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THE ANALYTICS OF INVESTMENT, q, AND CASH FLOW

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ABSTRACT

In this paper I analyze the relationships among investment, q, and cash flow in a tractable stochastic model in which marginal q and average q are identically equal. After analyzing the impact of changes in the distribution of the marginal operating profit of capital, I extend the model to include measurement error and analyze the cash-flow coefficient in regressions of investment on q and cash flow. In empirical studies, the estimated cash-flow coefficient is generally positive and larger for rapidly growing firms. Such findings are typically interpreted as evidence of financial frictions facing firms. I derive closed-form expressions for the cash-flow coefficient that are positive and larger for faster growing firms, yet there are no financial frictions in the model.

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Empirically estimated investment equations typically find that Tobin's q has a positive effect on capital investment by firms, and that even after taking account of the effect of Tobin's q on investment, cash flow has a positive effect on investment. Intepretations of these results, of course, rely on some theoretical model of investment. Typically, the theoretical model that underlies the relationship between Tobin's q and investment is based on convex capital adjustment costs.[1] In such a framework, marginal q is a sufficient statistic for investment, and, as a consequence, other variables, in particular, cash flow, should not have any explanatory power for investment, once account is taken of marginal q. The fact that cash flow has a positive impact on investment, even after taking account of q, is interpreted by many researchers as evidence of financing constraints facing firms. That interpretation is bolstered by the finding that the cash-flow coefficient is larger for firms, such as rapidly growing firms, that are likely to be financially constrained.

In this paper, I develop and analyze a tractable stochastic model of investment, q, and cash flow and use it to interpret the empirical results described above. As in Lucas (1967), I specify the net profit of the firm, after deducting all costs associated with investment, to be a linearly homogeneous function of capital, labor, and investment. Lucas showed that this linear homogeneity implies that the growth rate of the firm is independent of its size. More relevant to the analysis in this paper, Hayashi (1982) showed that this linear homogeneity implies that Tobin's q, often called *average* q, is identically equal to marginal q. This equality of marginal q and average q is particularly powerful, because average q, which is in principle observable, can be used to measure marginal q, which is the appropriate shadow value of capital that determines the optimal rate of investment. In addition, this linearly homogeneous framework relates the investment-capital ratio to q and most empirical analyses, in fact, use the investment-capital ratio as the dependent variable in regressions.

To analyze the response of investment to q requires a framework with variation in q and in optimal investment. In this paper, I develop a model in which an exoge-

[1] Lucas and Prescott (1971) and Mussa (1977) first demonstrated the link between investment and securities prices, which are related to Tobin's q, in an adjustment cost framework.

nous Markov regime-switching process generates stochastic variation in the marginal operating profit of capital, which leads to stochastic variation in q and in the optimal investment-capital ratio. With this stochastic specification and the linear homogeneity of the profit function described above, the model is tractable enough to permit straightforward analysis of the effects on q and investment of changes in the marginal operating profit for a particular firm. The model can also be used to compare q and investment across firms that face different interest rates, different depreciation rates, and different stochastic processes for the exogenous marginal operating profit of capital. I apply this framework to analyze the impact on marginal q and investment of a mean-preserving spread in the unconditional distribution of the marginal operating profit of capital, as well as the impact of a change in the persistence of the Markov regime-switching process generating these marginal operating profits.

As mentioned earlier, a common feature of adjustment cost models of investment is that marginal q is a sufficient statistic for investment. Since average q and marginal q are identically equal in the linearly homogeneous framework used here, average q is also a sufficient statistic for investment. In particular, cash flow should not add any explanatory power for investment after taking account of average q. This feature holds in the model I present here and might appear to be an obstacle to accounting for the empirical cash-flow effect on investment described above. To overcome that obstacle, I introduce classical measurement error in Section 5 and derive a closed-form expression for the plim of the cash-flow coefficient in an investment regression. That expression indicates that if q is measured with error, then, as found empirically by Devereux and Schiantarelli (1990), the cash-flow coefficient can be larger for firms that grow more rapidly. However, because the model has perfect capital markets, without any financial frictions, the finding of positive cash-flow coefficients that are larger for faster-growing firms cannot be taken as evidence of financing constraints.

The interpretation of cash-flow coefficients and the role of measurement error in this paper build on and tie together two strands of the literature. One strand, represented by Cooper and Ejarque (2003), Gomes (2001), Alti (2003), and Abel and Eberly (2011), develops formal theoretical models of capital investment decisions and uses the formal model as an environment in which to analyze the cash-flow coefficient. The second strand, represented by Erickson and Whited (2000) and Gilchrist and Himmelberg (1995), focuses on the role of measurement error in inducing a spuriously positive cash-flow coefficient. All of the aforementioned papers in the first strand provide examples in which the cash-flow coefficient can be positive even when capital markets are frictionless. Thus, the finding of a positive cash-flow coefficient cannot be viewed as evidence of financial constraints. In all of these studies, however, average q is not identically equal to marginal q, so investment regressions using average q in place of marginal q suffer from misspecification. In the current paper I avoid this specification error by adhering to the Hayashi condition so that average q is identically equal to marginal q. Therefore, average q is an appropriate regressor in a linear investment equation when the adjustment cost function is quadratic. In this case, a nonzero cash-flow coefficient will arise if average q is measured with error, as emphasized in the second strand of the literature. Relative to that strand, the contribution of the current paper is to use a formal model of investment to show analytically that cash-flow coefficients arising from measurement error will be larger for firms that grow more rapidly.

Because the analysis of the model relies on the equality of marginal q and average q, I begin, in Section 1, by re-stating, and extending to a stochastic framework, the Hayashi condition under which average q and marginal q are equal. Section 2 introduces the model of the firm and analyzes the valuation of a unit of capital and the optimal investment decision in the case in which the marginal operating profit of capital is known to be constant forever. More than simply serving as a warm up to the stochastic model, Section 2 introduces a function that facilitates the analysis of the stochastic model that follows in later sections. I introduce a Markov regime-switching process for the marginal operating profit of capital in Section 3 to generate stochastic variation in q and optimal investment. In Section 4, I analyze the impact of changes in the stochastic properties of the marginal operating profit of capital, specifically, changes to the unconditional distribution and changes to the persistence of this exogenous random variable. In order to account for the positive impact of cash flow on investment, even after taking account of q, I introduce classical measurement error in Section 5. I derive the plim of the coefficients on q and cash flow in investment regressions and I show that if q is measured with error, the cash-flow coefficient is larger for firms that are growing more rapidly. Concluding remarks are in Section 6. The proofs of lemmas, propositions, and corollaries are in the Appendix.

1 The Hayashi Condition

Before describing the specific framework that I analyze in this paper, it is useful to begin with a simple, yet more general, description of the conditions under which average q and marginal q are equal. Consider a competitive firm with capital stock K_t at time t, where time is continuous. The firm accumulates capital by undertaking gross investment I_t at time t, and capital depreciates at rate δ, so the capital stock evolves according to

$$\frac{dK_t}{dt} = I_t - \delta K_t. \tag{1}$$

The firm uses capital, $K_t > 0$, and labor, $L_t \geq 0$, to produce and sell output at time t. I assume that the price of capital goods is constant and normalize it to be one. Define $\pi_t(K_t, I_t) = \max_{L_t} R_t(K_t, L_t, I_t) - I_t$, where $R_t(K_t, L_t, I_t)$ is revenue net of wage payments to labor and net of any investment adjustment costs. For now, I will simply assume that $\pi_t(K_t, I_t)$ is concave in K_t and I_t. Letting $M(t, s)$ be the stochastic discount factor used to discount cash flows at time $s \geq t$ back to time t, the value of the firm at time t is

$$V_t(K_t) = \max_{\{K_s, I_s\}} E_t\left\{\int_t^\infty \pi_s(K_s, I_s) M(t, s)\, ds\right\}, \tag{2}$$

subject to equation (1). The following proposition presents conditions for the equality of average q and marginal q, which are essentially the same as in Hayashi (1982), though the method of proof is different from Hayashi's proof and the framework is generalized to include uncertainty and possible non-separability of costs of adjustment from other components of the revenue function.

Proposition 1 *(extension of Hayashi) If $\pi_s(K_s, I_s)$ is linearly homogeneous in K_s and I_s, then for any $\omega \geq 0$, $V_t(\omega K_t) = \omega V_t(K_t)$, i.e., the value function is linearly homogeneous in K_t, so that average q, $\frac{V_t(K_t)}{K_t}$, and marginal q, $V_t'(K_t)$, are identically equal.*

For the remainder of this paper, I will assume that $\pi_s(K_s, I_s)$ is linearly homogeneous in K_s and I_s so that average q and marginal q are equal.

2 Model of the Firm

Consider a competitive firm that faces convex costs of adjustment that are separable from the production function. The firm uses capital, $K_t > 0$, and labor, $L_t \geq 0$, to produce non-storable output, Y_t, at time t according to the production function $Y_t = A_t f(K_t, L_t)$, where $f(K_t, L_t)$ is linearly homogeneous in K_t and L_t, and A_t is the exogenous level of total factor productivity. If the amount of labor is costlessly and instantaneously adjustable, the firm chooses L_t at time t to maximize instantaneous revenue less wages $p_t A_t f(K_t, L_t) - w_t L_t$, where p_t is the price of the firm's output at time t and w_t is the wage rate per unit of labor at time t. The linear homogeneity of $f(K_t, L_t)$ and the assumption that the firm is a price-taker in the markets for its output and labor together imply that the maximized value of revenue less wages is $\Phi_t K_t$, where $\Phi_t \equiv \max_l [p_t A_t f(1, l) - w_t l]$. The marginal (and average) operating profit of capital, Φ_t, is a deterministic function of A_t, p_t, and w_t, all of which are exogenous to the firm and possibly stochastic. Therefore, Φ_t is exogenous to the firm and, henceforth, I will treat Φ_t as the fundamental exogenous variable facing the firm, comprising the effects of productivity, output price, and the wage rate.

Define $\gamma_t \equiv \frac{I_t}{K_t}$ to be the investment-capital ratio at time t. Therefore, equation (1) implies that the growth rate of the capital stock, g_t, is

$$g_t \equiv \frac{1}{K_t}\frac{dK_t}{dt} = \gamma_t - \delta, \tag{3}$$

so that for $s \geq t$

$$K_s = K_t \exp\left(\int_t^s g_u du\right). \tag{4}$$

Finally, I will specify the stochastic discount factor, $M(t,s)$, to be simply $\exp(-r(s-t))$, so that net cash flows are discounted at the constant rate r.

At time t, the firm chooses gross investment, I_t. The cost of this investment has two components. The first component is the cost of purchasing capital at a price per unit that I assume to be constant over time and normalize to be one. Thus, this component of the cost of gross investment at rate I_t is simply $I_t = \gamma_t K_t$, which, of course, would be negative if the firm sells capital so that $I_t < 0$.

The second component is the cost of adjustment, $c(\gamma_t) K_t$, which is linearly homogeneous in I_t and K_t. I assume that $c(\gamma_t) \geq 0$ is strictly convex, at least twice differentiable, and attains its minimum value at γ_0, so that $c'(\gamma_0) = 0$. I assume that the minimum value of the adjustment cost is zero, that is, $c(\gamma_0) = 0$. The most common specifications of γ_0 are $\gamma_0 = 0$ and $\gamma_0 = \delta$.[2] When $\gamma_0 = 0$, the adjustment cost function is viewed as a function of gross investment relative to the capital stock; when $\gamma_0 = \delta$, the adjustment cost function is viewed as a function of net investment relative to the capital stock. Here I will simply assume that $\gamma_0 < r + \delta$, which implies $c'(r+\delta) > 0$. In addition, I assume that for some γ^m, $c'(\gamma^m) = -1$. The strict convexity of $c(\gamma_t)$ implies that $\gamma^m < \gamma_0$.

The total cost of investment, which comprises the purchase cost of capital and the cost of adjustment, is $[\gamma_t + c(\gamma_t)] K_t$. For a given K_t, the total cost of investment is strictly convex in γ_t and attains its minimum value at γ^m. After choosing the optimal usage of labor, the amount of revenue less wages and less the total cost of investment is

$$\pi_t(K_t, I_t) \equiv [\Phi_t - \gamma_t - c(\gamma_t)] K_t. \tag{5}$$

2.1 Constant Φ_t

Consider the case in which the marginal operating profit of capital, Φ_t, is known to be constant forever. The analytic apparatus developed in the case of certainty will prove

[2] Cooper and Ejarque (2003) and Zhang (2005) choose $\gamma_0 = 0$; Alti (2003) and Treadway (1969) choose $\gamma_0 = \delta$ (though Treadway's formulation of adjustment costs is simply $C\left(\dot{K}_t\right)$ which is minimized at $\dot{K}_t = 0$, i.e., when $\gamma_t = \delta$); Hall (2004) considers both $\gamma_0 = 0$ and $\gamma_0 = \delta$; and Chirinko (1993) and Summers (1983) consider arbitrary γ_0.

to be useful in later sections when Φ_t evolves according to a Markov regime-switching process.

I begin by defining G, which is an admissible set of values for Φ, the constant marginal operating profit of capital, as[3]

$$G \equiv \{\Phi : c(\gamma^m) + \gamma^m < \Phi < c(r+\delta) + r + \delta\}. \tag{6}$$

The lower bound on G ensures that there is a value of γ_t such that $\pi_t(K_t, I_t) > 0$ when $K_t > 0$, so that the value of the firm is positive.[4] The upper bound on Φ keeps the value of the firm finite when it is positive.[5]

2.1.1 The Value of a Unit of Capital

With a constant marginal operating profit of capital, Φ, constant depreciation rate, δ, and constant discount rate, r, the optimal investment-capital ratio, γ_t, is constant also. In this case, the value of the firm in equation (2) can be written as

$$V_t(K_t) = \max_{\gamma} \int_t^{\infty} [\Phi - \gamma - c(\gamma)] K_s e^{-r(s-t)} ds. \tag{7}$$

Dividing both sides of equation (7) by K_t and using equation (4) with $g_u = \gamma - \delta$ yields an expression for the average value of a unit of capital, $v \equiv \frac{V_t(K_t)}{K_t}$, which is[6]

$$v = \max_{\gamma} \frac{\Phi - \gamma - c(\gamma)}{r + \delta - \gamma}. \tag{8}$$

The value of a unit of capital shown in equation (8) equals $\pi_t(K_t, I_t)/K_t$ divided by the excess of the interest rate, r, over the growth rate, $\gamma - \delta$.

