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THE WEITZMAN MODEL REVISITED

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### **ABSTRACT**

We show how technical change, measured as a shift in the GDP function, is combined with net income to track welfare change. This provides a bridge between the productivity literature and the welfare-related literature that tends to reason in terms of net product functions: although the relevant income measure is net of depreciation, productivity is measured based on gross output. We show that net product, net income, net expenditure and productivity change are complements, not substitutes. We also examine whether holding gains and losses should be part of depreciation and conclude that in a general equilibrium setting, either productivity change or holding gains should be part of an extended Weitzman-type net income measure, but not both.

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## 1. Introduction

Weitzman (1976) was first to provide a rigorous formulation of the link between net income/product and consumption-based economic welfare. He showed that in a closed economy with no government, no autonomous technical change and under competitive conditions, net income/product can be seen as the stationary-equivalent flow to discounted future consumption. This result is one way of formalizing Hicks' (1939) third income measure that defines income as "[...] the maximum amount of money which the individual can spend this week, and still expect to be able to spend the same amount in real terms in each ensuing week" (p.174) and further: "[...] the calculation of income consists in finding some sort of standard stream of values whose present capitalized value equals the present value of the stream of receipts which is actually in prospect" (p. 184). Not surprisingly, Weitzman's work found an important application in the analysis of exhaustible resource depletion and environmentally sustainable development<sup>1</sup>.

Weitzman's original model did not include an explicit role for technical change.<sup>2</sup> This is a potentially important omission, because costless advances in technical efficiency are welfare-enhancing, and may mitigate the problem of exhaustible resources (along with product-oriented technological advances). Nordhaus (1995) was among the first to raise this issue by demonstrating that net product by itself is not a sufficient indicator of welfare change, and that it

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<sup>1</sup> An overview of this literature can be found in Heal and Kriström (2005).

<sup>2</sup> It was also based on a time-invariant rate of interest, but subsequent research generalized the analysis to allow for a time-varying rate (Asheim 1994).

needs to be augmented by the effects of disembodied technical change. Subsequently, Weitzman (1997) made the same point:

“The consequences of technical change being absent from the standard time-autonomous model might be quite serious for the basic welfare interpretation of Green NNP. We know that future growth is largely driven by the rate of technological progress, however it is conceptualized. Since Green NNP theoretically equals annuity-equivalent future consumption possibilities *without* the "Solow residual", the proper measure of annuity-equivalent future consumption possibilities *with* the "Solow residual" might conceivably call for a sizable upward adjustment of Green NNP.” (Weitzman 1997, p.2).

The use of net product (NP) as an indicator of welfare is a key feature of this literature. While there is nothing analytically wrong with approaching the welfare problem this way, it does diverge from the dominant stream of economic results based on the circular flow of inputs and outputs through markets. In the circular flow framework, the output of the production sector is measured gross of depreciation, and represented in the aggregate by Gross Domestic Product (GDP). Gross output is also the basis for the Solow residual mentioned above by Weitzman, which has evolved as the shift in the gross output production function, following Jorgenson and Griliches (1967), and is now the standard for a vast empirical literature. The net approach is relatively unusual in the productivity literature (with some exceptions, for instance Diewert and Fox 2005).

The first objective of this paper is to extend the original Weitzman result using the conventional

concept of technical change based on gross output. This result allows us to link the welfare implications of technical change to standard growth theory (e.g., the Solow 1956 model of steady-state growth). In our extension of the Weitzman model, we draw a distinction between net income (NI) and net product (NP), and show that NP originates in a different quadrant of the economy than NI and that the two are not necessarily equal in any given year.

In a further generalization, we allow for an open economy. Terms of trade effects are potentially important in an open economy, since changes in import and export prices have effects that are similar to those of autonomous gross technical change. When these elements are incorporated into our modified version of the Weitzman model, welfare changes are shown to be linked to changes in wealth via three main elements: net savings; the rate of gross-output based technical change; and the rate of change in the terms of trade.

The introduction of technical change into the analysis of welfare allows us to examine another important issue. There has been a debate in the literature on income measurement as to whether capital gains and losses should be included or excluded from net income (see, for example, Eisner 1988). Sefton and Weale (2006) conclude that they should not be included in income as long as the economy is closed, whereas Asheim (1996) reaches the opposite conclusion. Our results agree with Sefton and Weale so long as there is no technical change or openness to trade, but we also find that expected holding gains and losses *do* enter the Hicksian definition of income via technical change, i.e., that these gains need to be added to net income to obtain Hicksian income.

Our model also speaks to the relation between depreciation and holding gains. Hill and Hill (2003), Diewert (2005) advocate a composite measure that includes these two effects, on the grounds that changes in technology affect depreciation. We argue that, in our general equilibrium framework, these two effects are best kept separate because they represent different economic phenomena.

The following section sets out the traditional circular flow model of an economy, in order to lay the groundwork for the use of gross-output productivity change in our analysis of income and welfare. We then set out our basic model, first for producers and then consumers. We then use this model to analyze net income, expenditure and welfare, and derive our version of the Weitzman model with gross output-based productivity change. We next turn to the analysis of holding gains and losses within our model, and then to a discussion of depreciation. Finally, we discuss the link of model with the Solow residual, and briefly compare with related work in the literature. A final section sums up.

## **2. The Circular Flow Model of Economic Activity**

Knight's circular flow model of economic activity (CFM) is a schematic device that indicates how the input and output markets of an economy fit together (Patinkin 1973). As depicted in Figure 1, the CFM tracks the flow of commodities and payments between consumers and producers as they pass through markets. Much of standard economic theory rests on this intuitive foundation, as does conventional national income and product accounting theory. The

CFM is thus a useful starting point for a discussion of the appropriate concepts of income, expenditure and production to use in a Weitzman-like analysis of welfare.

[Figure 1 about here]

Figure 1 divides economy activity into two sectors, production and consumption. Production is characterized by a gross output production function for each commodity that links the flow of inputs of labor and capital in Quadrant III to the output flows in Quadrant IV. On the consumer side, behavior is determined by a utility function whose arguments are consumption and leisure. The flows in Quadrant I are determined by the maximization of utility subject to the income of Quadrant II, which is determined by the work-leisure decision. Inputs and output flow clockwise through markets to and from sectors. The flow of payments runs counter clockwise, and a balance prevails in which expenditure, revenue, cost and gross income are equal. In national accounting practice, gross domestic product (GDP) is the flow from Quadrant I to IV, and gross domestic income (GDI) is the flow from Quadrant III to II. The fundamental national accounting identity is then:

$$P_C(t)C(t) + P_I(t)I(t) = W(t)L(t) + P_K(t)K(t) \quad (1)$$

where  $C(t)$  and  $I(t)$  represent consumption and investment at time  $t$  and  $L(t)$  and  $K(t)$  are labor and capital inputs with  $W(t)$  and  $P_K(t)$  as the corresponding prices. GDP is on the left-hand side of this equation and GDI on the right.

Consumer welfare is determined by the utility function, but since the CFM in Figure 1 is only a

snap-shot of the economy at one point in time, more structure needs to be added. The economy unfolds over time as a sequence of annual CFM flows, connected on the consumer side by an intertemporal utility function that encompasses both current and future consumption. The annual CFM's are also linked by the capital accumulation equation

$$\frac{\partial K(t)}{\partial t} \equiv \dot{K}(t) = I(t) - d(t)K(t) \quad (2)$$

The variable  $d(t)$  is the average rate at which the capital stock loses its productive capacity (a process termed “deterioration”). Each new capital good starts off at full productive capacity with an efficiency index  $\phi = 1$ , and end its useful life with  $\phi = 0$  where  $\phi$  is a continuous, declining function of the age of the asset. The variable  $d(t)$  is the average rate at which the  $\phi$ 's decline for the stock as a whole<sup>3</sup> (Feldstein and Rothschild 1974, Jorgenson 1973, Hulten 1990).

