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ABSTRACT

Mean-variance portfolio theory can apply to the streams of payoffs such as dividends following an initial investment, in place of one-period returns. This description is especially useful when returns are not independent over time and investors have non-marketed income. Investors hedge their outside income streams, and then their optimal payoff is split between an indexed perpetuity – the risk-free payoff – and a long-run mean-variance efficient payoff. "Long-run" moments sum over time as well as states of nature. In equilibrium, long-run expected returns vary with long-run market betas and outside-income betas. State-variable hedges do not appear in optimal payoffs or this equilibrium.

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1 Introduction and central results

This paper examines long-horizon portfolio problems and corresponding equilibria. I allow asset-return dynamics, dynamic trading, non-market wealth such as wages, businesses, or real estate, and I allow preference shocks. Markets are incomplete, so investors may not be able completely to hedge outside-income or state-variable shocks.

I focus on the optimal stream of final payoffs, rather than focus on the composition and dynamics of portfolio returns. I find that the stream of final payoffs obeys a classic mean-variance characterization and CAPM equilibrium pricing.

Let $x = \{x_t\}$ denote a stream of payoffs, costing $p(x)$ at time 0. (The Appendix summarizes all notation.) These payoffs can be coupons, dividends, or the payouts from dynamic trading strategies. I define a "long-run expectation" $\tilde{E}$ that sums over time, weighted by a number $\beta < 1$, as well as summing over states of nature, weighted by probabilities,

$$\tilde{E}(x) \equiv \frac{1-\beta}{\beta} E \sum_{t=1}^{\infty} \beta^t x_t. \tag{1}$$

Thus, I write the price of a payoff stream $\{x_t\}$, given a scaled discount factor $m = \{m_t\}$, as

$$p(x) = E \sum_{t=1}^{\infty} \beta^t m_t x_t = k\tilde{E}(mx); \quad k \equiv \frac{\beta}{1-\beta}.$$

With this notation, we can transparently see how the simple ideas that flow from $p = E(mx)$ in one-period models apply to the payoff streams of a multi-period, dynamic environment. We replace E with $\tilde{E}$, and reinterpret the results. For example, 1 is the risk-free payoff. Interpretation: the risk-free payoff to a long-run investor is an indexed perpetuity, which pays a constant real coupon in every date and state of nature. Of course, one can solve the same problems without the $\tilde{E}$ notation. The notation just clarifies intuition by showing how the intertemporal results all map to familiar one-period results.

I apply this notation to portfolio theory. We can write the long-horizon investor's objective $\tilde{E}[u(x)]$ and constraint $W = p(x) = k\tilde{E}(mx)$. This investor chooses an optimal payoff $\hat{x}$ by $u'(\hat{x}) = \lambda m$, where λ is the Lagrange multiplier on initial wealth and m is a stochastic discount factor or set of contingent claim prices.

I specify quadratic utility and hence linear marginal utility $u'(\hat{x}) = c^b - \hat{x}$. As in discrete-time one-period analysis, quadratic utility is necessary for mean-variance results, given that we want to allow dynamic trading and thus non-normal payoffs. Thus, the optimal payoff is $\hat{x} = c^b 1 - \lambda m$ where 1 denotes the riskless payoff. There is a unique traded discount factor $x^* = m$, and linear combinations of traded payoffs are traded. Thus, the optimal (traded) payoff is $\hat{x} = c^b 1 - \lambda x^*$ where 1 denotes the riskless payoff. Proposition 1 derives this statement more carefully, allows non-market income, a stochastic bliss point, and the absence of a riskfree payoff.

Then, I rearrange the terms of this general characterization to express long-run counterparts to traditional results. These more intuitive characterizations are the paper's central point.

An investor who does not have outside income buys a payoff that is split between an indexed perpetuity and a payoff on the long-run mean-variance frontier (Proposition 2). This "long-run" frontier is defined using $\tilde{E}(x)$, $\tilde{E}(x^2)$ and the stream of payoffs to a one-dollar investment (which I call "yields") in the place of $E(R)$, $E(R^2)$ and one-period returns. Long-run variance measures variation over time as well as variation across states of nature.

If all investors are of this type, then the equilibrium market payoff, which is a claim to the aggregate consumption stream, is also on the long-run mean-variance frontier. In this case, each investor's optimal payoff is a linear combination of an indexed perpetuity and the market payoff. The investor's optimal payoff is weighted more towards the perpetuity if his risk aversion is large relative to average risk aversion and vice-versa (Proposition 4).

That a risk-averse long-run investor should hold the indexed perpetuity, that the average investor holds the market payoff, and that others lie somewhere in between (or beyond) are all results that hold for general utility functions. Quadratic utility delivers the idea that optimal payoffs are a linear time-invariant combination of these two payoffs, and the link to long-run mean-variance efficiency.

In this equilibrium, a long-run CAPM holds: each asset's long-run expected yield (payoff divided by initial price) is proportional to its long-run market beta (Proposition 5). Long-run beta measures the long-run covariance of each asset's payoff with the market's payoff (dividend) stream, also summing over dates and states. It is thus a pure "cashflow beta," not a "discount-rate beta."

When investors have outside income, we can define an outside-income hedge payoff by a long-run regression. This hedge payoff is the traded payoff stream which is closest, in the $\tilde{E}(x^2)$ sense, to the investor's outside income stream.

I characterize the resulting optimal risky payoff in several equivalent ways: First, we can say that the investor's total payoff should be long-run mean-variance efficient, where his "total payoff" consists of his asset market payoff plus the hedge payoff he holds implicitly by his ownership of the outside income stream (Proposition 6). Second, we can say that the investor first shorts the outside-income hedge payoff, and then buys a long-run mean-variance efficient asset payoff (equations (39) and (42)). Third, and perhaps most insightfully, the investor modifies his holding of the long-run mean-variance efficient payoff to account for changes in effective risk aversion due to the market and perpetuity exposure of his outside-income stream, and then shorts a zero-cost zero-expected-yield payoff that hedges the idiosyncratic component of his outside income (Proposition 7).

There are similarly several equivalent ways to characterize optimal payoffs in a market of such investors (quadratic utility, outside income). First, each investor's total payoff combines the riskfree payoff and the market total payoff (Proposition 8). Second, each investor's optimal asset payoff combines the market asset payoff, the average outside-

income hedge payoff, and an idiosyncratic outside-income hedge payoff (Proposition 9, and orthogonalized as Proposition 10). The weights in each case contrast the individual with average values. Investors with average outside income and average risk aversion just hold the market asset payoff, despite dynamics in returns and the presence of outside income. Investors with no outside income buy the average outside-income hedge payoff, earning a premium for taking on others' risks.

In equilibrium, long-run expected yields obey a one-factor CAPM using the aggregate total payoff as reference (Proposition 11). We can also break the total market payoff into its asset market component and the hedge payoff for outside income, to obtain a long-run two-factor equilibrium pricing model: long-run expected yields are proportional to long-run asset-market betas and long-run betas with respect to the average outside-income hedge payoff (Proposition 12).

These results sound familiar from one-period mean-variance analysis with the words "yield" and "payoff" in place of "return" and "portfolio." The novelty is that the same results hold as a characterization of the payoff streams in a dynamic world.

The other novelty is what's missing. The standard Merton (1969, 1971a) characterization describes investments in portfolios whose returns hedge shocks to asset-return or outside-income state variables, as well as a conditionally mean-variance efficient portfolio. The composition of and allocations to these portfolios change constantly and must be rebalanced. In equilibrium, since the average investor must hold the market, the state-variable hedging portfolios become a Merton (1971b) conditional multifactor asset pricing model that describes conditional expected returns.

By contrast, the payoff characterization retains all the simple features of classical one-period mean-variance analysis. The optimal payoff does not include Mertonian state-variable hedging demands, pricing does not depend on state-variable pricing factors, and the payoff description is static, not even requiring rebalancing.

2 Why is this interesting?

Dynamic incomplete-market portfolio theory is important. The fact that returns are not i.i.d. means that investors can potentially exploit time-variation and cross-sectional variation in asset return moments, and should hedge state variables. By doing so, they provide socially useful risk transfer to the agents whose demands create return dynamics in the first place. In typical calibrations, these effects are large.

Dynamic incomplete-market portfolio theory is hard. Consider the simple question of how one should optimally invest in a stock index vs. a riskfree rate, given that index returns are predictable from variables such as the dividend-price ratio. This classic simple Merton (1969, 1971a) dynamic programming problem has been attacked by Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1997), Campbell and Viceira (1999), Barberis (2000), Brennan and Xia (2000, 2002), Wachter (2003), Sangvinatsos and

Wachter (2005), Liu (2007), Jurek and Viceira (2010) and many others. These papers are technically complex. There is no closed-form solution, even for the simplest case: power utility over consumption, an AR(1) forecasting variable whose shocks are not perfectly correlated with return shocks. An easily-summarized practical consensus on the size and nature of market-timing and hedging demands remains more elusive. The literature has not even tried to characterize the final payoffs.

Dynamic incomplete-market portfolio theory is widely ignored in practice, though it has been around half a century. Even highly sophisticated hedge funds typically form portfolios with one-period mean-variance optimizers – despite the fact that mean-variance optimization for a long-run investor assumes i.i.d. returns, while the funds' strategies are based on complex models of time-varying expected returns, variances, and correlations. Beyond formal portfolio construction, their informal thinking and marketing is almost universally based on one-period mean-variance analysis, ignoring Mertonian state variables. Institutions, endowments, wealthy individuals, and regulators struggle to use even the discipline of mean-variance analysis in place of name-based buckets, let alone to implement Mertonian state-variable hedging.

Well, calculating partial derivatives of unknown value functions is hard, and more importantly nebulous. People sensibly distrust model-dependent black boxes.

2.1 Separating payoffs from portfolio strategy

In this context, this paper's contribution is to provide a simple and intuitive benchmark characterization of optimal payoffs, the final dividend stream the consumer/investor receives from asset markets, *without* deriving and characterizing the dynamic portfolio strategy that supports those payoffs in a given market structure.

The payoff characterization is "simple," because market-timing and state-variable hedging demands do not appear. The results look just like familiar one-period mean-variance analysis. It is a "benchmark," because quadratic utility is undoubtedly unrealistic for many investors, and the results, although qualitatively sensible, can be a poor quantitative approximation to the results that obtain from other utility functions. It is a "characterization" because it describes what the optimal payoffs look like, without describing in detail how to support them by dynamic trading in a particular incomplete-market structure.

Focusing on payoffs helps us to separate the investor's risk and return decisions from the financial engineering of how to achieve optimal payoffs in a given market structure. That separation may be a useful step in getting investors and their advisers to understand and *use* dynamic portfolio theory. The portfolio approach has to be re-solved for every change in asset market structure, even if the final payoffs are no different. And the hedging demands emphasized by the portfolio approach are really means to an end – an optimal consumption stream – rather than the end itself. But this payoff characterization does not advance the art of solving dynamic portfolio problems. That's a feature, not a bug.

It's the whole point.

Two examples should clarify these points. First, consider an investor with a 10-year horizon. Examining *payoffs*, it is immediately obvious that a 10-year zero-coupon indexed bond is the riskless asset for this investor. Likewise, it is immediately obvious that an indexed perpetuity is the riskless asset for the standard infinitely-lived consumer-investor.

By contrast, in the standard dynamic-programming approach, long-term bonds are assets that happen to hedge changes in investment opportunity sets: long bond prices go up when interest rates go down, so they hedge reinvestment risk. We have to evaluate value-function derivatives and bond-return correlation models to decide how much and which bonds to buy in order to hedge optimally. It takes hard work for Campbell and Viceira (2001) and Wachter (2003) to prove that the resulting optimal portfolio converges to long-term bonds as risk aversion rises to infinity.

Furthermore, for less risk-averse investors, the optimal Mertonian portfolio mixes market-timing, alpha-chasing, dynamic hedging, and risk-taking motives. Nowhere does the portfolio approach state the obvious: "Long term bonds, not the short rate, are your riskless asset. Split your investment between them and risky securities."

Think about explaining to a 10-year investor whose 10-year bonds have just nosedived, that he didn't really "lose" anything, that this was part of a state-variable hedging program. If you focus instead on payoffs, it's a much easier conversation. This fact seems obvious, yet reflect how many long-run investors and their investment advisers regard money-market investments, not indexed perpetuities, as their core "riskless" asset, evaluate bond managers by one-year returns relative to simple benchmarks or peers, and rebalance their bond investments frequently.

As a second example, consider an investor who says, "I want to invest in bonds and stocks. I have a one-year horizon, and I can not lose more than 20%." If you look at *payoffs*, the answer is obvious: buy one-year, 20% out-of-the-money protective put options. If you ask the Mertonian stock vs. bond portfolio allocation question, you will rediscover the dynamic trading strategy that replicates put options. To the investor, that synthesis is a complex dynamic strategy, buying and selling as the market goes up and down. It looks suspiciously like technical-analysis alpha-chasing, and is in fact inexorably mixed with the dynamic alpha-exploitation that dynamic portfolio theory recommends on top of the put option.

Furthermore, the portfolio strategy will change depending on whether the investor can trade options, where bid/ask spreads and other transactions costs are larger, what strikes and expiries are available, which index he wants to follow and what options on it are available (not all indices and semi-passive strategies have the rich set of options available for the S&P500), and so forth. The financial engineering of the cheapest way to synthesize a given payoff gets mixed up with the economic question of which payoffs to deliver in the first place. In an incomplete market, much of what may seem like interesting dynamic portfolio theory is merely economically uninteresting (but technically challenging) synthesis of the same payoff (say, a put option) by other means (dynamic

trading) when the desired payoff is not directly traded.

Similarly, coupon-only after-tax inflation-protected bonds don't exist. In current markets, they have to be synthesized. A cost-effective synthesis might involve a fairly complicated combination of Treasuries, TIPS, their strips, municipal bonds and inflation swaps. It would depend on the nature of the risky portfolio as well, whether it already includes long-run corporate or municipal bonds. It's easy for an investor and his adviser to agree that indexed coupon-only bonds are conceptually the right "riskfree asset" on which to found the investment program. The investor really doesn't have to know much about the synthesis of this payoff in order to make the important policy decision. Likewise, "synthesize a coupon-only indexed strip" is a clearer direction for the adviser to give to the financial engineering department than is "implement Merton portfolio theory for this investor."

This separation of payoffs from the market-dependent dynamic portfolio and payout strategies that generate payoffs corresponds to a sensible separation of functions in asset management. The investor needs to think about risk and return, and find (with help) the optimal *payoff*. The manager then worries about the financial engineering of how to construct that payoff by dynamic trading and state-variable hedging in the currently-available set of liquid assets. That financial engineering is undoubtedly hard, and worth its fee.

