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THE DYNAMICS OF OPTIMAL RISK SHARING

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ABSTRACT

We study a dynamic-contracting problem involving risk sharing between two parties — the Proposer and the Responder — who invest in a risky asset until an exogenous but random termination time. In any time period they must invest all their wealth in the risky asset, but they can share the underlying investment and termination risk. When the project ends they consume their final accumulated wealth. The Proposer and the Responder have constant *relative* risk aversion R and r respectively, with $R>r>0$. We show that the optimal contract has three components: a non-contingent flow payment, a share in investment risk and a termination payment. We derive approximations for the optimal share in investment risk and the optimal termination payment, and we use numerical simulations to show that these approximations offer a close fit to the exact rules. The approximations take the form of a myopic benchmark plus a dynamic correction. In the case of the approximation for the optimal share in investment risk, the myopic benchmark is simply the classical formula for optimal risk sharing. This benchmark is endogenous because it depends on the wealths of the two parties. The dynamic correction is driven by counterparty risk. If both parties are fairly risk tolerant, in the sense that $2>R>r$, then the Proposer takes on more risk than she would under the myopic benchmark. If both parties are fairly risk averse, in the sense that $R>r>2$, then the Proposer takes on less risk than she would under the myopic benchmark. In the mixed case, in which $R>2>r$, the Proposer takes on more risk when the Responder's share in total wealth is low and less risk when the Responder's share in total wealth is high. In the case of the approximation for the optimal termination payment, the myopic benchmark is zero. The dynamic correction tells us, among other things, that: (i) if the asset has a high return then, following termination, the Responder compensates the Proposer for the loss of a valuable investment opportunity; and (ii) if the asset has a low return then, prior to termination, the Responder compensates the Proposer for the low returns obtained. Finally, we exploit our representation of the optimal contract to derive simple and easily interpretable sufficient conditions for the existence of an optimal contract.

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1. Introduction

This paper considers a risk-sharing problem in which two investors pool their resources to invest in a common risky venture. Investment returns are assumed to follow a geometric Brownian motion, and the investors' risk preferences are represented by utility functions exhibiting constant *relative* risk-aversion (CRRA). The two investors have different coefficients of relative risk-aversion and different initial wealth endowments. They can write a long-term insurance contract specifying a division of final output contingent on the sample path of output of the venture. The venture may end at any time with positive probability, and when it ends the two investors consume their final accumulated wealth.

To keep the analysis tractable we have stripped out of the model many features which would make it more realistic. For example, our model allows for only two investors, only one risky asset, and investors only consume at the end. In addition, we simplify the formulation of the optimal contracting problem by letting one individual, the Proposer, make a take-it-or-leave-it contract offer to the other, the Responder. Even so, the analysis of this optimal contracting problem is sufficiently complex that we are only able to approximate the optimal risk-sharing rule. For reasonable parameter configurations, however, this approximation is a good fit for the numerically determined optimal risk-sharing rule.

Optimal risk-sharing between two parties was first analyzed by Borch (1962), in the context of a reinsurance problem. He considers an optimal contract to share risk between an insurance and a reinsurance company (or between two insurance companies). While his framework is more general in many respects than the one we have just described, he only derives a necessary condition for optimal co-insurance between two risk-averse investors, the well known Borch condition.

In this paper we push the analysis further and derive explicit risk-sharing formulae that approximate the optimal risk-sharing rule. We do this by reformulating the risk-sharing problem as a recursive problem in which the Proposer offers the Responder spot contracts, each of which has three components: (i) a fixed transfer f to the Responder; (ii) a share s of spot investment returns; and (iii) a final transfer b to be paid to the Responder in the event that the venture terminates. We then derive relatively simple formulae for s and b that approximate the optimal risk-sharing spot contract. Thus, a central contribution of this paper is to derive (approximate) formulae for optimal risk-sharing for the CRRA case.

As each investor's aversion to risk and capacity to insure the other investor varies with its wealth, the optimal shares s and b vary with the underlying wealth distribution. Thus, one advantage of our recursive formulation is that it brings out explicitly the underlying dynamics of the risk-sharing problem. These dynamics can be understood as follows. Whenever the two investors engage in risk-sharing, the optimal spot contract will specify a division of total investment returns that is different from each investor's share in total wealth. As a result, the wealth distribution in the next period will be different from the wealth distribution in this period. For example, if the Responder insures the Proposer, by taking on a share of risk bigger than his share in wealth, then his wealth share will increase when there is a high investment return and decrease when there is a low investment return. Either way, the wealth distribution changes and consequently each investor's attitude towards risk and capacity to insure changes. This change in each investor's capacity to insure introduces endogenous *counterparty risk*, and forward-looking investors will take this risk into account in deciding on the optimal spot contract.[1]

To gain insight into how this counterparty risk can affect optimal risk-sharing, consider the extreme case in which the Responder is risk-neutral and the Proposer is risk-averse. It is well known that optimal risk-sharing in a one-shot insurance contracting problem in this case requires that the Responder insure the Proposer perfectly. But if the Responder were to do this repeatedly, then he would be sure to go bankrupt at some point, and then the Proposer would no longer be able to get any insurance at all. Foreseeing this, the Proposer would want to hold back from getting perfect insurance. Only when the Responder is relatively wealthy would the Proposer seek perfect insurance. When the Responder is relatively poor, the Proposer may optimally limit the amount of insurance she gets to preserve future insurance opportunities.

We are able to extend this insight to the general case, in which both investors are risk averse, by assuming that both the Proposer and the Responder are close to myopic, and by taking approximations around the myopic optimum. This approximation therefore takes the form of a *myopic benchmark* plus a *dynamic correction*.

Consider first the optimal rule for the sharing of investment risk, namely s. The myopic benchmark for s requires that the Responder take on a share in total investment risk equal to the well known ratio of the Proposer's coefficient of *absolute* risk aversion

[1] The role played by counterparty risk in our model is analogous to the role that it plays in futures markets. There, traders are required to maintain margin accounts as a way of eliminating default. Although these requirements prevent any default as such, they bear witness to the profound role played by counterparty risk, and they constrain the amount of hedging a counterparty can offer.

to the sum of both investors' coefficients of *absolute* risk aversion. As for the dynamic correction, if we denote by R and r the coefficients of relative risk aversion of the Proposer and Responder respectively, then:

1. When both investors are fairly risk tolerant, in the sense that $R, r < 2$, it is optimal for the less risk-averse investor to take on *less* risk in the dynamic-contracting problem than the myopic rule would specify. This is because the less risk averse investor is willing to take on risk on relatively unfavourable terms, so transferring more risk to that investor tends to reduce the stock of insurance available to the more risk averse investor in the future.

2. When both investors are fairly risk averse, in the sense that $R, r > 2$, it is optimal for the less risk-averse investor to take on *more* risk in the dynamic-contracting problem than the myopic rule would specify. This is because the less risk averse investor is only willing to take on risk on relatively favourable terms, so transferring more risk to that investor tends to increase the stock of insurance available to the more risk averse investor in the future.

3. When one investor (say the Responder) is fairly risk tolerant but the other investor (say the Proposer) is fairly risk averse, in other words when $R > 2 > r$, then the Proposer takes on less risk when she is relatively poor and more risk when she is relatively wealthy. This is because, when she is poor, her aversion to bearing risk outweighs her concern that the Responder may run out of money; but when she is rich, the opposite is true.

Consider next the optimal termination payment b. The myopic benchmark for b is zero. This is because there is no termination risk in the myopic limit. As for the dynamic correction, we show that: (i) if the venture has a high return then, *following* termination, the less risk averse investor compensates the more risk averse investor for the loss of a valuable investment opportunity; and (ii) if the venture has a low return then, *prior to* termination, the less risk averse investor compensates the more risk averse investor for the low returns obtained.

Although our model is highly stylized, it may be relevant to a number of applications. We have already mentioned reinsurance as one application. Insurance companies are obviously capital constrained and they rely on each other to share common risk. Our analysis sheds light on how these companies should structure their risk sharing to take account of counterparty risk. Another application, which was our initial motivation, is to portfolio- or fund-management contracts. In practice, the contract between a representative client and a fund manager often takes the simple form of a share of portfolio returns for the client equal to the client's share of investments in the fund minus a management fee, which is equal to a small percentage of the funds under management. We recognize that the main concern in portfolio management generally is the manager's incentive to run the fund in the client's best interest. Still, we believe that our analysis may be relevant if there are also dynamic risk-sharing considerations involved in the long-term relation between the client and the manager.[2]

Besides Borch (1962) and the large literature on optimal risk sharing that it has spawned (see Eeckhoudt, Gollier and Schlesinger, 2005) our paper is most closely related to the dynamic asset pricing problem with two classes of investors considered by Dumas (1989). He analyzes the equilibrium investment and consumption choices of two classes of investor with different coefficients of relative risk aversion in an otherwise standard competitive economy with aggregate shocks. Although Dumas mainly focuses on equilibrium asset pricing, his analysis proceeds via a planning problem. One key difference between his setup and ours is that he allows for ongoing consumption, while we only have consumption upon termination. Another is that we have termination risk, while he only considers an infinitely lived economy. Finally, Dumas' solution method only works in the case in which one investor has a log utility function.

There is by now accumulating evidence that consumers differ substantially in their risk preferences. Indeed, Barsky, Juster, Kimball, and Shapiro (1997), in their experimental study on risk-taking decisions, found that the behaviour of 5% of subjects was consistent with a coefficient of relative risk aversion of 33 or higher, that the behaviour of another 5% was consistent with a coefficient of 1.3 or lower, and that the median coefficient was about 7. Similarly, Guiso and Paiella (2008) and Chiappori and Paiella (2008) among others find evidence of heterogeneous risk preferences in households' actual portfolio allocations. In addition, using panel data on individual portfolio allocations between risky and riskless assets, Chiappori and Paiella (2008) are able to determine that the elasticity of the risky-asset share with respect to wealth in their sample is small and statistically insignificant, which is consistent with CRRA risk preferences. In another panel study on household

[2]For continuous-time models of portfolio-mangement contracts with moral-hazard and/or asymmetric information see Ou-Yang (2003), Cvitanic and Zhang (2007) and Cvitanic, Wan, and Zhang (2008).

portfolio choices, however, Paravisini, Rappoport and Ravina (2009) use investor fixed effects and find that the within-household elasticity of risk taking with respect to changes in household wealth is negative and quite large. They also find substantial heterogeneity in relative risk aversion in their sample, with an average coefficient of 2.85 and a median coefficient of 1.62.

The present paper is organized as follows. Section 2 describes the two investors' preferences and the investment technology. Section 3 derives the value function of the Responder under autarky. Section 4 formulates the long-run contracting problem between the Proposer and Responder. Section 5 formulates the spot-contracting problem, and Section 6 derives the associated Bellman equation of the Proposer. Section 7 establishes that any long-run contract for which the participation constraint of the Responder binds can be replicated by a flow of spot contracts for which the spot participation constraint of the Responder likewise binds. Section 8 shows that the Bellman equation of the Proposer under spot contracting can be reduced to a partial differential equation. Section 9 provides a first characterization of the optimal risk-sharing rule and termination payment in terms of the value functions of the Proposer and Responder. Section 10 uses asymptotic expansions to derive risk-sharing formulae which approximate the optimal risk-sharing rule and termination payment. Section 11 shows how the Bellman equation for the Proposer can be further reduced to a pair of ordinary differential equations on $(0, 1)$. Section 12 identifies sufficient conditions under which a solution to these equations extends continuously to $[0, 1]$. Section 13 then solves the resulting two-point boundary-value problem numerically. The numerical solutions show how well the formulae derived in Section 10 predict the qualitative shape of the optimal risk-sharing rule and termination payment, suggesting that these formulae contain most of the analytical insight into optimal risk sharing that can be obtained for our model. Section 14 offers some concluding comments.

2. Preferences and Technology

The initial wealths of the Proposer and the Responder are $W_0, w_0 > 0$. There is an exogenous termination time T, which is distributed exponentially with parameter $\beta > 0$. At any time $t \in [0, T]$ the parties have access to the same constant-stochastic-returns-to-scale investment opportunity, but they cannot consume. For an investment x, this investment opportunity yields flow returns

$$dx = x\,(\mu\,dt + \sigma\,dz),$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and z is a standard Wiener process (i.e. the shock dz at time t is normally distributed with mean 0 and variance dt and is independent of the shocks at all earlier times). Following termination, both parties consume their accumulated wealths W_{T+} and w_{T+}.[3]

The risk preferences of the Proposer and the Responder are represented by the strictly increasing and strictly concave utility functions U and u. In what follows it will sometimes be helpful to avoid imposing specific functional forms on U and u, but we shall frequently assume that they take the constant relative risk aversion (CRRA) form

$$U(W) = C_R(W) = \left\{ \begin{array}{ll} \dfrac{W^{1-R}-1}{1-R} & \text{if } R \neq 1 \\ \log(W) & \text{if } R = 1 \end{array} \right\}$$

and

$$u(w) = C_r(w) = \left\{ \begin{array}{ll} \dfrac{w^{1-r}-1}{1-r} & \text{if } r \neq 1 \\ \log(w) & \text{if } r = 1 \end{array} \right\},$$

where $R, r > 0$. When we do not assume this,

$$R = R(W) = -\frac{W\,U''(W)}{U'(W)}$$

and

$$r = r(w) = -\frac{w\,u''(w)}{u'(w)}$$

will denote the coefficients of relative risk aversion of U and u respectively.

Remark 1. *Throughout the paper, the subscript R will denote the Proposer and the subscript r will denote the Responder.*

3. AUTARKY FOR THE RESPONDER

Consider first the case in which the Responder invests on his own. His value function $\widetilde{v} : (0, \infty) \times \{0, 1\} \to \mathbb{R}$ for this case will provide his reservation value in the bilateral

[3] The notation here reflects the idea that W_{T+} and w_{T+} are the wealths of the two parties at the end of period T. In what follows, these need to be distinguished from W_T and w_T, which are the wealths of the two parties at the beginning of period T.

contracting problems described below. It satisfies the Bellman equation

$$\widetilde{v}(w,\chi) = \left\{ \begin{array}{ll} \mathrm{E}\left[\,\widetilde{v}(w+dw,\chi+d\chi)\,\right] & \text{if } \chi = 0 \\ u(w) & \text{if } \chi = 1 \end{array} \right\} \tag{1}$$

where: w is the accumulated wealth of the Responder; χ is an indicator taking the value 0 if the problem has not yet terminated and the value 1 if the problem has terminated;[4]

$$dw = w\,(\mu\,dt + \sigma\,dz);$$

and

$$d\chi = \left\{ \begin{array}{ll} 0 & \text{if the problem does not terminate} \\ 1 & \text{if the problem terminates} \end{array} \right\}.$$

Putting $v = \widetilde{v}(\cdot,0)$ in equation (1), we obtain

$$\begin{aligned} v(w) &= \mathrm{E}\left[v(w) + v'(w)\,dw + \tfrac{1}{2}\,v''(w)\,dw^2 + (u(w)-v(w))\,d\chi\right] \\ &= v(w) + \left(v'(w)\,\mu\,w + \tfrac{1}{2}\,v''(w)\,\sigma^2\,w^2 + (u(w)-v(w))\,\beta\right)dt \end{aligned}$$

or

$$0 = \tfrac{1}{2}\,\sigma^2\,w^2\,v'' + \mu\,w\,v' + \beta\,(u(w) - v), \tag{2}$$

where we have suppressed the dependence of v on w.

If r is constant, then equation (2) has an explicit solution. Indeed, given that wealth follows a geometric Wiener process and that utility is CRRA, it is natural to conjecture that v will take the form

$$v(w) = C_r(\rho_r\,w),$$

where ρ_r is the certainty-equivalent rate of return of the Responder under autarky. This conjecture is correct, and leads to the following Proposition:

Proposition 2. *Suppose that the Responder has constant relative risk aversion. Then equation (2) has a solution of the form*

$$v(w) = C_r(w) + w\,C_r'(w)\,\psi_r,$$

[4]More precisely, the random function $\chi : [0,\infty) \to \{0,1\}$ is given by the formula $\chi_t = 0$ if $t \le T$ and $\chi_t = 1$ if $t > T$. In particular, in a reversal of the usual convention, χ is continuous on the left.

where $\psi_r = \dfrac{\mu - \frac{1}{2}\, r\, \sigma^2}{\beta_r}$ *and* $\beta_r = \beta - (1-r)\,(\mu - \frac{1}{2}\, r\, \sigma^2)$ *are the normalized value function of the Responder and the effective discount rate of the Responder respectively. Moreover*

$$v'(w) = \gamma_r\, C_r'(w),$$

where $\gamma_r = \dfrac{\beta}{\beta_r}$ *is the normalized marginal value of wealth of the Responder.*

In particular, under autarky, the normalized value function of the Responder ψ_r is simply the risk-adjusted rate of return $\mu - \frac{1}{2}\, r\, \sigma^2$ divided by the effective discount rate β_r.

Proof. Suppose that $v(w) = C_r(\rho_r\, w)$, and put $\psi_r = C_r(\rho_r)$. Then

$$v(w) = C_r(\rho_r\, w) = C_r(w) + w\, C_r'(w)\, C_r(\rho_r) = C_r(w) + w\, C_r'(w)\, \psi_r,$$

$$u(w) - v(w) = C_r(w) - (C_r(w) + w\, C_r'(w)\, \psi_r) = -w\, C_r'(w)\, \psi_r,$$
$$v'(w) = \rho_r\, C_r'(\rho_r\, w) = \rho_r\, C_r'(\rho_r)\, C_r'(w) = (1 + (1-r)\, \psi_r)\, C_r'(w),$$
$$v''(w) = (1 + (1-r)\, \psi_r)\, C_r''(w) = -\frac{r}{w}\, v'(w).$$

Hence, substituting in equation (2) and solving for ψ_r, we obtain

$$\begin{aligned} 0 &= \tfrac{1}{2}\, \sigma^2\, w^2\, v'' + \mu\, w\, v' + \beta\, (u(w) - v) \\ &= \left(\mu - \tfrac{1}{2}\, r\, \sigma^2\right) w\, v' - \beta\, w\, C_r'(w)\, \psi_r \\ &= \left(\left(\mu - \tfrac{1}{2}\, r\, \sigma^2\right) (1 + (1-r)\, \psi_r) - \beta\, \psi_r\right) w\, C_r'(w) \\ &= \left(\mu - \tfrac{1}{2}\, r\, \sigma^2 - \beta_r\, \psi_r\right) w\, C_r'(w), \end{aligned}$$

where $\beta_r = \beta - (1-r)\, (\mu - \frac{1}{2}\, r\, \sigma^2)$. Dividing through by $w\, C_r'(w)$ then yields $\psi_r = \frac{\mu - \frac{1}{2}\, r\, \sigma^2}{\beta_r}$. Hence $1 + (1-r)\, \psi_r = \frac{\beta}{\beta_r}$. Hence $v'(w) = \frac{\beta}{\beta_r}\, C_r'(w)$. ∎

Since the marginal value of wealth and the normalized marginal value of wealth must be positive, we see in particular that a basic requirement for our contracting problem to make sense is that $\beta_r > 0$, i.e. that:

Condition I. $\beta > (1-r)\, (\mu - \frac{1}{2}\, r\, \sigma^2)$.