The age-efficiency index  $\phi$  also plays a role in linking the asset price<sup>4</sup>  $P_I(t)$  to the future stream of Jorgenson's (1963) user costs  $P_K(t)$ . For an asset of age  $a$ , the former is the discounted present value of the latter, adjusted for the decline in efficiency:

$$P_I(t, a) = \int_t^\infty e^{-\int_t^s r(\tau) d\tau} P_K(s) \phi(a + s - t) ds \quad (3)$$

This equation can be solved for the current period user cost of an asset of age  $a$  to yield:

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<sup>3</sup> The capital stock itself is measured as  $K(t) = \int_{-\infty}^t \phi(t-s)I(s)ds$ . Then,  $\dot{K}(t) = I(t) - d(t)K(t)$  with  $d(t) \equiv \int_{-\infty}^t \frac{\phi(t-s)I(s)}{K(t)} \frac{\phi'(t-s)}{\phi(t-s)} ds$ . Thus,  $d(t)$  is a weighted average of the rate of efficiency decline as the asset ages.

The shares of each vintage in the total stock constitute the weights.

<sup>4</sup>  $P_I(t)$  is the purchase price of a new asset. Correspondingly,  $P_K(t)$  is the user cost per unit of a new or ‘new equivalent’ asset. We make this precision here as the distinction between net income and net product in this section requires a vintage-specific price structure.



$$\phi(a)P_K(t) = P_I(t, a)(r(t) + \delta(t, a) - \hat{P}_I(t)) \quad (4)$$

The variable  $r(t)$  is the nominal rate of return to capital and  $\hat{P}_I(t) \equiv \partial \ln P_I(t, a) / \partial t$  is the rate of change of investment price<sup>5</sup>  $P_I(t)$ , of which more will be said later in the paper. The variable  $\delta(t, a)$  is the average rate at which the *value* of an asset of age  $a$  declines as a consequence of ageing (wear, tear, obsolescence) i.e., the rate of depreciation. The total value of user costs or remuneration of capital is obtained by aggregating across vintages  $a$ :

$$P_K(t)K(t) = \int_0^\infty P_K(t)\phi(a)I(t-a)da = (r(t) + \delta(t) - \hat{P}_I(t))S(t) \quad (5)$$

Vintage accounting requires the introduction of a second capital stock variable,  $S(t) \equiv \int_0^\infty P_I(t, a)I(t-a)da$ , the net or wealth stock that represents the market value in used asset prices  $P_I(t, a)$  of the economy's capital stock. In general, the value of the wealth stock is not equal to the physical stock at 'as new prices',  $P_I(t)K(t)$ . As  $K(t)$  is measured in  $\theta$ -efficiency units,  $P_I(t)K(t)$  represents the cost of replacing the existing collection of investment goods with the equivalent quantity of new capital. In (5),  $\delta(t)$  represents the average rate of depreciation across all vintages.<sup>6</sup>

Having defined depreciation and deterioration in a general way, we can now discuss measures of

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<sup>5</sup> Without much loss of generality, we assume that, as opposed to the *level* of the investment good price, its *rate of change* is independent of the age of the capital good.

<sup>6</sup> Akin to the average rate of deterioration  $d(t)$ , the average rate of depreciation is a weighted average of vintage-specific rates of depreciation with each vintage's share in the total wealth stock as weights:

$$\delta(t) = \int_0^\infty \frac{P_I(t, a)I(t-a)}{S(t)} \delta(t, a) da .$$

net product, net income and net expenditure. Much of the literature in the Weitzman tradition has worked with the concept of net *product* (NP) as the annual indicator of long-run consumer welfare. In the CFM, this role is usually assigned to net *income* (NI) or consumer *expenditure* plus net savings (NE), which is directly associated with the consumers' consumption decisions. The distinction between the two concepts is concealed in national accounting practice, because NP is set to equal NI by deducting depreciation from both gross product and gross income. However, from the standpoint of the CFM, NP and NI are different economic concepts. Net product is part of the product flow in Quadrant IV, while net income is part of the income flow in Quadrant II and net expenditure NE is part of Quadrant I. Net expenditure with its saving component provides the explicit link to intertemporal considerations.

In the framework at hand, NP is defined as value of gross output on the left-hand side of equation (1) net of potential replacement investment  $d(t)P_I(t)K(t)$  needed to compensate for the deterioration of the stock. In view of the accumulation equation (2), NP is

$$NP(t) = P_C(t)C(t) + P_I(t)\dot{K}(t) \quad (6)$$

NI is gross income on the right-hand side of (1) less the depreciation of the value of the stock:

$$NI(t) = W(t)L(t) + P_K(t)K(t) - \delta(t)S(t) = W(t)L(t) + (r(t) - \hat{P}_I(t))S(t) \quad (7)$$

Combining the last three equations yields a relation between the two net concepts:

$$NI(t) = NP(t) + [d(t)P_I(t)K(t) - \delta(t)S(t)] \quad (8)$$

The terms in square brackets (call it  $\Psi(t)$ ) is the consumption loan implicit in the mismatch between the rate at which an asset's productive capacity erodes and the rate at which its value erodes. This equation indicates that a fraction of NP must (in general) be allocated to future

consumption, either as a debit or a credit, in order to reflect under- (over-) funded replacement requirements. In the case where an asset's capacity deteriorates more slowly than its value in the early years of asset life, NP exceeds NI by an amount that must be repaid by the time the asset is retired from service.<sup>7</sup> We further note that current-price NE is always equal to NI and thus differs from the NP expression by the consumption loan term  $\Psi(t)$ .

The literature of growth and welfare has been dominated by the special case of geometric depreciation. In this case, and only in this case, the rate of deterioration,  $d(t)$  equals the rate of depreciation,  $\delta(t)$ , and both are constant over time. The processes of deterioration and depreciation are thus characterized by a single constant,  $\delta$ , and there is only a single stock variable as  $P_I(t)K(t)=S(t)$ . This simplifies the analysis considerably. The age structure of the capital stock in the accumulation and asset pricing equations need not be taken into account, and, moreover, the implicit loan parameter,  $\Psi(t)$ , is uniformly zero. This last condition is particularly helpful in the Weitzman literature, since the need to analyze a second type of asset/liability is avoided. And, as an empirical matter, the available evidence supports the geometric form.<sup>8</sup> For these reasons, we will continue in the tradition of the Weitzman literature and impose this assumption on our analysis. We note, however, that the assumption of geometric depreciation,

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<sup>7</sup> In the "one-hoss shay" form of deterioration, an asset retains its full productive capacity up to the date at which is retired from service (thus, the efficiency index equals one until retirement, at which point it drops to zero). The rate of deterioration  $d(t)$  is therefore zero until the point of retirement. The rate of depreciation  $\delta(t)$  is non-zero, because the value of the assets falls as it approaches retirement. The replacement cost of the asset's capacity remains constant at  $p_I(t)K(t)$  even as the value declines because of approaching retirement. In other words, consumer wealth declines as the price at which the consumer can sell the asset falls, even though the current quantity of capital, as measured by its productive capacity, remains unchanged. In this example, current NP overstates the consumers' true welfare position.

<sup>8</sup> See, for example, Hulten and Wyckoff (1996) and Hulten (2007). We also note that even when the  $\theta_t$  is non-zero, it will tend to be small if investment programs are relatively smooth and growing slowly. A rapidly growing stock increases its importance.

and the resulting equality of NP and NI or NE, encourages the view that NP is, *per se*, the starting point for welfare calculations. This is indirectly true when depreciation is geometric, but not in general. As we shall demonstrate below, neither NP nor NI or NE alone are sufficient statistics for welfare measurement in the presence of autonomous technical change. But we conclude that, because of the existence of the implicit loan  $\Psi(t)$ , NP may be a biased indicator of long-run welfare even in the simplest case of a closed economy and in the absence of technical change. The full analysis of the general case where NP does not equal NI and NE is an area for future research. For the purpose at hand, NI will be our entry point to measuring welfare change.

