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ABSTRACT

The question of whether and how mutual fund managers provide valuable services for their clients motivates one of the largest literatures in finance. One candidate explanation is that funds process information about future asset values and use that information to invest in high-valued assets. But formal theories are scarce because information choice models with many assets are difficult to solve as well as difficult to test. This paper tackles both problems by developing a new attention allocation model that uses the state of the business cycle to predict information choices, which in turn, predict observable patterns of portfolio investments and returns. The predictions about fund portfolios' covariance with payoff shocks, cross-fund portfolio and return dispersion, and their excess returns are all supported by the data. These findings offer new evidence that some investment managers have skill and that attention is allocated rationally.

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> "What information consumes is rather obvious: It consumes the attention of its recipients. Hence a wealth of information creates a poverty of attention, and a need to allocate that attention efficiently among the overabundance of information sources that might consume it." Simon (1971)

The question of whether and how mutual fund managers provide valuable services for their clients motivates one of the largest literatures in empirical finance. A natural candidate explanation is that funds process information about future asset values and use that information to invest in high-valued assets. But few such theories have been written because information choice models with many assets are difficult to solve and difficult to test. This paper tackles both of these problems by developing a new model that uses an observable variable–the state of the business cycle–to predict information choices and that links those information choices to observable patterns in portfolio investments and returns.

We use business cycle variation as our observable state because of recent empirical evidence suggesting that the way funds provide value changes over the cycle (Kosowski (2011), Glode (2011), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014)). We explore a fund manager's choice of what information to process in different states of the business cycle. We find that fund managers optimally choose to process information about aggregate shocks in recessions and idiosyncratic shocks in booms. The resulting fund portfolios exhibit the same kind of "time-varying skill" as do those in the data.

To understand how fund information strategies depend on the cycle, we build a new model. Existing mutual fund theories explain fund flows and fees, but do not tell us how funds add value.[1] Existing models of information processing and portfolio choice either prohibit managers from choosing between aggregate or idiosyncratic information (Van Nieuwerburgh and Veldkamp 2010), or require that there are only two assets (Mondria 2010), rendering all shocks aggregate. Therefore, we develop a new methodology that can accommodate N assets and information choices with a more general asset payoff and signal structure.

The model's solution offers a rich set of predictions, which we test with mutual fund data. Just as importantly, the model is a building block. It can be extended to allow for asymmetric initial information across investors, multiple countries with home and foreign

[1] For theoretical models of fees and flows, asset price effects, manager incentive problems, and other aspects of mutual funds, see e.g., Mamaysky and Spiegel (2002), Berk and Green (2004), Kaniel and Kondor (2013), Cuoco and Kaniel (2011), Chien, Cole, and Lustig (2011), Chapman, Evans, and Xu (2010), and Pástor and Stambaugh (2012). A number of recent papers in the empirical mutual fund literature also find that some managers have skill, e.g., Kacperczyk, Sialm, and Zheng (2005, 2008), Kacperczyk and Seru (2007), Cremers and Petajisto (2009), Huang, Sialm, and Zhang (2011), Koijen (2014), Baker, Litov, Wachter, and Wurgler (2010).

funds, high and low-capacity funds, a choice over the quantity of information capacity, etc. The framework provides a new lens through which to analyze the empirical literature and to study which empirical patterns are consistent with optimal information-processing behavior.

In the model, a fraction of investment managers have skill. These skilled managers can observe a fixed number of signals about asset payoffs and choose what fraction of those signals will contain aggregate versus stock-specific information. We think of aggregate signals as macroeconomic data that affect future cash flows of all firms, and of stock-specific signals as firm-level data that forecast the part of firms' future cash flows that is independent of the aggregate shocks. Based on their signals, skilled managers form portfolios, choosing larger portfolio weights for assets that are more likely to have high returns. In the data, recessions are times when aggregate volatility rises and the price of risk surges. When we embed these two forces in our model, both govern attention allocation.