[3]Since $c'(\gamma^m) = -1$ and $c'(r+\delta) > 0$, the strict convexity of $c(\gamma_t)$ implies that $\gamma^m < r + \delta$ and that $c(\gamma_t) + \gamma_t$ is strictly increasing in γ_t for all $\gamma_t > \gamma^m$. Therefore, $c(r+\delta) + r + \delta > c(\gamma^m) + \gamma^m$ so that G is non-empty.

[4]Since γ^m minimizes $\gamma + c(\gamma)$, it maximizes $\Phi - c(\gamma) - \gamma$. The restriction $\Phi > c(\gamma^m) + \gamma^m$ implies that $\Phi - c(\gamma) - \gamma > 0$ for some γ, whereas if Φ were less than or equal to $c(\gamma^m) + \gamma^m$, then $\Phi - c(\gamma) - \gamma$ would not be positive for any γ.

[5]If (1) $\Phi - c(\gamma) - \gamma > 0$ and (2) $\gamma - \delta > r$, the value of the firm, with $K_t > 0$, would be positive and infinite because the growth rate of $[\Phi - c(\gamma) - \gamma] K_t$, which is $\gamma - \delta$, would exceed r. The upper bound on $\Phi \in G$ implies that if $\gamma \geq r + \delta$, then $c(\gamma) + \gamma \geq c(r+\delta) + r + \delta$, so $\Phi - c(\gamma) - \gamma < 0$.

[6]The expression in equation (8) holds only if the integral in equation (7) is finite, which requires the growth rate of capital, $\gamma - \delta$, to be less than r. The optimal value of γ is smaller than $r + \delta$.

Differentiate the maximand on the right-hand side of equation (8) with respect to γ and set the derivative equal to zero to obtain[7]

$$1 + c'(\gamma) = \frac{\Phi - \gamma - c(\gamma)}{r + \delta - \gamma}. \tag{9}$$

Rewriting equation (9) yields

$$\Phi - (r + \delta) - c(\gamma) - (r + \delta - \gamma) c'(\gamma) = 0. \tag{10}$$

To characterize the optimal investment-capital ratio, it will be useful to define a function $H(\gamma, \Phi, \rho)$ as

$$H(\gamma, \Phi, \rho) \equiv \Phi - \rho - c(\gamma) - (\rho - \gamma) c'(\gamma), \tag{11}$$

where ρ is an arbitrary constant greater than or equal to $r + \delta$. The optimal value of γ, characterized by equation (10), satisfies

$$H(\gamma, \Phi, r + \delta) = 0, \tag{12}$$

where the value of ρ in $H(\gamma, \Phi, \rho)$ is set equal to $r + \delta$. The following lemma presents several useful properties of $H(\gamma, \Phi, \rho)$.

Lemma 1 *Define $H(\gamma, \Phi, \rho) \equiv \Phi - \rho - c(\gamma) - (\rho - \gamma) c'(\gamma)$ and assume that $\Phi \in G$ and $\rho \geq r + \delta$. Then:*

1. *$H(\gamma, \Phi, \rho)$ is an increasing, linear function of Φ.*
2. *$H(\gamma, \Phi, \rho)$ is a decreasing, linear function of ρ for $\gamma > \gamma^m$, where $c'(\gamma^m) = -1$.*
3. *$H(\gamma, \Phi, \rho)$ is strictly quasi-convex in γ.*
4. *$H(\gamma_0, \Phi, \rho) = \Phi - \rho$, where $\gamma_0 \equiv \arg\min_\gamma c(\gamma)$.*
5. *$H(\gamma^m, \Phi, \rho) > 0$, where $c'(\gamma^m) = -1$.*

[7] Define $h(\gamma) \equiv \frac{\Phi - \gamma - c(\gamma)}{r+\delta-\gamma}$ and observe that $h'(\gamma) = \frac{1}{r+\delta-\gamma}[-1 - c'(\gamma) + h(\gamma)]$ and that $h''(\gamma) = \frac{1}{r+\delta-\gamma} h'(\gamma) + \frac{1}{r+\delta-\gamma}[-c''(\gamma) + h'(\gamma)]$. Therefore, if $h'(\widehat{\gamma}) = 0$, then $h''(\widehat{\gamma}) = \frac{-1}{r+\delta-\gamma} c''(\widehat{\gamma})$. If $h'(\widehat{\gamma}) = 0$ and $\widehat{\gamma} < r + \delta$, then $h''(\widehat{\gamma}) < 0$ so $h(\widehat{\gamma})$ is a local maximum. However, if $h'(\widehat{\gamma}) = 0$ and $\widehat{\gamma} > r + \delta$, then $h''(\widehat{\gamma}) > 0$ so $h(\widehat{\gamma})$ is a local minimum.

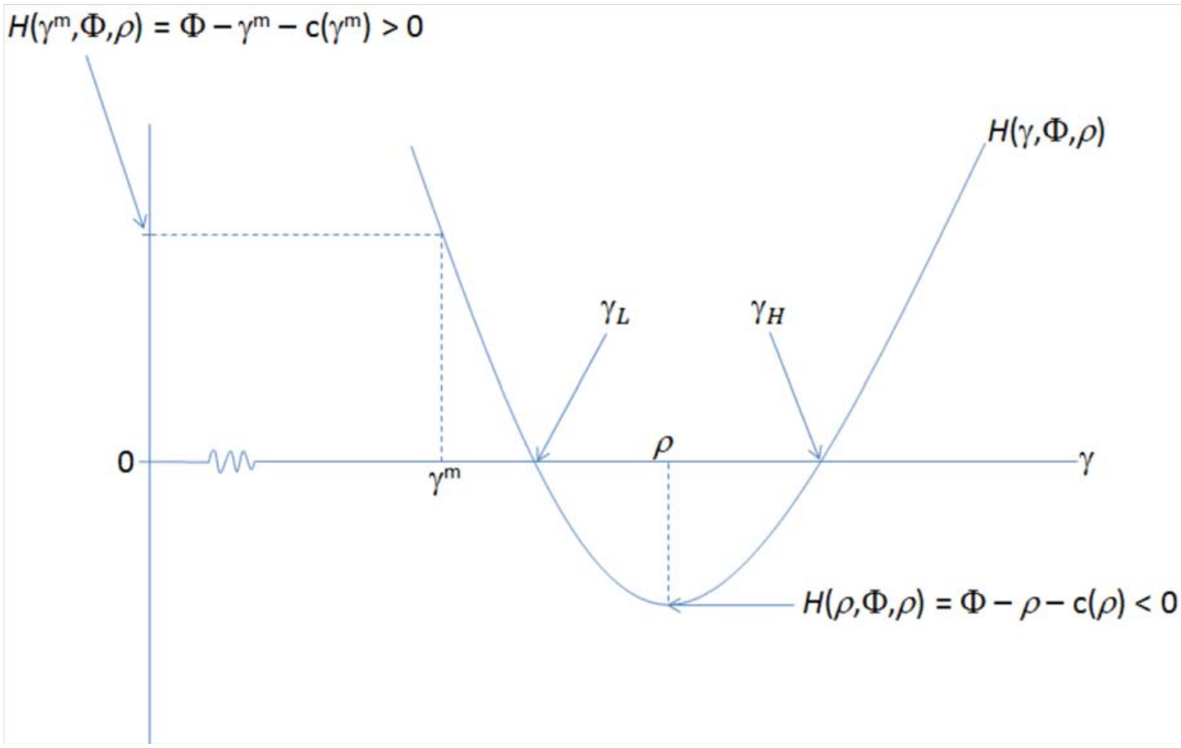

Figure 1: $H(\gamma, \Phi, \rho)$

6. $\min_\gamma H(\gamma, \Phi, \rho) = H(\rho, \Phi, \rho) = \Phi - \rho - c(\rho) < 0$.

7. *There is a unique* $\gamma_L \in (\gamma^m, \rho)$ *and a unique* $\gamma_H \in (\rho, \infty)$ *such that* $H(\gamma_L, \Phi, \rho) = H(\gamma_H, \Phi, \rho) = 0$. *Also,* $\Phi - \gamma_L - c(\gamma_L) > 0 > \Phi - \gamma_H - c(\gamma_H)$.

Figure 1 illustrates $H(\gamma, \Phi, \rho)$ as a function of γ for given values of $\Phi \in G$ and $\rho \geq r + \delta$. If the adjustment cost function $c(\gamma)$ is quadratic, then $H(\gamma, \Phi, \rho)$ is quadratic in γ and is a convex function of γ. In general, however, $H(\gamma, \Phi, \rho)$ need not be convex in γ, but it is strictly quasi-convex in γ. This figure shows that $H(\gamma^m, \Phi, \rho) > 0 > H(\rho, \Phi, \rho)$, and that $H(\gamma_L, \Phi, \rho) = H(\gamma_H, \Phi, \rho) = 0$. Since $\Phi - \gamma_H - c(\gamma_H) < 0$ (statement 7 of Lemma 1), the optimal investment-capital ratio cannot equal γ_H.[8]

The following proposition characterizes the optimal investment-capital ratio γ^c, where the superscript "c" indicates the optimal value of γ under *certainty* with a

[8] Statement 7 of Lemma 1 implies that if $\gamma = \gamma_H$, then $\pi_t(K_t, I_t) = \pi_t(K_t, \gamma_H K_t) = [\Phi - \gamma_H - c(\gamma_H)] K_t < 0$ and hence the value of the firm would be negative. Therefore, consistent with footnote 6, the optimal value of γ cannot equal $\gamma_H > r + \delta$.

constant value of Φ_t.

Proposition 2 *If $\Phi_t \in G$ is known with certainty to be constant and equal to Φ forever, then the optimal investment-capital ratio, $\gamma^c(\Phi, r+\delta)$, is the unique value of $\gamma \in (\gamma^m, r+\delta)$ that satisfies $H(\gamma, \Phi, r+\delta) = 0$.*

Corollary 1 $\frac{\partial \gamma^c(\Phi, r+\delta)}{\partial \Phi} = \frac{1}{(r+\delta-\gamma^c)c''(\gamma^c)} > 0$ *and* $\frac{\partial \gamma^c(\Phi, r+\delta)}{\partial (r+\delta)} = -\frac{1+c'(\gamma^c)}{(r+\delta-\gamma^c)c''(\gamma^c)} < 0.$

Corollary 1 states that a firm with a higher deterministic constant value of marginal operating profit of capital, Φ, will have a higher optimal value of the investment-capital ratio. It also states that a firm with a higher user cost of capital, $r+\delta$, will have a lower optimal value of the investment-capital ratio.[9]

Define $v^c(\Phi, r+\delta)$ as the common value of marginal q and average q if Φ_t is known with certainty to be constant and equal to Φ.

Corollary 2 $\frac{\partial v^c(\Phi, r+\delta)}{\partial \Phi} = \frac{1}{r+\delta-\gamma^c} > 0$, $\frac{\partial^2 v^c(\Phi, r+\delta)}{(\partial \Phi)^2} = \frac{1}{(r+\delta-\gamma^c)^3} \frac{1}{c''(\gamma^c)} > 0$, *and* $\frac{\partial v^c(\Phi, r+\delta)}{\partial (r+\delta)} = -\frac{1+c'(\gamma^c)}{r+\delta-\gamma^c} < 0.$

Corollary 2 states that $v^c(\Phi, r+\delta)$ is an increasing convex function of the marginal operating profit of capital, Φ, and a decreasing function of the user cost of capital, $r+\delta$.

Corollary 3 $\gamma^c(\Phi, r+\delta) \gtreqless \gamma_0 \equiv \arg\min_\gamma c(\gamma)$ *as* $\Phi \gtreqless r+\delta$.

The parameter γ_0 is the value of the investment-capital ratio γ at which the adjustment cost $c(\gamma)$ takes on its minimum value, which is zero. Corollary 3 states that in the case in which the marginal operating profit Φ is known with certainty to be constant, the optimal value of γ will equal γ_0 if and only if $\Phi = r+\delta$.[10] If $\Phi > r+\delta$, the optimal investment-capital ratio exceeds γ_0, and if $\Phi < r+\delta$, the optimal investment-capital ratio is less than γ_0.

[9] See equation (25) in subsection 3.4 for a general expression for the user cost of capital. In the absence of adjustment costs, $q(\phi_t)$ and $\overline{q}$ in that equation would both equal one and the user cost of capital would equal $r+\delta$, which is the Jorgensonian user cost $(r+\delta)p_{K,t} - \frac{dp_{K,t}}{dt}$, when $p_{K,t}$ is constant and equal to one.

[10] If $\Phi = r+\delta$, the firm will maintain a constant level of the capital stock if and only if $\gamma_0 = \delta$, so that the adjustment cost function $c(\gamma)$ attains its minimum when *net* investment is zero. Alternatively, if $\Phi = r+\delta$ and $\gamma_0 = 0$, the firm will undertake zero *gross* investment and the capital stock will shrink at rate δ.

3 Markov Regime-Switching Process for Φ_t

In this section I develop and analyze a model of a firm facing stochastic variation in the marginal operating profit of capital, Φ_t, governed by a Markov regime-switching process. Specifically, a regime is defined by a constant value of Φ_t. If the marginal operating profit of capital at time t, Φ_t, equals ϕ, it remains equal to ϕ until a new regime arrives. The arrival process for new regimes is a Poisson process with probability λdt of a new arrival during a short interval of time dt. When a new regime arrives, a new value of the marginal operating profit of capital, Φ, is drawn from a distribution with c.d.f $F(\Phi)$, where the support of $F(\Phi)$ is in G, defined in equation (6). $F(\Phi)$ can be continuous or not continuous, so the random variable Φ can be continuous, discrete, or mixed. The values of Φ are drawn independently across regimes.

