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THE FORWARD PREMIUM PUZZLE IN A TWO-COUNTRY WORLD

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ABSTRACT

I explore the behavior of asset prices and the exchange rate in a two-country world. When the large country has bad news, the relative price of the small country's output declines. As a result, the small country's bonds are risky, and uncovered interest parity fails, with positive excess returns available to investors who borrow at the large country's interest rate and lend at the small country's interest rate. I use a diagrammatic approach to derive these and other results in a calibration-free way.

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An extensive literature has documented the fact that interest-rate differentials across countries are not, on average, counteracted by offsetting currency movements. As a result, it is possible to earn excess returns by investing in high-interest-rate currencies and borrowing in low-interest-rate currencies.

This paper presents a simple two-country model in which this failure of uncovered interest parity (UIP)—also known as the forward premium puzzle—emerges in equilibrium. The outputs of the two countries are imperfect substitutes for one another, so units matter, and there are two separate term structures of interest rates, one for each good. Global financial markets are assumed to be perfectly integrated; assets are real rather than nominal, and are priced as if by a representative global investor with power utility; there are no non-tradable goods, no liquidity issues, no portfolio constraints.

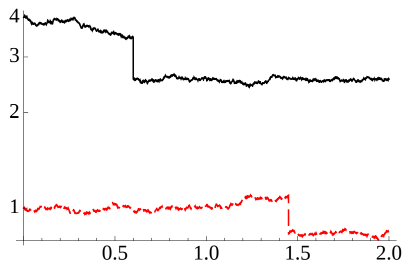
Even so, UIP fails to hold. Suppose, for example, that both countries have the same distribution of output growth, and that one country is much smaller than the other. In equilibrium, the smaller country has a higher short-term real interest rate. (Hassan (2009) documents that small countries tend to have higher nominal interest rates.) As a result, UIP fails, since the small country's exchange rate does not depreciate enough, on average, to offset the interest-rate differential. In fact, on the contrary, the exchange rate is expected to appreciate—an example of Siegel's (1972) "paradox"—so UIP fails in the strong sense that this expected appreciation actually *increases* the expected excess return on the carry trade.

Why does the smaller country have a higher interest rate? Any risk-based explanation must provide a story for why the small country's bond underperforms in bad states of the world. In the model considered here, bad states are those in which the large country experiences low output growth, since its output contributes the majority of the representative investor's consumption. But bad news for the large country corresponds to an increase in the relative supply of the small country's good, and so to a depreciation in its exchange rate. This depreciation also causes the small-country bond to underperform; hence the risk premium.

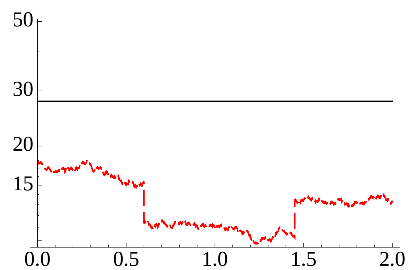
Figure 1 provides an illustration. It shows a sample realization over a two-year period, in a numerical example in which output growth is i.i.d. across the two countries.[1] Panel 1a plots the paths of exogenous fundamentals: the outputs, or dividends, produced by the two countries. Initially, the larger country (black line) contributes 80% of global output.[2] After about 0.6 years, it experiences a

[1] Although the paper's theoretical results cover the case in which dividends are correlated across countries, I use this i.i.d. example throughout the paper so that all correlations and asymmetries that emerge do so endogenously.

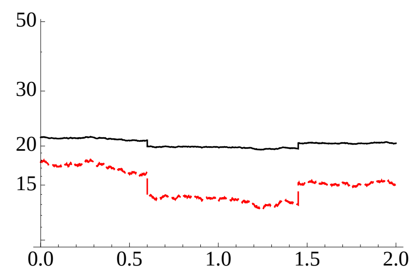
[2] The example is set up so that the exchange rate initially equals 1: thus the large country's time-0 output share, in common units, is $4/(4+1) = 0.8$. In the notation that will be introduced below—and labelling the small country as



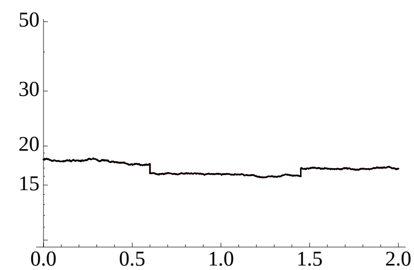
(a) Dividends



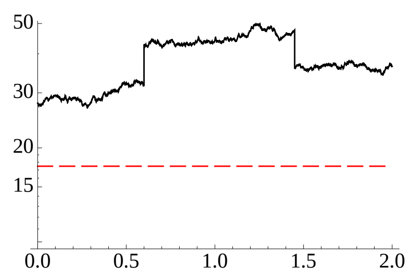
(b) Bond prices (large currency, $\eta = 1$)



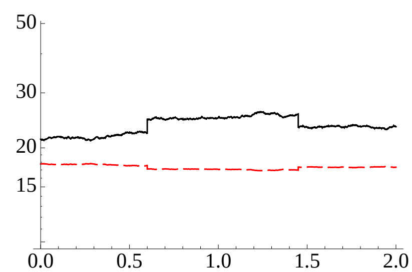
(c) Bond prices (large currency, $\eta = 2$)



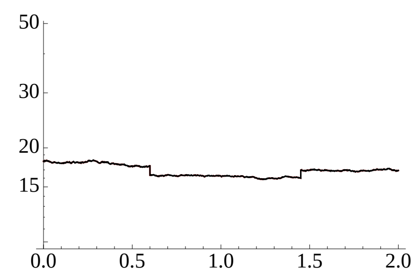
(d) Bond prices (large currency, $\eta = \infty$)



(e) Bond prices (small currency, $\eta = 1$)



(f) Bond prices (small currency, $\eta = 2$)



(g) Bond prices (small currency, $\eta = \infty$)

Figure 1: Sample paths of perpetuity prices in each currency, for different values of η. Each figure is plotted on a log scale, and is based on the same underlying path of fundamentals, shown in panel (a).

disaster that causes its output to drop; the smaller country experiences a similar disaster after about 1.5 years.

The panels below show how the behavior of perpetuity prices depends on the elasticity of substitution between goods, η, with the large country's bond in black and the small country's bond in red. The left-hand column sets $\eta = 1$, so that consumption is a Cobb-Douglas aggregator of the two goods. Panels 1b and 1e illustrate the well-known feature of the Cobb-Douglas setup that bond prices are constant in their own currency, despite the large shocks each country experiences. On the other hand, the exchange rate is extremely volatile, so for example the price of the large country's bond, denominated in the small country's units, jumps up when the large country experiences its output disaster. The right-hand column shows the other extreme, in which the two goods are perfect substitutes. The exchange rate effect disappears: panels 1d and 1g show that bond prices in the perfect substitutes case are the same—and time-varying—in each set of units. Put crudely, in the Cobb-Douglas case all the action is in exchange rates and none in valuation ratios, in conflict with the empirical evidence that movements in valuation ratios are a major driver of movements in asset prices;[3] and in the perfect substitutes case all the action is in valuation ratios and none in exchange rates.

In between, both effects are present. Panel 1c shows that in large country units, exchange rate movements exacerbate the poor performance of the small country's bond when the large country suffers its disaster. As a result, the small bond is riskier than the large bond, and so the overall level of the small bond's price is lower, reflecting higher interest rates in the small country and hence the emergence of a carry trade. Notice, also, that the carry trade experiences severe underperformance at times of large-country disaster. Panel 1f shows the corresponding plots viewed in small country units. The large country's disaster reduces the relative supply of its good, so its currency appreciates. In small-country units, the large country's bond therefore outperforms at the time of disaster and hence is a hedge, so earns a negative excess return.

The price-dividend ratios of the output claims associated with each of the two countries also depend on the relative size of the two economies, as in Cochrane, Longstaff and Santa-Clara (2008) and Martin (2011a). If, say, the large country experiences bad output news then the other country's output share increases; its output claim is now riskier—more correlated with consumption growth—so requires a higher excess return and a lower price-dividend ratio. In this way, shocks to one country

country 1, with a 20% output share—$s_0 = w = 0.2$.

[3] See, for example, Campbell and Ammer (1993), or Cochrane (2008) for a recent survey.

affect asset valuations in the other country. Furthermore, if goods are imperfect substitutes, then the small country's currency depreciates when the large country experiences bad output news, as before. This amplifies the underperformance of the small-country output claim in large-country units. Figure 5, in the Appendix, illustrates this, using the same sample paths for output as Figure 1.

To what extent are these effects dependent on the particular numerical example chosen? Less than one might think: the failure of UIP occurs in *any* calibration of the model, and I also provide economically interpretable nonparametric conditions under which I am able to sign the direction in which UIP fails—that is, to show when it is the small country whose bonds are risky, as in the example above. An unusual feature of the paper is that it develops a method of demonstrating that this relationship holds in a diagrammatic way via visual proofs. (These visual proofs were the original route to the more conventional algebraic proofs which are also provided.) I apply this nonparametric approach to show that various other relationships hold within the model, for example between risk premia on the large and small countries' output claims, in own and foreign units.

Frankel (1980), Hansen and Hodrick (1980), and Fama (1984) are amongst the early contributions to the literature on the forward premium puzzle. More recently, Brunnermeier, Nagel and Pedersen (2008) and Jurek (2009) have emphasized the fact that realized returns on carry trade strategies are negatively skewed. Burnside, Eichenbaum, Kleshchelski and Rebelo (2008) suggest that the excess returns apparently available on the currency carry trade may reflect a peso problem; but Jorda and Taylor (2009) argue that the forward premium puzzle returns once one considers more sophisticated carry trade strategies that exploit information about fundamentals. More generally, Dumas, Harvey and Ruiz (2003) present evidence that international markets are well integrated; Lustig, Roussanov and Verdelhan (2009) provide support for the idea that the behavior of international asset prices is amenable to a risk-based explanation; and Hollifield and Yaron (2001) argue that models of currency risk premia should focus on real, as opposed to nominal, risk. Alvarez, Atkeson and Kehoe (2007), working in a lognormal framework, emphasize the contrast between the data and the predictions of conventional macroeconomic models.

Various theoretical models have been offered in response to this evidence. Verdelhan (2010) considers a model with habit formation. Colacito and Croce (2010) and Bansal and Shaliastovich (2010) argue that long-run risk, together with Epstein-Zin preferences, can explain the forward premium puzzle. Farhi, Fraiberger, Gabaix, Ranciere and Verdelhan (2009) construct a model featuring rare disasters, and present supporting evidence from FX option data. Hassan (2009) linearizes a two-period lognormal model with nontraded goods and introduces money via a cash-in-advance constraint. Plantin and Shin (2008) present a game theoretic analysis in an environment with trading frictions; their model endogenously generates sudden losses on the carry trade. Backus, Gavazzoni, Telmer, and Zin (2010) explore the failure of UIP in a nominal, lognormal, framework with exogenously specified Taylor rules.

This paper makes comparatively simple assumptions about preferences and cashflows and emphasizes, instead, the interaction between intratemporal and intertemporal prices. Although the ingredients of the model are fairly standard, it has not previously been solved. I compute asset prices by extending the approach of Martin (2011a) to the imperfect substitution case; this avoids the need to rely on log-linearizations, on log utility, or on a unit elasticity of substitution between goods. The framework does allow for jumps, but none of the qualitative results depends on their presence. However, allowing for the possibility of jumps shows that the results hold independently of Merton's (1973) ICAPM, since the ICAPM does not hold once asset prices can jump.[4] Moreover, there is an increasingly large body of work that emphasizes the importance of nonlognormality in financial markets (Rietz (1988), Barro (2006), Martin (2011b)). The challenge is to allow for nonlognormality without sacrificing tractability; my approach lets me do so in a model that is hard to solve even in the lognormal case. Cole and Obstfeld (1991), Zapatero (1995), Pavlova and Rigobon (2007), and Stathopoulos (2009) also explore the consequences of intratemporal price adjustment, but all four papers rely on assumptions of log utility and unit elasticity of substitution between goods in deriving their analytical results. To my knowledge, no other papers in this literature derive calibration-free results of the type derived here.

The next section discusses UIP and the forward premium puzzle, and lists some necessary ingredients of any model in which UIP fails to hold. Section 2 sets up the model, which contains these ingredients. Section 3 characterizes asset prices, the exchange rate, and expected returns. Section 4 considers the small-country limit. Section 5 concludes. All proofs are in the Appendix.

1 UIP and the forward premium puzzle

UIP is a conjectured relationship between next year's spot exchange rate between two countries, e_{t+1}, today's spot exchange rate, e_t, and 1-year interest rates in each country, $i_{1,t}$ and $i_{2,t}$:

$$\mathbb{E}\,\log e_{t+1} = \log e_t + i_{1,t} - i_{2,t}\,. \tag{1}$$

[4] See Adler and Dumas (1983) for an early application of ICAPM logic to international finance.

The thought behind (1) is this: if country 2 has a lower interest rate than country 1, surely this should be compensated by the expected appreciation of its currency? Unfortunately this natural idea is decisively rejected by the data: in the relationship

$$\log e_{t+1} - \log e_t = a_0 + a_1(i_{1,t} - i_{2,t}) + \varepsilon_{t+1}\,, \tag{2}$$

UIP holds if $a_0 = 0$ and $a_1 = 1$. Typically, however, a_1 is estimated to be close to zero, or even negative.

An equivalent formulation of the UIP relationship exploits covered interest parity, which is the no-arbitrage relationship between today's 1-year forward exchange rate, f_t, today's spot exchange rate, and the two countries' 1-year rates: $\log f_t = \log e_t + i_{1,t} - i_{2,t}$. This shows that (1) is equivalent to $\mathbb{E} \log e_{t+1} = \log f_t$. The failure of UIP can therefore be rephrased as the failure of forward exchange rates to be unbiased predictors of future exchange rates.