To this point, the discussion has related to nominal values only. Welfare and productivity measurement have, however, to be based on real values and on volumes. We shall use *real* in the sense that a value has been expressed in equivalents of consumption units. We shall use *volume* to designate the quantity component that, along with a price component, makes up the value of an economic transaction. Volumes are in particular relevant in the computation of productivity, the ratio of the volume of outputs to the volume of inputs. The theory of productivity measurement is based on production functions or production possibility frontiers as in Jorgenson (1966). Much of the recent literature in the Weitzman tradition does work directly with a production function based on net, not gross, output (for instance Sefton and Weale 2006). Technical change is then defined in terms of the shift in the net function. This approach puts welfare considerations directly into the structure of production, thereby ‘folding’ the income and expenditure side of the CFM over its product side. There is no mathematical objection to this approach, but our critique about the lack of generality of NP raises questions about its relevance

for analysis of growth and wealth. Also, it would appear more difficult to interpret technical change in the case of a net production function than in the more familiar case of a gross production function in part because it is hard to think of inventions that lead to more output net of replacement investment without first having led to more gross output itself<sup>9</sup>. Thus, if the Weitzman welfare result can be derived within the standard gross-output framework in which technical change augments a gross production function, the Weitzman result is put on a more conventional footing with standard growth theory. Also, by keeping the distinction between real NI and the volumes of outputs and inputs, the theoretical discussion matches up with the CFM and with the sequence of accounts as they appear in statistical practice<sup>10</sup>. The following sections show how this can be accomplished and how measures of real NI, augmented by measures of gross-output based productivity change track movements in welfare.

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<sup>9</sup> There are other issues. Sectoring the aggregate flow of NP into its industrial and firm components is problematic (how would for example a net input-net output table be constructed?).

<sup>10</sup> Jorgenson (2009) shows how a national accounting framework relates to productivity measurement and welfare-related indicators. Jorgenson focuses on NE as the key indicator to link up to welfare considerations. This is broadly consistent with our own approach that emphasizes the role of NI as a welfare indicator. Quadrant II NI equals Quadrant I NE, and the two measures reveal different aspects of the welfare problem. Since our focus is on the literature inspired by Weitzman and the problem of holding gains, we examine the role played by NI.

### 3. Producers

The economy under consideration produces a consumption good  $C$  and an investment good  $I$  by way of capital<sup>11</sup> and labor inputs  $K$  and  $L$ . Denote the feasible combinations of outputs and inputs, the economy's production possibilities set, by  $T(t)$ . Then, a combination of outputs can be produced with a given set of inputs<sup>12</sup> if  $(C, I, K, L) \in T$ . We assume that the technology is subject to constant returns to scale, implying that the value of outputs exactly equals the remuneration of inputs for each period. The time dependence of  $T(\cdot)$  allows for non-stationary technology and more will be said later about the nature of the technical change at hand. We use a nominal GDP function<sup>13</sup>:

$$G(P_C, P_I, K, L, t) \equiv \max_{C, I} \{P_C C + P_I I : (C, I, K, L) \in T\} \quad (9)$$

Thus,  $G$  is the maximum value of final goods (GDP) that the economy can produce given output prices  $P_C$  and  $P_I$ , a set of primary inputs  $K, L$  and technology  $T$ .  $G$  integrates the fact that the output of consumer and investment goods has been chosen in a revenue-maximising way. We note here that this formulation is a well-established representation of technology in the productivity literature. The output of investment goods is measured as gross investment and the accumulation constraint in the economy was given in (2).

Assuming differentiability of the GDP function, and revenue-maximizing behavior of producers,

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<sup>11</sup> The capital stock  $K$  figures as an element in the production possibility frontier but the actual capital input into production is a flow of capital services that we take to be in fixed proportion to the capital stock.

<sup>12</sup> In what follows, we drop explicit indications of time periods with variables unless there is danger of ambiguity.

<sup>13</sup>  $G$  is known as the national product function in the international trade literature (see Kohli (1978)). It was introduced into the economics literature by Samuelson (1953). Properties of  $G$  are linear homogeneity and convexity in  $P_C$  and  $P_I$  and linear homogeneity and concavity in  $K$  and  $L$ ; see Diewert (1974).

the quantity of outputs will be equal to the marginal change in nominal output with respect to output prices:

$$\frac{\partial G}{\partial P_c} = C; \frac{\partial G}{\partial P_l} = I \quad (10)$$

Also, the partial derivatives of the GDP function with respect to capital and labor inputs will equal their respective prices, given competitive factor markets<sup>14</sup>:

$$\frac{\partial G}{\partial K} = P_K; \frac{\partial G}{\partial L} = W \quad (11)$$

Thus, the value of the marginal product of capital input  $\partial G/\partial K$  equals the price of capital services  $P_K$ . The nominal rate of return corresponds to the discount rate that producers would apply in an inter-temporal framework.  $r$  is time-varying and in our set-up equals the market interest rate. Intuitively speaking, the three components – required rate of return, depreciation charges and expected asset price changes – are the cost elements that an owner of a capital good would add up when fixing a rental rate for a capital good under competitive conditions. Similar to capital, the value of the marginal product of labor  $\partial G/\partial L$  equals the nominal wage rate  $W$  under competitive conditions.

The GDP function is also a convenient tool to measure productivity change. More specifically, we define productivity change as a shift of the GDP function over time.<sup>15</sup> The rate of productivity change  $\lambda$  is treated as an exogenous, time-dependent variable:

$$\frac{\partial G}{\partial t} / G \equiv \lambda \quad (12)$$

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<sup>14</sup> Diewert (1974).

<sup>15</sup> This is a continuous-time version of Diewert and Morrison's (1986) discrete measure of productivity change.

To lend more structure to the productivity variable (12), we differentiate the GDP function totally and express it in terms of growth rates:

$$dG = \frac{\partial G}{\partial P_C} dP_C + \frac{\partial G}{\partial P_I} dP_I + \frac{\partial G}{\partial K} dK + \frac{\partial G}{\partial L} dL + \frac{\partial G}{\partial t} dt \quad (13)$$

$$\hat{G} - \frac{P_C C}{G} \hat{P}_C - \frac{P_I I}{G} \hat{P}_I = \frac{P_K K}{G} \hat{K} + \frac{WL}{G} \hat{L} + \frac{\partial G}{\partial t} / G$$

By definition, nominal GDP is the value of final products, i.e.,  $G = P_C C + P_I I$ , so it must also be the case that

$$\dot{G} - C\dot{P}_C - I\dot{P}_I = P_C \dot{C} + P_I \dot{I} \quad (14)$$

Combine (13) and (14):

$$\frac{P_C C}{G} \hat{C} + \frac{P_I I}{G} \hat{I} = \frac{P_K K}{G} \hat{K} + \frac{WL}{G} \hat{L} + \frac{\partial G}{\partial t} / G \quad (15)$$

The left hand side in the second line of (13) is the difference between nominal GDP growth and the growth rate of a Divisia output price index, the share-weighted rate of change of consumption and investment good prices. This difference is thus deflated or volume GDP. From (15) it is apparent that this equals a share-weighted volume measure of production of the consumption and of the investment good. On the right hand side of (13) we find a volume index of labor and capital inputs and our measure of productivity change. Consequently, (15) is a standard growth accounting equation where the change in volume output is either attributed to variations in inputs or to variations in residual productivity.

