NBER WORKING PAPER SERIES

RATIONAL HOUSING BUBBLE

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Working Paper 19354
http://www.nber.org/papers/w19354

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2013

I should thank helpful comments from seminar participants at Federal Reserve Bank of Minneapolis, University of Queensland, and the 24th NBER EASE conference. I also thank Li Chao for excellent research assistance. All errors are my own. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

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JEL No. D21,E13,E21,R21

ABSTRACT

This paper studies an economy inhabited by overlapping generations of homeowners and investors, with the only difference between the two being that homeowners derive utility from housing services whereas investors do not. Tight collateral constraint limits the borrowing capacity of homeowners and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. As long as the rental market friction is high enough, the investors will hold a positive number of vacant houses in equilibrium. A housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and without the expectation of receiving a dividend either in terms of utility or rent. The model can be applied to China, where the housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

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Housing assets play a dual role. These assets are not only an investment good but also a consumption good. With the first property alone, housing assets, such as fiat money, can have a positive value in the overlapping generation model developed by Samuelson (1958). People are willing to hold housing assets as a store of value. Housing assets have a rational bubble because their intrinsic value is zero. However, with the second property alone, housing assets, such as a Lucas tree, cannot have a rational bubble in Samuelson's model for the following reason: with a positive population growth rate, the model economy has two stationary equilibria with an interest rate that is either above or below the population growth rate.[1] In equilibrium, the growth rate of the bubble is equal to the interest rate, and the size of the bubble cannot grow more rapidly than the economy does. Therefore, only the lower interest rate is possible in equilibrium. Moreover, positive dividends (either in terms of rent or in terms of utility) rule out a negative equilibrium interest rate. Hence, the growth rate of the bubble must be positive and lower than the population growth rate, which implies that the size of the bubble as a proportion of the economy approaches zero in the stationary equilibrium.

My research question is the following: can housing assets have a rational bubble with both properties described above? This paper departs from the two-period consumption-loan model developed by Samuelson (1958) with only one twist: the economy consists of two types of households, homeowners and investors, with the only difference between the two being that homeowners derive utility from housing services whereas investors do not. With two types of households coexisting in the model, the equilibrium can have two possible outcomes, which depend on the degree of collateral constraint.

If the collateral constraint is loose, the model economy ultimately arrives at a bubbleless equilibrium, in which investors lend to workers at an interest rate that is higher than the population growth rate. Because the equilibrium interest rate is higher than the return rate to housing assets (which is equal to the population growth rate), investors have no incentives to hold the housing assets.

[1] If the population growth rate is zero, there is only one equilibrium with a positive interest rate.

Tight collateral constraint limits the borrowing capacity of homeowners and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. There is an excess supply of funds from the investors and asset shortage because homeowners are borrowing-constrained at the equilibrium interest rate. In the equilibrium, investors use the excess funds to purchase houses that are useless to them and expect that the future young investors will purchase the housing assets from them.

As long as the rental housing market friction is high enough, the rental market cannot absorb all of the housing assets bought by investors and the investors will hold some empty houses in the equilibrium. This behavior occurs because high rental market friction implies a higher rental-price-to-housing-price ratio, which has homeowners substitute rental housing for owner-occupied housing. However, investors are always indifferent between leaving houses empty or renting them out in a bubbly equilibrium. This suggests that the elasticity of rental houses supply is infinitely elastic and the amount of housing that are rented out in the equilibrium is completely determined by the demand of homeowners. Therefore, a housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent.

The main contribution of the paper is the extension of Samuelson (1958) to include two types of agents with preference heterogeneity and to show that a housing bubble is possible even if only part of the population derives dividends from housing assets. The presence of a bubble is robust to the production sector and to the rental housing market. The sufficient and necessary conditions for the existence of bubbly equilibrium are tight collateral constraint and high rental market friction.

The second contribution of the paper is the demonstration that a housing bubble can still exist in a production economy à la Diamond (1965). Tirole (1985) uses that framework to study the existence of a bubble in the presence of a Lucas tree that pays a fixed dividend. Tirole shows that a bubble absorbs the excess savings and helps achieve efficiency as long as the economy would become dynamically inefficient if there was no bubble. This paper extends Tirole (1985) to the study of housing assets, the rent value of which is endogenous and grows as rapidly as the economy does. In a similar vein, my paper shows that a housing bubble absorbs the excess savings from investors and removes dynamic efficiency although the sources of dynamic inefficiency are different. The dual role of housing assets remove the multiplicity of equilibria and the stationary equilibrium achieved is unique.

There is extensive literature on asset bubbles. My paper is related to rational bubbles under symmetric information. (See Brunnermeier (2009) for other forms of bubbles). In terms of causes of a bubble, recent studies on bubbles focus on financial friction and credit constraint. Kocherlakota (2009), Miao and Wang (2011), Farhi and Tirole (2012), and Martin and Ventura (2012) introduce credit constraint and investor heterogeneity. Bubbles serve as a collateral asset that helps alleviate the financial constraint of productive firms. Caballero and Krishnamurthy (2006) and Caballero, Farhi and Gourinchas (2008) argue that speculative bubbles alleviate the asset scarcity problem in an emerging market and explain global imbalance. Instead of focusing on the role of bubbles in alleviating the borrowing constraint of investment, this paper focuses on the roles of bubbles as a store of value for household consumption. In other words, previous studies hold that households purchase bubble assets to borrow (and invest). In my paper, it is argued that households purchase bubble assets to save(and consume).

The theoretical model of Arce and Lopez-Salido (2011) is the most similar to that presented in my paper. Arce and Lopez-Salido (2011) introduces housing assets in a three-period OLG model, in which multiple stationary equilibria exist depending on the financial constraint. My paper constructs a two-period overlapping-generation model with two types of agents and a production sector. It shows that multiple equilibria do not necessarily appear in the overlapping generation model. In some sense, the bubble that arises may show strong stability. Arce and Lopez-Salido (2011) does not consider the production sector and therefore is silent about investment and capital accumulation.

In terms of model predictions, the investment-related demand for a store of value can generate positive co-movement between investment and asset prices. The consumption-related demand for a store of value usually crowds out savings and reduces investment. However, my model is able to generate the right correlation based on exogenous shocks to the liquidity supply. In the empirical section, I apply the model to China, where the housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

In terms of welfare implications, all previous studies hold that bubble is Pareto improving and efficient if it does not burst. In my paper, it is argued that a bubble is good for investors because it is a good substitute for consumption loans. However, bubble reduces the welfare of homeowners. Moreover, it raises the borrowing rate and reduces the amount of housing services consumed.

The structure of this paper is organized as follows. Section 1 constructs an overlapping generation model with exogenous endowment growth to illustrate the existence of housing bubble. Section 2 discusses the model extension which includes the rental housing market and production sector. Section 3 considers a policy experiment of pension reform that may cause the merge of housing bubble. It uses empirical evidence from China to test the implications of theoretical model. Concluding remarks are provided in Section 5.

1 Benchmark Model

The benchmark model is a two-period overlapping generation model based on the consumption-loan model by Samuelson (1958).

1.1 Preference and Endowment

The economy is inhabited by two types of households: investors and homeowners. Both types live for two periods. Investors have the Cobb-Douglas utility function

$$u^I\left(c_t^t, c_{t+1}^t\right) = \ln c_t^t + \beta \ln c_{t+1}^t \tag{1}$$

where $\beta > 0$. Let c_t^t and c_{t+1}^t denote the non-durable consumption of households born at t at time t and $t+1$, respectively. The homeowners derive utility not only from non-durable consumption but also from housing services.

$$u^H\left(c_t^t, c_{t+1}^t, h_{t+1}^t\right) = \ln c_t^t + \beta\left(1-\zeta\right)\ln c_{t+1}^t + \beta\zeta \ln h_{t+1}^t \tag{2}$$

where $0 < \zeta < 1$. Because of the homothetic preference, both types of households spend $1/\left(1+\beta\right)$ of their total wealth in the first-period consumption in absence of borrowing constraint.

Both investors and homeowners receive y_t^t when young and 0 when old.[2] Denote the growth rate of output per capita by g. Hence,

$$\frac{y_{t+1}^{t+1}}{y_t^t} = 1+g \tag{3}$$

In each period, there are $N_t\omega$ young homeowners and $N_t\left(1-\omega\right)$ young investors, $0 < \omega < 1$. The population growth rate is

$$\frac{N_{t+1}}{N_t} = 1+n \tag{4}$$

[2] Section 2 includes the production sector and endogenous wage rate. Since I introduce pay-as-you-go social security in the model, the old will receive positive pension benefit. Hence, I can normalize the labor income of the elderly to zero without loss of generality.

1.2 Social Security

The government is running a pay-as-you-go (PAYG) social security plan. It collects τy_t^t from each young individual at period t and pays $\tau\ (1+n)\ y_t^t$ to each old generation, where $\tau\ \geq\ 0$. Hence, the gross return on PAYG system is given by $(1+g)\,(1+n)$. There is no government consumption. The government budget constraint is balanced each period.

1.3 Asset Market

The price of owner-occupied houses in terms of non-durable consumption goods is given by p_t. Housing assets are completely divisible. For simplicity, I assume away rental market in the benchmark model. It can be considered as the extreme case where rental market friction is too high. See the extension of the model in section 2 for the active rental market.

Both homeowners and investors are subject to the same borrowing constraint

$$a_{t+1}^t \geq -\,(1-\theta)\ p_t h_{t+1}^t \tag{5}$$

where housing is the only collateral in this economy. The downpayment ratio θ satisfies $0\ <\ \theta\ <\ 1$.

The model abstracts from housing construction. It assumes the total stock of housing in the economy is H_t, which is a continuous and differentiable function of p_t. Incoporating the housing construction by government or investors will not affect the qualitative conclusion of the paper.

1.4 Investors' Problem

The problem of investors who are born after time $t\ \geq\ 1$ can be written as

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta \ln c_{t+1}^t \tag{6}$$

subject to the following constraints

$$\begin{aligned}
c_t^t + a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau)\, y_t^t \\
c_{t+1}^t &= \tau\,(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t \\
a_{t+1}^t &\geq -(1-\theta)\, p_t h_{t+1}^t \\
c_t^t, c_{t+1}^t, h_{t+1}^t &\geq 0
\end{aligned}$$

The solution to the investors' problem is given in the appendix. In proposition 1, we have the following sufficient conditions for investors' optimal allocations.

Proposition 1 *Given* $\tau, g, n, \left\{R_t, p_t, y_t^t\right\}_{t=1}^{\infty}$, *the optimal decisions of investors are the followings:*

1. *If* $R_{t+1} = \frac{p_{t+1}}{p_t}$, *then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t \\
c_{t+1}^t &= \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t \\
a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau)\, y_t^t - c_t^t \\
a_{t+1}^t &> -(1-\theta)\, p_t h_{t+1}^t \\
h_{t+1}^t &\geq 0
\end{aligned}$$

2. *If* $R_{t+1} > \frac{p_{t+1}}{p_t}$, *then*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t \\
c_{t+1}^t &= \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t \\
a_{t+1}^t &= (1-\tau)\, y_t^t - c_t^t > 0 \\
h_{t+1}^t &= 0
\end{aligned}$$

3. If $R_{t+1} < \frac{p_{t+1}}{p_t}$, then

$$
\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\gamma_{\,t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t+1}}\right]y_t^t \\
a_{t+1}^t &= -\,(1-\theta)\;p_t h_{t+1}^t \\
p_t h_{t+1}^t &= \frac{\beta\gamma_{\,t+1}\,(1-\tau)-\tau\,(1+n)\,(1+g)}{\theta\gamma_{\,t+1}\,(1+\beta)}y_t^t \\
h_{t+1}^t &> 0
\end{aligned}
$$

where $\gamma_{\,t+1} \equiv \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$

1.5 Homeowners' Problem

The problem of homeowners who are born after time $t \geq 1$

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta\,(1-\zeta)\ln c_{t+1}^t + \beta\zeta\ln h_{t+1}^t \tag{7}$$

subject to the following constraints

$$
\begin{aligned}
c_t^t + a_{t+1}^t &= (1-\tau)\,y_t^t - p_t h_{t+1}^t \\
c_{t+1}^t &= \tau\,(1+n)\,y_{t+1}^{t+1} + R_{t+1}a_{t+1}^t + p_{t+1}h_{t+1}^t \\
a_{t+1}^t &\geq -\,(1-\theta)\;p_t h_{t+1}^t \\
c_t^t, c_{t+1}^t, h_{t+1}^t &\geq 0
\end{aligned}
$$

Workers' problem is solved in the appendix. The optimal decision rules are given by the following proposition 2.

Proposition 2 *Given $\tau, g, n, \left\{R_t,\, p_t,\, y_t^t\right\}_{t=1}^{\infty}$, the optimal decisions of homeowners are the followings*

1. *If homeowners are not borrowing constrained, the optimal allocations are*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\,(1-\zeta)\,R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{1}{1-\frac{p_{t+1}}{p_t R_{t+1}}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y_t^t \\
a_{t+1}^t &= (1-\tau)\,y_t^t - p_t h_{t+1}^t - c_t^t
\end{aligned}$$

2. *If homeowners are borrowing constrained, the optimal allocations are*

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta\,(1-\zeta)\,\gamma_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{t+1}}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{\Psi_t+\Phi_t}{2\theta\varphi\,(1+\beta)} \\
a_{t+1}^t &= -\,(1-\theta)\,p_t h_{t+1}^t
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{t+1} &\equiv \frac{\lambda_1}{\lambda_2} = \frac{b+\frac{\Psi_t+\Phi_t}{2\theta(1+\beta)}}{\beta\,(1-\zeta)\left(a-\frac{\Psi_t+\Phi_t}{2\varphi(1+\beta)}\right)} \\
\Psi_t &\equiv a\varphi\beta - b\theta\,(1+\beta\zeta) \\
\Phi_t &\equiv \sqrt{\Psi_t^2+4ab\theta\beta\zeta\varphi\,(1+\beta)} \\
\varphi &\equiv \frac{p_{t+1}}{p_t}-(1-\theta)\,R_{t+1} \\
a &\equiv (1-\tau)\,y_t^t \\
b &\equiv \tau\,(1+n)\,(1+g)\,y_t^t
\end{aligned}$$

1.6 Competitive Equilibrium

Definition 3 *Given the financial asset* $a_1^{1,i}$ *and housing stocks* $h_1^{1,i}$ *for the initial old, the distribution of households* $\left\{\mu_t^i\right\}_{t=1}^{\infty}$ *with total mass equal to the population size, the initial interest rate* R_1*, pension system* τ*, housing stock* $\{H_t\}_{t=1}^{\infty}$*, the competitive equilibrium consists of the endowment sequences* $\left\{y_t^{t,i}\right\}_{t=1}^{\infty}$*, prices* $\{p_t, R_{t+1}\}_{t=1}^{\infty}$*, allocations* $\left\{c_t^{t,i}, c_{t+1}^{t,i}, h_{t+1}^{t,i}\right\}_{t=1}^{\infty}$*, and the initial consumption* $c_1^{0,i}$*,* $i = I, H$ *such that*

1. *The allocations solve the problem of investors (6) and homeowners (7)*

2. *The housing market, financial market, and goods market clear*

$$\begin{aligned}\int h_{t+1}^{t,i} d\mu_t^i &= H_{t+1} \\ \int a_{t+1}^{t,i} d\mu_t^i &= 0 \\ \int c_t^{t,i} d\mu_t^i + \int c_t^{t-1,i} d\mu_{t-1}^i + p_t \int h_{t+1}^{t,i} d\mu_t^i &= \int y_t^{t,i} d\mu_t^i + p_t \int h_t^{t-1,i} d\mu_{t-1}^i\end{aligned}$$

In order to characterize the existence and uniqueness of the stationary equilibrium, we first study the properties of optimal decision rules. Lemma 4 describes the shapes of the supply curve and demand curve in the loan market.