The model generates six main predictions. It predicts how volatility and the price of risk each affect attention allocation, portfolio dispersion, and portfolio returns. The first pair of predictions tell us that attention should be reallocated over the business cycle. In recessions, more volatile aggregate shocks should draw more attention, because it is more valuable to pay attention to more uncertain outcomes. The elevated price of risk amplifies this reallocation: Since aggregate shocks affect a large fraction of the portfolio's value, paying attention to aggregate shocks resolves more portfolio risk than learning about stock-specific risks. When the price of risk is high, such risk-minimizing attention choices become more valuable. While the idea that it is more valuable to shift attention to more volatile shocks is straightforward, whether changes in the price of risk would amplify or counteract this effect is not obvious.

The remaining predictions do not come from the reallocation of attention. Rather, they help to distinguish this theory from non-informational alternatives and support the idea that at least some portfolio managers are engaging in value-maximizing behavior. The second pair of results predict business cycle effects on cross-fund portfolio and profit dispersion. Since recessions are times when large aggregate shocks to asset payoffs create more comovement in asset payoffs, passive portfolios would have returns that also comove more in recessions, which would imply less dispersion. In contrast, when investment managers learn about asset payoffs and manage their portfolios according to what they learn, fund returns comove *less* and dispersion increases in recessions. One reason is that when aggregate shocks become more volatile, managers who learn about aggregate shocks put less weight on their common prior beliefs, which have less predictive power, and more weight on their heterogeneous signals.

This generates more heterogeneous beliefs in recessions and therefore more heterogeneous investment strategies and fund returns. The other reason is that a higher price of risk induces managers to take less risk, which makes prices less informative. Like prior beliefs, information in prices is common information. When prices contain less information, this common information is weighted less and heterogeneous signals are weighted more, resulting in more heterogeneous portfolio returns.

Finally, the fifth and sixth predictions describe the effect of risk and the price of risk on fund performance. Since the average fund can only outperform the market if there are other, non-fund investors who underperform, the model also includes unskilled non-fund investors. Both the heightened uncertainty about asset payoffs and the elevated price of bearing risk in recessions make information more valuable. Therefore, the informational advantage of the skilled over the unskilled increases and generates higher returns for informed managers. The average fund's outperformance rises.

We test the model's predictions on the universe of actively managed U.S. equity mutual funds. To test the first prediction, a key insight is that managers can only choose portfolios that covary with shocks they pay attention to. Thus, to detect cyclical changes in attention, we should look for changes in covariances. We estimate the covariance of each fund's portfolio holdings with the aggregate payoff shock, proxied by innovations in industrial production growth. This covariance measures a manager's ability to time the market by increasing (decreasing) her portfolio positions in anticipation of good (bad) macroeconomic news. This timing covariance rises in recessions. We also calculate the covariance of a fund's portfolio holdings with asset-specific shocks, proxied by innovations in firms' earnings. This covariance measures managers' ability to pick stocks that subsequently experience unexpectedly high earnings. Consistent with the theory, this stock-picking covariance increases in expansions. The idea that one can test rational inattention models by looking for changes in covariances is similar to that in Maćkowiak, Moench, and Wiederholt (2009). Our paper exploits time-series rather than cross-sectional variation in the covariance of shocks and economic outcomes and uses mutual fund portfolios instead of firm-level pricing data.

Second, we test for cyclical changes in portfolio dispersion. We find that, in recessions, funds hold portfolios that differ more from one another. As a result, their cross-sectional return dispersion increases, consistent with the theory. In the model, much of this dispersion comes from taking different bets on market outcomes, which should show up as dispersion in CAPM betas. We find evidence in the data for higher beta dispersion in recessions.

Third, we document fund outperformance in recessions, extending earlier results in the literature. Risk-adjusted excess fund returns (alphas) are around 1.6 to 4.6% per year higher in recessions, depending on the specification. Gross alphas (before fees) are not statistically different from zero in expansions, but they are significantly positive in recessions.[2] These cyclical differences are statistically and economically significant.

Fourth, we decompose effects of recessions on covariance, dispersion, and performance, by separating them into price of risk and volatility. When we use both price of risk and aggregate volatility as explanatory variables, we find that both contribute about equally to our three main results. Showing that these results are truly business-cycle phenomena–as opposed to merely high volatility phenomena–is interesting because it connects these results with the existing macroeconomics literature on rational inattention (e.g., Sims (2003), Maćkowiak and Wiederholt (2009, 2015)).