We can also use the fact that $e_{t+1}/e_t = M_{2,t+1}/M_{1,t+1}$—where $M_{i,t+1}$ is the stochastic discount factor that prices assets denominated in the units of country i, $i = 1, 2$—to gain some understanding of necessary ingredients of models that generate the violation of UIP. For, we can take logs then expectations to conclude that, as an identity,

$$\mathbb{E}_t \Delta \log e_{t+1} = \mathbb{E}_t \log M_{2,t+1} - \mathbb{E}_t \log M_{1,t+1}\,. \tag{3}$$

Now, if $M_{1,t+1}$ and $M_{2,t+1}$ were roughly constant—as would be the case if either there were little risk in the economy, or if investors were roughly risk-neutral—then we could approximate (3) by

$$\mathbb{E}_t \Delta \log e_{t+1} \approx \log \mathbb{E}_t M_{2,t+1} - \log \mathbb{E}_t M_{1,t+1} = i_{1,t} - i_{2,t}\,. \tag{4}$$

That is, UIP can be formally justified in economies in which either the price or quantity of risk is very low. Empirically, however, high Sharpe ratios—the equity premium puzzle—tell us that the stochastic discount factors $M_{i,t+1}$ are volatile. Thus the move from (3) to (4) was not justified, and we must take the effects of Jensen's inequality into account, arriving at the identity

$$\mathbb{E}_t \Delta \log e_{t+1} = i_{1,t} - i_{2,t} + \underbrace{\log \mathbb{E}_t M_{1,t+1} - \mathbb{E}_t \log M_{1,t+1}}_{L_t(M_{1,t+1})} - \underbrace{\left(\log \mathbb{E}_t M_{2,t+1} - \mathbb{E}_t \log M_{2,t+1}\right)}_{L_t(M_{2,t+1})}\,. \tag{5}$$

The terms $L_t(M_{i,t+1})$ measure the variability of the SDFs; following Backus, Chernov and Martin (2011), I call $L_t(M_{i,t+1})$ the *entropy* of $M_{i,t+1}$. High SDF entropy translates into high attainable expected risk-adjusted returns (Bansal and Lehmann (1997), Alvarez and Jermann (2005)), much as high SDF volatility translates into high attainable Sharpe ratios (Hansen and Jagannathan (1991)).

The identity (5) reveals some necessary ingredients of any model in which UIP fails to hold (a point originally made by Fama (1984), and later revisited by Backus, Foresi and Telmer (2001)). First, as discussed, the entropies of $M_{1,t+1}$ and $M_{2,t+1}$ must be non-zero and economically significant: risk must matter. Second, there must be an asymmetry: if the entropies were equal they would cancel out, returning us to a world in which UIP held. Third, to generate the patterns found when estimating (2), $L_t(M_{1,t+1}) - L_t(M_{2,t+1})$ should be small at times when $i_{1,t} - i_{2,t}$ is high: if a country has high interest rates then assets denominated in its currency should earn relatively low risk premia. The model that follows has these properties.

2 Setup

There are two countries with output streams $\{D_{1t}\}$ and $\{D_{2t}\}$ respectively, at least one of which is nondeterministic. I assume that global markets are perfectly integrated, so assets are priced by a representative global investor with expected utility

$$\mathbb{E} \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma}\, dt\,,$$

where C_t is the consumption aggregator

$$C_t \equiv \left[w^{1/\eta} D_{1t}^{\frac{\eta-1}{\eta}} + (1-w)^{1/\eta} D_{2t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}\,.$$

Here w controls the relative importance of goods 1 and 2, and $\eta \in [1,\infty)$ is the elasticity of intratemporal substitution between the goods of the two assets. Write $u(D_{1t}, D_{2t}) \equiv C_t^{1-\gamma}/(1-\gamma)$ for the instantaneous felicity function, $u_i(D_{1t}, D_{2t})$ for the marginal utility of good i, $e_t \equiv u_2(D_{1t}, D_{2t})/u_1(D_{1t}, D_{2t})$ for the intratemporal price of a unit of good 2 in units of good 1, and $M_{i\tau} \equiv e^{-\rho(\tau-t)} u_i(D_{1\tau}, D_{2\tau})/u_i(D_{1t}, D_{2t})$ for the stochastic discount factor that prices time-τ claims to good i. The stochastic discount factors $M_{1\tau}$ and $M_{2\tau}$ and the relative price e_t are linked by the equation

$$\frac{e_\tau}{e_t} = \frac{M_{2\tau}}{M_{1\tau}}\,,$$

which appears as Proposition 1 of Backus, Foresi and Telmer (2001) and as equation 1 of Brandt, Cochrane and Santa-Clara (2006). For consistency with these papers, I will refer to e_t as the exchange rate. (International economists would refer to it as the terms of trade.) When the relative supply of country 2's good declines, its relative price—the exchange rate e_t—increases.

The price at time t of asset i, in units of good i ("in i-units") is

$$P_{it} = \mathbb{E} \int_t^\infty M_{i\tau} D_{i\tau}\, d\tau \tag{6}$$

from Lucas's (1978) Euler equation. It is important to emphasize that the perturbation logic underlying (6) implies that the price P_{it} is *denominated in units of good i*.[5] I use stars to indicate a price *not* expressed in its own units: $P^*_{2t} = P_{2t} e_t$ and $P^*_{1t} = P_{1t}/e_t$.

Similarly, the price of a perpetuity, or consol bond, that pays out a constant stream of good i dividends at a rate of one per unit time, is (in own units)

$$B_{it} = \mathbb{E} \int_t^\infty M_{i\tau}\, d\tau \,. \tag{7}$$

The outputs of the two countries, D_{1t} and D_{2t}, are taken as exogenous—though they could also be thought of as being determined by a production side of the economy that is left unmodelled here—and are assumed to have dividend growth that is i.i.d. over time, though not necessarily across countries. Formally, $\widetilde{y}_{it} \equiv y_{it} - y_{i0} \equiv \log D_{it} - \log D_{i0}$ is a Lévy process for $i = 1, 2$. The relevant properties of the dividend growth processes are conveniently summarized by the cumulant-generating function (CGF) $\boldsymbol{c}(\theta_1, \theta_2) \equiv \log \mathbb{E}\left[\exp\left\{\theta_1(y_{1,t+1} - y_{1,t}) + \theta_2(y_{2,t+1} - y_{2,t})\right\}\right]$.[6]

One of the main themes of the paper is that by exploiting general properties of CGFs, it is possible to establish features of asset prices in this economy that hold not just for a particular calibration but for a whole family of driving stochastic processes. The most important such property is that CGFs are always *convex*. But I also find it helpful to introduce three nonparametric properties, each of which the CGF may or may not possess. The first is the *exchangeability* property,[7] which holds if $\boldsymbol{c}(\theta_1, \theta_2) = \boldsymbol{c}(\theta_2, \theta_1)$. This can be thought of as imposing a *cet. par.* assumption: it ensures the two countries have the same means and volatilities of output growth, the same arrival rates of jumps, and so on. It therefore focusses attention on the underlying economic mechanism and on the consequences of asymmetry in country size alone. The second is the *convex difference* property, which is a restriction on the higher cumulants of log output growth. It holds in the lognormal case, and more generally it ensures, roughly speaking, that output growth in each country is not

[5] In theory, a natural approach would be to work in different units—perhaps the price of the international consumption bundle C_t. My approach is intended to mirror how financial economists think, and work, in practice, i.e. in terms of dollars and yen and so on, rather than some notion of a global unit of purchasing power.

[6] Cumulants and cumulant-generating functions are also discussed by Backus, Foresi and Telmer (2001), Backus, Chernov and Martin (2011) and Martin (2010).

[7] Two random variables X_1, X_2 are said to be *exchangeable* if the joint distribution of (X_1, X_2) is the same as that of (X_2, X_1).

positively skewed. The third is the *linked fundamentals* property. In the lognormal case, it is natural to consider imposing the economically plausible assumption that the correlation between the two countries' output growth is nonnegative. The linked fundamentals property, which is the appropriate generalization of this idea to arbitrary Lévy processes, requires that the CGF is supermodular.

I illustrate the results with parametric examples. The framework permits these to be flexibly specified; I assume that the log outputs have a correlated Brownian motion component with drifts μ_i, volatilities σ_i, $i = 1, 2$, and correlation κ. There are also two kinds of jumps. The first kind affects country i idiosyncratically. The second hits both countries simultaneously; the sizes of such jumps may or may not be correlated. Jumps arrive at times dictated by Poisson processes with arrival rates ω_1 and ω_2 for the idiosyncratic jumps, and ω for the simultaneous jumps. Jump sizes are lognormal: when country i experiences an idiosyncratic jump, the size of that jump is lognormal with mean $\mu_{J,i}$ and volatility $\sigma_{J,i}$. When there is a simultaneous jump, the sizes of jumps in the two countries are jointly lognormal, with means ν_i, volatilities τ_i and correlation ξ. The resulting CGF is

$$\begin{aligned}\boldsymbol{c}(\theta_1, \theta_2) \quad = \quad & \mu_1\theta_1 + \mu_2\theta_2 + \frac{1}{2}\sigma_1^2\theta_1^2 + \kappa\sigma_1\sigma_2\theta_1\theta_2 + \frac{1}{2}\sigma_2^2\theta_2^2 + \\ & + \omega_1\left(e^{\mu_{J,1}\theta_1 + \frac{1}{2}\sigma_{J,1}^2\theta_1^2} - 1\right) + \omega_2\left(e^{\mu_{J,2}\theta_2 + \frac{1}{2}\sigma_{J,2}^2\theta_2^2} - 1\right) + \\ & + \omega\left(e^{\nu_1\theta_1 + \nu_2\theta_2 + \frac{1}{2}\tau_1^2\theta_1^2 + \xi\tau_1\tau_2\theta_1\theta_2 + \frac{1}{2}\tau_2^2\theta_2^2} - 1\right). \end{aligned} \tag{8}$$

For notational convenience, let $\chi = (\eta - 1)/\eta$ and $\widehat{\gamma} = (\gamma + \chi - 1)/\chi$. The variable χ ranges between 0 (the Cobb-Douglas case) and 1 (the perfect substitutes case). I make the following assumptions:

Assumption 1. $\gamma \geq 1$ *and* $\eta \geq 1$.

Assumption 2 (Finiteness conditions)**.** *Tastes and technologies are such that*

$$\begin{aligned} \rho - \boldsymbol{c}[\chi(1 - \widehat{\gamma}/2), -\chi\widehat{\gamma}/2] &> 0 \\ \rho - \boldsymbol{c}[-\chi\widehat{\gamma}/2, \chi(1 - \widehat{\gamma}/2)] &> 0 \\ \rho - \boldsymbol{c}[\chi(1 - 1/\chi - \widehat{\gamma}/2), -\chi\widehat{\gamma}/2] &> 0 \\ \rho - \boldsymbol{c}[-\chi\widehat{\gamma}/2, \chi(1 - 1/\chi - \widehat{\gamma}/2)] &> 0. \end{aligned}$$

Typical estimates in the literature put both γ and η somewhere in the range 2–10, so Assumption 1 is very mild. It implies that $\widehat{\gamma} \geq 0$. Assumption 2 requires that the discount rate is sufficiently high. It ensures that expected utility is finite, as I show in the course of proving Result 2.

3 Prices, interest rates and expected returns

Price-dividend ratios, interest rates, expected returns, and the exchange rate depend on the relative sizes of the two countries. The appendix shows that country 1's share of world output denominated in common units, $s_t \equiv D_{1t}/(D_{1t} + e_t \cdot D_{2t}) \in (0, 1)$, can be used as the state variable for the economy. In some respects, though, it is more natural to use $\widehat{u}_t$, a monotonic transformation of s_t, as the state variable: $\widehat{u}_t \equiv \log[(1 - s_t)/s_t]$. I express formulas in terms of $\widehat{u}_t$, but plot graphs against the more easily interpreted s_t. I drop subscripts when referring to the current (time-0) value of either variable, so $s = s_0$ and $\widehat{u} = \widehat{u}_0$. If country 1 has a small share of output, s is small and $\widehat{u}$ is large.

Result 1 (Exchange rate)**.** *The exchange rate e_t can be expressed in terms of $\widehat{u}_t$ via*

$$e_t = [(1 - w)/w]^{(1-\chi)/\chi} \cdot e^{-[(1-\chi)/\chi]\widehat{u}_t}. \tag{9}$$

The expected appreciation in good 1's relative price, FX_1^*, is

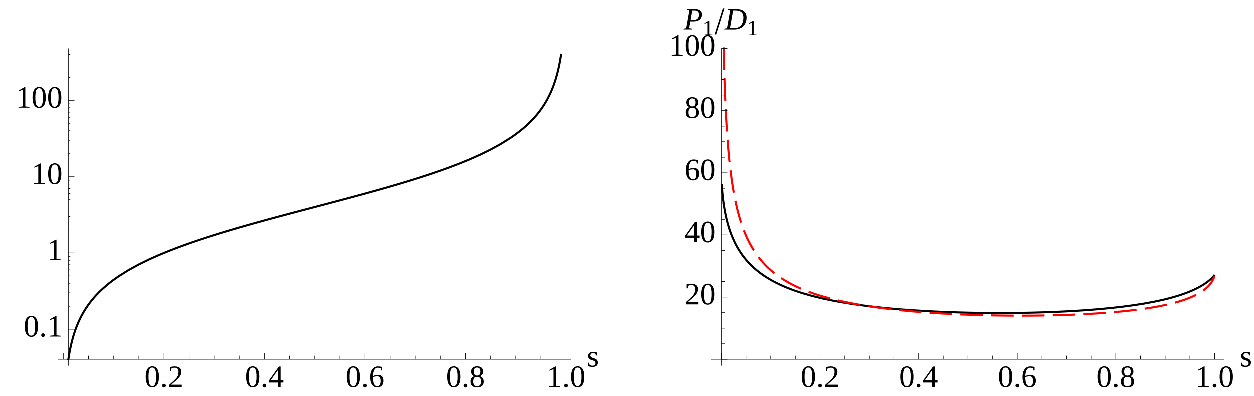
$$FX_1^* \, dt \equiv \frac{\mathbb{E}d(1/e_t)}{1/e_t} = \boldsymbol{c}(\chi - 1, 1 - \chi) \, dt, \tag{10}$$

and the expected appreciation in good 2's relative price, FX_2^*, is

$$FX_2^* \, dt \equiv \frac{\mathbb{E}de_t}{e_t} = \boldsymbol{c}(1 - \chi, \chi - 1) \, dt. \tag{11}$$

The average expected appreciation, $(FX_1^* + FX_2^*)/2$, is positive—an example of Siegel's (1972) "paradox".