Lemma 4 *The loan demand (loan supply) of homeowners (investors) is always a strictly decreasing (increasing) function of interest rate.*

Proof. See appendix. ■

We can detrend the allocations and prices using their growth rate along the balanced growth path. We can define $\tilde{y}_t^t \equiv \frac{y_t^t}{(1+g)^t}$, $\tilde{c}_t^t \equiv \frac{c_t^t}{(1+g)^t}$, $\tilde{c}_t^{t-1} \equiv \frac{c_t^{t-1}}{(1+n)(1+g)^t}$, $\tilde{a}_{t+1}^t \equiv \frac{a_{t+1}^t}{(1+g)^t}$, $\tilde{p}_t \equiv \frac{p_t}{(1+n)^t(1+g)^t}$, $\tilde{R}_{t+1} \equiv \frac{R_{t+1}}{(1+n)(1+g)}$, $\tilde{h}_{t+1}^t \equiv h_{t+1}^t (1+n)^t$, $\tilde{H}_{t+1} \equiv H_{t+1}$, $\tilde{\varphi} \equiv \frac{\varphi}{(1+n)(1+g)}$. Without loss of generality, I assume $g = n = 0$ from now on. Keep in mind that all the variables are detrended.

The following lemma 5 actually states that the dynamic inefficiency, i.e., $R^* < n+g$, can not happen in the equilibrium. The intuition is the following. As long as there are positive measure of homeowners, the model economy is similar to the Samuelson model with a Lucas tree, which rules out negative net interest rate. However, it can not rule out zero net interest rate because of the collateral constraint and the presence of investors.

Lemma 5 *If* $0 < \omega, \theta < 1$, *there is no stationary equilibrium with gross interest rate* $R^* < 1$

Proof. See Appendix. ■

The proposition 6 characterizes the uniqueness of stationary equilibrium. Intuitively, tighter borrowing constraint (higher θ) reduces the loan demand from homeowners and drives the equilibrium interest rate down. Homeowners are more likely to be borrowing constrained under low interest rate.

Proposition 6 *There exists a unique stationary equilibrium.*

1. *If* $\theta \leq \theta_L$, *there are unconstrained homeowners and unconstrained investors holding zero housing assets*

2. *If* $\theta_L < \theta \leq \theta_H$, *there are borrowing-constrained homeowners and unconstrained investors holding zero housing assets*

3. *If* $\theta > \theta_H$, *then there are constrained homeowners and unconstrained investors holding housing assets*

   where

$$\theta_L = \omega$$

   and θ_H *is determined by*

$$(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y - \omega\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi+\Phi}{2\theta_H(\beta+1)} = 0$$

   Ψ *and* Φ *are defined in proposition 2.*

Proof. See Appendix. ■

Figure 1 shows the stationary equilibrium in three cases. The dotted line is the loan supply of investors. The minimum equilibrium gross interest rate is 1. The solid line is the loan demand from homeowners. As proved by Lemma 1, it is a decreasing function of interest rate. It is kinked because it consists of two parts. The flatter part is the loan demand of unconstrained homeowners. The steeper part is the loan demand of borrowing-constrained homeowners. The intersection point pins down the equilibrium interest rate.

Proposition 7 *The third case of stationary equilibrium, i.e., constrained homeowners and unconstrained investors with empty housing, is a bubbly equilibrium for investors, but not for homeowners.*

Proof. See Appendix. ■

The proposition 7 describes the special feature of the equilibrium with bubble, i.e., it is a bubble from investor's point of view only. It may seem strange. However, in order to understand the intuition, let me quote a paragraph from Tirole (1985). He described two views of money: the fundamentalist view and the bubbly view of money. The fundamentalist view argues that "money is held to finance transactions (or to pay taxes or to satisfy a reserve requirement). To this purpose, money must be a store of value. However, it is not held for speculative purposes as there is no bubble on money." The bubbly view argues that"money is a pure store value à la Samuelson (1958). It does not serve any transaction purpose at least in the long run. This view implies that price of money (bubble) grows at the real rate of interest, and that money is held entirely for speculation"."The two representations are in the long run inconsistent."

This paper combines the two views together in one model through different preferences on housing assets. Homeowners derive utility from housing assets. This is similar to the fundamentalist view. Investors treat housing assets as investment tools and a store of value. This is same as the bubbly view. Therefore, it shows that the two representations can be consistent when we study two types of agents and a special type of asset: housing assets.

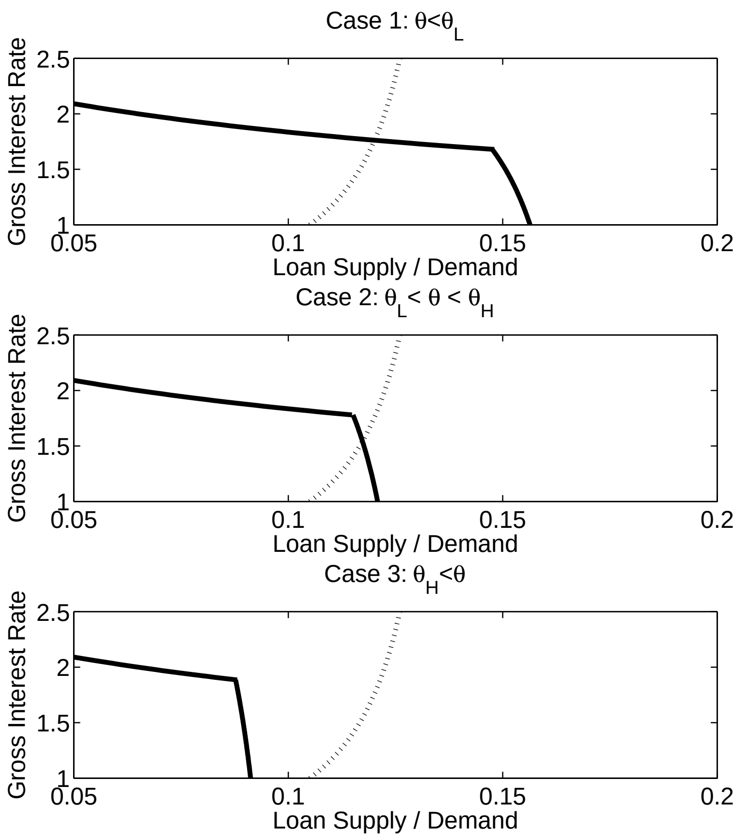

Figure 1: Three Cases of Stationary Equilibrium

The fraction of homeowners $\omega = 0.65$, payroll tax $\tau = 0.2$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$.

2 Model Extension

This section extends the benchmark model to include the rental market and production sector. It shows that the qualitative results in the previous section still hold.

2.1 Model with Rental Market

In this section, I construct a two-period model with rental market. The investors' problem can be written as

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^t, h_{t+1}^R, a_{t+1}^t} \ln c_t^t + \beta \ln c_{t+1}^t \tag{8}$$

subject to the following constraints

$$\begin{aligned}
c_t^t + a_{t+1}^t + p_t h_{t+1}^t &= (1-\tau)\, y_t^t + p_t^r h_{t+1}^R \\
c_{t+1}^t &= \tau\,(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r\, p_{t+1} h_{t+1}^R \\
h_{t+1}^t &\geq h_{t+1}^R \\
a_{t+1}^t &\geq -(1-\theta)\, p_t h_{t+1}^t \\
c_t^t, c_{t+1}^t, h_{t+1}^t, h_{t+1}^R &\geq 0
\end{aligned}$$

where h_{t+1}^R denotes the amount of houses that are rent out. $\delta_r > 0$ denotes the depreciation rate of rental housing. I assume frictional rental market in this paper, in the sense that owner-occupied housing will have a smaller depreciation rate than rental housing. This can be interpreted as the moral hazard problem of tenants. I normalize the depreciation rate of owner-occupied housing to zero.

Because of the assumption that investors can not derive utility flow directly from rental housing, the investors will not rent houses in the model. Since all the homeowners are homogenous, they will not provide positive rental housing in the equilibrium. Hence, the homeowners are the demand side of rental market. The homeowners' optimization problem becomes

$$\max_{c_t^t, c_{t+1}^t, h_{t+1}^r, h_{t+1}^t, a_{t+1}^t} \ln c_t^t + \beta (1-\zeta) \ln c_{t+1}^t + \beta\zeta \ln \left(h_{t+1}^r + h_{t+1}^t\right) \tag{9}$$

subject to the following constraints

$$\begin{aligned} c_t^t + a_{t+1}^t &= (1-\tau)\, y_t^t - p_t h_{t+1}^t - p_t^r h_{t+1}^r \\ c_{t+1}^t &= \tau (1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t \\ a_{t+1}^t &\geq -(1-\theta)\, p_t h_{t+1}^t \\ c_t^t, c_{t+1}^t, h_{t+1}^t, h_{t+1}^r &\geq 0 \end{aligned}$$

where h_{t+1}^r is the amount of housing rent by homeowners. We can similarly define the competitive equilibrium.

Definition 8 *Given the financial assets $a_1^{1,i}$ and housing stocks $h_1^{1,i}$ for the initial old, the distribution of households $\left\{\mu_t^i\right\}_{t=1}^{\infty}$ with total mass equals to the population size, the initial interest rate R_1, pension system τ, housing stocks $\{H_t\}_{t=1}^{\infty}$, the competitive equilibrium is the sequence of endowment $\left\{y_t^{t,i}\right\}_{t=1}^{\infty}$, prices $\left\{p_t, R_{t+1}, p_t^r\right\}_{t=1}^{\infty}$, allocations $\left\{c_t^{t,i}, c_{t+1}^{t,i}, h_{t+1}^{t,i}, h_{t+1}^{R,i}, h_{t+1}^{r,i}\right\}_{t=1}^{\infty}$, and the initial consumption $c_1^{0,i}$,$i = I, H$ such that*

1. *The allocations solve the problem of investors (8) and homeowners (9)*

2. *The housing market, financial market, rental market, and goods market*

clear

$$\begin{aligned}
\int h_{t+1}^{t,i} d\mu_t^i &= H_{t+1} \\
\int a_{t+1}^{t,i} d\mu_t^i &= 0 \\
\int h_{t+1}^{R,i} d\mu_t^i &= \int h_{t+1}^{r,i} d\mu_t^i \\
\int c_t^{t,i} d\mu_t^i + \int c_t^{t-1,i} d\mu_{t-1}^i + p_t \int h_{t+1}^{t,i} d\mu_t^i &= \int y_t^{t,i} d\mu_t^i + p_t \int h_t^{t-1,i} d\mu_{t-1}^i
\end{aligned}$$

The policy functions for the problem of investors (8) and homeowners (9) are solved in the Appendix. The following lemma 9 can simplify the our analysis a lot.

Lemma 9 *Unconstrained homeowners will not rent houses in the stationary equilibrium.*

Proof. See Appendix. ■

We are interested in wether the rental market can remove the bubbly stationary equilibrium. To simplify the analysis, I assume away pension system, i.e., let $\tau = 0$. The the following proposition 10 states that a housing bubble exists in the equilibrium after the pension reform if the collateral constraint θ and the rental market friction δ_r are large enough.

Proposition 10 *If the collateral constraint $\theta > \omega$ and the rental market friction δ_r is large enough, there exists a bubble equilibrium after the pension reform. More precisely,*

1. *If $\delta_r \geq \theta\zeta$, then homeowners will not rent houses and investors will hold empty houses. There exists a housing bubble for investors.*

2. *If $\theta\zeta > \delta_r \geq \omega\zeta$, then homeowners will rent some houses and investors will still hold some empty houses. There exists a housing bubble for investors.*

3. *If $\delta_r < \omega\zeta$, investors will rent all the houses to homeowners and there is no housing bubble.*

Proof. See Appendix. ■

2.2 Model with Production Sector

The benchmark model can be extended to include the production sector à la Diamond (1965). Suppose there exists a production sector with production function written as

$$Y_t = F\left(K_t, A_t L_t\right) \tag{10}$$

where the growth rate of labor-augmented technology is given by $A_{t+1}/A_t = 1+g$. Suppose $F\left(K_t, A_t L_t\right) = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$, the profit maximization of the firm implies that

$$\begin{aligned} R_t &= 1+\alpha K_t^{\alpha-1}\left(A_t L_t\right)^{1-\alpha}-\delta \\ w_t &= \left(1-\alpha\right) A_t K_t^{\alpha}\left(A_t L_t\right)^{-\alpha} \end{aligned}$$

Now the investors' problem becomes

$$\max \ln c_t^t + \beta \ln c_{t+1}^t \tag{11}$$

subject to the following constraints

$$\begin{aligned} c_t^t + a_{t+1}^t + p_t h_{t+1}^t &= \left(1-\tau\right) w_t \\ c_{t+1}^t &= \tau\left(1+n\right) w_{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t \\ c_t^t, c_{t+1}^t, h_{t+1}^t &\geq 0 \end{aligned}$$

The households' problem becomes

$$\max \ln c_t + \beta\left(1-\zeta\right) \ln c_{t+1}^t + \beta\zeta \ln h_{t+1}^t \tag{12}$$

subject to the following constraints

$$\begin{aligned} c_t^t + a_{t+1}^t &= (1-\tau)\, w_t - p_t h_{t+1}^t \\ c_{t+1}^t &= \tau\,(1+n)\, w_{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t \\ a_{t+1}^t &\geq -(1-\theta)\, p_t h_{t+1}^t \\ c_t^t, c_{t+1}^t, h_{t+1}^t &\geq 0 \end{aligned}$$

We can similarly define the competitive equilibrium.

Definition 11 *Given the financial assets $a_1^{1,i}$ and housing stocks $h_1^{1,i}$ for the initial old, the distribution of households $\left\{\mu_t^i\right\}_{t=1}^{\infty}$ with total mass equals to the population size, the initial interest rate R_1, pension system τ, housing stocks $\{H_t\}_{t=1}^{\infty}$, the competitive equilibrium consists of prices $\{p_t, R_{t+1}\}_{t=1}^{\infty}$, allocations $\left\{c_t^{t,i}, c_{t+1}^{t,i}, h_{t+1}^{t,i}, K_{t+1}\right\}_{t=1}^{\infty}$, and the initial consumption $c_1^{0,i}$,$i = I, H$ such that*

1. *The allocations solve the problem of investors (11) and homeowners (12)*

2. *Firm rent capital and hire labor from households to maximize profit.*

3. *The housing market, financial market, labor market, and goods market clear*

$$\begin{aligned} \int h_{t+1}^{t,i} d\mu_t^i &= H_{t+1} \\ \int a_{t+1}^{t,i} d\mu_t^i &= K_{t+1} \\ N_t &= L_t \\ \int c_t^{t,i} d\mu_t^i + \int c_t^{t-1,i} d\mu_{t-1}^i + p_t \int h_{t+1}^{t,i} d\mu_t^i + K_{t+1} &= Y_t + p_t \int h_t^{t-1,i} d\mu_{t-1}^i \end{aligned}$$

We can normalize all economic variable by their growth rate along the balance growth path. Denote $\tilde{y}_t^t \equiv \frac{y_t^t}{(1+g)^t}$, $\tilde{c}_t^t \equiv \frac{c_t^t}{(1+g)^t}$, $\tilde{c}_t^{t-1} \equiv \frac{c_t^{t-1}}{(1+n)(1+g)^t}$, $\tilde{a}_{t+1}^t \equiv \frac{a_{t+1}^t}{(1+g)^t}$, $\tilde{k}_{t+1} \equiv \frac{k_{t+1}}{(1+g)^t(1+n)^t}$, $\tilde{p}_t \equiv \frac{p_t}{(1+n)^t(1+g)^t}$, $\tilde{R}_{t+1} \equiv \frac{R_{t+1}}{(1+n)(1+g)}$, $\tilde{h}_{t+1}^t \equiv h_{t+1}^t\,(1+n)^t$, $\tilde{H}_{t+1} \equiv H_{t+1}$, $\tilde{\varphi} \equiv \frac{\varphi}{(1+n)(1+g)}$

We are interested in the stationary equilibrium with production sector. To simplify the analysis, I assume away the pension system, i.e., let $\tau = 0$. The proposition 12 proves that a housing bubble can exist even under a production sector. It is essentially a dynamic inefficiency condition for the economy with housing assets. Housing bubble solves the dynamic inefficiency problem by absorbing the excess supply of loan in the market.