Related theories of mutual funds Many mutual fund theories account for some of the facts we document. But they do not explain all our facts jointly or answer our main question: How do funds go about adding value for investors? One strand of the literature focuses on changes in fund performance that arise when fund managers change. While manager turnover and sample selection effects may be important for the measurement of many mutual fund facts, they do not change the nature of the puzzles our model aims to explain. In the Supplementary Appendix (Section S.10), we re-estimate our main results using managers, instead of funds, as the unit of observation, and include manager fixed effects. We find the same results as at the fund level. Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014) document that it is the same managers who pick stocks well in booms that also time the market in recessions, and check that there are no systematic differences in age, educational background, or experience of fund managers in recessions versus expansions. Similarly, Chevalier and Ellison (1999) show that young managers with career concerns may have an incentive to herd. It would seem logical that the concern for being fired would be greatest in recessions. But if that were the case, herding should be most prevalent in recessions and it should make the dispersion in portfolios decline. Instead, our results show that portfolio dispersion rises in recessions. The convex relationship between mutual fund performance and fund inflows can explain outperformance and higher portfolio dispersion in recessions (Kaniel and Kondor 2013). Likewise, Glode (2011) argues that funds outperform

[2]Net alphas (after fees) for the average fund are negative in expansions (-0.6%) and positive (1.0%) in recessions for our most conservative metric. Gross alphas are higher by about 1% point per year. Since funds do not set fees in our model, we have no predictions about after-fee alphas. For a theory about why we should expect net alphas to be zero, see Berk and Green (2004). For recent empirical work, see Berk and van Binsbergen (2015).

in recessions because their investors' marginal utility is highest then. Neither mechanism explains why performance and dispersion also rise in times of high macro volatility or why skill measures are cyclical. Each of these theories likely captures an important feature of the mutual fund market. But the set of facts we present, taken together, are supportive of our explanation for the information-based origins of mutual fund skill.

The rest of the paper is organized as follows. Section 1 lays out our model. After describing the setup, we characterize the optimal information and investment choices of skilled and unskilled investors. We show how equilibrium asset prices are formed. We derive theoretical predictions for funds' attention allocation, portfolio dispersion, and performance. Section 2 explains how we bring the model to the data. Section 3 tests the model's predictions using the context of actively managed equity mutual funds. Section 4 concludes.

1 Model

We develop a model whose purpose is to understand how the optimal attention allocation of investment managers depends on the business cycle, and how attention affects asset holdings and asset prices. The model builds on the information choice model in Van Nieuwerburgh and Veldkamp (2010), but with a new solution methodology that allows us to consider signals about any linear combination of assets, a generalization advocated by Sims (2006). Much of the complexity of the model comes from the fact that it is an equilibrium model. But in order to study the effects of attention on asset holdings, asset prices, and fund performance, having an equilibrium model is a necessity. In particular, an equilibrium model ensures that for every investor that outperforms the market, there is someone who underperforms.

1.1 Setup

The model has three periods. At time 1, skilled investment managers choose how to allocate their attention across different assets. At time 2, all investors choose their portfolios of risky and riskless assets. At time 3, asset payoffs and utility are realized.

Assets The model features 1 riskless and n risky assets. The price of the riskless asset is normalized to 1 and it pays off r at time 3. Risky assets $i \in \{1, ..., n-1\}$ have random payoffs f_i with respective loadings $b_i, ..., b_{n-1}$ on an aggregate shock z_n, and face stock-specific shocks $z_1, ..., z_{n-1}$. The n-th asset, is a composite asset whose payoff has no stock-specific shock and a loading of one on the aggregate shock z_n. We use this composite asset as a stand-in for all other assets. Formally,

$$
\begin{aligned}
f_i &= \mu_i + b_i z_n + z_i, \qquad i \in \{1, \ldots, n-1\} && (1) \\
f_n &= \mu_n + z_n && (2)
\end{aligned}
$$

where the risk factors $z_i \sim N(0, \sigma_i)$, are mutually independent for $i \in \{1, \ldots, n-1, n\}$. We define the $n \times 1$ vector $f = [f_1, f_2, ..f_n]'$.

