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ASSET PRICES IN A TIME SERIES MODEL WITH DISPARATELY INFORMED, COMPETITIVE TRADERS

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ABSTRACT

This paper examines the time series properties of the price of a risky asset implied by a model in which competitive traders are heterogeneously informed about the underlying sources of uncertainty in the economy. Traders do not observe the shocks in the period they occur. However, traders are imperfectly and heterogeneously informed about these shocks for three reasons: (1) the shocks are serially correlated and hence partially forecastable from their past history, (2) each trader receives private signals about the current values of a subset of the shocks, and (3) the equilibrium price conveys information about the private signals and beliefs of other traders. Since prices convey information in this economy, traders will face an infinite regress problem in expectations associated with their desire to forecast the beliefs of others, the beliefs of others about average beliefs, etc. The equilibrium time series representation for the price of the risky security is deduced in various imperfect information environments. Then the volatility and autocorrelations of prices in this model are compared to the corresponding statistics for a model in which agents are homogeneously informed.

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1. Introduction

Surveys of the participants in organized securities markets indicate that traders hold widely different beliefs about the future course of economic activity. Expectations not only differ, but they evidently respond significantly to unexpected movements in such variables as real economic growth and the weekly changes in the money stock [see Cornell (1983) for a review of some of this literature]. Furthermore, French and Roll (1984) have found that the variance of stock prices is greater over periods when the stock market is open than when it is closed. Together, these observations suggest that disparate beliefs and the sharing of information through the trading process may be important ingredients in modeling asset price determination. The purpose of this paper is to explore the implications of disparate expectations for the time series properties of asset prices in the context of a simple model with competitive traders facing serially correlated shocks.

While the models examined are partial equilibrium in nature, this exploration is motivated in part by the apparent inconsistency of representative agent, dynamic equilibrium models with the behavior of asset prices. The variances and autocorrelation functions of asset returns seem to be inconsistent with the implications of both linear expectations models [see, e.g., Shiller (1979, 1981), Singleton (1980, 1985), Scott (1985)] and the nonlinear models studied by Hansen and Singleton (1982), Ferson (1983), Dunn and Singleton (1985), and Eichenbaum and Hansen (1985), among others. Now all of these models assume that agents have a common information set. In light of the evidence to the contrary, it seems worthwhile to explore the consequences for the time series properties of asset returns of introducing heterogeneity in the form of disparate information sets. This paper compares the implied variances and autocorrelations of prices for alternative specifications of agents' information sets in the context of a simple, partial equilibrium asset pricing model.

There is an extensive literature on asset pricing in the presence of disparately informed traders. Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), Kyle (1984), Admati (1985), and Altug (1984), among others, investigate the role of equilibrium security prices as aggregators of individual traders' information. Attention is restricted to one-period models in these studies, and the focus is primarily on the extent to which prices reveal private information to all traders.[1] None of these studies have considered the implications of disparate expectations for the time series properties of security prices. Indeed, the models investigated to date have typically not been designed to address this issue.

Hellwig (1982) considers a time series model for stock prices in which agents condition their expectations on past rather than current prices. He shows that as the length of the trading interval decreases, the equilibrium price process obtained from conditioning on past prices approximates the fully revealing equilibrium price that emerges in his model when agents condition on current prices. Hellwig does not examine the time series properties of his model. More recently, Shiller (1984) has studied a simple dynamic model of stock prices in which there are two groups of traders: one group responds to expected returns optimally forecasted and the other responds myopically. Agents do not explicitly have different information sets in this model. Furthermore, neither Hellwig (1982) or Shiller (1984) examine equilibria where agents condition on current prices and prices are not fully revealing.

This paper takes a first step toward filling this void by examining a model with a continuum of traders in which the underlying sources of uncertainty in the economy exhibit serial dependence over time. An economy with a single risky asset, in which traders have a one-period investment horizon, is studied. By restricting attention to a one-period investment horizon, I am able to isolate the consequences for equilibrium asset prices of serially dependent shocks in a model with disparate expectations. Although I do not attempt to model formally the price of an existing risky security, the specifications of both the objective functions and information sets of the traders are motivated partially by the structure of U.S. government bond markets.

The model and the equilibrium time series representation for the price of the risky security in the presence of disparate information and serially correlated shocks are described in section two. The equilibrium price depends on the average forecast across disparately informed traders of the next period's price. This dependence gives rise to an infinite regress problem in expectations that is similar to the problem discussed by Townsend (1983a,b). Following Townsend (1983b), a technique of undetermined coefficients is used to solve this infinite regress problem and obtain an expression for the equilibrium price. Details of this derivation are presented in the Appendix.

In section three the price relation is compared to the corresponding relations obtained from several related models. First, I consider a model in which agents have complete current information. It is shown that the disturbances have a more persistent effect on prices in the model with disparate expectations than the model with complete information. To help interpret this finding, these models are also compared briefly with a model in which agents are imperfectly, but homogeneously, informed. Additionally, I examine the consequences for prices of setting the variances of some of the disturbances to zero under disparate expectations.

In section four, two models are solved numerically to obtain the time series representations of the equilibrium price of the risky security for various sets of hypothetical values of the parameters characterizing preferences, uncertainty, and the supply of the security. First, the general model set forth in section two is solved. Then the solutions of a model in which traders are imperfectly, but homogeneously, informed are calculated. Using these price representations, the consequences of subsets of agents having better information than others or different levels of risk aversion are investigated. In addition, the implications for prices of different specifications of the stochastic process for the underlying uncertainty in the model are examined.

Concluding comments are presented in section five.

2. A Model with a Continuum of Traders with One-period Investment Horizons

Suppose there are a continuum of investors indexed by $i \varepsilon [0,1]$. Each investor has the opportunity to invest in a single risky security with price p_t and stochastic coupon payment c_t at date t. Purchases of these securities may be financed by borrowing at the constant rate $\bar{r}$. Then the wealth of the ith investor evolves according to the relation

$$(1) \qquad w_{it+1} = z_{it}(p_{t+1}+c_{t+1}) - (z_{it}p_t - w_{it})(1+\bar{r}),$$

where w_{it} is the level of wealth and z_{it} denotes the holdings of the risky security at date t. Relation (1) does not constrain the net worth of each trader to exceed some minimum level, nor are short sales limited. Incorporating minimum capital requirements, constraints on short sales, and other limitations on trading that are present in U.S. stock and bond markets is of interest, but is beyond the scope of this analysis.

The ith investor is assumed to have a one-period investment horizon and to rank alternative investment strategies using the exponential utility function

$$(2) \qquad E_t^i \, -e^{-\gamma_i W_{it+1}},$$

where E_t^i denotes the expectation of investor i conditioned on his information set (sigma-algebra) Φ_t^i at date t and γ_i is the constant coefficient of absolute risk aversion. The assumption that investors have a one-period investment horizon simplifies the analysis by removing the intertemporal dependence between investment decisions at dates t and t+j, $j \geq 1$, that would be induced by multi-period horizons [see Pfleiderer (1984)]. In this manner I am able to focus on the implications of disparate expectations for the time series properties of p_t in models in which the persistence of shocks is the primary source of dynamics in the model.

The coupon stream $\{c_t\}$ is assumed to be normally distributed and to follow a first-order autoregressive process

$$c_t = \bar{c} + \psi c_{t-1} + u_t, \; Eu_t = 0, \; \mathrm{Var}\, u_t = \sigma_u^2; \; |\psi| \leq 1.$$

The disturbance $\{u_t\}$ is assumed to be independent of other sources of uncertainty in the model. If the risky security is a bond (stock), then $\{c_t\}$ represents a stochastic coupon (dividend). Some implications for prices of altering the stochastic process for coupons are discussed in section three.

In order to deduce an equilibrium expression for p_t under disparate expectations, the underlying sources of uncertainty facing investors are assumed to be normally distributed. This assumption implies that the distribution of the price of the risky security at date t+1, conditional on Φ_t^i, is normal with mean $E_t^i p_{t+1}$ and variance $Var_t^i p_{t+1}$, and the conditional variance $\delta_i = Var_t^i(p_{t+1}+c_{t+1})$ is a constant. Furthermore, the first-order conditions for the maximization of (2), subject to the wealth equation (1), are

$$-\gamma_i E_t^i(p_{t+1}-\alpha p_t) - \gamma_i \psi c_t + \gamma_i^2 z_{it}\, \delta_i - \gamma_i \bar{c} = 0, \tag{3}$$

where $\alpha \equiv [1+\bar{r}] > 1$. Solving for z_{it} gives the demand for the risky security by the ith trader:

$$z_{it} = \{E_t^i p_{t+1} - \alpha p_t\}/(\gamma_i \delta_i) + (\bar{c}+\psi c_t)\ /(\gamma_i \delta_i),\ i \in [0,1]. \tag{4}$$

A negative value of z_{it} indicates short selling by the ith investor.