In other words, the rate of discounting β must exceed the rate of growth of utility when wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\, r\, \sigma^2$.

Remark 3. *When $r < 1$, the utility function of the Responder is unbounded above. The main point of Condition I is then to ensure that the wealth of the Responder cannot grow too fast. When $r > 1$, the utility function of the Responder is unbounded below, and the main point of Condition I is to ensure that the wealth of the Responder cannot shrink too fast.*

It is also natural to assume that the analogue of Condition I for the Proposer holds, namely:

Condition II. $\beta > (1 - R)\,(\mu - \frac{1}{2}\,R\,\sigma^2)$.

4. THE LONG-RUN-CONTRACTING PROBLEM

Suppose that the Proposer offers the Responder a long-run contract q, according to which the two parties will pool their wealths until termination, after which the wealth pool will be shared between them. More precisely: let Ω denote the set of pairs (X, T) such that $X : [0, \infty) \to (0, \infty)$ is continuous on the left,[5] $T \in [0, \infty)$ and X is constant on (T, ∞);[6] and let $q : \Omega \to \mathbb{R}$ be a bounded measurable function such that $0 < q(X, T) < X_{T+}$ for all $(X, T) \in \Omega$.[7] If the Responder accepts q then: the initial wealth pool will be

$$X_0 = W_0 + w_0; \tag{3}$$

[5]We assume that the timepaths of all of our variables are continuous on the left. This is a departure from the usual convention, which is to take the timepaths of variables to be continuous on the right. We do this because, in order to write our various Bellman equations in a compact way, it is helpful to have W, w and χ, which are short for W_t, w_t and χ_t, denote the values of the Proposer's wealth, the Responder's wealth and the termination indicator at the beginning of period t; and to have $W + dW$, $w + dw$ and $\chi + d\chi$ denote the values of these variables at the end of period t.

[6]In view of our convention that the timepaths of variables are continuous on the left, the timepath of a variable over the stochastic interval $[0, T]$ tells us the value of the variable at the beginning of every period $t \in [0, T]$ and — by taking limits on the right — the value of the variable at the end of every period $t \in [0, T)$. In particular, it tells us the initial value of the variable, i.e. the value at the beginning of period 0, but not the final value of the variable, i.e. the value at the end of period T. (If we adopted the usual convention, namely that the timepaths of variables are continuous on the right, then the timepath of a variable over the stochastic interval $[0, T]$ would tell us the final value of the variable at the end of period T but not the initial value of the variable at the beginning of time 0.) We therefore need to supply the final value. We do this by requiring that the variable be defined but constant on (T, ∞), and by interpreting the value there as the final value. This convention has the advantage that the value at the end of period T can — like the values at the end of any other period $t \in [0, T)$ — be found by taking the limit on the right.

[7]Notice that the information conveyed by the pair (X, T) consists of: T, which is the termination time; the restriction of X to the interval $[0, T]$, which tells us the value X_t of X at the beginning of each period $t \in [0, T]$ and the value X_{t+} of X at the end of each period $t \in [0, T)$; and the value of X on the interval (T, ∞), which tells us the value X_{T+} of X at the end of each period T. The final payment $q(X, T)$ can depend on all this information.

the wealth pool will evolve according to the equation

$$dX = \left\{ \begin{array}{cc} X\,(\mu\,dt + \sigma\,dz) & \text{if } t \in [0,T] \\ 0 & \text{if } t \in (T,\infty) \end{array} \right\}; \tag{4}$$

and the final wealths of the Proposer and the Responder will be

$$W_{T+} = X_{T+} - q(X,T) \quad \text{and} \quad w_{T+} = q(X,T). \tag{5}$$

If the Responder rejects q then both parties will operate under autarky until termination.[8]

In the long-run contracting problem, the Proposer's problem is therefore to choose q to maximize her expected utility

$$\mathrm{E}[U(W_{T+})] \tag{6}$$

subject to the dynamics (3-5) and the participation constraint of the Responder, namely

$$\mathrm{E}[u(w_{T+})] \geq v(w_0), \tag{7}$$

where v is the value function of the Responder under autarky.

Three points should be noted. First, taken together, Conditions I and II ensure that the expected payoffs of both parties are well defined in the event of disagreement. Second, the Proposer can always do at least as well in the long-run contracting problem as she can under autarky. This is because she can reproduce the autarky outcome by offering the contract

$$q(X,T) = \frac{w_0}{W_0 + w_0}\,X_{T+}.$$

Third, the participation constraint of the Responder always binds in the long-run contracting problem. For, if it did not, then the Proposer could just scale down q until the participation constraint of the Responder did bind. This would have the effect of transferring a strictly positive – albeit stochastic – amount of the final total wealth from the Responder to the Proposer, and would therefore make the Proposer strictly better off.

The main challenge in establishing the existence of an optimal contract is therefore to show that the expected payoff of the Proposer is bounded above. One of the many advantages of the spot-contracting problem introduced below is that it allows us to find

[8] Since X is continuous, total wealth X_{T+} at the end of period T is equal to total wealth X_T at the beginning of period T. We write X_{T+} in the interests of notational consistency.

transparent sufficient conditions under which this is the case.

5. THE SPOT-CONTRACTING PROBLEM

The easiest way to solve the long-run contracting problem is to show that it can be reduced to a spot contracting problem. In a spot contracting problem, the two parties start out with their initial wealths W_0 and w_0. Then, in each period $t \in [0,T]$, the Proposer offers the Responder a spot contract

$$(f, s, b) \in \mathbb{R} \times \mathbb{R} \times \left(-\tfrac{w}{W+w}, \tfrac{W}{W+w}\right).$$

If the Responder accepts then he receives:

1. a non-contingent transfer $(W + w)\, f\, dt$, which is an up-front payment for his participation in the risk-sharing arrangement;
2. a contingent transfer $(W + w)\, s\, (\mu\, dt + \sigma\, dz)$, which is his share in the total returns on investment; and
3. a contingent transfer $(W + w)\, b\, d\chi$, which is an insurance payment in the event that he loses the investment opportunity as a result of termination.

More explicitly, if the Responder accepts, then the changes in the wealths of the Proposer and the Responder are

$$dW = (W + w)\,(-f\, dt + (1 - s)\,(\mu\, dt + \sigma\, dz) - b\, d\chi)\,, \tag{8}$$

$$dw = (W + w)\,(f\, dt + s\,(\mu\, dt + \sigma\, dz) + b\, d\chi)\,. \tag{9}$$

If the Responder rejects the spot contract, then both parties invest under autarky for the current period, and the changes in the wealths of the Proposer and the Responder are

$$dW = W\,(\mu\, dt + \sigma\, dz), \tag{10}$$

$$dw = w\,(\mu\, dt + \sigma\, dz). \tag{11}$$

In each period $t \in (T, \infty)$, $dW = dw = 0$. Finally, at the end of period T, both parties consume their accumulated stock of wealth to obtain utilities $U(W_{T+})$ and $u(w_{T+})$.

6. THE BELLMAN EQUATION OF THE PROPOSER

In this section, we consider the case of the spot contracting problem in which the value function of the Responder is his value function under autarky, namely $\widetilde{v}$, and the spot contract offered by the Proposer is always accepted. As will become clear in the next section, this is the only case that we shall need. In this case, the Bellman equation of the Proposer can be derived as follows. Suppose that $\chi = 0$ and, for any given spot contract (f, s, b), put

$$\begin{aligned} dW^S &= (W + w)\ (-f\,dt + (1 - s)\,(\mu\,dt + \sigma\,dz) - b\,d\chi)\,, \\ dw^S &= (W + w)\ (f\,dt + s\,(\mu\,dt + \sigma\,dz) + b\,d\chi)\,, \\ dW^A &= W\,(\mu\,dt + \sigma\,dz), \\ dw^A &= w\,(\mu\,dt + \sigma\,dz). \end{aligned}$$

In other words, let dW^S and dw^S be the changes in the wealth of the Proposer and the Responder if (f, s, b) is accepted; and let dW^A and dw^A be the changes in the wealth of the Proposer and the Responder if (f, s, b) is rejected.

Further, let $A(w)$ denote the set of (f, s, b) such that:

1. the participation constraint of the Responder, namely

$$\mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right] \geq \mathrm{E}\left[\,\widetilde{v}(w + dw^A, d\chi)\,\right], \tag{12}$$

   is satisfied; and

2. the Bellman equation of the Responder, namely

$$\widetilde{v}(w, 0) = \mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right], \tag{13}$$

   is satisfied.

Then the Bellman equation of the Proposer is the equation

$$\widetilde{V}(W, w, \chi) = \left\{ \begin{array}{ll} \max\limits_{(f,s,b)\in A(w)} \mathrm{E}\left[\widetilde{V}(W + dW^S, w + dw^S, \chi + d\chi)\right] & \text{if } \chi = 0 \\ U(W) & \text{if } \chi = 1 \end{array} \right\}, \tag{14}$$

where $\widetilde{V} : (0, \infty)^2 \times \{0, 1\} \to \mathbb{R}$.

Here inequality (12) says that the Responder weakly prefers to accept (f, s, b) rather than proceed under autarky when his continuation utility is given by $\widetilde{v}$; equation (13) says that $\widetilde{v}(w, 0)$ is the expected utility to the Responder from accepting (f, s, b) when his continuation utility is given by $\widetilde{v}$; and equation (14) says that if $\chi = 0$ then $\widetilde{V}(W, w, \chi)$ is the expected utility to the Proposer from choosing the best feasible (f, s, b) when her continuation utility is given by $\widetilde{V}$, and that if $\chi = 1$ then $\widetilde{V}(W, w, \chi)$ is simply $U(W)$.

Notice that the Proposer can always ensure that the participation constraint of the Responder is satisfied by choosing $f = 0$, $s = \frac{w}{W+w}$ and $b = 0$. In other words, the Proposer can always reproduce the autarky outcome by a suitable choice of spot contract. Notice too that the only reason why dW^A does not feature explicitly in these equations is that $\widetilde{v}$ does not depend on W. Notice finally that, in the special case with which we are concerned (namely the case in which the value function of the Responder under spot contracting is simply his value function under autarky), the participation constraint holds as an equality:

Lemma 4. *The following three statements are equivalent:*

1. $(f, s, b) \in A(w)$, *i.e. both the participation constraint and the Bellman equation of the Responder hold;*
2. $\mathrm{E}\left[\widetilde{v}(w + dw^S, d\chi)\right] = \mathrm{E}\left[\widetilde{v}(w + dw^A, d\chi)\right]$, *i.e. the participation constraint of the Responder holds as an equality;*
3. $\widetilde{v}(w, 0) = \mathrm{E}\left[\widetilde{v}(w + dw^S, d\chi)\right]$, *i.e. the Bellman equation of the Responder holds.*

Proof. We show first that statement 3 implies statement 2. Indeed, since $\widetilde{v}$ is the value function of the Responder under autarky, we have

$$\widetilde{v}(w, 0) = \mathrm{E}\left[\widetilde{v}(w + dw^A, d\chi)\right]. \tag{15}$$

Combining this with statement 3 leads immediately to statement 2. We show next that statement 2 implies statement 1. Indeed, if statement 2 holds then, a fortiori, the participation constraint of the Responder must hold. On the other hand, combining statement 2 with equation (15) shows that the Bellman equation of the Responder is satisfied. That statement 1 implies statement 3 is trivial. ■

7. Replicating a Long-Run Contract

In this section, we show that any long-run contract for which the participation constraint of the Responder holds as an equality can be replicated by a flow of spot contracts for which the participation constraint of the Responder again holds as an equality. More precisely: recall that $v = \widetilde{v}(\cdot, 0)$ is the value function of the Responder under autarky prior to termination; suppose that we are given a long-run contract $q : \Omega \to (0, \infty)$ such that $\mathrm{E}[u(q(X,T))] = v(w_0)$; let $\mathcal{F}_t$ denote the information available up to the beginning of period $\min\{t, T\}$;[9,10] and put

$$m_t = \mathrm{E}\left[\, u(q(X,T)) \mid \mathcal{F}_t \,\right].$$

Then: $m_0 = v(w_0)$ by choice of q; m is a martingale; and we may apply the martingale representation theorem to show that there exist coefficients η and θ such that

$$dm = \eta \, dz + \theta \, (d\chi - \beta \, dt).$$

Here: dz and $d\chi$ are the innovations to information at time t; η and θ depend only on information available at the beginning of period t; and, by subtracting $\beta \, dt$ from $d\chi$, we ensure that $\mathrm{E}\left[dm \mid \mathcal{F}_t \right] = 0$. Moreover: m is continuous on $[0, T]$; m may jump at T; and m is constant and equal to $u(q(X,T))$ on (T, ∞).

Next, define the certainty-equivalent wealth process c of the Responder by the formula

$$c_t = \left\{ \begin{array}{ll} v^{-1}(m_t) & \text{if } t \in [0, T] \\ u^{-1}(m_t) & \text{if } t \in (T, \infty) \end{array} \right\}.$$

In other words, let c_t be the unique solution of the equation $\widetilde{v}(c_t, \chi_t) = m_t$. Then:

[9] Up to now we have largely suppressed the time subscipt. However, the argument given at the end of the section makes explicit use of two different times, namely t and $t + dt$, and it is therefore helpful to make the time subsscript explicit on the four variables that are involved in that argument, namely $\mathcal{F}$, m, c and χ.

[10] The underlying stochastic drivers of our model are the standard Wiener process z and the exponentially distributed termination time T. If $t > T$, then the information available up to the beginning of time t includes the timepath of z over the interval $(T, t]$. By conditioning only on information available at the beginning of period $\min\{t, T\}$, we exclude the use of this additional stochastic information.

$c_0 = v^{-1}(m_0) = w_0$; it follows from Itô's Lemma that, for $t \in [0, T]$, we have

$$\begin{aligned} dc &= \left(\tfrac{1}{2}\,\eta^2\, g''(m_t) - \beta\,\theta\, g'(m_t)\right) dt + \eta\, g'(m_t)\, dz \\ &\quad + \left(u^{-1}(m_t + \theta) - g(m_t)\right) d\chi, \end{aligned} \tag{16}$$

where $g = v^{-1}$; and c is constant and equal to $q(X, T)$ on (T, ∞).

Now, if we match the coefficients of dt, dz and $d\chi$ in equation (9) for the dynamics of the wealth of the Responder in the spot-contracting problem with the coefficients of dt, dz and $d\chi$ in equation (16) for the dynamics of the certainty-equivalent wealth of the Responder in the long-run contracting problem, then we get

$$X\,(f + s\,\mu) = \tfrac{1}{2}\,\eta^2\, g''(m_t) - \beta\,\theta\, g'(m_t), \tag{17}$$

$$X\, s\,\sigma = \eta\, g'(m_t), \tag{18}$$

$$X\, b = u^{-1}(m_t + \theta) - g(m_t). \tag{19}$$

Solving this system of linear equations for (f, s, b) yields

$$f = \frac{\frac{1}{2}\,\sigma\,\eta^2\, g''(m_t) - (\sigma\,\beta\,\theta + \mu\,\eta)\, g'(m_t)}{X\,\sigma}, \tag{20}$$

$$s = \frac{\eta\, g'(m_t)}{X\,\sigma}, \tag{21}$$

$$b = \frac{u^{-1}(m_t + \theta) - g(m_t)}{X}. \tag{22}$$

In other words, q can be reproduced by the flow of spot contracts given by the formulae (20-22). Furthermore, for all $t \in [0, \infty)$, we have

$$\begin{aligned} \mathrm{E}\left[\,\widetilde{v}(c_t + dc, \chi_t + d\chi) \mid \mathcal{F}_t\,\right] &= \mathrm{E}\left[\,\widetilde{v}(c_{t+dt}, \chi_{t+dt}) \mid \mathcal{F}_t\,\right] \\ &= \mathrm{E}\left[\,m_{t+dt} \mid \mathcal{F}_t\,\right] \\ &= \mathrm{E}\left[\,m_t + dm \mid \mathcal{F}_t\,\right] \\ &= m_t \\ &= \widetilde{v}(c_t, \chi_t). \end{aligned}$$

In other words, the Bellman equation of the Responder holds. By Lemma 4, this is equivalent to saying that the Participation constraint of the Responder holds as an equality.

8. The Reduced Bellman Equation

In the long-run contracting problem, the Proposer can always do better by choosing a contract for which the participation constraint of the Responder holds as an equality. Furthermore, any such contract can be replicated by a flow of spot contracts for which the participation constraint of the Responder again holds as an equality. The Proposer can therefore always do at least as well in the spot-contracting problem as in the long-run contracting problem. It is therefore of considerable interest to solve the spot-contracting problem.