The Markovian nature of Φ implies that the value of the firm at time t depends only on the capital stock at time t, K_t, and the value of the marginal operating profit at time t, ϕ. The value of the firm $V(K_t, \phi)$ is

$$\begin{aligned} V(K_t, \phi) &= \max_{\gamma_t} \int_t^{t+dt} [\phi - \gamma_t - c(\gamma_t)] K_s e^{-r(s-t)} ds \\ &\quad + e^{-\lambda dt} e^{-rdt} V(K_{t+dt}, \phi) \\ &\quad + \left(1 - e^{-\lambda dt}\right) e^{-rdt} \int_G V(K_{t+dt}, \Phi) \, dF(\Phi), \end{aligned} \tag{13}$$

which is the maximized sum of three terms. The first term is the present value of $\pi(K_s, I_s) = [\phi - \gamma_s - c(\gamma_s)] K_s$ over the infinitesimal interval of time from t to $t+dt$, which, ignoring the infinitesimal probability that Φ_s and hence γ_s change during that interval, is $[\phi - \gamma_t - c(\gamma_t)] K_s$. The second term is the present value of the firm at time $t + dt$, conditional on Φ remaining equal to ϕ at time $t + dt$, weighted by the probability, $e^{-\lambda dt}$, that $\Phi_{t+dt} = \phi$. The third term is the present value of the expected value of the firm at time $t + dt$ conditional on a new regime for Φ at time $t + dt$, weighted by the probability that a new regime will arrive by time $t + dt$.

The Hayashi conditions in Proposition 1 hold in this framework so that the value of the firm is proportional to the capital stock. Therefore, the average value of the capital stock, $\frac{V(K_t,\phi)}{K_t}$, is independent of the capital stock and depends only on ϕ. I will define $v(\phi) \equiv \frac{V(K_t,\phi)}{K_t}$ to be Tobin's q, or equivalently, the average value of capital. Since average q and marginal q are identically equal in this framework, $v(\phi)$ is also marginal q.

Use the definition $v(\phi) \equiv \frac{V(K_t,\phi)}{K_t}$, again ignore the infinitesimal probability that $g_s = \gamma_s - \delta$ changes during the interval $[t, t+dt]$ so that $K_s = e^{(\gamma_t-\delta)(s-t)}K_t$, and perform the first integration on the right-hand side of equation (13) to obtain

$$\begin{aligned} v(\phi) &= \max_{\gamma_t} [\phi - \gamma_t - c(\gamma_t)] \frac{1 - e^{-(r+\delta-\gamma_t)dt}}{r + \delta - \gamma_t} \\ &\quad + e^{-\lambda dt} e^{-rdt} e^{(\gamma_t-\delta)dt} v(\phi) \\ &\quad + \left(1 - e^{-\lambda dt}\right) e^{-rdt} e^{(\gamma_t-\delta)dt} \int_G v(\Phi)\, dF(\Phi). \end{aligned} \tag{14}$$

In the limit as dt approaches zero, equation (14) becomes[11]

$$0 = \max_{\gamma} \phi - \gamma - c(\gamma) - (r + \delta + \lambda - \gamma) v(\phi) + \lambda \overline{v}, \tag{15}$$

where

$$\overline{v} \equiv \int_G v(\Phi)\, dF(\Phi) \tag{16}$$

is the unconditional expected value of a unit of capital, which is also the unconditional expected value of both average q and marginal q.

The maximization in equation (15) has the first-order condition

$$1 + c'(\gamma) = v(\phi). \tag{17}$$

Thus, the optimal value of γ equates the marginal cost of investment, comprising the purchase price of capital and the marginal adjustment cost, with marginal q and average q.

[11] Use the definition of $\overline{v}$ in equation (16) and the fact that for small x, $e^x \cong 1 + x$ to rewrite equation (14) as $v(\phi) = \max_{\gamma_t} [\phi - \gamma_t - c(\gamma_t)]\, dt + (1 - (r + \delta + \lambda - \gamma_t)\, dt)\, v(\phi) + (\lambda dt)(1 - (r + \delta - \gamma_t)\, dt)\, \overline{v}$. Next subtract $v(\phi)$ from both sides of the equation, and then divide both sides of the resulting equation by dt to obtain $0 = \max_{\gamma_t} [\phi - \gamma_t - c(\gamma_t)] - (r + \delta + \lambda - \gamma_t)\, v(\phi) + \lambda(1 - (r + \delta - \gamma_t)\, dt)\, \overline{v}$. Taking the limit as dt approaches zero, and replacing γ_t by simply γ yields equation (15).

3.1 Marginal q and Average q

In this subsection I present alternative expressions for marginal q and average q. Because the model presented here is a special case of Proposition 1, marginal q and average q are identically equal. Nevertheless, it is helpful to examine different expressions for marginal q and average q and to understand why these expressions, which at first glance look different, are equivalent.

Marginal q at time t is commonly expressed as the expected present value of the stream of contributions to revenue, less wages and investment costs, of the remaining undepreciated portion of a unit of capital installed at time t, which is

$$q(\phi) = E\left\{\int_t^\infty \frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-(r+\delta)(s-t)} ds | \Phi_t = \phi\right\}. \tag{18}$$

Average q at time t is the value of the firm at time t divided by K_t. Dividing both sides of equation (2) by K_t, using the linear homogeneity of $\pi_s(K_s, I_s)$, and using equation (4) and $M(t,s) = \exp(-r(s-t))$ yields

$$v(\phi) = E\left\{\int_t^\infty \pi_s(1, \gamma_s) \exp\left(-\int_t^s (r - g_u)\, du\right) ds | \Phi_t = \phi\right\}. \tag{19}$$

Proposition 3 *The value of marginal q is*

$$q(\phi) = \frac{\phi - c(\gamma) + \gamma c'(\gamma)}{r + \delta + \lambda} + \frac{\lambda}{r + \delta + \lambda}\overline{q},$$

where γ is the optimal value of γ when $\Phi_t = \phi$ and $\overline{q} \equiv \int_G q(\Phi)\, dF(\Phi)$ is the unconditional expected value of marginal q. The value of average q is

$$v(\phi) = \frac{\phi - \gamma - c(\gamma)}{r + \delta + \lambda - \gamma} + \frac{\lambda}{r + \delta + \lambda - \gamma}\overline{v},$$

where $\overline{v} \equiv \int_G v(\Phi)\, dF(\Phi)$ is the unconditional expected value of average q .

Proposition 1 implies that $q(\phi) \equiv v(\phi)$. However, at first glance, the expressions for $q(\phi)$ and $v(\phi)$ in Proposition 3 do not appear to be equivalent. The first term in the expression for $q(\phi)$ is $\phi - c(\gamma) + \gamma c'(\gamma)$ discounted at rate $r + \delta + \lambda$ and the first term in the expression for $v(\phi)$ is $\phi - \gamma - c(\gamma)$ discounted at rate $r + \delta + \lambda - \gamma$. To see that these expressions are equivalent, multiply $v(\phi)$ by $r+\delta+\lambda-\gamma$ and subtract the result from $q(\phi)$ multiplied by $r+\delta+\lambda$ to obtain

$$(r+\delta+\lambda)\left[q(\phi)-v(\phi)\right]+\gamma v(\phi)=\gamma\left[1+c'(\gamma)\right]+\lambda\left(\overline{q}-\overline{v}\right). \tag{20}$$

Since Proposition 1 implies that $q(\phi)\equiv v(\phi)$ and hence $\overline{q}=\overline{v}$, equation (20) implies $v(\phi)=1+c'(\gamma)$ if $\gamma\neq 0$, which is the first-order condition in equation (17).[12]

3.2 Optimal Investment

In this section I exploit the first-order condition for optimal investment in equation (17) to analyze several properties of optimal γ. The optimal value of γ depends on $v(\phi)$, which depends on $\overline{v}$. For now, I will treat $\overline{v}$ as a parameter and defer further analysis of $\overline{v}$ to subsection 3.3.

To analyze optimal γ, substitute the first-order condition for optimal γ from equation (17) into equation (15) to obtain

$$0=\phi-c(\gamma)-(r+\delta+\lambda)-(r+\delta+\lambda-\gamma)\,c'(\gamma)+\lambda\overline{v}. \tag{21}$$

Using the definition of $H(\gamma,\Phi,\rho)$ in equation (11), rewrite equation (21) as

$$H(\gamma,\phi,r+\delta+\lambda)=-\lambda\overline{v}. \tag{22}$$

Equation (22) characterizes the optimal value of γ when there is a constant instantaneous probability, λ, of a regime switch. Of course, when $\lambda=0$, this equation is equivalent to equation (12), which characterizes the optimal value of γ under certainty. The optimal value of γ when $\lambda=0$ is shown in Figure 2 as point A where $H(\gamma,\phi,r+\delta)=0$. The introduction of a positive value of λ, which introduces stochastic variation in the future values of $\pi_s(1,\gamma_s)$ and $\frac{\partial\pi_s(1,\gamma_s)}{\partial K_s}$, has two opposing effects on optimal γ in equation (22). First, the introduction of a positive value of λ increases ρ, from $r+\delta$ to $r+\delta+\lambda$, which reduces the value of $H(\gamma,\phi,\rho)$ by $\lambda(1+c'(\gamma))$

[12] If $\gamma=0$, the expressions for $q(\phi)$ and $v(\phi)$ in Proposition 3 become $q(\phi)=\frac{\phi-c(0)+\lambda\overline{q}}{r+\delta+\lambda}$ and $v(\phi)=\frac{\phi-c(0)+\lambda\overline{v}}{r+\delta+\lambda}$, respectively, so $\overline{v}=\overline{q}$ and $v(\phi)$ and $q(\phi)$ are consistent with each other.

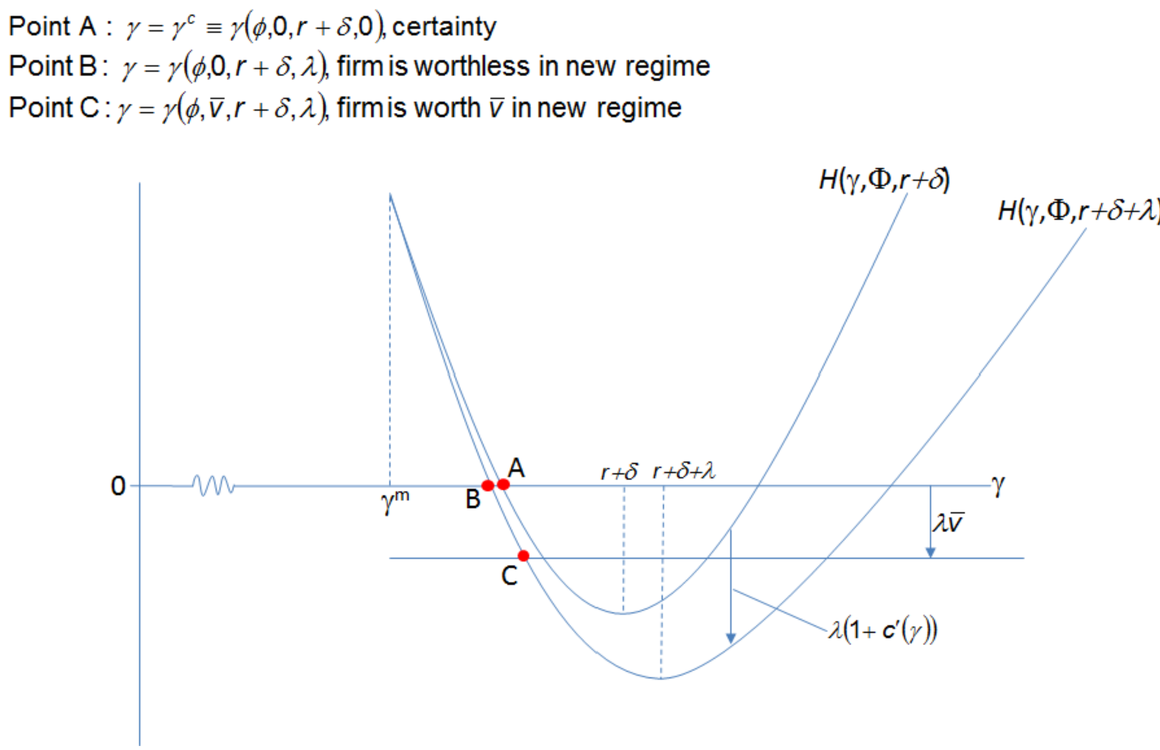

Figure 2: Optimal Investment under Uncertainty

at each value of γ, which induces the downward shift of the curve shown in Figure 2. This downward shift of the curve reduces the value of γ for which $H(\gamma, \phi, \rho) = 0$, as illustrated by the movement from point A to point B. The value of γ for which $H(\gamma, \phi, r + \delta + \lambda) = 0$ is the optimal value of γ that would arise if the firm were to disappear, with zero salvage value, when the regime switches. Thus, not surprisingly, the introduction of the possibility of a stochastic death of the firm reduces the value of a unit of capital and reduces the optimal investment-capital ratio. However, if the new regime does not eliminate the firm, there is a second impact on optimal γ of the introduction of a positive value of λ. Specifically, if the firm receives a new draw of Φ from the unconditional distribution $F(\Phi)$ when the regime changes, then $\overline{v}$ is the expected value of a unit of capital in the new regime. With $\overline{v} > 0$, the term $-\lambda\overline{v}$ on the right-hand side of equation (22) is negative, so that $H(\gamma, \phi, r + \delta + \lambda) < 0$ at the optimal value of γ. Reducing the value of $H(\gamma, \phi, r + \delta + \lambda)$ from zero to a negative value requires an increase in γ, as shown in Figure 2 by the movement from point B to point C. To summarize, the introduction of stochastic variation in future Φ has two opposing effects on the optimal value of γ. For some values of ϕ the introduction of uncertainty will increase the optimal value of γ, and for other values of ϕ it will decrease the optimal value of γ.

Define $\gamma(\phi, \kappa, r + \delta, \lambda)$ to be the optimal value of γ for given values of $r + \delta$ and λ if $\Phi = \phi$ and $\overline{v} = \kappa$. Formally, $\gamma(\phi, \kappa, r + \delta, \lambda)$ is defined by

$$H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa, \tag{23}$$

and $\gamma(\phi, \kappa, r + \delta, \lambda) < r + \delta + \lambda$. Of course, this definition is meaningful only if $\min_\gamma H(\gamma, \phi, r + \delta + \lambda) < -\lambda\kappa$. The following lemma identifies an interval of non-negative values of κ for which this definition is meaningful.

Lemma 2 *If $0 \leq \kappa \leq 1 + c'(r + \delta)$, $\lambda > 0$, and $\phi \in G$, then there exists a unique $\gamma(\phi, \kappa, r + \delta, \lambda) \in (\gamma^m, r + \delta + \lambda)$ for which $H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa$.*

Note that $\gamma(\phi, 0, r + \delta, 0) = \gamma^c(\phi, r + \delta)$, which is the optimal value of the investment-capital ratio, γ, in the case in which $\Phi_t = \phi$ with certainty forever.