In addition to the continuum of risk averse traders with demand functions (4), I assume that there is a class of traders whose net supply of securities at date t has the linear form (an interpretation of this class is provided subsequently).

$$z_t^a = \theta_t + \varepsilon_t + \xi p_t. \tag{5}$$

The disturbance θ_t is serially correlated and is assumed to follow either an autoregressive process of order one or a moving average process of order two:

$$(6)\qquad \theta_t = \rho\theta_{t-1} + \nu_t; \quad \text{or } \theta_t = \nu_t + \phi_1\nu_{t-1} + \phi_2\nu_{t-2}.$$

In both representations ν_t is distributed as a normal random variate with mean zero and variance σ_ν^2, and in the AR representation $|\rho| < 1$. The choices of AR(1) and MA(2) processes allow for a variety of serial correlation patterns in θ_t; modification of the following analysis for more general processes is conceptually straightforward. The disturbance ε_t, is serially independent and normally distributed with mean zero and variance σ_ε^2.

It remains to specify the information set for agent $i\varepsilon[0,1]$. Speculative traders observe current and past prices and coupon payments. Furthermore, there is no private information about future coupon payments. In this respect the model differs from previous models of information aggregation in security markets, which assumed that there is private information about the future dividend. Imperfect and private information is introduced into the model here as follows. The speculative traders are assumed to observe ν and ε with a two-period lag. The only information they have at date t about ε_t and ε_{t-1} is the information that can be extracted from p_t and p_{t-1}, so traders are equally, imperfectly informed about the process $\{\varepsilon_t\}$. In contrast, the ith trader is assumed to receive a private signal $s_{it} = \theta_t + \eta_{it}$ about θ_t. The $\{\eta_{it}\}$ are mutually and serially independent processes that are independent of the other disturbances in the model and have mean zero and variances $\sigma_{\eta i}^2$, $i \varepsilon [0,1]$.[2] Thus, at date t, traders are disparately informed about the disturbance θ_t underlying z_t^a and, accordingly, will have disparate expectations about future security prices. Combining these assumptions leads to the following information set for the ith investor:

$$(7)\qquad \Phi_t^i = \sigma a\ \{s_{it-j}, p_{t-j}, c_{t-j} : j \geq 0;\ \nu_{t-j},\ \varepsilon_{t-j} : j \geq 2\},$$

where $\sigma a\{\cdot\}$ denotes the sigma-algebra generated by the variables in brackets.[3] The two-period lag in acquiring information about ν and ε was chosen arbitrarily; any informational lag of at least one period for θ leaves traders disparately informed.

While this partial equilibrium model is intended primarily to be illustrative of the consequences of disparate expectations for the temporal behavior of security prices, the specification adopted is motivated by the structure of trading in U.S. government bonds. The number of active traders in bonds is large, so the assumption of competitive traders seems like a useful starting point. Furthermore, many traders finance their trading activity through repurchase agreements or short-term borrowing, which is consistent with the presence of the term $(z_{it}p_t - w_{it})(1+\bar{r})$ in the wealth equation (1).

The net supply z_t^a in (5) can be interpreted as the trading activity of agents who do not seek primarily to maximize wealth through speculative trading, but rather who trade for non-speculative purposes. Candidates for such traders in the U.S. bond markets are the U.S. Treasury, the Federal Reserve, (to a lesser extent) financial intermediaries, and those who are classified as "liquidity" traders in several previous models of trading under heterogeneous information. Pursuing this interpretation of the model, suppose that collectively the "non-speculative" agents trade both to satisfy certain "macroeconomic" objectives related to movements in output or unemployment, and for technical reasons related to the intermediation process or to changes in the monetary base due to activities of foreign official institutions. If these traders have more information than the speculative traders or there are systematic random components to their objective functions, then a component like θ_t , which is unobserved by the speculative traders, will appear in the trading rule z_t^a. At the same time, additional shocks that are unobserved by speculative traders and that relate to the intermediation process will affect the trading rules of intermediaries and the Federal Reserve. A second disturbance like ε_t might capture the affects on trading of such shocks.

Finally, the specification of a trader's information set, Φ_t^i, also captures some features of the actual information sets of bond traders. There is typically little uncertainty about the value of the coupon payment one period in the future, so there is not private information about $\{c_t\}$ in this model. On the other hand, there is substantial uncertainty about the current motives for the trading activity of the Federal Reserve and financial intermediaries. After some time has elapsed both information about the trading rule of the central bank and the balance sheets of intermediaries are released (open market committee minutes are published for example).[4] The introduction of the signal s_{it} and the two-period information lag are motivated by these observations.[5]

The idiosyncratic shocks η_{it} are the formal manifestation of the assumption that traders are differentially informed. One interpretation of the η_{it} is as follows. No trader can infer the Federal Reserve's operating procedures perfectly from available information, yet some information is available. Furthermore, traders differ in their innate ability to perceive the truth. Endowing agents with different measurement errors is one way of representing these different forecasting abilities. In particular, having the η_{it} drawn from distributions with different variances induces different forecast error variances across traders. The ability of trader i to extract the truth from the available data depends on the relative magnitudes of $\sigma_{\eta i}^2$ and σ_θ^2, where σ_θ^2 is the variance of θ_t. Traders are assumed not to share their assessment (s_{it}) of the value of θ_t with other traders. Also, this analysis abstracts completely from the possibility that high quality information (signals) may be available at a price; there are no costs to acquiring the signal s_{it}.[6]

The remainder of this section is devoted to describing a procedure for solving the model for the equilibrium time series representation of the price of the risky security in the presence of disparate information (p_t^D). Henceforth constant terms will be suppressed, so the price process derived will be in deviation from mean form. Suppose initially that all investors have the same coefficient of absolute risk aversion (γ) and the variance of the η_{it} are the same (σ_η^2), for all $i\epsilon[0,1]$. Then the net aggregate demand of the speculative traders is[7]

$$(8) \qquad z_t^d = \int_0^1 z_{it} d\mu(i) = \left[\int_0^1 E_t^i p_{t+1}^D \, d\mu(i) - \alpha p_t^D + \psi c_t \right] / (\gamma \delta^D),$$

where μ denotes Lebesgue measure on the interval $[0,1]$ and δ^D denotes the (common) conditional variance of $(p_{t+1}^D + c_{t+1})$. Equating supply and demand and solving for the price gives

$$(9) \qquad p_t^D = \lambda^D \{ \int_0^1 E_t^i p_{t+1}^D d\mu(i) \} + \lambda^D \psi c_t - \delta^D \gamma \lambda^D \{\theta_t + \varepsilon_t\}, \; \lambda^D \equiv 1/[\xi \gamma \delta^D + \alpha].$$

Notice that the equilibrium price depends on the average forecast across traders of p_{t+1}^D. Thus, each trader's forecast of p_{t+1}^D depends on his forecast of the market-wide average forecast of p_{t+2}^D, so the market-wide average forecast of p_{t+1}^D depends on the market-wide average forecast of the market-wide average forecast of p_{t+2}^D. Pursuing this logic, it is apparent that there is an infinite regress problem in expectations of the type discussed by Townsend (1983a,b). Following Townsend (1983b), this section uses a technique of undetermined coefficients to solve this problem and obtain an expression for the equilibrium price. This approach restricts attention to equilibrium price processes that are linear functions of the underlying disturbances in the economy.