Markowitz (1952) mean-variance theory and Merton's (1969, 1971a) dynamic extension really function as "benchmark characterizations" as well. Though Markowitz derived the mean-variance frontier more than 60 years ago, we still have no settled way to *compute* that frontier in real-world situations. Yes, we have a simple formula $w = \Sigma^{-1}\mu/\gamma$ (w = weights, Σ = excess return covariance matrix, μ = mean excess returns, γ = risk aversion), but this formula is essentially useless in practice. The hurdles of estimating large covariance matrices, overcoming the curse of $\sigma/\sqrt{T}$ in estimating mean returns, and dealing with parameter uncertainty and drift, are not minor matters. Computation is approached differently for different asset classes, trading restrictions, data sets, time horizons, conditioning information sets, parameter priors, and all the other peculiarities of a given application. The results of $\Sigma^{-1}\mu$ are so sensitive to input assumptions that commercial optimizers and both regulatory and commercial risk management calculations add layers of ad-hoc constraints, so many layers that the answers barely merit being called implementations of $\Sigma^{-1}\mu$. Much of the money management industry amounts to selling one or another attempted solution to estimating and computing $\Sigma^{-1}\mu$, at fees commensurate with the challenge of the problem. The weights $\Sigma^{-1}\mu$ can change dramatically when you include or exclude assets, though the characterization does not.

Why then is mean-variance theory then so famous? Classical one-period mean-variance analysis is really a brilliantly useful *characterization* of an optimal portfolio, useful for final investors to understand and think hard about risk allocations, and especially to avoid mistakes like bucket investing, closet indexing, and underdiversification, even though it is not a particularly useful guide to *computation*. Classical mean-variance analysis continues to dominate portfolio applications, even when investors are contemplating highly

non-normal payoffs, such as with options, credit risk, or dynamic trading, and even when investors have nonlinear marginal utility, especially including leverage, drawdown or other constraints limiting their acceptable losses. Why? Well, it is a simple conceptual benchmark, and benchmarks are useful. It's a good robust first step, even when the second step is fairly large as well.

Merton-style dynamic portfolio theory is an especially clear case of a benchmark characterization. The actual portfolio weights depend on value function derivatives, which are not known in most environments. Textbooks repeated the optimal portfolio in terms of unknown value-function derivatives for 30 years before the first attempts, cited above, to actually compute those derivatives for simple environments. Evidently, the Merton characterization of portfolios was and remains a conceptually valuable benchmark even without full solutions or realistic computations.

2.2 Dynamics and traded assets

On the other hand, the dynamic portfolio strategies that investors use to implement optimal payoffs can be trivial. If indexed perpetuities and a long-run mean-variance efficient payoff were traded – or if financial firms created them, as they created index funds – then investors would simply buy those payoffs. Investors would not even rebalance over time. Dynamic portfolio theory is only hard when, for some reason, the supply side of the market does not directly provide the payoffs of interest to investors.

This observation gives an extreme example of how sensitive portfolio strategies are to changes in the market structure, and suggests again why calculating dynamic strategies is an (important) financial engineering question rather than the way to generate economic intuition.

This observation also suggests that thinking about optimal payoffs will be useful for creating securities, or for explaining existing market structures. If more investors were asking for coupon-only real bonds, more securities firms would sell them, and the investor's "dynamic portfolio theory" would become the firm's risk management problem. More basically, why do stocks pay dividends and bonds pay coupons? Consumers could in principle synthesize such securities from dynamic trading of stocks that repurchase shares instead of paying dividends and zero-coupon bonds. But if dividends and coupons reflect the optimal final payouts many consumers desire, it may be less surprising that these are the basic marketed securities.

2.3 The market portfolio

Expressing portfolio theory relative to the market is an especially useful discipline. Portfolio theory seems to apply to everyone. But the average investor must hold the market portfolio, and consume from the market payoff, ignoring all tempting dynamics or additional factors. All deviations from market weights are a zero-sum game. For every investor

who earns positive market alpha, another one must earn negative alpha. We can't collectively market-time. We can't even collectively rebalance. Therefore, any deviation from market weights that we recommend to A must be mirrored in opposite advice given to B. If the advice applies to everyone, the equilibrium expected returns and covariances underlying the advice must change as soon as any measure of investors take the advice, and the advice will evaporate. If the advice seems to apply to everyone, maybe it's wrong.

Phrasing portfolio advice in terms of *deviations* from the market portfolio, driven by *deviations* of the investor from average, along described dimensions of heterogeneity such as risk aversion or exposure to outside risks, helps to preserve portfolio advice from this average-investor conundrum. By phrasing portfolio advice based on differences between individual and average preferences, outside risk exposures, and other characteristics, we implicitly trace back seemingly attractive prices to economic fundamentals of why the rest of the market seems to offer our investor a good deal.

Such a description may also be of practical use. Investors may not be able to answer well "what's your risk aversion?" but they may be able to answer "are you more or less risk averse than the average investor?" (Of course, they must answer that question with a bit more consistency than "are you smarter or better informed than the average investor?") Even in standard one-period mean-variance problems, phrasing portfolio maximization in terms of deviations from the market portfolio is a standard way to get vaguely sensible answers, i.e. answers that don't deviate enormously from market weights.

2.4 Outside income

Incorporating outside income or assets into portfolio theory is important. Real investors have houses, jobs, businesses, illiquid assets, assets that cannot be sold for tax, regulatory, or other reasons, or similar non-marketed liabilities (pension funds, endowments). These investors should start by hedging non-marketed risks. The zero-price, zero-premium hedge payoff most correlated with their outside income represents free insurance, risk reduction without any reduction in expected return.

Outside-income hedging by the average investor leads to changes in the market premium and the emergence of additional priced factors. Non-exposed investors can profit by adapting their portfolios and payoffs, effectively selling insurance.

In this way, incorporating outside income leads to an interesting and fruitful portfolio theory, one that predicts substantial heterogeneity in portfolios without violating the average-investor conundrum or attempting to chase zero-sum "inefficiencies." And, since hedging outside income, and understanding the priced risks that emerge from other's hedging, is not easy, incorporating outside income represents a privately and socially beneficial activity for the financial advice industry.

These ideas have been present since the early days of modern portfolio theory and asset pricing Mayers (1972), (1973) for example. On the idea that labor income looks like a bond, investors are often advised to move away from stocks as they approach retirement.

Roll and Ross (1984) advocated that individuals and institutions calibrate APT exposures to their specific risks or liability streams, and that professional advice might be helpful in doing so. Fama and French (1996) attribute the value premium to investors whose human capital is correlated with value returns. Those investors shun value stocks, generating the premium for the lucky investors whose outside income is not so exposed, and who then buy value stocks.

Incorporating outside income is hard. Most fundamentally, we can observe the *stream* of outside income, but we cannot easily observe its *value*. To apply standard return-based portfolio theory, one has to turn the income stream into a price. For example, Jagannathan and Wang (1996) assume an AR(1) labor income process and a constant discount rate, so that labor income *growth* measures human capital *returns*. However, price changes unrelated to current cash flows dominate high-frequency asset returns for assets whose prices we do observe, so they are likely to do so for non-traded assets as well. The AR(1), constant discount rate procedure would do a poor job of replicating observed stock returns from the stock dividend stream.

In addition, outside-income growth is not likely to be i.i.d., so outside-income state variables need to defined, modeled, and hedged as well as the income stream itself in order to find the optimal dynamic portfolio. Assets are valuable if their returns are negatively correlated with shocks to forecasts of future outside income or outside-income discount rates, both tricky concepts to measure.

These difficulties notwithstanding, a large literature incorporates outside income or assets into intertemporal portfolio theory, typically further generalizing the Mertonian dynamic-programming instantaneous-portfolio approach. Here too, we do not have an analytic solution to the most basic problem: power utility, lognormal i.i.d. returns, and a lognormal diffusion for outside income. (Duffie, Fleming, Soner, and Zariphopoulou (1997) and Koo (1998) characterize this problem.) A bit more progress has been made with CARA utility (Svensson and Werner (1993), Teplá (2000), Henderson (2002, 2005)), which alas is not much more realistic than quadratic. Most of the applied literature studies numerical or approximate solutions to particular calibrations. Highlights include Campbell (1996), Heaton and Lucas (2000a), (2000b), Davis and Willen (2000, 2001), Munk (2000), Viceira (2001), Lynch (2001), Flavin and Yamashita (2002), Gomes and Michaelides (2005), Yao and Zhang (2005), Davis, Kubler and Willen (2006), Benzoni, Collin-Dufresne, and Goldstein (2007) and Lynch and Tan (2011). A robust consensus is even more remote than it is for intertemporal portfolio theory.

Even in this extensive literature, most attention is still devoted to the simple stock/bond and long bond/short bond split. The important question, characterizing which risky portfolios hedge the risky components of outside income, remains essentially untouched. In part, that question is hard because it must recognize the heterogeneity of outside income across investors: If we all have the same outside income, the average-investor theorem says that we just hold the market portfolio.

Outside-income hedging is almost as widely ignored in practice as is Mertonian state-variable hedging. Steel workers, and their pension funds, do not short the steel industry

portfolio, or even the auto industry. Asset managers are only beginning emerge with expertise in this hedging, selling individual-specific hedging portfolio services. Academic research has focused almost entirely on finding "priced" factors, alphas for the one lonely (and typically myopic) mean-variance investor who has no outside income. It has ignored finding *non-priced* factors well-correlated with typical outside income streams, that, by providing free insurance, are potentially the most important deviations from market weights for typical investors.

In this context, it is attractive that the *payoff* characterization with outside income is quite simple. We just add the hedge payoff for outside income, constructed by long-run regression. The long-run regression is a regression of outside income streams or payoffs on asset payoffs – pure "cashflow beta." Intermediate prices – which we do not observe for the outside income stream – do not appear. State variables for outside income dynamics do not appear in optimal payoffs or in equilibrium pricing.

Background risk effects are also absent here. Breaking outside income to a tradeable (hedge payoff) and nontradeable (idiosyncratic) component, the latter does not appear. In general, agents who have to hold an unspanned component of outside income risk may act in a more risk-averse manner. By the assumption of quadratic utility, this effect is absent in mean-variance settings. Again, the point of a simple benchmark is to start with the most basic effects, and to avoid complex, though potentially important, refinements.

3 Literature: Complete markets and quadratic utility

This long-run mean-variance and equilibrium analysis builds on three other strands of literature.

First, the final-payoff view of dynamic intertemporal portfolio theory, the equivalence of static and dynamic optimization emphasized by my notation $\tilde{E}(x)$ in (1), and the potential to characterize final payoffs quickly without solving for the dynamic strategy that supports them, are of course well known in the complete-markets tradition.

In a complete market, there is a unique discount factor $\{m_t\}$ which summarizes the available asset-payoff space, and optimal payoffs consist simply of inverting $u'(c_t) = \lambda m_t$, where λ is the Lagrange multiplier on the wealth constraint. Cox and Huang (1989), He and Pearson (1991), He and Pagès (1993), Schroder and Skiadas (1999), Wachter (2003), Sangvinatsos and Wachter (2005) and many others follow this approach, which really traces back to Arrow and Debreu (1954).

This approach has not as yet been of widespread use in incomplete-market setups, however. In incomplete markets, there are an infinite number of potential discount factors m_t that are consistent with the available prices and payoffs. To complete the solution for optimal payoffs, one must search for which choice of m_t produces a *tradeable* payoff c_t from $u'(c_t) = \lambda m_t$. This search can be as hard as solving the dynamic program that constructs c_t as the payoff of a dynamic portfolio strategy.

The restriction to linear marginal utility underlying mean-variance analysis avoids this central problem. The discount factor x_t^* that is also a traded payoff is unique. If $u'(c_t)$ is linear, then inverting $u'(c_t) = \lambda x_t^*$ expresses c_t as a linear function of a traded payoff, so c_t is also a traded payoff. Classical mean-variance analysis has always handled market incompleteness transparently, and by mapping classical mean-variance analysis to payoff streams, this extension handles market incompleteness transparently in the same way.

Most modern finance theory is recast in contingent-claim language, in which state- and time-contingent payoffs replace moments such as mean, variance, and betas as the basic commodities over which preferences and budget constraints are defined. Hansen and Richard (1987) made the key connection between contingent claims and the mean-variance frontier, and valid in infinite-dimensional payoff space such as results from dynamic trading. As a minor pedagogical contribution, one can replace all the $\tilde{E}$ back to E, and this paper derives the propositions of classical mean-variance theory in one place, including outside income and preference shocks, and from this contingent-claims perspective.

Second, the basic idea of treating a discounted sum as a generalized "expectation" in (1) comes from Hansen (1987), who includes an extensive analysis of asset pricing with quadratic utility. In particular his Equation (1.8), p. 212 introduces the conditional inner product

$$\langle x|y\rangle \equiv E\sum_{t=1}^{\infty}\beta^t x_t y_t \tag{2}$$

which is proportional to the long-run second moment $\tilde{E}(xy)$ as I have defined it in (1). Hansen introduces the Hilbert space of payoffs with $\langle x|y\rangle < \infty$. He posits complete markets, solves the quantity dynamics, and studies allocations in equilibrium. My Proposition 1 is a generalization of his Equation (2.13), p. 218. Hansen shows that individual consumption is proportional to aggregate consumption in equilibrium (Equation (2.33) p. 222). Hansen and Sargent (2004) extends this asset pricing framework (see p. 55. and p. 108).

Magill and Quinzii (2000) is the precursor most directly related to this paper. Magill and Quinzii also specify a quadratic utility investor. Like Hansen (1987), they define the inner product (2) and consequent Hilbert space of payoffs. With complete markets, they also show (their Proposition 1) that in equilibrium each investor's payoff (consumption) is a combination of the risk free rate and the aggregate endowment stream. They generalize this statement (their Proposition 2) to a market that is missing a risk-free payoff, though it is still complete enough that investors can perfectly hedge idiosyncratic income shocks. They show that a "least-variable" payoff, the projection of the unit payoff on the set of traded payoffs, takes the place of the riskfree payoff in this equilibrium representation.

At one level, one can regard this paper as a substantial extension of Hansen (1987) and Magill and Quinzii's (2000) results. I characterize portfolios out of equilibrium, e.g. when other agents do not have quadratic utility. I allow arbitrary market completeness including unhedged idiosyncratic income. I allow preference shocks. I define the long-run mean-variance frontier, and I show that optimal payoffs lie on it. I express long-run pricing as well as optimal payoffs, showing how the long-run CAPM and long-run two-

factor model emerge, with average outside income becomes the second factor.

Most of all, I express the portfolio results in terms of risk aversion, and relative to market averages, connecting the quadratic utility representation to standard mean-variance portfolio theory.

On the other hand, the major point of this paper is that none of these results are, in retrospect, big theoretical contributions. Once you write down the definition of long-run mean (1), all of standard one-period mean-variance theory applies directly and essentially trivially to payoff streams. That notation makes the theory come alive, and it paints a novel picture of long-run portfolios and equilibrium. But it reveals that this whole endeavor is a relabeling of classic mean-variance analysis rather than a basic theoretical advance.

Third, this effort is part of a larger new interest in payoff streams and long-run analysis in asset pricing, for example Menzly, Santos and Veronesi (2004), Bansal and Yaron (2004), Bansal, Dittmar and Lundblad (2005), Lettau and Wachter (2007), Gabaix (2007), Hansen Heaton and Li (2008), and many others. These authors try to account for *prices* rather than expected returns, based on the long-run correlations of cash flow *streams*, rather than one-period return betas. This paper's characterization of long-run expected yields by a long-run (cashflow) beta is a simple expression of these ideas.