In this section, we make a start by showing that the Bellman equation of the Proposer under spot contracting, namely equation (14), can be reduced to a partial differential equation, namely equation (23) below. To this end: put $V = \widetilde{V}(\cdot, 0)$; denote the partial derivatives of V by V_W, V_w, V_{WW}, V_{Ww} and V_{ww}; and let $V_P = V_W - V_w$, $V_{WP} = V_{WW} - V_{Ww}$ and $V_{PP} = V_{WW} - 2\,V_{Ww} + V_{ww}$. Then:

Proposition 5. *V satisfies the equation*

$$\begin{aligned} 0 = \max_{(s,b)\in\mathbb{R}\times\left(-\frac{w}{W+w},\frac{W}{W+w}\right)} \Bigg\{ & \mu\,(W+w)\,V_W \\ & + \tfrac{1}{2}\,\sigma^2\,(W+w)^2 \left(V_{WW} - 2\,s\,V_{WP} + s^2\,V_{PP} + \frac{V_P}{v'}\,s^2\,v'' \right) \\ & + \beta \left(U(W-(W+w)\,b) - V + \frac{V_P}{v'}\,(u(w+(W+w)\,b) - v) \right) \Bigg\}. \end{aligned} \tag{23}$$

We shall refer to equation (23) as the reduced Bellman equation of the Proposer. The maximand in this equation involves three main terms. The first term is

$$\mu\,(W+w)\,V_W. \tag{Term 1}$$

In order to bring out the analogy with the other terms in the equation, it is helpful to separate this term into two parts. The first part is

$$(1-s)\,\mu\,(W+w)\,V_W + s\,\mu\,(W+w)\,V_w. \tag{Term 1a}$$

This is the direct benefit to the Proposer of the expected return on total wealth when it is shared according to the sharing rule s. It consists of: the Proposer's share $1-s$ in the expected return $\mu\,(W+w)$ times the shadow value V_W to the Proposer of wealth in the hands of the Proposer; plus the Responder's share s in the expected return $\mu\,(W+w)$ times the shadow value V_w to the Proposer of wealth in the hands of the Responder. The second part is

$$s\,\mu\,(W+w)\,V_P \tag{Term 1b}$$

This is the indirect benefit to the Proposer of the expected return on total wealth when it is shared according to the sharing rule s. In this part: $s\,\mu\,(W+w)$ is the Responder's share in the expected return $\mu\,(W+w)$; and V_P is the shadow value to the Proposer of transfers from the Responder to the Proposer. Notice that: the shadow value of transfers $V_P = V_W - V_w$ takes into account both the impact of a transfer on the Proposer's own wealth (as measured by V_W) and the impact of a transfer on the Responder's wealth (as measured by V_w); and the impact of a transfer on the Responder's wealth must be taken into account since (by making the Responder poorer) a transfer may worsen the terms on which the Proposer can get insurance from the Responder in the future.

The second term consists of two parts. The first part (with sign reversed) is

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW} - 2\,s\,V_{WP} + s^2\,V_{PP}\right). \tag{Term 2a}$$

This is the direct cost to the Proposer of the investment shocks when they are shared according to the sharing rule s. It can be written more explicitly as

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,\left((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\right).$$

Notice that the Proposer cares about shocks to her own wealth, about shocks to the Responder's wealth (since these affect the terms on which she can obtain insurance) and about the correlation between the shocks to her own wealth and those to the wealth of the Responder. The second part of the second term (with sign reversed) is

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,\frac{V_P}{v'}\,s^2\,v''. \tag{Term 2b}$$

This is the indirect cost to the Proposer of the shocks to W and w when they are shared according to the sharing rule s. In this part: $-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,s^2\,v''$ is the cost to the Responder of the shocks; v' is the shadow value to the Responder of wealth in the hands of the Responder; and V_P is the shadow value to the Proposer of transfers from the Responder to the Proposer. The cost to the Responder is initially measured in units of the Responder's utility. Dividing it by v' converts it into money terms, at which point its value to the Proposer can be found by multiplying by V_P. The second part of the second term can also be written

$$\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,V_P\left(-\frac{v''}{v'}\right)s^2,$$

which emphasizes the role played by the absolute risk aversion of the Responder (namely $-\frac{v''}{v'}$). Notice that the absolute risk aversion of the Responder is endogenous (it depends on w).

The third term likewise consists of two parts. The first part (with sign reversed) is

$$\beta\,(V - U(W-(W+w)\,b)). \tag{Term 3a}$$

This is the direct cost to the Proposer of termination when it is insured using the payment b. The second part of the third term (with sign reversed) is

$$\frac{V_P}{v'}\,\beta\,(v - u(w+(W+w)\,b)). \tag{Term 3b}$$

This is the indirect cost to the Proposer of termination when it is insured using the payment b. In this part: $\beta\,(v - u(w+(W+w)\,b))$ is the cost to the Responder of the possibility of termination; v' is, as above, the shadow value to the Responder of wealth in the hands of the Responder; and V_P is, as before, the shadow value to the Proposer of transfers from the Responder to the Proposer.

To summarize, under spot contracting, the Proposer takes into account the expected return obtained by both parties, the costs to both parties of the shocks to wealth as mitigated by the risk-sharing rule s and the costs to both parties of the termination risk as mitigated by the risk-sharing rule b. The maximand of the Proposer involves three terms: a term in μ; a term in σ^2; and a term in β. The term in μ does not involve either of the control variables s or b; the term in σ^2 involves only s; and the term in β involves only b. The problem of optimizing s is therefore separable from the problem of optimizing b.

Proof. In view of Lemma 4, $(f, s, b) \in A(w)$ iff equation (13) holds. Putting $v = \widetilde{v}(\cdot, 0)$

in that equation, we obtain

$$v(w) = \mathrm{E}\left[v(w) + v'(w)\,\delta w + \tfrac{1}{2}\,v''(w)\,\delta w^2 + (u(w+\Delta w) - v(w))\right],$$

where

$$\begin{aligned} \delta w &= (W+w)\,(f\,dt + s\,(\mu\,dt + \sigma\,dz)), \\ \Delta w &= (W+w)\,b\,d\chi. \end{aligned}$$

Hence

$$\begin{aligned} v = v + \Big(&(W+w)\,(f + s\,\mu)\,v' + \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,s^2\,v'' \\ &+ \beta\,(u(w + (W+w)\,b) - v)\Big)\,dt, \end{aligned}$$

where we have suppressed the dependence of v and its derivatives on w, or

$$(W+w)\,(f + s\,\mu) = -\tfrac{1}{2}\,(W+w)^2\,\sigma^2\,s^2\,\frac{v''}{v'} - \beta\,\frac{u(w + (W+w)\,b) - v}{v'}. \tag{24}$$

We conclude that $A(w)$ can be characterized as the set of (f,s,b) such that equation (24) holds.

Next, putting $V = \widetilde{V}(\cdot, 0)$ in equation (14), we obtain

$$\begin{aligned} V(W,w) = \max_{(f,s,b)\in A(w)} \mathrm{E}\Big[&V(W,w) + V_W(W,w)\,\delta W + V_w(W,w)\,\delta w \\ &+ \tfrac{1}{2}\left(V_{WW}(W,w)\,\delta W^2 + 2\,V_{Ww}(W,w)\,\delta W\,\delta w + V_{ww}(W,w)\,\delta w^2\right) \\ &+ (U(W + \Delta W) - V(W,w))\Big], \end{aligned}$$

where

$$\begin{aligned} \delta W &= (W+w)\,(-f\,dt + (1-s)\,(\mu\,dt + \sigma\,dz)), \\ \Delta W &= -(W+w)\,b\,d\chi \end{aligned}$$

and δw is as above. Hence

$$V = \max_{(f,s,b)\in A(w)} \Big\{ V + \Big((W+w)\,(-f+(1-s)\,\mu)\,V_W + (W+w)\,(f+s\,\mu)\,V_w$$
$$+ \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,\big((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\big)$$
$$+ \beta\,(U(W-(W+w)\,b) - V)\Big)\,dt \Big\},$$

where we have suppressed the dependence of V and its derivatives on (W, w), or

$$0 = \max_{(f,s,b)\in A(w)} \Big\{ (W+w)\,\mu\,V_W - (W+w)\,(f+s\,\mu)\,(V_W - V_w) +$$
$$+ \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,\big((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\big)$$
$$+ \beta\,(U(W-(W+w)\,b) - V)\Big\}.$$

Using equation (24) to substitute for $(W+w)\,(f+s\,\mu)$, taking advantage of the notation V_P, V_{WP} and V_{PP} and rearranging, we obtain equation (23). ■

9. First-Order Conditions for s and b

In this section, we give a preliminary characterization of the optimal sharing rule s and the optimal termination payment b in terms of the value functions V and v of the Proposer and the Responder.

Proposition 6. *The optimal sharing rule s takes the form*

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{wP}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}. \tag{25}$$

Proof. Maximizing the maximand in the reduced Bellman equation of the Proposer, namely equation (23), with respect to s boils down to maximizing the quadratic

$$V_{WW} - 2\,V_{WP}\,s + \left(V_{PP} + \frac{V_P}{v'}\,v''\right)s^2$$

with respect to s. Assuming that $V_{PP} + \frac{V_P}{v'} v'' > 0$,[11] this yields

$$s = \frac{V_{WP}}{V_{PP} + \frac{V_P}{v'} v''}.$$

Noting that $V_{WP} = V_{PP} + V_{wP}$ and dividing through by $-V_P$, we obtain the desired expression. ■

Expression (25) for the optimal *dynamic* sharing rule summarizes the main economic issues underlying our risk-sharing problem. In order to understand it better, it is helpful to compare it with the optimal *static* sharing rule

$$s^S = \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}.$$

Compared with this rule, the optimal dynamic rule exhibits three complications. First, the exogenous utility functions U and u are replaced with the endogenous value functions V and v. Second, the risk aversion of the Proposer is evaluated not with respect to her own wealth W, but instead with respect to the difference between her own wealth and that of the Responder, namely $P = W - w$.[12] Third, there is an additional term $-\frac{V_{wP}}{V_P}$ in the numerator. This term captures the idea that current changes in the Responder's wealth have implications for the price at which the Proposer will be able to obtain insurance in the future.[13]

Proposition 7. *The optimal termination payment b is the unique solution of*

$$\frac{U'(W - (W + w)\, b)}{u'(w + (W + w)\, b)} = \frac{V_P}{v'}. \tag{26}$$

Proof. Maximizing the maximand in the reduced Bellman equation of the Proposer with respect to b boils down to maximizing

$$U(W - (W + w)\, b) - V + \frac{V_P}{v'} \left(u(w + (W + w)\, b) - v\right)$$

[11] It can be shown quite generally that $V_{PP} + v'' \frac{V_P}{v'} \geq 0$, and our later analysis will confirm that this inequality is strict when both parties have constant relative risk aversion.

[12] The risk aversion of the Responder is still evaluated with respect to his own wealth since v does not depend on W.

[13] The analogous term for the Responder does not occur since v does not depend on W.

with respect to b. This expression is strictly concave in b, and the first-order condition for this maximization is

$$0 = (W + w)\left(-U'(W - (W + w)\, b) + \frac{V_P}{v'}\, u'(w + (W + w)\, b)\right).$$

Rearranging, we obtain the desired equation. ■

The optimality condition (26) is akin to the familiar Borch condition. The optimal final transfer is set so that the ratio of the Proposer's and the Responder's marginal utility of wealth in the event that termination occurs is equal to the ratio of the Proposer's and the Responder's marginal value of transfers (in the event that termination does not occur). The close analogy between this optimality condition and the Borch condition suggests that V_P and v' can be interpreted as the welfare weights of the Proposer and the Responder in a welfare maximization problem.

10. Asymptotic Expansions

A first approach to understanding optimal risk sharing is to consider what happens when β is large, i.e. when the future is heavily discounted. More precisely, we look for approximations to V, v, s and b in the form $V^{(0)} + \frac{1}{\beta} V^{(1)}$, $v^{(0)} + \frac{1}{\beta} v^{(1)}$, $s^{(0)} + \frac{1}{\beta} s^{(1)}$ and $b^{(0)} + \frac{1}{\beta} b^{(1)}$. A striking feature of these approximations is that they give a qualitatively accurate picture of the behaviour of V, v, s and b even when β takes on much more moderate values, as is demonstated by our numerical simulations in Section 13 below.

10.1. Myopic Terms. We begin by indentifying the myopic components of V, v, s and b, namely $V^{(0)}$, $v^{(0)}$, $s^{(0)}$ and $b^{(0)}$.

Proposition 8. $V^{(0)} = U$, $v^{(0)} = u$, $s^{(0)} = \dfrac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}$ *and* $b^{(0)} = 0$.

These expressions can be explained as follows. First, at order 0, the relationship ends immediately. The myopic value functions $V^{(0)}$ and $v^{(0)}$ are therefore simply the respective utilities U and u of consuming current wealth. Second, the myopic sharing rule $s^{(0)}$ is the familiar ratio of the Proposer's coefficient of absolute risk aversion to the sum of the two parties coefficients of absolute risk aversion. Third, the myopic termination payment $b^{(0)}$ is zero because, when β is very large, termination is essentially certain and it is not therefore possible to insure against it.

Proof. Dividing the Bellman equation of the Responder under autarky, namely (2), through by β and rearranging, we obtain

$$0 = u(w) - v + \tfrac{1}{\beta}\left(\tfrac{1}{2}\,\sigma^2\,w^2\,v'' + \mu\,w\,v'\right).$$

Hence, putting $v = v^{(0)} + \frac{1}{\beta}v^{(1)}$, denoting the first and second derivatives of $v^{(0)}$ by $v^{(0)}_w$ and $v^{(0)}_{ww}$ and rearranging, we obtain

$$0 = u(w) - v^{(0)} + \tfrac{1}{\beta}\left(\tfrac{1}{2}\,\sigma^2\,w^2\,v^{(0)}_{ww} + \mu\,w\,v^{(0)}_w - v^{(1)}\right) + \mathrm{O}\Big(\tfrac{1}{\beta^2}\Big). \tag{27}$$

Hence, equating terms of order 0, we obtain $v^{(0)} = u(w)$.

Second, dividing the Bellman equation of the Responder under spot contracting, namely (24), through by β and rearranging, we obtain

$$0 = u(w + (W+w)\,b) - v + \tfrac{1}{\beta}\Big(\tfrac{1}{2}\,(W+w)^2\,\sigma^2\,s^2\,v'' + (W+w)\,(f + s\,\mu)\,v'\Big)$$

Hence, putting $v = v^{(0)} + \frac{1}{\beta}\,v^{(1)}$, $f = f^{(0)} + \frac{1}{\beta}\,f^{(1)}$, $s = s^{(0)} + \frac{1}{\beta}\,s^{(1)}$ and $b = b^{(0)} + \frac{1}{\beta}\,b^{(1)}$, and equating terms of order 0, we obtain $v^{(0)} = u(w + (W+w)\,b^{(0)})$. But we have already shown that $v^{(0)} = u(w)$. It follows that $b^{(0)} = 0$.

Third, dividing the reduced Bellman equation of the Proposer, namely (23), through by β and rearranging, we obtain

$$\begin{aligned} 0 \;=\; & \max_{(s,b)\in\mathbb{R}\times\left(-\frac{w}{W+w},\frac{W}{W+w}\right)} \Bigg\{ U(W - (W+w)\,b) - V \\ & + \frac{V_P}{v'}\Big(u(w + (W+w)\,b) - v\Big) \;+\; \tfrac{1}{\beta}\Bigg(\mu\,(W+w)\,V_W \\ & + \tfrac{1}{2}\,\sigma^2\,(W+w)^2\Big(V_{WW} - 2\,s\,V_{WP} + s^2\,V_{PP} + \frac{V_P}{v'}\,s^2\,v''\Big)\Bigg)\Bigg\}. \end{aligned}$$

Hence, putting $V = V^{(0)} + \frac{1}{\beta}V^{(1)}$, $v = v^{(0)} + \frac{1}{\beta}v^{(1)}$, $s = s^{(0)} + \frac{1}{\beta}\,s^{(1)}$ and $b = b^{(0)} + \frac{1}{\beta}b^{(1)}$, bearing in mind the envelope principle (which tells us that – in calculating first-order terms – we need not consider first-order variations in s and b), denoting the first derivative of

$v^{(1)}$ by $v_w^{(1)}$ and rearranging, we obtain

$$0 = U(W - (W+w)\,b^{(0)}) - V^{(0)} + \frac{V_P^{(0)}}{v_w^{(0)}}\left(u(w + (W+w)\,b^{(0)}) - v^{(0)}\right)$$

$$+\tfrac{1}{\beta}\left(-V^{(1)} - v^{(1)}\,\frac{V_P^{(0)}}{v_w^{(0)}} + \left(u(w+(W+w)\,b^{(0)}) - v^{(0)}\right)\frac{V_P^{(1)}v_w^{(0)} - V_P^{(0)}v_w^{(1)}}{\left(v_w^{(0)}\right)^2}\right.$$

$$+\,\mu\,(W+w)\,V_W^{(0)}$$

$$\left.+\,\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW}^{(0)} - 2\,s^{(0)}\,V_{WP}^{(0)} + \left(s^{(0)}\right)^2\,V_{PP}^{(0)} + \frac{V_P^{(0)}}{v_w^{(0)}}\left(s^{(0)}\right)^2\,v_{ww}^{(0)}\right)\right)$$

$$+\mathrm{O}\left(\tfrac{1}{\beta^2}\right).$$

Hence, taking advantage of the fact that $b^{(0)} = 0$ and $v^{(0)} = u$,

$$0 = U(W) - V^{(0)} + \tfrac{1}{\beta}\left(-V^{(1)} - v^{(1)}\,\frac{V_P^{(0)}}{u'} + \mu\,(W+w)\,V_W^{(0)}\right.$$

$$\left.+\,\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW}^{(0)} - 2\,s^{(0)}\,V_{WP}^{(0)} + \left(s^{(0)}\right)^2\,V_{PP}^{(0)} + \left(s^{(0)}\right)^2\,\frac{u''}{u'}\,V_P^{(0)}\right)\right)$$

$$+\mathrm{O}\left(\tfrac{1}{\beta^2}\right). \tag{28}$$

Hence, equating terms of order 0, $V^{(0)} = U(W)$.

Fourth, the first-order condition for the optimal sharing rule, namely (25), takes the form

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{wP}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}.$$

Hence, putting $V = V^{(0)} + \frac{1}{\beta} V^{(1)}$, $v = v^{(0)} + \frac{1}{\beta} v^{(1)}$ and $s = s^{(0)} + \frac{1}{\beta} s^{(1)}$, and equating terms of order 0, we obtain:

$$s^{(0)} = \frac{-\frac{V_{PP}^{(0)}}{V_P^{(0)}} - \frac{V_{wP}^{(0)}}{V_P^{(0)}}}{-\frac{V_{PP}^{(0)}}{V_P^{(0)}} - \frac{v_{ww}^{(0)}}{v_w^{(0)}}}.$$

Finally, recalling that $V^{(0)} = U$ and $v^{(0)} = u$, we obtain the required expression. ∎

10.2. Dynamic Terms: $V^{(1)}$ and $v^{(1)}$. In this section we determine the dynamic corrections $V^{(1)}$ and $v^{(1)}$ by equating the terms of order 1 in $\frac{1}{\beta}$ in the relevant equations.

Proposition 9. *We have:*

1. $V^{(1)} = \left((\mu - \frac{1}{2} R\sigma^2) W + \frac{1}{2} \frac{(R-r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}} \right) U'(W);$
2. $v^{(1)} = \left((\mu - \frac{1}{2} r\sigma^2) w \right) u'(w).$

where we have suppressed the dependence of R and r on W and w respectively.