Figure 2a shows how the exchange rate varies as a function of country 1's output share. When country 1 is small, its goods are in short supply so command a high price. (In the perfect-substitutes case, $\eta = \infty$, the relative price would always equal 1, independent of country 1's output share.) The figure assumes that the representative investor has time preference rate $\rho = 0.04$ and risk aversion $\gamma = 4$; that the weight of country 1 in the consumption aggregator is $w = 0.2$, and the elasticity of substitution between goods is $\eta = 2$; and that the parameters governing technologies are, in the notation of (8), $\mu_1 = \mu_2 = 0.02$, $\sigma_1 = \sigma_2 = 0.1$, $\kappa = 0$, $\omega_1 = \omega_2 = 0.02$, $\mu_{J1} = \mu_{J2} = -0.2$, $\sigma_{J1} = \sigma_{J2} = 0.1$, and $\omega = 0$. I choose these parameter values to prove a clean illustration of the model because they ensure that countries have independent fundamentals and that the two countries have identically distributed output growth, so that any correlations or asymmetries that emerge are endogenous. The same calibration is used throughout the paper.



(a) e_t

(b) P_1/D_1

Figure 2: Left: The exchange rate—the relative price of good 2 in units of good 1—plotted on a logarithmic scale against s, the output share of country 1. Right: Price-dividend ratio on asset 1, plotted against s, in the imperfect substitution case $\eta = 2$ (black) and the perfect substitution case (dashed red).

The next result provides integral formulas for the price-dividend ratios of the two countries' output claims and for the prices of perpetuities that deliver a unit of good i per unit of time. Since the integrands in these formulas decay exponentially fast, the integrals can be numerically evaluated effectively instantaneously.[8]

Result 2 (Valuation ratios)**.** *The price-dividend ratios, P_{it}/D_{it}, of each country's output claim, and the perpetuities denominated in each good, B_{it}, $i = 1, 2$, are given by*

$$\begin{aligned} P_{1t}/D_{1t} &= V_{1,0}(\widehat{u}_t) && (12) \\ P_{2t}/D_{2t} &= V_{0,1}(\widehat{u}_t) && (13) \\ B_{1t} &= V_{1-1/\chi,0}(\widehat{u}_t) && (14) \\ B_{2t} &= V_{0,1-1/\chi}(\widehat{u}_t), && (15) \end{aligned}$$

where

$$V_{\alpha_1,\alpha_2}(\widehat{u}) \equiv \left(e^{\widehat{u}/2} + e^{-\widehat{u}/2}\right)^{\widehat{\gamma}} \int_{-\infty}^{\infty} \frac{e^{i\widehat{u}v}\mathscr{F}(v)}{\rho - \boldsymbol{c}[\chi(\alpha_1 - \widehat{\gamma}/2 - iv), \chi(\alpha_2 - \widehat{\gamma}/2 + iv)]}\, dv \qquad (16)$$

and $\mathscr{F}(v)$ is defined in terms of the Beta function, $\mathscr{F}(v) \equiv \frac{1}{2\pi} \cdot B(\widehat{\gamma}/2 - iv, \widehat{\gamma}/2 + iv)$.

[8]The integrals can also be expressed in closed form in terms of hypergeometric functions if output growth is lognormal, as in Martin (2011a).

These valuation ratios move around over time as dividends, and hence $\widehat{u}_t$ and s_t, move around. Figure 2b plots the price-dividend ratio of the claim to country 1's output stream, P_1/D_1, against country 1's output share, s. The solid line is the price-dividend ratio in the imperfect substitution case, using the same calibration as above, and the dashed line shows the price-dividend ratio in the perfect substitutes case.

The price-dividend ratio increases sharply as country 1's share of output declines in both the perfect and imperfect substitution cases, though the effect is muted in the latter case. To understand why, we must turn to the behavior of interest rates and risk premia.

Each good has its own set of zero-coupon bond prices, and attached to these bond prices are zero-coupon yields, which also move around over time, as shocks to the dividends of the two trees induce changes in $\widehat{u}$. Three measures of interest rates[9] are particularly natural: the riskless rates for each good, calculated from zero-coupon yields in the limit as $T \downarrow 0$; coupon yields on perpetuities, $1/B_{iT}$, provided by equations (14) and (15); and long rates, which are calculated from zero-coupon yields as $T \uparrow \infty$.

Result 3 (Interest rates)**.** *Writing $\mathscr{Y}_{T,i}(\widehat{u})$ for the continuously compounded T-period zero-coupon yield in i-units when the current state is $\widehat{u}$, we have*

$$\begin{aligned}
\mathscr{Y}_{T,1}(\widehat{u}) &= \frac{-1}{T}\log\left\{\left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\int_{-\infty}^{\infty} e^{i\widehat{u}v}\mathscr{F}(v)e^{-\{\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]\}T}\,dv\right\}\\
\mathscr{Y}_{T,2}(\widehat{u}) &= \frac{-1}{T}\log\left\{\left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\int_{-\infty}^{\infty} e^{i\widehat{u}v}\mathscr{F}(v)e^{-\{\rho-\boldsymbol{c}[\chi(-\widehat{\gamma}/2-iv),\chi(1-1/\chi-\widehat{\gamma}/2+iv)]\}T}\,dv\right\}.
\end{aligned}$$

The riskless rates, $R_{f,i}(\widehat{u}) = \lim_{T\downarrow 0}\mathscr{Y}_{T,i}(\widehat{u})$, $i = 1, 2$, are

$$\begin{aligned}
R_{f,1}(\widehat{u}) &= \left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\int_{-\infty}^{\infty} e^{i\widehat{u}v}\mathscr{F}(v)\left\{\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]\right\}dv\\
R_{f,2}(\widehat{u}) &= \left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\int_{-\infty}^{\infty} e^{i\widehat{u}v}\mathscr{F}(v)\left\{\rho-\boldsymbol{c}[\chi(-\widehat{\gamma}/2-iv),\chi(1-1/\chi-\widehat{\gamma}/2+iv)]\right\}dv.
\end{aligned}$$

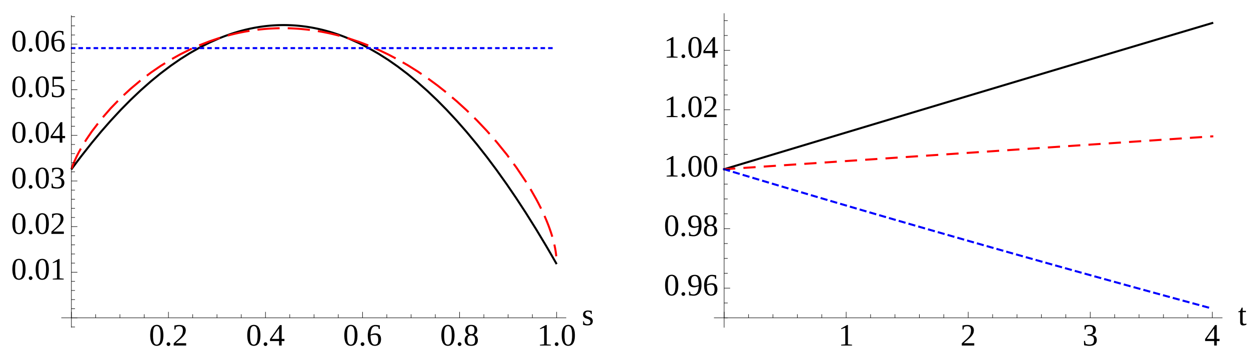
[9] These are "own-rates of interest"—a concept emphasized by Sraffa (1932), who also provides a picturesque description of covered interest parity: "[W]e need not stretch our imagination and think of an organised loan market amongst savages bartering deer for beavers. Loans are currently made in the present world in terms of every commodity for which there is a forward market. When a cotton spinner borrows a sum of money for three months and uses the proceeds to purchase spot, a quantity of raw cotton which he simultaneously sells three months forward, he is actually 'borrowing cotton' for that period. The rate of interest which he pays, per hundred bales of cotton, is the number of bales that can be purchased with the following sum of money: the interest on the money required to buy spot 100 bales, plus the excess (or minus the deficiency) of the spot over the forward prices of the 100 bales."

Long rates, $\mathscr{Y}_{\infty,i} \equiv \lim_{T\uparrow\infty} \mathscr{Y}_{T,i}(\widehat{u})$, are independent of $\widehat{u}$, hence constant over time:

$$\mathscr{Y}_{\infty,1} = \max_{\theta\in[0,\gamma+\chi-1]} \rho - \boldsymbol{c}(\theta-\gamma,-\theta) \tag{17}$$

$$\mathscr{Y}_{\infty,2} = \max_{\theta\in[1-\chi,\gamma]} \rho - \boldsymbol{c}(\theta-\gamma,-\theta)\,. \tag{18}$$

In this notation, the currently prevailing one-year rate in country j, as discussed in Section 1, is $i_{j,t} = \mathscr{Y}_{1,j}(\widehat{u})$. Figure 3a shows how the riskless rate (black solid line), perpetuity yield (red dashed line), and long rate (blue dotted line) depend on s. Riskless rates are low when s is close to 0 or to 1 due to precautionary savings demand in the face of an unbalanced—because poorly technologically diversified—economy, and high when s is close to 0.5. Country 1's yield curve can be upward-sloping (if its output share is close to 0 or to 1), downward-sloping (if its output share is close to about 0.45), or hump-shaped (for output shares close to 0.3 or to 0.6).



(a) Interest rates plotted against output share

(b) Forward and expected spot exchange rates

Figure 3: Left: The riskless rate (black solid), perpetuity yield (red dashed), and long rate (blue dotted) in 1-units plotted against s. Right: Forward price to time t of good 2 in 1-units ($F_{0\to t}$, black solid), expected future spot prices ($\mathbb{E}\, e_t = \mathbb{E}\, 1/e_t$, red dashed), and forward price of good 1 in 2-units ($1/F_{0\to t}$, blue dotted), plotted against t, assuming starting share $s = 20\%$.

The figure is not symmetric: the interest rate in good 1 is higher when country 1 is small than when it is large. In this example, as country 1's share of global output declines to zero, its interest rate approaches 3.25% while the large country's interest rate drops to 1.18%. From the perspective of an investor (or economist) thinking in large country units, this might suggest the following carry trade: borrow at the large-country interest rate of 1.18%, and invest in the small-country interest rate of 3.25%. We have not yet taken into account the effects of exchange-rate movements, however. Before doing so, remember that in this example the *exchangeability property* holds:

Property 1 (Exchangeability). *$\boldsymbol{c}(\theta_1, \theta_2) = \boldsymbol{c}(\theta_2, \theta_1)$ for all θ_1, θ_2.*

From Result 1, we know that $(FX_1^* + FX_2^*)/2$, is positive. But by the exchangeability property, we also know from equations (10) and (11) that $FX_1^* = FX_2^*$, and hence that both are strictly positive. Thus exchange rate movements actually work in favor of the carry trade. The reason for the carry trade's excess return is (of course) that it is risky: if the large country has bad news, the small country's exchange rate deteriorates, and the carry trade has a low return.

Returning to the general case, we can now see how the model generates the failure of uncovered interest parity in the regression (2). From equation (9) we see that, by construction, $\log e_{t+1} - \log e_t$ is independent of information known at time t, because $\widehat{u}_t$ inherits the independent increments property from the Lévy process that drives fundamentals. Therefore $\mathrm{cov}(\log e_{t+1} - \log e_t, i_{1,t} - i_{2,t}) = 0$; combining this with the fact that $\mathrm{var}(i_{1,t} - i_{2,t}) \neq 0$, we have

Result 4 (Failure of UIP). *Interest-rate differentials are totally uninformative about future movements of the exchange rate: $\mathrm{plim}(a_1) = 0$ for generic calibrations.*

Given that the random walk nature of $\log e_{t+1} - \log e_t$ was hard-wired in, the interesting aspect of the model is not that it generates $\mathrm{cov}(\log e_{t+1} - \log e_t, i_{1,t} - i_{2,t}) = 0$, but that interest rates can vary across countries, $\mathrm{var}(i_{1,t} - i_{2,t}) \neq 0$, despite the random walk character of exchange rates.

We can also interpret the failure of UIP in terms of forward rates. Define $F_{0\to t}$ to be the time-0 forward price of good 2 in 1-units, for settlement at t. A standard no-arbitrage argument implies that this forward exchange rate is determined by the spot exchange rate and t-period interest rates in the two countries: $F_{0\to t} = e_0 \cdot \exp\{(\mathscr{Y}_{t,1} - \mathscr{Y}_{t,2})t\}$. Figure 3b shows how the forward exchange rates $F_{0\to t}$ (black solid line) and $1/F_{0\to t}$ (blue dotted line) compare to expected future spot exchange rates $\mathbb{E}\, e_t$ and $\mathbb{E}\, 1/e_t$ (red dashed line) in the numerical example. The starting share of country 1 is $s = w = 0.2$, so the current spot exchange rate is $e_0 = 1$. Since the example features symmetric output growth processes, expected future spot exchange rates (shown as a dashed red line) are the same from the perspective of both countries—$\mathbb{E}\, e_t = \mathbb{E}\, 1/e_t$—and they lie above the spot price (Siegel's paradox again). The forward price of good 2 is even higher than its expected future spot price, while the forward price of good 1 moves in the opposite direction to its expected future spot price, because interest rates are higher in country 1 than in country 2. This is another manifestation of the violation of UIP.[10]

[10] I plot Figure 3b in levels rather than in logs to show the limited quantitative importance of Siegel's paradox. In logs, Figure 3b would show expected appreciation of the log exchange rate equal to zero at all time horizons, and the lines depicting forward prices would fan out symmetrically around it.

When the exchangeability property holds, long rates, unlike short rates, are equal across countries: $\mathscr{Y}_{\infty,1} = \mathscr{Y}_{\infty,2}$. Given that long rates are constant, this is economically obvious, but to see it formally, rewrite equation (18) as $\mathscr{Y}_{\infty,2} = \max_{\theta\in[0,\gamma+\chi-1]} \rho - \boldsymbol{c}(-\theta, \theta-\gamma)$. This equals $\mathscr{Y}_{\infty,1}$, given in equation (17), because $\boldsymbol{c}(-\theta, \theta-\gamma) = \boldsymbol{c}(\theta-\gamma, -\theta)$ by exchangeability. This has an implication that is interesting in its own right: $\mathscr{Y}_{\infty,1} = \mathscr{Y}_{\infty,2} = \rho - \boldsymbol{c}(-\gamma/2, -\gamma/2)$, so long rates are independent of the elasticity of substitution between goods.