Risk factors The vector of risk factor shocks, $z = [z_1, z_2, ..., z_{n-1}, z_n]'$, is normally distributed as: $z \sim \mathcal{N}(0, \Sigma)$ where Σ is a diagonal matrix. Stacking the equations above, we can write $f = \mu + \Gamma z$, where Γ is a $n \times n$ invertible matrix of loadings that map risk factors, z, into the mean-zero payoffs $(f - \mu)$. We define the payoff of the risk factors as $\tilde{f} \equiv \Gamma^{-1} f = \Gamma^{-1}\mu + z$. Thus, payoffs of risk factors are linear combinations of payoffs of the underlying assets. In other words, they are a payoff to a particular portfolio of assets. Working with risk factor payoffs and prices (denoted with tildes) allows us to solve the model in a tractable way.

Each risk factor has a stochastic supply given by $\bar{x}_i + x_i$, where noise x_i is normally distributed, with mean zero, variance σ_x, and no correlation with other noises: $x \sim \mathcal{N}(0, \sigma_x I)$. The vector of noisy asset supplies is $(\Gamma')^{-1}(\bar{x}+x)$. As in any (standard) noisy rational expectations equilibrium model, the supply is random to prevent the price from fully revealing the information of informed investors. An important assumption is that the supply of aggregate risk is large, relative to other risks: $\bar{x}_n >> \bar{x}_i$ for $i \neq n$. Its size is what makes aggregate risk fundamentally different from the other risks in the economy.

Portfolio Choice Problem There is a continuum of atomless investors. Each investor is endowed with initial wealth, W_0.[3] They have mean-variance preferences over time-3 wealth, with a risk-aversion coefficient, ρ. Let E_j and V_j denote investor j's expectations and variances conditioned on all information known at time 2, which includes prices and signals. Thus, investor j chooses how many shares of each asset to hold, q_j to maximize time-2 expected utility, U_{2j}:

$$U_{2j} = \rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j] \tag{3}$$

[3]Since there are no wealth effects in the preferences, the assumption of identical initial wealth is without loss of generality.

subject to the budget constraint: $W_j = rW_0 + q_j'(f - pr)$, where q_j and p are $n \times 1$ vectors of prices and quantities of each asset held by investor j. We can rewrite the budget constraint in terms of risk factor prices and quantities by defining $\tilde{p} \equiv \Gamma^{-1}p$, $\tilde{q}_j \equiv \Gamma' q_j$, and substituting $f = \Gamma\tilde{f}$ to get

$$W_j = rW_0 + \tilde{q}_j'(\tilde{f} - \tilde{p}r). \tag{4}$$

Prices Equilibrium prices are determined by market clearing:

$$\int \tilde{q}_j dj = \bar{x} + x, \tag{5}$$

where the left-hand side of the equation is the vector of aggregate demand and the right-hand side is the vector of aggregate supply of the risk factors.

Information, updating, and attention allocation At time 1, a skilled investment manager j chooses the precisions of signals that she will receive at time 2. Allocating attention to a risk factor means that a manager gets a more precise signal about that risky outcome. Mathematically, a manager j's vector of signals is $\eta_j = z + \varepsilon_j$, where the vector of signal noise is distributed as $\varepsilon_j \sim \mathcal{N}(0, \Sigma_{\eta j})$.[4] The variance matrix $\Sigma_{\eta j}$ is diagonal with ith diagonal element K_{ij}^{-1}. Thus, K_{ij} is the precision of investor j's signal about risk i. Private signal noise is independent across risks i and managers j. Note that these signals are about asset payoffs and contain no direct information about asset supply x. Managers combine signal realizations with priors and information extracted from asset prices to update their beliefs, using Bayes' law.

Signal precision choices $\{K_{ij}\}$ maximize time-1 expected utility, U_{1j}, of the fund's terminal wealth W_j. The objective is $-E[lnE_j[\exp(-\rho W_j)]]$, which is equivalent to maximizing

$$U_{1j} = E\left[\rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j]\right], \tag{6}$$

subject to three constraints.[5]

[4] This signal structure is similar to that in Mondria (2010) because signals are linear combinations of asset payoffs, plus normally-distributed noise. While Mondria allows for a choice over the linear combination, he only works with 2 assets and 1 signal. Appendix B shows how to use our method to solve the N-asset problem for signals that are about any linear combination of asset payoffs f of the form $\eta_j = \psi f + e_j$, where ψ is an invertible matrix and f and e_j are normally distributed with covariance matrices that need not be diagonal.