4 Asset pricing environment

This section sets up the framework to think about dynamic intertemporal portfolio problems in analogy to one-period problems, by treating date and state symmetrically.

To describe the asset pricing environment, the mean-variance frontier, and its connection to beta representations, I follow the Hansen-Richard (1987) approach as expanded in Cochrane (2005) Ch. 5 and 6. This approach makes clear the intimate link between marginal utility, mean-variance frontiers, and discount factors, and applies to infinite-dimensional payoff spaces generated by dynamic trading.

For readers anxious to get to optimal payoffs, the key results that I use later are the existence of a traded discount factor x^* (9) and its corresponding yield y^* (12), the characterization of the mean-variance frontier in (18) and (20), and the long-run Roll theorem connecting mean-variance frontiers and beta representations (21).

4.1 Payoffs and prices

I develop a notation that uses the same symbols to describe familiar one-period returns and to describe streams of payoffs in an intertemporal context. The Appendix summarizes all notation.

The symbol x denotes a *payoff*. In a one-period setting, the payoff is the amount x_1

that an investor receives at date 1, in each state of nature, for a time-zero price p_0. In an intertemporal setting, payoffs are the streams of dividends $\{x_t\} = \{x_1, x_2, ...\}$, or $\{x_t dt\}$ in continuous time, resulting from an initial purchase at price $p(\{x_t\})$, and indexed by state and date.

Returns are price-one payoffs. We form returns by dividing payoffs by their initial price,

$$y_t = x_t / p(\{x_t\}).$$

In the intertemporal setting, the "return" to a particular date – the dividends or coupons accruing to a one-dollar purchase – has the units of a yield, or coupon rate; it is a number like 0.04, not 1.04. In an intertemporal context, I use the notation y and the word "*yield*" rather than the letter r or word "return," to help keep the typical units in mind.

In a one-period model, the risk-free asset pays one unit in each state. The risk-free payoff in an intertemporal setting is one in all states and dates, a perpetuity,

$$x_t^f = 1.$$

The *risk free yield* is then naturally,

$$y_t^f = 1/p(\{1\}).$$

The riskfree yield is also a number like 0.01, not a number like the 1.01. (The presence of such a long-run risk free payoff does not imply that one-period riskless bonds are traded, or that a one-period riskless rate is constant.)

Excess yields are zero-price payoffs, which we can construct by differencing any two yields,

$$z_t = y_t^1 - y_t^2; \; p\left(\{z_t\}\right) = 0.$$

When not required for clarity, I drop the time subscripts and sequence notation, e.g. $y \equiv \{y_t\}$ and $p\left(\{x_t\}\right) = p(x)$.

I use the notation $\tilde{E}(x)$ to denote a sum over time as well as expectation over states of nature. The meaning depends on context:

$$\begin{aligned}
\text{One period:} \qquad & \tilde{E}(x) \equiv \frac{1}{\beta} E(\beta x_1). \\
\text{Intertemporal, discrete:} \qquad & \tilde{E}(x) \equiv \frac{1-\beta}{\beta} E \sum_{t=1}^{\infty} \beta^t x_t. \\
\text{Intertemporal, continuous:} \qquad & \tilde{E}(x) \equiv \rho E \int_{t=0}^{\infty} e^{-\rho t} x_t dt.
\end{aligned}$$

One can similarly define $\tilde{E}$ in environments with a terminal date, e.g.

$$\tilde{E}(x) \equiv \frac{\rho}{1 - e^{-\rho T}} E \int_{t=0}^{T} e^{-\rho t} x_t dt,$$

or with a separate lump-sum terminal payment.

The $\tilde{E}$ operator takes a sum over time, weighted by β^t or $e^{-\rho t}$, as well as a sum over states, weighted by probabilities. Weighting allows us to produce finite values for a larger set of payoff processes in an infinite-period environment. It will be useful to pick β or ρ as the agent's subjective discount factor, as it is useful to use the agent's subjective probabilities to take expectations. One can weight over time by different functions, as one can weight over states of nature by alternative probability measures.

I call $\tilde{E}(x)$ the "long run mean" and I call

$$\tilde{\sigma}^2(x) \equiv \tilde{E}(x^2) - \left[\tilde{E}(x)\right]^2 = \tilde{E}\left\{\left[x - \tilde{E}(x)\right]^2\right\}$$

the "long run variance" of the payoff stream x_t. This "variance" concept measures stability over time as well as stability across states of nature.

With this notation, I write the fundamental pricing equation as

$$p = k\tilde{E}(mx),$$

where m_t is a stochastic discount factor. Relative to standard notation, m_t is the discount factor scaled by the weighting function; m_t is $u'(c_t)$ not $\beta^t u'(c_t)$. In each context, this expression has a slightly different meaning and value of the constant k:

$$\begin{aligned}
\text{One period:} \qquad & p_0 = E(\beta m_1 x_1) = \beta\tilde{E}(m_1 x_1); \\
\text{Intertemporal, discrete:} \qquad & p = E\sum_{t=1}^{\infty}\beta^t m_t x_t = \frac{\beta}{1-\beta}\tilde{E}(mx); \\
\text{Intertemporal, continuous} \quad : \quad & p = E\int_0^{\infty} e^{-\rho t} m(t)x(t)dt = \frac{1}{\rho}\tilde{E}(mx).
\end{aligned}$$

The appearance of a constant k in the fundamental pricing equation is inelegant. However, we gain more in convenience by defining "long run mean" with weights that sum to one than we lose by introducing this constant in the pricing equation.

Next, I describe payoffs and prices. Let $\underline{X}$ denote the *payoff space,* the set of all payoff streams that investors can buy. Let $\underline{Y}$ denote the set of price-one yields, and let $\underline{Z}$ denote the set of price-zero excess yields,

$$\underline{Y} \equiv \{y \in \underline{X} : p(y) = 1\},$$

$$\underline{Z} \equiv \{z \in \underline{X} : p(z) = 0\}.$$

I limit the payoff space $\underline{X}$ to include only square–summable or square-integrable payoffs

$$\tilde{E}(x^2) < \infty. \tag{3}$$

In an infinite-period model, this requirement limits us to payoffs that do not grow too fast, i.e. that do not vary too much over time, as well as limiting variance in the usual sense. With this assumption $\tilde{E}(xy)$ is an inner product defining a Hilbert space as in Hansen (1987), Hansen and Richard (1987), Hansen and Sargent (2004) and Magill and

Quinzii (2000). Then, we can think of dividend streams x as vectors, $\tilde{E}(xy)$ as an inner product, $\tilde{E}(x^2)$ as the "size" of x and $\tilde{E}\left[(y-x)^2\right]$ as the "distance" between y and x.

I let investors buy any portfolio of payoffs, which means that $\underline{X}$ and $\underline{Z}$ are closed under linear combinations,

$$x, w \in \underline{X} \rightarrow ax + bw \in \underline{X}, \tag{4}$$

and I assume that prices and payoffs follow the law of one price, or that the pricing function is linear,

$$p(ax + bw) = ap(x) + bp(w). \tag{5}$$

In an intertemporal context, we also want to allow dynamic trading, or equivalently we want to allow investors to form and entrepreneurs to sell the payoffs of managed portfolios. We can capture this expansion of the payoff set by allowing state-contingent buying and selling. If $\{x_t\} \in \underline{X}$, we also allow any payoff stream created by selling an existing payoff,

$$\{x_0, ..., x_{t-1}, x_t + p_t(x)\} \in \underline{X} \tag{6}$$

or any stream created by buying a payoff in mid-stream,

$$\{0, ... - p_t(x), x_{t+1}, x_{t+2}...\} \in \underline{X}. \tag{7}$$

In the standard continuous-time context, given a set of assets with excess return process dr_t^e and riskfree rate r_t^f, we can include dynamic trading by allowing payoffs x_t generated by a cumulative value process V_t and a payout policy x_t,

$$dV_t = \left(r_t^f V_t - x_t\right) dt + w_t' dr_t^e. \tag{8}$$

As usual, we must limit dynamic trading so that the investor cannot generate arbitrage opportunities by trading too frequently, by rolling over debt forever or following doubling strategies. See, for example, Duffie (2001). (Section 8 gives an example.)

I do not assume that the payoff space $\underline{X}$ is complete, meaning that every stream of random variables can be traded. I explicitly allow for two sources of incompleteness: the investor may have a labor or business income stream that cannot be completely hedged with traded assets, and there may be state variables for investment opportunities whose shocks cannot be spanned by those of traded assets. I also consider the case that the riskless payoff is not traded.

4.2 Discount factor

The standard sufficient conditions (Ross (1978), Hansen (1987), Hansen and Richard (1987)) on the payoff space $\underline{X}$ guarantee the existence and uniqueness of a discount factor that is also a traded payoff. Given (3) and (4), linearity of the pricing function (5) holds if and only if there is a unique discount factor x^* that is also a traded payoff stream, i.e.

$$\exists x^* \in \underline{X} : p = k\tilde{E}\left(x^* x\right) \ \forall x \in \underline{X}. \tag{9}$$

When there is a finite vector of basis payoffs $\mathbf{x}$ with prices $\mathbf{p}$, and ignoring dynamic trading beyond what is included in the basis assets (the basis payoffs may themselves be payoffs from dynamically-managed portfolios), so the payoff space is $\underline{X} = \{\mathbf{c}'\mathbf{x}\}$, the usual discount factor construction applies. The payoff

$$x^* = \frac{1}{k}\mathbf{x}'\tilde{E}(\mathbf{x}\mathbf{x}')^{-1}\mathbf{p} \tag{10}$$

is a discount factor, i.e. it satisfies $p = k\tilde{E}(x^*x)$.

4.3 Mean-variance frontier

The *long-run mean-variance frontier* consists of payoffs that solve

$$\min_{\{y\in\underline{Y}\}} \tilde{E}(y^2) \text{ s.t. } \tilde{E}(y) = \mu.$$

The mean-variance frontier has a two-fund representation,

$$y^{mv} = y^* + \lambda z^*. \tag{11}$$

Here, y^* is defined by

$$y^* = \frac{x^*}{p(x^*)} = \frac{x^*}{k\tilde{E}(x^{*2})}, \tag{12}$$

and z^* is defined by

$$z^* = proj(1|\underline{Z}), \tag{13}$$

as the excess yield "closest" to the perpetuity payoff. λ is a number; as you vary λ you sweep out the frontier.

The payoffs y^* and z^* have the usual properties from the one period case, suitably reinterpreted. (Cochrane (2005) Ch. 6 gives a list of properties; y^* and z^* correspond to R^* and R^{e*} there.) The yield y^* is the discount-factor mimicking yield: For any discount factor m, we have $x^* = proj(m|\underline{X})$ and then (12). The yield y^* is the minimum long-run second moment yield,

$$y^* = \arg\min_{\{y\in\underline{Y}\}} \tilde{E}(y^2).$$

It lies on the lower half of the mean-variance frontier. Since y^* is proportional to x^* it can be used to price other payoffs:

$$\tilde{E}(y^*y) = \tilde{E}(y^{*2}) \quad \forall y \in \underline{Y}.$$

More generally, any mean-variance efficient yield carries pricing information.

An explicit formula for y^* for the finite-basis case follows from (10) and mimics the standard formulas for one-period mean-variance frontier returns:

$$y^* = \frac{\mathbf{1}'\tilde{E}(\mathbf{y}\mathbf{y}')^{-1}\mathbf{y}}{\mathbf{1}'\tilde{E}(\mathbf{y}\mathbf{y}')^{-1}\mathbf{1}}. \tag{14}$$

The excess yield z^* generates long-run means in the same way that x^* generates prices,

$$\tilde{E}(z) = \tilde{E}(z^* z) \forall\ z \in \underline{Z}. \tag{15}$$

Since z^* is a price-zero excess yield, y^* and z^* are orthogonal, $\tilde{E}(y^* z^*) = 0$. If a riskfree yield is traded ($1 \in \underline{X}$) then z^* is simply

$$z^* = \frac{y^f - y^*}{y^f}. \tag{16}$$

With a finite set of basis assets, we can also calculate z^* analogously to the calculation (10) of y^*:

$$z^* = \tilde{E}(\mathbf{z})' \tilde{E}\left(\mathbf{z}\mathbf{z}\right)^{-1} \mathbf{z} \tag{17}$$

where $\mathbf{z}$ is a vector of excess yields.

We can span the mean-variance frontier with any two efficient payoffs in place of y^* and z^* given by (11). When a risk-free payoff is traded, we can span the mean-variance frontier by y^* and y^f, rather than y^* and z^* as in (11). Substituting (16) in (11) and defining a new λ, we obtain

$$y^{mv} = y^f + \lambda\left(y^* - y^f\right). \tag{18}$$

When a riskfree payoff is not traded, one can use in place of y^f the minimum long-run variance yield, the yield $y^c = x^c / p(x^c)$ of the constant-mimicking payoff $x^c \equiv proj(1|\underline{X})$ or the zero-beta yield corresponding to y^*. See Cochrane (2005) Ch. 6.

The long-run mean-variance frontier of excess yields is defined by

$$\min_{\{z \in \underline{Z}\}} \tilde{E}(z^2) \ \text{ s.t. } \tilde{E}(z) = \mu. \tag{19}$$

This frontier is generated simply by

$$z^{mv} = \lambda z^*. \tag{20}$$

Equation (17) transparently solves (19) for a finite set of basis assets $\mathbf{z}$.

As in standard mean-variance analysis, the mean-variance frontier of *excess* (zero price) returns or yields is convenient, because it is always a V, not a hyperbola. Even if a riskfree payoff is not traded, the zero payoff is always traded – invest nothing, get nothing – so one does not need to keep track of riskfree asset special cases. The mean excess yield or return is also a real quantity, where asset yields and returns include inflation. Finally, excess returns and yields are full spaces so one does not need to keep track of the constraint that portfolio weights must sum to one (compare (17) to (14)) and $proj(x|\underline{Z})$ is meaningful, where projecting on yields or returns is not.

All of these results can be derived by following exactly the Hansen-Richard (1987) approach in one-period models, but using $\tilde{E}$ in the place of E. We show that any yield

y^i can be written as $y^i = y^* + \lambda^i z^* + \eta^i$, with $\tilde{E}(\eta^i) = 0$, $\tilde{E}(\eta^i y^*) = 0, \tilde{E}(\eta^i z^*) = 0$. The mean-variance frontier is then the set of yields with $\eta^i = 0$.

Traditional mean-variance formulas in terms of covariances also have straightforward translations. For example, in the case of a finite basis, we can write the mean-variance frontier of excess yields in the obvious analogue to the traditional formula $z^{mv} = \lambda \mu' \Sigma^{-1} z$ as $z^{mv} = \lambda \tilde{E}(\mathbf{z})' \widetilde{cov}\left(\mathbf{z}, \mathbf{z}\right)^{-1} \mathbf{z}$, rather than in second-moment form (17).

4.4 Expected yields and betas

As in one-period asset pricing, we can connect discount factors, pricing, mean-variance frontiers, and expected yield-beta models. As usual, the formulas look familiar, but the novelty is that standard analysis applies to long-run moments and streams of payoffs.