The expected return, ER, on an asset with price P and instantaneous dividend D is

$$ER\,dt \equiv \frac{\mathbb{E}dP}{P} + \frac{D\,dt}{P}.$$

This expected return is calculated in the asset's own units. The expected return on asset 2 in units of country 1 is

$$ER_2^*\,dt \equiv \frac{\mathbb{E}d(eP)}{eP} + \frac{eD\,dt}{eP} = \frac{\mathbb{E}d(eP)}{eP} + \frac{D\,dt}{P}.$$

The dividend yield component of expected returns is unit-free, but the expected capital gains component depends on exchange rate movements. To the extent that these are correlated with asset prices, there will be an associated risk premium.

Result 5 (Expected returns)**.** *Expected returns, $ER_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}(\widehat{u})$, are given by*

$$ER_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}(\widehat{u}) = \frac{1 + G_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}(\widehat{u})}{V_{\alpha_1,\alpha_2}(\widehat{u})} \tag{19}$$

where $G_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}(\widehat{u})$ is defined to equal

$$\sum_{m=0}^{\widehat{\gamma}} \binom{\widehat{\gamma}}{m} e^{(\widehat{\gamma}/2-m)\widehat{u}} \int_{-\infty}^{\infty} \frac{e^{i\widehat{u}v}\mathscr{F}(v)\boldsymbol{c}(\lambda_1 + m\chi - \widehat{\gamma}\chi/2 - i\chi v, \lambda_2 - m\chi + \widehat{\gamma}\chi/2 + i\chi v)}{\rho - \boldsymbol{c}[\chi(\alpha_1 - \widehat{\gamma}/2 - iv), \chi(\alpha_2 - \widehat{\gamma}/2 + iv)]}\,dv,$$

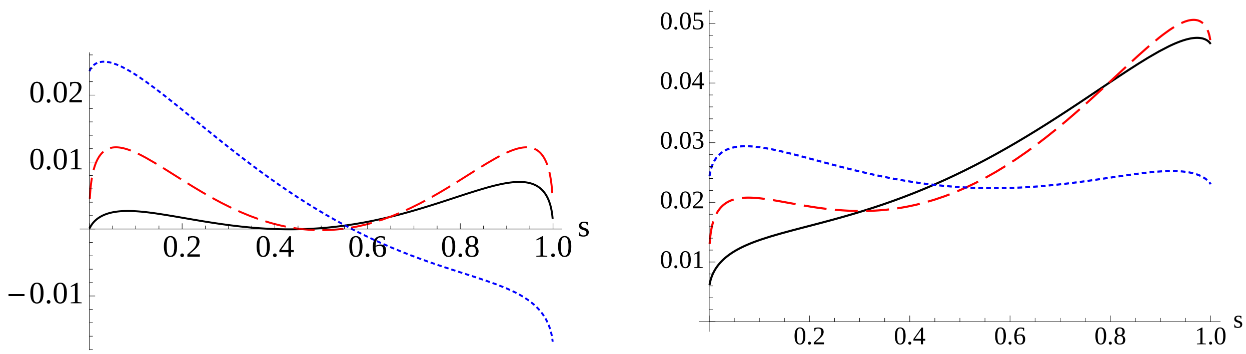
and the values of $\alpha_1, \alpha_2, \lambda_1, \lambda_2$, which depend on the asset and reference units of interest, are provided in Table 1.

Figure 4 shows how risk premia on the good-1 consol bond and on asset 1, in each set of units, depend on s. Own-unit risk premia are shown in black, and foreign-unit risk premia are shown as blue dotted lines. The risk premia that would prevail if the goods of the two countries were perfect substitutes are shown as red dashed lines.

Country 1's bond earns a risk premium in its own units because interest rates rise in bad times: if country 1 is small—s is small—then bad times correspond to bad news for country 2 and hence to a rise in s and (see Figure 3) country 1's riskless rate; and if country 1 is large—s is large—then

	Expected returns in 1-units				Expected returns in 2-units			
	α_1	α_2	λ_1	λ_2	α_1	α_2	λ_1	λ_2
tree 1	1	0	1	0	1	0	χ	$1-\chi$
tree 2	0	1	$1-\chi$	χ	0	1	0	1
bond 1	$1-1/\chi$	0	0	0	$1-1/\chi$	0	$\chi-1$	$1-\chi$
bond 2	0	$1-1/\chi$	$1-\chi$	$\chi-1$	0	$1-1/\chi$	0	0

Table 1: Values of $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ for Result 5.



(a) Good-1 bond risk premia

(b) Asset 1 risk premia

Figure 4: Left: Good-1 bond risk premia in 1-units (black solid) and in 2-units (blue dotted) and, for comparison, in the perfect substitutes case (red dashed). Right: Asset 1 risk premia in 1-units (black solid) and in 2-units (blue dotted) and in the perfect substitutes case (red dashed).

bad times correspond to bad news for country 1, and hence to a decline in s and a rise in country 1's riskless rate. Thinking in foreign units, the sign of the risk premium on country 1's bonds depends on the size of country 1. If it is small, then the risk premium on country 1's bond is even larger, due to the relative price effects discussed above: bad states of the world are those in which country 2 has bad news, the relative supply of country 1's good increases, and its relative price declines. This poor performance in bad states makes country 1's bonds risky, so they require a sizeable risk premium. If, on the other hand, country 1 is *large*, then bad states of the world are associated with a decline in the relative supply of its own good, associated with a favorable exchange rate adjustment. That is, country 1's bond is a hedge, so it earns a negative risk premium.

Figure 4b plots risk premia for asset 1 itself. In own units, the risk premium on asset 1 increases as country 1's output share—and hence its correlation with overall consumption—increases. In foreign units, the exchange rate effect described in the previous paragraph continues to operate, driving the risk premium up if country 1 is small, or down if country 1 is large.

4 The small-country limit

These complicated characterizations of riskless rates, price-dividend ratios, and expected returns simplify considerably in the *small-country limit* in which country 1 is very small and country 2 very large; the ability to take limits of the integral formulas is a major advantage of my analytical approach over the loglinearization approach. In the limit, several features of the model emerge more clearly, and those emphasized in the example above turn out to be characteristic of a whole family of possible calibrations.[11]

In this section, I assume that $\rho - \boldsymbol{c}(\chi, 1-\chi-\gamma) > 0$, so that the price-dividend ratio of each country's output claim is finite in the limit;[12] that $R_{f,1} > 0$ and $R_{f,2} > 0$, so that perpetuities have finite prices; and, strengthening Assumption 1, that $\gamma\eta > 2$.[13]

Result 6 (Asset pricing in the small-country limit)**.** *Interest rates are*

$$\begin{aligned} R_{f,1} &= \rho - \boldsymbol{c}(\chi - 1, 1-\chi-\gamma) && (20) \\ R_{f,2} &= \rho - \boldsymbol{c}(0, -\gamma). && (21) \end{aligned}$$

Since interest rates are constant, perpetuities are riskless when denominated in their own units, so do not earn a risk premium. But if returns are computed in foreign units, then a good-i perpetuity does earn a risk premium, written $XS^*_{B,i}$, where

$$\begin{aligned} XS^*_{B,1} &= \boldsymbol{c}(\chi-1, 1-\chi) + \boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) && (22) \\ XS^*_{B,2} &= \boldsymbol{c}(1-\chi, \chi-1) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(0,-\gamma). && (23) \end{aligned}$$

[11] By a continuity argument, the strict inequalities presented in Results 7, 8 and 9 of this section also hold away from the limit point, so long as country 1 is sufficiently small relative to country 2.

[12] Rather than merely for $s \in (0,1)$, as is ensured by previous assumptions. Strictly speaking, we could allow $\rho - \boldsymbol{c}(\chi, 1-\chi-\gamma) < 0$ so long as $\rho - \boldsymbol{c}(0, 1-\gamma) > 0$, because this is enough to ensure that the large country's output claim has a finite price-dividend ratio in the limit. But since this possibility was considered in detail in Martin (2011a), where it was described as the *supercritical* case, and since it is harder for the inequalities $\rho - \boldsymbol{c}(\chi, 1-\chi-\gamma) < 0$ and $\rho - \boldsymbol{c}(0, 1-\gamma) > 0$ to hold simultaneously if χ is close to zero—that is, if imperfect substitution is an important factor—I rule it out for simplicity.

[13] There is no real need for this last assumption; I make it because it seems uncontroversial, and because it reduces the number of cases to consider in Results 7, 8 and 9.

The dividend yields on the output claims are

$$
\begin{aligned}
D_1/P_1 &= \rho - \boldsymbol{c}(\chi, 1-\chi-\gamma) && (24)\\
D_2/P_2 &= \rho - \boldsymbol{c}(0, 1-\gamma). && (25)
\end{aligned}
$$

Excess returns denominated in own units, XS_i, are given by

$$
\begin{aligned}
XS_1 &= \boldsymbol{c}(1,0) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(\chi, 1-\chi-\gamma) && (26)\\
XS_2 &= \boldsymbol{c}(0,1) + \boldsymbol{c}(0, -\gamma) - \boldsymbol{c}(0, 1-\gamma). && (27)
\end{aligned}
$$

Excess returns denominated in foreign units, XS_i^*, are given by

$$
\begin{aligned}
XS_1^* &= \boldsymbol{c}(\chi, 1-\chi) + \boldsymbol{c}(0, -\gamma) - \boldsymbol{c}(\chi, 1-\chi-\gamma) && (28)\\
XS_2^* &= \boldsymbol{c}(1-\chi, \chi) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(0, 1-\gamma). && (29)
\end{aligned}
$$

The Gordon growth model holds: $D_i/P_i = XS_i + R_{f,i} - G_i$, where $G_1 \equiv \boldsymbol{c}(1,0)$ and $G_2 \equiv \boldsymbol{c}(0,1)$ are the (log) mean growth rates of output in each country.

In one sense, asset pricing in the large country is just closed-economy asset pricing: equations (21), (25) and (27) correspond directly to those derived in the one-tree economy of Martin (2010). But, for example, the risk premium on the large stock market in small-country units, given by equation (29), is a natural object of interest in a multi-country world that has no counterpart in a single closed economy.

As one would expect, the excess return on investment in a foreign country's bond can be decomposed as the sum of an interest-rate differential and an expected currency return, since we can rewrite equations (22) and (23) as $XS_{B,1}^* = FX_1^* + R_{f,1} - R_{f,2}$ and $XS_{B,2}^* = FX_2^* + R_{f,2} - R_{f,1}$.

To put these expressions in more familiar form, suppose that output growth is lognormal, and make the *cet. par.* assumption that the exchangeability property holds. Then $\boldsymbol{c}(\theta_1, \theta_2) = \mu\theta_1 + \mu\theta_2 + \sigma^2\theta_1^2/2 + \kappa\sigma^2\theta_1\theta_2 + \sigma^2\theta_2^2/2$, where μ is the mean, σ the volatility, and κ the cross-country correlation of log output growth in the two countries, and we have

$$\begin{aligned}
R_{f,1} &= \rho + \mu\gamma - \gamma^2\sigma^2/2 + \sigma^2(1-\kappa)(1-\chi)(\gamma+\chi-1) \\
R_{f,2} &= \rho + \mu\gamma - \gamma^2\sigma^2/2 \\
XS^*_{B,1} &= \gamma\sigma^2(1-\kappa)(1-\chi) \\
XS^*_{B,2} &= -(\gamma+2\chi-2)\sigma^2(1-\kappa)(1-\chi) \\
D_1/P_1 &= \rho + \mu(\gamma-1) - \sigma^2(\gamma-1)^2/2 - \sigma^2\chi(1-\kappa)(\gamma+\chi-1) \\
D_2/P_2 &= \rho + \mu(\gamma-1) - \sigma^2(\gamma-1)^2/2 \\
XS_1 &= \gamma\kappa\sigma^2 + \sigma^2(1-\kappa)(1-\chi) \\
XS_2 &= \gamma\sigma^2 \\
XS^*_1 &= \gamma\kappa\sigma^2 + \gamma\sigma^2(1-\kappa)(1-\chi) \\
XS^*_2 &= \gamma\sigma^2 - (\gamma+2\chi-1)\sigma^2(1-\kappa)(1-\chi).
\end{aligned}$$

In the perfect substitutes case, $\chi = 1$, there is no exchange-rate risk, so interest rates are equal in each country, bonds are riskless, and the small country's equity claim is risky only to the extent that its fundamentals are correlated with the large country's fundamentals. All this changes if the goods are imperfect substitutes ($\chi < 1$). The interest rate is higher in the small than in the large country; the excess return on the small country's bond in large units is positive, while that on the large country's bond in small units is negative; and the small country's dividend yield increases, and its equity risk premium increases—particularly when denominated in foreign units—as exchange-rate risk becomes important.[14]

Without the exchangeability assumption, the signs of most of these risk premia can be set arbitrarily even in the lognormal case (for example, by making the small country's output extremely volatile, and adjusting its correlation with the large country). The only risk premium for which this is not true is that of the large country's equity claim denominated in large units, which is always positive. In the lognormal case, this is obvious. Although it is no surprise that $XS_2 > 0$ in general, the proof that it holds illustrates how the convexity property of CGFs can be ex-

[14] Although this particular example should not be taken too literally, it is interesting to note how the extra risk premia that are in principle observable in a two-country world permit the model's deep parameters to be easily identified. For example, observing equity premia in each country in own units (XS_1 and XS_2) together with the small country's bond and equity premia in large units ($XS^*_{B,1}$ and XS^*_1) enables κ, γ, σ and χ to be identified. (This is not vacuous, since the equations are nonlinear.) Observing the riskless rate and dividend yield on either country's output claim would also enable ρ and μ to be identified.

ploited. For, $XS_2 = \boldsymbol{c}(0,1) + \boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(0,1-\gamma)$, and note that $(0,1)$ and $(0,-\gamma)$, considered as points in $\mathbb{R}^2$, are a midpoint-preserving-spread of $(0,1-\gamma)$ and $(0,0)$. Convexity then implies that $\boldsymbol{c}(0,1) + \boldsymbol{c}(0,-\gamma) > \boldsymbol{c}(0,1-\gamma) + \boldsymbol{c}(0,0)$, and the result follows because $\boldsymbol{c}(0,0) = 0$. Figure 6b in Appendix B illustrates this logic graphically. This graphical approach was essential in finding proofs of the more complicated results in the remainder of this section. See Appendix A for the conventional proofs, and Appendix B for the visual proofs.

To put some discipline on the model without making strong parametric assumptions, it is helpful to focus attention away from the details of the countries' output processes by assuming that the exchangeability property holds. We then have the following result.