[5] See Veldkamp (2011) for a discussion of the use of expected mean-variance utility in information choice problems. The Supplementary Appendix (Section S.2) proves versions of the main propositions for the

The first constraint is the budget constraint (4) that determines W_j as a function of investment decisions. The second constraint is *information capacity constraint.* It states that the sum of the signal precisions must not exceed the information capacity:

$$\sum_{i=1}^{n} K_{ij} \leq K. \tag{7}$$

In Bayesian updating with normal variables, observing one signal with precision K_i or two signals, each with precision $K_i/2$, is equivalent. Therefore, one interpretation of the capacity constraint is that it allows the manager to observe N signal draws, each with precision K_i/N, for large N. The investment manager then chooses how many of those N signals will be about each shock.[6] Note that our model holds each manager's total attention fixed and studies its allocation in recessions and expansions. Section 1.8 relaxes this assumption.

The third constraint is the *no-forgetting constraint*, which ensures that the chosen precisions are non-negative:

$$K_{ij} \geq 0 \quad i \in \{1, ..., n-1, n\} \tag{8}$$

It prevents the manager from erasing any prior information, to make room to gather new information about another shock.

Skilled and Unskilled Investors The only ex-ante difference between investors is that a fraction χ of them have *skill*, meaning that they can choose to observe a set of informative *private* signals about the risk factor shocks z_i. Unskilled investors are ones with zero signal precision: $\Sigma_{\eta j}^{-1} = 0$, or equivalently, $K_{ij} = 0$, $\forall i$. Both unskilled and skilled investors observe the information in prices, which are public signals, costlessly.[7]

When we bring the model to the data, we will call all skilled investors mutual funds. Furthermore, we will distinguish between two types of unskilled investors: unskilled mutual funds and non-fund investors.[8] In the model, the latter two types are identical. The reason

expected exponential utility model.

[6]The results are not sensitive to the exact nature of the information capacity constraint. The Supplementary Appendix (Section S.4) re-proves each one of our propositions for a model with an entropy constraint. The linear constraint (7) makes sense in our setting because additional fund analysts can be hired to process information. Twice as many analysts could produce twice the precision at twice the cost. To make information precision a continuous choice variable, let $\dot{k}\delta$ be the precision of each analyst and let $\dot{c}\delta$ be the cost of each analyst. Then take lim $\delta \to 0$. That problem with a continuous, linear cost function is a dual problem to our constrained maximization problem.

[7]If investors must expend capacity to learn from prices, the model predictions are unchanged. See Supplementary Appendix S.5.

[8]For our results to hold, it is sufficient to assume that the fraction of non-fund investors that are unskilled

for modeling non-fund investors is that without them, we cannot talk about average fund performance. The sum of all funds' holdings would have to equal the market by market clearing, and therefore, the average fund return would have to equal the market return. Without uninformed non-fund investors, the average fund could never systematically outperform the market return, in recessions or expansions.

Modeling recessions Since this is a static model, the investment world is either in the recession (R) or in the expansion state (E). The asset pricing literature identifies three principal effects of recessions: (1) returns are more volatile, (2) the price of risk is high, and (3) returns are unexpectedly low. Section 3 discusses the empirical evidence supporting the first two effects. The third effect of recessions, unexpectedly low returns, cannot affect attention allocation because attention must be allocated before returns are observed. Therefore, we abstract from it and consider only effects (1) and (2). To capture the return volatility effect (1) in the model, we assume that the prior variance of the aggregate shock in recessions (R) is higher than the one in expansions (E): $\sigma_n(R) > \sigma_n(E)$. To capture the varying price of risk (2), we vary the parameter that governs the price of risk, which is risk aversion. We assume $\rho(R) > \rho(E)$.We continue to use σ_n and ρ to denote aggregate shock variance and risk aversion in the current business cycle state.