The following lemma and its corollary list several properties of the optimal investment-capital ratio $\gamma(\phi, \kappa, r + \delta, \lambda)$ and $c'(\gamma(\phi, \kappa, r + \delta, \lambda))$.

Lemma 3 *Define $\rho \equiv r + \delta + \lambda$. If $\phi \in G$ and if $0 \leq \kappa \leq 1 + c'(r + \delta)$, then*

1. $\frac{\partial \gamma(\phi,\kappa,r+\delta,\lambda)}{\partial \phi} = \frac{1}{(\rho-\gamma)c''(\gamma)} > 0,$
2. $\frac{\partial \gamma(\phi,\kappa,r+\delta,\lambda)}{\partial \kappa} = \frac{\lambda}{(\rho-\gamma)c''(\gamma)} > 0,$
3. $\frac{\partial \gamma(\phi,\kappa,r+\delta,\lambda)}{\partial (r+\delta)} = -\frac{1+c'(\gamma)}{(\rho-\gamma)c''(\gamma)} < 0,$
4. $\frac{\partial \gamma(\phi,\kappa,r+\delta,\lambda)}{\partial \lambda} = -\frac{1+c'(\gamma)-\kappa}{(\rho-\gamma)c''(\gamma)}.$

Corollary 4 *Define $\rho \equiv r + \delta + \lambda$. If $\phi \in G$ and if $0 \leq \kappa \leq 1 + c'(r + \delta)$, then*

1. $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial \phi} = \frac{1}{\rho-\gamma} > 0,$
2. $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial \kappa} = \frac{\lambda}{\rho-\gamma} > 0,$
3. $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial (r+\delta)} = -\frac{1+c'(\gamma)}{\rho-\gamma} < 0,$

4. $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial \lambda} = -\frac{1+c'(\gamma)-\kappa}{\rho-\gamma}$,

5. $\frac{\partial^2 c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{(\partial \phi)^2} = \frac{1}{(\rho-\gamma)^3 c''(\gamma)} > 0$.

Lemma 3 and its corollary show that for any $\kappa \in [0, 1 + c'(r+\delta)]$ and $\phi \in G$, both $\gamma(\phi, \kappa, r+\delta, \lambda)$ and $c'(\gamma(\phi, \kappa, r+\delta, \lambda))$ are increasing functions of ϕ and κ, and decreasing functions of $r+\delta$. The impact of a higher value of λ depends on the size of ϕ. This result is easiest to describe for the case in which $\kappa = \overline{v}$, so that κ equals the unconditional expected value of a unit of capital. In this case, an increase in λ hastens the arrival of a new regime in which the expected value of a unit of capital is κ. For values of ϕ that are small enough that $1 + c'(\gamma(\phi, \overline{v}, r+\delta, \lambda)) < \kappa = \overline{v}$, hastening the arrival of a new regime increases the value of a unit of capital, thereby increasing optimal γ and the optimal value of $c'(\gamma)$. Alternatively, for values of ϕ that are large enough that $1 + c'(\gamma(\phi, \overline{v}, r+\delta, \lambda)) > \kappa = \overline{v}$, hastening the arrival of a new regime means an earlier end to the current regime with a high ϕ. As a result, capital is less valuable and the optimal values of γ and $c'(\gamma)$ decline. Finally, the corollary shows that $c'(\gamma(\phi, \kappa, r+\delta, \lambda))$ is strictly convex in ϕ. This convexity will be helpful in subsection 4.2 when I analyze the impact on the value of a unit of capital of a mean-preserving spread in the unconditional distribution $F(\Phi)$.

3.3 The Unconditional Expectation of a Unit of Capital

Equation (22) is a simple expression that characterizes the optimal value of γ. However, this expression depends on $\overline{v} \equiv \int_G v(\Phi)\, dF(\Phi)$, which is the unconditional expectation of the optimal value of a unit of installed capital. In this subsection, I prove that $\overline{v}$ is the unique fixed point of a particular function and show that this property helps analyze the impact on optimal investment of changes in the distribution $F(\Phi)$ and changes in λ.

Define

$$\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r+\delta, \lambda))\, dF(\Phi) \tag{24}$$

as the unconditional expectation of the marginal cost of investment, including the purchase cost of capital and the marginal adjustment cost, where $\gamma(\phi, \kappa, r+\delta, \lambda)$

is defined in equation (23) as the optimal value of the investment-capital ratio if $\Phi = \phi$ and $\overline{v} = \kappa$. Since the value of a unit of capital when $\Phi = \phi$ is $v(\phi) = 1 + c'(\gamma(\phi, \overline{v}, r + \delta, \lambda))$, optimal behavior by the firm implies that $\overline{v}$ satisfies $\alpha(\overline{v}) = \overline{v}$.

Lemma 4 *Suppose that the support of the distribution $F(\Phi)$ is contained in G. The function $\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r + \delta, \lambda))\, dF(\Phi)$ has the following three properties: (1) $\alpha(0) > 0$; (2) $\alpha(1 + c'(r + \delta)) < 1 + c'(r + \delta)$; and (3) $0 < \alpha'(\kappa) < 1$ for $\kappa \in [0, 1 + c'(r + \delta)]$.*

Lemma 4 together with the continuity of $\alpha(\kappa)$ leads to the following proposition.

Proposition 4 *Suppose that the support of the distribution $F(\Phi)$ is contained in G. Then $\overline{v}$ is the unique positive value of $\kappa \in (0, 1 + c'(r + \delta))$ that satisfies $\alpha(\kappa) = \kappa$.*

Lemma 4 also leads to the following corollary, which will prove useful in analyzing the effects of changes in the distribution $F(\Phi)$ and changes in λ.

Corollary 5 *For any $\kappa^* \in [0, 1 + c'(r + \delta)]$, sign $[\alpha(\kappa^*) - \kappa^*] =$ sign $[\overline{v} - \kappa^*]$.*

Corollary 5 helps determine the impact on $\overline{v}$ of changes in the distribution $F(\phi)$ or in λ. Let $\overline{v}_0$ be the initial value of $\overline{v}$ before the change in $F(\phi)$ or in λ. Then any change that increases $\alpha(\overline{v}_0)$ will increase $\overline{v}$, and any change that decreases $\alpha(\overline{v}_0)$ will decrease $\overline{v}$.

3.4 Marginal Profit and the User Cost of Capital

Jorgenson (1963) demonstrated that the optimal capital stock in a firm's dynamic optimization problem can be characterized by a static condition equating the marginal profit of capital and the user cost of capital. His seminal derivation was conducted in the absence of costs of adjustment. Here I illustrate the concepts of the marginal profit of capital and the user cost of capital in the presence of convex adjustment costs.

First, consider the user cost of capital. In the absence of adjustment costs (and in the absence of uncertainty), Jorgenson showed that the user cost of capital is $(r + \delta)\, p_{K,t} - \frac{dp_{K,t}}{dt}$, where $p_{K,t}$ is the purchase price of capital at time t. To extend the concept of user cost to the framework in this paper, I make two modifications to that concept: (1) replace $p_{K,t}$ by the shadow price of capital, $q(\phi_t)$, and (2) replace the change in $p_{K,t}$ by the expected change in the shadow price of capital, which is $\lambda(\overline{q} - q(\phi))$, since there is an instantaneous probability λ that the value of ϕ will change and induce a new value of q, which has an expected value of $\overline{q}$. Therefore, the user cost of capital, $u(\phi_t)$, is

$$u(\phi_t) = (r+\delta)\, q(\phi_t) - \lambda(\overline{q} - q(\phi_t)) = (r+\delta+\lambda)\, q(\phi_t) - \lambda\overline{q}. \tag{25}$$

The marginal profit of capital, $\frac{\partial \pi_t}{\partial K_t}$, is obtained by differentiating $\pi_t(K_t, I_t)$ in equation (5) partially with respect to K_t to obtain

$$\frac{\partial \pi_t(K_t, I_t)}{\partial K_t} = \phi_t - [c(\gamma_t) - \gamma_t c'(\gamma_t)]. \tag{26}$$

The marginal profit of capital in equation (26) comprises two components: (1) the marginal operating profit of capital, ϕ_t; and (2) the reduction in the adjustment cost $c(\gamma_t) K_t$ associated with an increase in K_t for given I_t. This reduction in the adjustment cost is $-\frac{d[c(\gamma_t)K_t]}{dK_t} = -[c(\gamma_t) - \gamma_t c'(\gamma_t)]$, where the derivative is computed holding I_t fixed.

Equating the marginal profit of capital, $\frac{\partial \pi_t(K_t, I_t)}{\partial K_t}$, in equation (26) and the user cost of capital, $u(\phi_t)$, in equation (25) yields

$$\phi_t - [c(\gamma_t) - \gamma_t c'(\gamma_t)] = (r+\delta+\lambda)\, q(\phi_t) - \lambda\overline{q}. \tag{27}$$

Use the first-order condition in equation (17) and the fact that $q(\phi) \equiv v(\phi)$ to replace $q(\phi_t)$ in equation (27) by $1 + c'(\gamma_t)$ to obtain

$$\phi_t - [c(\gamma_t) - \gamma_t c'(\gamma_t)] = (r+\delta+\lambda)\,(1 + c'(\gamma_t)) - \lambda\overline{q}. \tag{28}$$

Finally, use the definition of $H(\gamma, \Phi, \rho)$ in equation (11) to rewrite equation (28) as

$$H(\gamma_t, \phi_t, r+\delta+\lambda) = -\lambda\overline{q}, \tag{29}$$

which is identical (since $\overline{q} = \overline{v}$) to the first-order condition in equation (22).

4 Changing the Stochastic Properties of Φ

In this section I consider the impact of changing the stochastic properities of the marginal operating profit of capital, Φ. Specifically, I consider three changes: (1) replacing the original distribution $F(\phi)$ by a distribution that first-order stochastically dominates the original distribution; (2) introducing a mean-preserving spread on $F(\phi)$; and (3) increasing λ, the arrival rate of a new value of Φ, which reduces the persistence of Φ.

4.1 $F_2(\Phi)$ First-Order Stochastically Dominates $F_1(\Phi)$

In this subsection, I analyze a change in the distribution $F(\Phi)$ from $F_1(\Phi)$ to $F_2(\Phi)$, where $F_2(\Phi)$ first-order stochastically dominates $F_1(\Phi)$. Let $\overline{v}_i$ be the unconditional expected value of a unit of capital when the distribution of Φ is $F_i(\Phi)$, $i = 1, 2$. Also, let $\gamma(\Phi, \overline{v}_i, r + \delta, \lambda)$ be the optimal value of γ for given Φ when the distribution of Φ is $F_i(\Phi)$, and let $\Gamma_i(\gamma)$ be the induced distribution of the optimal value of γ when the distribution of Φ is $F_i(\Phi)$, $i = 1, 2$.

Proposition 5 *If $F_2(\Phi)$ strictly first-order stochastically dominates $F_1(\Phi)$, then $\overline{v}_2 > \overline{v}_1$ and $\Gamma_2(\gamma)$ strictly first-order stochastically dominates $\Gamma_1(\gamma)$.*

Proposition 5 states that moving to a more favorable distribution of Φ that first-order stochastically dominates the original distribution will increase $\overline{v}$, the average value of a unit of capital. The increase in $\overline{v}$ will increase the optimal value of γ at each value of Φ, and because the distribution of Φ becomes more favorable and optimal γ is increasing in Φ, the distribution of optimal γ also moves toward larger values in the sense of first-order stochastic dominance.

4.2 A Mean-Preserving Spread on $F(\Phi)$

Now consider the effect on optimal investment of a mean-preserving spread on the distribution $F(\Phi)$. This question was first addressed in a model with convex costs of adjustment by Hartman (1972) and then by Abel (1983). In both papers, the production function is linearly homogeneous in capital and labor and the firm is perfectly competitive, so that, as in this paper, the marginal operating profit of capital, Φ, is independent of the capital stock. Hartman and Abel both found that an increase in the variance of the price of output leads to an increase in the optimal rate of investment.[13] The channel through which this effect operates is the convexity of $\Phi_t \equiv \max_l \left[p_t A_t f(1,l) - w_t l\right]$ in $p_t A_t$ and w_t. This convexity implies that a mean-preserving spread on $p_t A_t$ or w_t at some future time t increases the expected value of future Φ_t and thus increases the expected present value of the stream of future Φ_t, which increases (marginal) q and hence increases investment. In the current paper, I analyze a different channel for increased uncertainty to affect investment. To focus on that channel, I analyze mean-preserving spreads on the distribution of Φ_t directly. Since the expected value of Φ_t remains unchanged by construction, any effects on the optimal value of γ will operate through a different channel than emphasized by Hartman (1972) and Abel (1983).

Proposition 6 *A mean-preserving spread of $F(\Phi)$ that maintains the support within G increases $\overline{v}$.*

The proof of Proposition 6 is in the Appendix, but it is helpful to examine a key step to get a sense for what is driving the result. As shown in the Appendix, this result relies on the fact that $c'(\gamma(\Phi,\kappa,r+\delta,\lambda))$ is convex in Φ, even though $c'(\gamma)$ may not be convex in γ and $\gamma(\Phi,\kappa,r+\delta,\lambda)$ may not be convex in Φ. Notice that $c'(\gamma(\Phi,\kappa,r+\delta,\lambda))$ will be convex in Φ if $\frac{\partial c'(\gamma(\Phi,\kappa,r+\delta,\lambda))}{\partial\Phi} = c''(\gamma(\Phi,\kappa,r+\delta,\lambda)) \times \frac{\partial\gamma(\Phi,\kappa,r+\delta,\lambda)}{\partial\Phi}$ is increasing in Φ. However, neither $c''(\gamma(\Phi,\kappa,r+\delta,\lambda))$ nor $\frac{\partial\gamma(\Phi,\kappa,r+\delta,\lambda)}{\partial\Phi} = \frac{1}{(\rho-\gamma(\Phi,\kappa,r+\delta,\lambda))c''(\gamma(\Phi,\kappa,r+\delta,\lambda))}$ is necessarily increasing in Φ. But their product, $\frac{1}{\rho-\gamma(\Phi,\kappa,r+\delta,\lambda)}$, is increasing in Φ, as in statement 5 of Corollary 4, so $c'(\gamma(\Phi,\kappa,r+\delta,\lambda))$ is convex in Φ. Therefore, a mean-preserving spread on Φ increases the unconditional expected value of $c'(\gamma(\Phi,\kappa,r+\delta,\lambda))$ and hence increases $\overline{v}$.

