NBER WORKING PAPER SERIES

LONG-TERM BEHAVIOR OF YIELD CURVES

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Working Paper No. 1789

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 1986

The research reported here is part of the NBER's research program in Financial Markets and Monetary Economics. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

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ABSTRACT

The flattening of yield curves at long-term maturities is proven to be approximately proportional to the reciprocal of the time to maturity under general conditions. This is a consequence of the persistence of earlier forward rates in the averaging process which produces yields from forward rates. This relationship suggests the use of a "reciprocal maturity yield curve" which significantly facilitates the interpretation of the behavior of long-term yields by linearizing them for display over a shorter interval. This is illustrated using a yield curve for U.S. Treasury bills.

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I. INTRODUCTION

Consider a class of similar securities with differing maturities, each involving an initial investment at time zero and providing a single known payment at maturity. Let R(m) denote the yield to maturity so that \$1.00 invested now would return an amount $\exp\{m\,R(m)\}$ at time m. We will be interested in the long term behavior of this yield curve R(m) for large m, and, in particular, in the approach of R(m) to its asymptotic value.

The forward rate function r(m) describes the (instantaneous) rate of return of a very short term investment at time m in the future which might be represented by a portfolio defined by selling securities which mature at time m and buying an equal (in present value) amount which mature at time $m + \delta$. Standard relationships (for example, Malkliel 1966, or Nelson 1972) represent the yield and forward rates in terms of one another:

$$(1.1) \qquad R(m) = \frac{1}{m} \int_0^m r(t)\,dt$$

$$(1.2) \qquad r(m) = R(m) + m\,R'(m).$$

Examination of (1.1) suggests that R(m) might decay towards its asymptotic value at a rate c/m for some constant c because short and medium term behavior of the forward rate will tend to persist at a rate 1/m in the averaging process.

Our primary objective is to show that this reciprocal maturity decay does indeed happen under fairly general assumptions and to propose a graphical technique based on these conclusions for examining long-term yields in actual data.

These results may be interpreted in the context of the Expectations Hypothesis, which states that the forward rates represent the yields expected at future time periods. Even if these expectations decay quickly towards some very long term value, the yield curve will decay slowly due to the persistence of earlier forward rate changes in the averaging process (1.1). Moreover, the yields will tend to decay like the reciprocal of the time to maturity and will not directly reflect the rate of decay of the expectations of future yields. A discussion of the Expectation Hypothesis and the term Structure of interest rates may be found in Cox, Ingersoll, and Ross (1981).

2. ASSUMPTIONS AND RESULTS.

We will require two assumptions. The first will guarantee existence of an asymptotic yield while the second will guarantee that the approach to this asymptotic value does not proceed too slowly. Precisely, the first assumption is that the forward rate has a finite limiting value:

$$(2.1) \qquad r(\infty) = \lim_{m \to \infty} r(m) \text{ exists and is finite.}$$

Livingston and Jain (1982) proved that yield curves must approach an asymptotic limit in the case of par bonds; our assumption is needed to deal with the more general case.

The second assumption requires that the forward rate approach this asymptotic value quickly enough; precisely:

$$(2.2) \qquad c = \int_0^\infty [r(t) - r(\infty)]\, dt \quad \text{exists and is finite.}$$

Implicit in this assumption is the mathematical consequence that the absolute integral is also finite. The assumption (2.2) is not overly restrictive:

exponential decay $r(m) - r(\infty) \sim \exp(-m/\tau)$ with positive τ easily satisfies the assumption, as does the much slower algebraic decay $r(m) - r(\infty) \sim m^{-\alpha}$ for large m with any $\alpha > 1$.

Under these assumptions (2.1 and 2.2) it follows that the asymptotic yield exists and is equal to the asymptotic forward rate. That is,

$$(2.3) \qquad R(\infty) = \lim_{m \to \infty} R(m) = r(\infty)$$

because

$$|R(m) - r(\infty)| \leq \frac{1}{m} \int_0^m |r(t) - r(\infty)| \, dt$$

where the right-hand side tends to zero as $m \to \infty$ because of assumption (2.2).