1.2 Equilibrium

This paper's methodological innovation is that its model relaxes an important assumption. Previous work assumed that assets and signals have the same principal components. Observing signals about aggregate and idiosyncratic shocks violates that assumption. Updating with such signals changes the conditional correlations of assets. So to solve the model, we perform a change of variables. We create linear combinations of assets (synthetic assets) such that the payoff of each synthetic asset is determined only by one shock (either aggregate or idiosyncratic). Then, we can choose information about, choose quantities of, and price these synthetic assets easily because each asset's payoff is independent of all the others and each signal is informative about one and only one asset. After we have a solution to the synthetic asset problem, we can invert the linear transformation to back out portfolios and prices of the original assets.

We begin by working through the mechanics of Bayesian updating. There are three types of information that are aggregated in time-2 posteriors beliefs: prior beliefs, price

is higher than that for the mutual funds.

information, and (private) signals. We conjecture and later verify that a transformation of prices $\tilde{p}$ generates an unbiased signal about the risk factor payoffs, $\eta_p = z + \epsilon_p$, where $\epsilon_p \sim N(0, \Sigma_p)$, for some diagonal variance matrix Σ_p. Then, by Bayes' law, the posterior beliefs about z are normally distributed with mean $\hat{z}_j = \hat{\Sigma}_j(\Sigma_{\eta j}^{-1}\eta_j + \Sigma_p^{-1}\eta_p)$ and posterior precision $\hat{\Sigma}_j^{-1} = \Sigma^{-1} + \Sigma_p^{-1} + \Sigma_{\eta j}^{-1}$. Using the definition $\tilde{f} = \Gamma^{-1}\mu + z$, we find that posterior beliefs about risk factor payoffs are $\tilde{f} \sim N(E_j[\tilde{f}], \hat{\Sigma}_j^{-1})$ where

$$E_j[\tilde{f}] = \Gamma^{-1}\mu + \hat{\Sigma}_j(\Sigma_{\eta j}^{-1}\eta_j + \Sigma_p^{-1}\eta_p). \tag{9}$$

Next, we solve the model in four steps.

Step 1: Solve for the optimal portfolios, given information sets.

Substituting the budget constraint (4) into the objective function (3) and taking the first-order condition with respect to $\tilde{q}_j$ reveals that optimal holdings are increasing in the investor's risk tolerance, precision of beliefs, and expected return:

$$\tilde{q}_j = \frac{1}{\rho}\hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r). \tag{10}$$

Step 2: Clear the asset market.

Substitute each agent j's optimal portfolio (10) into the market-clearing condition (5). Collecting terms and simplifying reveals that equilibrium asset prices are linear in payoff risk shocks and in supply shocks:

Lemma 1. $\tilde{p} = \frac{1}{r}(A + Bz + Cx)$

A detailed derivation of coefficients A, B, and C, expected utility, and the proofs of this lemma and all further propositions are in the Appendix.

In this model, agents learn from prices because prices are informative about the payoff shocks z. Next, we deduce what information is implied by Lemma 1. Price information is the signal about z contained in prices. The transformation of the price vector $\tilde{p}$ that yields an unbiased signal about z is $\eta_p \equiv B^{-1}(\tilde{p}r - A)$. Note that applying the formula for η_p to Lemma 1 reveals that $\eta_p = z + \varepsilon_p$, where the signal noise in prices is $\varepsilon_p = B^{-1}Cx$. Since we assume $x \sim N(0, \sigma_x I)$, the price noise is distributed $\varepsilon_p \sim N(0, \Sigma_p)$, where $\Sigma_p \equiv \sigma_x B^{-1}CC'B^{-1\prime}$. Substituting in the coefficients B and C from the proof of Lemma 1 shows that public signal precision Σ_p^{-1} is a diagonal matrix with ith diagonal element $\sigma_{pi}^{-1} = \frac{\bar{K}_i^2}{\rho^2\sigma_x}$, where $\bar{K}_i \equiv \int K_{ij}dj$ is the average capacity allocated to risk factor i.

Step 3: Compute ex-ante expected utility.