In other words, the dynamic correction $V^{(1)}$ is composed of three elements: the risk-adjusted rate of return on the Proposer's wealth, namely

$$(\mu - \tfrac{1}{2} R\sigma^2) W;$$

the monetary value of the gains from sharing investment risk, namely

$$\frac{1}{2} \frac{(R-r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}};$$

and the marginal utility of wealth $U'(W)$. The first two elements are measured in units of wealth. Multiplying them by $U'(W)$ converts them into units of the Proposer's utility. Similarly, $v^{(1)}$ is composed of two elements: the risk-adjusted rate of return on the Responder's wealth, namely

$$(\mu - \tfrac{1}{2} r\sigma^2) w;$$

and the marginal utility of wealth $u'(w)$.

Notice that there is no contribution to $V^{(1)}$ reflecting the monetary value of the gains from sharing termination risk. Such a contribution would be expected to arise at order 2. However, exploring higher-order terms in the expansions is beyond the scope of this paper.[14] Also, in accordance with the bargaining positions of the two parties, the Responder does not receive any share in the gains from sharing investment risk. Finally, these formulae do not depend on the assumption that U and u are CRRA. (The formulae for $s^{(1)}$ and $b^{(1)}$ below do.)

Proof. Equating terms of order 1 in equation (27), we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\,w^2\,v^{(0)}_{ww} + \mu\,w\,v^{(0)}_{w} - v^{(1)}.$$

Hence, using the fact that $v^{(0)} = u$ and rearranging, we obtain

$$\frac{v^{(1)}}{u'} = \left(\mu + \tfrac{1}{2}\,\sigma^2\,\frac{w\,u''}{u'}\right)\,w = \left(\mu - \tfrac{1}{2}\,r\,\sigma^2\right)\,w$$

as required. Next, equating terms of order 1 in equation (28),

$$0 = -V^{(1)} - v^{(1)}\,\frac{V^{(0)}_P}{u'} + \mu\,(W+w)\,V^{(0)}_W$$

$$+\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V^{(0)}_{WW} - 2\,s^{(0)}\,V^{(0)}_{WP} + \left(s^{(0)}\right)^2\,V^{(0)}_{PP} + \left(s^{(0)}\right)^2\,\frac{u''}{u'}\,V^{(0)}_P\right).$$

Hence, taking advantage of the fact that $V^{(0)} = U$ and rearranging,

$$\frac{V^{(1)}}{U'} = \mu\,(W+w) + \tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(\left(1-s^{(0)}\right)^2\,\frac{U''}{U'} + \left(s^{(0)}\right)^2\,\frac{u''}{u'}\right) - \frac{v^{(1)}}{u'}.$$

Finally, putting

$$\frac{v^{(1)}}{u'} = \left(\mu - \tfrac{1}{2}\,r\,\sigma^2\right)\,w, \quad s^{(0)} = \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}, \quad \frac{U''}{U'} = -\frac{R}{W} \quad \text{and} \quad \frac{u''}{u'} = -\frac{r}{w},$$

[14]Such contributions do occur in Conditions III and IV in Section 12 below.

and rearranging, we obtain

$$\frac{V^{(1)}}{U'} = (\mu - \tfrac{1}{2}\, R\, \sigma^2)\, W + \tfrac{1}{2}\, \frac{(R-r)^2\, \sigma^2}{\frac{R}{W} + \frac{r}{w}},$$

as required. ■

10.3. Dynamic Terms: $s^{(1)}$. In this section we determine the dynamic correction $s^{(1)}$ to the myopic sharing rule $s^{(0)}$. This correction is of interest for two reasons. First, it allows us to make qualitative predictions as to how the optimal risk-sharing rule s differs from the myopic sharing rule $s^{(0)}$. These predictions can then be compared with numerical simulations. Second, it offers some insight into why s differs from $s^{(0)}$ in the way that it does.

Proposition 10. *Suppose that both U and u are CRRA. Then*

$$s^{(0)} = \frac{\frac{R}{y}}{\frac{R}{y} + \frac{r}{1-y}}$$

and

$$s^{(1)} = \frac{\frac{1}{2}\, R\, r\, \sigma^2\, y^2\, (1-y)^2}{((1-y)\, R + y\, r)^5}\, (R-r)^3 \big((1-y)\, R + y\, r - 2\big),$$

where $y = \frac{W}{W+w}$ is the Proposer's share in aggregate wealth.

The main lessons that can be extracted from the formula for $s^{(0)}$ are as follows. First, the myopic sharing rule $s^{(0)}$ is — modulo normalization by multiplying the numerator and the denominator by $W+w$ — the ratio of the Proposer's absolute risk aversion, namely $\frac{R}{W}$, to the sum of the Proposer's and the Responder's absolute risk aversions, namely $\frac{R}{W} + \frac{r}{w}$. It is strictly decreasing in the Proposer's wealth share y. When $y = 0$, the Proposer is effectively infinitely risk averse, and $s^{(0)} = 1$. When $y = 1$, the Responder is effectively infinitely risk averse, and $s^{(0)} = 0$.

Second, notice that $s^{(0)}$ is the proportion of the investment risk on *total* wealth that the Responder bears. The proportion of the investment risk on his *own* wealth that he bears is therefore

$$\frac{W+w}{w}\, s^{(0)} = \frac{R}{(1-y)\, R + y\, r},$$

and his leverage is

$$\frac{R}{(1-y)\,R+y\,r}-1=\frac{y\,(R-r)}{(1-y)\,R+y\,r}.$$

If $R>r$ then his leverage is 0 when $y=0$, and rises to $\frac{R-r}{r}>0$ when $y=1$. In particular, his leverage is *greater* when his wealth is *smaller*. His leverage is 0 when $y=0$ because in that case he has all the wealth, and so risk sharing with the Proposer has a negligible impact. It is increasing in y because — from his point of view — the opportunities for risk sharing are increasing in y and, as the less risk averse party, taking advantage of these opportunites means increasing his leverage. If $R<r$, then his leverage is 0 when $y=0$ and falls to $\frac{R-r}{r}<0$ when $y=1$. In particular, his leverage is smaller when his wealth is smaller.

Third, if $R=r$, then $s^{(0)}=1-y$ and $s^{(1)}=0$. In other words, each party bears precisely the risk on their own wealth, and their wealth shares therefore remain unchanged.

Turning to the formula for the dynamic correction $s^{(1)}$, we begin with a definition:

Definition 11. *The Proposer is fairly risk tolerant if $R<2$ and fairly risk averse if $R>2$. Similarly, the Responder is fairly risk tolerant if $r<2$ and fairly risk averse if $r>2$.*

We go on to note that $s^{(1)}$ is the product of three terms, namely

$$\frac{\frac{1}{2}\,R\,r\,\sigma^2\,y^2\,(1-y)^2}{((1-y)\,R+y\,r)^5},\quad (R-r)^3,\quad (1-y)\,R+y\,r-2.$$

The first of these is always positive; the second has the same sign as $R-r$; and the third is affine in y. Hence, if we assume for concreteness that $R>r$, then we have three cases to consider:

The Risk-Tolerant Case When both parties are fairly risk tolerant, i.e. $2>R>r$, then $(1-y)\,R+y\,r-2<0$ for all $y\in[0,1]$. Hence $s^{(1)}<0$ for all $y\in(0,1)$. This suggests that $s-s^{(0)}<0$ for all $y\in(0,1)$, where s is the optimal risk-sharing rule. In other words, irrespective of the distribution of wealth, it is optimal for the Proposer to transfer *less* risk to the Responder than she would under the myopic risk-sharing rule. The risk-tolerant case is illustrated in Figure 1(a).

The Risk-Averse Case When both parties are fairly risk averse, i.e. $R>r>2$, then $(1-y)\,R+y\,r-2>0$ for all $y\in[0,1]$. Hence $s^{(1)}>0$ for all $y\in(0,1)$. This suggests that $s - s^{(0)} > 0$ for all $y \in (0, 1)$. In other words, irrespective of the distribution of wealth, it is optimal for the Proposer to transfer *more* risk to the Responder than she would under the myopic risk-sharing rule. The risk-averse case is illustrated in Figure 1(b).

The Mixed Case When the Proposer is fairly risk averse and the Responder is fairly risk tolerant, i.e. $R > 2 > r$, then $(1 - y) R + y r - 2 > 0$ for $y \in \left[0, \frac{R-2}{R-r}\right)$ and $(1 - y) R + y r - 2 < 0$ $y \in \left(\frac{R-2}{R-r}, 1\right]$. This suggests that $s - s^{(0)} > 0$ for $y \in \left(0, \frac{R-2}{R-r}\right)$ and $s - s^{(0)} < 0$ for $y \in \left(\frac{R-2}{R-r}, 1\right)$. In other words, when the Proposer has a small share in total wealth, it is optimal for her to transfer *more* risk to the Responder than she would under the myopic risk sharing rule; and, when she has a large share in total wealth, it is optimal for her to transfer *less* risk to him than she would under the myopic risk sharing rule. The mixed case is illustrated in Figure 1(c).

Figure 1 about here

The qualitative accuracy of these predictions can be demonstrated by plotting $s - s^{(0)}$, where s is the (numerically computed) optimal contract.[15] This is done in Figure 2. Figure 2(a) shows $s - s^{(0)}$ in the risk-tolerant case. This figure is very similar to Figure 1(a). The main difference is quantitative: the minimum in Figure 2(a) is somewhat lower than that in Figure 1(a). Figure 2(b) shows $s - s^{(0)}$ in the risk-averse case. This figure is very similar to Figure 1(b). The main difference is again quantitative: the maximum in Figure 2(b) is somewhat lower than that in Figure 1(b). Finally, Figure 2(c) shows $s - s^{(0)}$ in the mixed case. This figure is similar to Figure 1(c) in that the graph first rises to a positive maximum and then falls to a negative minimum. However, the balance between the left-hand hump and the right-hand hump is slightly different.

Figure 2 about here

The predictions are best understood in terms of counterparty risk. Indeed, consider the risk-tolerant case. Since $r < 2$, the Responder is willing to take on risk on relatively unfavourable terms, and the Proposer must bear in mind the possibility that he will eventually run out of wealth. This leads her to take on somewhat more risk than she

[15] See Section 13 below for more information on the numerically computed optimal contract.

Figure 1: the dynamic correction $\frac{1}{\beta}\, s^{(1)}$

Figure 1(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\sigma = 0.15$, $\beta = 0.05$)

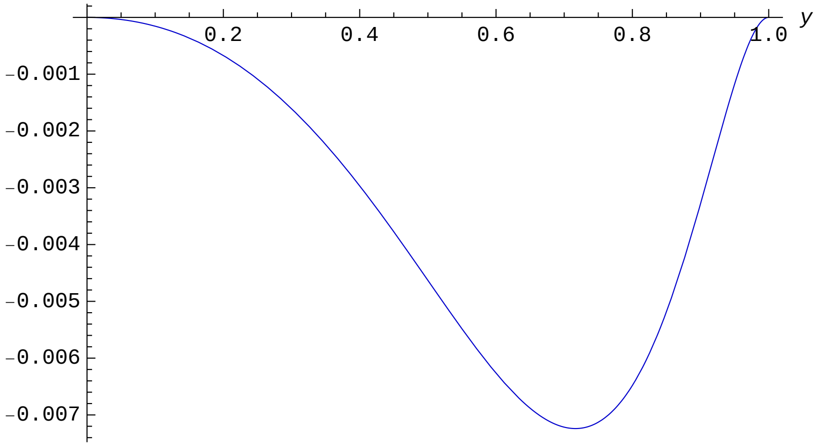


Figure 1(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\sigma = 0.15$, $\beta = 0.05$)

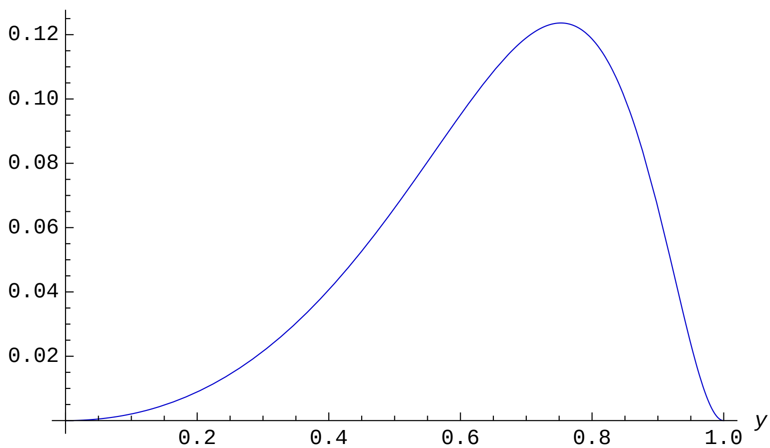


Figure 1(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\sigma = 0.15$, $\beta = 0.05$)

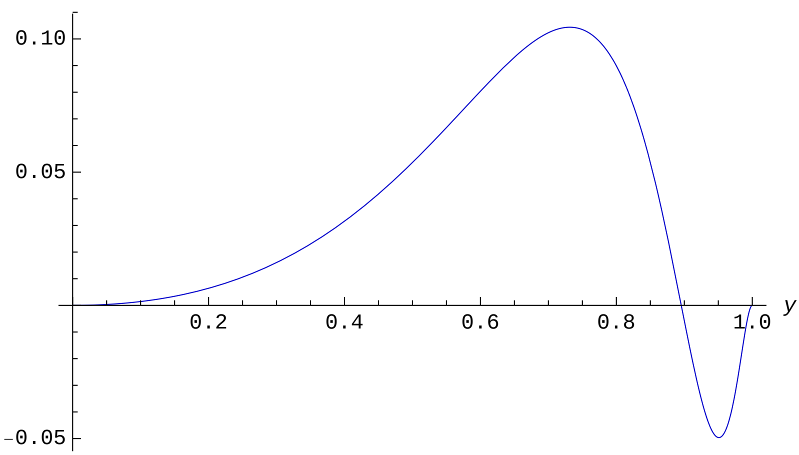


Figure 2: $s - s^{(0)}$, for comparison with $\frac{1}{\beta} s^{(1)}$

Figure 2(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

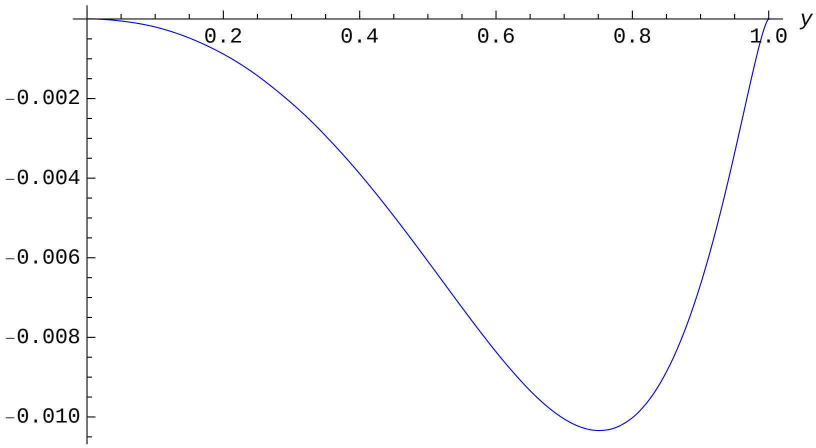


Figure 2(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

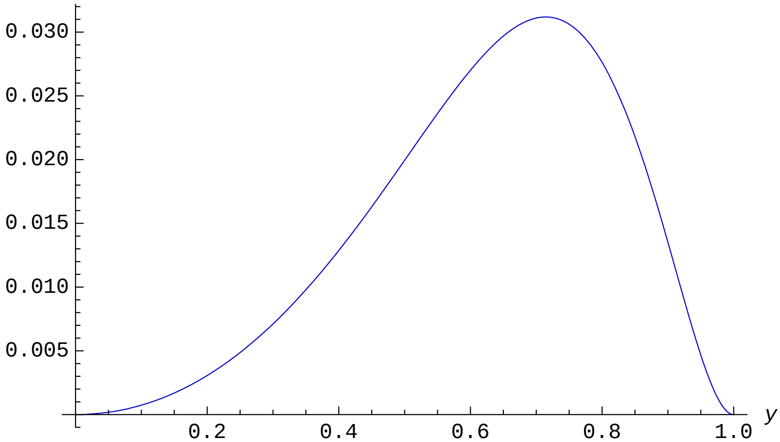
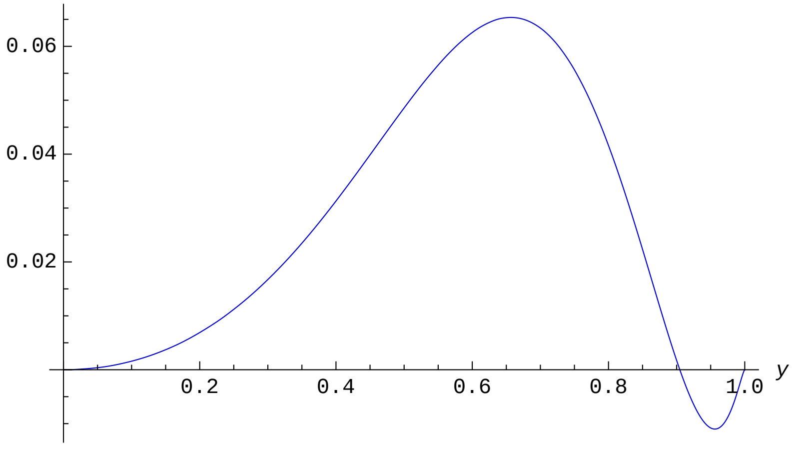


Figure 2(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)



would under the myopic benchmark, thereby delaying the time at which the Responder runs out of wealth. In effect, insurance is a scarce resource, and she chooses to husband it. Now consider the mixed case. The Responder is still willing to take on risk on relatively unfavourable terms, and the Proposer must still bear in mind the possibility that he will eventually run out of wealth. However, in this case the Proposer is only willing to take on the extra risk when she has a fairly large wealth share, i.e. when $y > \frac{R-2}{R-r}$. When $y < \frac{R-2}{R-r}$, the cost of bearing additional risk outweighs the benefit of husbanding insurance, and she transfers more risk to the Responder than she would under the myopic benchmark. Loosely speaking, the stock of insurance is measured by $1-y$, and should be exploited when $y < \frac{R-2}{R-r}$ and conserved when $y > \frac{R-2}{R-r}$. Finally, consider the risk-averse case. Since $r > 2$, the Responder is only willing to take on risk on relatively favourable terms. Transferring more risk to him therefore has the indirect effect of increasing the rate of growth of his wealth and therefore the stock of insurance. The Proposer therefore does not hesitate to transfer more risk to him than she would under the myopic benchmark.