The main result concerns the long-term behavior of the yield. Under the assumptions (2.1 and 2.2) we can prove that

$$(2.4) \qquad R(m) = R(\infty) + c/m + f(m)$$

where c was defined in assumption (2.2) and the remainder term $f(m)$ tends to zero at a rate faster than $1/m$ as $m \to \infty$ (that is, $mf(m) \to 0$). This result says that the yield $R(m)$ tends towards its asymptotic value $R(\infty)$ with decay dominated by c/m, with higher-order (faster) decay terms represented by the remainder term $f(m)$.

To prove (2.4), we need only show that $m\, f(m) \to 0$, where $f(m)$ is defined by (2.4) itself. To see this, substitute (2.3) and (2.2) into (2.4), then rearrange terms to obtain

$$m\, f(m) = \int_0^m [r(t) - r(\infty)]\, dt - \int_0^\infty [r(t) - r(\infty)]\, dt$$

$$= \int_m^\infty [r(t) - r(\infty)]\, dt$$

which tends to zero as $m \to \infty$ as a consequence of the integrability of $r(t) - r(\infty)$ assumed in (2.2).

3. THE RECIPROCAL MATURITY YIELD CURVE.

The asymptotic expansion (2.4) provides theoretical justification for a particular rescaling of the maturity axis in yield curves, which will reflect the greater difference one month makes in shorter as compared to longer term maturities. We define the "reciprocal maturity yield curve" as a display of yield plotted against $X = -1/m$, the negative reciprocal of the time to maturity. The minus sign is used to keep the data in the same increasing time order. Ignoring the smaller remainder term in (2.4), we have the approximation

(3.1) $\quad \text{Yield} \cong R(\infty) - c\, X, \quad \text{where } X = -1/m.$

This reciprocal maturity yield curve overcomes the difficulty inherent in understanding the behavior of long-term yields from a conventional yield curve. The conventional yield curve typically shows a curved arc extending over a long distance to the right in the plot. The reciprocal maturity yield curve simplifies this behavior in two ways: to a nearly straight line over a short distance. Because of the asymptotic expansion (2.4), use of the logarithm of the time to maturity (which is found in some sources) may not be as appropriate as the negative reciprocal re-expression we propose here.

Consider the yield curve for U.S. Treasury bills on December 24, 1981, shown in Figure 1. These data represent continuously compounded rates of return based on the asked discount price from Federal Reserve Bank of New York quote sheets. Note that there is no clear indication of just how far the yield curve might rise at a very large distance to the right. In particular, has it leveled off already at around 13%, or is it continuing to rise strongly beyond that?

Figure 2 shows the yield curve using the reciprocal maturity scale for this same date set. Note the clear tendency towards a regular steady linear increase for the longer term bills at the right. For comparison, note that the yield values are the same and are in the same order as in Figure 1. By compressing the time scale at longer maturities, the reciprocal maturity plot (Figure 2) expresses the expected decay towards an asymptotic yield as a simple straight line. In particular, we gain the clear impression that yields would actively continue their rise beyond 13%. This fact is difficult to ascertain from the conventional yield curve (Figure 1).

ACKNOWLEDGMENTS

This research was supported in part by the Center for the Study of Banking and Financial Markets at the University of Washington. We are grateful to Vance Roley for assistance in obtaining data, and to Ann Kremer for research assistance. Nelson's participation was also supported by the National Science Foundation under a grant to the National Bureau of Economic Research.