Result 7 (Strong failure of UIP for the small country)**.** *Suppose Property 1 holds. Then the interest rate in the small country is higher than the interest rate in the large country:* $R_{f,1} > R_{f,2}$*. But this higher interest rate is not offset by expected exchange rate movements. On the contrary, the expected appreciations in the relative price of each country's good are equal, and positive—Siegel's (1972) "paradox" once again. Thus uncovered interest parity (UIP) fails in a strong sense: not only do expected exchange rate movements not fully offset the small country's higher interest rate, they actually* increase *the expected return on the carry trade. That is,* $XS^*_{B,1} > FX^*_1 > 0$.

The corresponding result for the large country relies on a property that restricts the behavior of the higher cumulants of output growth.

Property 2 (Convex differences)**.** *The CGF* $\boldsymbol{c}(\cdot,\cdot)$ *has the* convex difference property (CDP) *if* $\boldsymbol{c}(\theta_1,\theta_2) - \boldsymbol{c}(\theta_1+t,\theta_2+t)$ *is convex in* (θ_1,θ_2) *for all* $t > 0$*,* θ_1*, and* θ_2 *such that* (θ_1,θ_2) *and* (θ_1+t,θ_2+t) *lie in the triangle* $\Delta \subset \mathbb{R}^2$ *whose corners are at* $(1,1)$, $(1,-\gamma-1)$ *and* $(-\gamma-1,1)$.

This property imposes a restriction that neither country has positively skewed log output growth. If, for example, output growth is independent in the two countries, so that $\boldsymbol{c}(\theta_1,\theta_2)$ can be expressed as $\boldsymbol{c}_1(\theta_1) + \boldsymbol{c}_2(\theta_2)$, then it is equivalent to $\boldsymbol{c}_i'''(\theta_i) \leq 0$, $i = 1,2$, in the triangle Δ. In particular, $\boldsymbol{c}_i'''(0) \leq 0$: that is, the third cumulant—skewness—cannot be positive. It is also satisfied if output growth is lognormal, in disaster calibrations of the type suggested in Barro (2006), and in the numerical example presented in this paper.[15]

When it holds, the large country's bond earns a negative risk premium from the perspective of investors thinking in small-country units—a type of "exorbitant privilege" (Gourinchas and Rey (2007), quoting Valéry Giscard d'Estaing).

[15] The visual proofs, specifically Figures 8b and 10b, reveal why the convex difference property is formulated as it is.

Result 8 (An exorbitant privilege)**.** *Suppose Properties 1 and 2 hold. Then UIP also fails for the large country, which has the "exorbitant privilege" of paying a negative risk premium on its bonds in small-country units:* $XS^*_{B,2} < 0$.

Result 7 showed only that the riskless rate is higher in the small country than in the large country. Result 8 is stronger: it can be rephrased as saying that the (unfavorable) riskless rate differential faced by an investor who borrows at the small country's interest rate and invests at the large country's interest rate is sufficiently large that it overcomes the favorable expected exchange rate movement: $R_{f,1} - R_{f,2} > FX^*_2$.

To characterize the risk premia on the two countries' output claims, I make a final assumption that the countries have *linked fundamentals*: in the lognormal case, for example, I want to rule out the possibility that the correlation between the two countries' output growth is negative so that the small country's output claim is a hedge. The following property turns out to be the appropriate way to capture this idea.

Property 3 (Linked fundamentals)**.** *The two countries have linked fundamentals if the CGF is* supermodular,[16] *meaning that for all* $\theta_1, \theta_2, \phi_1, \phi_2$ *in* Δ,

$$\boldsymbol{c}(\theta_1, \theta_2) + \boldsymbol{c}(\phi_1, \phi_2) \leq \boldsymbol{c}\left(\max\left\{\theta_1, \phi_1\right\}, \max\left\{\theta_2, \phi_2\right\}\right) + \boldsymbol{c}\left(\min\left\{\theta_1, \phi_1\right\}, \min\left\{\theta_2, \phi_2\right\}\right).$$

By Topkis's (1978) Characterization Theorem, a sufficient condition for Property 3 to hold is that

$$\frac{\partial^2 \boldsymbol{c}(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} \geq 0 \tag{30}$$

for all θ_1 and θ_2 in some open set containing Δ. It is immediate that the linked fundamentals property holds (with equality) if output growth is independent across countries. In any given parametric example, it is easy to check whether (30) holds. In the lognormal case, (30) shows that the linked fundamentals property is equivalent to the correlation between the two countries' log output growth being nonnegative.[17]

[16] Vives (1990), Milgrom and Roberts (1990a, 1990b), and Athey (2002) present various economic applications of supermodularity—in particular, to games with strategic complementarities. By Lemma 4 of Athey (2002), supermodularity of the CGF is implied by log-supermodularity of the probability density function of $(y_{1,t+1} - y_{1,t}, y_{2,t+1} - y_{2,t})$; this provides another way of generating examples in which the linked fundamentals property holds.

[17] As with the convex difference property, the easiest and best way to understand what supermodularity imposes is visually. See Figure 9c and the discussion in the caption.

Result 9. *Suppose Properties 1, 2 and 3 hold. Then there is a critical value* $\eta^* \in (1,\infty)$*—where* $\eta^* = 2$ *in the lognormal case—such that*

$$0 < XS_1 < XS_1^* < XS_2^* < XS_2 \qquad \text{if } \eta > \eta^*$$
$$0 < XS_1 < XS_2^* < XS_1^* < XS_2 \qquad \text{if } \eta < \eta^*.$$

We also have $XS_{B,1}^* \leq XS_1^*$.

If the countries have strictly linked fundamentals[18] *and* η *is sufficiently large then we have a total ordering of risk premia:* $XS_{B,2}^* < 0 < XS_{B,1}^* < XS_1 < XS_1^* < XS_2^* < XS_2$.

Result 9 extends the model's predictions regarding bond risk premia to risky assets. The risk premium on the small country's output claim is greater in foreign units than in own units, $XS_1^* > XS_1$, while the risk premium on the large country's claim is smaller in foreign units than in own units, $XS_2^* < XS_2$. The size of η indexes the amount of currency risk. If the goods of the two countries are sufficiently poor substitutes ($\eta < \eta^*$), then currency risk is so great that the risk premium on the small country's output claim in foreign units exceeds the risk premium on the large country's claim in foreign units, $XS_1^* > XS_2^*$, even though the small country contributes a negligible proportion of the representative agent's consumption.

5 Conclusion

The logic of this paper rests on a fundamental asymmetry: the representative agent cares more about the large country than the small country, since it provides a larger share of consumption. As a result, when the large country has bad news the relative price of its output increases, while the relative price of the small country's output decreases. As a result, interest rates differ across countries. Since, by construction, the exchange rate follows a random walk, UIP fails in *any* calibration.

In the small-country limit, some of the complications of the model evaporate, leaving behind the central intuition. It is then possible to characterize *how* UIP fails across whole families of possible calibrations that satisfy some rather general nonparametric restrictions on the cumulant-generating function. If, say, the random variables driving output growth in the two countries are exchangeable—loosely speaking, if the two countries have symmetric, but not necessarily independent, output growth distributions—then the small country's exchange rate is expected to appreciate. Even so, its bonds are riskier, and so earn a higher interest rate, than the large country's bonds.

[18] That is, if the inequality in the definition of Property 3 is strict.

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A Appendix

Recall the notation $\chi = (\eta - 1)/\eta$ and $\widehat{\gamma} = (\gamma + \chi - 1)/\chi$, and let $\widehat{y}_{1t} \equiv y_{1t} + [(1-\chi)/\chi] \log w$ and $\widehat{y}_{2t} \equiv y_{2t} + [(1-\chi)/\chi] \log(1-w)$. We also defined the state variable s_t: $s_t \equiv D_{1t}/(D_{1t} + e_t \cdot D_{2t}) \in (0,1)$ and $\widehat{u}_t$, a monotonic transformation of s_t: $\widehat{u}_t \equiv \log[(1-s_t)/s_t]$. This definition implies that $\widehat{u}_t = \chi(\widehat{y}_{2t} - \widehat{y}_{1t})$.

The consumption aggregator can be expressed as

$$C_t = \left[e^{\chi \widehat{y}_{10} + \chi \widetilde{y}_{1t}} + e^{\chi \widehat{y}_{20} + \chi \widetilde{y}_{2t}} \right]^{\frac{1}{\chi}} , \tag{31}$$

and the price of good 2 in 1-units is

$$e_t = \left(\frac{1-w}{w} \right)^{1-\chi} \left(\frac{D_{1t}}{D_{2t}} \right)^{1-\chi} = \left(\frac{1-w}{w} \right)^{(1-\chi)/\chi} \cdot e^{-[(1-\chi)/\chi]\widehat{u}_t}. \tag{32}$$

If $\eta = \infty$—the perfect substitutes case—then $\chi = 1$ so e_t is constant. If, on the other hand, $\eta = 1$, then C_t is a Cobb-Douglas aggregator of the two goods so that $C_t \propto D_{1t}^w D_{2t}^{1-w}$. It is easy to check that this implies that the price-dividend ratio of each asset is constant. Here I assume that $\eta \in (1, \infty)$. From (6), the (unit-free) price-dividend ratio of good 1 is

$$\frac{P_1}{D_{10}} = \mathbb{E} \int_0^\infty e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{-\chi \widehat{\gamma}} \left(\frac{D_{1t}}{D_{10}} \right)^{\chi} dt \,, \tag{33}$$

where $\widehat{\gamma} \equiv (\gamma + \chi - 1)/\chi$. So,

$$\frac{P_1}{D_{10}} = C_0^{\chi \widehat{\gamma}} \cdot \int_0^\infty e^{-\rho t} \, \mathbb{E} \left(\frac{e^{\chi \widetilde{y}_{1t}}}{\left[e^{\chi \widehat{y}_{10} + \chi \widetilde{y}_{1t}} + e^{\chi \widehat{y}_{20} + \chi \widetilde{y}_{2t}} \right]^{\widehat{\gamma}}} \right) dt, \tag{34}$$

and similarly, the price (in own units) of a zero-coupon bond that pays a unit of good 1 at time t is

$$
\begin{aligned}
Z_{t,1} &= \mathbb{E}e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\chi\widehat{\gamma}}\left(\frac{D_{1t}}{D_{10}}\right)^{\chi-1} \\
&= C_0^{\chi\widehat{\gamma}}e^{-\rho t}\cdot\mathbb{E}\left(\frac{e^{\frac{\chi-1}{\chi}\cdot\chi\widetilde{y}_{1t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right). \qquad (35)
\end{aligned}
$$

A perpetuity, or consol, that pays a constant stream of good 1, at rate 1 per unit time is just a portfolio of zero-coupon bonds, so integrating the above expression over t, we find the perpetuity price (which can also be thought of as a price-dividend ratio, since the dividend is fixed at 1 unit of the good)

$$
B_1 = C_0^{\chi\widehat{\gamma}}\cdot\int_0^\infty e^{-\rho t}\,\mathbb{E}\left(\frac{e^{\frac{\chi-1}{\chi}\cdot\chi\widetilde{y}_{1t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right)dt\,. \qquad (36)
$$

Correspondingly, the price-dividend ratio of asset 2 is

$$
\begin{aligned}
\frac{P_2}{D_{20}} &= \mathbb{E}\int_0^\infty e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\chi\widehat{\gamma}}\left(\frac{D_{2t}}{D_{20}}\right)^{\chi}dt \\
&= C_0^{\chi\widehat{\gamma}}\cdot\int_0^\infty e^{-\rho t}\,\mathbb{E}\left(\frac{e^{\chi\widetilde{y}_{2t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right)dt\,, \qquad (37)
\end{aligned}
$$

and the price, in 2-units, of the t-period zero-coupon good-2 bond is

$$
\begin{aligned}
Z_{t,2} &= \mathbb{E}e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\chi\widehat{\gamma}}\left(\frac{D_{2t}}{D_{20}}\right)^{\chi-1} \\
&= C_0^{\chi\widehat{\gamma}}e^{-\rho t}\cdot\mathbb{E}\left(\frac{e^{\frac{\chi-1}{\chi}\cdot\chi\widetilde{y}_{2t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right), \qquad (38)
\end{aligned}
$$

so the price-dividend ratio of the good-2 perpetuity is

$$
B_2 = C_0^{\chi\widehat{\gamma}}\cdot\int_0^\infty e^{-\rho t}\,\mathbb{E}\left(\frac{e^{\frac{\chi-1}{\chi}\cdot\chi\widetilde{y}_{2t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right)dt\,. \qquad (39)
$$

Equations (34)–(39) each feature an expectation of the form

$$
E(\alpha_1,\alpha_2)\equiv\mathbb{E}\left(\frac{e^{\alpha_1\chi\widetilde{y}_{1t}+\alpha_2\chi\widetilde{y}_{2t}}}{\left[e^{\chi\widehat{y}_{10}+\chi\widetilde{y}_{1t}}+e^{\chi\widehat{y}_{20}+\chi\widetilde{y}_{2t}}\right]^{\widehat{\gamma}}}\right), \qquad (40)
$$

where the values of α_1 and α_2 corresponding to the various assets are given in Table 2.

Proof of Result 2: Expression (40) can be rewritten

$$
E(\alpha_1,\alpha_2)=e^{-\chi\widehat{\gamma}(\widehat{y}_{10}+\widehat{y}_{20})/2}\cdot\mathbb{E}\left(\frac{e^{\chi(\alpha_1-\widehat{\gamma}/2)\widetilde{y}_{1t}+\chi(\alpha_2-\widehat{\gamma}/2)\widetilde{y}_{2t}}}{\left[e^{\chi(\widehat{y}_{20}+\widetilde{y}_{2t}-\widehat{y}_{10}-\widetilde{y}_{1t})/2}+e^{-\chi(\widehat{y}_{20}+\widetilde{y}_{2t}-\widehat{y}_{10}-\widetilde{y}_{1t})/2}\right]^{\widehat{\gamma}}}\right).
$$

	α_1	α_2
P_1/D_{10}	1	0
P_2/D_{20}	0	1
$B_1, Z_{T,1}$	$1-1/\chi$	0
$B_2, Z_{T,2}$	0	$1-1/\chi$

Table 2: Values of α_1 and α_2 in (40); all assets priced in own units.