Proof of Proposition 10. Put $A = -\frac{V_{PP}}{V_P}$, $a = -\frac{v''}{v'}$ and $\theta = -\frac{V_{wP}}{V_P}$. Then

$$s = \frac{A+\theta}{A+a}$$

and

$$\begin{aligned} A^{(0)} &= \tfrac{R}{W}, & A^{(1)} &= -\tfrac{R}{W^2}\,G + \tfrac{R}{W}\,G_P - G_{PP}, \\ a^{(0)} &= \tfrac{r}{w}, & a^{(1)} &= 0, \\ \theta^{(0)} &= 0, & \theta^{(1)} &= \tfrac{R}{W}\,G_w - G_{wP}, \end{aligned}$$

where

$$G = \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W}+\frac{r}{w}}.$$

Hence, using the fact that $\theta^{(0)} = a^{(1)} = 0$ and the fact that $s^{(0)} = \frac{A^{(0)}}{A^{(0)}+a^{(0)}}$,

$$\begin{aligned} \frac{s^{(1)}}{s^{(0)}} &= \frac{A^{(1)}+\theta^{(1)}}{A^{(0)}+\theta^{(0)}} - \frac{A^{(1)}+a^{(1)}}{A^{(0)}+a^{(0)}} = \frac{A^{(1)}+\theta^{(1)}}{A^{(0)}} - \frac{A^{(1)}}{A^{(0)}+a^{(0)}} \\ &= \left(1-s^{(0)}\right)\frac{A^{(1)}}{A^{(0)}} + \frac{\theta^{(1)}}{A^{(0)}}. \end{aligned}$$

Now, using the formulae for $A^{(0)}$, $A^{(1)}$, $\theta^{(1)}$ and G, we obtain

$$\frac{A^{(1)}}{A^{(0)}} = -\frac{\frac{1}{2}\, r\, \sigma^2\, y\, (R-r)^2\, ((1-y)\, R + y\, r - 2)}{((1-y)\, R + y\, r)^3}$$

and

$$\frac{\theta^{(1)}}{A^{(0)}} = \frac{\frac{1}{2}\, r\, \sigma^2\, y^2\, (R-r)^2\, ((1-y)\, R + y\, r - 2)}{((1-y)\, R + y\, r)^3} = -y\, \frac{A^{(1)}}{A^{(0)}}.$$

We therefore get

$$\begin{aligned} s^{(1)} &= s^{(0)} \left(1 - s^{(0)} - y\right) \frac{A^{(1)}}{A^{(0)}} \\ &= \frac{(1-y)\, R}{(1-y)\, R + y\, r} \left(-\frac{y\,(1-y)\,(R-r)}{(1-y)\, R + y\, r}\right) \frac{A^{(1)}}{A^{(0)}} \\ &= \frac{\frac{1}{2}\, R\, r\, \sigma^2\, y^2\, (1-y)^2\, (R-r)^3\, ((1-y)\, R + y\, r - 2)}{((1-y)\, R + y\, r)^5}, \end{aligned}$$

as required. ■

10.4. Dynamic Terms: $b^{(1)}$. In this section we determine the dynamic correction $b^{(1)}$ to the myopic termination payment $b^{(0)}$. Since $b^{(0)} = 0$, this correction leads directly to qualitative predictions about the optimal termination payment b. The principal prediction is that the sign of b will depend on whether the investment opportunity is good or bad, in the sense that the ratio $\frac{2\mu}{\sigma^2}$ is high or low relative to the other parameters of the model. For example, suppose that $R > r$. In this case, if the value of the investment opportunity is high, then we should have $b < 0$. In other words, the Responder should compensate the Proposer for the loss of the valuable investment opportunity when termination occurs. On the other hand, if the value of the investment opportunity is low, then we should have $b > 0$. In other words, the Responder should compensate the Proposer for the losses that she faces while the investment is ongoing, and receives in return a payment from the Proposer when the good state (namely termination) is reached.

There is, however, an important twist to the story: for intermediate values of the ratio $\frac{2\mu}{\sigma^2}$, the sign of b should depend on the Proposer's wealth share y. In the risk-tolerant case, we predict that b will be positive for small y and negative for large y. In the risk-averse case, we predict that b will be negative for small y and positive for large y. In the mixed case, the picture is more involved. However, the most interesting possibility is that in which b will be negative for y near 0 or 1 but positive for intermediate values of y.

Proposition 12. *Suppose that both U and u are CRRA. Then*

$$b^{(0)} = 0$$

and

$$b^{(1)} = \frac{y\,(1-y)}{(1-y)\,R + y\,r}\,(R-r)\left(\tfrac{1}{2}\,\sigma^2\,B(y) - \mu\right),$$

where

$$B(y) = r\,\frac{(R-r)^2\,y^2 - 3\,R\,(R-r)\,y + R\,(2\,R-1)}{((1-y)\,R + y\,r)^2}$$

and $y = \frac{W}{W+w}$ is the Proposer's share in aggregate wealth.

Now, $b^{(1)}$ is the product of three terms, namely

$$\frac{y\,(1-y)}{(1-y)\,R + y\,r}, \quad R-r, \quad \tfrac{1}{2}\,\sigma^2\,B(y) - \mu.$$

The first of these is always positive; the second has the same sign as $R-r$; and the third is linear in the core parameters μ and σ^2, but depends in an apparently complicated way on y. Fortunately, this complexity is more apparent than real: if we differentiate B with respect to y, then we obtain a formula that is highly reminiscent of the formula for $s^{(1)}$, namely

$$B'(y) = \frac{R\,r\,(R-r)}{((1-y)\,R + y\,r)^3}((1-y)\,R + y\,r - 2).$$

Assuming for concreteness that $R > r$, we therefore arrive at the three same cases that we encountered in the context of our discussion of $s^{(1)}$, namely:

The Risk-Tolerant Case If $2 > R > r$, then we have $B' < 0$ for all $y \in [0, 1]$. There are therefore three subcases to consider, namely

$$\tfrac{2\mu}{\sigma^2} < B(1), \quad \tfrac{2\mu}{\sigma^2} \in (B(1), B(0)), \quad \tfrac{2\mu}{\sigma^2} > B(0).$$

In the first subcase, $b^{(1)} > 0$ for all $y \in (0, 1)$; in the second, there exists $\overline{y} \in (0, 1)$ such that $b^{(1)} > 0$ for $y \in (0, \overline{y})$ and $b^{(1)} < 0$ for $y \in (\overline{y}, 1)$; in the third, $b^{(1)} < 0$ for all $y \in (0, 1)$. This case is illustrated in Figure 3(a).

The Risk-Averse Case If $R > r > 2$, then we have $B' > 0$ for all $y \in [0, 1]$. There are therefore again three subcases to consider, namely

$$\tfrac{2\mu}{\sigma^2} < B(0), \quad \tfrac{2\mu}{\sigma^2} \in (B(0), B(1)), \quad \tfrac{2\mu}{\sigma^2} > B(1).$$

In the first subcase, $b^{(1)} > 0$ for all $y \in (0, 1)$; in the second, there exists $\overline{y} \in (0, 1)$ such that $b^{(1)} < 0$ for $y \in (0, \overline{y})$ and $b^{(1)} > 0$ for $y \in (\overline{y}, 1)$; in the third, $b^{(1)} < 0$ for all $y \in (0, 1)$. This case is illustrated in Figure 3(b).

The Mixed Case If $R > 2 > r$, then B is inverse-U shaped: $B' > 0$ for $y \in \left[0, \frac{R-2}{R-r}\right)$ and $B' < 0$ for $y \in \left(\frac{R-2}{R-r}, 0\right]$. Putting

$$\overline{B} = \max\{B(y) \mid y \in [0, 1]\} = \tfrac{1}{4}\, r\,(4 + R),$$

there are therefore four subcases to consider, namely

$$\begin{aligned} &\tfrac{2\mu}{\sigma^2} < \min\{B(0), B(1)\}, && \tfrac{2\mu}{\sigma^2} \in \left(\min\{B(0), B(1)\}, \max\{B(0), B(1)\}\right), \\ &\tfrac{2\mu}{\sigma^2} \in \left(\max\{B(0), B(1)\}, \overline{B}\right), && \tfrac{2\mu}{\sigma^2} > \overline{B}. \end{aligned}$$

This case is illustrated in Figure 3(c).[16]

Figure 3 about here

Since $b^{(0)} = 0$, these observations concerning $b^{(1)}$ translate directly into predictions about the optimal termination payment b. These predictions are remarkable at three levels. First, they are qualitatively correct: every case and subcase described above occurs. Second, they are quantitatively correct in the sense that they even predict the parameter values for which the various cases will occur. For example, if we are looking for the third subcase of the mixed case, which involves $b^{(1)} < 0$ near the ends of the interval $(0, 1)$ but $b^{(1)} > 0$ in the middle, then we should choose

$$\tfrac{2\mu}{\sigma^2} \in \left(\max\left\{\tfrac{(2R-1)\,r}{R}, \tfrac{r^2-R+Rr}{r}\right\}, \tfrac{(R+4)\,r}{4}\right).$$

[16] The second subcase of the mixed case divides into two subsubcases depending on whether $B(0) < B(1)$ or $B(0) > B(1)$. In this connection, it is worth noting that $B(0) < B(1)$ iff the harmonic mean of R and r exceeds 2. This is another instance where the outcome depends on whether mean risk aversion lies above or below 2.

Figure 3: the function B

Figure 3(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$)

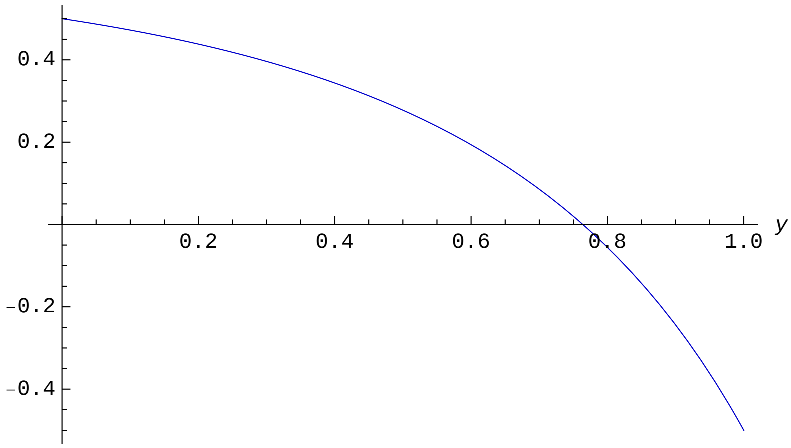


Figure 3(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$)

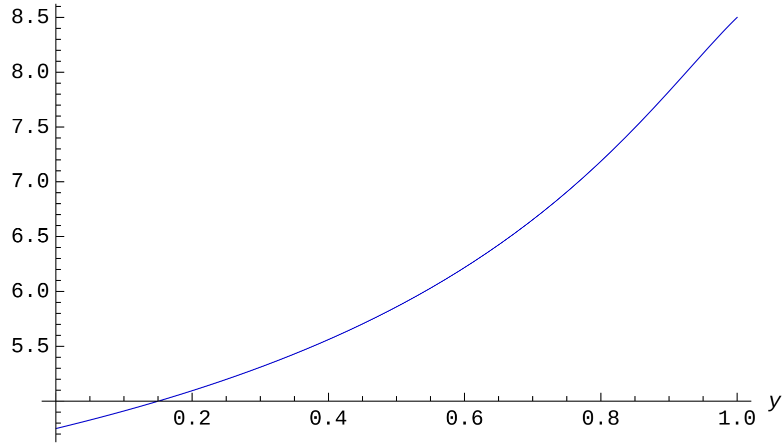
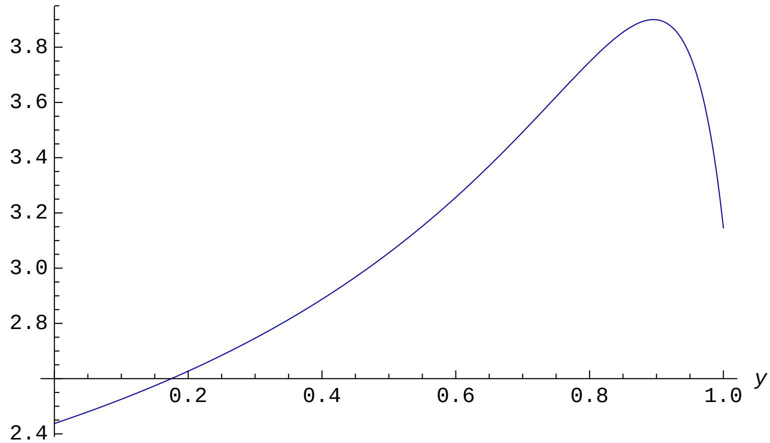


Figure 3(c): the Mixed Case (with $R = 8$, $r = 1.3$)



Third, they are correct for values of β as low as 0.05, even though the expansions are theoretically valid only for large β.[17]

Proof of Proposition 12. The first-order condition for b, namely (26), takes the form

$$\frac{U'(W-(W+w)\,b)}{u'(w+(W+w)\,b)}=\frac{V_P}{v'}.$$

Hence, using logarithmic differentiation and taking advantage of the fact that $b^{(0)}=0$,

$$-\frac{(W+w)\,U''(W)\,b^{(1)}}{U'(W)}-\frac{(W+w)\,u''(w)\,b^{(1)}}{u'(w)}=\frac{V_P^{(1)}}{V_P^{(0)}}-\frac{(v^{(1)})'}{(v^{(0)})'}.$$

Now

$$\begin{aligned} V_P^{(0)}&=U'(W), & V_P^{(1)}&=\left(-\tfrac{R}{W}\,(I+G)+(I+G)_P\right)U'(W),\\ v_w^{(0)}&=u'(w), & v_w^{(1)}&=\left(-\tfrac{r}{w}\,i+i_w\right)u'(w), \end{aligned}$$

where

$$I=(\mu-\tfrac{1}{2}\,R\,\sigma^2)\,W,\quad i=(\mu-\tfrac{1}{2}r\,\sigma^2)\,w\quad\text{and}\quad G=\tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W}+\frac{r}{w}}$$

denote the investment return of the Proposer, the investment return of the Responder and the monetary value of the gains from trade respectively. Hence

$$(W+w)\left(\frac{R}{W}+\frac{r}{w}\right)b^{(1)}=\left(-\frac{R}{W}\,(I+G)+(I+G)_P\right)-\left(-\frac{r}{w}\,i+i_w\right)$$

or

$$(W+w)\,b^{(1)}=-s^{(0)}\left((I+G)-\tfrac{W}{R}\,(I+G)_P\right)+\left(1-s^{(0)}\right)\left(i-\tfrac{w}{r}\,i_w\right).$$

In other words, when termination occurs: the Responder pays the Proposer a fraction $s^{(0)}$ of the total loss $I+G$ to the Proposer from termination; and the Proposer pays the Responder a fraction $1-s^{(0)}$ of the total loss i to the Responder from termination. These payments are, however, offset by terms reflecting the opportunity cost of buying the termination insurance ex ante. Finally, substituting for I, i and G and collecting

[17] The three parameters β, μ and σ^2 are not independent of one another: one of them can be scaled out of the problem. For the present purposes, it is convenient to scale out σ^2. In this way we arrive at the dimensionless parameter $\varepsilon=\frac{\sigma^2}{\beta}$. Our asymptotic expansions are premised on the assumption that this parameter is small. In all of the three baseline parameter constellations used in Section 13 below, we have $\sigma^2=0.0225$ and therefore $\varepsilon=0.45$. This is still less than 1, and cannot therefore be considered to be exceptionally large.

terms in μ and σ^2 yields the required formula for $b^{(1)}$. ■

11. The One-dimensional Bellman Equation

Asymptotic expansions in $\frac{1}{\beta}$ have yielded approximations for the optimal risk-sharing rule and the optimal termination payment when β is large. Do these approximations tell us most of what we want to know about the general case, or do new phenomena arise when β is not necessarily large? To answer this question, we need to compute numerical solutions for the Bellman equation and the optimal contract, and to compare these solutions with the approximations. In this section we undertake some of the preparatory analytical work that is needed before we can turn to the numerical simulations themselves. This involves two main steps: (i) we reduce the Bellman equation from a partial differential equation in (W, w)-space to a pair of ordinary differential equations in y-space; (ii) we normalize the value function with respect to the wealth share of both the Proposer and the Responder. In the process, we arrive at some new analytical insights. For example, we obtain a proof of the intuitively reasonable result that the Responder takes on more that his share in the total investment risk when $r < R$, and less than his share when $r > R$.[18] Or again, we obtain a much more concrete formula for s.[19] Nonetheless, the reader who is anxious to see the numerical results — and how they compare with the approximations — may wish to skip the remainder of this section, review Conditions III and IV in Section 12, and then proceed to Section 13.

11.1. Some Normalizations. Since both the Proposer and the Responder have CRRA utility, and since the returns to their investment follow a geometric Brownian motion, it is natural to look for a solution to the reduced Bellman equation of the Proposer in the form

$$V(W, w) = C_R(\rho(y)\, W),$$

where ρ is the Proposer's certainty-equivalent rate of return and $y = \frac{W}{W+w}$ is the Proposer's share in total wealth. However, as in the Responder's problem under autarky, we have

$$C_R(\rho(y)\, W) = C_R(W) + W\, C_R'(W)\, C_R(\rho(y)),$$

[18] See Corollary 17 below.

[19] Compare equation (25) above, which gives the formula for the general case in which U and u are not necessarily CRRA, with equation (39) below, which gives the formula obtained in the special case in which U and u are CRRA.

and it turns out to be more convenient to work in terms of

$$\psi(y) = C_R(\rho(y)),$$

which we call the normalized value function of the Proposer. It is also useful to note that

$$V_P(W, w) = \gamma(y)\, C'_R(W),$$

where

$$\gamma(y) = 1 + (1 - R)\,\psi(y) + y\,\psi'(y)$$

is what we call the normalized marginal value of transfers of the Proposer.