The procedure for obtaining a solution is as follows. First, I conjecture a value for δ^D (which determines λ^D) and a solution for p_t^D of the form

$$(10) \qquad p_t^D = A(L)\nu_t + B(L)\varepsilon_t + \frac{\psi\lambda^D}{1-\psi\lambda^D} c_t,$$

where $A(L) = \sum_{j=0}^{\infty} A_j L^j$ and $B(L) = \sum_{j=0}^{\infty} B_j L^j$ are polynomials in the lag operator L. Then, leading (10) one period and substituting this expression into (9) gives

$$(11) \qquad p_t^D = \lambda^D\{ \int_0^1 [A_1 E_t^i \nu_t + A_2 E_t^i \nu_{t-1} + A^*(L)\nu_{t-2}]d\mu(i)$$
$$+ \int_0^1 [B_1 E_t^i \varepsilon_t + B_2 E_t^i \varepsilon_{t-1} + B^*(L)\varepsilon_{t-2}]d\mu(i)\} + \frac{\psi\lambda^D}{1-\psi\lambda^D} c_t - \lambda^D\delta^D\gamma(\theta_t + \varepsilon_t),$$

where $A^*(L) = \sum_{j=0}^{\infty} A_{j+3}L^j$ and $B^*(L) = \sum_{j=0}^{\infty} B_{j+3}L^j$. In arriving at (11), I have used the fact that the optimal forecasts of ν_{t+1} and ε_{t+1} at date t are their unconditional means (zero), since $\{\nu_t\}$ and $\{\varepsilon_t\}$ are serially independent. Also, the coefficient on c_t in (10) and (11) comes from solving the recursion (9) forward to get p_t^D as a function of c_t with coefficient $\frac{\psi\lambda^D}{1-\psi\lambda^D}$.[8]

Inspection of (11) reveals that calculating the equilibrium price requires a solution for the optimal forecasts of ν_t, ν_{t-1}, ε_t, and ε_{t-1}, and the conditional variance of p^D_{t+1}. Only forecasts of the current and first lagged value of these disturbances are required, because ν_{t-s} and ε_{t-s}, for $s \geq 2$, are Φ^i_t measurable by assumption. Without this simplifying assumption, each trader would have to forecast the disturbances ν_{t-j} infinitely far into the past.

To determine the optimal forecasts of the unknown ν and ε, consider the variables

$$p^*_t = A_0\nu_t + A_1\nu_{t-1} + B_0\varepsilon_t + B_1\varepsilon_{t-1},$$

$$s^*_{it} = \nu_t + \phi_1\nu_{t-1} + \eta_{it},$$

which are observed by trader i at date t and embody information about the unobserved shocks. These variables, together with ε_{t-2} and ν_{t-2}, comprise the observer equation

$$(12) \qquad \begin{bmatrix} \varepsilon_{t-2} \\ \nu_{t-2} \\ s^*_{it} \\ p^*_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & L^2 \\ L^2 & 0 & 0 \\ (1+\phi_1 L) & 1 & 0 \\ (A_0+A_1L) & 0 & (B_0+B_1L) \end{bmatrix} \begin{bmatrix} \nu_t \\ \eta_{it} \\ \varepsilon_t \end{bmatrix}$$

or, more concisely, $y_t = M(L)\omega_t$, where $y'_t \equiv [\varepsilon_{t-2}, \nu_{t-2}, s^*_{it}, p^*_t]$, etc. Equation (12) cannot be used directly to determine the optimal forecast at date t of the unobserved shocks, because it is not a fundamental moving average representation. That is, the matrix M(z) viewed as a matrix of polynomials in the complex variable z is not of full rank for all z with $|z| \leq 1$. In the Appendix, a fundamental moving average representation for y_t is derived using the approach suggested by Townsend (1983a,b), suitably modified to apply to the model in this paper.

Having deduced the fundamental moving average representation, the forecasts of the unobserved shocks that appear in the price equation (11) can be derived using the Weiner-Kolmogorov optimal prediction formulas (Whittle 1963). Specifically, from (A.14) in the Appendix, it follows that $E_t^i\varepsilon_t, E_t^i\varepsilon_{t-1}$, $E_t^i\nu_t$ and $E_t^i\nu_{t-1}$ can be expressed as

$$E_t^i\nu_t = f_{\nu 0}(\nu_t,\nu_{t-1},\varepsilon_t,\varepsilon_{t-1},\eta_{it},\eta_{it-1})$$

$$E_t^i\nu_{t-1} = f_{\nu 1}(\nu_t,\nu_{t-1},\varepsilon_t,\varepsilon_{t-1},\eta_{it},\eta_{it-1}) \tag{13}$$

$$E_t^i\varepsilon_t = f_{\varepsilon 0}(\nu_t,\nu_{t-1},\varepsilon_t,\varepsilon_{t-1},\eta_{it},\eta_{it-1})$$

$$E_t^i\varepsilon_{t-1} = f_{\varepsilon 1}(\nu_t,\nu_{t-1},\nu_{t-2},\varepsilon_t,\varepsilon_{t-1},\varepsilon_{t-2},\eta_{it},\eta_{it-1},\eta_{it-2}),$$

where the f functions are linear under normality. In all cases, the coefficients are functions of the parameters characterizing s_{it}, z_t^a, and (A_0,A_1,B_0,B_1). Since the variance σ_η^2 is common to all traders, the functions in (13) are not indexed by i.

Returning to the equilibrium price relation (11), the market-wide average forecasts are obtained by integrating the optimal forecasts in (13) across all traders. For any fixed j, ν_{t-j} , ε_{t-j}, and c_t are constant functions of i. Furthermore, the η_{it} are independent across i and, therefore, $\int_0^1 \eta_{it-j}d\mu(i) \equiv 0$ (see footnote 7). Thus, the market-wide average forecasts will depend only on current and past values of ν_t, ε_t, and u_t (which confirms the conjecture in (10) that p_t^D is a function only of these variables). Next, the conditional variance of p^D_{t+1} is calculated as

$$(14) \qquad Var^i_t p^D_{t+1} = A_0^2 \sigma_\nu^2 + A_1^2 Var^i_t \nu_t + A_2^2 Var^i_t \nu_{t-1} + B_0^2 \sigma_\varepsilon^2 + B_1^2 Var^i_t \varepsilon_t + B_2^2 Var^i_t \varepsilon_{t-1} + [\psi\lambda^D/(1-\psi\lambda^D)]^2 \sigma_u^2,$$

where the conditional variances of the disturbances are also functions of the A_j, B_j, and the parameters characterizing the processes in (6) and $\{\eta_{it}\}$. The conditional variance of c_{t+1} is σ_u^2, so $\delta^D = Var^i_t(p^D_{t+1}+c_{t+1})$ equals the expression in (14) plus $[1 + 2\lambda^D\psi/(1-\lambda^D\psi)]\sigma_u^2$. The resulting value of δ^D is used to calculate a new value for λ^D.

Finally, the coefficients on the ν_{t-j} and ε_{t-j} in $\lambda^D\int_0^1 E^i_t p^D_{t+1} d\mu(i)$ and $-\lambda^D\delta^D\gamma(\theta_t + \varepsilon_t)$ are combined to get a new set of moving average parameters (the coefficients for the latter term will depend on whether the AR or MA representation of θ_t is adopted). Equating these coefficients to the corresponding coefficients in (9) and using (14) yields a set of (nonlinear) relations that can be solved for the undetermined coefficients A_j and B_j. This is accomplished by repeating the process just described until the conjectured values of the coefficients equal the values in the derived representation (11). When this occurs, a fixed point in the space of moving average representations for p^D_t has been reached. Numerical examples of such equilibrium price representations are presented in section four.

3. Properties of the Equilibrium Price Representation

It is instructive to compare the time series representations for prices from the model with disparate expectations to the representations implied by related models in which agents have partial or full information. Suppose that $\gamma_i = \gamma$ and $\mathrm{Var}(\eta_{it}) = \sigma_\eta^2$, for all $i \in [0,1]$. Also, suppose all agents have full current information, so their information set is $\Phi_t^F = \sigma a\ \{\nu_{t-j}, \varepsilon_{t-j}, c_{t-j}: j \geq 0\}$. Then, when $\{\theta_t\}$ follows an AR(1) process, the equilibrium price is given by

$$(15) \qquad p_t^F = \frac{-\lambda^F \delta^F \gamma}{(1-\lambda^F \rho)}\ \theta_t - \lambda^F \delta^F \gamma \varepsilon_t + \frac{\lambda^F \psi}{(1-\lambda^F \psi)}\ c_t,$$

where $\delta^F \equiv \mathrm{var}[p_t^F + c_t \mid \Phi_{t-1}^F]$ and $\lambda^F \equiv 1/[\xi \delta^F \gamma + \alpha]$. The price p_t^F is a function only of the current value of ε, since only ε_t affects the aggregate holdings of securities by nonspeculative traders (z_t^a). In contrast, the price p_t^D is affected by ε_t and ε_{t-1} (i.e., $B_0 \neq 0$, $B_1 \neq 0$), because $E_t^i \varepsilon_{t-j}$ and $E_t^i \nu_{t-j}$ $(j = 0,1)$ depend on ε_{t-1}, for all $i \in [0,1]$.[9] Longer lags of ε_t do not affect p_t^D, because $B_2 = 0$ in (11) for all values of the primitive parameters and, therefore, $E_t^i \varepsilon_{t-1}$ does not enter (11). Similarly, $B_j = 0$ for $j \geq 3$, so $B^*(L) = 0$. The number of additional MA terms in the representation for p_t^D (here one) is linked directly to the informational lag (two periods). An informational lag of k periods adds k-1 to the order of the MA component of p_t involving $\{\varepsilon_t\}$. This feature of time series models with disparately informed agents was stressed previously by Townsend (1983b).