A final step in our treatment of the producer side is to bring the first order conditions for producers in line with the equations relating to consumers later on. We express the former in real



terms, i.e., in units of the consumption good. For this purpose, lower case letters mark variables that have been divided by the price of the consumer good:  $p_I \equiv P_I/P_C$ ,  $w \equiv W/P_C$ . Also, we use  $r^* \equiv r - \hat{P}_C$  to designate a real rate of return. Then, the real rental price of capital and real wages are

$$\begin{aligned} (\partial G / \partial K) / P_C &= P_I(r - \hat{P}_C + \delta - \hat{P}_I + \hat{P}_C) / P_C \\ &= p_I(r^* + \delta - \hat{p}_I) \\ &\equiv p_K \\ (\partial G / \partial L) / P_C &= w \end{aligned} \tag{16}$$

Having set the stage on the supply side, we now turn to consumers.

#### 4. Households

Consumers, it is assumed, maximize the present value of discounted utility, which depends positively on consumption. As we are considering an open economy setting, it is necessary to distinguish between household consumption of the domestically-produced consumer good which shall be labeled  $C_D$ , and household consumption of an imported consumer good  $C_M$ , purchased at the price  $P_M$ . This price is expressed in domestic currency and exogenously determined. Utility depends negatively on the supply of labor  $L$ . We also take  $L$  as an exogenously determined variable and assume that wages fully adjust to inelastically supplied labor. Then, period-to-period utility<sup>16</sup> is presented by  $U[C_D, C_M, L]$ . Households choose their consumption pattern by

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<sup>16</sup> We use a utility function to represent the household's welfare maximization problem. To actually measure utility and its change over time, more structure is needed in particular to value the marginal utility of consumption which determines the factor of proportionality between welfare change and the enhanced net income measure. In particular, utility is an ordinal measure but quantifying its change requires cardinalization. One way of dealing this is by using a money metric scaling approach. Thereby, an expenditure function is used instead of the direct utility function and actual expenditures are equated with the reference price vector and a reference utility level in an initial equilibrium. Welfare changes can then be quantified relative to this point of reference. See for instance, Varian (1982).

maximizing the discounted value of future consumption flows. A constant rate  $\theta$  reflects households' time preferences:

$$V(t) = \int_t^{\infty} U[C_D(s), C_M(s), L(s)] e^{-\theta(s-t)} ds \quad (17)$$

In addition to wage income  $WL$ , households earn capital income in the form of interest payments on capital that they hold in the production sector. Interest payments amount to  $rP_I K$  where  $r$  has already been defined as the nominal market interest rate,  $P_I$  as the nominal price of capital goods and  $K$  as the net capital stock. The sum of labor and interest income minus consumer expenditure on domestically produced goods  $P_C C_D$  and on imported goods  $P_M C_M$  corresponds to net saving or the change in the value of capital held by households. The latter is gross investment minus depreciation plus revaluation of the capital goods. The household's budget constraint is then

$$P_I \dot{I} - \delta P_I K + \dot{P}_I K = WL + rP_I K - P_C C_D - P_M C_M \quad (18)$$

Several observations can be made with regard to this budget constraint. We start out by assuming that in the open economy at hand, trade is balanced so that the value of exported domestically-produced consumer goods  $P_C(C - C_D)$  just equals the value of imported goods for consumption:

$$P_M C_M = P_C(C - C_D) \quad (19)$$

When (18) and (19) are combined and rearranged, one obtains equation (1) that links domestic output  $P_C C + P_I I$  to gross domestic income<sup>17</sup>  $WL + P_K \dot{K}$ . Under geometric rates of depreciation and deterioration, net expenditure and net income in expression (7) simplify to

$$P_C C + P_I \dot{I} = WL + P_K \dot{K} - \delta P_I K \quad (20)$$

For what follows, (20) will be expressed in units of the domestically produced consumption

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<sup>17</sup> Implicit in this formulation is the assumption about constant returns to scale. It was assumed that the GDP function is linear homogenous in  $P_C$  and  $P_I$  so that Euler's theorem gives us  $G = (\partial G / \partial P_C) P_C + (\partial G / \partial P_I) P_I = P_C C + P_I I$ . Further, linear homogeneity of  $G$  in  $K$  and  $L$  implies  $G = (\partial G / \partial K) K + (\partial G / \partial L) L = P_K \dot{K} + WL$ .

good. With the same lower case notation as before, we obtain real NI

$$C + p_I \dot{K} = wL + (r^* - \hat{p}_I) p_I K. \quad (21)$$

Another observation concerning the budget constraint is that its integration gives rise to an interpretation in terms of stocks: the value of human capital, measured as the discounted flow of wage income plus the present stock of physical capital equals the discounted flow of the value of future consumption, or nominal consumption wealth:

$$p_I K + \int_t^\infty wL e^{-\int_t^s r^* d\tau} ds = \int_t^\infty (C_D + p_M C_M) e^{-\int_t^s r^* d\tau} ds \quad (22)$$

or

$$p_I K + HC = CW, \text{ where } HC \equiv \int_t^\infty wL e^{-\int_t^s r^* d\tau} ds \text{ and } CW = \int_t^\infty (C_D + p_M C_M) e^{-\int_t^s r^* d\tau} ds \quad (23)$$

The equations above define a simple but complete accounting system. The GDP function in (9) is essentially a production account. (21) to (23) provide the links between income generated in production, expenditure and wealth accounts.

Turning to household behavior on an optimal consumption path, first-order conditions are given by applying the Euler equation for (17), subject to (21):

$$\frac{d}{dt} \left\{ \frac{\partial U}{\partial C_D}(t) \right\} = \frac{\partial U}{\partial C_D}(t) [\theta - r^*] \quad (24)$$

$$-\frac{\partial U}{\partial L} = \frac{\partial U}{\partial C_D} w; \quad \frac{\partial U}{\partial C_D} p_M = \frac{\partial U}{\partial C_M}$$

The first expression (for a derivation see Annex A) indicates that on an optimal path, the percentage change in utility from a change in consumption is equal to the difference between the household's time preference rate and the real interest rate prevailing on financial markets. If the

former exceeds the latter, households will reduce consumption, save more and the marginal utility of consumption will rise. If real interest rates are below the time preference rate of households, the inverse happens.

The second expression states that wages adjust to the exogenously given labor such that the marginal disutility from work equals the marginal utility from extra consumption due to an additional unit of real wages.

The third expression indicates that consumers' choice between the domestically produced and the imported consumption good will be such that the marginal rate of substitution between the two products equals their relative prices.

## **5. Net expenditure and welfare**

This section will link our measures of net income to welfare, expanding on Weitzman (1976, 1997), Nordhaus (1995) and Sefton and Weale (2006). The right-hand side of equation (21) is real NE, the left-hand side is real net income which we shall label  $N$ . Under the assumption of geometric depreciation,  $N$  also equals real NP. As was shown in Section 2, the equality does not hold under a more general set-up.

$$N \equiv wL + (r^* - \hat{p}_1)p_1K = C + p_1\dot{K} \quad (25)$$

We can now link real NI to an optimal path of consumption and investment, by applying Weitzman's (1976, 1997) method in a situation where technology is non-stationary. This yields the following proposition whose derivation can be found in Annex B: on an optimal path, real

net domestic income *plus the discounted effects Z of technical change plus the discounted effects Q of changes in labor supply* are equal to a weighted average of present and future flows of consumption:

$$N + Z + Q = \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds \quad (26)$$

where  $Z \equiv \int_t^\infty (G_s / P_C) e^{-\int_t^s r^* d\tau} ds$  with  $\frac{\partial G}{\partial s} \equiv G_s$  and  $Q \equiv \int_t^\infty w L e^{-\int_t^s r^* d\tau} ds$ .