Proposition 12 *If $\tau = 0$ and the following condition holds, then there exists a housing bubble in the stationary equilibrium.*

$$\theta > \omega \frac{1}{1 - \alpha \frac{1+\beta}{\beta} \frac{n+g+1}{n+g+\delta}}$$

Proof. See Appendix. ■

3 Policy Experiment and Data

3.1 Pension Reform

We now consider a policy experiment. Suppose the government removes the PAYG system, i.e., $\tau = 0$. The removal of PAYG will always increase the supply of loan in the economy. It will reduce the borrowing of unconstrained homeowners. However, for the constrained homeowners, it will increase their loan demand. This is because the borrowing limit is increased by purchasing more housing assets using extra money from tax reduction.

Figure 2 is an illustration of pension reform in the endowment economy. The dotted line denotes the demand and supply of loans before the pension reform. The solid line denotes the loan demand and supply after the pension reform. Whether the new equilibrium interest rate will be pushed down towards zero depends on the tightness of collateral constraint. If the borrowing constraint is tight enough, the increase in the loan supply will surpass the increasing loan demand from constrained homeowners. Therefore, a housing bubble is possible. The proposition

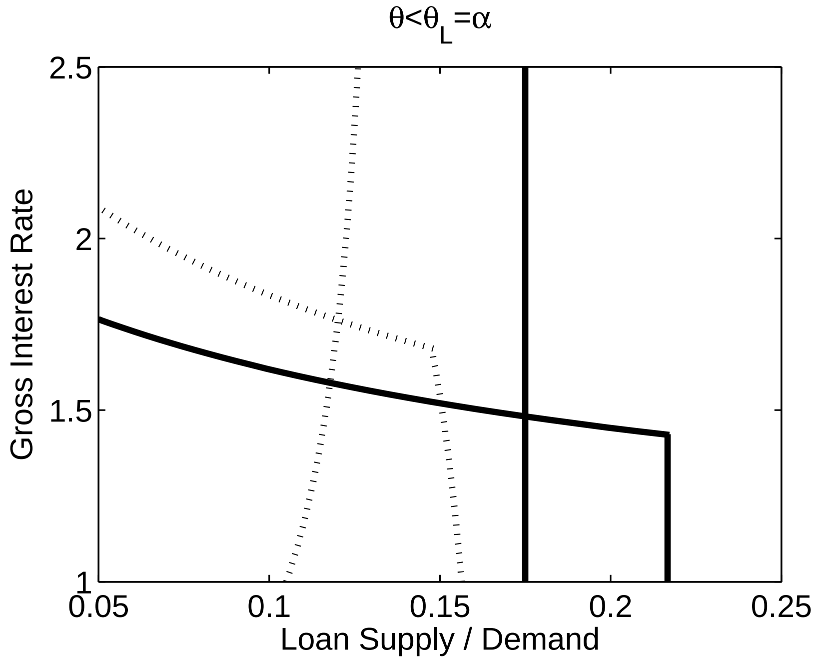

Figure 2: An Illustration of Pension Reform

The fraction of homeowners $\omega = 0.65$, payroll tax $\tau = 0.2$, downpayment ratio $\theta = 0.60$, income per capita $y = 1$, discount factor $\beta = 1$, and $\zeta = 0.5$. The dotted line denotes the loan demand and supply before the pension reform. The solid line denotes the loan demand and supply after the pension reform.

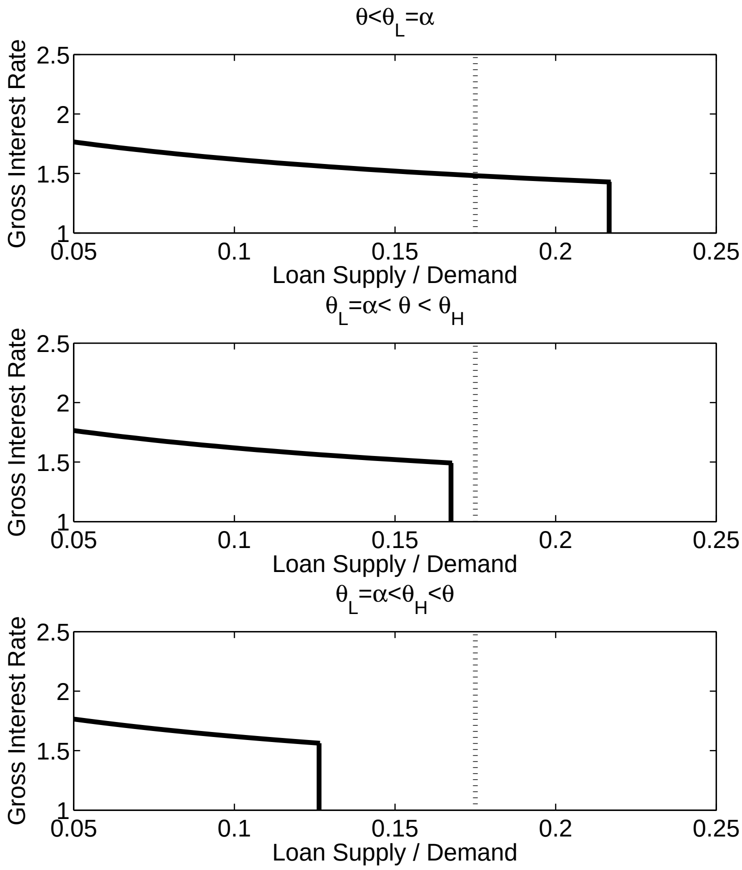

Figure 3: Stationary Equilibrium After the Pension Reform in Three Cases

The fraction of homeowners $\omega = 0.65$, payroll tax $\tau = 0$, downpayment ratio $\theta = 0.60, 0.66, 0.72$, income per capita $y = 1$,discount factor $\beta = 1$, and $\zeta = 0.5$.

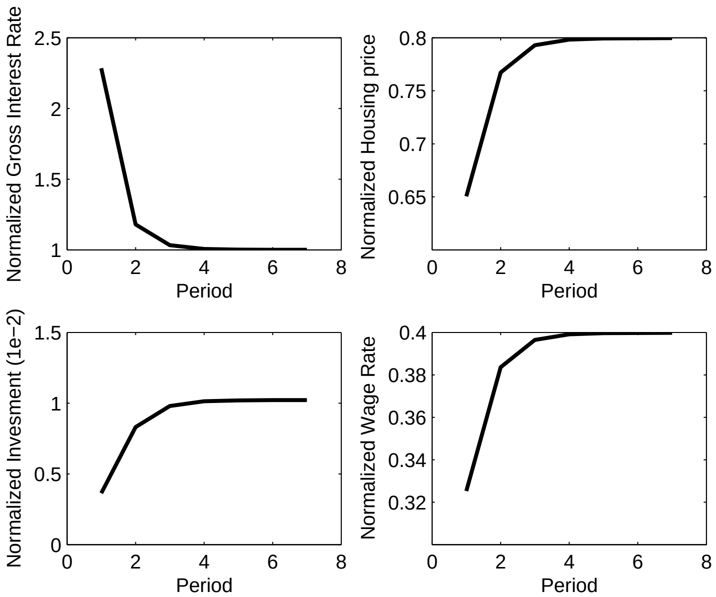

Figure 4: Transitional Dynamics after the Pension Reform

Model period equals 30 years. The fraction of homeowners $\omega = 0.33$, payroll tax decreases to zero from $\tau = 0.40$ after the reform, the downpayment ratio $\theta = 0.70$, discount factor $\beta = 1$, and $\zeta = 0.5$, the annual population growth rate is 2 percent and the productivity growth rate is 5 percent.

13 gives a sufficient condition for the housing bubble to exist in the endowment economy.

Proposition 13 *In the endowment economy, suppose the government removes the PAYG system. Bubble will arise if and only if* $\theta > \omega$*. A sufficient condition for housing wealth/GDP ratio to be higher than the pre-reform era is* $\tau > \frac{\theta-\omega}{1-\omega}$.

Proof. See Appendix. ■

Figure 3 exhibits the policy experiments in all three cases, i.e., $\theta < \theta_L$, $\theta_L < \theta < \theta_H$, and $\theta > \theta_H$. According to proposition 13, only pension reform in case 2 and case 3 (described in proposition 6) can trigger housing bubble.

Proposition 12 already states a sufficient condition for the housing bubble to exist in a production economy after the removal of pension system. The proposition 14 describes the transitional dynamics after the pension reform.

Proposition 14 *In the production economy, suppose the government removes the PAYG system and there exists a housing bubble in the new stationary equilibrium. Both housing price and interest rate converge monotonically to the unique new steady state.*

Proof. See Appendix. ■

Figure 4 shows the transition path after the pension reform in a production economy. The normalized interest rate is defined as the gross interest rate divided by the gross population growth rate plus the productivity growth rate. The normalized housing price growth rate is the housing price sequence divided by the current population and productivity level. The investment is normalized in the similar way. The normalized wage rate is defined as the wage rate divided by the current productivity. The proof of the proposition shows that the housing price growth rate is equal to the gross interest rate during the transition. Therefore, investors will hold housing assets right after the pension reform.

3.2 Data

In this section, I use China's experience as a test for the theoretical model. Although the US has already experienced a burst in housing bubble in 2008, housing prices in China have been increasing strongly over the past decade. The connected solid line in Figure 5 shows that the real land-selling price for the whole country increases at an annual rate 15.7 percent from 2000 to 2009. Unfortunately, there is no constant-quality official housing price index for China. I also draw the official average commodity building selling price for 35 large cities in China. It exhibits a slower annual growth rate, 7 percent, from year 2000 to 2009. Wu, Gyourko and Deng (2012) constructs constant quality price index for newly-built private housing in 35 major Chinese cities. According to their estimates, the annual price growth is nearly 10 percent from year 2000 to 2009.

The unprecedented housing boom in China encourages large increase real estate investment and the boom in the home ownerships. As shown by Figure 6, the share of real estate investment in total fixed investment increases from 13 percent at 1999 to 20 percent at 2010. The urban households home ownerships rate estimated from Urban Households Survey shows that China's home ownership rate is nearly 90 percent in 2010, among the highest in the world.[3] These two facts imply that a lot of households own more than one apartment.

Popular wisdoms claim that there is a housing bubble in China. According to the theoretical model, one evidence for the housing bubble is the high vacancy

[3] The urban home ownership rate increases from less than 30 percent to 70 percent during 1994-1999, a period when the housing reform takes place. Before the housing reform, it is the state-owned enterprises (SOE) that are responsible for providing employee housing to workers, with a little or no charge for rents. The government liberalizes the housing market in 1994 by selling the public housing to the current employee in state-owned enterprises at heavily subsidized price. Newly employed workers in SOE and workers in the private sectors have to purchase houses that are provided by private real estate developers. The transition into the new housing system ends around 1999, after which no SOE are allowed to provide employee housing to their workers. At the end of year 2010, the home ownership rate of urban households in China is 89.3 percent, which is among the highest in the world. 40.1 percent of them own privatized houses which previously are owned by the government or state-owned enterprises. 38 percent of households have bought houses that are provided at a market price.

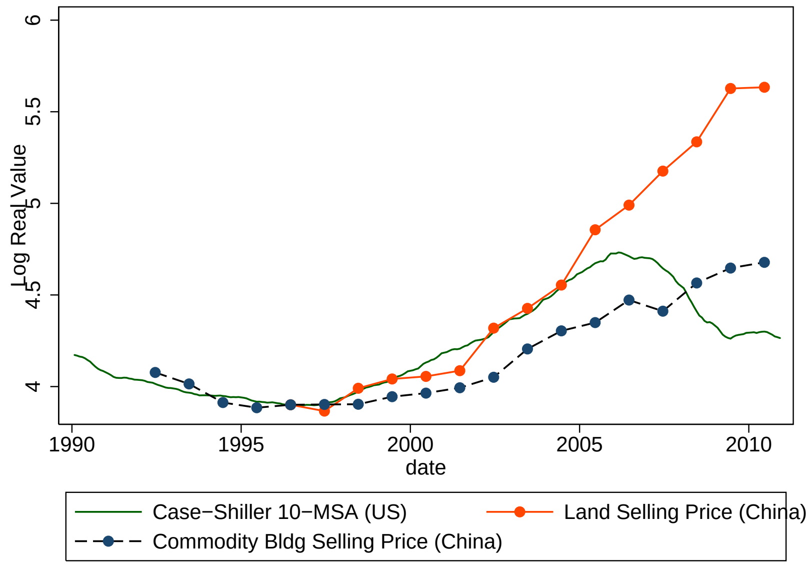

Figure 5: Housing Price and Land Price: China and the US

The US Housing price index is from S&P/Case-Shiller 10-MSA Index. The land selling price is computed by author using data from China Satistics Year Book. The land price is defined as total value of land purchased divided by total land space purchased. The commodity building sell prices is based on the 35-city average selling price series from National Bureau of Statistics. All series are in log real value deflated by CPI (Urban CPI for Chinese data) and normalized to the same level at year 1996.

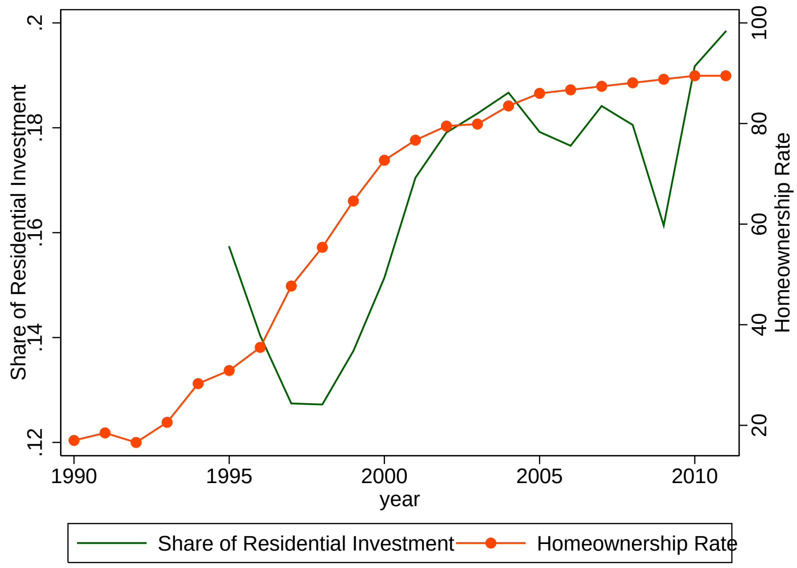

Figure 6: Urban Residential Investment and Homeownership Rate

The share of urban residential investment is defined as the real estate development (including land purchase) divided by the total investment in fixed assets in the whole country. Homeownership rate is from China urban households survey.

rate in China's housing market. A vacant house/apartment is a unit that has been sold but is not occupied by anybody. The vacancy rate is defined as all vacant units/all housing units (occupied + vacant). In the US, the gross vacancy rate rises from 12.7 to 14.5 during 2005-2010. In China, according to the China Family Panel Studies 2011, 22 percent of urban households own more than one apartment. Among them, only 25 percent households rent their apartments out. The vacancy rate in year 2010 is 11 percent according to author's estimate.

One of the reason that households hold empty apartment is lack of investment instruments and the need for a store of value. The conflict between the two is strengthened by the insufficient social security, which forces the middle-aged to buy empty houses as a store of value to finance their later-life consumption. Figure 7 plots the pension replacement rate and contribution rate in China. The pension reform starts in China from 1999, which shifts the traditional pay-as-you-go (PAYG) system to a mixture of PAYG system and fully-funded system. From then on, the replacement rate of pension system decreases from around 75 percent to only 45 percent in 2009. During the same period, the saving rate in China increases by 15 percent, which suggests that Chinese households increase savings partly to compensate the huge decline in the pension payment.