The main proposition I will use below is a simple case of Hansen and Richard's (1987) statement of Roll's (1977) theorem for infinite-dimensional payoff spaces: *If a riskfree rate is traded, then a single-beta representation holds for each asset i's long-run expected yield with respect to yield* y^{mv},

$$\tilde{E}(y^i) - y^f = \tilde{\beta}_{i,mv} \left[\tilde{E}(y^{mv}) - y^f \right], \tag{21}$$

if and only if y^{mv} *is on the long-run mean-variance frontier and* y^{mv} *is not the riskfree yield, i.e. if*

$$y^{mv} = y^f + \lambda^{mv} \left(y^* - y^f \right)$$

and $\lambda^{mv} \neq 0$.

Derivation. We can decompose any excess yield as

$$z^i = \lambda^i z^* + \eta^i,$$

where we choose λ^i so that $\tilde{E}(\eta^i z^*) = 0$. By (15), we then also have $\tilde{E}(\eta^i) = 0$, so $\lambda^i = \tilde{\beta}_{z^i, z^*}$ is the long-run regression coefficient of z^i on z^* with a constant. Equation (20) characterizes the mean-variance frontier of excess yields, $z^{mv} = \lambda^{mv} z^*$. So long as $\lambda^{mv} \neq 0$, we can then write

$$z^i = \frac{\lambda^i}{\lambda^{mv}} z^{mv} + \eta^i,$$

and $\tilde{\beta}_{z^i, z^{mv}} = \lambda^i / \lambda^{mv}$ is the long-run regression coefficient of z^i on z^{mv} with a constant. Taking long-run means, we have a single-beta representation with any long-run mean-variance efficient excess yield as reference,

$$\tilde{E}\left(z^i\right) = \tilde{\beta}_{z^i, z^{mv}} \tilde{E}\left(z^{mv}\right).$$

With a riskfree payoff we can take $z^{mv} = y^{mv} - y^f; z^i = y^i - y^f$ by (16) and (18), and $\tilde{\beta}_{z^i, z^{mv}} = \tilde{\beta}_{y^i, y^{mv}}$.

Long-run regression coefficients and long-run covariance sum over time as well as states, $\widetilde{cov}(x, y) = \tilde{E}(xy) - \tilde{E}(x)\tilde{E}(y)$; $\tilde{\beta}_{x,y} = \widetilde{cov}(x, y)/\tilde{\sigma}^2(x)$.

The case with no riskfree payoff works just as in the one-period setup, with a zero-beta rate taking the place of y^f and the exception that the mean-variance efficient reference yield cannot be the minimum long-run variance yield. Hansen and Richard's (1987), p. 600-611 proof works, translating the notation. The larger set of beta representations in Cochrane (2005) Ch. 5 and Ch 6 translate as well.

5 The general portfolio problem

An investor has initial wealth W, a stream of human or other non-marketable income $h = \{h_t\}$, and he can buy payoffs $x = \{x_t\} \in \underline{X}$ at prices p. I assume the law of one price (5) in the available prices and payoffs, so there is a discount factor m that satisfies $p = k\tilde{E}(mx)$. I specify time-separable expected-utility preferences. The investor's problem is then

$$\max_{\{x \in \underline{X}\}} \tilde{E}\left[u(c)\right] \quad \text{s.t. } W = k\tilde{E}(mx),\ c = h + x. \tag{22}$$

As a reminder, though the symbols look like a one-period problem, they stand for long-run portfolio problems, for example

$$\max_{\{x_t \in \underline{X}\}} E\sum_{t=1}^{\infty} \beta^t u(c_t) \ \text{ s.t. } W = p(\{x_t\}) = E\sum_{t=1}^{\infty} \beta^t m_t x_t;\ c_t = h_t + x_t,$$

or

$$\max_{\{x_t \in \underline{X}\}} E\int_{t=0}^{\infty} e^{-\rho t} u(c_t)dt \ \text{ s.t. } W = p(\{x_t\}) = E\int_{t=0}^{\infty} e^{-\rho t} m_t x_t dt;\ c_t = h_t + x_t.$$

The first-order conditions state that at an optimum $\hat{x}$,

$$u'(\hat{x} + h) = \lambda m, \tag{23}$$

where λ represents a Lagrange multiplier on the wealth constraint. Marginal utility is proportional to a discount factor.

Inverting (23), the solution to the portfolio problem is characterized by

$$\hat{x} = u'^{-1}(\lambda m) - h. \tag{24}$$

I use a hat $\hat{x}$ to denote optimal values.

The payoff (24) has a simple intuition: The investor consumes more $c = \hat{x} + h$ in "cheap" (low m) states and dates, and less in "expensive" (high m) states and dates, with u'^{-1} dictating how much or little to respond to this relative date- and state-price. The traded payoff $\hat{x}$ then offsets the effects of outside income h. The Lagrange multiplier λ scales the optimal payoff up and down to match initial asset-market wealth W.

If markets are complete, the discount factor $m = x^* \in \underline{X}$ is unique, and traded. Also, the nonlinear transformation and addition described by (24) lead to traded payoffs, so

the constraint $\hat{x} \in \underline{X}$ is satisfied. All we have to do is find the Lagrange multiplier λ to satisfy the initial wealth constraint.

I focus on the case that markets are not complete. Now, condition (24) is necessary, but not sufficient. There are many discount factors m that price assets, and for only one of them does the construction (24) produce a traded payoff $\hat{x} \in \underline{X}$. We still have to find that m.

To solve this problem, I specialize to quadratic utility, so that marginal utility is linear. The payoff space $\underline{X}$ is closed under linear transformations (portfolio formation, equation (4)), so once we construct the traded discount factor x^*, we know that $u'^{-1}(\lambda x^*)$ is also in the space of payoffs $\underline{X}$, and this is the optimal payoff.

Analytically, I specialize to

$$u(c_t) = -\frac{1}{2}(c_t^b - c_t)^2. \tag{25}$$

c_t^b is a potentially time-varying stochastic bliss point. A time-varying or stochastic preference shift can help to accommodate growth, life-cycle and household-composition effects, or to give a better approximation to nonlinear utility functions. (Hansen and Sargent (2004), Heaton (1993), Cochrane (2012b).) I do not allow free disposal.

The optimal payoff is then characterized by

Proposition 1. *The optimal payoff for the investor (25) is given by*

$$\hat{x} = \left(\underline{c}^b - \underline{h}\right) - \left[p(\underline{c}^b - \underline{h}) - W\right] y^*, \tag{26}$$

where the hedge payoffs $\underline{c}^b$, $\underline{h}$ are the projections of the preference shock and outside income on the set of traded payoffs, e.g.

$$\underline{c}^b \equiv proj\left(c^b|\underline{X}\right),\ \underline{h} \equiv proj\left(h|\underline{X}\right), \tag{27}$$

W is initial financial wealth, and y^* is the discount-factor mimicking and minimum second-moment yield (12).

Derivation. With the quadratic utility function (25), the first order condition (23) reads

$$c^b - \hat{x} - h = \lambda m.$$

Solving for $\hat{x}$ and projecting both sides on the set of traded assets $\underline{X}$ yields

$$\hat{x} = -\lambda x^* + \underline{c}^b - \underline{h}. \tag{28}$$

The wealth constraint states

$$W = p(\hat{x}) = -\lambda p(x^*) + p(\underline{c}^b - \underline{h}).$$

Solving for λ, substituting λ in (28), and using the definition $y^* = x^*/p(x^*)$ we obtain the optimal payoff (26).

The expression (26) offers a simple interpretation in the quadratic utility context: The investor starts by buying a payoff $\underline{c}^b - \underline{h}$ that gets consumption as close to the bliss point as traded assets allow. We can also think of the payoff $\underline{c}^b - \underline{h}$ as the optimal hedge for preference shocks and labor income risk. It is formed by a long-run regression of the streams c^b and h on the yields or dividend streams of the traded assets.

Typically, initial wealth W is lower than the cost $p(\underline{c}^b - \underline{h})$ of this ideal hedge payoff. Then, the investor shorts an optimal risky payoff y^* in order to buy the individual hedge payoff. The yield y^* is proportional to contingent claims prices, so by shorting y^* the investor is shorting the "most expensive" payoff, in order to generate the largest funds possible at minimum risk. The yield y^* is also on the mean-variance frontier.

In sum, *each investor's optimal payoff is a combination of a labor-income and preference-shock hedge payoff, plus an investment in a long-run mean-variance efficient yield.*

This statement makes no restriction on the set of traded assets. In particular, the proposition holds whether or not a riskfree payoff is traded.

6 The classic case with no outside income

Equation (26) is the most general statement of the formal portfolio results in this paper. However, equation (26) is a long way from traditional statements of mean-variance analysis, and the intuition that it expresses is closely tied to quadratic utility. Since intuition is most of the point of the paper, I rewrite (26) as a long-run version of standard statements of mean-variance analysis: I write the result in terms of the yield ("return") of the optimal payoff; I characterize preferences by risk aversion rather than by a bliss point; I express the optimal payoff in reference to a mean-variance efficient payoff on the upper part of the frontier, rather than the minimum second moment yield y^* which is on the lower part of the frontier; I express the optimal payoff as a set of distortions to mean-variance efficiency induced by nontraded income and preference shocks; and I express the optimal payoff relative to the market yield. I also describe the equilibrium pricing that results as CAPM or two-factor model.

6.1 Mean-variance frontier

I start with the special case that the investor has no hedgeable outside income or preference shocks. This classic special case simplifies the formulas a great deal, and shows the structure of the main ideas. The case with outside income and preference shocks then follows as a natural generalization.

Proposition 2 offers a more familiar expression of mean-variance portfolio ideas for this case:

Proposition 2. *If the bliss point hedge payoff $\underline{c}^b$ is constant, $\underline{c}^b = \bar{c}^b$, the outside-income hedge payoff $\underline{h} = 0$ and the riskfree yield y^f is traded, then the yield $\hat{y}$ of the investor's optimal payoff is on the long-run mean-variance frontier,*

$$\hat{y} = y^f + \frac{1}{\gamma}\left(y^f - y^*\right), \tag{29}$$

where γ is the investor's coefficient of risk aversion,

$$\frac{1}{\gamma} \equiv \frac{\bar{c}^b - y^f W}{y^f W}. \tag{30}$$

Derivation. Since $\bar{c}^b$ is constant and a riskfree rate is traded, $p(\bar{c}^b) = \bar{c}^b / y^f$. Then, from (26),

$$\begin{aligned}
\hat{x} &= \bar{c}^b - \left[\frac{\bar{c}^b}{y^f} - W\right] y^* \\
\hat{y} &= \frac{\hat{x}}{W} = \frac{\bar{c}^b}{y^f W} y^f - \left[\frac{\bar{c}^b}{y^f W} - 1\right] y^* \\
\hat{y} &= y^f + \left[\frac{\bar{c}^b}{y^f W} - 1\right]\left(y^f - y^*\right),
\end{aligned}$$

and (29) follows. From the characterization of the long-run mean-variance frontier given in Equation (18), equation (29) describes a long-run mean-variance efficient payoff.

For quadratic utility, the relative risk aversion coefficient at consumption c is

$$\frac{1}{\gamma(c)} \equiv -\frac{u'(c)}{cu''(c)} = \frac{c^b - c}{c}.$$

Thus, we interpret γ as defined by (30) as the local risk aversion coefficient, evaluated at a value of consumption which can be obtained by investing all wealth in the riskfree payoff.

Since y^* as minimum long-run second-moment yield is on the lower portion of the mean-variance frontier, expression (29) places the reference mean-variance efficient payoff on the more familiar, upper, portion of the frontier. We see a greater investment in this risky payoff for investors with lower risk aversion. The yield y^* is a particularly nice reference point for the mean-variance frontier, because the risky payoff weight is then the inverse of the relative risk aversion coefficient. The yield $-y^*$ is not the yield on the market payoff, however, unless average risk aversion happens to be one.

With a finite set of basis assets, (17) gives us an explicit expression of the optimal payoff,

$$\hat{y} = y^f + \frac{y^f}{\gamma} z^* = y^f + \frac{y^f}{\gamma} \tilde{E}(\mathbf{z})' \tilde{E} \left(\mathbf{z}\mathbf{z}'\right)^{-1} \mathbf{z}.$$

Proposition 2 is a result about long-horizon portfolio theory, in an environment with time-varying investment opportunities and incomplete markets. Yet the dynamic trading, rules for re-allocating wealth to securities based on state variables, and hedging demands for shocks to those state variables are absent from this representation. These demands can appear, in the *synthesis* of the mean-variance efficient payoff, if the market does not already offer the optimal payoffs, just as dynamic trading may be used to synthesize options that are not directly traded. But those demands do not appear in the description of the payoff itself. Similarly, the "risk aversion" governing payoff allocation in (29) is constant through time, though actual consumption and local risk aversion vary through time. The investor does not rebalance in response to such changes.

If a riskfree payoff is not traded, we can span the mean-variance frontier with the constant-mimicking payoff in its place:

Proposition 3. *If the bliss point c^b is constant $c^b = \bar{c}^b$ and the outside-income hedge payoff $\underline{h} = 0$, then the yield of the optimal payoff is on the long-run mean-variance frontier*

$$\hat{y} = y^c + \frac{1}{\gamma} \left(y^c - y^*\right),$$

where γ is the investor's coefficient of risk aversion,

$$\frac{1}{\gamma} \equiv \frac{\bar{c}^b - W/p(x^c)}{W/p(x^c)},$$

$x^c \equiv proj(1|\underline{X})$ is the constant-mimicking payoff, and $y^c \equiv x^c/p(x^c)$ is its yield

Derivation. Since c^b is constant, but a riskfree payoff is not traded, the projection $\underline{c}^b$ is proportional to the constant-mimicking payoff, $\underline{c}^b = c^b x^c$. Then, from (26),

$$\begin{aligned}
\hat{x} &= \bar{c}^b x^c - \left[\bar{c}^b p(x^c) - W\right] y^* \\
\hat{y} &= \frac{\bar{c}^b p(x^c)}{W} y^c - \left[\frac{\bar{c}^b p(x^c)}{W} - 1\right] y^* \\
\hat{y} &= y^c + \left[\frac{\bar{c}^b - W/p(x^c)}{W/p(x^c)}\right] \left(y^c - y^*\right)
\end{aligned}$$

6.2 Payoffs relative to the market

As in one-period setups, the average investor must hold the market payoff. Therefore, it is useful to express the optimal payoff relative to the market payoff, in an equilibrium in which investors have limited and described forms of heterogeneity. This expression is also useful because it's easy to name and construct the market payoff.