With stationary technology and fixed labor supply ( $Z=Q=0$ ), this reduces to one of Sefton and Weale's (2006) propositions. If, in addition, the real interest rate is constant, (26) leads to Weitzman's (1976) original result, namely that  $N$  is the return to consumption wealth. If the real interest rate is constant and equal to households' time preference rate, but allowing for autonomous technical change, expression (26) corresponds to Weitzman's (1997) result with technical change. Equation (26) shows that in the presence of technical change and variable labor supply, real net expenditure cannot be directly interpreted as a weighted average of current and future consumption, a point first made by Nordhaus (1995).

The above proposition provides a relationship between net income and consumption flows but not yet a link between  $N$  and consumer welfare. This link is established next. To do so, we follow the method described in Sefton and Weale (2006) and examine the change in discounted utility over time by differentiating discounted consumer utility  $V(t)$  with respect to time. This leads to the following result whose detailed derivation can be found in Annex C: *on an optimal path, welfare change is proportional to real savings  $p_1 \dot{K}$  minus the discounted effects of changes in the terms of trade,  $T$ , plus the discounted effects  $Z$  of technical change:*

$$\frac{dV}{dt} = \frac{\partial U}{\partial C_D} \{N - C - T + Z\} = \frac{\partial U}{\partial C_D} \{p_I \dot{K} - T + Z\}, \quad (27)$$

where  $T \equiv \int_t^\infty \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds$  and

$$Z \equiv \int_t^\infty (G_S / P_C) e^{-\int_t^s r^* d\tau} ds$$

Again, for  $Z=0$  and  $T=0$ , this result reduces to Sefton and Weale's (2006) proposition. Our model has been set in an open economy context with autonomous technical change, and it is intuitively plausible that consumer welfare will increase when there is more rapid technical change and/or when the relative price of imported consumer goods  $p_M$  declines, i.e., when the terms of trade improve.

We note that (27) is only intended to establish proportionality between welfare changes and the term  $N-C-T+Z$ . The fact that utility is only defined up to a monotonic transformation of  $U$  is consistent with a statement about proportionality and no ordinal utility measure is required.

Sefton and Weale (2006) define Hicksian income as the maximum amount a household can consume while keeping discounted utility constant.  $dV/dt$  will equal zero when consumption equals  $N+Z-T$ , i.e., real net expenditure adjusted for the effects of technical change and terms of trade. Thus, when a household spends  $N+Z-T$  on consumption, it is just as well off at the end as at the beginning of the period. The welfare-relevant measure is therefore what could be termed *Hicksian income (HI)*, or  $HI=N+Z-T$ . HI can be measured empirically.

## 6. Holding gains and losses

A longstanding issue in the debate about the measurement of income has been whether (expected) holding gains and losses from capital or from natural resources should be considered part of income (Eisner 1988). Sefton and Weale (2006) conclude that within their model which measures real income as the weighted discounted sum of future consumption there is no room to include capital gains in income as long as the economy is closed. Our framework confirms their conclusion concerning income but we show that (expected) holding gains and losses *are* part of the *welfare* measure. This arises because in a general equilibrium setting, expected holding gains and losses enter the welfare measure via the technical change effect that needs to be added to net income to obtain Hicksian income. This conclusion is based on the following relation (Annex D):

$$\dot{p}_I K = Z - \int_t^\infty \dot{r} * p_I K e^{-\int_t^s r^* d\tau} ds - \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds \quad (28)$$

Expression (28) states that on an optimal growth path, real holding gains on physical assets (the left-hand side expression) equal the expected discounted effects of technical change (first term on the right-hand side) minus the expected discounted effects of real interest changes and real wage changes (second and third terms on the right-hand side). In a world without autonomous technical change ( $Z=0$ ), real holding gains on capital can only arise if discounted real wages or interest rates fall or vice versa: expected real wage rises or real interest rate rises will lead to holding losses on capital. In such a zero-sum situation, there is no room to augment real net domestic product by a measure of holding gains. This is essentially Sefton and Weale's (2006) conclusion on the treatment of holding gains in a closed economy in the absence of technical

change. This is no longer true in a situation with autonomous technical change. From (28) it is apparent that  $Z$  corresponds to holding gains/losses plus the discounted value of expected factor price changes. Thus, in the presence of technical change, long-run holding gains on assets should be part of the income measure.

A slightly different interpretation of (28) is that real holding gains (or losses) are the results of two opposing forces: technical advance that shifts outward the production possibility frontier and so raises the present value of productive capital, and expected changes in real wages and interest rates. If the latter are expected to rise, this will reduce the present value of capital. In the simplest case of static expectations about real interest rates and real wages, capital gains will exactly match the discounted effects of technical change. Since we have already shown that  $Z$  is an element in Hicksian income, it follows that expected holding gains or losses are implicitly captured by  $Z$ . There is thus no need to add them on to the expression for Hicksian income, and doing so would involve double-counting. It is an open question whether one estimates  $Z$  – say on the basis of long-run trends in multi-factor productivity- or prefers to estimate trends in holding gains and losses plus factor prices and uses the result in the HI computations. However, only one of the two adjustments is warranted, not both.

## **7. Depreciation and Technical Change**

The discussion on holding gains and losses sheds also light on the definition of economic depreciation. The main issue<sup>18</sup> that has arisen recently is whether depreciation measures should

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<sup>18</sup> The question how to define and measure depreciation has recently been discussed in conjunction with the revision



be limited to the conventional idea<sup>19</sup> that depreciation should be measured as the erosion in asset value due to age or whether the definition should be expanded to include expected losses (and gains) on holding the asset. The question of how to define depreciation is ultimately a question about the nature of net income and net expenditure<sup>20</sup>. If tracking welfare is the objective, how should depreciation be defined? In equation (28) above, the welfare-based expression for Hicksian income  $HI=N+Z-T$ , uses a net measure of income  $N$  and depreciation that is not adjusted for holding gains and losses. However, we have demonstrated that holding gains and losses enter via  $Z$ . Also, the terms of trade effect  $T$  is nothing but a real holding gain or loss, albeit measured with regard to international trade.  $Z-T$  could be measured directly, or they could be measured as the expected rate of asset price changes, making use of (28). In the latter case,  $HI$  amounts to a conventional net measure plus a correction for real holding gains and losses. This is most obvious in the simplified case where  $T=0$  and  $Z = \dot{p}_I K$ . Then,  $HI$  reads as  $HI = N + \dot{p}_I K$ . Remembering that  $N$  equals gross expenditure or product  $(C + p_I I)$  minus conventionally-defined depreciation  $\delta p_I K$ , we get

$$\begin{aligned}
 HI &= N + \dot{p}_I K \\
 &= C + p_I I - \delta p_I K + \dot{p}_I K \\
 &= C + p_I I - (\delta p_I K - \dot{p}_I K)
 \end{aligned} \tag{29}$$

The term in brackets is the augmented measure for depreciation, i.e., conventional depreciation corrected for price changes. In this sense, the result confirms the welfare-relevance of the augmented expression for depreciation in the presence of productivity growth. However, the

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of the 1993 System of National Accounts (Hill and Hill 2003, Diewert 2005, Schreyer 2009, Diewert and Wykoff 2006).

<sup>19</sup> See Jorgenson (1996), Hulten (1990) and Triplett (1996) for a discussion of the traditional measure of depreciation.

<sup>20</sup> Diewert (2006) traces this discussion about net income back to a debate between Pigou (1924, 1941), Clark (1940) and Hayek (1941).

analysis would not carry through had net expenditure  $N$  been defined with the augmented depreciation measure in the first place. Also, in the case where there is no expected technical change, no adjustment of net measures should be made for price changes. The conventional expression  $\delta p_t K$  would thus appear to be the preferred way to go about defining and measuring depreciation.