We take the following proposition to be economically obvious:

Proposition 13. $\gamma > 0$. ■

Next, we shall need an appropriate normalization of ψ. To this end, we consider the Proposer's problem under autarky. By analogy with the Responder's problem under autarky, we see at once that the Proposer's value function under autarky takes the form $C_R(W) + W\, C'_R(W)\, \psi_R$, where

$$\psi_R = \frac{\mu - \frac{1}{2}\, R\, \sigma^2}{\beta_R}$$

and

$$\beta_R = \beta - (1 - R)\,(\mu - \tfrac{1}{2}\, R\, \sigma^2).$$

Moreover her marginal value of wealth under autarky takes the form $\gamma_R\, C'_R(W)$, where

$$\gamma_R = \frac{\beta}{\beta_R}.$$

The required normalization of ψ is then

$$\chi(y) = \frac{\psi(y) - \psi_R}{(1 - y)\,\gamma_R}.$$

The motivation for looking at the difference $\psi(y) - \psi_R$ should be clear: we want to see what the Proposer gains by sharing risk with the Responder. The motivation for dividing through by $1 - y$ is that the gain from trade is necessarily small when y is near 0 or 1, and we would like to measure the gain from trade relative to the wealth of the poorer of the two parties. Now, the initial normalization that we made in moving from V to ψ took care of the possibility that the Proposer might have little wealth. So it remains only to take care of the possibility that the Responder might have little wealth. This explains the presence of the factor $1-y$ in the denominator. The factor γ_R serves to simplify the algebra which follows.

It is also convenient to introduce analogous normalizations for the contracting variables s and b, namely

$$z = \frac{s-(1-y)}{y\,(1-y)} \quad \text{and} \quad g = \frac{b}{y\,(1-y)}.$$

From an economic perspective, z is simply the ratio of the Responder's leverage to the Principal's share in total wealth. Indeed, the total investment risk borne by the Responder is $s\,(W+w)$, and his leverage is therefore

$$\frac{s\,(W+w)-w}{w} = \frac{s-(1-y)}{1-y}.$$

From a mathematical perspective, notice that the Responder takes on a share in the total investment risk different from his autarky share, namely $1-y$. This difference will be small when y is near 0, since then the Responder has to bear almost all the risk (because he owns almost all the wealth). It will also be small when y is near 1, since then the Responder bears almost none of the risk (because he owns almost none of the wealth). Dividing through by $y\,(1-y)$ therefore normalizes the departure of the Responder's share under bilateral contracting from his share under autarky. Similarly, the proportion b of total wealth transferred between the two parties in the event of termination is small when y is near 0 or 1. It is therefore helpful to normalize b so that it is measured relative to the Proposer's wealth when y is near 0 and the Responder's wealth when y is near 1.

Finally, recall that the gross benefit from sharing termination risk is captured by the term

$$\beta \left(U(W-(W+w)\,b) - V + \frac{V_P}{v'}\,(u(w+(W+w)\,b)-v) \right)$$

in the reduced Bellman equation of the Proposer (namely (23)). Taking advantage of the fact that $V = C_R(\rho\,W)$, $v = C_r(\rho_r\,w)$, $V_P = \gamma\,C_R'(W)$ and $v' = \gamma_r\,C_r'(w)$, this term can be rewritten in the form

$$\beta\,(W+w)\,C_R'(W)\,y\,(1-y) \left(\Phi(g,y,\gamma) - \frac{\psi}{1-y} - \frac{\gamma}{\gamma_r}\,\frac{\psi_r}{y} \right),$$

where the function Φ is defined in two steps. The first step is to put

$$\Phi(g, y, \gamma) = \frac{C_R(1 - (1 - y)\, g)}{1 - y} + \frac{\gamma}{\gamma_r} \frac{C_r(1 + y\, g)}{y}$$

for all $(y, \gamma) \in (0, 1) \times (0, \infty)$ and all $g \in (-y^{-1}, (1 - y)^{-1})$. However, since $C_R(1) = C_r(1) = 0$ and $C_R'(1) = C_r'(1) = 1$, Φ extends continuously to include the case $(g, y, \gamma) \in (-\infty, 1) \times \{0\} \times (0, \infty)$ by means of the formula

$$\Phi(g, 0, \gamma) = C_R(1 - g) + \frac{\gamma}{\gamma_r}\, g,$$

and to include the case $(g, y, \gamma) \in (-1, \infty) \times \{1\} \times (0, \infty)$ by means of the formula

$$\Phi(g, 1, \gamma) = -g + \frac{\gamma}{\gamma_r}\, C_r(1 + g).$$

We shall also need the function ϕ given by the formula

$$\phi(y, \gamma) = \max_{g \in (-y^{-1}, (1-y)^{-1})} \{\Phi(g, y, \gamma)\}\,.$$

for all $(y, \gamma) \in (0, 1) \times (0, \infty)$, and by the formulae

$$\phi(0, \gamma) = \max_{g \in (-\infty, 1)} \{\Phi(g, 0, \gamma)\} \quad \text{and} \quad \phi(1, \gamma) = \max_{g \in (-1, \infty)} \{\Phi(g, 0, \gamma)\}$$

for all $\gamma \in (0, \infty)$.

We shall refer to χ as the normalized gain from trade and to ϕ as the normalized gain from sharing termination risk. Also, since the overall gain from trade is made up of the gain from sharing termination risk and the gain from sharing investment risk, we refer to $\chi - \phi$ as the normalized gain from sharing investment risk.

11.2. A Pair of One-Dimensional Equations. We are now in a position to take the first major step in the derivation of the one-dimensional version of the reduced Bellman equation of the Proposer:

Proposition 14. *The reduced Bellman equation of the Proposer, namely equation (23),* *can be written equivalently as a pair of one-dimensional equations for γ and χ, namely*

$$
\begin{aligned}
0 \;=\; & \max_{(z,g)\in\mathbb{R}\times(-y^{-1},(1-y)^{-1})} \left\{ \frac{\beta}{\gamma\,\sigma^2}\left(\Phi(g,y,\gamma)-\chi\right) \right. \\
& \left. + (R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \right\} \qquad (29)
\end{aligned}
$$

and

$$
y\,(1-y)\,\chi' = \frac{\gamma-\gamma_R}{\gamma_R} - ((1-R)-(2-R)\,y)\,\chi, \qquad (30)
$$

where we have suppressed the dependence of γ and χ on y.

Notice that choosing g reduces to maximizing $\Phi(g,y,\gamma)$ with respect to g, and that choosing z reduces to maximizing

$$
(R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \qquad (31)
$$

with respect to z. Also, the maximand (31) for z involves two terms: a linear incentive and a quadratic penalty. The linear incentive is simply $R-r$: the more the risk aversion of the Proposer exceeds that of the Responder, the greater will be the Responder's normalized leverage z. The quadratic penalty is (the multiplicative factor $\frac{W+w}{W\,w}$ aside) total endogenous risk aversion.[20] It is this which puts a brake on leverage.

[20] The endogenous absolute risk aversion of the Proposer is $-\frac{V_{PP}}{V_P}$ and the endogenous absolute risk aversion of the Responder is $-\frac{v''}{v'}$. Hence total endogenous risk aversion is

$$
-\frac{V_{PP}}{V_P} - \frac{v''}{v'} = \frac{W+w}{W\,w}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right).
$$

Cf. Section 11.3 below. Proposition 15 below shows that this is strictly positive.

Proof. Elementary calculations show that

$$\begin{aligned}
V_W &= (\gamma - y\,(\gamma - 1 - (1-R)\,\psi))\,C_R'(W), \\
V_P &= \gamma\,C_R'(W), \\
V_{WW} &= \left((1-y)^2\,\gamma' - \frac{R}{y}\,\gamma + R\,y\,(\gamma - 1 - (1-R)\,\psi)\right)\frac{C_R'(W)}{W+w}, \\
V_{WP} &= \left((1-y)\,\gamma' - \frac{R}{y}\,\gamma\right)\frac{C_R'(W)}{W+w}, \\
V_{PP} &= \left(\gamma' - \frac{R}{y}\,\gamma\right)\frac{C_R'(W)}{W+w}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\frac{U(W-(W+w)\,b)-V}{(W+w)\,C_R'(W)} &= \frac{W}{W+w}\,\frac{C_R(W-(W+w)\,b)-C_R(\rho\,W)}{W\,C_R'(W)} \\
&= \frac{W}{W+w}\left(C_R\!\left(1-\frac{W+w}{W}\,b\right)-C_R(\rho)\right) \\
&= y\,(C_R(1-(1-y)\,g)-\psi)
\end{aligned}$$

and

$$\begin{aligned}
\frac{V_P}{v'}\,\frac{u(w+b\,(W+w))-v}{(W+w)\,C_R'(W)} &= \frac{\gamma\,C_R'(W)}{\gamma_r\,C_r'(w)}\,\frac{C_r(w+(W+w)\,b)-C_r(\rho_r\,w)}{(W+w)\,C_R'(W)} \\
&= \frac{\gamma}{\gamma_r}\,\frac{w}{W+w}\,\frac{C_r(w+(W+w)\,b)-C_r(\rho_r\,w)}{w\,C_r'(w)} \\
&= \frac{\gamma}{\gamma_r}\,\frac{w}{W+w}\left(C_r\!\left(1+\frac{W+w}{w}\,b\right)-C_r(\rho_r)\right) \\
&= \frac{\gamma}{\gamma_r}\,(1-y)\,(C_r(1+y\,g)-\psi_r)\,.
\end{aligned}$$

Substituting into equation (23), dividing through by $(W+w)\,C_R'(W)$, taking advantage of the notation Φ, putting $s = (1-y)\,(1+y\,z)$, collecting terms in β, z and z^2, dividing through by $\gamma\,\sigma^2\,y\,(1-y)$ and rearranging therefore yields equation (29). Finally, we have

$$\gamma = 1 + (1-R)\,\psi + y\,\psi'$$

and

$$\psi = \psi_R + (1-y)\,\gamma_R\,\chi.$$

Differentiating the latter equation to get ψ' in terms of χ', substituting in the former equation and rearranging, we obtain equation (30). ∎

Next we establish that total endogenous risk aversion, and therefore the quadratic penalty on leverage, is strictly positive. The proof proceeds in two steps. The first step shows that total endogenous risk aversion must be non-negative, because otherwise the two parties would use the Wiener noise to construct bets, and thus arbitrage away the infinite gains from trade implicit in strictly negative total risk aversion.

Proposition 15. *Suppose that $R \neq r$. Then $(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma} > 0$.*

Proof. Since the maximand in equation (29) is additively separable in g and z, equation (29) can be written equivalently as

$$0 = \max_{z \in \mathbb{R}} \left\{ -\frac{\beta}{\gamma\,\sigma^2}\,(\chi - \phi) + (R - r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \right\}. \quad (32)$$

However, as it stands, this equation is not fully precise: z can take any real value, and therefore the coefficients of the equation are unbounded. To obtain a precise version of the equation, we need to normalize by dividing through by $1+z^2$.[21] Cf. Krylov [9]. Doing so yields

$$\begin{aligned} 0 \;=\; \sup_{z \in \mathbb{R}} & \left\{ -\frac{\beta}{\gamma\,\sigma^2}\,(\chi - \phi)\,\frac{1}{1+z^2} + (R - r)\,\frac{z}{1+z^2} \right. \\ & \left. -\tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right)\frac{z^2}{1+z^2} \right\}. \end{aligned}$$

In particular, the objective must be non-positive for all $z \in \mathbb{R}$. Letting $z \to \infty$ therefore yields

$$-\tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) \leq 0.$$

[21] Note that the normalization should in principle be applied consistently throughout the paper. However, we have suppressed it for expositional convenience. We make it explicit here since this is the one place where it plays an important role.

It therefore remains only to show that this inequality cannot hold as an equality. Suppose for a contradiction that $(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma} = 0$. Then we must have

$$\frac{1}{1+z^2}\left(-\frac{\beta}{\gamma\,\sigma^2}\,(\chi-\phi) + (R-r)\,z\right) \leq 0$$

for all $z \in \mathbb{R}$. Moreover, since $R \neq r$, the expression in parentheses is a non-constant affine function of z. It must therefore be strictly positive for some choice of z. This is the required contradiction. ■

Proposition 16. *The reduced Bellman equation of the Proposer, namely equation (23), can be written equivalently as a pair of one-dimensional equations for* γ *and* χ*, namely*

$$y\,(1-y)\,\gamma' = \left((1-y)\,R + y\,r - \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2\,\gamma}{\beta\,(\chi-\phi)}\right)\gamma \tag{33}$$

and

$$y\,(1-y)\,\chi' = \frac{\gamma-\gamma_R}{\gamma_R} - ((1-R)-(2-R)\,y)\,\chi. \tag{34}$$

Moreover the optimal risk-sharing rule takes the form

$$s = (1-y)\left(1 + y\,\frac{\beta\,(\chi-\phi)}{\frac{1}{2}\,(R-r)\,\sigma^2\,\gamma}\right). \tag{35}$$

We refer to the pair of equations (33-34) as the one-dimensional Bellman equation of the Proposer.

Proof. Rearranging (32), we obtain

$$\frac{\beta}{\gamma\,\sigma^2}\,(\chi-\phi) = \max_{z\in\mathbb{R}}\left\{(R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right)z^2\right\}. \tag{36}$$

The maximum is attained when the control variable

$$z = \frac{R-r}{(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}}, \tag{37}$$

and the maximum is

$$\frac{\frac{1}{2}\,(R-r)^2}{(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}}.$$

Hence

$$\frac{\beta}{\gamma\,\sigma^2}(\chi-\phi)=\frac{\frac{1}{2}\,(R-r)^2}{(1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}}. \tag{38}$$

In particular, since $R\neq r$ and $(1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}>0$, we have $\chi-\phi>0$.

Now, rearranging (38) yields (33). Rearranging (38) also yields

$$(1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}=\frac{\frac{1}{2}\,(R-r)^2\,\sigma^2\,\gamma}{\beta\,(\chi-\phi)}.$$

Substituting for the denominator of the RHS of (37) therefore gives

$$z=\frac{\beta\,(\chi-\phi)}{\frac{1}{2}\,(R-r)\,\sigma^2\,\gamma}.$$

Hence, noting that $s=(1-y)\,(1+y\,z)$, we obtain (35). Finally, (34) is the same as (30). ■

One simple but important implication of the expression (31) for the Proposer's objective is that: if $R>r$, then we will have $z>0$ for all $y\in[0,1]$ (i.e. the Responder takes on more than his autarky share of the risk); and, if $R<r$, then we will have $z<0$ for all $y\in[0,1]$ (i.e. the Responder takes on less than his autarky share of the risk). We formalize this observation in the following Corollary.

Corollary 17.

1. *If $R>r$ then $s>1-y$ for all $y\in(0,1)$.*
2. *If $R<r$ then $s<1-y$ for all $y\in(0,1)$.* ■

The proof of Proposition 16 also establishes that the normalized gain from sharing investment risk is strictly positive. Combining this with the fact that $\phi(y,\gamma)\geq\Phi(0,y,\gamma)=0$, we obtain a second Corollary:

Corollary 18. *Suppose that $R\neq r$. Then $\chi>\phi\geq 0$.* ■

Notice that $\phi=0$ iff $g=0$. Moreover, on the basis of the asymptotic expansions given in Section 10 above, we would expect that g will indeed take the value 0 for some choices of the parameter values.[22] This is confirmed by Figure 7 below.

[22] Specifically, it would be expected to happen for intermediate values of the ratio $\frac{2\mu}{\sigma^2}$.

11.3. Revisiting the General Formula for s. The changes of variable made in this section also shed light on the general formula for s given in Section 9, namely

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{wP}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}.$$

Indeed, using the formulae given in the proof of Proposition 14, we obtain

$$-\frac{V_{PP}}{V_P} = \frac{1}{W+w}\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right)$$

and

$$-\frac{V_{wP}}{V_P} = -\frac{V_{WP} - V_{PP}}{V_P} = \frac{1}{W+w}\, y\, \frac{\gamma'}{\gamma}.$$

Moreover

$$-\frac{v''}{v'} = \frac{r}{w} = \frac{1}{W+w}\,\frac{r}{1-y}.$$

Hence

$$s = \frac{\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right) + y\,\frac{\gamma'}{\gamma}}{\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right) + \frac{r}{1-y}}. \tag{39}$$

In particular, if $\frac{\gamma'}{\gamma} > 0$, then there are two effects. First, the endogenous risk aversion $-\frac{V_{PP}}{V_P}$ of the Proposer is lower than her exogenous risk aversion $-\frac{U''}{U'}$. Second, the elasticity $-\frac{V_{wP}}{V_P}$ of the shadow value of transfers V_P with respect to changes in the Responder's wealth w is strictly positive. The first effect tends to lower the Responder's share in investment risk: the Proposer can afford to take on more risk because, if she receives a negative shock, then this is partially offset by a reduction in the opportunity cost of obtaining insurance; and, if she receives a positive shock, then this is partially offset by an increase in the opportunity cost of obtaining insurance. The second effect tends to raise the Responder's share in investment risk: obtaining more insurance from the Responder tends to increase the Responder's wealth, and increasing the Responder's wealth decreases the Proposer's opportunity cost of obtaining insurance.

It turns out that the second effect always outweighs the first. Indeed, we have

$$s - s^{(0)} = \frac{(R-r)\,\frac{\gamma'}{\gamma}}{\left(\frac{R}{y} + \frac{r}{1-y}\right)\left(\frac{R}{y} + \frac{r}{1-y} - \frac{\gamma'}{\gamma}\right)}.$$

Moreover Proposition 15 tells us that $\frac{R}{y} + \frac{r}{1-y} - \frac{\gamma'}{\gamma} > 0$. Hence:

Proposition 19. *Suppose that $R > r$. Then $s - s^{(0)} > 0$ iff $\gamma' > 0$.* ■

In other words, the Responder takes on more risk under the optimal risk-sharing rule than he would under the myopic benchmark iff the shadow value of transfers γ is decreasing in the Responder's share in total wealth.

12. THE CASE OF LARGE WEALTH

One of the advantages of the one-dimensional Bellman equation derived in the previous section is that it allows us to investigate what happens when the wealth of one party becomes very large relative to the wealth of the other. We can understand what happens when $w \to \infty$ by examining the limit of the solution of the one-dimensional Bellman equation as $y \to 0$, and what happens when $W \to \infty$ by examining the limit of the solution of the one-dimensional Bellman equation as $y \to 1$. In this section, we formulate sufficient conditions under which the relevant limits exist. These conditions are also sufficient conditions for the existence of an optimal constract.

It is natural to begin by requiring that both parties have finite payoffs under autarky. Indeed, from a risk-sharing perspective, the Responder is effectively on his own when $y = 0$, and the Proposer is effectively on her own when $y = 1$. This is what Conditions I and II ensure. However, when $y \in (0, 1)$, there is a gain from risk sharing. Moreover, when normalized with respect to the wealth of the Responder, this gain will be largest when $y = 1$; and, when normalized for the wealth of the Proposer, it will be largest when $y = 0$. In order to ensure that the normalized gain from risk sharing does not become infinite, it therefore suffices to ensure that it is finite when $y = 1$ and $y = 0$.