Proposition 4. *If all investors are of this type, (quadratic utility, constant bliss point, no outside income) the yield of each investor's optimal payoff is split between the riskfree yield and the yield of the market payoff,*

$$\hat{y}_i = y^f + \frac{\gamma_a}{\gamma_i}\left(\hat{y}_a - y^f\right). \tag{31}$$

The yield on the market payoff is defined by a wealth-weighted average

$$\hat{y}_a \equiv \frac{\sum_j W_j \hat{y}_j}{\sum_j W_j} = \frac{\sum_j \hat{x}_j}{\sum_j W_j} = \frac{\hat{x}_a}{W_a},$$

and it is a claim to the aggregate consumption stream. Aggregate risk aversion is defined as a wealth-weighted average of individual risk aversions,

$$\frac{1}{\gamma_a} \equiv \frac{\sum_j W_j \frac{1}{\gamma_j}}{\sum_j W_j}.$$

Derivation. Start with (29),

$$\hat{y}_i = y^f + \frac{1}{\gamma_i}\left(y^f - y^*\right).$$

Sum over investors, and divide by wealth,

$$\begin{aligned} \frac{\sum_j W_j \hat{y}_j}{\sum_j W_j} &= y^f + \frac{\sum_j W_j \frac{1}{\gamma_j}}{\sum_j W_j}\left(y^f - y^*\right) \\ \hat{y}_a &= y^f + \frac{1}{\gamma_a}\left(y^f - y^*\right) \\ y^f - y^* &= \gamma_a\left(\hat{y}_a - y^f\right). \end{aligned} \tag{32}$$

Substitute this result in the right hand side of (29).

I use subscripts such as y_i and y_a to distinguish who holds a payoff, and superscripts such as y^i, y^*, y^{mv} to differentiate assets.

The infinitely risk-averse investor holds the perpetuity y^f, as pointed out by Campbell and Viceira (2001) and Wachter (2003). An investor whose risk aversion is the same as that of the average investor just holds the market payoff, as he must, and that market payoff pays aggregate consumption as its dividend.

The infinite risk aversion and average-investor theorems hold in more general models. The novelty here is a way of drawing the line between these two extremes: investors with risk aversion greater than average, but still finite, purchase a payoff that is a linear combination of the market payoff and the real perpetuity. This linear interpolation is what requires quadratic utility.

If a riskfree payoff is not traded, Proposition 3 leads to the same representation as Proposition 4 (31), with the constant-mimicking portfolio yield y^c in place of the riskfree rate y^f. This result is a central point of Magill and Quinzii (2000), who call y^c the "least variable" stream. I use a different language because constant-mimicking payoff yield y^c is not the minimum long-run variance yield. We can also describe the frontier with any other frontier security, such as the minimum long-run variance yield, or the zero-beta yield for y^a. The remaining propositions generalize to the case that a riskfree yield is not traded in the same way.

The market payoff is always traded. By definition, the market payoff is the average payoff that each investor chooses among the traded payoffs. Thus, these statements otherwise encompass general market incompleteness.

6.3 A long-run CAPM

In an equilibrium of investors who are all of the same type, but vary by risk aversion, equilibrium prices follow a long-run version of the CAPM.

Proposition 5. *For each asset i, the long-run expected yield follows a long-run CAPM,*

$$\tilde{E}(y^i) - y^f = \tilde{\beta}_{i,a}\left[\tilde{E}(\hat{y}^a) - y^f\right] \tag{33}$$

where $\tilde{\beta}_{i,a}$ is a long-run regression coefficient of yield i on the market yield.

This proposition follows simply from the long-run Roll theorem (21) and the fact (32) that the market payoff is long-run mean-variance efficient.

Mertonian state variables for time-varying investment opportunities disappear from long-run expected yields, as they disappear from the optimal payoff, even though there can be a complex ICAPM representation of one-period returns.

Long-run betas are all "cashflow betas," not "valuation betas," or "discount-rate" betas which dominate short-run return correlations. Long-run betas are thus a bit more plausibly "exogenous" than one-period betas. If we regard cashflows as "exogenous," then the initial price is really the "endogenous" variable: $y^i = x^i/p(\{x_i\})$, and (33) describes how $p(\{x_i\})$ is formed. I use quotes because everything is endogenous in general equilibrium production (not Lucas-tree or endowment) economies, but the informal habit of reading causality from betas to expected returns does have a quite different flavor in the long-run context, since the betas are pure cashflow betas.

7 Outside income and preference shocks

Next, I allow outside income and preference shocks, potentially correlated with traded asset payoffs. I separate the preference-shock hedge payoff into a constant and a variable component,

$$\underline{c}^b = \bar{c}^b \times 1 + \tilde{c}^b; \tilde{E}\left(\tilde{c}^b\right) = 0.$$

Proposition 1, (26), now reads

$$\hat{x} = \left(\bar{c}^b + \tilde{c}^b - \underline{h}\right) - \left[p(\bar{c}^b + \tilde{c}^b - \underline{h}) - W\right] y^*. \tag{34}$$

The variable part $\tilde{c}^b$ of the bliss point hedge payoff $\underline{c}^b$ and the outside income hedge payoff $\underline{h}$ enter together in everything that follows from this equation, so to save some space I combine them in what follows; I use the symbol $\underline{h}$ to denote the quantity $\underline{h} - \tilde{c}^b$, I write only "outside income hedge payoff" to refer to both components, and I start with (34) written as

$$\hat{x} = \left(\bar{c}^b - \underline{h}\right) - \left[p(\bar{c}^b - \underline{h}) - W\right] y^*. \tag{35}$$

Now, the investor sells the hedge payoff for outside income $(-\underline{h})$ and then invests the proceeds $p(\underline{h})$, along with any wealth W, in the long-run mean-variance efficient payoff y^*.

There are several different, equivalent representations of the optimal payoff and pricing with outside income, each of which offers a different intuition.

7.1 Mean-variance frontier and risk aversion

With outside income, we can say that the mean-variance characterization now applies to the yield of the "total payoff" $\hat{y}^T$, which consists of the payoff $\hat{x}$ that the investor holds directly in asset markets, plus the outside-income hedge payoff $\underline{h}$ that the investor holds implicitly:

Proposition 6. *The total yield is on the long-run mean-variance frontier,*

$$\hat{y}^T = y^f + \frac{1}{\gamma}\left(y^f - y^*\right). \tag{36}$$

The total yield and risk aversion are defined here as

$$\hat{y}^T \equiv \frac{\hat{x} + \underline{h}}{W + p(\underline{h})} \tag{37}$$

$$\frac{1}{\gamma} \equiv \frac{\bar{c}^b - y^f\left[W + p(\underline{h})\right]}{y^f\left[W + p(\underline{h})\right]}. \tag{38}$$

Derivation. Since $\bar{c}^b$ is constant and a riskfree rate is traded, $p(\bar{c}^b) = \bar{c}^b/y^f$. Then, from (34),

$$\begin{aligned} \hat{x} &= \left(\bar{c}^b - \underline{h}\right) - \left[p(\bar{c}^b - \underline{h}) - W\right] y^* \\ \frac{\hat{x} + \underline{h}}{W + p(\underline{h})} &= \frac{\bar{c}^b}{y^f \left[W + p(\underline{h})\right]} y^f - \left[\frac{\bar{c}^b}{y^f \left[W + p(\underline{h})\right]} - 1\right] y^* \\ \hat{y}^T &= y^f + \left[\frac{\bar{c}^b}{y^f \left[W + p(\underline{h})\right]} - 1\right] \left(y^f - y^*\right), \end{aligned}$$

and (36) follows.

For this representation, we interpret γ as defined by (38) as the local risk aversion coefficient, evaluated at a bliss point $\bar{c}^b$ and a value of consumption in which the investor invests all wealth, including the proceeds from shorting the outside-income hedge payoff, in the riskfree asset.

Time-varying investments and hedging demands for state-variable shocks are still absent in this representation, now including state variables for outside income, though such hedging demands may well appear in the dynamic strategies required to synthesize optimal payoffs. The outside income stream, represented in its hedge payoff $\underline{h}$, does appear in the optimal payoff, as it appears in familiar one-period mean-variance problems.

7.2 Portfolio distortions

Saying the total payoff is long-run mean-variance efficient is helpful, but in order to figure out what assets to buy $\hat{x}$ the investor needs to subtract the outside-income hedge payoff from the long-run mean-variance efficient total payoff. Buying the frontier and then shorting a hedge payoff will mean buying and selling the same assets, which may be costly. (On the other hand, it may not: One could imagine an asset market with a common "frontier" portfolio sold to everybody, and specialized "hedging" portfolios separately tailored to people with different outside-income risk characteristics.) For these reasons, it is useful to characterize the asset payoff directly. Finally, it's useful to characterize the optimal asset payoff as a set of distortions relative to the mean-variance frontier and then relative to the market payoff.

To characterize the asset payoff, we can start by explicitly subtracting the outside-income hedge payoff from the mean-variance efficient total payoff. In equations, we can write that the yield $\hat{y} = \hat{x}/W$ of the optimal asset payoff is

$$s_W \left(\hat{y} - y^f\right) = \frac{1}{\gamma} \left(y^f - y^*\right) - s_h \left(y^h - y^f\right), \tag{39}$$

where y^h denotes the yield of the outside-income hedge payoff

$$y^h \equiv \frac{\underline{h}}{p(\underline{h})},$$

and the wealth shares s_W, s_h are defined as

$$s_W \equiv \frac{W}{W + p(\underline{h})}; \; s_h \equiv \frac{p(\underline{h})}{W + p(\underline{h})}.$$

We do not have $p(h) = p(\underline{h})$; we do not know how to assign prices for nontraded payoffs. Hence, though I call $W + p(\underline{h})$ "total wealth," it really is only "asset wealth plus the value of the outside-income hedge payoff." Equation (39) follows quickly from (36) by recognizing that $\hat{y}^T = s_W \hat{y} + s_h y^h$.

Equation (39) nicely generalizes mean-variance advice: just subtract the outside-income hedge payoff from the mean-variance payoff. I found it prettiest to express this result in zero-cost form, but corresponding expressions for $\hat{y}$ itself are straightforward.

Next, the outside-income hedge payoff $\underline{h}$ will usually contain some riskfree yield y^f and some of the mean-variance yield $\left(y^f - y^*\right)$. Expression (39) still contains a lot of buying something and simultaneously selling it, a potentially inefficient way to describe a portfolio. It is therefore interesting to describe the investor's asset yield $\hat{y}$ or payoff $\hat{x}$, more directly, as follows:

Proposition 7: *Break the outside-income hedge yield into three components, defined by the long-run regression*

$$y^h = y^f + \tilde{\beta}_h \left(y^f - y^*\right) + \eta^h. \tag{40}$$

Then the yield of the optimal payoff in the presence of outside income can be expressed as

$$s_W \left(\hat{y} - y^f\right) = \left(\frac{1}{\gamma} - s_h \tilde{\beta}_h\right) \left(y^f - y^*\right) - s_h \eta^h. \tag{41}$$

Derivation: Substitute (40) into (39).

Proposition 7 naturally directs the investor to lower his exposure to the mean-variance efficient payoff, as if he were more risk averse, if his outside-income hedge payoff is highly long-run correlated $\tilde{\beta}_h$ with the mean-variance efficient payoff.

The payoff η^h is the orthogonalized and idiosyncratic $\tilde{E}(y^* \eta^h) = 0$, zero-cost $p(\eta^h) = 0$ and zero-mean $\tilde{E}(\eta^h) = 0$ component of the outside-income hedge payoff. It constitutes free insurance against outside income risks. Selling (or buying) this payoff ought to be the *first* thing every investor does. Characterizing such payoffs is an important task for academic portfolio advice.

The idiosyncratic component η^h varies over time as well as across states of nature. For example, if the investor has a certain wage stream and retires at a given date with certainty, then η^h goes from a positive to a negative loading on the indexed perpetuity on the retirement date, and "short η^h" generates the usual shift from stocks to bonds at that date.

7.3 Shares and the low-wealth limit

In (39) and (41), as wealth W and hence the share of asset wealth s_W declines, the investor becomes less risk averse, allocating larger shares to the risky assets. Similarly, when the investor has no outside wealth $p(\underline{h}) = 0$, risk aversion itself in (30) or (38) declines as wealth declines, and the risky payoff share explodes.

This behavior is really an effect of units. From (39), the asset payoff $\hat{x}$ consists of the sale of the outside-income hedge payoff $\underline{h}$, and then investment of the proceeds $p(\underline{h})$ along with asset wealth W in the long-run mean-variance efficient payoff,

$$\hat{x} = [W + p(\underline{h})] \left[y^f + \frac{1}{\gamma} \left(y^f - y^* \right) \right] - \underline{h} \tag{42}$$

As $W \to 0$, the asset payoff $\hat{x}$ becomes a pure zero-cost payoff. The investor just shorts the outside income hedge payoff and invests the proceeds in a mean-variance payoff, with the risk aversion that an investor with wealth $p(\underline{h})$ would display. But the value of the asset payoff remains $p(\hat{x}) = W$. By dividing the asset payoff by its value, asset wealth W, the yield on this payoff appears enormously risky.

One might reexpress the results just in terms of payoffs $\hat{x}$ rather than yields as in (42), or scale the asset payoff $\hat{x}$ by total wealth $W + p(\underline{h})$. But I think that expression would be just as misleading, and the tradition of expressing portfolio results in terms of returns (yields) is a good one. In the real world we – investors, financial institutions, and regulators – often do look at the value and riskiness of asset portfolios in isolation, without trying to assess the value $p(\underline{h})$ or the risk characteristics $\underline{h}$ of outside income. Investors with substantial outside income and little asset wealth *should* hold nearly zero-cost hedging payoffs that, taken on their own, look very risky. Taking out a mortgage and investing in the stock market is a classic example.

The $W = 0$ and hence $\gamma = 0$ limit without outside income is less interesting. Looking back at Propositions 1 and 2, a quadratic-utility investor with no financial wealth $W = 0$ and no human wealth $p(\underline{h}) = 0$ invests anyway, in a zero-cost payoff

$$\hat{x} = \frac{\bar{c}^b}{y^f} \left(y^f - y^* \right).$$

The price $p(\hat{x}) = 0$, so the yield $\hat{y} = \hat{x}/p(\hat{x})$ of this zero-price payoff is not defined. This payoff is on the mean-variance frontier of zero-cost payoffs (20). To incorporate this case, we could write the optimal payoff in zero-cost form as

$$\hat{x} = Wy^f + \left[\frac{\bar{c}^b}{y^f} - W \right] \left(y^f - y^* \right) \tag{43}$$

However, realistic utility functions, which do not extend to negative consumption, prescribe zero investment $\hat{x} = 0$ for $W = 0$ and $p(\underline{h}) = 0$, and not zero risk aversion.

7.4 Payoffs relative to the market

Again, we want to express the optimal payoff relative to the well-defined market payoff. That translation is especially useful in this case, as the market payoff is likely not to be long-run mean-variance efficient.

Consider a market of investors who are all of the same type (quadratic utility), but with heterogeneous risk aversion (bliss point, initial wealth) and also heterogeneous outside income streams. Now investors think about how their outside income stream differs from the market average outside income stream, as well as how their risk aversion differs from the market average risk aversion. Again, several equivalent representations give different kinds of intuition.

First, the two-fund theorem still applies. The investor's *total* payoff, including the outside-income hedge payoff he holds implicitly, is split between the index perpetuity and the market's *total* payoff $\hat{y}_a^T$, which adds outside-income hedge payoff of the average investor to the actual market payoff $\hat{y}_a$. Total payoffs are mean-variance efficient. Actual asset payoffs are not.