Martin (2011a) shows the Fourier transform result that for $\omega \in \mathbb{R}$ and $\widehat{\gamma} > 0$,

$$\frac{1}{\left(e^{\omega/2}+e^{-\omega/2}\right)^{\widehat{\gamma}}} = \int_{-\infty}^{\infty} e^{i\omega v} \mathscr{F}(v)\, dv\,, \tag{41}$$

where i is the complex number $\sqrt{-1}$ and $\mathscr{F}(v) \equiv \frac{1}{2\pi} \cdot B(\widehat{\gamma}/2 + iv, \widehat{\gamma}/2 - iv)$ defines $\mathscr{F}(v)$ in terms of the Euler beta function. Applying this, we find that

$$\begin{aligned} E(\alpha_1, \alpha_2) &= e^{-\chi\widehat{\gamma}(\widehat{y}_{10}+\widehat{y}_{20})/2} \cdot \mathbb{E}\left(e^{\chi(\alpha_1-\widehat{\gamma}/2)\widetilde{y}_{1t}+\chi(\alpha_2-\widehat{\gamma}/2)\widetilde{y}_{2t}} \int_{-\infty}^{\infty} e^{i\chi(\widehat{y}_{20}+\widetilde{y}_{2t}-\widehat{y}_{10}-\widetilde{y}_{1t})v} \mathscr{F}(v)\, dv \right) \\ &= e^{-\chi\widehat{\gamma}(\widehat{y}_{10}+\widehat{y}_{20})/2} \int_{-\infty}^{\infty} e^{i\widehat{u}v} \mathscr{F}(v) \cdot e^{\boldsymbol{c}(\chi(\alpha_1-\widehat{\gamma}/2-iv),\chi(\alpha_2-\widehat{\gamma}/2+iv))t}\, dv \end{aligned} \tag{42}$$

where $\widehat{u} \equiv \chi(\widehat{y}_{20} - \widehat{y}_{10})$.

The generic expression we want to evaluate is

$$\begin{aligned} V_{\alpha_1,\alpha_2}(\widehat{u}) &= C_0^{\chi\widehat{\gamma}} \int_0^{\infty} e^{-\rho t} \mathbb{E} \frac{e^{\alpha_1\chi\widetilde{y}_{1t}+\alpha_2\chi\widetilde{y}_{2t}}}{\left[e^{\chi(\widehat{y}_{10}+\widetilde{y}_{1t})}+e^{\chi(\widehat{y}_{20}+\widetilde{y}_{2t})}\right]^{\widehat{\gamma}}}\, dt \\ &= \frac{\left[e^{\chi\widehat{y}_{10}}+e^{\chi\widehat{y}_{20}}\right]^{\widehat{\gamma}}}{e^{\chi\widehat{\gamma}(\widehat{y}_{10}+\widehat{y}_{20})/2}} \cdot \int_0^{\infty} e^{-\rho t} \cdot e^{\chi\widehat{\gamma}(\widehat{y}_{10}+\widehat{y}_{20})/2} \cdot E(\alpha_1,\alpha_2)\, dt \\ &= \left[e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right]^{\widehat{\gamma}} \int_{t=0}^{\infty} \int_{v=-\infty}^{\infty} e^{-\{\rho-\boldsymbol{c}[\chi(\alpha_1-\widehat{\gamma}/2-iv),\chi(\alpha_2-\widehat{\gamma}/2+iv)]\}t} e^{i\widehat{u}v} \mathscr{F}(v)\, dv\, dt \\ &= \left[e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right]^{\widehat{\gamma}} \int_{-\infty}^{\infty} \frac{e^{i\widehat{u}v}\mathscr{F}(v)}{\rho-\boldsymbol{c}\left[\chi(\alpha_1-\widehat{\gamma}/2-iv),\chi(\alpha_2-\widehat{\gamma}/2+iv)\right]}\, dv \end{aligned} \tag{43}$$

using (42) and assuming $\rho - \boldsymbol{c}\left[\chi(\alpha_1-\widehat{\gamma}/2), \chi(\alpha_2-\widehat{\gamma}/2)\right] > 0$. Using the results of Appendix A.3 of Martin (2011a), this ensures that for all $v \in \mathbb{R}$, Re $\rho - \boldsymbol{c}\left[\chi(\alpha_1-\widehat{\gamma}/2-iv), \chi(\alpha_2-\widehat{\gamma}/2+iv)\right] > 0$, as required. □

Proof of Result 1: For arbitrary constants w_1 and w_2,

$$\mathbb{E}\left[d\left(e^{w_1\widehat{y}_{1t}+w_2\widehat{y}_{2t}}\right)\right] = \boldsymbol{c}(w_1,w_2)e^{w_1\widehat{y}_{1t}+w_2\widehat{y}_{2t}}\, dt\,. \tag{44}$$

The result follows by applying (44) to equation (9), using the definition of $\widehat{u}$; the proof of Siegel's paradox follows by the convexity of the CGF, which implies that $\boldsymbol{c}(\chi-1,1-\chi)+\boldsymbol{c}(1-\chi,\chi-1)>0$. For a visual proof, see Appendix B. □

Proof of Result 3: Using the preceding results in equations (35) and (38), we have

$$Z_{t,1}=\left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\cdot\int_{-\infty}^{\infty}e^{i\widehat{u}v}\mathscr{F}(v)e^{-\{\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]\}t}\,dv$$

and

$$Z_{t,2}=\left(e^{\widehat{u}/2}+e^{-\widehat{u}/2}\right)^{\widehat{\gamma}}\cdot\int_{-\infty}^{\infty}e^{i\widehat{u}v}\mathscr{F}(v)e^{-\{\rho-\boldsymbol{c}[\chi(-\widehat{\gamma}/2-iv),\chi(1-1/\chi-\widehat{\gamma}/2+iv)]\}t}\,dv.$$

The zero-coupon yields follow immediately; and using l'Hôpital's rule to take the limit as $t\downarrow 0$, the riskless rate expressions follow too.

To calculate long rates, we use the method of steepest descent. In the case of the long rate in 1-units, which is

$$\mathscr{Y}_{\infty,1}(\widehat{u})=\lim_{t\uparrow\infty}\frac{-1}{t}\log\left\{\int_{-\infty}^{\infty}e^{i\widehat{u}v}\mathscr{F}(v)e^{-\{\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]\}t}\,dv\right\},$$

we are interested in a stationary point of $\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]$, considered as a function of $v\in\mathbb{C}$. If $v=ix$ is pure imaginary, this function is concave when considered as a function of $x\in\mathbb{R}$ (Martin (2011a)), so has a stationary point at some ix^*, $x^*\in\mathbb{R}$. If $|x^*|<\gamma/2$ then the contour of integration can be continuously deformed to pass through the stationary point without crossing a pole. It follows by the method of steepest descent that $\mathscr{Y}_{\infty,1}(\widehat{u})=\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2+x^*),\chi(-\widehat{\gamma}/2-x^*)]$. If, on the other hand, the stationary point occurs for $x^*>\widehat{\gamma}/2$, then there is a residue to take into account at $v=(\widehat{\gamma}/2)i$; it turns out that $\mathscr{Y}_{\infty,1}(\widehat{u})=\rho-\boldsymbol{c}[\chi-1,1-\chi-\gamma]$. Similarly, if the stationary point occurs at $x^*<-\widehat{\gamma}/2$ then $\mathscr{Y}_{\infty,1}(\widehat{u})=\rho-\boldsymbol{c}[-\gamma,0]$.

These cases can be summarized by writing $\mathscr{Y}_{\infty,1}=\max_{\theta\in[-\widehat{\gamma}/2,\widehat{\gamma}/2]}\rho-\boldsymbol{c}[\chi(1-1/\chi-\widehat{\gamma}/2+s),\chi(-\widehat{\gamma}/2-s)]$, or equivalently, $\mathscr{Y}_{\infty,1}=\max_{\theta\in[0,\gamma+\chi-1]}\rho-\boldsymbol{c}(\theta-\gamma,-\theta)$.

By exchangeability, the long rate in 2-units is $\mathscr{Y}_{\infty,2}=\max_{\theta\in[0,\gamma+\chi-1]}\rho-\boldsymbol{c}(-\theta,\theta-\gamma)$, which can also be rewritten as $\mathscr{Y}_{\infty,2}=\max_{\theta\in[1-\chi,\gamma]}\rho-\boldsymbol{c}(\theta-\gamma,-\theta)$. □

Proof of Result 4: A proof was provided in the text. □

Proof of Result 5: The dividend yield component of the expected return is the reciprocal of the valuation ratio, so it remains to calculate $\mathbb{E}dP/P$.

The general problem we face has

$$P=e^{\lambda_1\widehat{y}_{1t}+\lambda_2\widehat{y}_{2t}}\left(e^{\chi(\widehat{y}_{1t}-\widehat{y}_{2t})/2}+e^{\chi(\widehat{y}_{2t}-\widehat{y}_{1t})/2}\right)^{\widehat{\gamma}}\int_{-\infty}^{\infty}\frac{e^{i\chi(\widehat{y}_{2t}-\widehat{y}_{1t})v}\mathscr{F}(v)}{\rho-\boldsymbol{c}[\chi(\alpha_1-\widehat{\gamma}/2-iv),\chi(\alpha_2-\widehat{\gamma}/2+iv)]}\,dv\,,$$

where $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ vary from asset to asset and are supplied in Table 1. This can be rewritten as

$$P = \sum_{m=0}^{\widehat{\gamma}} \binom{\widehat{\gamma}}{m} \int_{-\infty}^{\infty} \frac{\mathscr{F}(v) e^{\widehat{y}_{1t}(\lambda_1 + m\chi - \widehat{\gamma}\chi/2 - i\chi v) + \widehat{y}_{2t}(\lambda_2 - m\chi + \widehat{\gamma}\chi/2 + i\chi v)}}{\rho - \boldsymbol{c}[\chi(\alpha_1 - \widehat{\gamma}/2 - iv), \chi(\alpha_2 - \widehat{\gamma}/2 + iv)]} \, dv \, ,$$

so using (44), we find that $\mathbb{E}dP$ equals

$$\sum \binom{\widehat{\gamma}}{m} \int \frac{\mathscr{F}(v) \boldsymbol{c}(\lambda_1 + m\chi - \widehat{\gamma}\chi/2 - i\chi v, \lambda_2 - m\chi + \widehat{\gamma}\chi/2 + i\chi v) e^{\widehat{y}_{1t}(\lambda_1 + m\chi - \widehat{\gamma}\chi/2 - i\chi v) + \widehat{y}_{2t}(\lambda_2 - m\chi + \widehat{\gamma}\chi/2 + i\chi v)}}{\rho - \boldsymbol{c}[\chi(\alpha_1 - \widehat{\gamma}/2 - iv), \chi(\alpha_2 - \widehat{\gamma}/2 + iv)]} \, dv \, .$$

Dividing $\mathbb{E}dP$ by P and rearranging, the result follows, after defining $G_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}(\widehat{u})$ to equal

$$\sum_{m=0}^{\widehat{\gamma}} \binom{\widehat{\gamma}}{m} e^{(\widehat{\gamma}/2 - m)\widehat{u}} \int_{-\infty}^{\infty} \frac{e^{i\widehat{u}v} \mathscr{F}(v) \boldsymbol{c}(\lambda_1 + m\chi - \widehat{\gamma}\chi/2 - i\chi v, \lambda_2 - m\chi + \widehat{\gamma}\chi/2 + i\chi v)}{\rho - \boldsymbol{c}[\chi(\alpha_1 - \widehat{\gamma}/2 - iv), \chi(\alpha_2 - \widehat{\gamma}/2 + iv)]} \, dv.$$

□

Proof of Result 6: The proof is very similar to the proofs of Propositions 6 and 7 in Martin (2011a), so is omitted. □

Proof of Result 7: To show that $R_{f,1} > R_{f,2}$, I must show that $\boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) > 0$. Invoking exchangeability, this is equivalent to showing that $\boldsymbol{c}(0,-\gamma) + \boldsymbol{c}(-\gamma,0) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(1-\chi-\gamma, \chi-1) > 0$. But this is an immediate consequence of the fact that $\boldsymbol{c}(\cdot,\cdot)$, as a cumulant-generating function, is strictly convex.

In Result 1, I showed that $(FX_1^* + FX_2^*)/2 > 0$. Since we are assuming here that Property 1 holds, we have $FX_1^* = FX_2^*$. Combining these two facts, it follows that $FX_1^* > 0$ and $FX_2^* > 0$. □

Proof of Result 8: $XS_{B,2}^* < 0$ if and only if $\boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(1-\chi, \chi-1) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) > 0$. By exchangeability, this is equivalent to showing that $\boldsymbol{c}(0,-\gamma) + \boldsymbol{c}(-\gamma,0) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(1-\chi-\gamma, \chi-1) - \boldsymbol{c}(1-\chi, \chi-1) - \boldsymbol{c}(\chi-1, 1-\chi) > 0$. By strict convexity of the CGF, this is true so long as $[\boldsymbol{c}(\chi-1-\gamma/2, 1-\chi-\gamma/2) - 2\boldsymbol{c}(-\gamma/2, -\gamma/2) + \boldsymbol{c}(1-\chi-\gamma/2, \chi-1-\gamma/2)] - [\boldsymbol{c}(1-\chi, \chi-1) - 2\boldsymbol{c}(0,0) + \boldsymbol{c}(\chi-1, 1-\chi)] \geq 0$. (Here I am using the fact that $\gamma + 2\chi - 2 > 0$, which follows because of the assumption that $\gamma\eta > 2$.) But this holds because the convex difference property ensures that the first term in square brackets is (weakly) greater than the second term in square brackets. □

Proof of Result 9: To show that $XS_1 > 0$, I must show that $\boldsymbol{c}(1,0) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(\chi, 1-\chi-\gamma) > 0$. This is equivalent to $[\boldsymbol{c}(1,0) - \boldsymbol{c}(\chi,0) - \boldsymbol{c}(0,0) + \boldsymbol{c}(\chi-1,0)] + [\boldsymbol{c}(\chi,0) - \boldsymbol{c}(\chi-1,0) - \boldsymbol{c}(\chi, 1-\chi-\gamma) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma)] > 0$. This holds because the first term in square brackets is positive due to strict convexity of the CGF, and the second is nonnegative by supermodularity.

