exists $\bar{\lambda}$ such that, when $\lambda > \bar{\lambda}$, $\hat{c}_\lambda(a) < \frac{2\lambda}{\underline{\sigma}}a$, for all $a \in (\underline{a}, \bar{a})$. Let $f(a) = \frac{2\lambda}{\underline{\sigma}}a$ and $g(a) = \hat{c}_\lambda(a)$. With $\lambda > \underline{\sigma}r$, $f(\underline{a}) = \frac{2\lambda}{\underline{\sigma}} > r\underline{a}$. We just need to verify that if $f(a) = g(a)$ then $f'(a) = \frac{2\lambda}{\underline{\sigma}} > g'(a) = \hat{c}'_\lambda(a)$. Indeed, by differentiating, the first order condition (6c) with respect to a ,

$$\hat{c}'_\lambda(a) = \frac{V''_\lambda(a)}{U''_1(\hat{c}_\lambda(a))}.$$

To get $V''_\lambda(a)$, differentiating (6a) with respect to a and use the first order condition for c , we obtain

$$\begin{aligned}
V_\lambda''(a)(ra - \hat{c}_\lambda(a)) &= (\rho + \lambda - r) V_\lambda'(a) - \lambda W_\lambda'(a) \\
&= (\rho + \lambda - r) U_1'(\hat{c}_\lambda(a)) - \lambda W_\lambda'(a).
\end{aligned}$$

Therefore, because $W_\lambda' \geq 0$ as shown in Lemma 10,

$$\begin{aligned}
\hat{c}'_\lambda(a) &= \frac{(\rho + \lambda - r) U_1'(\hat{c}_\lambda(a)) - \lambda W_\lambda'(a)}{-U_1''(\hat{c}_\lambda(a)) (\hat{c}_\lambda(a) - ra)} \leq \frac{(\rho + \lambda - r) U_1'(\hat{c}_\lambda(a))}{-U_1''(\hat{c}_\lambda(a)) (\hat{c}_\lambda(a) - ra)} \\
&= (\rho + \lambda - r) \frac{1}{\sigma(U_1, \hat{c}_\lambda(a))} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)} < \frac{\lambda}{\underline{\sigma}} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)} \\
&= \frac{\lambda}{\underline{\sigma}} \frac{\frac{2\lambda}{\underline{\sigma}}}{\frac{2\lambda}{\underline{\sigma}} - ra}.
\end{aligned}$$

By choosing $\bar{\lambda}$ sufficiently large, for all $\lambda > \bar{\lambda}$, $\frac{\frac{2\lambda}{\underline{\sigma}}}{\frac{2\lambda}{\underline{\sigma}} - ra} < 2$ for all $a \in (\underline{a}, \bar{a})$. Therefore, by

Lemma 7, $\hat{c}_\lambda(a) < \frac{2\lambda}{\underline{\sigma}} a$.

Now, going back to inequality (58),

$$\lambda (V_\lambda(a) - W_\lambda(a)) \leq U_1(\hat{c}_\lambda(a)) - \rho \bar{V}_\lambda(\underline{a}) < U_1\left(\frac{2\lambda}{\underline{\sigma}} \bar{a}\right) - \rho \bar{V}_\lambda(\underline{a}).$$

By the INADA conditions

$$\lim_{\lambda \rightarrow \infty} \frac{U_1\left(\frac{2\lambda}{\underline{\sigma}} \bar{a}\right)}{\lambda} = 0.$$

It is easy to show that $\lim_{\lambda \rightarrow \infty} \frac{\bar{V}_\lambda(\underline{a})}{\lambda} = 0$. Thus we obtain the desired convergence (57). \square