Notice, however, that the more persistent effect of ε_t on p_t^D is due to the assumption that all agents have incomplete information about ν_t and ε_t for two periods, and not to the presence of disparate expectations per se. For suppose all agents are partially, but homogeneously, informed and receive a common signal s_t about θ_t. Then the equilibrium price in this economy (p_t^H) will be given by a version of (11), with the expectations in (13) augmented to include the η_{t-j} (since they are now common shocks). Furthermore, calculating the optimal forecasts in this case proceeds exactly as in the model with disparate expectations.

Digressing briefly, these observations imply that an observational equivalence property across alternative information structures is satisfied by an important class of linear models. Specifically, suppose aggregate demand is given exogenously by a linear function of prices and that the decision rules of firms depend linearly on expected future prices, but not on higher order moments of prices. Then the coefficients in the decision rules of firms on the aggregate shocks in a model with disparately informed firms will be identical to those on the same shocks in the corresponding model with homogeneously and imperfectly informed firms. In particular, the response of the equilibrium price to shocks in Townsend's (1983a,b) linear-quadratic, symmetric information models of firm behavior are identical to the responses that would be obtained from the corresponding model with homogeneous information. An analogous observational equivalence obtains for models of consumers facing linear supply functions.

The asset pricing model described in section two is not a member of the class of models for which this observational equivalence obtains. The reason for this is that the time series representation of the equilibrium price depends on the conditional second moment of next period's price, which is not invarient to the information structure. More precisely, p_t^H is a function of the common measurement error η_t, while p_t^D is not a function of the idiosyncratic errors η_{it}. Consequently, $Var_t^i p_t^D \neq Var_t^i p_t^H$ and, hence, $\delta^D \neq \delta^H$. The moving average coefficients of p_t^D and p_t^H depend on δ^D and δ^H, respectively, as follows. When θ_t follows an MA(2) process, then the coefficients on the ν's in the expression for p_t^D are $A_j = 0$ for $j \geq 3$

and $A_2 = -\lambda^D \gamma \delta^D \phi_2$. On the other hand, when θ_t follows an AR(1) process, then $A_j = -\lambda^D \delta^D \gamma \rho^j / (1-\lambda^D \rho)$ for $j \geq 3$ and $A_2 = \lambda^D A_3 - \lambda^D \delta^D \gamma \rho^2$. The corresponding coefficients for p_t^H have the same form with λ^H and δ^H in place of λ^D and δ^D, respectively. In sum, the shocks ε_t and ν_t may have very different effects on prices in the full, partial, and disparate information economies. For comparison, the coefficients of the time series representation of p_t^H are also calculated in section four.

There are two related partial information economies that are also of interest. First, suppose that $\sigma_\varepsilon^2 = 0$ so that only ν_t affects aggregate nonspeculative supply, z_t^a. (As in the original model, agents receive different signals s_{it}.) For this case, ε_t can be removed from (12) and p_t^* can be replaced by $\nu_t + \phi_1 \nu_{t-1}$. Then in solving for the undetermined coefficients in (10) we can set $B_j = 0$, for $j \geq 0$. Furthermore, given the entire past history of p_t, it is now possible to infer ν_t. In other words, $\{p_{t-j}^D : j \geq 0\}$ and $\{\nu_{t-j} : j \geq 0\}$ span the same linear space and all agents are fully informed (the signals s_{it} are redundant). Consequently, p_t^D is given by a special case of the full information solution (15) with $\varepsilon_t = 0$, for all t. Evidently, a nondegenerate distribution for ε_t prevents prices from being fully revealing of the unobserved disturbances.

Second, it is instructive to examine the role of uncertain coupon payments in the context of the model (9). If $\sigma_u^2 = 0$ and $c_t = \bar{c}$, for all t, then the traded security is no longer risky. That is, a constant price $\bar{p} = (\bar{c} / \bar{r})$ solves the first-order condition (3) and, hence, is an equilibrium price in this setting. The traded security is a riskless consol that is a perfect substitute for the riskless security with return $\bar{r}$ and, therefore, in equilibrium $\bar{r} = (\bar{c} / \bar{p})$, the one-period return on the consol. It follows that with $\sigma_u^2 = 0$ and $\bar{p} = (\bar{c} / \bar{r})$, the fact that agents are disparately informed has no effect on the equilibrium price $\bar{p}$.

The constant price equilibrium is not the only rational expectations equilibrium solution to this model, however. If all agents believe that past prices convey information about current and past shocks, and hence future prices, then prices will in fact have a nontrivial moving average representation in terms of current and past shocks ν and ε. Thus, there are multiple rational expectations equilibria for this model. Note that the equilibrium in which p_t^D has a nonzero variance is not a "bubble" equilibrium of the type discussed by Taylor (1977) and McCallum (1984). Agents are not conditioning on non-fundamental information. The presence of multiple equilibria is a "knife-edge" phenomenon in the context of (9). If $\sigma_u^2 > 0$ (c_{t+1} is not perfectly forecastable), then $\delta^D = \mathrm{Var}_t^i(p_{t+1}^D + c_{t+1}) > 0$. Consequently, a constant price is not an equilibrium price. Indeed, even if σ_u^2 is small, the variance of p_t^D may be relatively large, because θ_t and ε_t may dominate the behavior of prices. The price p_t^D has a nontrivial time series representation because, with $\sigma_u^2 > 0$, p_t^D is a function of θ_t and ε_t and past prices convey information about future prices. Of course, even if $\sigma_u^2 > 0$, there are bubble equilibria for this model, since traders are "myopically rational" with a one-period investment horizon (see Tirole 1982). In section four, attention is restricted to the non-bubble solutions for p_t^D.

4. Time Series Implications of the Model

A minor modification of the model with disparately informed agents will be useful for presenting numerical examples. Suppose that there are two distinct subsets of traders each with positive measure, that have different risk characteristics or qualities of information (as measured by $\sigma_{\eta_i}^2$). Then the equilibrium price is obtained by integrating over the subspaces of agents:

$$(16) \qquad p_t^D = \lambda^*[\int_0^\Lambda E_t^i p_{t+1}^D d\mu(i)/(\gamma_1\delta_1) + \int_\Lambda^1 E_t^i p_{t+1}^D d\mu(i)/(\gamma_2\delta_2)]$$

$$- \lambda^*(\theta_t + \varepsilon_t) + \lambda^*\psi c_t,$$

where $0 \le \Lambda \le 1$; traders indexed by $i \in [0,\Lambda]$ have parameters γ_1, δ_1, and $\sigma^2_{\eta_1}$; traders indexed by $i \in (\Lambda,1]$ have parameters γ_2, δ_2, and $\sigma^2_{\eta_2}$; and

$$\lambda^* = \left[\xi + \alpha[\frac{\Lambda}{\gamma_1\delta_1} + \frac{(1-\Lambda)}{\gamma_2\delta_2}] \right]^{-1}.$$

To shed more light on the quantitative properties of the price process under disparate expectations, the equilibrium moving average representations and several population moments were calculated using hypothetical parameter values. These calculations differ in several important respects from the comparative static analysis in Hellwig (1980). Perhaps the most important difference is that δ^D and δ^H represent the conditional variances of endogenous variables (p_t^D and p_t^H), whereas the payoff from the security in Hellwig's model is drawn from an exogenous distribution. A change in any parameter that alters the time series representation for price in the model in this paper will also change the conditional variance. Thus, unlike in many previous studies, the conditional variance cannot be held fixed when comparing models.