By exchangeability, $XS_1 < XS_1^*$ is equivalent to $[\boldsymbol{c}(-\gamma,0) + \boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(1-\chi-\gamma, \chi-1)] - [\boldsymbol{c}(1,0) + \boldsymbol{c}(0,1) - \boldsymbol{c}(\chi, 1-\chi) - \boldsymbol{c}(1-\chi, \chi)] > 0$. By strict convexity of the CGF, it suffices to show that $[\boldsymbol{c}(-\gamma/2 - 1/2, -\gamma/2 + 1/2) + \boldsymbol{c}(-\gamma/2 + 1/2, -\gamma/2 - 1/2) - \boldsymbol{c}(-\gamma/2 + 1/2 -$

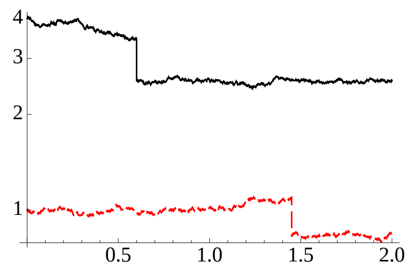
$\chi, -\gamma-1/2+\chi) - \boldsymbol{c}(-\gamma/2-1/2+\chi, -\gamma/2+1/2-\chi)] - [\boldsymbol{c}(1,0)+\boldsymbol{c}(0,1)-\boldsymbol{c}(\chi, 1-\chi)-\boldsymbol{c}(1-\chi, \chi)] \geq 0$, which holds by the convex difference property. The proof that $XS_1 < XS_2^*$ is very similar, so is omitted.

By exchangeability, $XS_1^* < XS_2$ is equivalent to $[\boldsymbol{c}(\chi, 1-\chi-\gamma) + \boldsymbol{c}(1-\chi-\gamma, \chi) - \boldsymbol{c}(0, 1-\gamma) - \boldsymbol{c}(1-\gamma, 0)] + [\boldsymbol{c}(1,0)+\boldsymbol{c}(0,1)-\boldsymbol{c}(\chi, 1-\chi)-\boldsymbol{c}(1-\chi, \chi)] > 0$. But each of the terms in square brackets is positive by strict convexity of the CGF. The proof that $XS_2^* < XS_2$ is very similar, so is omitted.

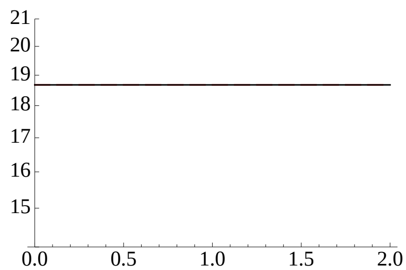
The final inequality that holds independent of conditions on η is that $XS_{B,1}^* \leq XS_1^*$. This is equivalent to $\boldsymbol{c}(\chi, 1-\chi) - \boldsymbol{c}(\chi, 1-\chi-\gamma) - \boldsymbol{c}(\chi-1, 1-\chi) + \boldsymbol{c}(\chi-1, 1-\chi-\gamma) \geq 0$, which follows from supermodularity.

By exchangeability, the sign of $XS_2^* - XS_1^*$ is the same as the sign of the (strictly convex) function $Q(\chi) \equiv \boldsymbol{c}(\chi-1, 1-\chi-\gamma) + \boldsymbol{c}(1-\chi-\gamma, \chi-1) - \boldsymbol{c}(0, 1-\gamma) - \boldsymbol{c}(1-\gamma, 0) - \boldsymbol{c}(0, -\gamma) - \boldsymbol{c}(-\gamma, 0) + \boldsymbol{c}(\chi, 1-\chi-\gamma) + \boldsymbol{c}(1-\chi-\gamma, \chi)$. Now, $Q(0) = \boldsymbol{c}(-1, 1-\gamma) + \boldsymbol{c}(1-\gamma, -1) - \boldsymbol{c}(0, -\gamma) - \boldsymbol{c}(-\gamma, 0) < 0$ and $Q(1) = -\boldsymbol{c}(0, 1-\gamma) - \boldsymbol{c}(1-\gamma, 0) + \boldsymbol{c}(1, -\gamma) + \boldsymbol{c}(-\gamma, 1) > 0$ by strict convexity of $\boldsymbol{c}(\cdot, \cdot)$. $Q(\chi)$ is also continuous; thus, $Q(\chi^*) = 0$ for some unique $\chi^* \in (0,1)$. Defining η^* via $\chi^* = 1 - 1/\eta^*$, we have $XS_2^* = XS_1^*$ if $\eta = \eta^*$; $XS_2^* < XS_1^*$ for $\eta < \eta^*$; and $XS_2^* > XS_1^*$ for $\eta > \eta^*$.

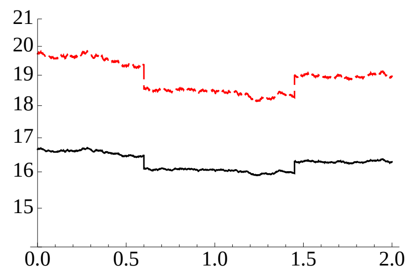
It remains to be shown that $XS_{B,1}^* < XS_1$ if the CGF is strictly supermodular and η is sufficiently large; equivalently, $\boldsymbol{c}(\chi-1, 1-\chi) + \boldsymbol{c}(0, -\gamma) - 2\boldsymbol{c}(\chi-1, 1-\chi-\gamma) - \boldsymbol{c}(1,0) + \boldsymbol{c}(\chi, 1-\chi-\gamma) < 0$. In the case $\chi = 1$, we must show that $\boldsymbol{c}(0, -\gamma) + \boldsymbol{c}(1, 0) - \boldsymbol{c}(1, -\gamma) > 0 = \boldsymbol{c}(0,0)$. This follows by strict supermodularity. Therefore by continuity, $XS_{B,1}^* < XS_1$ for χ in some neighborhood of 1; equivalently, $XS_{B,1}^* < XS_1$ for sufficiently large η. $\square$



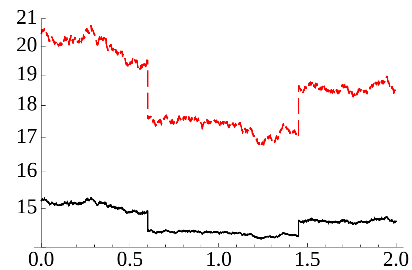
(a) Dividends, D_i



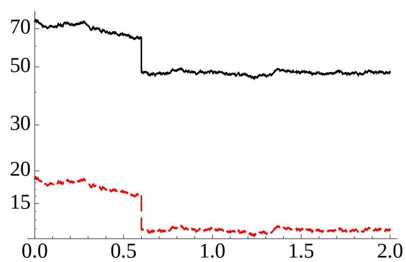
(b) P_i/D_i ($\eta = 1$)



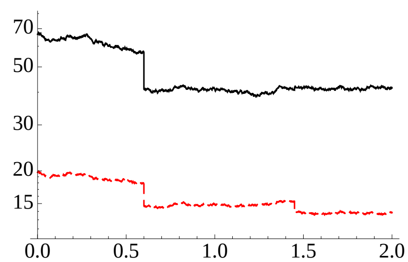
(c) P_i/D_i ($\eta = 2$)



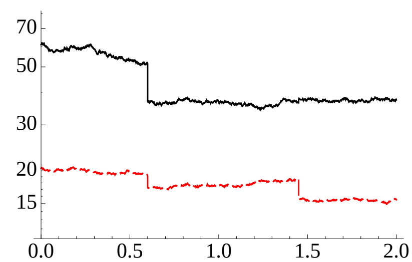
(d) P_i/D_i ($\eta = \infty$)



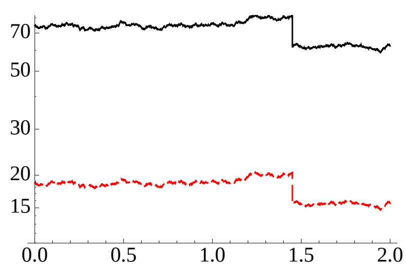
(e) P_i (large units, $\eta = 1$)



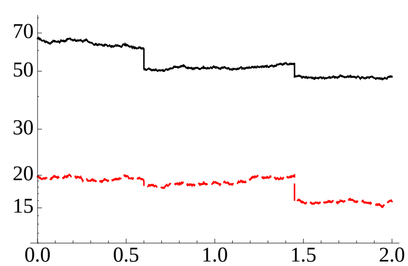
(f) P_i (large units, $\eta = 2$)



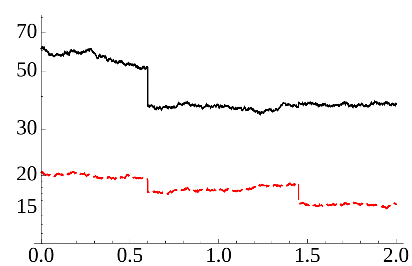
(g) P_i (large units, $\eta = \infty$)



(h) P_i (small units, $\eta = 1$)



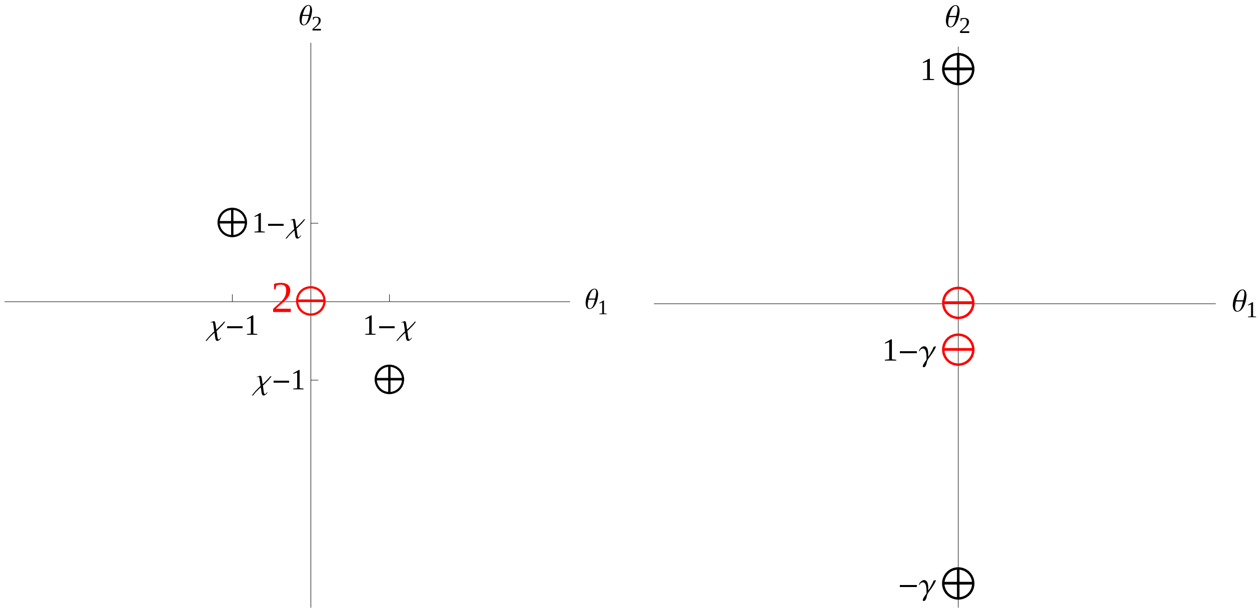
(i) P_i (small units, $\eta = 2$)



(j) P_i (small units, $\eta = \infty$)

Figure 5: Sample paths of price-dividend ratios, P_i/D_i, and prices, P_i of output claims of each country, for different values of η. All paths are based on the same underlying path of fundamentals, shown in panel (a).

B Visual proofs



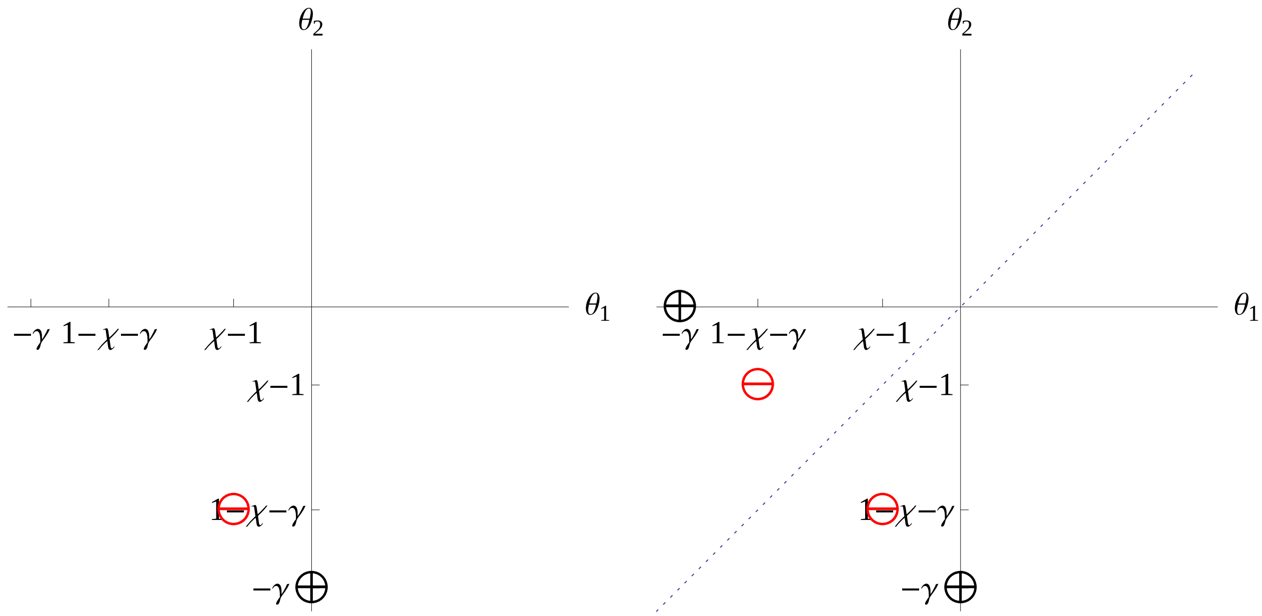
(a) This is positive, by convexity

(b) This is positive, by convexity

Figure 6: Left: A visual proof of Siegel's paradox. We must show that $\boldsymbol{c}(\chi-1,1-\chi)+\boldsymbol{c}(1-\chi,\chi-1) > 0$, or equivalently, because $\boldsymbol{c}(0,0) = 0$, that $\boldsymbol{c}(\chi - 1, 1 - \chi) - 2\boldsymbol{c}(0,0) + \boldsymbol{c}(1 - \chi, \chi - 1) > 0$. Panel (a) represents the left-hand side of this inequality in the sense that the signed sum of values taken by the CGF at the indicated points (with signs indicated by black plus and red minus signs) is the expression on the left-hand side. I will say that the panel is positive if the signed sum that it represents is positive. So, the aim is to show that the panel is positive. Convexity of the CGF gives the result.