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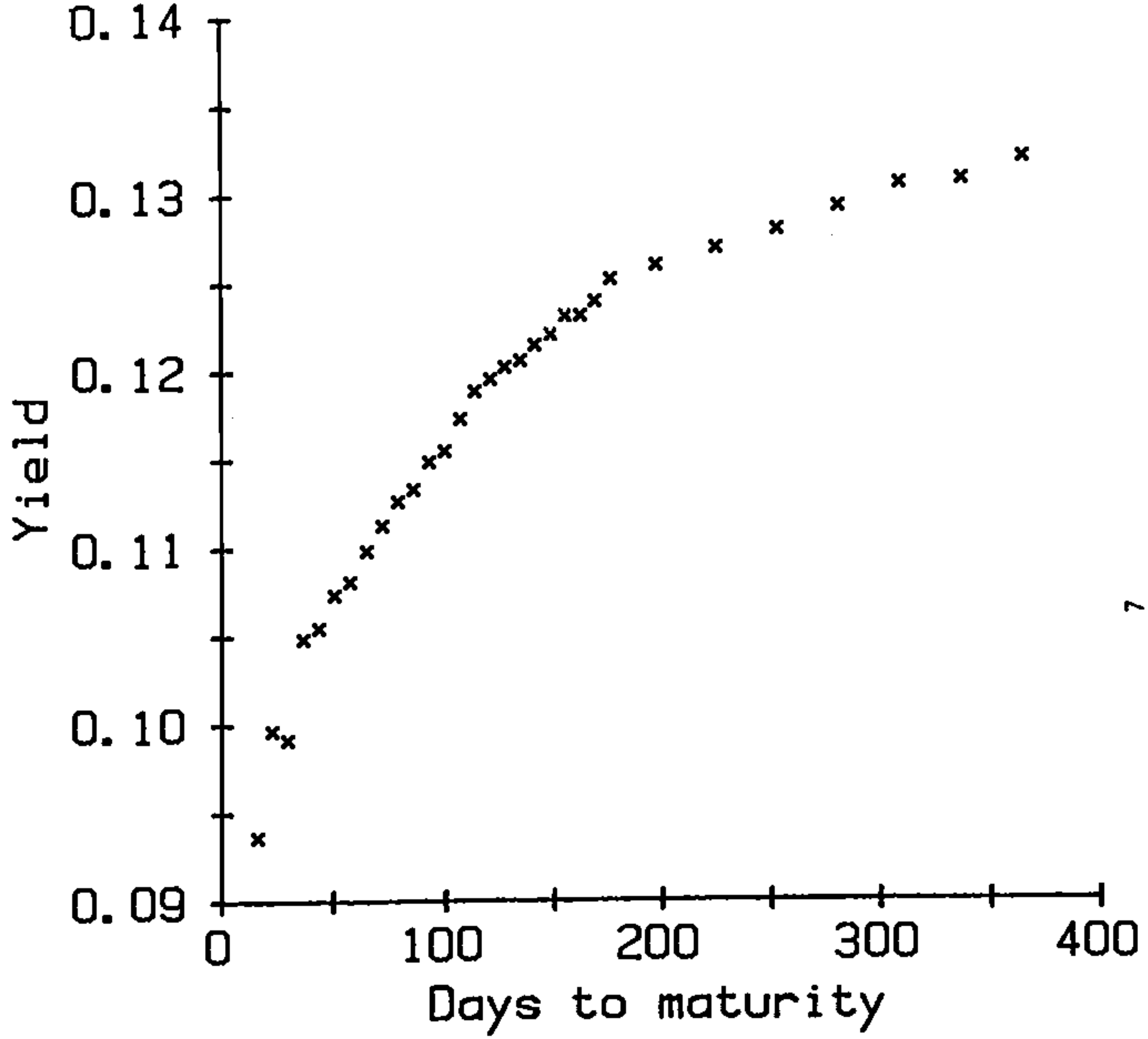
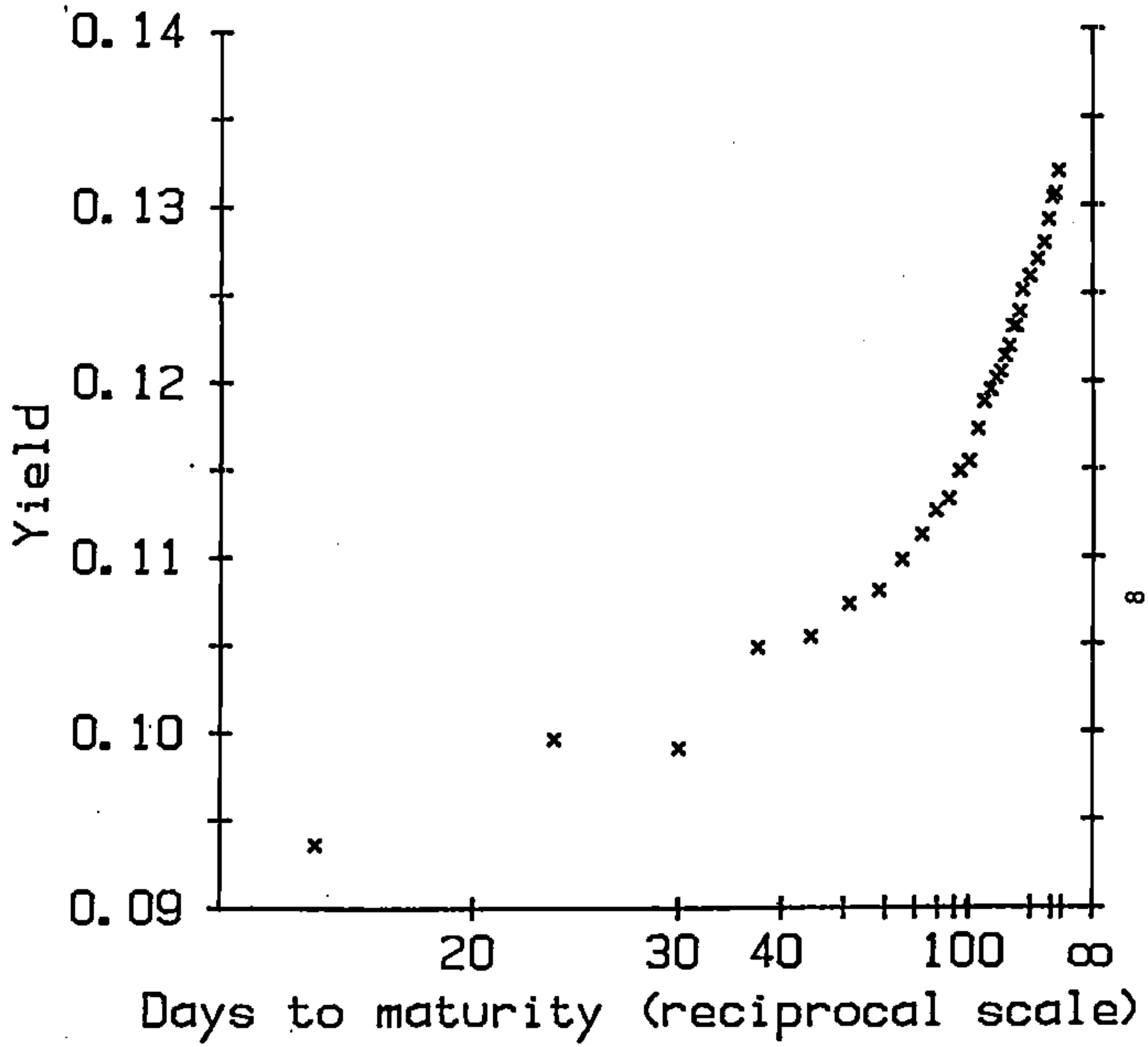


Figure 1 The yield curve for U.S. treasury bills on December 21, 1981. Note the difficulty in assessing the degree to which the yields have levelled off at long maturities.



<u>Figure 2</u>. The reciprocal maturity yield curve for US treasury bills on December 21, 1981. Note, in comparison to Figure 1, the simple interpretation of long-term yields as a linear (rather than curved) approach over a shortened interval.