Table 1 displays the parameters for the various models considered. For all the models with θ_t following an MA process (Models 1-11), σ_ν=1., α=1.02, and c_t follows an AR(1) process with autoregressive parameter ψ = .7 and innovation variance σ^2_u = .1. For the first four models, the parameter σ_ε was set at unity and θ_t was assumed to follow a moving average process with parameters $\phi_1 = .8$ and $\phi_2 = .64$. This MA process is a truncated version of an AR(1) process with decay parameter $\rho = .8$. The implied variance of θ_t is 2.05, since $\sigma_\nu^2 = 1$. The quality of an individual trader's signal depends on the relative values of σ_η^2 and 2.05; the smaller is $\sigma_\eta^2 / \sigma_\theta^2$, the better informed is the trader. The variation in the coupon payment is made small relative to the variation in $\{\theta_t\}$ and $\{\varepsilon_t\}$ in order to focus on the implications for price movements of disparate beliefs about the actions of "non-speculative" traders. This is consistent with the fact that the profits of bond traders are determined largely by the movements in prices and not by variation in coupon income (McCurdy 1979).

The parameter ξ was set at 1.5. With $\xi > 0$, upward pressure on prices induces an increase in nonspeculative supply. This supply, in turn, attenuates the upward pressure on prices. I would expect ξ to be positive if z_t^a represents a trading rule of the Federal Reserve and the Open Market Committee gives positive weight to low variation in interest rates in their objective function. On the other hand, the sign of ξ might be changed by the presence of other types of nonspeculative suppliers. In the absence of a more detailed model of z_t^a, I shall proceed under the assumption that $\xi > 0$.

The nonlinear equations that must be solved for the undetermined coefficients (A_0, A_1, B_0, B_1) are cubic equations. Accordingly there may be multiple equilibria for some configurations of the parameters. The possibility of multiple equilibria in rational expectation models in which conditional variances enter demand equations was illustrated by McCafferty and Driskill (1980) in the context of a speculative model of inventory holdings. I have been unable to demonstrate that there are unique solutions for the models displayed in Table 1. However, for these models, the successive approximations method always converged rapidly to a moving average representation with coefficients that are insensitive to their initial values over a broad range of initial values. Furthermore, for some other combinations of parameters, the successive approximations method did not converge, which suggests both that some models will have either no or multiple linear equilibria and that for these models the solution method fails. Finally, when $\sigma_u^2 = 0$ and there are (at least) two equilibria, the successive approximations method found two different linear equilibria depending on which initial values were used. These observations provide some assurance that I have found unique linear equilibria for the models displayed in Table 1.

The parameters of the equilibrium price and several descriptive statistics for Models 1 through 11 are displayed in Table 2. For the ℓth model, the row ℓ.D in Table 2 displays the moving average coefficients for p_t^D; Var p_t^D and (δ_1^D, δ_2^D); the first two autocorrelation coefficients of p_t^D; and the variances of the stochastic "inputs" into the price process (see, e.g., (11)). The row ℓ.H displays the corresponding statistics for p_t^H. The objective here is to compare the quantitative features of alternative specifications of the model and, therefore, the parameters were chosen simply to provide benchmark sets of results for the purpose of comparisons.

For Model 1, Λ is set at unity so there is only one type of trader (i.e., traders have common values of γ and σ_η^2), $\gamma = 2.$, and $\sigma_\eta = 2.$ Model 2 is identical to Model 1 except for the value of the risk aversion parameter. A lower value of γ leads to a smaller variance for p_t^D. Providing an interpretation of this finding is complicated by the dependence of net speculative demand and nonspeculative supply on p_t^D. Intuitively, an increase in nonspeculative supply due to say an increase in ν_t puts downward pressure on prices. With $\xi > 0$, the price pressure induces a reduction in supply which partially offsets the supply shock ν_t. Now a reduction in γ makes demand more sensitive to price changes (see (4)). Thus, as γ falls, prices must respond less to a given supply shock in order to clear the market, and, therefore, price is less variable. Lower risk aversion also leads to smaller autocorrelations for p_t^D.

Next, consider the consequences of reducing σ_η^2. In Model 3, σ_η is set at .75 (with γ=1.), which implies that $\sigma_\eta^2/\sigma_s^2 = .215$ ($\sigma_s^2 \equiv \mathrm{Var}\, s_{it}$) and $\mathrm{Var} p_t^D = .463$. The noise to signal ratio for the comparable Model 2 is $\sigma_\eta^2/\sigma_s^2 = .66$, and Var p_t^D is .481 for this model. Thus, a uniform increase in the quality of each trader's signal leads to a decrease in the variance of the equilibrium price. This result is explained by the decline in δ_1^D with a reduction in σ_η. A smaller conditional variance also leads to relatively more sensitive demands for a given expected price change and, hence, to lower volatility in prices. The roles of γ and δ are not entirely symmetric, however. A smaller value of σ_η also increases the autocorrelations in p_t^D. Put differently, for this θ_t process, a uniform increase in the quality of traders' information leads to a less choppy price process, while decreasing risk aversion leads to a more choppy process.

In Model 4 there are two types of investors. Ninety percent of the investors have preferences with $\gamma_1 = 2$ and $\sigma_{\eta 1}=2$, while the remaining ten percent have $\gamma_2 = 1$ and $\sigma_{\eta 2}=.75$. Consistent with previous results, the price is less volatile relative to the comparable Model 1 where all investors have $\gamma = 2.$ and $\sigma_\eta=2$. The decline in the variance of p_t^D (from 1.02 to .971) is quite small, however. Solving the comparable model with (γ_1= 1., $\sigma_{\eta 1}$= 2) and (γ_2 = .5, $\sigma_{\eta 2}$ = .75), gives $\mathrm{Var} p_t^D = .348$. Comparing this to $\mathrm{Var} p_t^D$=.481 from Model 2, it follows that halving the coefficient of absolute risk aversion for ten percent of the traders leads to a larger percentage decline in $\mathrm{Var}p_t^D$ at low levels of risk aversion. In both cases, the effects on the autocorrelations are small.

In sum, the presence of a small group of relatively well informed and more risk tolerant traders reduces the volatility of the price of the risky asset. A similar result is obtained in the "large market" models in Hellwig (1980) and Admati (1985). This finding is interesting in light of the recent analyses of Kyle (1984) and Altug (1984) of one-period models in which a finite number of agents behave strategically. They found that a trader with inside information may act to conceal this information and in doing so reduce the variance of the price. Here it has been shown that a model with competitive traders leads to a qualitatively similar result, but for different reasons. Furthermore, in the competitive model, the primary source of reduced volatility is the lower risk aversion, and not the better information.[10]

In Models 5-7, $\{\theta_t\}$ follows an MA(2) process with coefficients $\theta_1 = -.8$ and $\theta_2 = .64$ (and $\sigma_\varepsilon = 1.$). This MA process is a truncated version of an AR(1) process with autoregressive parameter $\rho = -.8$. The two MA representations considered lead to θ processes with the same variances, but different autocorrelations. The relatively choppy MA process $(\phi_1=-.8,\ \phi_2=.64)$ leads to a smaller variance for p_t^D (compare Model 1.D and 5.D). Another difference across the two MA processes is that the first autocorrelation coefficient is negative when $\phi_1=-.8$. Qualitatively, the effects on the variances of prices of changing γ or σ_η are the same as in the first four models.

Consider next the properties of p_t^H, obtained when agents are homogeneously, but imperfectly informed. The equilibrium time series representations for p_t^H involve both the coefficients (A_0, A_1, B_0, B_1) and the coefficients F_0 and F_1 on η_t and η_{t-1}, respectively. For Models 1 through 4, the variance of p_t^H is greater than or equal to the variance of p_t^D. However, a notable feature of the results is that in all cases the differences between these variances are small. This similarity is a consequence of the similarity in the corresponding MA coefficients and the relatively small values of F_0 and F_1. Note also that the autocorrelations for p_t^D and p_t^H are also very similar. Thus, for the models examined, the time series properties of p_t^D are induced largely by the incompleteness of information rather than the disparate nature of information.

In Models 5 - 7, the variance of p_t^D is greater than the variance of p_t^H, but again the differences are small. Evidently, when nonspeculative supply is choppy, differential information leads to a more volatile price process than homogeneous, partial information, in this economic environment.