Right: A visual proof that the large country requires a positive risk premium in own units, $XS_2 > 0$. We must show that $\boldsymbol{c}(0,1) + \boldsymbol{c}(0,-\gamma) - \boldsymbol{c}(0,1-\gamma) - \boldsymbol{c}(0,0) > 0$. The result follows by convexity of the CGF.

Note: Part of the advantage of the visual approach, once you're used to it, is that it is immediately clear if an expression is positive by virtue of convexity. We require the four points to be arranged (i) on a straight line, (ii) symmetrically about their midpoint, and (iii) with this sign pattern.

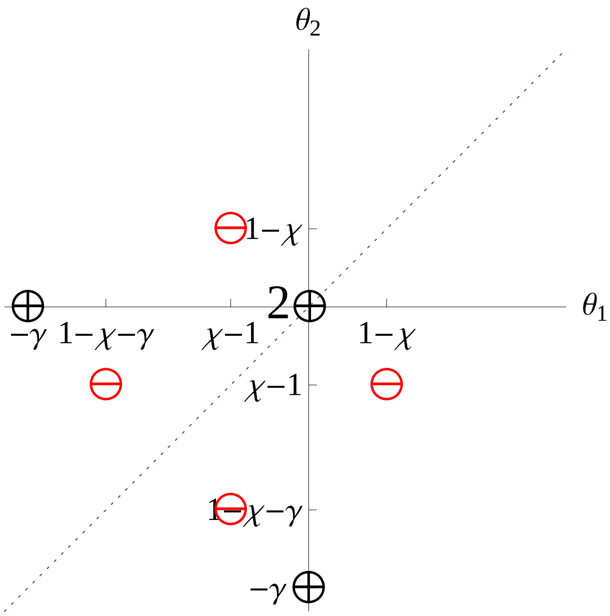


(a) To prove this is positive...

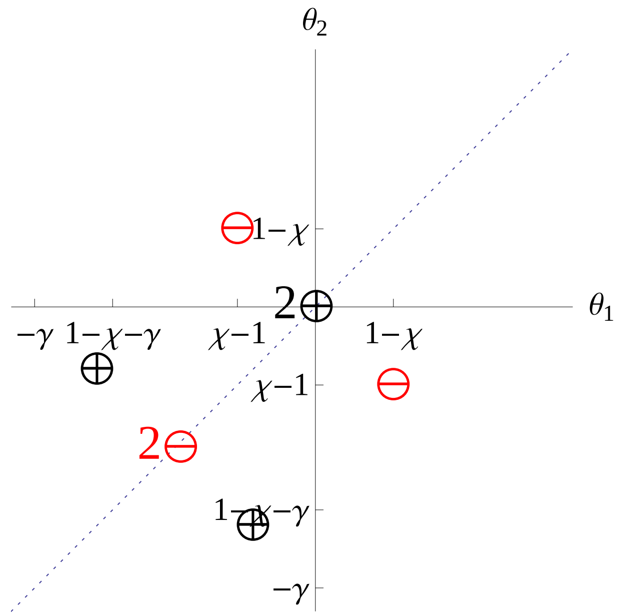
(b) ...use exchangeability then convexity

Figure 7: A visual proof that the small country's bond earns an excess return in foreign units that is greater than the expected currency return, $XS^*_{B,1} > FX^*_1$. We must show that $\boldsymbol{c}(0, -\gamma) - \boldsymbol{c}(\chi - 1, 1 - \chi - \gamma) > 0$, i.e. that panel (a) is positive. By exchangeability, this is equivalent to $\boldsymbol{c}(0, -\gamma) + \boldsymbol{c}(-\gamma, 0) - \boldsymbol{c}(\chi - 1, 1 - \chi - \gamma) - \boldsymbol{c}(1 - \chi - \gamma, \chi - 1) > 0$. Graphically, this corresponds to reflecting in the 45 degree line, as shown in panel (b). Panel (b) is positive by convexity, so the result follows.

Note: from now on, I will exploit exchangeability without further comment. When I do, I plot the 45 degree line.



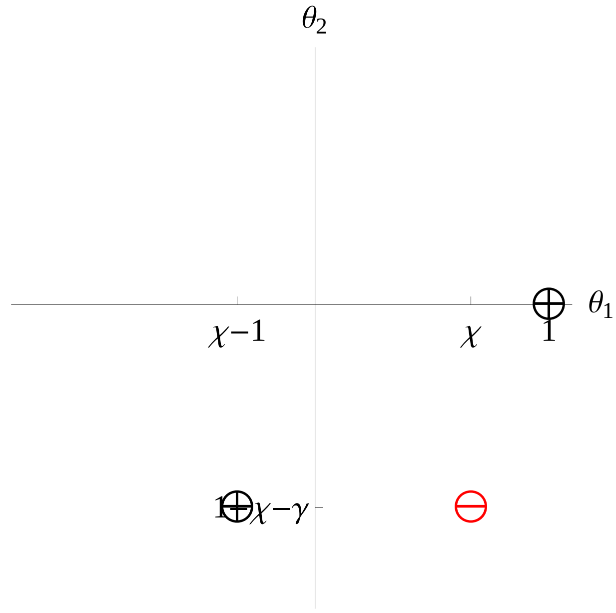
(a) To prove this is positive. . .



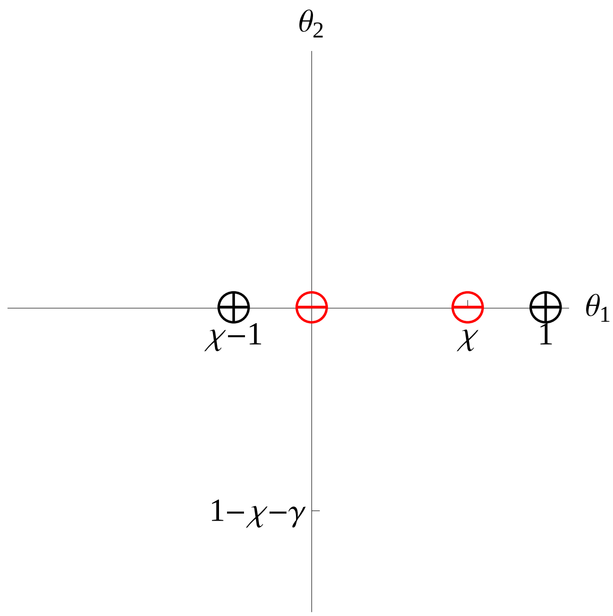
(b) . . . use convexity then CDP

Figure 8: A visual proof that the large country's bond earns a negative risk premium in foreign units, $XS^*_{B,2} < 0$. Exchangeability has already been used in the left panel. In each panel, there are two groups of points, running from north-west to south-east. In the right panel, the four south-west-most points have been compressed towards their mid-point. By convexity, this makes the sum smaller. Even so, it remains positive, by the convex difference property, and thus the left panel was also positive.

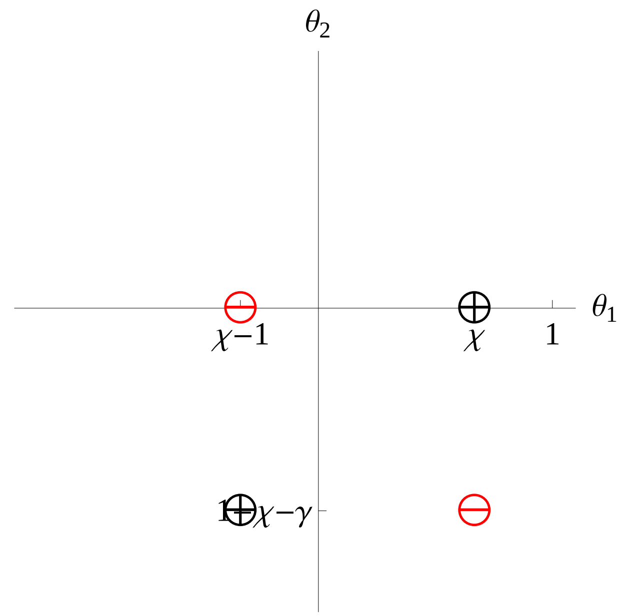
Note: the convex difference property was formulated as it was so that sign patterns like panel (b) would be positive.



(a) This is positive because. . .



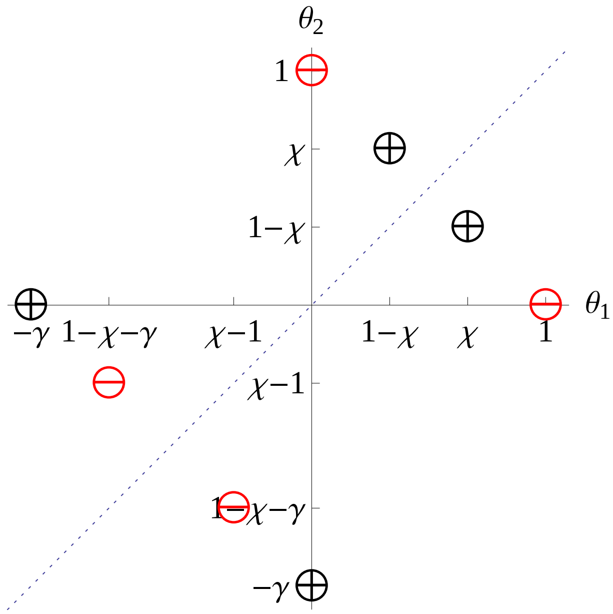
(b) . . . this is, by convexity. . .



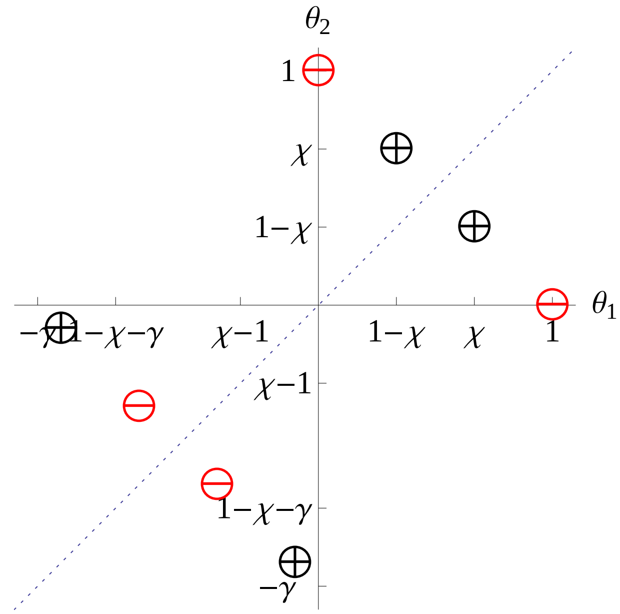
(c) . . . and this is, by supermodularity

Figure 9: A visual proof that the risk premium on the small country's output claim is positive in own units, $XS_1 > 0$. The top panel, which represents XS_1, is the sum of the bottom two panels. Now, panel (b) is strictly positive by convexity of the CGF and panel (c) is nonnegative by supermodularity. The CGF is zero at the origin, so the top panel really is the sum of the bottom two panels.

Note: panel (c) illustrates what supermodularity "is", namely the property that every rectangle with this sign pattern is positive.

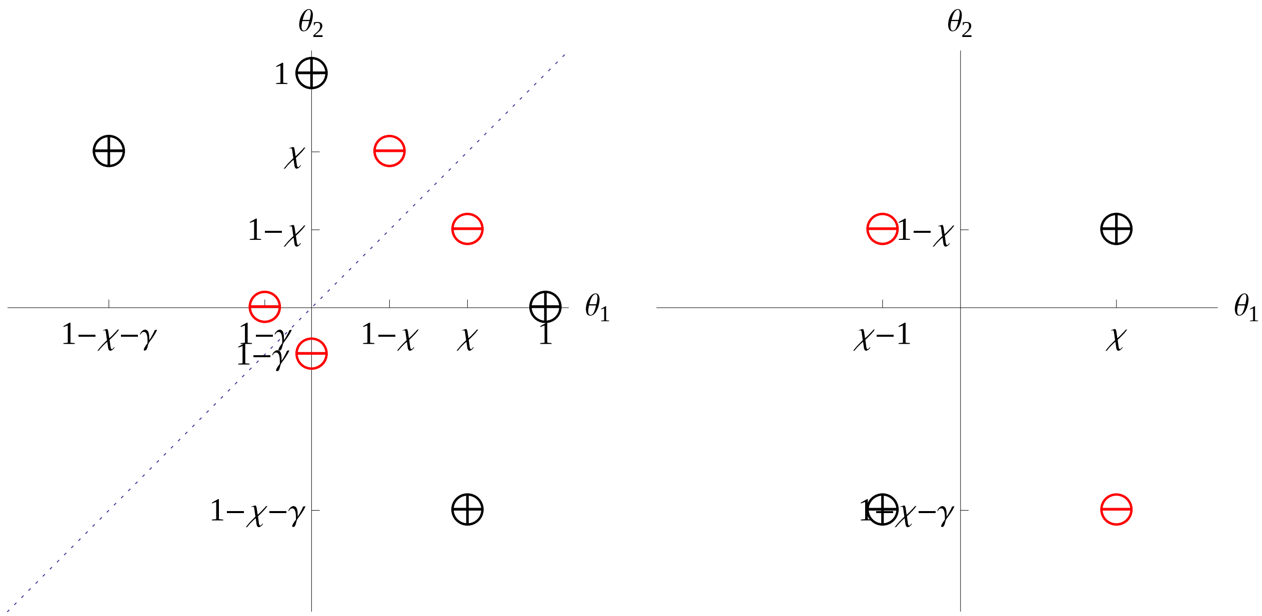


(a) To prove this is positive...



(b) ...use convexity then CDP

Figure 10: A visual proof that the risk premium on the small country's output claim is higher in foreign units than it is in own units, $XS_1 < XS_1^*$. The logic is as in Figure 8.



(a) Positive by convexity

(b) Positive by supermodularity

Figure 11: Left: A visual proof that the risk premium on the large country's output claim, in large-country units, is higher than the risk premium on the small country's output claim, in large-country units, $XS_1^* < XS_2$. Exchangeability has already been used. The two groups of points are both positive by convexity of the CGF.

Right: A visual proof that in large-country units, the small country's output claim requires a (weakly) higher risk premium than its bond, $XS_{B,1}^* \leq XS_1^*$. The result follows immediately by supermodularity.