The conventional expression for depreciation also fits intuitively with the present framework where welfare is defined in terms of discounted flows of consumer goods: depreciation is central to obtain a measure of net expenditure i.e., whatever can be spent in a period while keeping wealth intact. With consumption in the utility function, ‘keeping wealth intact’ means keeping a capital stock intact that ensures the flow of consumption. If capital goods become cheaper as a consequence of technical change, fewer resources need to be put aside to ensure the productive capacity of the capital stock, even if the consequence is that wealth in consumption units declines due to declining relative prices of the capital good.

This does not invalidate the economic rationale of accounting for holding gains and losses (or technical change - see discussion in Section 3) when deriving welfare-relevant income measures. However, it is preferable not to make this adjustment while *defining* depreciation and to keep revaluation and depreciation as separate effects.

## **8. Technical change and the Solow residual**

In the discussion of the supply side of our model, productivity change was defined as a shift in

the GDP function over time. In expressions (13) and (15) it was demonstrated that this formulation leads directly to the standard formulation of the Solow residual, measured as the difference between a Divisia index of the volume of outputs and a Divisia index of the volume of inputs:

$$\lambda \equiv G_t / G = \frac{P_C C}{G} \hat{C} + \frac{P_I I}{G} \hat{I} - \left( \frac{P_K K}{G} \hat{K} + \frac{W L}{G} \hat{L} \right) \quad (30)$$

The link between multi-factor productivity and the change in consumer welfare can now be made explicit. The discounted effects of technical change are captured by the term

$$\begin{aligned} Z &= \int_t^{\infty} (G_s / P_C) e^{-\int_t^s r^* d\tau} ds \\ &= \int_t^{\infty} (G / P_C) (G_s / G) e^{-\int_t^s r^* d\tau} ds \\ &= \int_t^{\infty} (G / C P_C) C \lambda e^{-\int_t^s r^* d\tau} ds \\ &= \int_t^{\infty} C \lambda / s_C e^{-\int_t^s r^* d\tau} ds \end{aligned} \quad (31)$$

The last line in (31) merits special attention. Consider first the expression  $\lambda/s_C$  where  $s_C \equiv (P_C C/G)$  is the share of consumption in GDP. Productivity change  $\lambda$  operates on the GDP function as a whole, raising the output of the domestic production of consumer goods and of investment goods for a given set of primary inputs. Note that if production technology had not been formulated as a GDP function but as a consumption-augmenting production possibility curve<sup>21</sup>, as for example in Weitzman (1976), and if productivity change for such a consumption-augmenting technology were labeled  $\mu$ , one would obtain  $\mu = \lambda/s_C$ .

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<sup>21</sup> Technology would then presented by a function  $C=F(I, K, L, t)$  where  $F$  would be decreasing in  $I$ , and increasing in  $K$  and  $L$ . Productivity change would be measured as  $\mu = [\delta F / \delta t] / F$ .

This lends additional structure to our result. Assume constancy of the expected rate of consumption-enhancing technical change  $\mu=\lambda/s_C$ . Then,  $Z$  gives rise to another useful presentation, namely as the expected rate of (consumption augmenting) productivity change applied to consumption wealth:

$$\begin{aligned} Z &= \int_t^{\infty} C\lambda / s_C e^{-\int_t^s r^* d\tau} ds \\ &= \mu \int_t^{\infty} C e^{-\int_t^s r^* d\tau} ds \\ &= \mu CW \end{aligned} \tag{32}$$

Consumption wealth,  $CW$ , was defined as the discounted flow of future consumption in the inter-temporal budget constraint (22). Equations (26) and (31) imply that technical change enhances consumption wealth  $CW$ . And it is clear from equation (30) that  $\lambda$  is the standard Solow residual derived from the gross output production function. This result reinforces one of the main points of this paper: both gross and net measures of economic activity are important statistics of an economy. They are complements, not substitutes, and they inform about different aspects of economic growth. Intuitively, the rate of technical change  $\lambda$  measures the change in the constraint that limits current and future consumption (consumption and gross investment), while (26) links the benefit of a shift in the constraint to welfare.

Proceeding as in (31), the discounted terms-of-trade effects can be presented as the expected rate of real price change of imported goods,  $\xi$ , multiplied by consumption wealth from imported goods,  $CW_M$ .

$$\begin{aligned}
T &= \int_t^{\infty} \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds \\
&= \int_t^{\infty} (\dot{p}_M / p_M) p_M C_M e^{-\int_t^s r^* d\tau} ds \\
&= \xi \int_t^{\infty} p_M C_M e^{-\int_t^s r^* d\tau} ds \\
&= \xi CW_M
\end{aligned} \tag{33}$$

We can now re-state Hicksian income and welfare changes from equation (27) in the following way:

$$\begin{aligned}
\frac{dV}{dt} &\propto HE - C = p_I \dot{K} + Z - T \\
&= p_I \dot{K} + CW\mu - \xi CW_M \\
&= CW \left( \frac{p_I K}{CW} \hat{K} + \mu - \frac{CW_M}{CW} \xi \right)
\end{aligned} \tag{34}$$

The first line re-states the earlier result: welfare change is proportional to the difference between Hicksian income and present consumption, itself equal to net saving plus the discounted effects of productivity change and minus the discounted effects of future changes in the terms-of-trade. The third line shows that welfare change is also proportional to a ‘return’ on consumption wealth  $CW$ . As  $CW$  is the sum of human capital and physical capital, it can also be considered as an expression of total wealth. The ‘rate of return’ on  $CW$  has three components: the rate of net investment, weighted by the share of physical capital in consumption wealth; the rate of technical changes in the form of a scaled-up Solow residual and the rate of change of real import prices weighted by the share of consumption wealth from imported goods in total consumption wealth.

## 9. Links to Other Approaches

The proportionality of welfare change and multi-factor productivity change is actually a familiar result in a different form. Basu and Fernald (2002) establish that the change in utility is proportional to the change in the Solow residual, in an optimizing framework. Their conclusion differs from ours, however, because neither real savings nor terms of trade effects figure in their result. They reason that “welfare benefit is proportional to productivity growth, not to output growth since the consumer subtracts the welfare cost of supplying any extra capital and labor” (page 971). The last point is not supported by our model here. Net savings (not output) *do* enter as a determinant of welfare change in our model because they contribute to future productive capacity of the economy and are not exactly offset by present consumption foregone. The results coincide in a situation of steady growth when there is no net capital formation, and when there are no secular terms of trade effects. Then welfare growth is exclusively driven by productivity growth<sup>22</sup>.

We also note that under certain assumptions, the parameter  $\mu$  can be interpreted as the Harrodian rate of technical change. In the neoclassical steady-state growth model of Solow (1956), all variables, including consumption per capita, grow at the rate  $\mu$ . Moreover, the Harrodian rate  $\mu$  is a special case of the *dynamic residual* of Hulten (1979), which is itself the weight sum of the annual gross-output Solow residuals (the  $\lambda$  above). In this framework, capital is treated as an intertemporal intermediate good, and the dynamic residual is defined as the shift in the intertemporal consumption possibility (the intertemporal counterpart of the standard production

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<sup>22</sup> See also Diewert’s (2007) discussion of the Basu and Fernald results.

possibility frontier), measured along the optimal utility path. These results also tie gross-output technical change to welfare maximization.

## **10. Conclusion**

Our paper has ranged over several related issues, and the main results can be summarized as follows: First, in the presence of disembodied technical change and in an open economy, real net savings are no longer a sufficient gauge of movements in economic welfare. Productivity and terms of trade effects need to be taken into account, and we provide an explicit decomposition of consumption-welfare and show how the separate effects can be measured. However, real net income remains a necessary ingredient to gauge changes in welfare.