There is another interesting feature of the variances. The column of Table 2 labeled Var(INP) displays the variances of the stochastic "inputs" into the difference equation for p_t^D and p_t^H [see (16)]. A question that is often asked in the context of linear rational expectations models is whether the variance of the "output" variable (here price) exceeds the variance of the "input" variable (here $\lambda^*(\theta_t + \varepsilon_t) - \lambda^*\psi c_t$). In some contexts it can be shown that as long as the inputs are not explosive processes, the variance of the output must be less than the variance of the input. This observation underlies the volatility tests of, for instance, expectations theories of the term structure of interest rates (Shiller 1979, Singleton 1980) and the monetary model of exchange rate determination (Meese and Singleton 1983). Interestingly, $\mathrm{Var}p_t^D$ is not always less than the sum of the variances of the "inputs" in the security model being investigated here. For the MA process ($\phi_1 = -.8$, $\phi_2 = .64$) the sum of $\mathrm{Var}(\lambda^*\theta_t)$, $\mathrm{Var}(\lambda^*\varepsilon_t)$, and $\mathrm{Var}(\lambda^*\psi c_t)$ is greater than $\mathrm{Var}(p_t^D)$. However, for the MA process ($\phi_1 = .8$, $\phi_2 = .64$) the variances of p_t^D and p_t^H exceed the sum of the variances of the inputs. Thus, under disparate expectations, a smooth trading rule by nonspeculative traders leads to a price process that is more volatile than the input process, while the variability of the price process is attenuated for a choppy trading rule.

That $\mathrm{Var}p_t^D$ may exceed $\mathrm{Var}[\lambda^*(\theta_t+\varepsilon_t-\psi c_t)]$ is attributable to the risk aversion of traders. Shiller (1981) showed that the variance of the price of a security is less than or equal to the variance of the dividend under the implicit assumption of risk neutrality. In the context of a model with logarithmic utility, Michener (1982) demonstrated that Shiler's bound may be violated if agents are risk averse. The results in Table 2 provide another example of this fact in the context of a model with exponential utility and differential or partial information. Notice that for this model the ratio of $\mathrm{Var}\ p_t^D$ to $\mathrm{Var}[\lambda^*(\theta_t+\varepsilon_t-\psi c_t)]$ increases as γ decreases (compare models 1 and 2).

The time series representations of prices for several models with $\{\theta_t\}$ following autoregressive processes are displayed in Table 3. The parameters for these models are different than those for Models 1-7. Specifically, $\sigma_\nu^2 = 2.$ and $\sigma_\varepsilon^2 = .5$, which gives $\sigma_\theta^2 = 4.$ Also, $\sigma_u^2 = .2$ and $\psi = .8$. These parameters were chosen to induce larger autocorrelations in prices and to assure that a majority of the variation in prices is attributable to variation in θ.

Models 8, 9, and 10 correspond to Models 1, 2, and 4 in Table 2, and Models 11 and 12 correspond to Models 5 and 6. Qualitatively, the implications for prices of lowering risk aversion or increasing the quality of traders' signals are the same. One difference across the two sets of results in that $(\mathrm{Var}_t^i p_{t+1}^D / \mathrm{Var}\ p_{t+1}^D)$ is much smaller for the AR models than for the MA models. Evidently, the conclusions drawn from Models 1-4 are insensitive to this ratio.

Finally, a comparison of the variances of the full information price p_t^F to the corresponding variances of p_t^D is interesting. For the MA representation $(\phi_1 = .8,\ \phi_2 = .64)$, Var p_t^D is larger than Var p_t^F for $\gamma = 2$, but Var $p_t^D <$ Var p_t^F for $\gamma = 1$.[11] Thus, for low levels of risk aversion, disparate expectations does not lead to a more volatile price in this model. On the other hand, Var $p_t^D >$ Var p_t^F for $\gamma \geq 1$ when θ follows the AR(1) model with $\rho = .8$. Thus, if much of the variation in nonspeculative supply is due to persistent shocks about which there is partial information, the price may be much more volatile than what would be observed in a full information economy even if speculative traders do not exhibit much risk aversion.

5. Discussion and Extensions

In the context of a simple model of security prices, it has been shown that both the variance and autocorrelations of prices are affected by the presence of disparately informed traders. Precisely how prices are affected depends critically on whether the trading rule of the nonspeculative traders is smooth or choppy. When the persistent shock to the positions of nonspeculative traders is "smooth" and these positions are positively related to the current price, security prices may be much more volatile than the variables which determine equilibrium prices. Moreover, the price has a larger variance and is more choppy than the price in the corresponding model with full current information. Finally, introducing traders with relatively low levels of absolute risk aversion and high quality information leads to a decline in price variability. These orderings among second moments are typically reversed in the case of "choppy" forcing variables. Many of the aggregate economic variables that affect trading over the business cycle (e.g. output, unemployment, inflation) are in fact quite smooth. If the time series characteristics of these variables are inherited by the trading rules of the nonspeculative traders, then the case of smooth forcing variables may be relevant for modeling security markets in the U.S. However, in light of the sensitivity of the results to the specification of nonspeculative traders, a more systematic analysis of their objective functions is an important topic for future research.

Another interesting finding is that the equilibrium prices for the models with disparate information and partial, homogeneous information follow very similar time series processes. It remains to be seen whether this similarity carries over to alternative parameterizations and information structures. Based on the findings to date, however, it appears that disparate information per se in a competitive market does not significantly effect the equilibrium price process. That is, imperfect information is the primary source of the difference between models with complete current information and imperfect and disparate information.

There are many extensions of the simple model considered here that warrant investigation. The assumption that traders have a one-period investment horizon is certainly restrictive. Traders are concerned about their inventory holdings over time. Furthermore, they surely are trying to anticipate economic developments several periods ahead when making their investment decisions in the current period. As noted above, another important extension is the development of a more complete model of the trading activity of nonspeculative traders. This would lead to more readily interpretable disturbances and possibly allow for a more direct link of the quantitative properties of the model to the properties of the data for U.S. securities markets.

Appendix

Derivation of Fundamental Moving Average Representation for the Model in Section 2

This appendix derives the fundamental moving average for y_t used in sections two through four. The basic method used to obtain this representation is the one suggested by Townsend (1983a). The details differ because the model he considers is different, and Townsend assumes that all of the shocks have unit variances whereas I allow the shocks to have different variances.

The autocovariance generating function for y_t, $g_y(z)$, can be expressed as

$$(A.1) \qquad g_y(z) = M(z)\Omega M(z)^*,$$

where z is a complex number, Ω is the diagonal covariance matrix of ω_t, and "*" denotes transposition and conjugation. We seek a representation of y_t of the form $y_t = M^{\#}(L)\omega_t^{\#}$ such that $\omega_t^{\#}$ is fundamental for y_t and $g_y(z) = M^{\#}(z)M^{\#}(z)^*$.

Toward this end, rewrite y_t as $N(L)\tilde{\omega}_t$; where

$$(A.2) \qquad N(L) = M(L)\Omega^{\frac{1}{2}} = \begin{bmatrix} 0 & 0 & L^2\sigma_\varepsilon \\ L^2\sigma_\nu & 0 & 0 \\ (1+\phi_1 L)\sigma_\nu & \sigma_\eta & 0 \\ (A_0+A_1L)\sigma_\nu & 0 & (B_0+B_1L)\sigma_\varepsilon \end{bmatrix}$$

$$(A.3) \qquad \tilde{\omega}_t = \Omega^{-\frac{1}{2}}\omega_t = \begin{bmatrix} \nu_t/\sigma_\nu \\ \eta_t/\sigma_\eta \\ \varepsilon_t/\sigma_\varepsilon \end{bmatrix}$$

The rank of $N(z)$ evaluated at $z=0$ is less than three. A matrix of full rank at all z inside the unit circle is obtained by postmultiplying $N(z)$ by a series of Blashke matrices. More precisely, let

$$\text{(A.4)} \qquad B(z) = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let W and $\hat{W}$ be 3x3 matrices satisfying $WW'=I$ and $\hat{W}\hat{W}'=I$. Then it will be shown that W and $\hat{W}$ can be chosen such that

$$\text{(A.5)} \qquad M^{\#}(L) = N(L)WB(L)\hat{W}B(L)$$

$$\text{(A.6)} \qquad \omega_t^{\#} = B(L^{-1})'\hat{W}'B(L^{-1})'W'\tilde{\omega}_t$$

gives the desired fundamental moving average representation.