Proposition 8. *When all investors are of this type (quadratic utility, outside income), each investor's total payoff is proportional to the aggregate total payoff, which is a claim to the traded component of aggregate consumption:*

$$\hat{y}_i^T = y^f + \frac{\gamma_a}{\gamma_i}\left(\hat{y}_a^T - y^f\right), \tag{44}$$

where the yield on the total aggregate payoff is

$$\hat{y}_a^T \equiv \frac{\sum_j \left[W_j + p(\underline{h}_j)\right] \hat{y}_j^T}{\sum_j \left[W_j + p(\underline{h}_j)\right]} = \frac{\sum_j \hat{x}_j + \underline{h}_j}{\sum_j \left[W_j + p(\underline{h}_j)\right]} = \frac{\hat{x}_a + \underline{h}_a}{W_a + p(\underline{h}_a)},$$

and aggregate risk aversion is defined as a wealth-weighted average of individual risk aversion,

$$\frac{1}{\gamma_a} \equiv \frac{\sum_j \left[W_j + p(\underline{h}_j)\right] \frac{1}{\gamma_j}}{\sum_j \left[W_j + p(\underline{h}_j)\right]}.$$

Derivation. Start with (36)

$$\hat{y}_i^T = y^f + \frac{1}{\gamma_i}\left(y^f - y^*\right). \tag{45}$$

Sum over people and divide by total wealth,

$$\begin{aligned}
\frac{\sum_j \left[W_j + p(\underline{h}_j)\right] \hat{y}_j^T}{\sum_j \left[W_j + p(\underline{h}_j)\right]} &= y^f + \frac{\sum_j \left[W_j + p(\underline{h}_j)\right] \frac{1}{\gamma_j}}{\sum_j \left[W_j + p(\underline{h}_j)\right]}\left(y^f - y^*\right) \\
\hat{y}_a^T &= y^f + \frac{1}{\gamma_a}\left(y^f - y^*\right) \\
y^f - y^* &= \gamma_a\left(\hat{y}_a^T - y^f\right).
\end{aligned}$$

Substitute this result in the right hand side of (45).

The interpretation of the aggregate total payoff as a claim to the traded component of aggregate consumption follows from the aggregated budget constraint $c_a = x_a + h_a$, and the definition $\underline{h}_a = proj(h_a|\underline{X})$, so

$$\begin{aligned} \hat{y}_a^T &= (\hat{x}_a + \underline{h}_a) / [W_a + p(\underline{h}_a)] \\ &= proj(c_a|\underline{X})/p[proj(c_a|\underline{X})]. \end{aligned}$$

Equation (44) is an optimal partial risk-sharing result: the payoff $\hat{y}^T$ is split according to risk aversion, with less risk-averse people getting a more variable stream. Asset payoffs are reduced when individuals get a large outside-income hedge payoff. It's only a partial risk sharing result: Investors also bear the risks of the non-hedgeable components of outside income. Optimal payoffs engineer as much risk sharing as is possible in the traded asset markets.

To understand the optimal asset payoffs – what the investor should buy, in the end – it is again useful to break the optimal payoff up into components with different economic functions.

Proposition 9. *The yield of the investor's optimal asset payoff can be written in terms of the market asset yield, the market average outside-income hedge yield, and the individual outside-income hedge yield as*

$$s_{Wi}\left(\hat{y}_i - y^f\right) = \frac{\gamma_a}{\gamma_i} s_{Wa}\left(\hat{y}_a - y^f\right) + \frac{\gamma_a}{\gamma_i} s_{ha}\left(y_a^h - y^f\right) - s_{hi}\left(y_i^h - y^f\right) \tag{46}$$

where

$$\begin{aligned} y_a^h &\equiv \frac{\underline{h}_a}{p(\underline{h}_a)}; y_i^h \equiv \frac{\underline{h}_i}{p(\underline{h}_i)} \\ s_{Wa} &\equiv \frac{W_a}{W_a + p(\underline{h}_a)}; s_{Wi} \equiv \frac{W_i}{W_i + p(\underline{h}_i)} \\ s_{ha} &\equiv \frac{p(\underline{h}_a)}{W_a + p(\underline{h}_a)}; s_{hi} \equiv \frac{p(\underline{h}_i)}{W_i + p(\underline{h}_i)}. \end{aligned}$$

Derivation. From (44),

$$\gamma_i\left(\hat{y}_i^T - y^f\right) = \gamma_a\left(\hat{y}_a^T - y^f\right)$$

$$\gamma_i\left[\left(\frac{\hat{x}_i + \underline{h}_i}{W_i + p(\underline{h}_i)}\right) - y^f\right] = \gamma_a\left(\frac{\hat{x}_a + \underline{h}_a}{W_a + p(\underline{h}_a)} - y^f\right)$$

$$\gamma_i\left[s_{Wi}\hat{y}_i + s_{hi}y_i^h - y^f\right] = \gamma_a\left[s_{Wa}\hat{y}_a + s_{ha}y_a^h - y^f\right]$$

$$\gamma_i\left[s_{Wi}\left(\hat{y}_i - y^f\right) + s_{hi}\left(y_i^h - y^f\right)\right] = \gamma_a\left[s_{Wa}\left(\hat{y}_a - y^f\right) + s_{ha}\left(y_a^h - y^f\right)\right].$$

Equation (46) follows.

The first term in (46) describes how the investor takes on market risk. As before, investors who are temperamentally more risk averse than average $\gamma_i < \gamma_a$ take on less market risk. The units effect of wealth shares s_W again affects market risk. The *yield* of the investor's market portfolio will appear very risky, and his effective risk aversion very low, if he is predominantly financing investment from the sale of an outside-income hedge payoff. But now, we relate individual to average: if the average investor is also financing most of his asset payoff by selling an outside-income hedge payoff, then the individual investor will again just buy the market payoff. The market payoff itself will have a very risky yield in this circumstance, an intriguing observation given the puzzlingly large risk of stock market investment.

Next, the investor moves away from the market payoff towards the average outside-income hedge payoff. Here, the investor provides insurance to the average investor, and typically earns a premium for doing so. An individual who is very risk averse, either intrinsically or because he has a very large share of asset wealth in his portfolio, will display a yield that reflects less outside-income insurance to the market.

Finally, the investor shorts his own outside-income hedge payoff. Here risk aversion is irrelevant; the investor shorts the whole thing.

The latter two effects can offset, if the investor's hedge payoff is similar to the average payoff. For example, if the investor has average adjusted risk aversion, $s_{Wi}\gamma_i = s_{Wa}\gamma_a$, and an average split between the value of outside and asset wealth $s_i^h = s_a^h$, then (46) reduces to

$$s_W \left(\hat{y}_i - \hat{y}_a\right) = s_h \left(y_a^h - y_i^h\right).$$

The yield on this investor's optimal payoff starts with the market yield, then holds only the *difference* between aggregate and individual outside income hedge payoff. He buys from the average investor the payoff that the average investor would like to short, and shorts the payoff that best hedges his own outside income.

As in Proposition 7, it is interesting to orthogonalize the payoffs, in part to avoid the latter simultaneous buying and selling of hedge payoffs. Define components by successive long-run regressions to define orthogonalized yields ε_a, the component of the aggregate outside-income hedge payoff orthogonal to the market, and η_i^h, the component of the individual outside-income hedge payoff orthogonal to both market and aggregate outside-income hedge payoff, by

$$y_a^h - y^f = \tilde{\beta}_{ha,a}\left(\hat{y}_a - y^f\right) + \varepsilon_a,$$

$$\begin{aligned} y_i^h - y^f &= \tilde{\beta}_{hi,a}\left(\hat{y}_a - y^f\right) + \tilde{\beta}_{hi,ha}\varepsilon_a + \eta_i^h. \\ &= \tilde{\beta}_{hi,a}\left(\hat{y}_a - y^f\right) + \tilde{\beta}_{hi,ha}\left(y_a^h - y^f\right) + \eta_i^h. \end{aligned}$$

The three payoffs $\left(\hat{y}_a - y^f\right)$, ε_a, and η_i^h are all zero cost. In addition, $\tilde{E}(\eta_i^h) = 0$. The payoff η_i^h is again a zero-price, zero-mean, "idiosyncratic" component of the outside income hedge payoff – the part of the individual outside-income hedge payoff not spanned by a constant, the market payoff or the average outside-income hedge payoff. The last two

lines deliver the same $\tilde{\beta}$ and definition of η_i^h, and offer different ways to think about the orthogonalization.

Substituting these definitions in (46), we obtain

Proposition 10. *The investor's asset portfolio can be written in terms of the market asset yield, the orthogonalized market average outside-income hedge yield, and the idiosyncratic individual outside-income hedge yield as*

$$\begin{aligned} s_{Wi}\left(\hat{y}_i - y^f\right) &= \left[\frac{\gamma_a}{\gamma_i}s_{Wa} + \frac{\gamma_a}{\gamma_i}s_{ha}\tilde{\beta}_{ha,a} - s_{hi}\tilde{\beta}_{hi,a}\right]\left(\hat{y}_a - y^f\right) \\ &\quad + \left[\frac{\gamma_a}{\gamma_i}s_{ha} - s_{hi}\tilde{\beta}_{hi,ha}\right]\varepsilon_a - s_{hi}\eta_i^h. \end{aligned} \tag{47}$$

The first term in brackets now adjusts the yield of the investor's optimal payoff for different market exposures of the individual vs. the average outside income hedge payoff, in addition to already-explored dimensions of heterogeneity. For example, if the individual's outside income hedge payoff is highly long-run correlated with the market payoff, $\tilde{\beta}_{hi,a}$ is large, we might think that the individual should scale back his market exposure. But if everyone else's outside-income hedge payoff is also highly correlated with the market payoff, $\tilde{\beta}_{ha,a}$ is also large, and the individual is otherwise average ($\gamma_i = \gamma_a, s_{Wi} - s_{Wa}$), then the individual just holds the market payoff.

The second term in brackets adjusts the individual's exposure to the orthogonalized average outside income hedge payoff, ε_a. The investor with no or uncorrelated outside-income, $\tilde{\beta}_{hi,ha} = 0$, buys the market average outside income hedge payoff, deviating from market weights to do so, and earning a premium for taking this risk. But now we recognize that by selling his own outside-income hedge payoff, the investor may end up undoing this investment. An investor whose outside-income hedge payoff is highly correlated with the average, $\tilde{\beta}_{hi,ha}$ is large, will have to avoid this opportunity. This is a cleaner statement than Proposition 9, which would have this investor simultaneously buy and sell the same thing.

Finally, the investor sells the completely idiosyncratic, zero-price, zero-mean component of his outside-income hedge payoff η_i^h.

7.5 Equilibrium with outside income

Next, I characterize the equilibrium of investors with heterogenous outside income streams as well as heterogenous risk aversion. Since the average total payoff $\hat{y}_a^T$ is long-run mean-variance efficient in Proposition 8, and using the long-run Roll theorem (21), we have a long-run version of the total-wealth CAPM,

Proposition 11. *The expected long-run yield of each asset i follows a CAPM using the aggregate total yield as reference payoff:*

$$\tilde{E}(y^i) - y^f = \tilde{\beta}_{i,T}\left[\tilde{E}(\hat{y}_a^T) - y^f\right]. \tag{48}$$

Again, $\hat{y}^T$ defined in (37) is the yield on the asset payoff plus the hedge payoff for outside income. Thus, it's not quite correct to say that "total wealth" or the "total income stream" must appear in the CAPM reference portfolio. What appears is the *hedge payoff* for total income.

The total yield is not the traded asset-market yield. Therefore, it's interesting to separate out the asset market yield as a first factor, and therefore to express pricing in a two-factor model in place of the total-income CAPM (48):

Proposition 12. *The expected long-run yield of each asset i follows a multifactor model, with the market payoff and average outside-income hedge payoff as factors,*

$$\tilde{E}(y^i) - y^f = \tilde{\beta}_{i,a}\left[\tilde{E}(\hat{y}_a) - y^f\right] + \tilde{\beta}_{i,h}\left[\tilde{E}(y_a^h) - y^f\right], \tag{49}$$

where $\tilde{\beta}_{i,a}$ and $\tilde{\beta}_{i,h}$ are long-run multiple regression coefficients.

Proposition 12 follows directly from Proposition 11, and the fact that the total yield is a linear combination of the asset yield and average outside income hedge yield, $\hat{y}_a^T = s_{Wa}\hat{y}_a + s_{ha}y_a^h$.

In pricing as in portfolios, Mertonian state variables for outside income and investment opportunities disappear. When the tradeable component of outside income does not average to zero $y_a^h \neq 0$, a second pricing factor emerges in addition to the market payoff. Assets have higher long-run expected yields if their cashflows have a higher long-run covariance with aggregate outside income. For example, if as Fama and French (1996) speculate, the average investor's outside income is correlated with the payouts of a class of "distressed" securities, then those securities will require higher long-run expected yields, and they will receive lower prices. We see a "value" effect in prices and a "value factor" in long-run expected yields.

We also can represent pricing with orthogonalized factors, which are possibly more interesting. Define again ε_a by a long-run regression.

$$y_a^h - y^f = \tilde{\beta}_{ha,a}\left(\hat{y}_a - y^f\right) + \varepsilon_a$$

Then, our multifactor model can use ε_a in place of y_a^h and become:

$$\tilde{E}(y^i) - y^f = \tilde{\beta}_{i,a}\left[\tilde{E}(\hat{y}_a) - y^f\right] + \tilde{\beta}_{i,\varepsilon}\tilde{E}(\varepsilon_a) \tag{50}$$

Comparing the pricing results (49) and (50) to the optimal payoff expressions (46) and (47), we see the same right hand variables. The payoff expressions (46) and (47) tell

the investor how much to put in to the "priced assets" corresponding to the aggregate market portfolio and aggregate outside-income-hedge payoff, and then to perfectly hedge residual, zero-price, mean-zero, risk. The payoff shares advocated by (46) and (47) are given by "risk aversions" which combine true risk aversion, and aversions induced by the character of outside income.

The resulting picture is similar to that painted by Fama (1996), (illustrated in three dimensions in Cochrane (2011)) who describes portfolio choice between "multifactor efficient" portfolios which are also the pricing factors.

8 Towards calculations

In this section, I use two simple environments – the standard permanent income model, and the lognormal i.i.d. environment – to investigate the connections between portfolios and payoffs.

These are completely standard and well-understood environments. The point is to view these environments through the concepts in this paper: I characterize the payoff spaces (dividend streams) created by dynamic trading, I find discount factors in the payoff spaces, I characterize the hedge payoff for outside income, and I find the optimal payoff. Connecting views, I characterize the dynamic trading strategies that synthesize optimal payoffs when those payoffs are not directly marketed.

This investigation admittedly runs a bit counter to the philosophy articulated in the introduction, that one should not focus much attention on payoff engineering. However, building up payoff streams from underlying investment opportunities is not as simple as it seems, and giving a clear example of the connection between payoffs and portfolios in well known environments is a natural step.