[13] Caballero (1991) showed that the positive impact of uncertainty on optimal investment can be reversed by relaxing the assumption of perfect competition or by relaxing the linear homogeneity of the production function in capital and labor.

Corollary 6 *If $c'''(\gamma) \leq 0$ for $\gamma^m \leq \gamma \leq r + \delta + \lambda$, then a mean-preserving spread of $F(\Phi)$ that maintains the support within G increases $\overline{\gamma} \equiv \int_G \gamma(\Phi, \overline{v}, r + \delta, \lambda)\, dF(\Phi)$, the unconditional expected value of γ.*

As shown in the proof of Corollary 6 in the Appendix, a mean-preserving spread of $F(\Phi)$ increases the unconditional expected value of the investment-capital ratio through two channels. First, if $c'''(\gamma) \leq 0$, then $\gamma(\Phi, \kappa, r + \delta, \lambda)$, is (weakly) convex in Φ, so that a mean-preserving spread of $F(\Phi)$ (weakly) increases the expected value of $\gamma(\Phi, \kappa, r + \delta, \lambda)$. Second, a mean-preserving spread of $F(\Phi)$ increases $\overline{v}$, which also increases $\overline{\gamma} \equiv \int_G \gamma(\Phi, \overline{v}, r + \delta, \lambda)\, dF(\Phi)$. In the special case of quadratic adjustment costs, $c'''(\gamma) \equiv 0$, so the first channel is absent, but the second channel is sufficient for a mean-preserving spread of $F(\Phi)$ to increase the unconditional expected value of the investment-capital ratio.

4.3 A Change in Persistence of Regimes

Now consider a change in the persistence of regimes governing Φ. With a constant instantaneous probability λ of a switch in the regime, the expected life of a regime is $\frac{1}{\lambda}$, so an increase in λ reduces the persistence of the regime.

Proposition 7 *If $F(\Phi)$ is non-degenerate, then $\frac{\partial \overline{v}}{\partial \lambda} < 0$, so that an increase in the persistence of regimes (which is a reduction in λ) increases $\overline{v}$.*

The following informal argument may provide some helpful intuition. An increase in λ increases the frequency with which new values of Φ are drawn from $F(\Phi)$ and thus reduces the unconditional variance of the average value Φ over a given horizon. This reduction in variability reduces $\overline{v}$, somewhat analogously to Proposition 6. The formal proof, which is in the Appendix, does not rely on this loose analogy but is less intuitive.

The following corollary exploits the fact that in the case of quadratic adjustment costs, the optimal value of γ is an increasing linear function of $v(\phi)$.

Corollary 7 *If $F(\Phi)$ is non-degenerate and if $c(\gamma)$ is quadratic, then $\frac{\partial \overline{\gamma}}{\partial \lambda} < 0$.*

5 Measurement Error and the Cash Flow Effect on Investment

The model developed in this paper focuses on three variables that are often used in empirical studies of investment, specifically, the investment-capital ratio, γ, the value of a unit of capital, v, which is Tobin's q, and cash flow per unit of capital, Φ. This model, like most existing models, uses the first-order condition for optimal investment, $1 + c'(\gamma) = v(\phi)$ (equation (17)), to draw a tight link between γ and v. This link is often described by saying that v is a sufficient statistic for γ, meaning that if an observer knows the adjustment cost function and the value of v, then the value of γ can be computed in a straightforward manner without any additional information or knowledge of the values of any other variables. Indeed, if the adjustment cost function, $c(\gamma)$, is quadratic, the marginal adjustment cost function is linear, and optimal γ is a linear function of v.

The empirical literature has a long history of finding that v is not a sufficient statistic for γ. In particular, at least since the work of Fazzari, Hubbard, and Petersen (1988), researchers have found that in a regression of γ on v and Φ, estimated coefficients on both v and Φ tend to be positive and statistically significant. The finding of a positive significant coefficient on cash flow, Φ, is often interpreted as evidence that firms face financing constraints or some other imperfection in financial markets. This interpretation of financial frictions, as they are sometimes known, is bolstered by the finding that for firms that one might suspect to be more likely to face these frictions, the cash flow effect tends to be more substantial. For instance, as the argument goes, firms that are growing rapidly may encounter more substantial financial frictions, and it turns out that the cash-flow coefficient is often larger for such firms.[14]

In this section, I will offer a different interpretation of the cash-flow coefficient. I will demonstrate that if v is observed with classical measurement error, then the coefficient on v is biased toward zero and, more importantly, the coefficient on cash

[14] For instance, Devereux and Schiantarelli (1990) state "The perhaps surprising result from table 11.7 is that the coefficient on cash flow is greater for firms operating in growing sectors." (p. 298).

flow, Φ, will be positive, even though the coefficient on Φ would be zero in the absence of measurement error in v. The fact that measurement error in v can affect the coefficient estimates in this way been pointed out by Erickson and Whited (2000) and Gilchrist and Himmelberg (1995), though the particular simple expressions I present in this paper appear to be new. More novel, however, is the analytical demonstration that the cash-flow coefficient will be larger for firms that grow more rapidly.

The finding that measurement error in v can lead to a positive cash-flow coefficient does not use the particular model in this paper, other than the result that v and Φ are positively correlated with each other. I derive this result in subsection 5.1. Then in subsection 5.2, I use the model in this paper to demonstrate that cash-flow coefficients are larger for firms that have higher growth rates. Although the literature interprets the empirical finding of larger cash-flow coefficients for more rapidly growing firms as evidence of financial frictions, the model here has no financial frictions whatsoever, and yet leads to the same finding. Therefore, the finding of positive cash-flow coefficients, including larger coefficients for firms that are growing more rapidly, does not necessarily show that financial frictions are important or operative.

5.1 Coefficient Estimates under Measurement Error

In this subsection I analyze the impact of measurement error on the estimated coefficients on q and cash flow in investment regressions. To isolate measurement error from specification error that might arise by fitting a linear function to a nonlinear relationship, I assume that the adjustment cost function is quadratic so that optimal γ is a linear function of v. In particular, the adjustment cost function is

$$c(\gamma_t) = \frac{1}{2}\theta(\gamma_t - \gamma_0)^2, \tag{30}$$

where $\theta > 0$ and, as discussed earlier, $\gamma^m < \gamma_0 < r + \delta$. The first-order condition for optimal γ in equation (17) implies that

$$\gamma_t = \gamma_0 + \frac{v_t - 1}{\theta}. \tag{31}$$

Assume that the manager of the firm can observe v, Φ, and γ without error, but people outside the firm, including the econometrician, observe these variables with classical measurment error. Specifically, the econometrician observes the value of a unit of capital as $\widetilde{q} = v + \varepsilon_q$, the investment-capital ratio as $\widetilde{\gamma} = \gamma + \varepsilon_\gamma = \gamma_0 + \frac{v-1}{\theta} + \varepsilon_\gamma$, and cash flow as $\widetilde{c} = \Phi_t + \varepsilon_c$, where the observation errors ε_q, ε_γ, and ε_c, are mean zero, mutually independent, and independent of v, Φ, and γ. Erickson and Whited (2000) offer a useful taxonomy of reasons for measurement error in q, and except for differences between marginal q and average q (which are non-existent in the model presented here), those reasons could apply here.

Consider a linear regression of $\widetilde{\gamma}$ on $\widetilde{q}$ and $\widetilde{c}$, after all variables have been de-meaned. Let b_q and b_c be the probability limits of the estimated coefficients on $\widetilde{q}$ and $\widetilde{c}$, respectively, so

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \begin{bmatrix} Var(\widetilde{q}) & Cov(\widetilde{q}, \widetilde{c}) \\ Cov(\widetilde{q}, \widetilde{c}) & Var(\widetilde{c}) \end{bmatrix}^{-1} \begin{bmatrix} Cov(\widetilde{q}, \widetilde{\gamma}) \\ Cov(\widetilde{c}, \widetilde{\gamma}) \end{bmatrix}. \tag{32}$$

The variance-covariance matrix, A, of $(\widetilde{q}, \widetilde{c}, \widetilde{\gamma})$ conveniently displays the variances and covariances in equation (32), where

$$A = \begin{bmatrix} Var(v) + Var(\varepsilon_q) & Cov(v, \Phi) & \frac{1}{\theta} Var(v) \\ Cov(v, \Phi) & Var(\Phi) + Var(\varepsilon_c) & \frac{1}{\theta} Cov(v, \Phi) \\ \frac{1}{\theta} Var(v) & \frac{1}{\theta} Cov(v, \Phi) & \frac{1}{\theta^2} Var(v) + Var(\varepsilon_\gamma) \end{bmatrix}. \tag{33}$$

Substituting the relevant second moments from equation (33) into equation (32), and performing the indicated matrix inversion and matrix multiplication yields

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \frac{\frac{1}{\theta}}{[Var(v) + Var(\varepsilon_q)][Var(\Phi) + Var(\varepsilon_c)] - [Cov(v, \Phi)]^2} \times \begin{bmatrix} [Var(\Phi) + Var(\varepsilon_c)] Var(v) - Cov(v, \Phi) Cov(v, \Phi) \\ [Var(v) + Var(\varepsilon_q)] Cov(v, \Phi) - Cov(v, \Phi) Var(v) \end{bmatrix}. \tag{34}$$

Define $s_q^2 \equiv \frac{Var(\varepsilon_q)}{Var(v)}$ as the variance of the measurement error in $\widetilde{q}$ normalized by $Var(v)$, which is the variance of the true value of q; $s_c^2 \equiv \frac{Var(\varepsilon_c)}{Var(\Phi)}$ as the variance of the measurement error in cash flow normalized by the variance of the true value of cash flow; and $R^2 = \frac{[Cov(v,\Phi)]^2}{Var(\Phi)Var(v)}$ as the squared correlation between the true values of q and cash flow. Dividing both the numerators and denominators of b_q and b_c in equation (34) by $Var(\Phi)Var(v)$ yields

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \frac{\frac{1}{\theta}}{1 - R^2 + s_c^2 + s_q^2 + s_q^2 s_c^2} \begin{bmatrix} 1 - R^2 + s_c^2 \\ s_q^2 \frac{Cov(v,\Phi)}{Var(\Phi)} \end{bmatrix}. \tag{35}$$

Equation (35) shows the impact of measurement error in q. If q is perfectly measured, then $s_q^2 = 0$ and, regardless of whether cash flow is measured with error, equation (35) immediately yields $b_q = \frac{1}{\theta}$ and $b_c = 0$. Thus, if q is perfectly measured, b_q equals the derivative of the optimal value of γ with respect to v in the first-order condition in equation (31). In addition, the estimated effect of cash flow on investment, b_c, is zero. Erickson and Whited (2000) use measurement-error consistent GMM estimators and find empirically that the cash-flow coefficient is zero and that investment is well explained by q, when properly removing the effects of measurement error.

If q is measured with error, so that $s_q^2 > 0$, then, b_q, the estimated coefficient on q, is smaller than $\frac{1}{\theta}$, the true derivative of γ with respect to v. Moreover, if $s_q^2 > 0$, then b_c, the estimated coefficient on cash flow can be nonzero; in fact, if q and cash flow are positively correlated, the estimated cash-flow coefficient, b_c, is positive. Much of the investment literature interprets a significantly positive coefficient on cash flow in a regression of investment on q and cash flow as evidence of financing constraints.[15] Yet equation (35) demonstrates that measurement error in q will lead to a positive coefficient on cash flow, provided that q and cash flow are positively correlated, even if there are no financial frictions. This argument is not restricted to the particular specification of the firm in this model, and has been made less formally

[15] As discussed earlier, notable exceptions include Abel and Eberly (2011), Alti (2003), Cooper and Ejarque (2003), Erickson and Whited (2000), Gilchrist and Himmelberg (1995), and Gomes (2001).

by, for example, Gilchrist and Himmelberg (1995).[16] The model in this paper allows the analysis to go one step further and to account for differences in the estimated cash-flow coefficients for firms with different growth rates, as I discuss in the next subsection.[17]

5.2 Larger Cash-Flow Coefficients for More Rapidly Growing Firms

Proponents of the view that positive cash-flow coefficients are evidence of financing constraints bolster their view by showing that firms that are likely to face binding financing constraints are likely to exhibit larger, more significant positive cash-flow coefficients. For instance, they argue that firms that are growing more quickly are more likely to face binding financing constraints. Empirical evidence that rapidly growing firms have larger, significant positive cash-flow coefficients is then presented as evidence of financing constraints. However, the model in this paper offers an alternative interpretation. Equation (35) shows that the cash-flow coefficient is proportional to $\frac{Cov(v,\Phi)}{Var(\Phi)}$, which is the population regression coefficient of v on Φ. The analog of this coefficient in the model is $\frac{dv(\Phi)}{d\Phi}$, which equals $\frac{\partial c'(\gamma(\Phi,\overline{v},r+\delta,\lambda))}{\partial \Phi}$ because $v(\Phi) = 1 + c'(\gamma(\Phi, \overline{v}, r+\delta, \lambda))$. Since $\frac{\partial c'(\gamma(\Phi,\overline{v},r+\delta,\lambda))}{\partial \Phi} = \frac{1}{r+\delta+\lambda-\gamma}$ (statement 1 of Corollary 4), $\frac{dv(\Phi)}{d\Phi} = \frac{1}{r+\lambda-(\gamma-\delta)}$, which is increasing in the growth rate of capital, $\gamma - \delta$, for given $r + \lambda$. Therefore, the cash-flow coefficient is increasing in the growth rate of the firm.

To use the model to compare the investment behaviors of a slowly growing firm and a rapidly growing firm, I will consider firms that face different unconditional distributions, $F(\Phi)$, of Φ, that endogenously lead to different growth rates. The following proposition states that the firm with the more favorable $F(\Phi)$ in the sense of strict first-order stochastic dominance will grow more rapidly and will have the

[16] Gilchrist and Himmelberg, p. 544, state "More generally, anything that systematically reduces the signal-to-noise ratio of Tobin's Q (for example, measurement error or 'excess volatility' of stock prices) will shift explanatory power away from Tobin's Q toward cash flow, thus making such firms appear to be financially constrained when in fact they are not."

[17] Erickson and Whited (2000) derive an alternative expression for the estimated cash-flow coefficient and demonstrate that to the extent that measurement error in q imparts downward bias in b_q, the cash-flow coefficient will be higher for firms with more variable cash flows.

higher cash-flow coefficient, which is proportional to $\frac{dv(\Phi)}{d\Phi}$, even though there are no financial frictions in the model.