The matrix W is chosen such that the first column is given by $[\sigma_\eta/s \;\; -\sigma_\nu/s \;\; -A_0\sigma_\nu\sigma_\eta/(B_0\sigma_\varepsilon s)]'$, where

$$s = [\sigma_\eta^2 + \sigma_\nu^2 + \frac{A_0^2\sigma_\nu^2\sigma_\eta^2}{B_0^2\sigma_\varepsilon^2}]^{\frac{1}{2}}.$$

Using this vector as the first column of W, a Gram-Schmidt orthogonalization procedure can be used to construct the remaining columns. This procedure yields the matrix

$$\text{(A.7)}\quad W = \begin{bmatrix} \sigma_\eta/s & d & 0 \\ -\sigma_\nu/s & \dfrac{\sigma_\eta\sigma_\nu}{s^2 d} & \dfrac{A_0\sigma_\nu\sigma_\eta}{B_0\sigma_\varepsilon s d} \\ \dfrac{-A_0\sigma_\nu\sigma_\eta}{B_0\sigma_\varepsilon s} & \dfrac{\sigma_\eta^2 A_0\sigma_\nu}{B_0\sigma_\varepsilon s^2 d} & \dfrac{-\sigma_\nu}{sd} \end{bmatrix},$$

where $d = (1 - \sigma_\eta^2/s^2)^{\frac{1}{2}}$. Next, the matrix $N^{\#}(z) = N(z)WB(z)$ is formed:

$$\text{(A.8)}\quad N^{\#}(z) = \begin{bmatrix} \dfrac{-zA_0\sigma_\nu\sigma_\eta}{B_0 s} & \dfrac{z^2 A_0\sigma_\nu\sigma_\eta^2}{B_0 s^2 d} & \dfrac{-z^2\sigma_\nu\sigma_\varepsilon}{sd} \\ z\sigma_\nu\sigma_\eta/s & z^2\sigma_\nu d & 0 \\ \phi_1\sigma_\nu\sigma_\eta/s & \dfrac{\sigma_\nu+\phi_1 z\sigma_\nu d}{d} & \dfrac{A_0\sigma_\nu\sigma_\eta^2}{B_0\sigma_\varepsilon sd} \end{bmatrix}$$

The rank of the matrix $N^{\#}(0)$ is still less than three, although one of the zeros of the determinant of the lower 3x3 matrix has now been "flipped" outside the unit circle.

Similar steps lead to a matrix $M^{\#}(z)$ that has full rank at $z=0$. Evaluating $N^{\#}(z)$ at $z=0$ gives

$$N^{\#}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_1\sigma_\nu\sigma_\nu/s & \sigma_\nu/d & \dfrac{A_0\sigma_\nu\sigma_\eta^2}{B_0\sigma_\varepsilon sd} \\ \dfrac{\sigma_\nu\sigma_\eta}{sB_0}(B_0A_1-A_0B_1) & \dfrac{A_0\sigma_\nu}{d} & \dfrac{-B_0\sigma_\nu\sigma_\varepsilon}{sd} \end{bmatrix} .$$

We seek a matrix $\hat{W}$ satisfying $\hat{W}\hat{W}' = I$ and the first column of $N^{\#}(0)\hat{W}$ is the zero vector. Let

$$\zeta = [\frac{\sigma_\eta}{sB_0}(A_1B_0-A_0B_1) + \frac{B_0^2\sigma_\varepsilon^2\phi_1}{A_0\sigma_\eta s}]/[\frac{A_0}{d} + \frac{B_0^2\sigma_\varepsilon^2}{A_0\sigma_\eta^2 d}],$$

$$\zeta^* = \frac{B_0\sigma_\varepsilon\phi_1 d}{A_0\sigma_\eta} - \frac{B_0\sigma_\varepsilon s}{A_0\sigma_\eta^2}\zeta ,$$

$$\hat{s}^2 = 1 + \zeta^2 + \frac{B_0^2\sigma_\varepsilon^2 s^2 d^2}{A_0^2\sigma_\nu^2\sigma_\eta^4}[\frac{\phi_1\sigma_\nu\sigma_\eta}{s} - \frac{\sigma_\nu}{d}\zeta]^2 .$$

Then a matrix that satisfies these conditions is given by

$$\text{(A.9)}\qquad \hat{W} = \begin{vmatrix} -1/\hat{s} & (1-1/\hat{s}^2)^{\frac{1}{2}} & 0 \\ \zeta/\hat{s} & \dfrac{\zeta/\hat{s}^2}{(1-1/\hat{s}^2)^{\frac{1}{2}}} & \left|1 - \dfrac{\zeta^2/\hat{s}^2}{(1-1/\hat{s}^2)}\right|^{\frac{1}{2}} \\ \zeta^*/\hat{s} & \dfrac{\zeta^*/\hat{s}^2}{(1-1/\hat{s}^2)^{\frac{1}{2}}} & -\dfrac{\zeta}{\zeta^*}\left|1 - \dfrac{\zeta^2/\hat{s}^2}{(1-1/\hat{s}^2)}\right|^{\frac{1}{2}} \end{vmatrix}$$

Forming $M^{\#}(z) = N^{\#}(z)\hat{W}B(z)$ gives a fairly complicated matrix. What is needed for interpreting the equilibrium price relation are the orders of the elements of the first two row of $N^{\#}(z)$. The elements of the first two rows take the form:

$$\text{(A.10)} \qquad \begin{vmatrix} M_{11}^{0}+M_{11}^{1}z & M_{12}^{1}z+M_{12}^{2}z^{2} & M_{13}^{2}z^{2} \\ M_{21}^{0}+M_{21}^{1}z & M_{22}^{1}z+M_{22}^{2}z^{2} & M_{23}^{2}z^{2} \end{vmatrix} .$$

The vector $\omega_t^{\#}$ is given by $B(L^{-1})'\hat{W}'B(L^{-1})W'\tilde{\omega}_t$. Upon working out the expressions for the elements of $\omega_t^{\#}$ in terms of the primitive parameters, it is seen that $\omega_t^{\#}$ takes the form (ignoring the terms involving η_{it}, η_{it-1}, and η_{it-2}):

$$\text{(A.11)} \qquad \omega_t^{\#} = \begin{vmatrix} c_{\nu 1}^{1}\nu_{t-1} + c_{\nu 2}^{1}\nu_{t-2} + c_{\varepsilon 1}^{1}\varepsilon_{t-1} + c_{\varepsilon 2}^{1}\varepsilon_{t-2} \\ c_{\nu 0}^{2}\nu_{t} + c_{\nu 1}^{2}\nu_{t-1} + c_{\varepsilon 0}^{2}\varepsilon_{t} + c_{\varepsilon 1}^{2}\varepsilon_{t-1} \\ c_{\nu 0}^{3}\nu_{t} + c_{\varepsilon 0}^{3}\varepsilon_{t} \end{vmatrix}$$

Using (A.10), (A.11), and the Wiener-Kolmogorov prediction formulas

$$\text{(A.12)} \qquad E_t^i y_{t+1} = \left[\frac{M^{\#}(L)}{L}\right]_+ \omega_t^{\#} \quad ;$$

and

$$\text{(A.13)} \qquad E_t^i y_{t+2} = \left[\frac{M^{\#}(L)}{L^2}\right]_+ \omega_t^{\#} \; ,$$

where $[\ \ ]_+$ is the annihilation operator which says ignore negative powers of L, it can be shown that

$$E_t^i \nu_t = f_{\nu 0}(\nu_t, \nu_{t-1}, \varepsilon_t, \varepsilon_{t-1}, \eta_{it-1}, \eta_{it-1})$$

$$\text{(A.14)} \qquad E_t^i \nu_{t-1} = f_{\nu 1}(\nu_t, \nu_{t-1}, \nu_{t-2}, \varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \eta_{it-1}, \eta_{it-2})$$

$$E_t^i \varepsilon_t = f_{\varepsilon 0}(\nu_t, \nu_{t-1}, \varepsilon_t, \varepsilon_{t-1}, \eta_{it}, \eta_{it-1})$$

$$E_t^i \varepsilon_{t-1} = f_{\varepsilon 1}(\nu_t, \nu_{t-1}, \nu_{t-2}, \varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \eta_{it}, \eta_{it-1}, \eta_{it-2}),$$

where the f functions are linear.