8.1 Permanent income model and standard results

Consider the standard quadratic utility permanent-income equilibrium model, with an AR(1) income stream and interest rate equal to discount rate $\rho = r^f$:

$$\max E \int_{t=0}^{\infty} e^{-r^f t} \left(-\frac{1}{2}\right) (c^b - c_t)^2 dt \quad \text{s.t.} \tag{51}$$

$$\begin{aligned} dW_t &= \left(r^f W_t + h_t - c_t\right) dt; \ \lim_{t\to\infty} E\left(e^{-r^f t} W_t\right) = 0 \qquad (52)\\ dh_t &= -\phi h_t dt + \sigma dB_t. \end{aligned}$$

A quick recap of the standard results: The equilibrium consumption process follows the familiar permanent-income rule,

$$c_t = r^f W_t + \frac{r^f}{r^f + \phi} h_t \tag{53}$$

and hence a random walk, whose innovations are the innovations to that present value,

$$dc_t = \frac{r^f}{r^f + \phi}\sigma dB_t. \tag{54}$$

More explicit solutions for income and consumption are

$$h_t = h_0 e^{-\phi t} + \int_{s=0}^{t} e^{-\phi(t-s)}\sigma dB_s, \tag{55}$$

$$c_t = r^f W_0 + \frac{r^f}{r^f + \phi}h_0 + \frac{r^f \sigma}{r^f + \phi}\int_{s=0}^{t} dB_s. \tag{56}$$

As a general-equilibrium model, we think of W_t as the capital stock, and the interest rate and budget constraint (52) represent a linear production technology with constant marginal product of capital. Alternatively, they can represent international borrowing and lending at a constant real rate with costless transport.

8.1.1 Payoff space, discount factor, and frontier

So far, this is a representative-agent general equilibrium model. We can introduce a variety of more or less complete asset markets. For example, in the textbook use of this model, we specify complete markets, and we find from marginal utility the equilibrium prices of consumption and the outside income stream h_t, just sufficient that the investor chooses to consume c_t and not to market h_t. These prices reflect interesting time-varying risk premia.

Here, it is more interesting to specify that there is a single traded asset, the constant riskfree rate, and that h_t is a non-marketed income stream. Doing so lets me characterize a hedge payoff for the nontraded income stream.

Though the interest rate is the only available asset return, the investor can create payoffs by dynamic saving and dissaving. The payoff space $\underline{X}$ generated in this way is any payout stream $\{x_t\}$ consistent with

$$dV_t = \left(r^f V_t - x_t\right) dt \tag{57}$$

and

$$V_t < \infty;\ \lim_{t\to\infty}\left(e^{-r^f t}V_t\right) = 0 \tag{58}$$

or equivalently

$$V_0 = \int_{t=0}^{\infty} e^{-r^f t}x_t dt < \infty. \tag{59}$$

The latter conditions do not have expectations: In a T period model, we have $V_T = 0$ in each state. The corresponding limit must hold for each path.

Equation (57) represents a *payout policy*. The payoff x_t is the amount saved or removed from wealth built up in the riskfree investment. With a single asset this example does

not have an interesting *portfolio policy*, state-contingent allocation of wealth to different investment opportunities.

We are used to thinking that riskfree investment means nonstochastic payoffs. But by changing the payout x_t in response to events – for example income shocks dB_t – the investor can create state- and time-dependence in the payoff stream x_t that is not present in the asset market. But the investor cannot create arbitrary contingent claims. A strategy that withdraws x_t today must pay it back in present value sense, or the trading conditions (58)-(59) are violated. Similarly, since $y_t = x_t/V_0$, (59) implies that traded yields must obey

$$\int_{t=0}^{\infty} e^{-r^f t} y_t dt = 1, \tag{60}$$

ex-post as well as ex-ante. The "riskfree" nature of the asset, and the incompleteness of the market, implies that the *ex-post* integrals (59)-(60) must be nonstochastic, not that each x_t or y_t cannot vary.

The integrals in (59)-(60) are the "long run" payoffs inside the long-run mean operator $\tilde{E}$. Thus, the constant interest rate requires that long-run payoffs do not vary across states of nature, much as a constant interest rate induces one-period returns which do not vary across states of nature. This fact does not imply that long-run variance is zero, however. The variance of long-run returns is a different object from the long-run variance of returns. A path that borrows and then repays has a larger long-run variance than a constant payoff. The quantity $\int_{t=0}^{\infty} e^{-r^f t} y_t^2 dt$ may vary across states of nature even though $\int_{t=0}^{\infty} e^{-r^f t} y_t dt$ does not.

The traded discount factor payoff is simply $x_t^* = \exp[(\rho - r^f)t] = 1$. This x^* is the dividend process of a trading strategy of the form (57): $dV_t = (r^f V_t - 1)dt$ implies $V_t = 1/r^f$. Be careful: You can only price *traded* payoffs with this discount factor. In particular you cannot price $\{h_t\}$ or $\{c_t\}$ with $x^* = 1$, because those payoffs are not traded. (Although $dW_t = \left(r^f W_t + h_t - c_t\right) dt$ leads to a convergent W_t path, $dV_t = (r^f V_t - c_t)dt$ or $dV_t = (r^f V_t - h_t)dt$ separately do not.) The familiar equations for the price of the consumption or endowment stream in this model use marginal utility (non traded, here) as a discount factor, and generate risk premiums that $x_t^* = 1$ does not generate.

Next, with $x^* = 1$, $p(x^*) = 1/r^f$ so $y^* = r^f = y^f$ and the long-run mean-variance efficient payoff space collapses to the point y^f. Traded payoffs generated by dynamic trading of the risk-free rate must obey risk-neutral pricing, which means a degenerate frontier. Directly, (60) implies that for any traded $\{y_t\}$, $\tilde{E}(y) = r^f$. A spectrum of assets then lies on a flat line in long-run mean-variance space to the right of the riskfree payoff.

8.1.2 Optimal payoff and outside-income hedge payoff

In this simple model, we can find the optimal asset payoff – which consists of withdrawals or additions to wealth or capital stock – as a consequence of the familiar solutions,

$$\hat{x}_t = c_t - h_t,$$

with c_t and h_t solutions as given above. The investor saves $\hat{x}_t < 0$ when income h_t is greater than consumption c_t and vice versa.

The point is to understand this result in the long-run portfolio theory framework. Proposition 1 characterized the optimal payoff as

$$\hat{x} = \left(\underline{c}^b - \underline{h}\right) - \left[p(\underline{c}^b - \underline{h}) - W_0\right] y^*. \tag{61}$$

Here, c^b is a constant, so tradeable and $\underline{c}^b = \bar{c}^b$ with $p(\underline{c}^b) = \bar{c}^b/r^f$. We also have $y^* = y^f = r^f$. The interesting component of (61) is the hedge payoff for outside income $\underline{h}$. It is given by

$$\underline{h}_t = h_t - \frac{r^f}{r^f + \phi} \int_{s=0}^{t} \sigma dB_s, \tag{62}$$

or, directly,

$$\underline{h}_t = h_0 e^{-\phi t} + \int_{s=0}^{t} \left(e^{-\phi(t-s)} - \frac{r^f}{r^f + \phi} \right) \sigma dB_s. \tag{63}$$

We can also write the hedge payoff $\underline{h}_t$ in state-variable form,

$$\underline{h}_t = h_t - \left[r^f \left(W_t - W_0 \right) + \frac{r^f}{r^f + \phi} \left(h_t - h_0 \right) \right]. \tag{64}$$

To see the intuition of these results, it helps to write the differential,

$$d\underline{h}_t = dh_t - \frac{r^f \sigma}{r^f + \phi} dB_t = dh_t - d\left\{ r^f E_t \int_{s=0}^{\infty} e^{-rs} h_{t+s} ds \right\}.$$

Thus, from any change in actual income dh_t, we form the change in the tradeable component by subtracting the change in the permanent component of income, the part which cannot be self-insured by saving/borrowing and then repaying. This subtraction produces a transitory component of income, which can be completely self-insured, and which is tradeable. Likewise, expression (62) shows that the tradeable component of income subtracts off the amount by which permanent income has changed since time zero.

To verify that the hedge payoff $\underline{h}_t$ is tradeable, you can evaluate[1] $\int_{t=0}^{\infty} e^{-r^f t} \underline{h}_t dt = h_0/(r^f + \phi)$ for any $\{dB_t\}$, thus satisfying condition (59) and giving us $p(\underline{h}) = h_0/(r^f + \phi)$

[1] From (63),

$$\begin{aligned}
\int_{t=0}^{\infty} e^{-r^f t} \underline{h}_t dt &= \int_{t=0}^{\infty} e^{-r^f t} h_0 e^{-\phi t} dt + \int_{t=0}^{\infty} e^{-r^f t} dt \left[\int_{s=0}^{t} \left(e^{-\phi(t-s)} - \frac{r^f}{r^f + \phi} \right) \sigma dB_s \right]. \\
\int_{t=0}^{\infty} e^{-r^f t} \underline{h}_t dt &= \frac{h_0}{r^f + \phi} + \int_{s=0}^{\infty} \left[\int_{t=s}^{\infty} \left(e^{\phi s} e^{-\left(r^f + \phi\right)t} - \frac{r^f}{r^f + \phi} e^{-r^f t} \right) dt \right] \sigma dB_s \\
\int_{t=0}^{\infty} e^{-r^f t} \underline{h}_t dt &= \frac{h_0}{r^f + \phi} + \int_{s=0}^{\infty} \left[e^{\phi s} \frac{e^{-\left(r^f + \phi\right)s}}{r^f + \phi} - \frac{e^{-r^f s}}{r^f + \phi} \right] \sigma dB_s \\
\int_{t=0}^{\infty} e^{-r^f t} \underline{h}_t dt &= \frac{h_0}{r^f + \phi} + 0.
\end{aligned}$$

as well. The same evaluation shows that the actual outside income payoff h_t is not tradeable: $\int_{t=0}^{\infty} e^{-r^f t} h_t dt$ includes terms in dB_t that can reach arbitrarily large values. Thus, $\underline{h}_t$ is an interesting example of a payoff that varies across time and states of nature, yet its integrals $\int_{s=0}^{\infty} e^{-r^f s} \underline{h}_{t+s} ds$ are nonstochastic, because withdrawals are always paid back in ex-post present value terms.

Thus, we have broken up nontradeable outside income h_t into a tradeable component $\underline{h}_t$ and a non-traded residual, $\eta^h \equiv h_t - \underline{h}_t$ (the notation η^h corresponds to the decomposition of Proposition 7.) The components are long-run orthogonal, $\tilde{E}(\eta^h \underline{h}) = 0$, and the residual is orthogonal to any traded payoff including $x^* = 1$, $\tilde{E}(\eta^h 1) = 0$.

Having found $y^* = y^f$, $p(\underline{c}^b) = \underline{c}^b / y^f$ we can can write the optimal payoff from (61) as

$$\hat{x}_t = -\underline{h}_t + \left[\frac{h_0}{r^f + \phi} + W_0 \right] y^f. \tag{65}$$

In sum, then, Proposition 1 interprets this standard permanent income situation as follows: The investor sells an asset equal to the tradeable component $\underline{h}$ of his income stream, given by (62), (63) or (64). If the payoff $\underline{h}$ is directly marketed, he simply sells it as a contingent claim. If $\underline{h}$ must be synthesized by him or his adviser, he starts by borrowing $p(\underline{h}_t) = V_0 = h_0/(r^f + \phi)$, and repaying this debt stochastically in the amount $\underline{h}_t$ at each date, so the account balance follows $dV_t = \left(r^f V_t - \underline{h}_t \right) dt$. Then, he invests the proceeds of the sale, V_0, plus his initial wealth W_0, in the long-run mean-variance efficient payoff, which is here the riskfree rate.

At this point, we can verify that $\hat{x}_t$ given by (65) satisfies $\hat{x}_t = c_t - h_t$. Substitute (62) into (65) to obtain

$$\hat{x}_t = -h_t + \frac{r^f \sigma}{r^f + \phi} \int_{s=0}^{t} dB_s + \left[\frac{h_0}{r^f + \phi} + W_0 \right] y^f,$$

and using (56) we recognize c_t in the second two terms. I worked backwards to derive (62) in this way, which takes more algebra.

8.2 Lognormal i.i.d.

The lognormal i.i.d. case is a standard environment for power-utility portfolio theory. There is a constant riskfree rate r^f, risky assets have excess returns

$$dr_t^e = dr_t - r^f dt = \mu dt + \sigma dB_t \tag{66}$$

with covariance matrix

$$\Sigma = \sigma \sigma',$$

there are no preference shocks (c^b is a constant) or outside income ($h = 0$). In general equilibrium, we think of the assets as as linear production technologies with stochastic marginal products of capital. I find optimal payoffs in this environment, both by traditional methods and by applying the concepts of this paper. I study how optimal payoffs,

if not directly traded as contingent claims, are synthesized from dynamic trading in the underlying assets (technologies) of (66). Cochrane (2012b) presents a more detailed investigation of long-run mean-variance analysis in this environment.

The investor's wealth (capital) W_t follows

$$dW_t = (r^f W_t - x_t)dt + W_t w_t' dr_t^e. \tag{67}$$

Any tradeable stream of payoffs and $x_t = c_t$ in particular is synthesized by a portfolio strategy w_t and a payout rule x_t.

The power utility investor can synthesize his optimal payoff and consumption stream with an instantaneously mean-variance efficient portfolio

$$w_t = \frac{1}{\gamma}\Sigma^{-1}\mu, \tag{68}$$

and following a payout rule proportional to wealth,

$$c_t = \hat{x}_t = \frac{1}{\gamma}\left[\rho + (\gamma - 1)\left(r^f + \frac{1}{2}\frac{1}{\gamma}\mu'\Sigma^{-1}\mu\right)\right] W_t.$$

The quadratic utility investor can similarly synthesize his optimal payoff and consumption stream by holding a mean-variance efficient portfolio

$$w_t = \frac{\left(c^b - r^f W_t\right)}{r^f W_t}\Sigma^{-1}\mu = \frac{1}{\gamma_t}\Sigma^{-1}\mu \tag{69}$$

and following a payout rule in which consumption rises with wealth,

$$c_t = \hat{x}_t = c^b - \left[2r^f - \mu'\Sigma^{-1}\mu - \rho\right]\left(\frac{c^b}{r^f} - W_t\right). \tag{70}$$

However, the quadratic utility investor's effective risk aversion and allocation to risky assets varies over time, becoming more risk averse as wealth rises (69), and his payout rule includes an intercept, which can send wealth and consumption to negative values.

Following the representation of this paper, however, the same optimal payoff can be equivalently represented by a *static* sum of *two* investments, the riskfree asset (indexed perpetuity) and the long-run mean-variance efficient payoff, rather than dynamically synthesizing a single investor-specific portfolio. To find this representation, I follow the standard steps.