Substitute optimal risky asset holdings from equation (10) into the first-period objective function (6) to get: $U_{1j} = rW_0 + \frac{1}{2}E_1\left[(E_j[\tilde{f}] - \tilde{p}r)\hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r)\right]$. Note that the expected excess return $(E_j[\tilde{f}] - \tilde{p}r)$ depends on signals and prices, both of which are unknown at time 1. Because asset prices are linear functions of normally distributed shocks, $E_j[\tilde{f}] - \tilde{p}r$, is normally distributed as well. Thus, $(E_j[\tilde{f}] - \tilde{p}r)\hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r)$ is a non-central χ^2-distributed variable. Computing its mean yields:

$$U_{1j} = rW_0 + \frac{1}{2}trace(\hat{\Sigma}_j^{-1}V_1[E_j[\tilde{f}] - \tilde{p}r]) + \frac{1}{2}E_1[E_j[\tilde{f}] - \tilde{p}r]'\hat{\Sigma}_j^{-1}E_1[E_j[\tilde{f}] - \tilde{p}r]. \tag{11}$$

Step 4: Solve for information choices.

Note that in expected utility (11), the choice variables K_{ij} enter only through the posterior variance $\hat{\Sigma}_j$ and through $V_1[E_j[\tilde{f}] - \tilde{p}r] = V_1[\tilde{f} - \tilde{p}r] - \hat{\Sigma}_j$. Since there is a continuum of investors, and since $V_1[\tilde{f} - \tilde{p}r]$ and $E_1[E_j[\tilde{f}] - \tilde{p}r]$ depend only on parameters and on aggregate information choices, each investor takes them as given.

Since $\hat{\Sigma}_j^{-1}$ and $V_1[E_j[\tilde{f}] - \tilde{p}r]$ are both diagonal matrices and $E_1[E_j[\tilde{f}] - \tilde{p}r]$ is a vector, the last two terms of (11) are weighted sums of the diagonal elements of $\hat{\Sigma}_j^{-1}$. Thus, (11) can be rewritten as $U_{1j} = rW_0 + \sum_i \lambda_i \hat{\Sigma}_j^{-1}(i,i) - n/2$, for positive coefficients λ_i. Since rW_0 is a constant and $\hat{\Sigma}_j^{-1}(i,i) = \Sigma^{-1}(i,i) + \Sigma_p^{-1}(i,i) + K_{ij}$, the information choice problem is:

$$\max_{K_{1j},\ldots,K_{nj}} \sum_{i=1}^{n} \lambda_i K_{ij} + \text{constant} \tag{12}$$

$$s.t. \quad \sum_{i=1}^{n} K_{ij} \leq K \tag{13}$$

$$\text{where} \quad \lambda_i = \bar{\sigma}_i[1 + (\rho^2\sigma_x + \bar{K}_i)\bar{\sigma}_i] + \rho^2\bar{x}_i^2\bar{\sigma}_i^2, \tag{14}$$

where $\bar{\sigma}_i^{-1} = \int \hat{\Sigma}_j^{-1}(i,i)dj$ is the average precision of posterior beliefs about risk i. Its inverse, average variance $\bar{\sigma}_i$ is decreasing in $\bar{K}_i$. Equation (14) is derived in the Appendix.

To maximize a weighted sum (12) subject to an unweighted sum (13), the skilled manager optimally assigns all capacity to the risk(s) with the highest weight. If there is a unique $i^* = \text{argmax}_i \lambda_i$, then the solution is to set $K_{i^*j} = K$ and $K_{lj} = 0$, $\forall l \neq i^*$.

In many cases, there will be multiple risks with identical λ weights. That is because λ_i is decreasing in $\bar{K}_i$, the average investor's signal precision. When there exist risks i, l s.t. $\lambda_i = \lambda_l$, then investors are indifferent about which risk to learn about. The next result shows that this indifference is not a knife-edge case. It arises whenever the aggregate amount of information capacity is sufficiently high.

Lemma 2. *If $\bar{x}_i$ is sufficiently large $\forall i$ and $\sum_i \sum_j K_{ij} \geq \underline{K}$, then there exist risks l and l' such that $\lambda_l = \lambda_{l'}$.*

This is strategic substitutability in information acquisition, an effect first noted by Grossman and Stiglitz (1980). The more other investors learn about a risk, the more informative prices are and the less valuable it is for other investors to learn about the same risk. If one risk has the highest marginal utility for signal precision, but capacity is high, then many investors will learn about that risk, causing its marginal utility to fall and equalize with the next most valuable risk. With more capacity, the highest two λ_i's will be driven down until the equate with the next λ, and so forth. This type of equilibrium is called a "waterfilling" solution (see, Cover and Thomas (1991)). The equilibrium uniquely pins down which risk factors are being learned about in equilibrium, and how much is learned about them, but not which investor learns about which risk factor. For simplicity, we restrict attention to the unique symmetric equilibrium where all skilled investors choose the same allocation of information precision. However, only the dispersion results (Propositions 3 and 4) depend on this restriction.