Putting $y = 1$ in equation (33) and rearranging yields

$$\chi(1) - \phi(1, \gamma(1)) = \frac{\frac{1}{2}(R-r)^2 \sigma^2}{r \beta} \gamma(1). \tag{40}$$

Similarly, putting $y = 1$ in equation (34) and rearranging yields

$$\chi(1) = \frac{\gamma_R - \gamma(1)}{\gamma_R}. \tag{41}$$

We also have

$$\phi(1, \gamma(1)) = \max_{g \in (-1, \infty)} \left\{ -g + \frac{\gamma(1)}{\gamma_r} C_r(1+g) \right\}. \tag{42}$$

Now, if we denote the maximizer in equation (42) by $g(1)$, then the first-order condition yields

$$\frac{\gamma(1)}{\gamma_r}\, C_r'(1+g(1)) = 1, \tag{43}$$

and equation (42) itself becomes

$$\phi(1,\gamma(1)) = -g(1) + \frac{\gamma(1)}{\gamma_r}\, C_r(1+g(1)).$$

Hence

$$\begin{aligned}
\frac{\gamma(1)}{\gamma_r} + (1-r)\,\phi(1,\gamma(1)) &= \frac{\gamma(1)}{\gamma_r}\left(1 + (1-r)\, C_r(1+g(1))\right) - (1-r)\, g(1) \\
&= \frac{\gamma(1)}{\gamma_r}(1+g(1))\, C_r'(1+g(1)) - (1-r)\, g(1) \\
&= (1+g(1)) - (1-r)\, g(1) \\
&= 1 + r\, g(1).
\end{aligned} \tag{44}$$

Next, note that the three equations (40), (41) and (44) are linear in the three unknowns $\chi(1)$, $\phi(1,\gamma(1))$ and $g(1)$. Solving them yields

$$\frac{r}{\gamma(1)}\,(1+g(1)) = \frac{1}{\gamma_r} - (1-r)\left(\frac{1}{\gamma_R} + \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{r\,\beta}\right). \tag{45}$$

Now, $1+g(1)$ must be in the domain of the function C_r, i.e. we must have $1+g(1) > 0$. The right-hand side of equation (45) must therefore also be strictly positive. Putting $\gamma_R = \frac{\beta}{\beta_R}$, $\gamma_r = \frac{\beta}{\beta_r}$ and $\beta_r = \beta - (1-r)\,(\mu - \frac{1}{2}\, r\,\sigma^2)$, and rearranging, shows that this is the case iff:

Condition III. $\beta > (1-r)\left(\mu - \frac{1}{2}\, r\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{r} + \beta_R\right)$.

The best way to understand Condition III is to compare it with Condition I. Condition I requires that the rate of discounting β must exceed the rate of growth of the Responder's utility when his wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\, r\,\sigma^2$, whereas Condition III requires that the rate of discounting β must exceed the rate of growth of utility when wealth grows at the higher rate $\mu - \frac{1}{2}\, r\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{r} + \beta_R$. The latter rate is composed of three terms: the certainty-equivalent rate of return derived from investment, namely $\mu - \frac{1}{2}\, r\, \sigma^2$, the certainty-equivalent rate of return deriving from sharing investment risk, namely $\frac{1}{2}\, \frac{(R-r)^2 \sigma^2}{r}$, and the certainty-equivalent rate of return deriving from sharing termination risk, namely β_R.[23] Moreover Condition II requires precisely that $\beta_R > 0$. Hence, assuming that Condition II holds, Condition III is stronger than Condition I when $r < 1$: the Responder's utility function is unbounded above, and his certainty-equivalent rate of return is larger, so we need more discounting if his expected payoff is to be finite. Similarly, again assuming that Condition II holds, Condition I is stronger than Condition III when $r > 1$: the Responder's utility function is unbounded below, and his certainty-equivalent rate of return is larger, so we now need less discounting than before in order to ensure that his expected payoff is finite.

We also need the corresponding condition derived from the case $y = 0$. Proceeding analogously to the case $y = 1$, we obtain the four equations

$$\chi(0) - \phi(0, \gamma(0)) \;=\; \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2}{R\,\beta}\,\gamma(0), \tag{46}$$

$$(1-R)\,\chi(0) \;=\; \frac{\gamma(0) - \gamma_R}{\gamma_R}, \tag{47}$$

$$C_R'(1 - g(0)) \;=\; \frac{\gamma(0)}{\gamma_r}, \tag{48}$$

$$1 + (1-R)\,\phi(0, \gamma(0)) \;=\; (1 - R\,g(0))\,\frac{\gamma(0)}{\gamma_r}. \tag{49}$$

Now, if $R \neq 1$, then we can solve the three equations (46), (47) and (49) for the three unknowns $\chi(0)$, $\phi(0, \gamma(0))$ and $g(0)$. Doing so yields

$$\frac{R}{\gamma_r}\,(1 - g(0)) = \frac{1}{\gamma_R} - (1-R)\left(\frac{1}{\gamma_r} + \frac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{R\,\beta}\right). \tag{50}$$

[23] According to Proposition 9, the investment return on the Responder's wealth and the monetary value of the gains from sharing investment risk are

$$(\mu - \tfrac{1}{2}\, r\, \sigma^2)\, w \quad \text{and} \quad \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W} + \frac{r}{w}}.$$

Normalizing these expressions with respect to the Responder's wealth, i.e. dividing through by w, yields

$$\mu - \tfrac{1}{2}\, r\, \sigma^2 \quad \text{and} \quad \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2\, y}{(1-y)\,R + y\,r}.$$

Putting $y = 1$ then yields the first two expressions in the text.

Similarly, if $R = 1$, then one can solve the two equations (47) and (48) for the two unknowns $\gamma(0)$ and $g(0)$. Doing so yields the appropriate special case of (50). Finally, putting $\gamma_R = \frac{\beta}{\beta_R}$, $\gamma_r = \frac{\beta}{\beta_r}$ and $\beta_R = \beta - (1-R)\,(\mu - \frac{1}{2}\,R\,\sigma^2)$ in (50) and rearranging, we find that $1 - g(0) > 0$ iff

Condition IV. $\beta > (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R} + \beta_r\right)$.

Condition IV is completely analogous to Condition III. In particular, assuming that Condition I holds, Condition IV is stronger than Condition II when $R < 1$, and Condition II is stronger than Condition IV when $R > 1$. Furthermore, it can be shown that, in the case in which both $R < 1$ and $r < 1$, Conditions III and IV together imply Conditions I and II. The discussion of the present section can therefore be summarized as follows.

Proposition 20. *A sufficient condition for the one-dimensional Bellman equation of the Proposer, namely (33-34), to have a solution that is continuous on* $[0,1]$ *is that both of the following conditions hold:*

$$\beta > \left\{ \begin{array}{cl} (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R} + \beta_r\right) & \text{if } R \leq 1 \\ (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2\right) & \text{if } R \geq 1 \end{array} \right\}$$

and

$$\beta > \left\{ \begin{array}{cl} (1-r)\left(\mu - \frac{1}{2}\,r\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{r} + \beta_R\right) & \text{if } r \leq 1 \\ (1-r)\left(\mu - \frac{1}{2}\,r\,\sigma^2\right) & \text{if } r \geq 1 \end{array} \right\}.$$

In particular, these conditions are sufficient for the existence of an optimal contract. ■

Remark 21. *If one is interested primarily in the original problem, in which* $W > 0$ *and* $w > 0$*, but not in the extended problem, in which the solution of the one-dimensional Bellman equation of the Proposer is required to be continuous right up to the boundary, then Conditions I and II should probably still be regarded as minimal conditions: they ensure that the outside options of both parties are well defined. However, it may be possible to weaken Conditions III and IV: if the Proposer's normalized value diverges as* $y \to 0$*, then this may be offset by the fact that her wealth* $W \to 0$*; and, if the Responder's normalized value diverges as* $y \to 1$*, then this may be offset by the fact that his wealth* $w \to 0$*. Investigating this possibility is, however, beyond the scope of the current paper.*

13. Numerical Solutions

In this section we solve the system (33-34) numerically, and use these solutions to compute the optimal contract. The numerical solutions are obtained using the MatLab program *bvp4c*.

The most striking feature of the numerical solutions is the degree to which they conform to the predictions of the asymptotic expansions. The first prediction is quantitative: there should be three main cases, namely the risk-tolerant case ($2 > R > r$), the risk-averse case ($R > r > 2$) and the mixed case ($R > 2 > r$). The second set of predictions is qualitative:

1. in the risk-tolerant case, we should have $s - s^{(0)} < 0$ for all y;
2. in the risk-averse case, we should have $s - s^{(0)} > 0$ for all y;
3. in the mixed case we should have $s - s^{(0)} > 0$ for low y and $s - s^{(0)} < 0$ for high y.

The third set of predictions is again quantitative. Putting $B(0) = 2\,r - \frac{r}{R}$, $B(1) = R + r - \frac{R}{r}$ and $\overline{B} = \frac{1}{4}\,r\,(4+R)$:

1. in the risk-tolerant case, we should have $b > 0$ for all y if $\frac{2\mu}{\sigma^2} < B(1)$; $b > 0$ for low y and $b < 0$ for high y if $\frac{2\mu}{\sigma^2} \in (B(1), B(0))$; $b < 0$ for all y if $\frac{2\mu}{\sigma^2} > B(0)$.
2. in the risk-averse case, we should have $b > 0$ for all y if $\frac{2\mu}{\sigma^2} < B(0)$; $b < 0$ for low y and $b > 0$ for high y if $\frac{2\mu}{\sigma^2} \in (B(0), B(1))$; $b < 0$ for all y if $\frac{2\mu}{\sigma^2} > B(1)$.
3. in the mixed case we should have $b > 0$ for all y if $\frac{2\mu}{\sigma^2} < \min\{B(0), B(1)\}$; $b < 0$ for low y, $b > 0$ for intermediate y and $b < 0$ for high y if $\frac{2\mu}{\sigma^2} \in \big(\max\{B(0), B(1)\}, \overline{B}\big)$; $b < 0$ for all y if $\frac{2\mu}{\sigma^2} > \overline{B}$.[24]

These predictions are borne out by all the simulations presented here and by numerous unreported simulations. Moreover it seems likely that the first prediction and the second set of predictions taken together are in fact a theorem.

[24] We omit the subcase in which $\frac{2\mu}{\sigma^2} \in (\min\{B(0), B(1)\}, \max\{B(0), B(1)\})$ since it divides into two subsubcases depending on whether $B(0) < B(1)$ or $B(0) > B(1)$.

In this section we shall focus mainly on three baseline parameter constellations:

Case	R	r	μ	σ	β
Risk-Tolerant	1	0.5	0.025	0.15	0.05
Risk-Averse	10	2.5	0.12	0.15	0.05
Mixed	8	1.3	0.10	0.15	0.05

In all three cases: $\sigma = 0.15$, which is in line with estimates of the volatility of the US stock market: and $\beta = 0.05$, which is loosely calibrated on estimates of subjective discount rates. The coefficients of relative risk aversion R and r are then chosen — from within the range that has been found in empirical studies — to balance two competing objectives: on the one hand, we do not want the dynamic effects to be swamped by the myopic effects; but, on the other, we want our simulations to be completely stable.[25] The final parameter μ is then chosen in such a way that Conditions I-IV are satisfied. For example, in the risk-tolerant case, the cost of risk is low and μ must not be too large; and, in the risk-averse case, the cost of risk is high and μ must not be too small.

For such a small value of β, we cannot expect our asymptotic expansions — which are based on the assumption that β is large — to be quantitatively accurate in all respects.[26] However, as we shall see, the qualitative predictions — and some of the quantitative predictions — obtained from these expansions are remarkably accurate even though the β we actually use is rather small.

We begin by noting that $\frac{s^{(0)}}{1-y}$ is the fraction of the investment risk on his own wealth that the Responder bears under the myopic contract (and $\frac{s^{(0)}}{1-y}-1$ is his leverage). Similarly, $\frac{s}{1-y}$ is the fraction of the investment risk on his own wealth that he bears under the optimal contract (and $\frac{s}{1-y}-1$ is his leverage). Figure 4 plots $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$ as a function of y in the three baseline cases. In all cases, both $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$ increase from 1 to $\frac{R}{r}$ as y increases from 0 to 1. They take the value 1 when $y = 0$, because in that case the Responder has all the wealth, and so risk sharing with the Proposer has a negligible impact on his leverage. They are increasing in y because — from the point of view of the Responder — the opportunities for risk sharing are increasing in y and, as the less risk averse party, taking advantage of these opportunites means increasing his leverage. Notice that the leverage of the Responder can be quite substantial when $y = 1$: in the risk-tolerant case

[25] For examples of such studies, see Barsky, Juster, Kimball, and Shapiro (1997), Guiso and Paiella (2008), Chiappori and Paiella (2008) and Paravisini, Rappoport and Ravina (2009).

[26] For further discussion of the magnitude of β, see footnote 17 above.

he is 100% levered when $y = 1$; in the risk-averse case he is 300% levered; and in the mixed case he is about 515% levered.

Figure 4 about here

The relationship between $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$ is exactly as predicted by the dynamic correction $\frac{1}{\beta}\frac{s^{(1)}}{1-y}$: in the risk-tolerant case, the Responder takes on less risk than he would under the myopic benchmark, i.e. $\frac{s}{1-y} < \frac{s^{(0)}}{1-y}$; in the risk-averse case, the Responder takes on more risk than he would under the myopic benchmark, i.e. $\frac{s}{1-y} > \frac{s^{(0)}}{1-y}$; and, in the mixed case, the Responder takes on more risk when his wealth is high and less risk when his wealth is low, i.e. $\frac{s}{1-y} > \frac{s^{(0)}}{1-y}$ when y is low and $\frac{s}{1-y} < \frac{s^{(0)}}{1-y}$ when y is high.

The size of the difference $\frac{s}{1-y} - \frac{s^{(0)}}{1-y}$ between the Responder's leverage under the optimal contract and his leverage under the myopic benchmark can be seen more clearly in Figure 5. In the risk-tolerant case, this difference troughs for y around 0.92, with a value of about -0.07. At this point the myopic benchmark is around 1.85, so the difference is about -3.7% of the benchmark. A somewhat larger effect (reflecting the greater difference in risk aversion) is obtained for the risk-averse case: the difference peaks for y around 0.79, with a value of about 0.13. At this point the myopic benchmark is around 2.44, so the difference is about 5.5% of the benchmark. The largest effects are obtained for the mixed case. In this case, the difference first peaks for y around 0.70, with a value of about 0.21, and then troughs for y around 0.98, with a value of about -0.38. So, in the mixed case, the extreme differences are about 8.7% and -6.7% of the benchmark respectively.

Figure 5 about here

Next, $\frac{b}{1-y}$ is the fraction of the Responder's wealth that the Proposer pays to the Responder on termination. Figure 6 shows that, in all three of our baseline cases, this fraction is 0 when $y = 0$ and decreases as y increases from 0 to 1. It is zero at $y = 0$ because the Responder has all the wealth, and any payment from the Proposer to the Responder is therefore negligible relative to the Responder's wealth. It is negative for $y > 0$ because the investment opportunity is valuable, and it is therefore the Responder who compensates the Proposer when it terminates. What is perhaps most striking is the sheer size of the payments made by the Responder: in the risk-tolerant case, he pays out about 62% of his wealth when $y = 1$; in the risk-averse case, he pays out about 26%;

Figure 4: the myopic and optimal risk-sharing rules $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$

Figure 4(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

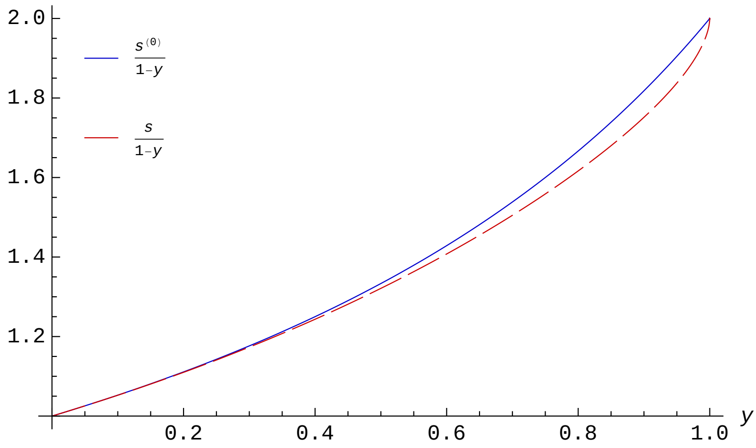


Figure 4(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

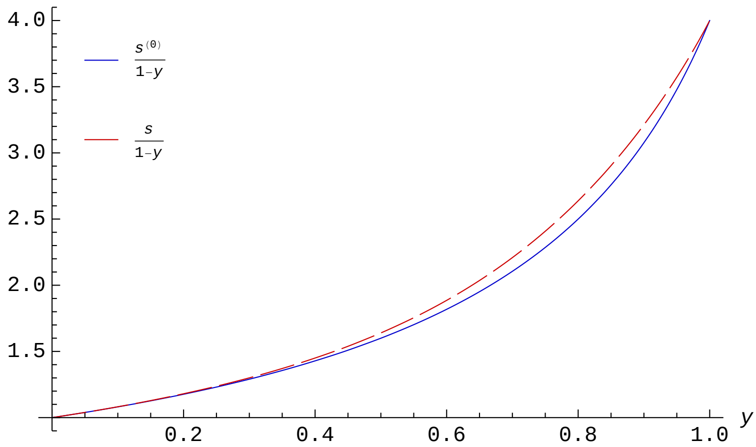


Figure 4(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)