What if those households just invest their pension in terms of stocks and other investment tools? Because the poor development in the financial market, the average return on the stock market over the past twenty years in very low (the average real return on shanghai stock market index is only 2 percent from year 2000 to 2009) and median households can only access to risk-free bond which delivers almost zero interest actually. Therefore, the missing social security is accompanied by the dynamic inefficiency in China. Figure 8 shows that the real interest rate is China is much lower than the real GNP growth rate, which makes risk-free bond unattractive relative to housing assets.

Although there is studies documenting that the capital return in China is very high, however, those projects are not accessible to normal households in China. In fact, Chinese government itself has accumulated great amount of foreign as-

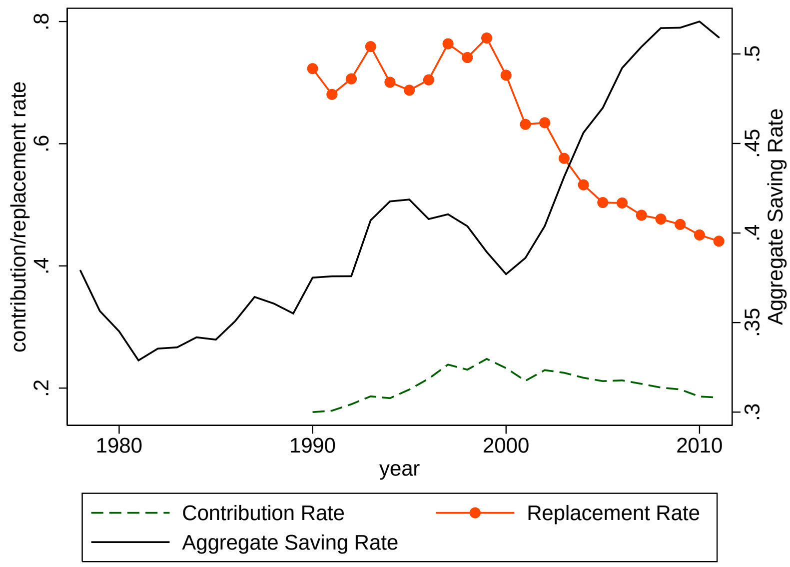

Figure 7: Social Security Replacement Rate and Contribution Rate

Data are from China Statistics Year Books 1990-2010. Replacement Rate is defined as the total pension benefit payment per urban retiree covered in the pension system divided by the average urban wage rate. The contribution rate is the total contribution per urban worker covered in the pension system divided by the average urban wage rate.

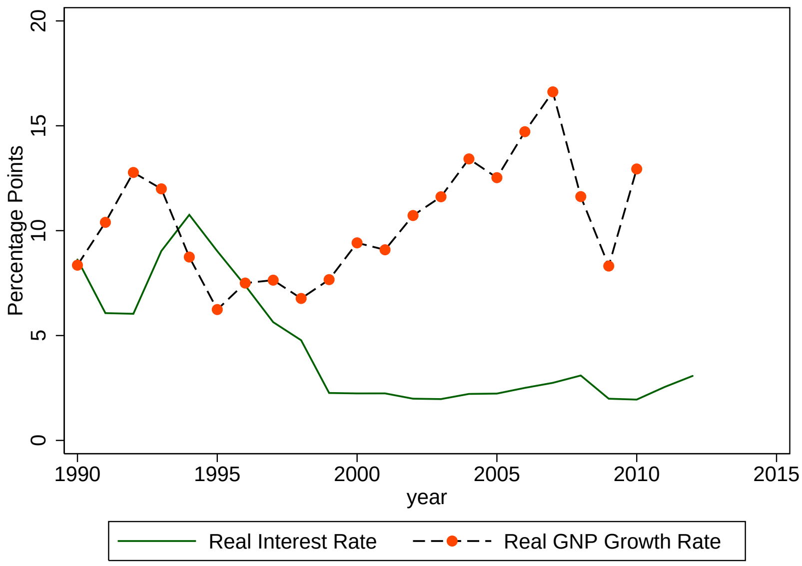

Figure 8: Dynamic Inefficiency

The real interest rate is the benchmark interest rate set by the central bank for one-year fixed-term deposit deflated by CPI. The Real GNP annual growth rate is also deflated by CPI.

sets and implicitly issue collateralized bonds to Chinese citizens. The low return of government bonds reflects the huge demand for assets or investment tools in China. There are many reasons for causing the dynamic inefficiency problem, e.g., the poor financial development, the absence of social security system, etc. If the capital account were fully open, Chinese households would have purchased huge amount of assets abroad directly. This dynamic inefficiency creates excess supply of liquidity which allows for speculative bubble.

3.3 Test of the Model

The pension system in China mainly operates at city or province levels. Each city or province has its own pension fund account and replacement rate. Therefore, we can exploit the regional variations in pension system to identify its effect on regional housing prices. According to the model prediction, we would observe larger housing price appreciation for the province where the pension contribution rate declines most. I first compute the theoretical contribute rate $\tau^{i,t}$, which is the tax rate that would balance the budget constraint of pay-as-you-go pension system for province i at year t.

$$\tau^{i,t} = \frac{\text{expenditure}^{i,t}}{\text{worker}^{i,t} \times \text{wage}^{i,t}}, i = 1, \ldots, 35, t = 2001, \ldots, 2011$$

where expenditure^{i,t} the total expenditure of pension fund at province i and year t. worker^{i,t} is the number of workers covered by the pension fund and wage^{i,t} is the average wage rate of workers.

Figure 9 plots the changes in housing prices across 35 cities against the changes in the theoretical contribution rates over year 2001-2011. There is a clear negative correlation, which confirms the prediction of the theory. The simple OLS univariate regression has a coefficient -2.84, which is significant at 1 percent confidence level. The R squared is .24.

In order to estimate the effect of theoretical contribution rate on housing price, I run the following regression.

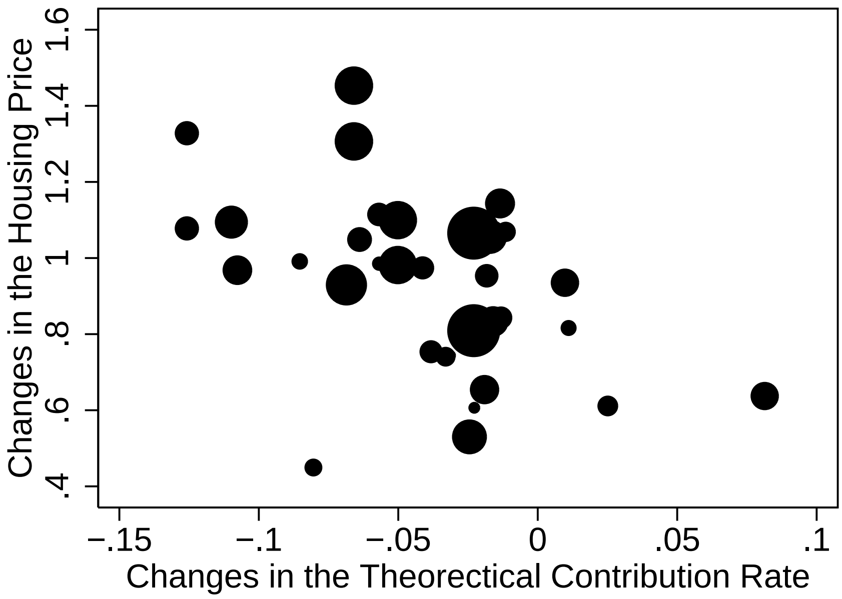

Figure 9: Cross-section 10-year Changes in the Housing prices and Changes in the Theoretical Contribution Rate 2001-2011

The commodity building sell prices is based on the 35-city average selling price series from National Bureau of Statistics. All series are in log real value deflated by CPI (Urban CPI for Chinese data). Most 35 cities are the capital cities. Since the pension system is mainly operating at provincial level, I merge the capital city with their province and compute theoretical contribution rate using provincial data. The size of dot represents the size of the population in that province.

Table 1: Regression Result on the Effect of Theoretical Contribution Rate on Housing Prices

Depend Var. log(housing price)	Pooled regression	Random Effect	Fixed Effect
log(gdp)	.0335	.157 ***	.0335
	(0.29)	(3.61)	(0.35)
Theoretical Contribution Rate	-1.91***	-1.88***	-1.49***
	(-3.41)	(-5.39)	(-4.04)
Wealth Effect	1.42***	1.58***	1.42***
	(3.11)	(4.12)	(3.69)
Year Effect	Yes	Yes	Yes
Province Dummies	Yes	Yes	Yes
R-Squared	0.92	.68	.60
No. of Obs.	385	385	385

$$ln\left(P_{i,t}\right) = \alpha Z^{i,t} + \beta \tau^{i,t} + \gamma \, \text{wealth\_effect}^{i,t}$$

where $Z^{i,t}$ include the city dummies, year dummies and the $log\left(gdp^{i,t}\right)$ for province i at year t. Because the actual pension contribution is usually higher than the theoretical contribution rate, I define the wealth effect as the difference between the theoretical contribution rate minus the actual contribution rate. This measures the other channel which pension reform can affect the households behavior and housing prices.

The regression results are given by Table 1. The coefficient before the theoretical contribution rate is smaller than the slope in Figure 9. The fixed effect model shows that a 10 percentage points decline in the theoretical contribution rate contributes to a 14.2 percent increase in real housing price level.

4 Conclusion

This paper studies an economy inhabited by overlapping generations of homeowners and investors, with the only difference between the two being that homeowners derive utility from housing services whereas investors do not. Tight collateral constraint limits the borrowing capacity of homeowners and drives the equilibrium interest rate level down to the housing price growth rate, which makes housing attractive as a store of value for investors. As long as the rental market friction is high enough, the investors will hold a positive number of vacant houses in equilibrium. A housing bubble arises in an equilibrium in which investors hold houses for resale purposes only and without the expectation of receiving a dividend either in terms of utility or rent. The model can be applied to China, where the housing bubble can be attributed to the rapid decline in the replacement rate of the pension system.

This paper also shed some lights on the issue of government debt. If the government lends too much when the collateral constraint is high, it will drive the interest too low and investors will start to accumulate bubble assets. The Chinese government passed a stimulus package after the financial crisis hits the US in 2008, which triggered a further wave of housing price boom in China.

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A Mathematical Appendix

A.1 Proof of Proposition 1

The Lagrangian function is

$$
\begin{aligned}
& L = \ln c_t^t + \beta \ln c_{t+1}^t \\
& +\lambda_1 \left[(1-\tau)\, y_t^t - c_t^t - a_{t+1}^t - p_t h_{t+1}^t \right] \\
& +\lambda_2 \left[\tau\, (1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t - c_{t+1}^t \right] \\
& +\mu_1 \left[a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t \right] \\
& +\nu_1 h_{t+1}^t
\end{aligned}
$$

The FOCs become

$$
\begin{aligned}
c_t^t &: \quad \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \quad \frac{\beta}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: \quad -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h_{t+1}^t &: \quad -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t + \nu_1 = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &\geq 0, \text{ if } a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0, \text{ then } \mu_1 = 0 \\
\nu_1 &\geq 0, \text{ if } h_{t+1}^t > 0, \text{ then } \nu_1 = 0
\end{aligned}
$$

The life-time budget constraint for the investors is

$$
c_t^t + \frac{c_{t+1}^t}{R_{t+1}} = (1-\tau)\, y_t^t + \frac{\tau\, (1+n)\, y_{t+1}^{t+1}}{R_{t+1}} + \left(\frac{p_{t+1}}{R_{t+1}} - p_t \right) h_{t+1}^t
$$

1. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0$, i.e., the borrowing constraint of the investors is

not binding; $h^t_{t+1} > 0$,i.e., the unconstrained investors hold positive amount of housing. Therefore $\mu_1 = \nu_1 = 0$. Plug them into the FOCs

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \end{aligned}$$

The following equality holds$R_{t+1} = \frac{p_{t+1}}{p_t}$ and the optimal consumption rules are

$$\begin{aligned} c^t_t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y^t_t \\ c^t_{t+1} &= \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y^t_t \end{aligned}$$

The allocation between the loans and housing assets are indeterminate. The total saving is determined by

$$a^t_{t+1} + p_t h^t_{t+1} = (1-\tau_t)\, y^t_t - c^t_t$$

2. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} > 0$, i.e., the borrowing constraint of investor is not binding; $h^t_{t+1} = 0$, i.e., the investor holds zero amount of housing. Therefore, $\mu_1 = 0$, $\nu_1 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \nu_1 &= 0 \end{aligned}$$

Hence, $R_{t+1} \geq \frac{p_{t+1}}{p_t}$

   (a) If $\nu_1 = 0$, then we go back to case 1

   (b) If $\nu_1 > 0$, then $R_{t+1} > \frac{p_{t+1}}{p_t}$. The purchase of housing are less attractive than the lending to the others.

$$\begin{aligned} a_{t+1}^{t} &= (1-\tau)\, y_t^t - c_t^t \\ h_{t+1}^{t} &= 0 \end{aligned}$$

3. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0$, i.e., the borrowing constraint of the investors is binding; $h_{t+1}^t > 0$, i.e., the constrained investors hold positive amount of housing. Therefore, $\mu_1 \geq 0, \nu_1 = 0$.

   (a) If $\mu_1 = \upsilon_1 = 0$, we go back to case 1. If $\mu_1 > 0, \nu_1 = 0$, then

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &> R_{t+1} \\ \frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &= \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} \end{aligned}$$

   Suppose $\frac{p_{t+1}}{p_t} < R_{t+1} < \frac{\lambda_1}{\lambda_2}$, then $R_{t+1} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t} < \frac{p_{t+1}-(1-\theta)p_{t+1}}{\theta p_t} = \frac{p_{t+1}}{p_t}$,a contradiction! Therefore,

$$R_{t+1} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t}$$

   Let $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t}$. Rewrite the budget constraints as

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t - \theta p_t h_{t+1}^t \\ c_{t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t + \theta \gamma_t p_t h_{t+1}^t \end{aligned}$$

   Solve for $p_t h_{t+1}^t$

$$p_t h_{t+1}^t = \frac{\beta\gamma_t\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta\gamma_t\,(1+\beta)} y_t^t$$

Therefore

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{(1+n)(1+g)}{\gamma_t}\right]y_t^t \\ c_{t+1}^t &= \frac{\beta\gamma_t}{1+\beta}\left[1-\tau+\frac{(1+n)(1+g)}{\gamma_t}\right]y_t^t \\ a_{t+1}^t &= -(1-\theta)\,p_t h_{t+1}^t \\ p_t h_{t+1}^t &= \frac{\beta\gamma_t(1-\tau)-\tau(1+n)(1+g)}{\theta\gamma_t(1+\beta)}y_t^t \end{aligned}$$

4. $a_{t+1}^t + (1-\theta)\,p_t h_{t+1}^t = 0$,i.e., the borrowing constraint of the investors is binding; $h_{t+1}^t = 0$,i.e., the investors hold positive amount of housing

$$\begin{aligned} c_t^t &= (1-\tau)\,y_t^t \\ c_{t+1}^t &= \tau(1+n)(1+g)\,y_t^t \end{aligned}$$

Then $\mu_1, \upsilon_1 \geq 0$.

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta)\,p_t + \nu_1 &= 0 \end{aligned}$$

(a) If $\mu_1, \nu_1 > 0$, either investors have too little endowment when they are young and do not want to save

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - R_{t+1}(1-\theta)\,p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} > R_{t+1}$$

or investors' borrowing cost is too large

$$\frac{\lambda_1}{\lambda_2} > R_{t+1} > \frac{p_{t+1}}{p_t} > \frac{p_{t+1} - R_{t+1}(1-\theta)\,p_t}{\theta p_t}$$

In this article, I assume the young has enough endowment and wants to save. Therefore, I rule out the case $\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1}-R_{t+1}(1-\theta)p_t}{\theta p_t} > \frac{p_{t+1}}{p_t} >$

R_{t+1}.