First, we find the traded discount factor. I start with the conventional diffusion representation for a discount factor x_t^* which prices the returns $r^f dt$ and dr_t^e,

$$\frac{dx_t^*}{x_t^*} = \left(\rho - r^f\right)dt - \mu'\Sigma^{-1}\sigma dB_t.$$

You can check that this construction satisfies the defining properties $E(dx_t^*/x_t^*) = \left(\rho - r^f\right) dt$ and $E(dr_t^e) = -E(dr_t^e dx_t^*/x_t^*)$. Next, a bit of algebra[2] reexpresses the discount factor as

$$dx_t^* = \left(r^f x_t^* - y_t^*\right) dt - x_t^* \mu' \Sigma^{-1} dr_t^e. \tag{73}$$

This equation says many things: x^* is a discount factor, it is a traded payoff (since y_t^* is proportional to x_t^*), and it is a value process that generates the yield $y^* = x^*/p(x^*)$ as a traded payoff. The dynamic strategy that generates the payoff y^* is a constantly-rebalanced constant-weight *short* position in a mean-variance efficient investment, $w = -\mu' \Sigma^{-1} dr_t^e$, that pays out a constant fraction $1/p(x^*)$ of its value.

By Proposition 1, the quadratic utility investor consumes a constant linear function of y^*, with an intercept that represents investment in the riskfree payoff:

$$c_t = \hat{x} = c^b - \left[\frac{c^b}{y^f} - W_0\right] y_t^*.$$

Proposition 2 rewrites this result as

$$\hat{y}_t = \frac{c_t}{W_0} = y^f + \frac{1}{\gamma}\left(y^f - y_t^*\right);\ \frac{1}{\gamma} \equiv \frac{c^b - y^f W_0}{y^f W_0} \tag{74}$$

These are the same optimal payoffs and consumption streams as described by (69)-(70). In place of the time-varying portfolio and payout weights in (69), we express the optimal payoff as a static sum of two payoffs. One is the riskfree payoff. The other–the long-run mean-variance efficient payoff–if not directly traded can be synthesized by shorting a payoff y^* that is short a constant fraction of its value in a mean-variance efficient portfolio, as described by (73).

The yield that is halfway between y^* (or the market payoff) and the riskfree payoff is not generated by a portfolio that constantly rebalances to 50/50 weights. Constant rebalancing gives rise to payoffs that are nonlinear functions of underlying payoffs.

For investors who differ by wealth W_0 and hence risk aversion γ, expressions (69)-(70) require each investor to undertake an investor-specific dynamic allocation. By contrast

[2]The price of x^* is

$$p(x^*) = k\tilde{E}(x^{*2}) = \frac{1}{2r^f - \rho - \mu' \Sigma^{-1} \mu} \tag{71}$$

Then we can rewrite x^* as a value process in the form (67),

$$\begin{aligned}
\frac{dx_t^*}{x_t^*} &= \left[r^f - \left(2r^f - \rho - \mu' \Sigma^{-1} \mu\right)\right] dt - \mu' \Sigma^{-1} \left(\mu dt + \sigma dB_t\right) \\
\frac{dx_t^*}{x_t^*} &= \left[r^f - \frac{1}{p(x^*)}\right] dt - \mu' \Sigma^{-1} dr_t^e \\
dx_t^* &= \left(r^f x_t^* - y_t^*\right) dt - x_t^* \mu' \Sigma^{-1} dr_t^e.
\end{aligned} \tag{72}$$

the two funds in (74) are the same for all investors. Investors only differ in the initial allocation of wealth across the two funds.

This observation opens the way to a different, but equivalent, market structure. Rather than market the two funds, r_t^f and $\mu'\Sigma^{-1}dr_t^e$ – the riskfree and tangency portfolios – but then require each investor to dynamically adjust between these securities, in different amounts depending on initial risk aversion (wealth and bliss point), the indexed perpetuity y^f and the long-run mean-variance efficient *payoff* $\left(y^f - y_t^*\right)$ could be marketed. Then investors would achieve the same end result by simply buying different amounts of these two funds, statically eating the dividends, and never rebalancing. If r^f and dr^e represent the economy's technological opportunities, then intermediaries do the dynamic investing and disinvesting.

8.2.1 Quadratic utility, the CAPM, and approximations

When the quadratic-utility investor implements optimal payoffs by the dynamic strategy described by (67), (69) and (70), wealth follows

$$\left(\frac{c^b}{r^f} - W_t\right) = \left(\frac{c^b}{r^f} - W_0\right) e^{-\left[(r^f-\rho)+\frac{1}{2}\mu'\Sigma^{-1}\mu\right]t-\mu'\Sigma^{-1}\sigma\int_0^t dB_s}.$$

Wealth at time t is a negative lognormal, capped above at c^b/r^f and extending downward. This is a natural result of the portfolio rule (69): As wealth rises, the investor pulls out of risky assets and increasingly seeks to fund bliss point consumption c^b forever. The consumption process c_t is also lognormal capped above at c^b.

This result has important implications for the nature of the market portfolio. Since the average investor pulls investment out of the risky technologies as wealth rises, the distribution of the market return can be completely different from that of the underlying technologies – in this case, even reversing the direction of the lognormal tail from right to left. Thus, it would be a mistake to apply long-run mean-variance analysis by specifying an investor who splits between riskfree and "market" opportunities, and model the latter as a lognormal diffusion.

This example also addresses the Dybvig and Ingersoll (1982) paradox. Dybvig and Ingersoll criticized CAPM / quadratic utility models, pointing out that *if* the market return can attain sufficiently high values, then the discount factor, as a linear function of the market return, must take on a negative value, implying arbitrage opportunities in complete markets. But the distribution of the market return is endogenous. In the permanent income model without disposal above, market returns do attain sufficiently high values, and consumption exceeds the bliss point leading to negative marginal utility. In this environment, however, the quadratic utility investor rebalances away from risky assets, so the market wealth never attains the high enough values, and marginal utility and discount factor are positive in every state of nature. The quadratic utility CAPM *can* exist with dynamic trading and non-normal returns, without offering arbitrage opportunities.

A wealth process and consumption process that are capped above are unrealistic. This aspect is relatively easy to handle by adding growing bliss points, or habits or other temporal nonseparabilities. Cochrane (2012b) discusses this possibility at more length. Hansen and Sargent (2004), Heaton (1993), and Cochrane (2012a) also show how to extend quadratic quantity dynamics and asset pricing to include habits, durability, and recursive utility.

Equation (70) reveals problems when $2r^f - \rho - \mu'\Sigma^{-1}\mu \leq 0$. With lognormal i.i.d. investment opportunities and quadratic utility, and for parameters $2r^f - \rho - \mu'\Sigma^{-1}\mu \leq 0$, investors finance early consumption by huge falls in consumption late in life or in states of nature when markets fall. As the lifetime increases, this repayment is indefinitely extended, so consumption becomes arbitrarily close to the bliss point. (Cochrane (2012b) provides details.) With a market Sharpe ratio $\sqrt{\mu'\Sigma^{-1}\mu}$ of about 1/3, these are not unreasonable parameters. Stochastic bliss points or temporal nonseparabilities do not resolve the central issue. The lesson is simply that combination of quadratic utility (even with generalized bliss points), lognormal i.i.d. investment opportunities, and a large risk premium doesn't give interesting results.

This observation is not a "fatal flaw" for quantitative application (as opposed to merely useful conceptual benchmark) of these ideas. Actual index returns are far from lognormal. But it does mean that a quantitatively realistic calculation must incorporate ingredients such as stochastic volatility, mean-reversion, or additional state variables to generate such departures from lognormality, exceeding by far the back of any envelope.

Quadratic utility approximations to more realistic utility functions and standard distributional assumptions do not work well for large (realistic) risk premia. The answer may be to look directly for similar representations based directly from other utility functions. What *do* the payoffs look like, and what is the nonlinear line between the indexed perpetuity and the market payoff?

Alternatively, log approximations may work better in practice. For example, the Campbell-Shiller identity can be written

$$\sum_{j=1}^{\infty} \rho^{j-1} r_t = \sum_{j=1}^{\infty} \rho^{j-1} \Delta d_t - (p_0 - d_0)$$

where $r = \log$ return, $\Delta d = \log$ dividend growth, $p = \log$ price, and $\rho \approx 0.96$ is a constant of approximation. Simply applying mean-variance and beta pricing ideas to long-run returns so defined – which all come from long-run dividend growth – may be more fruitful than literal use of the long-run expectation of the level of payoffs investigated here.

8.3 Finite basis

In conventional mean-variance analysis, we typically do not try directly to compute mean-variance frontiers of thousands of U.S. stocks, let alone the plethora of other available assets. Instead, we typically form a much smaller number of portfolios first. Implicitly,

we assume that the interesting opportunities in the larger set of securities is spanned by the much smaller number of portfolios.

The same approach is attractive in addressing intertemporal issues. Rather than try to find the exact optimum in an infinite-dimensional space of portfolio weights and payout rules, we can simply include a finite number of well-chosen dynamic trading strategies– rules for portfolio weights w_t and payout rules x_t as a function of state variables. Brandt and Santa-Clara (2006) advocate and implement this strategy for conventional dynamic portfolio return optimization. Their approach adapts transparently here.

If we make this simplification, most of the technical difficulties vanish. With a finite vector of basis payoffs $\mathbf{x}$, we can easily find a traded discount factor $x^* = \frac{1}{k}\mathbf{p}'\tilde{E}(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x}$. We can then find the optimal payoffs, which will be of the form $\mathbf{c}'\mathbf{x}$. The basis payoffs can include dynamic trading rules. For example, if returns are predictable, $r^e_{t+1} = a+bz_t+\varepsilon_{t+1}$, we can include the managed-portfolio return $z_t r^e_{t+1}$ and payout rules that depend on wealth and z_t.

9 Concluding remarks

This article has shown that the familiar mean-variance characterization for one-period returns applies straightforwardly to the stream of payoffs following an initial investment in a dynamic, intertemporal environment with incomplete markets and outside income.

Dynamic trading based on state variables that change over time is really just a different way of constructing a more interesting cross-section of long-run payoff or return opportunities. Conventional one-period asset pricing then applies to the stream of payoffs following an initial investment. Thinking about streams of payoffs rather than dynamic trading strategies may help us to apply intertemporal portfolio theory in practice.

The best 5 word summary may be Robert Shiller's often-offered insight, "Buy stocks for the dividends," i.e. rather than for short-run price gain. However, recognize that payouts from dynamic strategies are "dividends," and that measures of a given dividend stream's covariance with market dividend streams matters for putting together a good portfolio of dividends.

10 Appendix: Notation

x_t: A payoff. Its stream $\{x_t\}$ or x.

$p(x), p(\{x_t\})$: Price of the payoff stream $\{x_t\}$.

y, y_t: Yield, payoff to a one dollar investment, $y_t = x_t/p(x)$.

x^f, y^f: Riskfree payoff and yield, $x^f = 1$, $y^f = 1/p(x^f)$.

z: Excess yield, payoff to costless investment, e.g. $z = y - y^f$.

β, ρ: Discount factor and rate.

m_t: Scaled discount factor, e.g. $p(x) = E\sum_{t=1}^{\infty} \beta^t m_t x_t$

$\tilde{E}(x)$: Long-run mean, e.g. $\tilde{E}(x) = \rho E \int_{t=0}^{\infty} e^{-\rho t} x_t dt$.

$\tilde{\sigma}^2(x) \equiv \tilde{E}(x^2) - \left[\tilde{E}(x)\right]^2$: Long-run variance.

$\tilde{\beta}, \widetilde{cov}$: Long-run regression coefficient and covariance, $\widetilde{cov}(x,y) = \tilde{E}(xy) - \tilde{E}(x)\tilde{E}(y)$; $\tilde{\beta}_{x,y} = \widetilde{cov}(x,y)/\tilde{\sigma}^2(x)$.

$\underline{X}, \underline{Y}, \underline{Z}$: Sets of available payoffs, yields, excess yields.

$\mathbf{x}, \mathbf{y}, \mathbf{z}$: Vectors of N basis payoffs, yields, or excess yields.

x^*: Discount-factor mimicking payoff, i.e. $x^* \in \underline{X}$, $p(x) = k\tilde{E}(x^* x)$.

$y^* = x^*/p(x^*)$: Minimum long-run second moment yield.

$z^* = proj(1|\underline{Z})$: Mean-generating excess yield. When y^f is traded $z^* = \left(y^f - y^*\right)/y^f$.

y^{mv}, z^{mv}: Mean-variance efficient yield, excess yield.

η: Idiosyncratic component of a yield, $y = y^* + \lambda z^* + \eta$, and $p(\eta) = 0, \tilde{E}(\eta) = 0$.

$x^c \equiv proj(1|\underline{X})$; $y^c \equiv x^c/p(x^c)$. Constant-mimicking payoff and its yield.

x^i, y^i, z^i: Generic ith asset.

$\hat{x}, \hat{y}$: Optimal payoff, yield of optimal payoff.

$\hat{y}_i, \hat{y}_a$: Yield on investor i's optimal payoff, and market average.

W, W_t: Initial wealth and wealth at time t.

c^b: Bliss point of quadratic utility, $u(c) = -1/2\left(c^b - c\right)^2$.

$\underline{c}^b$: Hedge payoff for the bliss point, $\underline{c}^b = proj(c^b|\underline{X})$.

$\bar{c}^b$, $\tilde{c}^b$: Constant and variable components of the bliss point hedge payoff, $\underline{c}^b = \bar{c}^b \times 1 + \tilde{c}^b$; $\tilde{E}\left(\tilde{c}^b\right) = 0$.

$\gamma, \gamma_i, \gamma_a$: Relative risk aversion coefficient, defined locally at c, $1/\gamma = (c^b - c)/c$. γ_i = Investor i's risk aversion, γ_a = wealth-weighted market average risk aversion.

h, h_t: Stream of outside labor or business income.

$\underline{h}$: Hedge payoff for outside income , $\underline{h} = proj(h|\underline{X})$.

y^h, y_i^h, y_a^h: Yield on outside income hedge payoff, $y^h = \underline{h}/p(\underline{h})$, individual i value, and market a average.

$\hat{y}^T, \hat{y}_i^T, \hat{y}_a^T$: Optimal total payoff, $\hat{y}^T \equiv (\hat{x} + \underline{h})/\left[W + p(\underline{h})\right]$, $\hat{y}_i^T$ = individual value, and $\hat{y}_a^T$ = market average.

$\tilde{\beta}_h$, η^h: Regression coefficient and idiosyncratic component of outside-income hedge yield. $y^h = y^f + \tilde{\beta}_h\left(y^f - y^*\right) + \eta^h$.

$\tilde{\beta}_{ha,a}$, ε_a: Regression coefficient, and orthogonalized outside-income hedge factor payoff for the average investor, $y_a^h - y^f = \tilde{\beta}_{ha,a}\left(\hat{y}_a - y^f\right) + \varepsilon_a$.

$\tilde{\beta}_{hi,a}$, $\tilde{\beta}_{hi,ha}$, η_i^h Regression coefficients, and idiosyncratic component of outside-income hedge yield, relative to market and average outside-income hedge, $y_i^h - y^f = \tilde{\beta}_{hi,a}\left(\hat{y}_a - y^f\right) + \tilde{\beta}_{hi,ha}\left(y_a^h - y^f\right) + \eta_i^h$.

s_W, s_h: Share of asset and outside wealth, $s_W = W/\left[W + p(\underline{h})\right], s_h = p(\underline{h})/\left[W + p(\underline{h})\right]$.

$s_{Wa}, s_{ha}, s_{Wi}, s_{hi}$: Market average and individual shares.

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