The conditional variances of the prices were calculated using the standard formulas for the conditional variances of normal random variables.

Footnotes

1. Williams (1977) and Jarrow (1980) have taken a different approach to modeling with heterogeneous beliefs in their extensions of the capital asset pricing model. At least some of the agents in their models do not know all of the parameters of the price process and, therefore, must learn about the parameters over time. Unlike the studies of aggregation of information in securities markets, these studies of capital asset pricing models are not concerned with solving for an endogenous price that agents condition on when forming expectations about future prices.

   A third and closely related literature is represented by the work of Feldman (1983), Detemple (1983), and Gennotte (1984). These studies investigate the implications for asset pricing models of partial information about the underlying state variables. It is shown that, in continuous time and under normality, the unobserved state variables can simply be replaced by agents' best forecast of these variables in deducing equilibrium price representations. These studies also do not address the properties of prices in models where equilibrium prices convey imperfectly the private information to other traders.

2. These independence assumptions could easily be relaxed, although the computations in section four would be somewhat more complicated.

3. Subsequent examples could be modified to incorporate a larger common information set. In particular, the price of the risky security need not be the only commonly observed variable at date t. For illustrative purposes, I shall work with Φ_t^i in (7).

4. A similar structure of uncertainty could be rationalized using the analysis in Siegel (1985). Specifically, measures of real economic growth are published with a lag, but there are economic indicators published before the announcement of the growth figures that are correlated with real growth. Siegel (1985) discusses a signaling interpretation of the effects of money stock announcements on interest rates that is based on such a correlation between money and real income.

5. In practice, some information about the trading activity of the Federal Reserve is available to traders. Given the structure of uncertainty considered here, current knowledge of z_t^a would fully reveal θ_t and ε_t to traders. Full revelation would not be present if there were additional sources of uncertainty, however.

6. The model is not, however, inconsistent with a desire on the part of traders to acquire information, since prices are not fully revealing of θ_t and ε_t.

7. Formally, the integrals in (8) may not be well defined, since realizations of the process $\{E_t^i p_{t+1}^D\}_{i\varepsilon[0,1]}$ need not be measurable [but see Judd 1985]. Throughout this analysis I shall assume that the integrals are well defined and that a version of the strong law of large numbers holds. Specifically, the integral $\int_0^1 \eta_{it} d\mu(i) = 0$, since the η_{it} $(i\varepsilon[0,1])$ are independent.

8. This simple expression for the coefficient on c_t obtains because c_t is common information at date t and is independent of θ_t and ε_t.

9. Of course, if $\{\varepsilon_t\}$ were serially correlated than longer lags of ε_t would appear in (15).

10. If $\xi < 0$, then the variances of p_t^D are much larger than in the corresponding models with $\xi > 0$. Furthermore, Var p_t^D is inversely related to the values of γ and σ_η when $\xi < 0$.

11. What makes this possible, of course, is the fact that the equilibrium prices are functions of their respective conditional variances. Consequently, arguments based on the law of iterated expectations cannot be applied to order the variances for all possible representation of $\{\theta_t\}$ and $\{\varepsilon_t\}$.

TABLE 1

Description of the Models Solved for Equilibrium Prices

Model	Λ	γ_1	γ_2	$\sigma_{\eta 1}$	$\sigma_{\eta 2}$	ξ	ρ	ϕ_1	ϕ_2
1	1	2.	*	2.	*	1.5	*	.8	.64
2	1	1.	*	2.	*	1.5	*	.8	.64
3	1	1.	*	.75	*	1.5	*	.8	.64
4	.9	2.	1.	2.	.75	1.5	*	.8	.64
5	1	2.	*	2.	*	1.5	*	-.8	.64
6	1	1.	*	2.	*	1.5	*	-.8	.64
7	.9	2	1.	2.	.75	1.5	*	-.8	.64
8	1	2.	*	2.	*	1.5	.8	*	*
9	1	1.	*	2.	*	1.5	.8	*	*
10	.9	2.	1.	2.	.75	1.5	.8	*	*
11	1	2.	*	2.	*	1.5	-.8	*	*
12	1	1.	*	2.	*	1.5	-.8	*	*

TABLE 2

Time Series Representations of Equilibrium Prices

Moving Average θ Processes

Model	A_0	A_1	A_2	B_0	B_1	C_0	F_0	F_1	$Var(p_t)$	$Var^i_t p_{t+1}$	Corr (p_t,p_{t-1})	Corr (p_t,p_{t-2})	$Var(INP)^a$	$Var\ p^F_t$
1.D	-.572	-.470	-.322	-.558	-.097	.544	*	*	1.02	.839	.456	.208	.796	.867
1.H	-.574	-.470	-.322	-.558	-.010	.545	-.019	-.026	1.03	.842	.460	.208	.798	*
2.D	-.410	-.337	-.192	-.385	-.067	.266	*	*	.481	.403	.447	.178	.284	.570
2.H	-.420	-.338	-.195	-.382	-.093	.271	-.038	-.037	.500	.419	.455	.178	.292	*
3.D	-.431	-.336	-.185	-.342	.003	.254	*	*	.463	.371	.465	.186	.263	.570
3.H	-.439	-.339	-.188	-.333	.011	.258	-.106	-.063	.475	.382	.473	.187	.271	*
4.D	-.564	-.463	-.310	-.540	.083	.513	*	*	.971	.796/.765	.458	.206	.738	.876
4.H	-.566	-.462	-.310	-.536	-.091	.514	-.030	-.032	.971	.803/.766	.460	.207	.740	*
5.D	-.327	.232	-.247	-.340	-.025	.370	*	*	.365	.292	-.291	.257	.467	.351
5.H	-.327	.227	-.244	-.330	-.025	.364	.004	-.027	.357	.282	-.291	.259	.458	*
6.D	-.090	.051	-.073	-.097	-.018	.086	*	*	.027	.022	-.203	.205	.040	.244
6.H	-.093	.050	-.072	-.091	-.018	.086	-.002	-.014	.027	.021	-.200	.273	.040	*
7.D	-.264	.183	-.209	-.289	-.025	.297	*	*	.252	.203	-.273	.258	.336	.351
7.H	-.277	.176	-.205	-.275	-.024	.289	.007	-.033	.238	.190	-.275	.264	.323	*

a. $VAR(INP) = Var\ (\lambda^*[\theta_t + \varepsilon_t + \psi c_t])$.

TABLE 3

Time Series Representations of Equilibrium Prices

Autoregressive θ Processes

Model	A_0	A_1	A_2	B_0	B_1	C_0	F_0	F_1	$Var(p_t)$	$Var^i_t p_{t+1}$	Corr (p_t, p_{t-1})	Corr (p_t, p_{t-2})	Var(INP)[a]	$Varp^F_t$
8.D	-.648	-.570	-.931	-.641	-.045	.873	*	*	5.59	1.80	.633	.463	2.17	2.24
8.H	-.649	-.569	-.932	-.638	-.047	.874	-.011	-.038	5.60	1.81	.633	.463	2.18	*
9.D	-.627	-.569	-.746	-.612	-.057	.666	*	*	3.99	1.57	.647	.436	1.60	2.10
9.H	-.629	-.566	-.748	-.601	-.667	.668	-.028	-.068	4.02	1.58	.646	.435	1.61	*
10.D	-.647	-.573	-.908	-.634	-.044	.846	*	*	5.37	1.77/1.66	.636	.460	2.10	2.24
10.H	-.648	-.572	-.910	-.630	-.047	.848	-.018	-.048	5.38	1.79/1.66	.636	.461	2.10	*
11.D	-.465	.389	-.148	-.471	.015	.732	*	*	1.39	.803	-.290	.329	1.79	.847
11.H	-.467	.388	-.148	-.467	-.017	.732	-.0006	-.039	1.40	.805	-.290	.328	1.79	*
12.D	-.177	.166	-.055	-.183	.023	.259	*	*	.244	.122	-.419	.351	.422	.666
12.H	-.190	.168	-.057	-.168	-.033	.266	-.022	-.054	.271	.135	-.385	.334	.442	*

a. $Var(INP) = Var\ (\lambda^*[\theta_t + \varepsilon_t + \psi c_t])$.

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