(b) If $\mu_1 > 0, \upsilon_1 = 0$, We go back to Case 3

(c) If $\mu_1 = 0, \nu_1 > 0$, We go back to Case 2

(d) If $\mu_1 = 0, \upsilon_1 = 0$, We go back to Case 1

A.2 Proof of Proposition 2

The Lagrangian function is

$$\begin{aligned}
L &= \ln c_t^t + \beta\zeta \ln\left(h_{t+1}^t\right) + \beta\left(1-\zeta\right)\ln c_{t+1}^t \\
&+\lambda_1\left[\left(1-\tau\right)y_t - p_t h_{t+1}^t - c_t^t - a_{t+1}^t\right] \\
&+\lambda_2\left[\tau\left(1+n\right)\left(1+g\right)y_t + R_{t+1}a_{t+1}^t + p_{t+1}h_{t+1}^t - c_{t+1}^t\right] \\
&+\mu_1\left[a_{t+1}^t + \left(1-\theta\right)p_t h_{t+1}^t\right]
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \frac{\beta\left(1-\zeta\right)}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h_{t+1}^t &: \frac{\beta\zeta}{h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\left(1-\theta\right)p_t = 0
\end{aligned}$$

where

$$\mu_1 \geq 0, \text{ if } a_{t+1}^t + \left(1-\theta\right)p_t h_{t+1}^t > 0, \text{then } \mu_1 = 0$$

and the life-time budget constraint is given by

$$c_t^t + \frac{c_{t+1}^t}{R_{t+1}} + \left(p_t - \frac{p_{t+1}}{R_{t+1}}\right)h_{t+1}^t = \left(1-\tau\right)y_t^t + \frac{\tau\left(1+n\right)y_{t+1}^{t+1}}{R_{t+1}}$$

1. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0$, i.e., the borrowing constraint of the homeowners is not binding. Therefore, $\mu_1 = 0$. Hence,

$$\frac{\lambda_1}{\lambda_2} = R_{t+1} = \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{p_t}$$

The optimal decision rules are

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
p_t h_{t+1}^t &= \frac{1}{1-\frac{p_{t+1}}{p_t R_{t+1}}}\frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
a_{t+1}^t &= (1-\tau)\,y_t^t - p_t h_{t+1}^t - c_t^t
\end{aligned}$$

2. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0$, i.e., the borrowing constraint of the homeowners is binding. Therefore, $\mu_1 \geq 0$

   (a) If $\mu_1 = 0$, then we go back to Case 1.

   (b) If $\mu_1 > 0$

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\
\frac{\beta\zeta}{h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1(1-\theta)\,p_t &= 0
\end{aligned}$$

   Hence, the condition for R_{t+1} is given by

$$R_{t+1} < \frac{\lambda_1}{\lambda_2}$$

   Let $\frac{\lambda_1}{\lambda_2} \equiv \gamma_t$, then from the budget constraint

$$c_t^t = (1-\tau)\,y_t^t - \theta p_t h_{t+1}^t$$

and

$$c_{t+1}^{t} = \tau\,(1+n)\,(1+g)\,y_t^t + \left(p_{t+1} - R_{t+1}\,(1-\theta)\,p_t\right) h_{t+1}^{t}$$

From the FOC w.r.t. h_{t+1}^{t}, we have

$$\frac{\beta\zeta}{h_{t+1}^{t}} - \lambda_1 \theta p_t + \lambda_2 \left(p_{t+1} - R_{t+1}\,(1-\theta)\,p_t\right) = 0$$

Use the expression for λ_1, λ_2, we have

$$\begin{aligned}
1 &= \lambda_1 (1-\tau)\, y_t^t - \lambda_1 \theta p_t h_{t+1}^{t} \\
\beta\,(1-\zeta) &= \lambda_2 \tau\,(1+n)\,(1+g)\,y_t^t + \lambda_2 \left(p_{t+1} - R_{t+1}\,(1-\theta)\,p_t\right) h_{t+1}^{t} \\
\beta\zeta &= \lambda_1 \theta p_t h_{t+1}^{t} - \lambda_2 \left(p_{t+1} - R_{t+1}\,(1-\theta)\,p_t\right) h_{t+1}^{t}
\end{aligned}$$

Therefore

$$1 + \beta = \lambda_1 (1-\tau)\, y_t^t + \lambda_2 \tau\,(1+n)\,(1+g)\,y_t^t$$

Note that

$$\begin{aligned}
1 + \beta = &\frac{(1-\tau)\, y_t^t}{(1-\tau)\, y_t^t - \theta p_t h_{t+1}^{t}} \\
&+ \beta\,(1-\zeta)\,\frac{\tau\,(1+n)\,(1+g)\,y_t^t}{\tau\,(1+n)\,(1+g)\,y_t^t + \left(p_{t+1} - R_{t+1}\,(1-\theta)\,p_t\right) h_{t+1}^{t}}
\end{aligned}$$

This is a quadratic equation for $p_t h_{t+1}^{t}$. Let

$$\begin{aligned}
x &= p_t h_{t+1}^{t} \\
\varphi &= \frac{p_{t+1}}{p_t} - (1-\theta)\, R_{t+1} \\
a &= (1-\tau)\, y_t^t \\
b &= \tau\,(1+n)\,(1+g)\,y_t^t
\end{aligned}$$

Then

$$1+\beta=\frac{a}{a-\theta x}+\frac{\beta\left(1-\zeta\right)b}{b+\varphi x}$$

It has a unique positive solution

$$p_th_{t+1}^t=x=\frac{\Psi_t+\Phi_t}{2\theta\varphi\left(1+\beta\right)}$$

where $\Psi_t = a\varphi\beta - b\theta\left(1+\beta\zeta\right), \Phi_t = \sqrt{\Psi_t^2+4ab\theta\beta\zeta\varphi\left(\beta+1\right)}$. We can define γ_t

$$\gamma_t=\frac{\lambda_1}{\lambda_2}=\frac{c_{t+1}^t}{\beta\left(1-\zeta\right)c_t^t}=\frac{b+\varphi x}{\beta\left(1-\zeta\right)\left(a-\theta x\right)}$$

and

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\left(1+n\right)\left(1+g\right)}{\gamma_t}\right]y_t^t\\
c_{t+1}^t &= \frac{\beta\left(1-\zeta\right)\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau\left(1+n\right)\left(1+g\right)}{\gamma_t}\right]y_t^t\\
p_th_{t+1}^t &= \frac{\beta\gamma_t\left(1-\tau\right)-\tau\left(1+n\right)\left(1+g\right)}{\theta\gamma_t\left(1+\beta\right)}y_t^t
\end{aligned}$$

A.3 Proof of Lemma 4

We start first by looking the saving function of the unconstrained homeowner/investor. It is obvious to see the saving function of the unconstrained homeowner/investor is a decreasing function of interest rate. When the investor is borrowing constrained, higher interest rate reduces γ_t and implies fewer housing bought. Hence, the amount investor can borrowing is a decreasing function of interest rate. When the homeowner is borrowing constrained, the loan demand function becomes complicated. Differentiate $p_th_{t+1}^t$ directly w.r.t. φ

$$
\begin{aligned}
p_t h_{t+1}^t &= \frac{\Psi_t + \sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)}}{2\theta\varphi\left(\beta+1\right)} \\
&= 2ab\beta\zeta \frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \Psi_t}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\partial p_t h_{t+1}^t}{\partial \varphi} = -2ab\beta\zeta \left(\frac{1}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \Psi_t}\right)^2 \\
&\times \left(\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)} - \frac{d}{d\varphi}\Psi_t\right)
\end{aligned}
$$

Note that $\Psi_t = a\varphi\beta - b\theta\left(1+\beta\zeta\right)$and $\frac{d}{d\varphi}\Psi_t = a\beta$ Also

$$
\begin{aligned}
&\frac{d}{d\varphi}\sqrt{\Psi_t^2 + 4ab\theta\beta\zeta\varphi\left(\beta+1\right)} \\
&= a\beta \frac{\left(a\varphi\beta - b\theta\left(1+\beta\zeta\right)\right) + 2b\zeta\theta\left(\beta+1\right)}{\sqrt{\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)}} < a\beta
\end{aligned}
$$

because

$$
\begin{aligned}
&\left(\left(a\varphi\beta - b\theta\left(1+\beta\zeta\right)\right) + 2b\zeta\theta\left(\beta+1\right)\right)^2 - \left(\Psi_t^2 + 4ab\beta\zeta\theta\varphi\left(\beta+1\right)\right) \\
&= -4b^2\zeta\theta^2\left(\beta+1\right)\left(1-\zeta\right) < 0
\end{aligned}
$$

Therefore,$\frac{\partial p_t h_{t+1}^t}{\partial \varphi} > 0$, $\frac{\partial p_t h_{t+1}^t}{\partial R} < 0$ The loan demand of constrained homeowner is an increasing function of interest rate.

A.4 Proof of Lemma 5

The stationary equilibrium is defined as the competitive general equilibrium in which all individual allocations and prices are time invariant. We need to further assume that $H_t = \bar{H}$ in the stationary equilibrium to get constant housing price. Denote the constant housing price by p^*. Obviously we have $p^* > 0$. Otherwise, workers would purchase infinite amount of houses. Suppose the equilibrium gross interest $R^* < 1$. The gross return of housing for the investors is 1, which is higher than the gross return R^* on consumption loans. From the previous decision rules, the borrowing constraint for both types of households would be binding. The total borrowing of workers is positive and the total borrowing of investors is non-negative. Therefore, the market for loans can not clear at $R^* < 1$. Equilibrium interest rate has to be higher and $R^* < 1$ cannot be a equilibrium interest rate. Note that if $\theta = 1$, both investors and households can not borrow in the equilibrium. Any $R^* < 1$ can be the equilibrium interest rate.

A.5 Proof of Proposition 6

The optimal demand and supply of loans are continuous. Lemma **??** proves that the demand of loans from homeowners is monotonically decreasing in the interest rate and the supply of loans from investors is a monotonically increasing function of interest rate. From Lemma 5, there exists a unique stationary equilibrium with $R^* \geq 1$.

Investors will not be borrowing constrained when $R^* \geq 1$. They supply loans in the market. θ will only affect the optimal decision of homeowners, who are the demand side of loan market. High θ reduces the borrowing limit of constrained homeowners. If θ is high enough, the total borrowing from homeowners become less than the total loan supply from investors. Net interest has to be lower in order to clear the consumption loan market. When the net interest rate drops to zero, investors would then invest extra cash in the housing market. Therefore, there are two threshold levels for collateral constraint, denoted by θ_L and θ_H and three different cases which we analyze one by one.

1. Unconstrained homeowners and unconstrained investors without housing. In the stationary equilibrium, $y_t^t = y$, $H_t = H$. The equilibrium prices $\left(p_1^*, R_1^*\right)$ are determined by

$$\begin{aligned} H &= \omega \frac{1}{p_1} \frac{R_1}{R_1 - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1}\right) y \\ 0 &= 1 - \tau - \frac{1}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1}\right) \left(1 + \omega \frac{\beta\zeta R_1}{R_1 - 1}\right) \end{aligned}$$

The second equation determines a unique $R_1^* > 1$.[4] Hence, housing price can be determined by

$$p_1^* = \omega \frac{y}{H} \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Note that θ can not affect either p_1^* or R_1^*. Now we can solve for the first threshold θ_L when homeowners is borrowing constrained

$$(1-\tau) - \frac{1}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right) = \theta_L \frac{R_1^*}{R_1^* - 1} \frac{\beta\zeta}{1+\beta} \left(1 - \tau + \frac{\tau}{R_1^*}\right)$$

Using the loan market clearing condition, we have $\theta_L = \omega$. Therefore $\frac{\partial \theta_L}{\partial \omega} = 1$. The intuition is that more homeowners will increase the equilibrium interest rate. When the interest rate becomes higher, homeowners will reduce the consumption and housing expenditure. They will be borrowing constrained under a stricter borrowing constraint.

2. Constrained homeowners and unconstrained investors without housing. The

[4] The other solution $R < 1$ cannot be an equilibrium interest rate.

equilibrium prices $\left(p_2^*, R_2^*\right)$ are determined by

$$\begin{aligned}
\omega \frac{1}{p_2} \frac{\Psi + \Phi}{2\theta\varphi\left(\beta + 1\right)} &= H \\
\left(1 - \omega\right) \left[1 - \tau - \frac{1}{1 + \beta} \left(1 - \tau + \frac{\tau}{R_2}\right)\right] y - \omega \left(1 - \theta\right) \frac{\Psi + \Phi}{2\theta\varphi\left(\beta + 1\right)} &= 0
\end{aligned}$$

The two equations imply two implicit functions $p_2^*\left(R_2^*, \theta\right)$ and $R_2^*\left(\theta\right)$. The effect of θ on equilibrium housing price is given by

$$\frac{dp_2^*\left(R_2^*, \theta\right)}{d\theta} = \frac{\partial p_2^*\left(R_2^*, \theta\right)}{\partial R_2^*} \frac{dR_2^*}{d\theta} + \frac{\partial p_2^*\left(R_2^*, \theta\right)}{\partial \theta}$$

On one hand, tighter credit constraint reduces the housing demand, which tends to reduce the price. However, tighter credit constraint also reduces interest rate, which in turns encourages housing consumption. Hence, the total effect is indeterminate.

3. Constrained homeowners and unconstrained investors with empty housing. When $R_3^* = \frac{p_{t+1}}{p_t} = 1$, The market clearing conditions become

$$\begin{aligned}
\omega \frac{1}{p_3} \frac{\Psi + \Phi}{2\theta\varphi\left(\beta + 1\right)} + \left(1 - \omega\right) \frac{I}{p_3} &= H \\
\left(1 - \omega\right) \left[\left(1 - \tau\right) y - \frac{1}{1 + \beta} y - I\right] - \omega \left(1 - \theta\right) \frac{\Psi + \Phi}{2\theta\varphi\left(\beta + 1\right)} &= 0
\end{aligned}$$

where I denotes the investor's purchase of housing assets. Combine the two conditions and note that $\varphi = \theta$ when $R = 1$.

$$\left(1 - \omega\right) \left(1 - \tau - \frac{1}{1 + \beta}\right) y + \omega \frac{\Psi + \Phi}{2\theta\left(\beta + 1\right)} = p_3 H$$

which suggests that p_3^* is independent of θ since $\left(\Psi + \Phi\right)/\theta$ does not depend on θ. The total amount of savings is invested in housing assets. The threshold θ_H for investors to hold housing assets is determined by

$$(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y-\omega\left(\frac{1-\theta_H}{\theta_H}\right)\frac{\Psi+\Phi}{2\theta_H(\beta+1)}=0$$

It is also true that $\frac{\partial\theta_H}{\partial\omega}>0$. This is because high ω implies fewer loan supply from investors. The collateral constraint has to be higher to clear the loan market.

A.6 Proof of Proposition 7

Suppose there is a useless asset called paper. In case 3, it has positive value in the equilibrium. This is because investor has excess supply of loan in the market, which can be invested in the paper. Since the equilibrium interest rate is 1, the price of paper remains constant in the equilibrium. The size of the paper bubble is given by

$$B=(1-\omega)\left(1-\tau-\frac{1}{1+\beta}\right)y-\omega\left(\frac{1-\theta}{\theta}\right)\frac{\Psi+\Phi}{2\theta(\beta+1)}>0 \text{ for } \theta>\theta_H$$

This is called pure bubble. However, the bubble can also take the form of housing assets. If the investors purchase the housing assets I instead, then

$$B=(1-\omega)I$$

which means bubble can shift from paper market to the housing market. If we define the bubble as the case in which investors hold houses for resale purposes only and not with the expectation of receiving a dividend either in terms of utility or rent, then the case 3 satisfies this definition because we rule out the rental market. The next question is whether there is bubble for homeowners? The answer is no. First of all, we define the fundamental value of housing assets to homeowners, and then we show that under properly adjusted interest rate, the housing price is equal to its fundamental value for homeowners in all three cases.