Proposition 8 *Consider two firms with identical quadratic adjustment cost functions but with different unconditional distributions of* Φ, $F_1(\Phi)$ *and* $F_2(\Phi)$, *which imply different unconditional values of capital,* $\overline{v}_1$ *and* $\overline{v}_2$. *If* $F_2(\Phi)$ *strictly first-order stochastically dominates* $F_1(\Phi)$, *then*

1. $\gamma(\Phi, \overline{v}_2, r+\delta, \lambda) > \gamma(\Phi, \overline{v}_1, r+\delta, \lambda)$,
2. $\int_G \gamma(\Phi, \overline{v}_2, r+\delta, \lambda)\, dF_2(\Phi) > \int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda)\, dF_1(\Phi)$,
3. $\frac{dv_2(\Phi)}{d\Phi} > \frac{dv_1(\Phi)}{d\Phi}$, *and*
4. $\int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) > \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi)$.

Proposition 8 states that the firm with distribution $F_2(\Phi)$ is the faster-growing firm, whether the speed of growth is measured by the investment-capital ratio at any given value of Φ (statement 1) or by the unconditional expectation of the investment-capital ratio (statement 2). This proposition also states that the firm with the distribution $F_2(\Phi)$ has the higher value of $\frac{dv(\Phi)}{d\Phi}$ for a given value of Φ (statement 3) and the higher unconditional expected value of $\frac{dv(\Phi)}{d\Phi}$ (statement 4). Therefore, the firm with distribution $F_2(\Phi)$ has the higher value of $\frac{Cov(v,\Phi)}{Var(\Phi)}$ and hence the higher cash-flow coefficient. To summarize, the firm that is growing more rapidly has the larger coefficient on cash flow, even though there are no financial frictions in this model.

6 Concluding Remarks

This paper develops a model of a competitive firm with constant returns to scale to provide a tractable stochastic framework to analyze the behavior and interrelationships among optimal investment, q, and cash flow. As first shown by Hayashi (1982), average q and marginal q are identically equal in this framework. Within the class of models for which average q and marginal q are equal, the model presented here places only one additional restriction on technology, namely that adjustment costs, which are a function of investment and the capital stock, are additively separable from the production function for output, which is a function of capital and labor. For convenience, the model specifies a constant discount rate and a constant depreciation rate of capital. Finally, the analysis of the stochastic model is greatly facilitated by the simple Markov regime-switching specification for the marginal operating profit of capital.

The model developed here is tractable enough to analyze various aspects of optimal investment behavior in a framework that is consistent with empirical analyses that use average q to measure marginal q and that specify the investment-capital ratio as a function of q. When the marginal operating profit of capital follows a Markov regime-switching process, I present analytic expressions for the optimal investment-capital ratio and the value of a unit of capital that are closed-form up to a single undetermined scalar constant.

After demonstrating various properties of optimal investment and q, I use the model to analyze the effects of three changes in the stochastic environment facing the firm. First, a favorable shift in the unconditional distribution of the marginal operating profit of capital, in the sense of first-order stochastic dominance, increases the expected value of a unit of capital, and shifts the distribution of the optimal investment-capital ratio in a first-order stochastically dominating way. Second, a mean-preserving spread of the unconditional distribution of the marginal operating profit of capital increases the average value of a unit of capital, as in the existing literature, though the channel through which the effect operates is different than in previous studies. Third, an increase in the persistence of regimes increases the average value of a unit of capital.

To address the common empirical finding of a positive coefficient on cash flow in a regression of the investment-capital ratio on q and cash flow, I introduce classical measurement error. Consistent with existing arguments, I show that measurement error in q can lead to a positive coefficient on cash flow. However, I use the model to go a step further and demonstrate that the model can account for the finding of larger cash-flow coefficients for firms that grow more rapidly. Proponents of the importance of financing constraints point to the positive coefficient on cash flow as evidence of the importance of these constraints. Moreover, they argue that larger cash-flow coefficients for firms likely to be constrained, such as rapidly growing firms, support the interpretation that positive cash-flow coefficients indicate the importance of financing constraints. However, the model presented here has no financing constraints at all, yet in the presence of classical measurement error, it predicts coefficients on cash flow that are both positive and larger for firms that grow more rapidly.

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Appendix: Proofs of Lemmas, Propositions and Corollaries

Proof. of Proposition 1: Assume that $K_t > 0$ and let $\left\{K_s^A, I_s^A\right\}_{s=t}^{s=\infty}$ satisfy the capital accumulation equation in (1) and attain the maximum on the right-hand side of (2). Let $\left\{K_s^B, I_s^B\right\}_{s=t}^{s=\infty} = \left\{\omega K_s^A, \omega I_s^A\right\}_{s=t}^{s=\infty}$ for an arbitrary $\omega > 0$ and note that $\left\{K_s^B, I_s^B\right\}_{s=t}^{s=\infty}$ satisfies the capital accumulation equation in (1). Then $V_t\left(\omega K_t^A\right) = V_t\left(K_t^B\right) \geq E_t\left\{\int_t^\infty \pi_s\left(K_s^B, I_s^B\right) M\left(t,s\right) ds\right\} = E_t\left\{\int_t^\infty \pi_s\left(\omega K_s^A, \omega I_s^A\right) M\left(t,s\right) ds\right\} = \omega E_t\left\{\int_t^\infty \pi_s\left(K_s^A, I_s^A\right) M\left(t,s\right) ds\right\} = \omega V_t\left(K_t^A\right)$. Since $V_t\left(\omega K_t\right) \geq \omega V_t\left(K_t\right)$ for any $\omega > 0$ and any $K_t > 0$, we have $V_t\left(\frac{1}{\omega}K_t^B\right) \geq \frac{1}{\omega}V_t\left(K_t^B\right)$, which implies $\omega V_t\left(K_t^A\right) = \omega V_t\left(\frac{1}{\omega}K_t^B\right) \geq V_t\left(K_t^B\right) = V_t\left(\omega K_t^A\right)$. Therefore, $V_t\left(\omega K_t^A\right) \geq \omega V_t\left(K_t^A\right) \geq V_t\left(\omega K_t^A\right)$, which implies $V_t\left(\omega K_t^A\right) = \omega V_t\left(K_t^A\right)$. ■

Proof. of Lemma 1: (1) Inspection of the definition of $H\left(\gamma, \Phi, \rho\right)$ in equation (11) immediately reveals that $H\left(\gamma, \Phi, \rho\right)$ is an increasing linear function of Φ. (2) Differentiating $H\left(\gamma, \Phi, \rho\right)$ with respect to ρ yields $H_\rho = -\left(1 + c'\left(\gamma\right)\right)$, which is negative for $\gamma > \gamma^m$. (3) Differentiating $H\left(\gamma, \Phi, \rho\right)$ with respect to γ yields $H_\gamma\left(\gamma, \Phi, \rho\right) = -\left(\rho - \gamma\right) c''\left(\gamma\right)$. Since $c\left(\gamma\right)$ is strictly convex, $c''\left(\gamma\right) > 0$. Therefore, $H\left(\gamma, \Phi, \rho\right)$ is strictly decreasing in γ for $\gamma < \rho$, strictly increasing in γ for $\gamma > \rho$, and minimized with respect to γ at $\gamma = \rho$. That is, $H\left(\gamma, \Phi, \rho\right)$ is strictly quasi-convex in γ. (4) The adjustment cost function $c\left(\gamma\right)$ attains its minimum value, which is zero, at $\gamma = \gamma_0$, so $c\left(\gamma_0\right) = c'\left(\gamma_0\right) = 0$. Therefore, $H\left(\gamma_0, \Phi, \rho\right) = \Phi - \rho - c\left(\gamma_0\right) - \left(\rho - \gamma_0\right) c'\left(\gamma_0\right) = \Phi - \rho$. (5) Since $c'\left(\gamma^m\right) = -1$, $H\left(\gamma^m, \Phi, \rho\right) = \Phi - \rho - c\left(\gamma^m\right) - \left(\rho - \gamma^m\right) c'\left(\gamma^m\right) = \Phi - \rho - c\left(\gamma^m\right) + \rho - \gamma^m = \Phi - c\left(\gamma^m\right) - \gamma^m > 0$ since $\Phi \in G$. (6) Since $H_\gamma\left(\gamma, \Phi, \rho\right) = -\left(\rho - \gamma\right) c''\left(\gamma\right)$ and $H\left(\gamma, \Phi, \rho\right)$ is strictly quasi-convex in γ, $H\left(\gamma, \Phi, \rho\right)$ is minimized with respect to γ at $\gamma = \rho$. Therefore, $\min_\gamma H\left(\gamma, \Phi, \rho\right) = H\left(\rho, \Phi, \rho\right) = \Phi - \rho - c\left(\rho\right) \leq \Phi - \left(r + \delta\right) - c\left(r + \delta\right) < 0$, where the first inequality follows from $c'\left(r + \delta\right) > 0$, $c''\left(\gamma\right) > 0$, and $\rho \geq r + \delta$, and the second inequality follows from the assumption that $\Phi \in G$. (7) $H\left(\gamma^m, \Phi, \rho\right) > 0 > H\left(\rho, \Phi, \rho\right)$ and the strict quasi-convexity of $H\left(\gamma, \Phi, \rho\right)$ in γ imply that $H\left(\gamma, \Phi, \rho\right) = 0$ for exactly two distinct values of γ, denoted γ_L and $\gamma_H > \gamma_L$, with $\gamma^m < \gamma_L < \rho$ and $\gamma_H > \rho$. $H\left(\gamma_i, \Phi, \rho\right) = 0$, for $i = L, H$, implies $\Phi - \gamma_i - c(\gamma_i) = (\rho - \gamma_i)[1 + c'(\gamma_i)]$, $i = L, H$. Since $1 + c'(\gamma) > 0$ for $\gamma > \gamma^m$, it follows that $\Phi - \gamma_L - c(\gamma_L) > 0$ for $\gamma_L \in (\gamma^m, \rho)$ and $\Phi - \gamma_H - c(\gamma_H) < 0$ for $\gamma_H \in (\rho, \infty)$. ■

Proof. of Proposition 2: The optimal value of the investment-capital ratio when Φ_t is known with certainty to be constant and equal to Φ, that is, $\gamma^c(\Phi, r + \delta)$, is a root of $H(\gamma, \Phi, r + \delta) = 0$. Statement 7 of Lemma 1 states that this equation has two roots: $\gamma_L \in (\gamma^m, r + \delta)$ and $\gamma_H \in (r + \delta, \infty)$ and that $\Phi - \gamma_L - c(\gamma_L) > 0 > \Phi - \gamma_H - c(\gamma_H)$. Since $\pi_t(K_t, \gamma_H K_t) = [\Phi_t - \gamma_H - c(\gamma_H)] K_t < 0$, the root γ_H cannot be the optimal value of γ. Therefore, the smaller root, γ_L, is the optimal value of the investment-capital ratio when Φ_t is known with certainty to be constant and equal to Φ. ■

Proof. of Corollary 1: Apply the implicit function theorem to $H(\gamma^c, \Phi, r + \delta) = 0$ and use the facts that $\gamma^c < r + \delta$, $1 + c'(\gamma^c) > 0$, and $c''(\gamma^c) > 0$ to obtain $\frac{\partial \gamma^c(\Phi, r+\delta)}{\partial \Phi} = -\frac{H_\Phi}{H_\gamma} = \frac{1}{(r+\delta-\gamma^c)c''(\gamma^c)} > 0$ and $\frac{\partial \gamma^c(\Phi, r+\delta)}{\partial (r+\delta)} = -\frac{H_\rho}{H_\gamma} = -\frac{1+c'(\gamma^c)}{(r+\delta-\gamma^c)c''(\gamma^c)} < 0$. ■

Proof. of Corollary 2: Differentiate the first-order condition for optimal investment, which holds at all points of time when Φ_t is known with certainty to be constant and equal to Φ, $v^c(\Phi, r + \delta) = 1 + c'(\gamma^c)$, with respect to Φ to obtain $\frac{\partial v^c(\Phi, r+\delta)}{\partial \Phi} = c''(\gamma^c) \frac{\partial \gamma^c(\Phi, r+\delta)}{\partial \Phi} = c''(\gamma^c) \frac{1}{(r+\delta-\gamma^c)c''(\gamma^c)} = \frac{1}{r+\delta-\gamma^c} > 0$. Differentiate the expression for $\frac{\partial v^c(\Phi, r+\delta)}{\partial \Phi}$ with respect to Φ to obtain $\frac{\partial^2 v^c(\Phi, r+\delta)}{(\partial \Phi)^2} = \frac{1}{(r+\delta-\gamma^c)^2} \frac{\partial \gamma^c(\Phi, r+\delta)}{\partial \Phi} = \frac{1}{(r+\delta-\gamma^c)^2} \frac{1}{(r+\delta-\gamma^c)c''(\gamma^c)} = \frac{1}{(r+\delta-\gamma^c)^3} \frac{1}{c''(\gamma^c)} > 0$. Differentiate $v^c(\Phi, r + \delta) = 1 + c'(\gamma^c)$, with respect to $r + \delta$ to obtain $\frac{\partial v^c(\Phi, r+\delta)}{\partial (r+\delta)} = c''(\gamma^c) \frac{\partial \gamma^c(\Phi, r+\delta)}{\partial (r+\delta)} = -c''(\gamma^c) \frac{1+c'(\gamma^c)}{(r+\delta-\gamma^c)c''(\gamma^c)} = -\frac{1+c'(\gamma^c)}{r+\delta-\gamma^c} < 0$. ■

Proof. of Corollary 3: $H(\gamma^c(\Phi, r + \delta), \Phi, r + \delta) = 0$ and (from statement 4 of Lemma 1) $H(\gamma_0, \Phi, r + \delta) = \Phi - (r + \delta)$, so $H(\gamma^c(\Phi, r + \delta), \Phi, r + \delta) - H(\gamma_0, \Phi, r + \delta) = -[\Phi - (r + \delta)]$. Since $H_\gamma(\gamma, \Phi, r + \delta) < 0$ for all $\gamma < r + \delta$, including $\gamma = \gamma^c(\Phi, r + \delta)$ and $\gamma = \gamma_0$ and all values between, $\gamma^c(\Phi, r + \delta) \gtreqless \gamma_0$ as $H(\gamma^c(\Phi, r + \delta), \Phi, r + \delta) \lesseqgtr H(\gamma_0, \Phi, r + \delta)$ as $\Phi \gtreqless r + \delta$. ■