Second, a measure of net income that has been augmented by the effects of technical change leaves no room for a supplementary, explicit term to capture real holding gains or losses on domestic assets. This is a result of the general equilibrium setting and any supplementary recognition of asset price changes in deriving welfare change would result in double counting. But the fact that expected holding gains do enter the Hicksian definition of income via technical change implies that these gains need to be added to net income in the absence of a term that captures anticipated productivity change. This effect is sometimes directly incorporated into the depreciation measure but we conclude that the effects of ageing and the effects of price changes should be kept separate because holding gains or losses should only be recognized in the presence of long-run productivity growth.

Third, we emphasize that both GDP and the appropriate measures of net income and expenditure are important statistics of actual economies as they evolve over time. They are complements, not substitutes, and they provide information about the supply, income and expenditure sides of the standard circular flow representation of an economy. Indeed, our model shows that elements from both the supply and the income side are needed to inform about welfare developments. The fact that depreciation is typically (and usefully) modeled as a constant rate of value loss blurs the distinction between net product, and net income and expenditure because in this special case they coincide in value terms. We show that the welfare perspective generates no specific requirement to formulate technology change as the shift of a net product function so that a gross output productivity measure can be used to augment real net expenditure as a welfare indicator. These conclusions provide a theoretical underpinning for the structure of the system of national accounts with its distinction between production, income and expenditure and bridge welfare aspects to a large body of growth accounting literature.



## Annex A: First-order conditions for a household optimum

The Euler equations defining the first order conditions for a consumer optimum are

$$e^{-\theta(s-t)} \frac{\partial U}{\partial K} - \frac{d}{ds} \left\{ e^{-\theta(s-t)} \frac{\partial U}{\partial \dot{K}} \right\} = 0 \quad (\text{A.1})$$

The budget constraint for households is

$$C_D = wL + (r^* - \hat{p}_I)p_I K - p_I \dot{K} - p_M C_M. \quad (\text{A.2})$$

Then,  $\frac{\partial U}{\partial K} = \frac{\partial U}{\partial C_D} \frac{\partial C_D}{\partial K} = \frac{\partial U}{\partial C_D} (r^* - \hat{p}_I)p_I$  and

$$\frac{\partial U}{\partial \dot{K}} = \frac{\partial U}{\partial C_D} \frac{\partial C_D}{\partial \dot{K}} = \frac{\partial U}{\partial C_D} (-p_I) \text{ so that (A.1) becomes}$$

$$\begin{aligned} e^{-\theta(s-t)} \frac{\partial U}{\partial C_D} (r^* - \hat{p}_I)p_I &= \theta e^{-\theta(s-t)} \frac{\partial U}{\partial C_D} p_I - e^{-\theta(s-t)} \frac{d}{ds} \left\{ \frac{\partial U}{\partial C_D} p_I \right\} \\ e^{-\theta(s-t)} \frac{\partial U}{\partial C_D} (r^* - \hat{p}_I)p_I &= \theta e^{-\theta(s-t)} \frac{\partial U}{\partial C_D} p_I - e^{-\theta(s-t)} \left\{ \frac{\partial^2 U}{\partial C_D^2} \dot{C}_D p_I + \frac{\partial^2 U}{\partial C_D \partial L} \dot{L} p_I + \frac{\partial U}{\partial C_D} \dot{p}_I \right\} \\ \frac{\partial U}{\partial C_D} [r^* - \hat{p}_I - \theta - \dot{\hat{p}}_I] p_I &= - \frac{\partial^2 U}{\partial C_D^2} \dot{C}_D p_I - \frac{\partial^2 U}{\partial C_D \partial L} \dot{L} p_I \\ \frac{d}{ds} \left\{ \frac{\partial U}{\partial C_D} \right\} &= \frac{\partial U}{\partial C_D} [\theta - r^*] \end{aligned} \quad (\text{A.3})$$

Expressed in terms of time  $t$ , one has

$$\frac{d}{dt} \frac{\partial U}{\partial C_D}(t) = \frac{\partial U}{\partial C_D}(t) [\theta - r^*].$$

Also,

$$\begin{aligned} \frac{dU}{dC_M} &= \frac{\partial U}{\partial C_D} \frac{\partial C_D}{\partial C_M} + \frac{\partial U}{\partial C_M} = - \frac{\partial U}{\partial C_D} p_M + \frac{\partial U}{\partial C_M} = 0 \text{ and therefore} \\ \frac{\partial U}{\partial C_D} p_M &= \frac{\partial U}{\partial C_M} \end{aligned} \quad (\text{A.4})$$

$$\frac{dU}{dL} = \frac{\partial U}{\partial C} \frac{\partial C}{\partial L} + \frac{\partial U}{\partial L} = \frac{\partial U}{\partial C} w + \frac{\partial U}{\partial L} = 0 \quad (\text{A.5})$$

## Annex B: Net income and welfare

To establish the link between net income and consumer welfare, we follow Weitzman (1976, 1997) and Sefton and Weale (2006). Recall the growth accounting equation (15) which we repeat for convenience:

$$\frac{P_C C}{G} \hat{C} + \frac{P_I I}{G} \hat{I} = \frac{P_K K}{G} \hat{K} + \frac{W L}{G} \hat{L} + \frac{\partial G}{\partial t} / G \quad \text{or} \quad (\text{B.1})$$

$$P_C \dot{C} + P_I \dot{I} = P_K \dot{K} + W \dot{L} + \frac{\partial G}{\partial t} \quad \text{after multiplication by } G \text{ and in real terms} \quad (\text{B.2})$$

$$\dot{C} + p_I \dot{I} = p_K \dot{K} + w \dot{L} + G_t / P_C \quad \text{after division by } P_C \text{ and using } \frac{\partial G}{\partial t} \equiv G_t \quad (\text{B.3})$$

Because a constant rate of depreciation has assumed, NI equals NP equals NE and in real terms one gets

$$N = C + p_I (I - \delta K) = C + p_I \dot{K} \quad (\text{B.4})$$

Its change over time is

$$\dot{N} = \dot{C} + \dot{p}_I \dot{K} + p_I \ddot{K}. \quad (\text{B.5})$$

Inserting (B.3) into (B.5) gives

$$\begin{aligned} \dot{N} &= (p_I r^* - \dot{p}_I) \dot{K} - p_I \ddot{K} + w \dot{L} + G_t / P_C + \dot{p}_I \dot{K} + p_I \ddot{K} \\ &= p_I r^* \dot{K} + w \dot{L} + G_t / P_C \\ &= r^* (N - C) + w \dot{L} + G_t / P_C \end{aligned} \quad (\text{B.6})$$

The last line of (B.6) is a first order differential equation in N whose solution is

$$\begin{aligned} N &= \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds - \int_t^\infty w \dot{L} e^{-\int_t^s r^* d\tau} ds - \int_t^\infty (G_s / P_C) e^{-\int_t^s r^* d\tau} ds \\ &= \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds - \int_t^\infty w \dot{L} e^{-\int_t^s r^* d\tau} ds - Z \end{aligned} \quad (\text{B.7})$$

Here  $Q \equiv \int_t^\infty w \dot{L} e^{-\int_t^s r^* d\tau} ds$  and  $Z \equiv \int_t^\infty (G_s / P_C) e^{-\int_t^s r^* d\tau} ds$  have been used as shorthand for the discounted effects of future changes in employment and future effects of technical change. (B.7)

informs us that real net income plus the effects of a changing labor force and technology,  $Q$  and  $Z$ , equals the real return to consumption wealth, or  $\int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds$ .