1. Unconstrained homeowners and unconstrained investors without housing. The fundamental value of housing is defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{R_{t+1}} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{R_{t+1}..R_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{R_{t+1}..R_{t+T-1}} \end{aligned}$$

Using the first order condition of homeowners

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{R_{t+1}..R_{t+\tau}} \left(p_{t+\tau} R_{t+\tau} - p_{t+\tau+1}\right) + \lim_{T\to\infty} p_{t+T} \frac{1}{R_{t+1}..R_{t+T-1}}$$

In the stationary equilibrium, $R_1^* > 1, \lim_{T\to\infty} p_1^* \frac{1}{(R_1^*)^T} = 0$

$$p^F = \sum_{\tau=0}^{\infty} \frac{1}{\left(R_1^*\right)^{\tau+1}} \left(p_1^* R_1^* - p_1^*\right) = p_1^* \sum_{\tau=0}^{\infty} \frac{R_1^* - 1}{\left(R_1^*\right)^{\tau+1}} = p_1^*$$

2. Constrained homeowners and unconstrained investors without housing. The fundamental value of housing can be defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta}\frac{c_{t+1}^t}{h_{t+1}^t}}{\hat{R}_t} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t..\hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t..\hat{R}_{t+T-1}} \end{aligned}$$

where $\hat{R}_t = \theta\frac{\lambda_1}{\lambda_2} + (1-\theta) R_{t+1}$. This measures the effective interest rate that households face. It takes into account the shadow value of borrowing constraint. If the borrowing constraint is not binding, $\lambda_1/\lambda_2 = R_{t+1} = \hat{R}_t$. If the borrowing constraint is binding, the effect interest rate is a weighted average of λ_1/λ_2 and R_{t+1}. Therefore, $R_{t+1} < \hat{R}_t < \lambda_1/\lambda_2$. Using the first order condition of constrained homeowners

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t .. \hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - \mu_1 (1-\theta) p_t}{\lambda_2} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t .. \hat{R}_{t+T-1}}$$

In the stationary equilibrium, $\hat{R}_2^* = \theta \frac{\lambda_1}{\lambda_2} + (1-\theta) R_2^* > 1$, $\lim_{T\to\infty} p_2^* \frac{1}{\left(\hat{R}_2^*\right)^T} = 0$

$$\begin{aligned} p^F &= \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \frac{\lambda_1 p_2^* - \lambda_2 p_2^* - \left(\lambda_1 - \lambda_2 R_2^*\right)(1-\theta) p_2^*}{\lambda_2} \\ &= p_2^* \sum_{\tau=0}^{\infty} \frac{1}{\left(\hat{R}_2^*\right)^{\tau+1}} \left(\frac{\lambda_1}{\lambda_2}\theta + R_2^* (1-\theta) - 1\right) \\ &= p_2^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_2^* - 1}{\left(\hat{R}_2^*\right)^{\tau+1}} = p_2^* \end{aligned}$$

3. Constrained homeowners and unconstrained investors with empty housing. The fundamental value of housing can be defined as

$$\begin{aligned} p_t^F &= \frac{p_{t+1} + \frac{\zeta}{1-\zeta} \frac{c_{t+1}^t}{h_{t+1}^t}}{\hat{R}_t} \\ &= \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t .. \hat{R}_{t+\tau}} \frac{\zeta}{1-\zeta} \frac{c_{t+\tau+1}^{t+\tau}}{h_{t+\tau+1}^{t+\tau}} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t .. \hat{R}_{t+T-1}} \end{aligned}$$

where $\hat{R}_3 = \theta \frac{\lambda_1}{\lambda_2} + 1 - \theta$. Using the first order condition of homeowners,

$$p_t^F = \sum_{\tau=0}^{\infty} \frac{1}{\hat{R}_t .. \hat{R}_{t+\tau}} \frac{\lambda_1 p_t - \lambda_2 p_{t+1} - (\lambda_1 - \lambda_2 R_{t+1})(1-\theta) p_t}{\lambda_2} + \lim_{T\to\infty} p_{t+T} \frac{1}{\hat{R}_t .. \hat{R}_{t+T-1}}$$

In the stationary equilibrium, $p_t = p_3^*$, $\hat{R}_3^* > 1$, $\lim_{T\to\infty} p_3^* \frac{1}{\left(\hat{R}_3^*\right)^T} = 0$

$$p^F = p_3^* \sum_{\tau=0}^{\infty} \frac{\hat{R}_3^* - 1}{\left(\hat{R}_3^*\right)^\tau} = p_3^*$$

A.7 Model Extension

A.7.1 Investor's Problem

The Lagrangian function is

$$\begin{aligned}
L &= \ln c_t^t + \beta \ln c_{t+1}^t \\
&+\lambda_1 \left[(1-\tau)\, y_t^t + p_t^r h_{t+1}^R - c_t^t - a_{t+1}^t - p_t h_{t+1}^t\right] \\
&+\lambda_2 \left[\tau\,(1+n)\, y_{t+1}^{t+1} + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r\, p_{t+1} h_{t+1}^R - c_{t+1}^t\right] \\
&+\mu_1 \left[a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t\right] \\
&+\mu_2 \left[h_{t+1}^t - h_{t+1}^R\right] \\
&+\nu_1 h_{t+1}^t \\
&+\nu_2 h_{t+1}^R
\end{aligned}$$

The FOCs become

$$\begin{aligned}
c_t^t &: \quad \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \quad \frac{\beta}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: \quad -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h_{t+1}^t &: \quad -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t + \mu_2 + \nu_1 = 0 \\
h_{t+1}^R &: \quad \lambda_1 p_t^r - \lambda_2 \delta_r\, p_{t+1} - \mu_2 + \nu_2 = 0
\end{aligned}$$

where

$$
\begin{aligned}
\mu_1 &\geq 0, \text{ if } a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0, \text{then } \mu_1 = 0 \\
\mu_2 &\geq 0, \text{ if } h_{t+1}^t - h_{t+1}^R > 0, \text{ then } \mu_2 = 0 \\
\nu_1 &\geq 0, \text{ if } h_{t+1}^t > 0, \text{then } \nu_1 = 0 \\
\nu_2 &\geq 0, \text{ if } h_{t+1}^R > 0, \text{ then } \nu_2 = 0
\end{aligned}
$$

The life-time budget constraint for the investors is

$$
c_t^t + \frac{c_{t+1}^t}{R_{t+1}} = (1-\tau)\, y_t^t + \frac{\tau\,(1+n)\, y_{t+1}^{t+1}}{R_{t+1}} + \left(\frac{p_{t+1}}{R_{t+1}} - p_t\right) h_{t+1}^t + \left(p_t^r - \frac{\delta_r\, p_{t+1}}{R_{t+1}}\right) h_{t+1}^R
$$

1. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t > 0$, $h_{t+1}^t - h_{t+1}^R > 0$, $h_{t+1}^t > 0$, $h_{t+1}^R > 0$, Then $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs

$$
\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} &= 0 \\
-\lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\
\lambda_1 p_t^r - \lambda_2 \delta_r\, p_{t+1} &= 0
\end{aligned}
$$

The following equality holds

$$
R_{t+1} = \frac{p_{t+1}}{p_t} = \frac{\delta_r\, p_{t+1}}{p_t^r} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r}
$$

and the optimal consumption rules are

$$
\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t \\
c_{t+1}^t &= \frac{\beta R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right] y_t^t
\end{aligned}
$$

and the private loans, housing assets, and rental housing are jointly determined by

$$a^t_{t+1} + p_t h^t_{t+1} - p^r_t h^R_{t+1} = (1-\tau)\, y^t_t - c^t_t$$

Note that

$$\frac{\delta_r\, p_{t+1}}{p^r_t} = R_{t+1} = \frac{p_{t+1}}{p_t} = \frac{(1-\delta_r)\, p_{t+1}}{p_t - p^r_t}$$

Then

$$R_{t+1} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} = \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta p_t - p^r_t}$$

2. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} > 0, h^t_{t+1} - h^R_{t+1} > 0, h^t_{t+1} > 0, h^R_{t+1} = 0,$ then $\mu_1 = \mu_2 = \nu_1 = 0,\ \nu_2 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} &= 0 \\ \lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} + \nu_2 &= 0 \end{aligned}$$

Hence,

$$R_{t+1} = \frac{p_{t+1}}{p_t} \leq \frac{\delta_r\, p_{t+1}}{p^r_t}$$

   (a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to the Case 1.

   (b) If $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 > 0$, then

$$\frac{\delta_r\, p_{t+1}}{p^r_t} > R_{t+1} = \frac{p_{t+1}}{p_t} > \frac{p_{t+1}\,(1-\delta_r)}{p_t - p^r_t}$$

   and

$$a^t_{t+1} + p_t h^t_{t+1} = (1-\tau)\, y^t_t - c^t_t$$

   Under this case, it is also true that

$$R_{t+1} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} > \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta p_t - p^r_t}$$

3. $a_{t+1}^{t}+(1-\theta)\,p_t h_{t+1}^{t}>0, h_{t+1}^{t}-h_{t+1}^{R}=0,\ h_{t+1}^{t}>0, h_{t+1}^{R}>0,$ then $\mu_1=\nu_1=\nu_2=0$, $\mu_2\geq 0$. Plug them into the FOCs,

$$\begin{aligned}-\lambda_1+\lambda_2 R_{t+1} &= 0\\ -\lambda_1 p_t+\lambda_2 p_{t+1}+\mu_2 &= 0\\ \lambda_1 p_t^r-\lambda_2\delta_r p_{t+1}-\mu_2 &= 0\end{aligned}$$

Hence,

$$\begin{aligned}R_{t+1} &\geq \frac{p_{t+1}}{p_t}\\ R_{t+1} &\geq \frac{\delta_r p_{t+1}}{p_t^r}\\ R_{t+1} &= \frac{p_{t+1}\,(1-\delta_r)}{p_t-p_t^r}\end{aligned}$$

(a) If $\mu_1=\mu_2=\nu_1=\nu_2=0$, then we go back to the Case 1.

(b) If $\mu_1=\nu_1=\nu_2=0$, $\mu_2>0$, then

$$R_{t+1}=\frac{p_{t+1}\,(1-\delta_r)}{p_t-p_t^r}>\frac{p_{t+1}}{p_t}>\frac{\delta_r p_{t+1}}{p_t^r}$$

and

$$\begin{aligned}a_{t+1}^{t}+\left(p_t-p_t^r\right)h_{t+1}^{t} &= (1-\tau)\,y_t^t-c_t^t\\ h_{t+1}^{R} &= h_{t+1}^{t}\end{aligned}$$

In this case, it is also true that

$$R_{t+1}=\frac{(1-\delta_r)\,p_{t+1}-R_{t+1}\,(1-\theta)\,p_t}{\theta p_t-p_t^r}>\frac{p_{t+1}-(1-\theta)\,R_{t+1}p_t}{\theta p_t}$$

4. $a_{t+1}^{t}+(1-\theta)\,p_t h_{t+1}^{t}>0, h_{t+1}^{t}=h_{t+1}^{R}=0$, then $\mu_1=0$, $\mu_2\geq 0$, $\nu_1\geq$

$0, \nu_2 \geq 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_2 + \nu_1 &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} - \mu_2 + \nu_2 &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} R_{t+1} &\geq \frac{p_{t+1}}{p_t} \\ R_{t+1} &\geq \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r} \end{aligned}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1

(b) If $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 > 0$, then we go back to Case 2

(c) If $\mu_1 = \nu_1 = \nu_2 = 0, \mu_2 > 0$, then we go back to Case 3

(d) If $\mu_1 = 0, \mu_2 + \nu_1 > 0, \nu_1 + \nu_2 > 0$, then $R_{t+1} > \frac{p_{t+1}}{p_t}$ and $R_{t+1} > \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r}$.

$$\begin{aligned} a_{t+1}^t &= (1-\tau)\, y_t^t - c_t^t \\ h_{t+1}^R &= h_{t+1}^t = 0 \end{aligned}$$

It is also true that

$$\begin{aligned} R_{t+1} &> \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta p_t - p_t^r} \\ R_{t+1} &> \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} \end{aligned}$$

5. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0, h_{t+1}^t - h_{t+1}^R > 0, h_{t+1}^t > 0, h_{t+1}^R > 0$, then

$\mu_1 \geq 0$, $\mu_2 = \nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &\geq R_{t+1} \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &= \frac{\delta_r p_{t+1}}{p_t^r} \end{aligned}$$

Discussion:

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1.

(b) If $\mu_1 > 0$, $\mu_2 = \nu_1 = \nu_2 = 0$, then

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$$

Use the equation $\frac{\lambda_1}{\lambda_2} = \frac{\delta_r p_{t+1}}{p_t^r}$ then we have an expression for R_{t+1}

$$R_{t+1} = \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta} < \frac{p_{t+1}}{p_t}$$

It follows that

$$R_{t+1}, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} = \frac{\delta_r p_{t+1}}{p_t^r}$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing constrained. Secondly, the investors are indifferent between constrained-borrow-to-empty and constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} = \frac{(1-\delta_r) p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t - p_t^r}$$

Let $x \equiv \left(p_t h_{t+1}^t - \frac{p_t^r}{\theta} h_{t+1}^R \right)$ and $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$. Rewrite the budget constraints as

$$\begin{aligned}
c_t^t + \theta p_t h_{t+1}^t &= (1-\tau) y_t + p_t^r h_{t+1}^R \\
c_{t+1}^t &= \tau (1+n)(1+g) y_t^t + \left(p_t h_{t+1}^t - \frac{p_t^r}{\theta} h_{t+1}^R \right) \theta \gamma_t
\end{aligned}$$

Then

$$\begin{aligned}
c_t^t &= (1-\tau) y_t^t - \theta x \\
c_{t+1}^t &= \tau (1+n)(1+g) y_t^t + \theta \gamma_t x
\end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_t (1-\tau) y_t^t - \tau (1+n)(1+g) y_t^t}{\theta \gamma_t (\beta + 1)}$$

Therefore

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t} \right] y_t^t \\
c_{t+1}^t &= \frac{\beta \gamma_t}{1+\beta} \left[1 - \tau + \frac{\tau (1+n)(1+g)}{\gamma_t} \right] y_t^t \\
a_{t+1}^t &= -(1-\theta) p_t h_{t+1}^t \\
p_t h_{t+1}^t - \frac{p_t^r h_{t+1}^R}{\theta} &= \frac{\beta \gamma_t (1-\tau) - \tau (1+n)(1+g)}{\theta \gamma_t (\beta + 1)} y_t^t
\end{aligned}$$

6. $a_{t+1}^t + (1-\theta) p_t h_{t+1}^t = 0$, $h_{t+1}^t - h_{t+1}^R > 0$, $h_{t+1}^t > 0$, $h_{t+1}^R = 0$, then

$\mu_1, \nu_2 \geq 0$, $\mu_2 = \nu_1 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta) p_t &= 0 \\ \lambda_1 p_t^r - \lambda_2 \delta_r p_{t+1} + \nu_2 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &\geq R_{t+1} \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &\leq \frac{\delta_r p_{t+1}}{p_t^r} \end{aligned}$$

(a) If $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 1

(b) If $\mu_1 > 0$, $\mu_2 = \nu_1 = \nu_2 = 0$, then we go back to Case 5

(c) If $\mu_1 = \mu_2 = \nu_1 = 0$, $\nu_2 > 0$, then we go back to Case 2

(d) If $\mu_1 > 0$, $\nu_2 > 0$, $\mu_2 = \nu_1 = 0$, then

$$\frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$$

Use the condition that $\frac{\lambda_1}{\lambda_2} < \frac{\delta_r p_{t+1}}{p_t^r}$, and the following inequality for R_{t+1} holds

$$R_{t+1} > \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r p_{t+1}}{p_t^r}}{1-\theta}$$