Proof. of Proposition 3: **Calculation of $q(\phi)$:** Suppose that $\Phi_s = \phi$ for all $t \leq s < x$ and the regime switches at time x, with a new drawing of Φ, say Φ_x, from the unconditional distribution $F(\Phi)$. The expression for $\pi_s(K_s, I_s)$ in equation (5) can be written as $\pi_s(K_s, I_s) = \left[\phi - c\left(\frac{I_s}{K_s}\right)\right] K_s - I_s$. Therefore, $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} =$

$\phi - c(\gamma_s) + \gamma_s c'(\gamma_s)$, which equals $\phi - c(\gamma_t) + \gamma_t c'(\gamma_t)$ for all $t \leq s < x$, so that the marginal contribution to profit at time s accruing to the remaining undepreciated portion of a unit of capital installed at time t is $e^{-\delta(s-t)}[\phi - c(\gamma_t) + \gamma_t c'(\gamma_t)]$. As for the stream of marginal contributions of capital accruing from time x onward to the remaining undepreciated portion of a unit of capital, their expected present value as of time x is $e^{-\delta(x-t)} q(\Phi_x)$; the expected present value of $e^{-\delta(x-t)} q(\Phi_x)$ as of time t is $e^{-(r+\delta)(x-t)}\overline{q}$, where $\overline{q} \equiv \int_G q(\Phi)\, dF(\Phi)$ is the unconditional expected value of $q(\Phi_x)$. Therefore, $q_{t|x}$, the value of q_t, conditional on the next regime switch occurring at time $x > t$ is

$$q_{t|x} = \frac{1 - e^{-(r+\delta)(x-t)}}{r+\delta}\left(\phi - c(\gamma_t) + \gamma_t c'(\gamma_t)\right) + e^{-(r+\delta)(x-t)}\overline{q}. \tag{36}$$

The first term on the right-hand side of equation (36) is the present value of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)}$ $= \frac{\partial \pi_s(1,\gamma_s)}{\partial K_s} e^{-\delta(s-t)}$ from time t to time x.[18] The second term is the expected present value, discounted to time t, of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)}$ from time x onward.

The probability density for the first switch in the regime after time t occurring at time $x > t$ is $\lambda e^{-\lambda(x-t)}$, so that

$$q(\phi) = \int_t^{\infty} \lambda e^{-\lambda(x-t)} q_{t|x}\, dx. \tag{37}$$

Substituting equation (36) into equation (37) and performing the integration yields

$$q(\phi) = \frac{\phi - c(\gamma) + \gamma c'(\gamma)}{r+\delta+\lambda} + \frac{\lambda}{r+\delta+\lambda}\overline{q}, \tag{38}$$

where γ in this equation is the optimal value of γ when $\Phi = \phi$.

Calculation of $v(\phi)$: Suppose that $\Phi_s = \phi$ for all $t \leq s < x$ and the regime switches at time x, with a new drawing of Φ from the unconditional distribution $F(\Phi)$. With $\pi_s(K_s, I_s)$ specified as in equation (5), $\pi_s(1, \gamma_s) = \phi - \gamma_t - c(\gamma_t)$ for all $t \leq s < x$ and the growth rate of the capital stock is $g_u = g_t \equiv \gamma_t - \delta$ for all $t \leq u < x$. Therefore, $v_{t|x}$, the value of v_t, conditional on the next regime switch

[18] The equality of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)}$ and $\frac{\partial \pi_s(1,\gamma_s)}{\partial K_s} e^{-\delta(s-t)}$ is an implication of the linear homogeneity of $\pi_s(K_s, I_s)$.

occurring at time $x > t$, is

$$v_{t|x} = \frac{1 - e^{-(r+\delta-\gamma_t)(x-t)}}{r + \delta - \gamma_t} \left(\phi - \gamma_t - c\left(\gamma_t\right)\right) + e^{-(r+\delta-\gamma_t)(x-t)} \overline{v}, \tag{39}$$

where $\overline{v} \equiv \int_G v\left(\Phi\right) dF\left(\Phi\right)$ is the unconditional expected value of $v\left(\Phi_x\right)$ defined in equation (16). The first term on the right-hand side of equation (39) is the present value of $\pi_s\left(1, \gamma_s\right) \exp\left(\int_t^s g_u du\right) = \pi_t\left(1, \gamma_t\right) e^{(\gamma_t-\delta)(s-t)}$ from time t to time x. The second term on the right-hand side of equation (39) is the expected present value, discounted to time t, of $\pi_s\left(1, \gamma_s\right) \exp\left(\int_t^s g_u du\right)$ from time x onward, $E\left\{\int_x^\infty \pi_s\left(1, \gamma_s\right) \exp\left(-\int_t^s \left(r - g_u\right) du\right) ds\right\} =$ $e^{-(r+\delta-\gamma_t)(x-t)} E\left\{\int_x^\infty \pi_s\left(1, \gamma_s\right) \exp\left(-\int_x^s \left(r - g_u\right) du\right) ds\right\} = e^{-(r+\delta-\gamma_t)(x-t)} \overline{v}$.

Since the probability density for the first switch in the regime after time t occurring at time $x > t$ is $\lambda e^{-\lambda(x-t)}$, the average value of capital is

$$v\left(\phi\right) = \int_t^\infty \lambda e^{-\lambda(x-t)} v_{t|x} dx. \tag{40}$$

Next substitute equation (39) into equation (40) and perform the integration to obtain

$$v\left(\phi\right) = \frac{\phi - \gamma - c\left(\gamma\right)}{r + \delta + \lambda - \gamma} + \frac{\lambda}{r + \delta + \lambda - \gamma} \overline{v}, \tag{41}$$

where γ in this equation is the optimal value of γ when $\Phi = \phi$. ■

Proof. of Lemma 2: Since $\kappa \leq 1 + c'\left(r + \delta\right)$ and $\lambda > 0$, $\lambda\kappa \leq \lambda + \lambda c'\left(r + \delta\right)$. The strict convexity of $c\left(\gamma\right)$ implies that $c\left(r + \delta + \lambda\right) - c\left(r + \delta\right) > \lambda c'\left(r + \delta\right)$, so $\lambda\kappa < \lambda + c\left(r + \delta + \lambda\right) - c\left(r + \delta\right) = r + \delta + \lambda + c\left(r + \delta + \lambda\right) - \left(r + \delta\right) - c\left(r + \delta\right) =$ $-\left[\left(r + \delta\right) + c\left(r + \delta\right) - \rho - c\left(\rho\right)\right]$, where $\rho = r + \delta + \lambda$. The proof of statement 6 of Lemma 1 implies $H\left(\rho, r + \delta + c\left(r + \delta\right), \rho\right) = r + \delta + c\left(r + \delta\right) - \rho - c\left(\rho\right) < 0$, so $\lambda\kappa < -H\left(\rho, r + \delta + c\left(r + \delta\right), \rho\right)$ or, equivalently, $-\lambda\kappa > H\left(\rho, r + \delta + c\left(r + \delta\right), \rho\right) >$ $H\left(\rho, \Phi, \rho\right)$ for any $\Phi \in G$. Since $H_\gamma\left(\gamma, \Phi, \rho\right) < 0$ for $\gamma < \rho$ and (from statement 5 of Lemma 1) for any $\Phi \in G$, $H\left(\gamma^m, \Phi, \rho\right) > 0 \geq -\lambda\kappa$, there is a unique $\gamma \in \left(\gamma^m, \rho\right)$ for which $H\left(\gamma, \Phi, \rho\right) = -\lambda\kappa \leq 0$. That value of γ is $\gamma\left(\Phi, \kappa, r + \delta, \lambda\right)$. ■

Proof. of Lemma 3: Let $\rho = r+\delta+\lambda$. (1) Differentiate $H\left(\gamma\left(\phi, \kappa, r + \delta, \lambda\right), \phi, r + \delta + \lambda\right)$ $= -\lambda\kappa$ with respect to ϕ to obtain $H_\gamma\left(\gamma\left(\phi, \kappa, r + \delta, \lambda\right), \phi, \rho\right) \frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\phi}$ +

$H_\phi\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=0$. Therefore, $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\phi}=-\frac{H_\phi(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho)}{H_\gamma(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho)}$. Use the facts that $H_\phi\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=1$ and $H_\gamma\left(\gamma,\phi,\rho\right)=-\left(\rho-\gamma\right)c''\left(\gamma\right)$ to obtain $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\phi}=\frac{1}{(\rho-\gamma)c''(\gamma)}>0$ since optimal $\gamma<\rho$.

(2) Differentiate $H\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=-\lambda\kappa$ with respect to κ to obtain $H_\gamma\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\kappa}=-\lambda$. Use the fact that $H_\gamma\left(\gamma,\phi,\rho\right)=-\left(\rho-\gamma\right)c''\left(\gamma\right)$ to obtain $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\kappa}=\frac{\lambda}{(\rho-\gamma)c''(\gamma)}>0$ since optimal $\gamma<\rho$.

(3) Differentiate $H\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,r+\delta+\lambda\right)=-\lambda\kappa$ with respect to $r+\delta$ to obtain

$H_\gamma\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial(r+\delta)}+H_\rho\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=0$. Use the facts that

$H_\rho\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=-\left(1+c'\left(\gamma\right)\right)$ and $H_\gamma\left(\gamma,\Phi,\rho\right)=-\left(\rho-\gamma\right)c''\left(\gamma\right)$ to obtain $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial(r+\delta)}=-\frac{H_\rho(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho)}{H_\gamma(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho)}=-\frac{1+c'(\gamma)}{(\rho-\gamma)c''(\gamma)}<0$ since optimal $\gamma<\rho$.

(4) Differentiate $H\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,r+\delta+\lambda\right)=-\lambda\kappa$ with respect to λ to obtain

$H_\gamma\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\lambda}+H_\rho\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=-\kappa$. Use the facts that $H_\rho\left(\gamma\left(\phi,\kappa,r+\delta,\lambda\right),\phi,\rho\right)=-\left(1+c'\left(\gamma\right)\right)$ and $H_\gamma\left(\gamma,\Phi,\rho\right)=-\left(\rho-\gamma\right)c''\left(\gamma\right)$ to obtain $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\lambda}=-\frac{(\kappa+H_\rho(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho))}{H_\gamma(\gamma(\phi,\kappa,r+\delta,\lambda),\phi,\rho)}=-\frac{1+c'(\gamma)-\kappa}{(\rho-\gamma)c''(\gamma)}$. ■

Proof. of Corollary 4: Use the chain rule to obtain $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial x}=\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial\gamma}\frac{\partial\gamma}{\partial x}$ $=c''\left(\gamma\right)\frac{\partial\gamma}{\partial x}$ for $x=\phi$, κ, $r+\delta$, and λ. Also use the chain rule to obtain $\frac{\partial^2 c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{(\partial\phi)^2}$ $=\frac{\partial}{\partial\phi}\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial\phi}=\frac{\partial}{\partial\phi}\frac{1}{\rho-\gamma}=\frac{1}{(\rho-\gamma)^2}\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\phi}=\frac{1}{(\rho-\gamma)^3c''(\gamma)}>0$ since optimal $\gamma<\rho$.

■

Proof. of Lemma 4: To prove property (1), let $\kappa=0$ and use the definitions of $\alpha\left(\kappa\right)$ and $\gamma\left(\phi,\kappa,r+\delta,\lambda\right)$ to obtain $\alpha\left(0\right)=1+\int_G c'\left(\gamma\left(\Phi,0,r+\delta,\lambda\right)\right)dF\left(\Phi\right)$. Lemma 2 implies that $\gamma\left(\Phi,0,r+\delta,\lambda\right)>\gamma^m$ and the convexity of $c\left(\gamma\right)$ implies $c'\left(\gamma\right)$ is strictly increasing so $\alpha\left(0\right)>1+\int_G c'\left(\gamma^m\right)dF\left(\Phi\right)=1+c'\left(\gamma^m\right)=0$.

To prove property (2), let $\kappa=1+c'\left(r+\delta\right)$ and use the definitions of $\alpha\left(\kappa\right)$ and $\gamma\left(\phi,\kappa,r+\delta,\lambda\right)$ to obtain $\alpha\left(1+c'\left(r+\delta\right)\right)=1+\int_G c'\left(\gamma\left(\Phi,1+c'\left(r+\delta\right),r+\delta,\lambda\right)\right)dF\left(\Phi\right)$. The definition of $H\left(\gamma,\phi,\rho\right)$ implies that $H\left(\gamma,\phi,r+\delta+\lambda\right)=H\left(\gamma,\phi,r+\delta\right)-\lambda\left(1+c'\left(\gamma\right)\right)$. In particular, this equation holds for $\gamma=r+\delta$, so that $H\left(r+\delta,\phi,r+\delta+\lambda\right)=H\left(r+\delta,\phi,r+\delta\right)-\lambda\left(1+c'\left(r+\delta\right)\right)$. Since $\Phi\in G\equiv\{\Phi:c\left(\gamma^m\right)+\gamma^m<\Phi<c\left(r+\delta\right)+r+\delta\}$,

$H(r+\delta,\phi,r+\delta) = \phi - (r+\delta) - c(r+\delta) < 0$. Therefore, $H(r+\delta,\phi,r+\delta+\lambda)$ $< -\lambda(1+c'(r+\delta)) \leq 0$, which, along with $H_\gamma(\gamma,\phi,r+\delta+\lambda) < 0$ for $\gamma < \rho$, implies $\gamma(\phi,1+c'(r+\delta),r+\delta,\lambda) < r+\delta$. Therefore, the convexity of $c(\gamma)$ implies $\alpha(1+c'(r+\delta)) = 1+\int_G c'(\gamma(\Phi,1+c'(r+\delta),r+\delta,\lambda))\,dF(\Phi) < 1+\int_G c'(r+\delta)\,dF(\Phi)$ $= 1+c'(r+\delta)$.