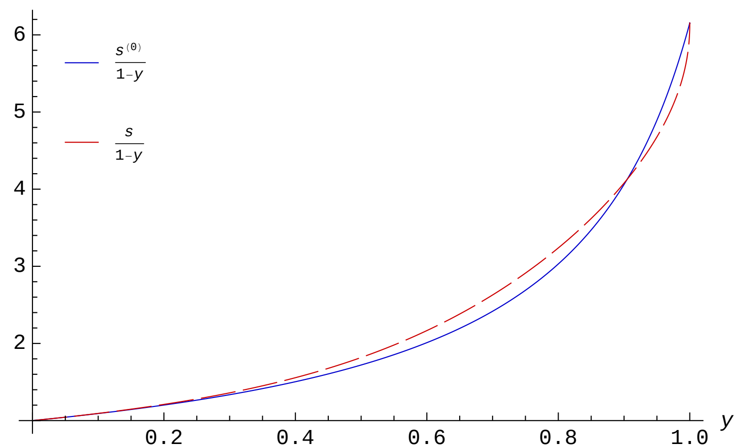


Figure 5: the difference $\frac{s}{1-y} - \frac{s^{(0)}}{1-y}$

Figure 5(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

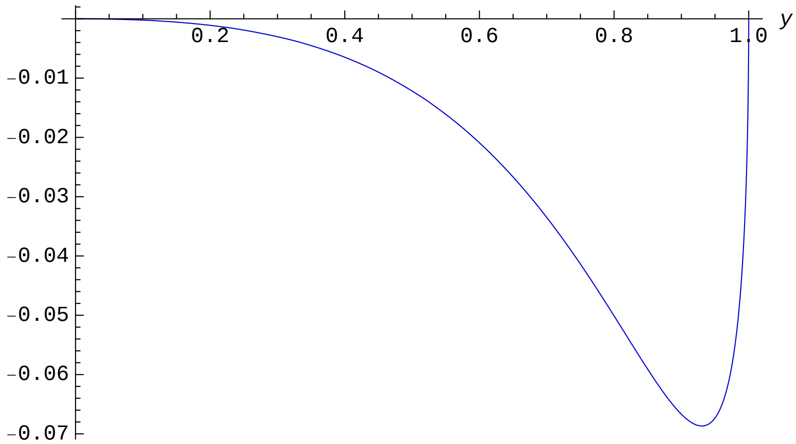


Figure 5(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

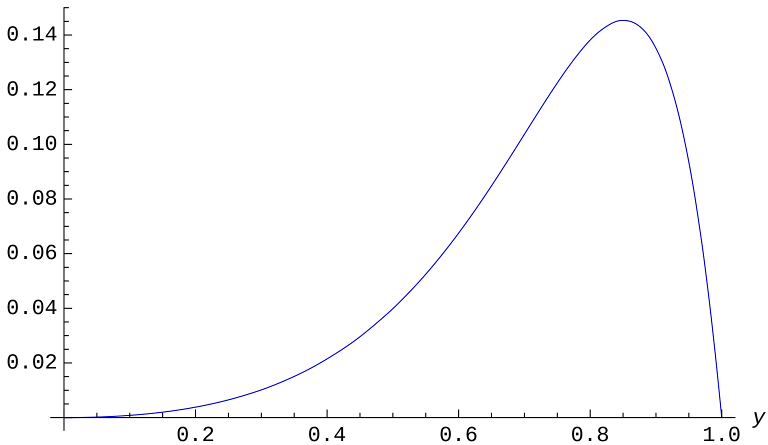
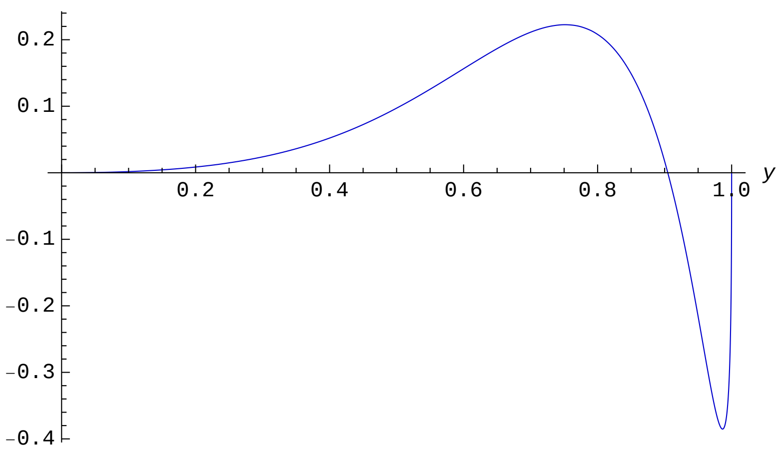


Figure 5(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)



and in the mixed case, he pays out almost 94%.[27] The magnitude of these payments underlines the importance of sharing termination risk.

Figure 6 about here

Finally, the asymptotic expansions for $\frac{b}{1-y}$ suggest that it will not always be the case that $\frac{b}{1-y} < 0$ or that $\frac{b}{1-y}$ is decreasing. In fact, those expansions make a number of predictions to which we turn in our final figure. For example, in the risk-tolerant case, we have $\frac{1}{2}\sigma^2 B(0) \approx 0.0056$ and $\frac{1}{2}\sigma^2 B(1) \approx -0.0056$. Hence, assuming that we keep $\sigma = 0.15$, we expect to have:

1. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ iff $\mu < -0.0056$;
2. $\frac{b}{1-y} > 0$ for low y and $\frac{b}{1-y} < 0$ for high y iff $\mu \in (-0.0056, 0.0056)$;
3. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ iff $\mu > 0.0056$.

Figure 7(a), which plots $\frac{b}{1-y}$ for $\mu \in \{-0.01, 0, 0.01\}$, is consistent with these predictions.

In the risk-averse case, we have $\frac{1}{2}\sigma^2 B(0) \approx 0.053$ and $\frac{1}{2}\sigma^2 B(1) \approx 0.096$. Hence we expect to have:

1. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ iff $\mu < 0.053$;
2. $\frac{b}{1-y} > 0$ for low y and $\frac{b}{1-y} < 0$ for high y iff $\mu \in (0.053, 0.096)$;
3. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ iff $\mu > 0.096$.

Figure 7(b), which plots $\frac{b}{1-y}$ for $\mu \in \{0.03, 0.08, 0.12\}$, is consistent with these predictions. (In this figure, β has to be chosen appropriately in order to ensure that Conditions I-IV are satisfied.)

In the mixed case, we have $\frac{1}{2}\sigma^2 B(0) \approx 0.027$, $\frac{1}{2}\sigma^2 B(1) \approx 0.035$ and $\frac{1}{2}\sigma^2 \max\{B(y) \mid y \in [0,1]\} \approx 0.044$. Hence we expect to have:

[27] If we multiply equations (43) and (45) together and rearrange, then we obtain

$$C_r(1+g(1)) = \frac{1}{r\,\beta_r}\left(\beta_r - \beta_P - \frac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{r}\right).$$

This equation can then be inverted to yield an explicit formula for $g(1)$. The numerical solutions for $\frac{b}{1-y}$ at $y = 1$ are in excellent agreement with this formula.

Figure 6: the optimal termination payment $\frac{b}{1-y}$

Figure 6(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

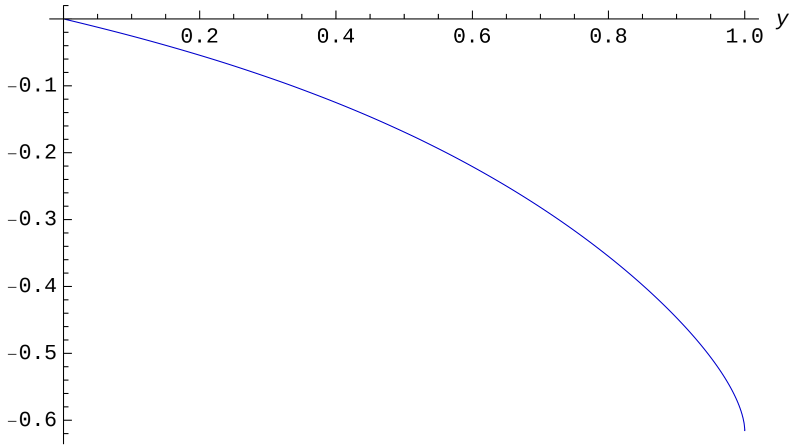


Figure 6(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

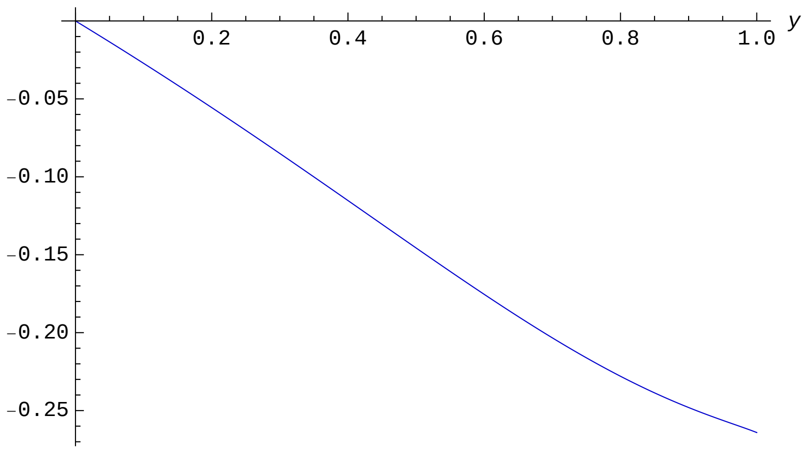
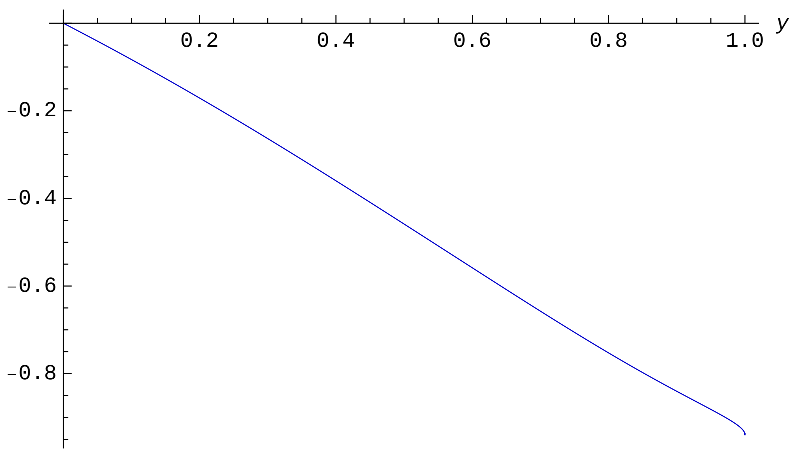


Figure 6(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)



1. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ iff $\mu < 0.027$;
2. $\frac{b}{1-y} < 0$ for low y, $\frac{b}{1-y} > 0$ for intermediate y and $\frac{b}{1-y} < 0$ for high y iff $\mu \in (0.035, 0.044)$;
3. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ iff $\mu > 0.044$.

Figure 7(c), which plots $\frac{b}{1-y}$ for $\mu \in \{0.02, 0.04, 0.05\}$, is consistent with these predictions. (In this figure, β must again be chosen appropriately in order to ensure that Conditions I-IV are satisfied.)

Figure 7 about here

What is remarkable about Figure 7 is the way in which the asymptotic expansions give a detailed guide as to what patterns to expect, and precise suggestions for the parameter values that will give rise to those patterns.

14. Conclusion

In this paper we have analyzed an optimal risk-sharing problem in which two parties invest in a common constant-returns-to-scale risky asset. The two parties have different coefficients of relative risk aversion, and they start with different wealth endowments. We have taken out many interesting features from the model to keep the analysis tractable. In particular, we have only allowed for consumption at the end, and we have only considered an extreme bargaining situation in which one of the parties can make take-it-or-leave-it offers to the other. Within this model we have, however, been able to push the characterization of optimal risk-sharing quite far.

For example, we have used asymptotic expansions to obtain approximations to the optimal risk-sharing rules. These approximations capture in a transparent way the main tradeoffs that the contracting parties face. Moreover numerical simulations confirm that the picture that they generate is qualitatively (and sometimes quantitatively) accurate.

The approximations can be decomposed into a *myopic benchmark* and a *dynamic correction*. In the case of the optimal rule s for the Responder's share in investment risk, the myopic benchmark $s^{(0)}$ is the classical ratio of the Proposer's absolute risk aversion to the sum of the Proposer's and the Responder's absolute risk aversions, namely

$$\frac{\frac{R}{W}}{\frac{R}{W}+\frac{r}{w}},$$

Figure 7: the optimal termination payment $\frac{b}{1-y}$ (other parameter values)

Figure 7(a): the Risk-Tolerant Case with $R = 1$, $r = 0.5$, $\sigma = 0.15$,

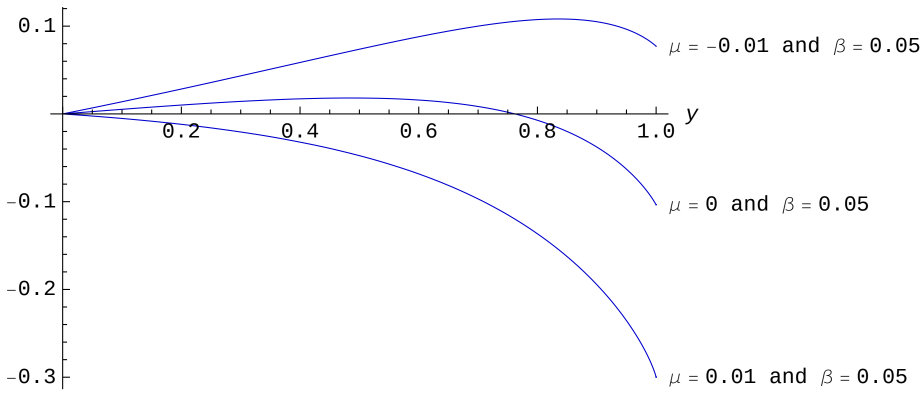


Figure 7(b): the Risk-Averse Case with $R = 10$, $r = 2.5$, $\sigma = 0.15$,

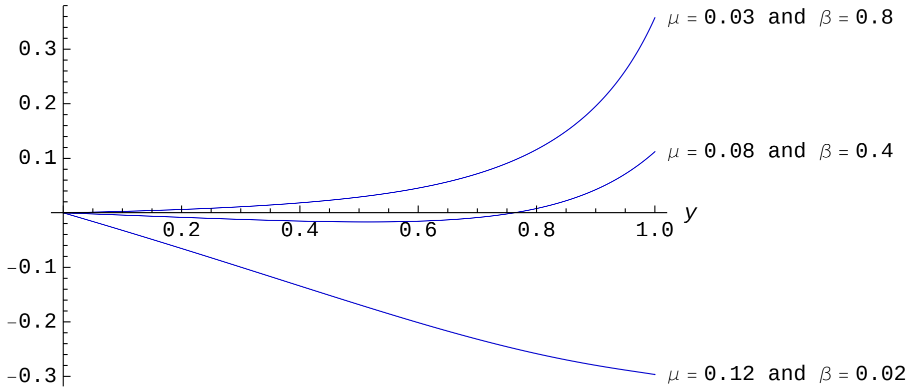
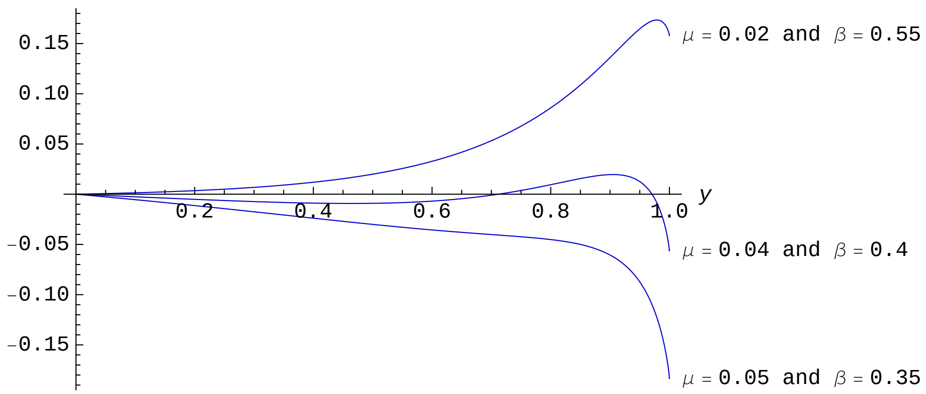


Figure 7(c): the Mixed Case with $R = 8$, $r = 1.3$, $\sigma = 0.15$,



where R and r are the coefficients of relative risk aversion of the Proposer and the Responder, and W and w are their wealths. This formula captures the basic aspects of risk sharing. For example, the wealthier or the less risk averse the Proposer, the less the investment risk taken on by the Responder.

The myopic benchmark does not, however, capture counterparty risk. For example, if the Responder is risk neutral (i.e. if $r = 0$), then it predicts that the Responder will take on all the investment risk. However, if the Responder took on all the investment risk, then he would run out of wealth in finite time. The Proposer would thereafter not be able to obtain any insurance. In other words, the Responder's insurance capacity is finite, and the Proposer should take this into account by adjusting the risk-sharing rule to conserve it as it begins to run low.

These ideas are captured by the dynamic correction $\frac{1}{\beta}\, s^{(1)}$, which can be written in the form

$$\frac{\frac{1}{2}\, R\, r\, \sigma^2\, y^2\, (1-y)^2}{\beta\, ((1-y)\, R + y\, r)^5}\, (R-r)^3\, ((1-y)\, R + y\, r - 2)\,,$$

where β is the hazard rate of termination, σ is the volatility of investment returns and $y = \frac{W}{W+w}$ is the Proposer's share in total wealth. This formula for the dynamic correction is the product of three terms. The first is always positive, so the sign of the dynamic correction is determined by two considerations: whether the Proposer is more risk averse than the Responder, in the sense that $R > r$; and whether the average risk aversion of the two parties is large, in the sense that $(1-y)\, R + y\, r > 2$. (As each party's coefficient of relative risk aversion is weighted by the other party's share in total wealth, it is the risk aversion of the poorer party that matters most in this inequality.)

The three key predictions from this formula for the dynamic correction are then as follows. First, if both investors are fairly risk tolerant (in the sense that $R, r < 2$), then the investor who is more risk averse takes on a *larger* share of total investment risk than she would under the myopic benchmark. Indeed, the investor who is less risk averse is willing to take on risk on relatively *unfavourable* terms. So the more risk he takes on, the sooner he will run out of wealth. The optimal dynamic contract therefore transfers *less* risk to him than the optimal myopic contract.

Second, if both investors are fairly risk averse (in the sense that $R, r > 2$), then the investor who is more risk averse takes on a *smaller* share of total investment risk than she would under the myopic benchmark. This is because the investor who is less risk averse is only willing to take on risk on relatively *favourable* terms. So taking on more risk actually delays the time at which he will run out of wealth. The optimal dynamic contract therefore transfers *more* risk to him than the optimal myopic contract.

Third, if one investor is fairly risk averse and the other is fairly risk tolerant (in the sense that $R > 2 > r$ or $r > 2 > R$), then the investor who is more risk averse takes on a *smaller* share of total investment risk when her wealth is *small* and a *larger* share when her wealth is *large*. This is because, while she would like to reduce the amount of risk transferred to her risk-tolerant counterparty, the cost of bearing the extra risk herself is too high when her wealth is low.

In sum, the approximations to the optimal risk-sharing rule s we have derived capture in a relatively simple way the tradeoff between getting more insurance coverage today and preserving future insurance options. Moreover, these rules are explicit and easy to apply.

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