It turns out that $\frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} > \frac{p_{t+1}}{p_t}$ implies $\frac{p_{t+1}}{p_t} > R_{t+1}$. Therefore, It follows that

$$R_{t+1}, \frac{p_{t+1}(1-\delta_r)}{p_t - p_t^r} < \frac{p_{t+1}}{p_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t} < \frac{\delta_r p_{t+1}}{p_t^r}$$

First of all, this suggests that the borrowing cost is smaller than the intertemporal rate of substitution Therefore, the investors must be borrowing constrained. Secondly, the investors prefer the constrained-borrow-to-empty to the constrained-borrow-to-rent, i.e.,

$$\frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t} > \frac{(1-\delta_r)\, p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t - p_t^r}$$

Let $x \equiv p_t h_{t+1}^t$ and $\gamma_{\,t} \equiv \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta) R_{t+1} p_t}{\theta p_t}$. Use the fact that

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t - \theta p_t h_{t+1}^t \\ c_{t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t - R_{t+1}\,(1-\theta)\, p_t h_{t+1}^t + p_{t+1} h_{t+1}^t \end{aligned}$$

Then

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t - \theta x \\ c_{t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t + \theta \gamma_{\,t} x \end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_{\,t}\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta \gamma_{\,t}\,(\beta+1)} y_t^t$$

Therefore

$$\begin{aligned} c_t^t &= \frac{1}{1+\beta}\left[1 - \tau + \frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t}}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta\gamma_{\,t}}{1+\beta}\left[(1-\tau) + \frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t}}\right] y_t^t \\ a_{t+1}^t &= -(1-\theta)\, p_t h_{t+1}^t \\ p_t h_{t+1}^t &= \frac{\beta \gamma_{\,t}\,(1-\tau) - \tau\,(1+n)\,(1+g)}{\theta \gamma_{\,t}\,(\beta+1)} y_t^t \\ h_{t+1}^R &= 0 \end{aligned}$$

7. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} = 0$, i.e., the borrowing constraint of the investors is binding

   $h^t_{t+1} - h^R_{t+1} = 0$, i.e., the investors rent all the houses out

   $h^t_{t+1} > 0, h^R_{t+1} > 0$, i.e., the investors hold positive amount of housing

   Therefore, $\mu_1, \mu_2 \geq 0$, $\nu_1 = \nu_2 = 0$. Plug them into the FOCs,

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ -\lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t + \mu_2 &= 0 \\ \lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

   Hence,

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &\geq R_{t+1} \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{p_{t+1}}{p_t} \\ \frac{\lambda_1}{\lambda_2} &\geq \frac{\delta_r p_{t+1}}{p^r_t} \end{aligned}$$

   Use the fact that

$$\begin{aligned} -\lambda_1 p_t + \lambda_2 p_{t+1} + (\lambda_1 - \lambda_2 R_{t+1})(1-\theta)\, p_t + \mu_2 &= 0 \\ \lambda_1 p^r_t - \lambda_2 \delta_r p_{t+1} - \mu_2 &= 0 \end{aligned}$$

   Solve for $\frac{\lambda_1}{\lambda_2}$

$$\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}(1-\theta)\, p_t}{\theta p_t - p^r_t}$$

   (a) If $\mu_1 = 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to Case 1.

   (b) If $\mu_1 > 0, \mu_2 = 0, \nu_1 = \nu_2 = 0$, then we go back to Case 5.

   (c) If $\mu_1 = 0, \mu_2 > 0, \nu_1 = \nu_2 = 0$, then we go back to Case 3.

(d) If $\mu_1 > 0$, $\mu_2 > 0$, $\nu_1 = \nu_2 = 0$, then we have

$$\begin{aligned}
\frac{\lambda_1}{\lambda_2} &> R_{t+1} \\
\frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}}{p_t} \\
\frac{\lambda_1}{\lambda_2} &> \frac{\delta_r\, p_{t+1}}{p_t^r}
\end{aligned}$$

Use the expression $\frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\, p_{t+1} - R_{t+1}(1-\theta)\, p_t}{\theta\, p_t - p_t^r}$, the above three inequalities implies

$$\begin{aligned}
R_{t+1} &< \frac{(1-\delta_r)\, p_{t+1}}{p_t - p_t^r} \\
R_{t+1} &< \frac{\frac{p_{t+1}}{p_t} - \theta \frac{\delta_r\, p_{t+1}}{p_t^r}}{1-\theta}
\end{aligned}$$

where I use the assumption $\theta\, p_t - p_t^r > 0$. Therefore

$$\frac{(1-\delta_r)\, p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta\, p_t - p_t^r} = \frac{\lambda_1}{\lambda_2} > \frac{\delta_r\, p_{t+1}}{p_t^r}, \frac{p_{t+1}}{p_t}, R_{t+1}$$

It is also true that

$$\begin{aligned}
\frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1} - R_{t+1}\,(1-\theta)\, p_t}{\theta\, p_t} \\
\frac{\lambda_1}{\lambda_2} &> \frac{p_{t+1}\,(1-\delta_r)}{p_t - p_t^r}
\end{aligned}$$

Recall that

$$\begin{aligned}
c_t^t &= (1-\tau)\, y_t^t + p_t^r h_{t+1}^R - \theta\, p_t h_{t+1}^t \\
c_{t+1}^t &= (1+n)\,(1+g)\, y_t^t + R_{t+1} a_{t+1}^t + p_{t+1} h_{t+1}^t - \delta_r\, p_{t+1} h_{t+1}^R
\end{aligned}$$

Let $x \equiv \left(p_t - \frac{p_t^r}{\theta}\right) h_{t+1}^t$, $\gamma_t \equiv \frac{\lambda_1}{\lambda_2} = \frac{(1-\delta_r)\frac{p_{t+1}}{p_t} - R_{t+1}(1-\theta)}{\theta - \frac{p_t^r}{p_t}}$. Then the above budget constraint becomes

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t - \theta x \\ c_{t+1}^t &= (1+n)\,(1+g)\, y_t^t + \theta \gamma_{\,t} x \end{aligned}$$

Solve for x

$$x = \frac{\beta \gamma_{\,t}\,(1-\tau)\, y_t^t - \tau_{t+1} y_{t+1}^t}{\theta \gamma_{\,t}\,(\beta+1)}$$

Therefore

$$\begin{aligned} \left(p_t - \frac{p_t^r}{\theta}\right) h_{t+1}^t &= \frac{\beta \gamma_{\,t}\,(1-\tau)\, y_t^t - (1+n)\,(1+g)\, y_t^t}{\theta \gamma_{\,t}\,(\beta+1)} \\ h_{t+1}^t &= h_{t+1}^R \\ c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t}}\right] y_t^t \\ c_{t+1}^t &= \frac{\beta \gamma_{\,t}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{\gamma_{\,t}}\right] y_t^t \end{aligned}$$

8. $a_{t+1}^t + (1-\theta)\, p_t h_{t+1}^t = 0, h_{t+1}^t - h_{t+1}^R = 0, h_{t+1}^t = h_{t+1}^R = 0,$ then $\mu_1, \mu_2, \upsilon_1, \upsilon_2 \geq 0.$

$$\begin{aligned} c_t^t &= (1-\tau)\, y_t^t \\ c_{t+1}^t &= \tau\,(1+n)\,(1+g)\, y_t^t \end{aligned}$$

A.7.2 Homeowner's Problem

The Lagrangian function is

$$
\begin{aligned}
L =\ & \ln c_t^t + \beta\left(1-\zeta\right)\ln c_{t+1}^t + \beta\zeta \ln\left(h_{t+1}^r + h_{t+1}^t\right) \\
& + \lambda_1\left[\left(1-\tau\right)y_t^t - p_t^r h_{t+1}^r - p_t h_{t+1}^t - c_t^t - a_{t+1}^t\right] \\
& + \lambda_2\left[\tau\left(1+n\right)\left(1+g\right)y_t + R_{t+1}a_{t+1}^t + p_{t+1}h_{t+1}^t - c_{t+1}^t\right] \\
& + \mu_1\left[a_{t+1}^t + \left(1-\theta\right)p_t h_{t+1}^t\right] \\
& + \nu_1 h_{t+1}^t \\
& + \nu_2 h_{t+1}^r
\end{aligned}
$$

The FOCs become

$$
\begin{aligned}
c_t^t &: \frac{1}{c_t^t} - \lambda_1 = 0 \\
c_{t+1}^t &: \frac{\beta\left(1-\zeta\right)}{c_{t+1}^t} - \lambda_2 = 0 \\
a_{t+1}^t &: -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 = 0 \\
h_{t+1}^t &: \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1\left(1-\theta\right)p_t + \nu_1 = 0 \\
h_{t+1}^r &: \frac{\beta\zeta}{h_{t+1}^r + h_{t+1}^t} - \lambda_1 p_t^r + \nu_2 = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &\geq 0,\ \text{if } a_{t+1}^t + \left(1-\theta\right)p_t h_{t+1}^t > 0, \text{then } \mu_1 = 0 \\
\nu_1 &\geq 0,\ \text{if } h_{t+1}^t > 0, \text{then } \nu_1 = 0 \\
\nu_2 &\geq 0,\ \text{if } h_{t+1}^r > 0,\ \text{then } \nu_2 = 0
\end{aligned}
$$

and the life-time budget constraint is given by

$$
c_t^t + \frac{c_{t+1}^t}{R_{t+1}} + p_t^r h_{t+1}^r + \left(p_t - \frac{p_{t+1}}{R_{t+1}}\right)h_{t+1}^t = \left(1-\tau\right)y_t^t + \frac{\tau\left(1+n\right)\left(1+g\right)y_t^t}{R_{t+1}}
$$

1. $a_{t+1}^{t}+(1-\theta)\,p_t h_{t+1}^{t}>0, h_{t+1}^{t}>0, h_{t+1}^{r}>0$, then $\mu_1=\nu_1=\nu_2=0$

$$\begin{aligned} -\lambda_1+\lambda_2 R_{t+1} &= 0\\ \frac{\beta\zeta}{h_{t+1}^{r}+h_{t+1}^{t}}-\lambda_1 p_t+\lambda_2 p_{t+1} &= 0\\ \frac{\beta\zeta}{h_{t+1}^{r}+h_{t+1}^{t}}-\lambda_1 p_t^{r} &= 0 \end{aligned}$$

Hence,

$$\frac{\lambda_1}{\lambda_2}=R_{t+1}=\frac{p_{t+1}}{p_t-p_t^{r}}=\frac{p_{t+1}-(1-\theta)\,R_{t+1}p_t}{\theta p_t-p_t^{r}}$$

The optimal decision rules are

$$\begin{aligned} c_t^{t} &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y_t^{t}\\ c_{t+1}^{t} &= \frac{\beta\,(1-\zeta)\,R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau\,(1+n)\,(1+g)}{R_{t+1}}\right]y_t^{t}\\ h_{t+1}^{r}+h_{t+1}^{t} &= \frac{\beta\zeta}{p_t^{r}}c_t^{t}\\ \left(p_t-p_t^{r}\right)h_{t+1}^{t}+a_{t+1}^{t} &= (1-\tau)\,y_t^{t}-(1+\beta\zeta)\,c_t^{t} \end{aligned}$$

2. $a_{t+1}^{t}+(1-\theta)\,p_t h_{t+1}^{t}>0, h_{t+1}^{t}>0, h_{t+1}^{r}=0$, then $\mu_1=\nu_1=0, \nu_2\geq 0$.If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1=\nu_1=0, \nu_2>0$,

$$\begin{aligned} -\lambda_1+\lambda_2 R_{t+1} &= 0\\ \frac{\beta\zeta}{h_{t+1}^{t}}-\lambda_1 p_t+\lambda_2 p_{t+1} &= 0\\ \frac{\beta\zeta}{h_{t+1}^{t}}-\lambda_1 p_t^{r}+\nu_2 &= 0 \end{aligned}$$

Hence

$$\frac{\lambda_1}{\lambda_2}=R_{t+1}<\frac{p_{t+1}}{p_t-p_t^{r}}<\frac{p_{t+1}-(1-\theta)\,R_{t+1}p_t}{\theta p_t-p_t^{r}}$$

This suggests that if the rental price is high enough, i.e., $p_t^r > p_t - \frac{p_{t+1}}{R_{t+1}}$, unconstrained workers will choose to own houses. The optimal policy rules are

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta(1-\zeta)R_{t+1}}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
h_{t+1}^t &= \frac{\beta\zeta}{p_t-\frac{p_{t+1}}{R_{t+1}}}c_t^t \\
a_{t+1}^t &= (1-\tau)y_t^t-\frac{(1+\beta\zeta)p_t-\frac{p_{t+1}}{R_{t+1}}}{p_t-\frac{p_{t+1}}{R_{t+1}}}c_t^t
\end{aligned}$$

3. $a_{t+1}^t+(1-\theta)p_t h_{t+1}^t > 0$, $h_{t+1}^t = 0$, $h_{t+1}^r > 0$, then $\mu_1 = 0$, $\nu_1 \geq 0$, $\nu_2 = 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to Case 1. If $\mu_1 = \nu_2 = 0$, $\nu_1 > 0$

$$\begin{aligned}
-\lambda_1+\lambda_2 R_{t+1} &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r+h_{t+1}^t}-\lambda_1 p_t+\lambda_2 p_{t+1}+\nu_1 &= 0 \\
\frac{\beta\zeta}{h_{t+1}^r+h_{t+1}^t}-\lambda_1 p_t^r &= 0
\end{aligned}$$

Hence
$$\frac{\lambda_1}{\lambda_2} = R_{t+1} > \frac{p_{t+1}}{p_t-p_t^r} > \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t-p_t^r}$$

The optimal policy rules are

$$\begin{aligned}
c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
c_{t+1}^t &= \frac{\beta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t \\
p_t^r h_{t+1}^r &= \frac{\beta\zeta}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{R_{t+1}}\right]y_t^t
\end{aligned}$$

4. $a^t_{t+1} + (1-\theta)\, p_t h^t_{t+1} = 0$, $h^t_{t+1} > 0, h^r_{t+1} > 0$, then $\mu_1 \geq 0, \nu_1 = \nu_2 = 0$.If $\mu_1 = \nu_1 = \nu_2 = 0$, then we go back to Case 1. If $\mu_1 > 0, \nu_1 = 0, \nu_2 = 0$

$$\begin{aligned} -\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\ \frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t &= 0 \\ \frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p^r_t &= 0 \end{aligned}$$

Hence, the condition for R_{t+1} is

$$R_{t+1} < \frac{p_{t+1}}{p_t - p^r_t} < \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t - p^r_t}$$

Because

$$\begin{aligned} c^t_t &= (1-\tau)\, y^t_t - \theta p_t h^t_{t+1} + p^r_t h^t_{t+1} - p^r_t \left(h^t_{t+1} + h^r_{t+1}\right) \\ c^t_{t+1} &= \tau\,(1+n)\,(1+g)\, y^t_t + (p_{t+1} - R_{t+1}(1-\theta)\, p_t)\, h^t_{t+1} \end{aligned}$$

Then we have

$$1 + \beta\zeta = \lambda_1 (1-\tau)\, y^t_t - \lambda_1 h^t_{t+1} \left(\theta p_t - p^r_t\right)$$

and

$$\beta\,(1-\zeta) = \lambda_2 \tau\,(1+n)\,(1+g)\, y^t_t + \lambda_2 (p_{t+1} - R_{t+1}(1-\theta)\, p_t)\, h^t_{t+1}$$

Combine the above two equations and let $\frac{\lambda_1}{\lambda_2} \equiv \gamma_t$, then we have

$$(1+\beta)\, c^t_t = \frac{\tau\,(1+n)\,(1+g)\, y^t_t}{\gamma_t} + (1-\tau)\, y^t_t$$