To prove property (3), it is helpful to first prove that $\gamma(\phi,\kappa,r+\delta,\lambda) < r+\delta$ for all κ in $[0,1+c'(r+\delta)]$. The proof of property (2) above includes a proof that $\gamma(\phi,1+c'(r+\delta),r+\delta,\lambda) < r+\delta$. Therefore, statement 2 of Lemma 3, i.e., $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\kappa} = \frac{\lambda}{(\rho-\gamma)c''(\gamma)} > 0$, implies that $\gamma(\phi,\kappa,r+\delta,\lambda) < r+\delta$ for all κ in $[0,1+c'(r+\delta)]$. Use the definition of $\alpha(\kappa)$ to obtain $\alpha'(\kappa) = \int_G \frac{\partial c'(\Phi,\kappa,r+\delta,\lambda)}{\partial\kappa}dF(\Phi)$. Use statement 2 in Corollary 4, i.e., $\frac{\partial c'(\gamma(\Phi,\kappa,r+\delta,\lambda))}{\partial\kappa} = \frac{\lambda}{r+\delta+\lambda-\gamma(\Phi,\kappa,r+\delta,\lambda)} > 0$, to obtain $\alpha'(\kappa) = \int_G \frac{\lambda}{r+\delta+\lambda-\gamma(\Phi,\kappa,r+\delta.\lambda)}dF(\Phi) > 0$. Since $\gamma(\Phi,\kappa,r+\delta,\lambda) < r+\delta$, $\frac{\lambda}{r+\delta+\lambda-\gamma(\Phi,\kappa,r+\delta,\lambda)} < 1$. Therefore, $\alpha'(\kappa) < \int_G dF(\Phi) = 1$, which completes the proof that $0 < \alpha'(\kappa) < 1$ for $\kappa \in [0,1+c'(r+\delta)]$. ■

Proof. of Proposition 4: The function $\alpha(\kappa)$ is continuous over the domain $[0,1+c'(r+\delta)]$ and has the three properties listed in Lemma 4. Therefore, there exists a unique positive value of $\kappa < 1+c'(r+\delta)$ that satisfies $\alpha(\kappa) = \kappa$. For that value of κ, $\int_G [1+c'(\gamma(\Phi,\kappa,r+\delta,\lambda))]\,dF(\Phi) = \kappa = \overline{v}$. ■

Proof. of Corollary 5: Suppose that $\alpha(\kappa^*) < \kappa^*$. Property 3 of Lemma 4 implies that $\alpha(\kappa) < \kappa$ for all $\kappa \in [\kappa^*,1+c'(r+\delta)]$. Therefore, the unique value of $\overline{v}$ for which $\alpha(\overline{v}) = \overline{v}$ is less than κ^*, so $\overline{v} - \kappa^* < 0$ when $\alpha(\kappa^*) - \kappa^* < 0$. A similar argument proves that $\overline{v} - \kappa^* > 0$ when $\alpha(\kappa^*) - \kappa^* > 0$. ■

Proof. of Proposition 5: Let $\kappa^* = \overline{v}_1$, which implies that $\overline{v}_1 = \int_G [1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda))]\,dF_1(\Phi)$. Since $c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda))$ is strictly increasing in Φ (statement 1 of Corollary 4), the assumption that $F_2(\Phi)$ strictly first-order stochastically dominates $F_1(\Phi)$ implies that $\kappa^* = \overline{v}_1 =$ $\int_G [1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda))]\,dF_1(\Phi) < \int_G [1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda))]\,dF_2(\Phi) \equiv \alpha_2(\kappa^*)$, using the definition in (24). Therefore, Corollary 5 implies that $\overline{v}_2 > \kappa^* = \overline{v}_1$.

Define $\omega(\gamma^*,\kappa,r+\delta,\lambda)$ to be the value of Φ for which $\gamma(\Phi,\kappa,r+\delta,\lambda) = \gamma^*$. Since $\gamma(\Phi,\kappa,r+\delta,\lambda)$ is strictly increasing in Φ and strictly increasing in κ, it follows that $\omega(\gamma^*, \kappa, r+\delta, \lambda)$ is strictly increasing in γ^* and is strictly decreasing in κ. Note that $\Gamma_1(\gamma^*) = F_1(\omega(\gamma^*, \overline{v}_1, r+\delta, \lambda)) \geq F_2(\omega(\gamma^*, \overline{v}_1, r+\delta, \lambda)) \geq F_2(\omega(\gamma^*, \overline{v}_2, r+\delta, \lambda))$ $= \Gamma_2(\gamma^*)$, where the first (weak) inequality follows from the assumption that $F_2(\Phi)$ first-order stochastically dominates $F_1(\Phi)$, and the second inequality follows from the facts that $\overline{v}_2 > \overline{v}_1$, $\omega(\gamma^*, \kappa, r+\delta, \lambda)$ is strictly decreasing in κ, and $F_2(\Phi)$ is increasing. Since $F_2(\Phi)$ strictly first-order stochastically dominates $F_1(\Phi)$, the inequality $\Gamma_1(\gamma^*) \geq \Gamma_2(\gamma^*)$, which holds for all γ^*, holds strictly for some γ^*. Therefore, $\Gamma_2(\gamma)$ strictly first-order stochastically dominates $\Gamma_1(\gamma)$. ■

Proof. of Proposition 6: Suppose that initially $\overline{v} = \kappa^*$. Since statement 5 of Corollary 4 states that $\frac{\partial^2 c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{(\partial\phi)^2} = \frac{1}{(\rho-\gamma)^3 c''(\gamma)} > 0$, $c'(\gamma(\phi, \kappa, r+\delta, \lambda))$ is a convex function of ϕ. Therefore, a mean-preserving spread of $F(\Phi)$ increases the value of $\int_G c'(\gamma(\Phi, \kappa^*, r+\delta, \lambda))\, dF(\Phi)$, which increases the value of $\alpha(\kappa^*) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa^*, r+\delta, \lambda))\, dF(\Phi)$, so that $\alpha(\kappa^*) > \kappa^*$. Corollary 5 implies that under the new distribution $\overline{v} > \kappa^*$. ■

Proof. of Corollary 6: Assume that $c'''(\gamma) \leq 0$ for $\gamma^m \leq \gamma \leq r+\delta+\lambda$. Differentiate $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\phi} = \frac{1}{(r+\delta+\lambda-\gamma)c''(\gamma)}$ (from statement 1 of Lemma 3) with respect to ϕ to obtain $\frac{\partial^2\gamma(\phi,\kappa,r+\delta,\lambda)}{(\partial\phi)^2} = \frac{c''(\gamma)-(r+\delta+\lambda-\gamma)c'''(\gamma)}{[(r+\delta+\lambda-\gamma)c''(\gamma)]^2} \geq 0$ for $\gamma^m \leq \gamma \leq r+\delta+\lambda$, since $c''(\gamma) > 0$ and $c'''(\gamma) \leq 0$. Let $F_1(\Phi)$ be the original distribution and let $F_2(\Phi)$ be the new distribution obtained from a mean-preserving spread on $F_1(\Phi)$ that maintains the support within G. Let $\overline{\gamma}_i = \int_G \gamma(\Phi, \overline{v}_i, r+\delta, \lambda)\, dF_i(\Phi)$ be the unconditional expected value of the investment-capital ratio under $F_i(\Phi)$, where $\overline{v}_i$ is the value of $\overline{v}$ under $F_i(\Phi)$. Therefore, $\overline{\gamma}_2 = \int_G \gamma(\Phi, \overline{v}_2, r+\delta, \lambda)\, dF_2(\Phi) \geq \int_G \gamma(\Phi, \overline{v}_2, r+\delta, \lambda)\, dF_1(\Phi)$ $> \int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda)\, dF_1(\Phi) = \overline{\gamma}_1$, where the first (weak) inequality follows from the facts that $\frac{\partial^2\gamma(\phi,\kappa,r+\delta,\lambda)}{(\partial\phi)^2} \geq 0$ and $F_2(\Phi)$ is a mean-preserving spread on $F_1(\Phi)$, and the second inequality follows from statement 2 in Lemma 3 that $\frac{\partial\gamma(\phi,\kappa,r+\delta,\lambda)}{\partial\kappa} > 0$ and the fact that $\overline{v}_2 > \overline{v}_1$, which follows from Proposition 6. ■

Proof. of Proposition 7: Since $\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r+\delta, \lambda))\, dF(\Phi)$, we have $\frac{\partial\alpha(\kappa)}{\partial\lambda} = \int_G \frac{\partial c'(\gamma(\Phi,\kappa,r+\delta,\lambda))}{\partial\lambda} dF(\Phi)$. Use statement 4 of Corollary 4 to obtain $\frac{\partial\alpha(\kappa)}{\partial\lambda}$ $= \int_G \frac{\kappa-(1+c'(\gamma(\Phi,\kappa,r+\delta,\lambda)))}{\rho-\gamma(\Phi,\kappa,r+\delta,\lambda)} dF(\Phi)$. Let $\kappa^* = \overline{v}$ and define ϕ^* as the unique value of Φ for which $1 + c'(\gamma(\phi^*, \kappa^*, r+\delta, \lambda)) = \kappa^*$, so $\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r+\delta, \lambda))) > 0$ if

$\Phi < \phi^*$ and $\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r+\delta, \lambda))) < 0$ if $\Phi > \phi^*$. Since $\gamma(\Phi, \kappa^*, r+\delta, \lambda)$ is increasing in Φ, $\frac{1}{\rho - \gamma(\Phi,\kappa^*,r+\delta,\lambda)} < \frac{1}{\rho - \gamma(\phi^*,\kappa^*,r+\delta,\lambda)}$ if $\Phi < \phi^*$, and $\frac{1}{\rho - \gamma(\Phi,\kappa^*,r+\delta,\lambda)} > \frac{1}{\rho - \gamma(\phi^*,\kappa^*,r+\delta,\lambda)}$ if $\Phi > \phi^*$. Therefore, $\frac{\kappa^* - (1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda)))}{\rho - \gamma(\Phi,\kappa^*,r+\delta,\lambda)} \leq \frac{\kappa^* - (1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda)))}{\rho - \gamma(\phi^*,\kappa^*,r+\delta,\lambda)}$ with strict inequality for $\Phi \neq \phi^*$. Therefore, if $F(\Phi)$ is non-degenerate, then $\frac{\partial \alpha(\kappa^*)}{\partial \lambda} < \int_G \frac{\kappa^* - (1+c'(\gamma(\Phi,\kappa^*,r+\delta,\lambda)))}{\rho - \gamma(\phi^*,\kappa^*,r+\delta,\lambda)} dF(\Phi) = \frac{1}{\rho - \gamma(\phi^*,\kappa^*,r+\delta,\lambda)} \int_G [\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r+\delta, \lambda)))] \, dF(\Phi)$ $= 0$. Since an increase in λ reduces $\alpha(\kappa^*)$ for κ^* equal to the original value of $\overline{v}$, Corollary 5 implies that an increase in λ also reduces $\overline{v}$. ■

Proof. of Corollary 7: If $c(\gamma)$ is quadratic and convex with a minimum value of 0 attained at $\gamma = \gamma_0$, it can be written as $c(\gamma) = \frac{1}{2}\theta(\gamma - \gamma_0)^2$ where $\theta > 0$. Therefore, $c'(\gamma) = \theta\gamma - \theta\gamma_0$. The first-order condition in equation (17) can be written as $1 + \theta\gamma - \theta\gamma_0 = v$, which implies that $1 + \theta\overline{\gamma} - \theta\gamma_0 = \overline{v}$. Therefore, $\frac{\partial \overline{\gamma}}{\partial \lambda} = \frac{1}{\theta}\frac{\partial \overline{v}}{\partial \lambda} < 0$, where the inequality follows from $\theta > 0$ and Proposition 7, which states that $\frac{\partial \overline{v}}{\partial \lambda} < 0$.
■

Proof. of Proposition 8: Proposition 5 states that $\overline{v}_2 > \overline{v}_1$ and statement 2 of Lemma 3 states that $\gamma(\phi, \kappa, r+\delta, \lambda)$ is increasing in κ. Therefore, $\gamma(\Phi, \overline{v}_2, r+\delta, \lambda)$ $> \gamma(\Phi, \overline{v}_1, r+\delta, \lambda)$, which proves statement 1. Statement 1 implies $\int_G \gamma(\Phi, \overline{v}_2, r+\delta, \lambda) \, dF_2(\Phi)$ $> \int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda) \, dF_2(\Phi)$ and statement 1 of Lemma 3 that $\gamma(\phi, \kappa, r+\delta, \lambda)$ is increasing in ϕ implies that $\int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda) \, dF_2(\Phi) > \int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda) \, dF_1(\Phi)$. Therefore, $\int_G \gamma(\Phi, \overline{v}_2, r+\delta, \lambda) \, dF_2(\Phi) > \int_G \gamma(\Phi, \overline{v}_1, r+\delta, \lambda) \, dF_1(\Phi)$, which proves statement 2.

Since $v(\Phi) = 1 + c'(\gamma(\Phi, \overline{v}, r+\delta, \lambda))$, $\frac{dv_i(\Phi)}{d\Phi} = \frac{\partial c'(\gamma(\phi,\overline{v}_i,r+\delta,\lambda))}{\partial \phi}$, $i = 1, 2$. Statement 1 of Corollary 4 is $\frac{\partial c'(\gamma(\phi,\kappa,r+\delta,\lambda))}{\partial \phi} = \frac{1}{\rho - \gamma}$, so $\frac{dv_i(\Phi)}{d\Phi} = \frac{1}{\rho - \gamma(\phi,\overline{v}_i,r+\delta,\lambda)}$. Therefore, statement 1 of Proposition 8 implies that $\frac{dv_2(\Phi)}{d\Phi} > \frac{dv_1(\Phi)}{d\Phi}$, which proves statement 3.

Statement 1 of Lemma 3 is that $\gamma(\phi, \kappa, r+\delta, \lambda)$ is increasing in ϕ, so that $\frac{1}{\rho - \gamma(\Phi,\overline{v}_2,r+\delta,\lambda)}$ is increasing in Φ. Therefore, $\int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) = \int_G \frac{1}{\rho - \gamma(\Phi,\overline{v}_2,r+\delta,\lambda)} dF_2(\Phi) >$ $\int_G \frac{1}{\rho - \gamma(\phi,\overline{v}_2,r+\delta,\lambda)} dF_1(\Phi)$. Since $\gamma(\phi, \kappa, r+\delta, \lambda)$ is increasing in κ, $\int_G \frac{1}{\rho - \gamma(\phi,\overline{v}_2,r+\delta,\lambda)} dF_1(\Phi)$ $> \int_G \frac{1}{\rho - \gamma(\phi,\overline{v}_1,r+\delta,\lambda)} dF_1(\Phi) = \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi)$. Putting together the inequalities in the two preceding sentences implies $\int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) > \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi)$, which proves statement 4. ■