### Annex C: Welfare changes

To derive the relation shown in equation (27), differentiate  $V$  with respect to time:

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt} \left\{ \int_t^\infty U(C_D, C_M, L) e^{-\theta(s-t)} ds \right\} \\ &= -U(t) + \int_t^\infty \theta U(s) e^{-\theta(s-t)} ds \\ &= -U(t) + U(t) + \int_t^\infty \frac{dU}{ds} e^{-\theta(s-t)} ds \\ &= \int_t^\infty \frac{dU}{ds} e^{-\theta(s-t)} ds \end{aligned} \tag{C.1}$$

where  $U(t)$  has been used as shorthand for  $U[C_D(t), C_M(t), L(t)]$ . Thus, present welfare change corresponds to the discounted effects of future changes in consumption and labor on utility.

Continuing from (C.1), one gets

$$\begin{aligned} \frac{dV}{dt} &= \int_t^\infty \frac{dU(s)}{ds} e^{-\theta(s-t)} ds \\ &= \int_t^\infty \left( \frac{\partial U}{\partial C_D} \dot{C}_D + \frac{\partial U}{\partial C_M} \dot{C}_M + \frac{\partial U}{\partial L} \dot{L} \right) e^{-\theta(s-t)} ds \\ &= \int_t^\infty \frac{\partial U}{\partial C_D} (\dot{C}_D + p_M \dot{C}_M - w \dot{L}) e^{-\theta(s-t)} ds \end{aligned} \tag{C.2}$$

The last line in (C.2) was obtained by inserting the first-order conditions for a household optimum as shown in (A.5). From the same set of conditions we have

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\partial U}{\partial C_D}(t) \right\} &= \frac{\partial U}{\partial C_D}(t) [\theta - r^*]. \text{ When integrated, the solution to this differential equation is} \\ \frac{\partial U}{\partial C_D}(s) &= \frac{\partial U}{\partial C_D}(t) e^{-\theta(s-t) + \int_t^s r^* d\tau}. \end{aligned} \tag{C.2}$$

(C.2) can then be transformed into

$$\begin{aligned}
\frac{dV}{dt} &= \int_t^\infty \frac{\partial U}{\partial C_D} (\dot{C}_D + p_M \dot{C}_M - w\dot{L}) e^{-\theta(s-t)} ds \\
&= \frac{\partial U}{\partial C_D} (t) \int_t^\infty (\dot{C}_D + p_M \dot{C}_M - w\dot{L}) e^{-\int_t^s r^* d\tau} ds \\
&= \frac{\partial U}{\partial C_D} \left\{ [C_D + p_M C_M] e^{-\int_t^s r^* d\tau} \right\}_t^\infty - \int_t^\infty (-r^* C_D - r^* p_M C_M + \dot{p}_M C_M + w\dot{L}) e^{-\int_t^s r^* d\tau} ds \quad (C.3) \\
&= \frac{\partial U}{\partial C_D} \left\{ -C + \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds - \int_t^\infty \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds - \int_t^\infty w\dot{L} e^{-\int_t^s r^* d\tau} ds \right\}.
\end{aligned}$$

From (26) we have  $N + Z + Q = \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds$  where  $Q$  stands for the discounted value of real

income from changes in labor input:  $Q \equiv \int_t^\infty w\dot{L} e^{-\int_t^s r^* d\tau} ds$ . When inserted into (D.3), one gets

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\partial U}{\partial C_D} \left\{ -C + \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds - \int_t^\infty \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds - \int_t^\infty w\dot{L} e^{-\int_t^s r^* d\tau} ds \right\} \\
&= \frac{\partial U}{\partial C_D} \left\{ -C + N + Z - \int_t^\infty \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds \right\} \\
&= \frac{\partial U}{\partial C_D} \{-C + N + Z - T\} \\
&= \frac{\partial U}{\partial C_D} \{p_1 \dot{K} + Z - T\} \quad (C.4)
\end{aligned}$$

where the terms of trade effect  $T$  has been defined as  $T \equiv \int_t^\infty \dot{p}_M C_M e^{-\int_t^s r^* d\tau} ds$ . The last line in (C.4) is the desired result.

#### Annex D: Holding gains and losses

To derive equation (28), we recall expression (27) that links real net income and discounted consumption

$$N + Z + Q = \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds \quad (D.1)$$

and start by showing that

$$N + Z = r * p_I K + wL + \int_t^\infty \dot{r} * p_I K e^{-\int_t^s r^* d\tau} ds + \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds. \quad (D.2)$$

In essence, real net income, corrected for future changes in productivity, equals primary factor incomes in the form of interest payments and real wages plus the discounted effects of future changes in real interest and real wages. The following proof for (D.1) follows Sefton and Weale (2006).

**Proof:**

The household's budget constraint in its form (22) implies that consumption wealth (CW) equals the sum of physical capital ( $p_I K$ ) and human capital HC:

$$\begin{aligned} p_I K + \int_t^\infty w L e^{-\int_t^s r^* d\tau} ds &= \int_t^\infty (C_D + p_M C_M) e^{-\int_t^s r^* d\tau} ds \\ p_I K + HC &= CW \end{aligned} \quad (D.3)$$

where  $HC \equiv \int_t^\infty w L e^{-\int_t^s r^* d\tau} ds$  and  $CW \equiv \int_t^\infty C e^{-\int_t^s r^* d\tau} ds$ .

Note that

$$\int_t^\infty r * C e^{-\int_t^s r^* d\tau} ds = - \int_t^\infty r * \frac{d}{ds} \left\{ C W e^{-\int_t^s r^* d\tau} \right\} ds \quad \text{and} \quad (D.4)$$

$$\frac{d}{ds} \left\{ H C e^{-\int_t^s r^* d\tau} \right\} = - w L e^{-\int_t^s r^* d\tau}. \quad (D.5)$$

Inserting (D.3) into (D.4) yields

$$\begin{aligned} \int_t^\infty r * C e^{-\int_t^s r^* d\tau} ds &= - \int_t^\infty r * \frac{d}{ds} \left\{ (p_I K + HC) e^{-\int_t^s r^* d\tau} \right\} ds \\ &= - \int_t^\infty r * \frac{d}{ds} \left\{ p_I K e^{-\int_t^s r^* d\tau} \right\} ds + \int_t^\infty r * w L e^{-\int_t^s r^* d\tau} ds \\ &= - \int_t^\infty r * \frac{d}{ds} \left\{ p_I K e^{-\int_t^s r^* d\tau} \right\} ds - \int_t^\infty w L \frac{d}{ds} \left( e^{-\int_t^s r^* d\tau} \right) ds \\ &= r * p_I K + wL + \int_t^\infty \dot{r} * p_I K e^{-\int_t^s r^* d\tau} ds + \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds + \int_t^\infty w L e^{-\int_t^s r^* d\tau} ds \end{aligned} \quad (D.6)$$

The last line in (D.6) was derived after integration by parts of the second-to-last expression.

Remembering that  $Q$  has been defined as  $Q = \int_t^\infty w L e^{-\int_t^s r^* d\tau} ds$ , and invoking (D.1), (D.6) can be

expressed as

$$\begin{aligned}
N + Z + Q &= \int_t^\infty r^* C e^{-\int_t^s r^* d\tau} ds \\
&= r^* p_l K + wL + \int_t^\infty \dot{r}^* p_l K e^{-\int_t^s r^* d\tau} ds + \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds + Q . \quad (D.7) \\
N + Z &= r^* p_l K + wL + \int_t^\infty \dot{r}^* p_l K e^{-\int_t^s r^* d\tau} ds + \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds
\end{aligned}$$

Real net income equals real wages and real net remuneration of capital. Therefore,

$$N = wL + r^* p_l K - \dot{p}_l K . \quad (D.8)$$

It follows that

$$Z = \dot{p}_l K + \int_t^\infty \dot{r}^* p_l K e^{-\int_t^s r^* d\tau} ds + \int_t^\infty \dot{w} L e^{-\int_t^s r^* d\tau} ds \quad (D.9)$$

which is the desired result.

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Figure 1