If we know γ_t, then we can express $c_t^t, c_{t+1}^t, h_{t+1}^t$ in terms of γ_t

$$1+\beta = \frac{(1-\tau)\,y_t^t}{(1-\tau)\,y_t^t - \theta p_t h_{t+1}^t - p_t^r h_{t+1}^r} +\beta\,(1-\zeta)\,\frac{\tau\,(1+n)\,(1+g)\,y_t^t}{\tau\,(1+n)\,(1+g)\,y_t^t + (p_{t+1} - R_{t+1}\,(1-\theta)\,p_t)\,h_{t+1}^t}$$

Use $\frac{1+\beta\zeta}{(1-\tau)y_t^t - (\theta p_t - p_t^r)h_{t+1}^t} = \lambda_1 = \frac{1}{c_t^t}$, the above equation can be simplified into

$$\begin{aligned} 1+\beta &= \frac{(1-\tau)\,(1+\beta\zeta)\,y_t^t}{(1-\tau)\,y_t^t - \left(\theta p_t - p_t^r\right)h_{t+1}^t} \\ &\quad +\beta\,(1-\zeta)\,\frac{\tau\,(1+n)\,(1+g)\,y_t^t}{\tau\,(1+n)\,(1+g)\,y_t^t + (p_{t+1} - R_{t+1}\,(1-\theta)\,p_t)\,h_{t+1}^t} \end{aligned}$$

This is a quadratic equation for $p_t h_{t+1}^t$. Let

$$\begin{aligned} x &= p_t h_{t+1}^t \\ \hat{\theta} &= \theta - \frac{p_t^r}{p_t} \\ \varphi &= \frac{p_{t+1}}{p_t} - (1-\theta)\,R_{t+1} \\ a &= (1-\tau)\,y_t^t \\ b &= \tau\,(1+n)\,(1+g)\,y_t^t \end{aligned}$$

$$1+\beta = \frac{(1+\beta\zeta)\,a}{a-\hat{\theta}x} + \frac{\beta\,(1-\zeta)\,b}{b+\varphi x}$$

with one solution is zero, the other solution is

$$x = \frac{a\varphi\beta\,(1-\zeta) - b\hat{\theta}\,(1+\beta\zeta)}{\hat{\theta}\varphi\,(1+\beta)}$$

We can still define γ_t

$$\begin{aligned}\gamma_t &= \frac{\lambda_1}{\lambda_2} = \frac{c_{t+1}^t}{\beta(1-\zeta)c_t^t} = \frac{(b+\varphi x)(1+\beta\zeta)}{\beta(1-\zeta)\left(a-\hat{\theta}x\right)} \\ &= \frac{\varphi}{\hat{\theta}} = \frac{p_{t+1}-(1-\theta)R_{t+1}p_t}{\theta p_t - p_t^r}\end{aligned}$$

which gives

$$\begin{aligned}c_t^t &= \frac{1}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\ c_{t+1}^t &= \frac{\beta(1-\zeta)\gamma_t}{1+\beta}\left[1-\tau+\frac{\tau(1+n)(1+g)}{\gamma_t}\right]y_t^t \\ p_t h_{t+1}^t &= \frac{p_t}{\theta p_t - p_t^r}\left[(1-\tau)y_t^t-(1+\beta\zeta)c_t^t\right] \\ h_{t+1}^r &= \frac{(1-\tau)y_t^t-\theta p_t h_{t+1}^t - c_t^t}{p_t^r} \\ a_{t+1}^t &= -(1-\theta)p_t h_{t+1}^t\end{aligned}$$

5. $a_{t+1}^t+(1-\theta)p_t h_{t+1}^t=0$, $h_{t+1}^t>0$, $h_{t+1}^r=0$, then $\mu_1\geq 0, \nu_1=0, \nu_2\geq 0$.If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1=0, \nu_1=0, \nu_2>0$, then we go back to Case 2. If $\mu_1>0, \nu_1=0, \nu_2=0$, then we go back to Case 4. If $\mu_1>0, \nu_1=0, \nu_2>0$, then the solution is the same as the benchmark model without rental market.

6. $a_{t+1}^t+(1-\theta)p_t h_{t+1}^t=0$, $h_{t+1}^t=0, h_{t+1}^r>0$, then $\mu_1\geq 0, \nu_1\geq 0, \nu_2=0$. If $\mu_1=\nu_1=\nu_2=0$, then we go back to Case 1. If $\mu_1>0, \nu_1=0, \nu_2=0$, then we go back to Case 4. If $\mu_1=0, \nu_1>0, \nu_2=0$,

then we go back to Case 3. If $\mu_1 > 0, \nu_1 > 0, \nu_2 = 0$, then

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} + \mu_1 &= 0 \\
\frac{\beta\zeta}{h^r_{t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \mu_1 (1-\theta)\, p_t + \nu_1 &= 0 \\
\frac{\beta\zeta}{h^r_{t+1}} - \lambda_1 p^r_t &= 0
\end{aligned}$$

Either

$$\frac{\lambda_1}{\lambda_2} > R_{t+1} > \frac{p_{t+1}}{p_t} > \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t - p^r_t}$$

or

$$\frac{\lambda_1}{\lambda_2} > \frac{p_{t+1} - (1-\theta)\, R_{t+1} p_t}{\theta p_t - p^r_t} > \frac{p_{t+1}}{p_t} > R_{t+1}$$

$$\begin{aligned}
a^t_{t+1} &= 0 \\
h^t_{t+1} &= 0 \\
c^t_{t+1} &= \tau\,(1+n)\,(1+g)\, y^t_t \\
c^t_t &= \frac{1}{1+\beta\zeta}(1-\tau)\, y^t_t \\
p^r_t h^r_{t+1} &= \frac{\beta\zeta}{1+\beta\zeta}(1-\tau)\, y^t_t
\end{aligned}$$

A.8 Proof of Lemma 9

Suppose homeowners is not borrowing constrained. The Focs of homeowners become

$$\begin{aligned}
-\lambda_1 + \lambda_2 R_{t+1} &= 0 \\
\frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p_t + \lambda_2 p_{t+1} + \upsilon_1 &= 0 \\
\frac{\beta\zeta}{h^r_{t+1} + h^t_{t+1}} - \lambda_1 p^r_t + \nu_2 &= 0
\end{aligned}$$

Suppose $h^r_{t+1} > 0$, then $\upsilon_2 = 0$,

$$\lambda_1 p^r_t - \lambda_1 p_t + \lambda_2 p_{t+1} + \upsilon_1 = 0$$

Therefore

$$R_{t+1} = \frac{\lambda_1}{\lambda_2} = \frac{p_{t+1} + \frac{\upsilon_1}{\lambda_1}}{p_t - p^r_t} \geq \frac{p_{t+1}}{p_t - p^r_t} > \frac{p_{t+1}(1-\delta_r)}{p_t - p^r_t}$$

This suggests that investors would not hold housing assets because the return of investment in housing assets is strictly less than the return on consumption loans. Hence, $h^r_{t+1} = 0$ if homeowners are borrowing constrained. This is a contradiction. Therefore, $h^r_{t+1} = 0$ if homeowners are unconstrained.

A.9 Proof of Proposition 10

Since our point of interest is to see whether frictional rental market can resolve the problem of vacant houses and prevent the rise of bubbles, I assume $\theta > \theta_L = \omega$, such that there exists a bubble after the pension reform when $\delta_r = 0$. From Lemma 10, we know that investors will hold housing assets only if homeowners are borrowing constrained. Therefore, I only consider the equilibrium where homeowners are borrowing constrained and investors lend to homeowners.

When there is a housing bubble, $R^* = 1$. For the investors to be indifferent between holding empty houses and renting them out, it must be $p^r = \delta_r p$. For the homeowners to rent positive amount of housing, the necessary condition is

$$R^* < \frac{p}{p - p^r} < \frac{\lambda_1}{\lambda_2} = \gamma = \frac{\theta}{\theta - \delta_r}$$

which is obviously satisfied when $R^* = 1$. The demand function for rental housing is given by

$$
\begin{aligned}
p^r h^r &= y - c - \theta p h \\
&= \frac{\beta}{1+\beta} y - \frac{\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y
\end{aligned}
$$

If $\delta_r \geq \theta\zeta$, then $p^r h^r < 0$. Homeowners demand zero rental housing if the rental market friction $\delta_r \geq \theta\zeta$.

Housing bubble can still exist even with active rental market. The loan supply is given by

$$
\int a^I d\mu^i = (1-\omega)\left(1 - \frac{1}{1+\beta}\right) y + p^r \int h^R d\mu^i - p \int h^I d\mu^i
$$

where$h^I \geq h^R$. Let's suppose $h^I = h^R + h^B$, where h^B is the amount of vacant houses.

$$
\int a^I d\mu^i = (1-\omega)\frac{\beta}{1+\beta} y + \left(p^r - p\right) \int h^R d\mu^i - p \int h^B d\mu^i
$$

The loan demand function can be written as

$$
\int a^H d\mu^i = -\omega \frac{1-\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y
$$

The loan market clearing condition requires that$\int a^I d\mu^i + \int a^H d\mu^i = 0$. Hence

$$
\begin{aligned}
& p \int h^B d\mu^i \\
= \ & (1-\omega)\frac{\beta}{1+\beta} y - \left(p - p^r\right) \int h^R d\mu^i - \omega \frac{1-\theta}{\theta - \delta_r} \frac{\beta(1-\zeta)}{1+\beta} y \\
= \ & \frac{\beta}{1+\beta} y \left(1 - \frac{\omega\zeta}{\delta_r}\right)
\end{aligned}
$$

where the second equality comes from the market clearing condition for rental market, $\int h^R d\mu^i = \int h^r d\mu^i$. If $\delta_r > \omega\zeta$, then $p \int h^B d\mu^i > 0$, i.e., there are empty housing held by investors even through the rental market is active.

A.10 Proof of Proposition 12

In the equilibrium, if $R_{t+1} \equiv (1+n)(1+g)$, then $\frac{K_t}{A_t L_t} = \left(\frac{n+g+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. We know that this is the lowest equilibrium interest rate. Hence, $K_{t+1} = \left(\frac{n+g+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} A_{t+1} L_{t+1}$ is maximum asset demand the production sector can absorb. If there exists bubble in the equilibrium, the the following condition holds

$$A_t (1-\omega) L_t \frac{\beta}{1+\beta} \left(\frac{K_t}{A_t L_t}\right)^{\alpha} > A_t \omega L_t \frac{1-\theta}{\theta} \frac{\beta}{1+\beta} \left(\frac{K_t}{A_t L_t}\right)^{\alpha} + K_{t+1}$$

Because $K_{t+1} = \frac{K_t}{A_t L_t} A_{t+1} L_{t+1}$, the above condition can be simplified as

$$(1-\omega) \frac{\beta}{1+\beta} \frac{n+g+\delta}{\alpha} > \omega \frac{1-\theta}{\theta} \frac{\beta}{1+\beta} \frac{n+g+\delta}{\alpha} + 1 + n + g$$

which implies

$$\theta > \omega \frac{1}{1 - \alpha \frac{1+\beta}{\beta} \frac{n+g+1}{n+g+\delta}}$$

A.11 Proof of Proposition 13

When $\tau = 0$, the total supply of loan by investors becomes $(1-\omega) \frac{\beta}{1+\beta} y$. The total loan demand from constrained homeowners becomes $\omega \frac{1-\theta}{\theta} \frac{\beta}{\beta+1} y$. Note that both the supply and demand does not depend on interest rate. Therefore, bubble will arise iff

$$(1-\omega) \frac{\beta}{1+\beta} y > \omega \frac{1-\theta}{\theta} \frac{\beta}{\beta+1} y$$

which is equivalent to $\theta > \theta_L = \omega$. Therefore, if the economy stays at the case 1 of stationary equilibrium, where both investors and homeowners are unconstrained, then the removal of pension system will not trigger a bubble equilibrium.

If the economy stays at case 2 of stationary equilibrium, we have

$$\frac{p_2 H}{y} = \frac{1-\omega}{1-\theta}\left[1-\tau-\frac{1}{1+\beta}\left(1-\tau+\frac{\tau}{R_2}\right)\right]$$

In the bubble equilibrium, the housing wealth/GDP ratio is $\frac{\beta}{1+\beta}$. If $\tau > \frac{\theta-\omega}{1-\omega}$, then

$$\frac{p_2 H}{y} < \frac{(1-\omega)(1-\tau)}{1-\theta}\frac{\beta}{1+\beta} < \frac{\beta}{1+\beta}$$

A.12 Proof of Proposition 14

We know that households are constrained and investor hold housing assets close to the neighborhood of new stationary equilibrium. From the financial market constraint, we can show that$K_{t+1} = (1-\omega) L_t a^I_{t+1} + \omega L_t a^H_{t+1}$. Because

$$\begin{aligned} a^I_{t+1} + p_t h^I_{t+1} &= \frac{\beta}{1+\beta} w_t \\ a^H_{t+1} &= -(1-\theta) p_t h^H_{t+1} \end{aligned}$$

Plug them to the expression for K_{t+1}, we have

$$\begin{aligned} K_{t+1} &= (1-\omega) L_t \frac{\beta}{1+\beta} w_t - (1-\theta) p_t h^H_{t+1} \omega L_t \\ &= (1-\omega) L_t \frac{\beta}{1+\beta} w_t + \omega L_t \frac{\beta}{1+\beta} w_t - p_t H \end{aligned}$$

Hence $p_t H + K_{t+1} = L_t \frac{\beta}{1+\beta} w_t$. Because

$$w_t = (1-\alpha) A_t K^\alpha_t (A_t L_t)^{-\alpha} L_t$$

then

$$\tilde{p}_t H + \tilde{k}_{t+1}(1+n+g) = \frac{\beta}{1+\beta}(1-\alpha)\tilde{k}^\alpha_t$$

where $p_t = \tilde{p}_t A_t L_t$, $K_{t+1} = \tilde{k}_{t+1} A_{t+1} L_{t+1}$.

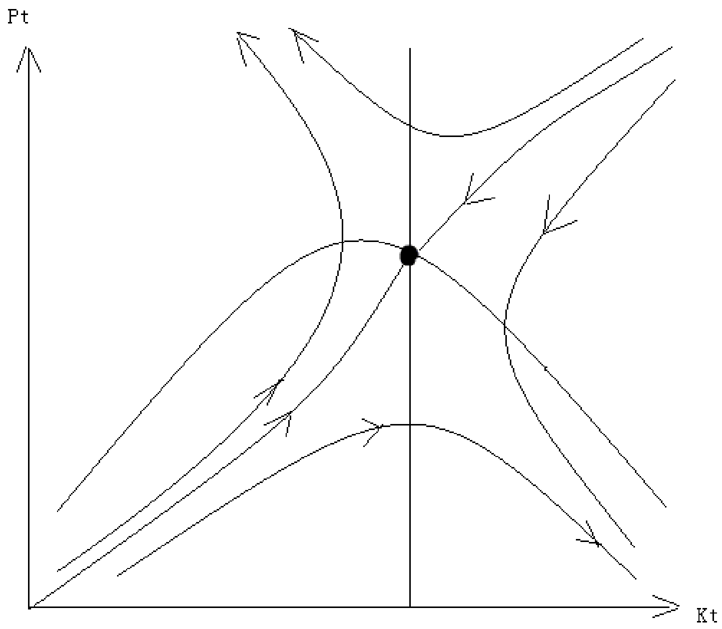

Figure 10: Phase Diagram for the Transitional Dynamics after the Pension Reform

When investor hold housing assets, we know that $p_{t+1}/p_t = R_{t+1}$, or equivalently,

$$\frac{\tilde{p}_{t+1}}{\tilde{p}_t} = \left(1 + \alpha \tilde{k}_{t+1}^{\alpha-1} - \delta\right) / \left(1 + n + g\right)$$

Therefore, those two equations determine a autonomous system of $\left(\tilde{p}_t, \tilde{k}_t\right)$ with $\tilde{p}_t > 0$ and $\tilde{k}_t > 0$. The phase diagram is shown by figure 10. Note that $\tilde{p}_t = 0$ cannot be a stationary equilibrium price because households will demand infinite amount.