The following sections explain the model's key predictions: attention allocation, dispersion in investors' portfolios, average performance, and the effect of recessions on these objects beyond that of aggregate volatility. For each prediction, we state and prove a proposition. The next section explains how we test the hypothesis in the data.

1.3 Cyclical Attention Reallocation

Recessions involve changes in the volatility of aggregate shocks and changes in the price of risk. In order to see the effect of the two recession aspects on the attention allocation strategies of skilled investors, we consider each separately, beginning with the rise in volatility.

Proposition 1. *For each skilled investor j, the optimal attention allocation for risk i (K_{ij}) is weakly increasing in its variance σ_i.*

The proof of this and subsequent propositions are in the Appendix.

The result tells us that investors prefer to learn more about any shock that has a high prior payoff variance. Information is most valuable about the most uncertain outcomes. The shift of attention to aggregate risk in recessions is just one application of this proposition, but it is the empirically relevant case. Since recessions are times when aggregate volatility increases (while idiosyncratic volatilities do not), it is a time when aggregate shocks are relatively more valuable to learn about. The converse is true in expansions.

The proposition takes into account not only the effect of a marginal increase of variance on the marginal value of learning about a risk, and hence on the capacity allocated to that risk, but also the offsetting equilibrium effect. In any interior equilibrium, attention is reallocated until the marginal values of learning about any risks that are learned about are equalized. Thus, when σ_n rises in recessions, the marginal value of learning more about the aggregate risk rises, more attention is allocated to the aggregate risk, which offsets the increase in marginal value until indifference in the marginal values across risk factors is restored. The net result is always a weakly increasing capacity devoted to the risk whose variance increases. As the proof shows, the "weakly" increasing refers to the cases where either all capacity is already allocated to the risk whose variance increases or no capacity is allocated to that risk and the marginal increase in variance does not change that. In all other cases, when risk i is one of the risks being learned about prior to the increase in σ_i, the increase in capacity devoted to i is strict.

Next, we consider the effect of an increase in the price of risk. An increase in the price of risk induces managers to allocate even more attention to the shock that is in the most abundant supply. We have assumed that the aggregate risk is the most abundant. The additional price of risk effect should show up as an effect of recessions on attention allocation, over and above what aggregate volatility alone can explain. The parameter that governs the price of risk in our model is risk aversion. The following result implies that an increase in risk aversion in recessions is an independent force driving the reallocation of attention from stock-specific to aggregate shocks.

Proposition 2. *If $\bar{x}_i$ is sufficiently large then, for each skilled investor j, the optimal attention allocation for risk i (K_{ij}) is weakly increasing in risk aversion ρ.*

The intuition for this result rests on the fact that a shock in abundant supply affects a large fraction of the value of an investor's portfolio. Therefore, a marginal reduction in the uncertainty about this shock reduces total portfolio risk by more than the same-sized reduction in the uncertainty about a less abundant shock. In other words, learning about the abundant shock, which is the aggregate shock, is the most efficient way to reduce portfolio risk. The more risk averse an agent is, the more attractive allocating attention to aggregate shocks becomes. Like the previous one, this result takes into account the equilibrium reallocation of capacity after the increase in risk aversion.

These results are robust to many model changes. In the Supplementary Appendix, we examine versions of the model in which agents learn about the payoffs of assets, rather than about risks directly (Section S.3) and in which information choices are governed by an entropy constraint rather than a linear capacity constraint (Section S.4). Both of our attention allocation results hold in these settings. When the aggregate shock variance rises or risk aversion increases, agents pay more attention to assets whose returns are most sensitive to aggregate shocks.

Investors' optimal attention allocation decisions are reflected in their portfolio holdings. In recessions, skilled investors predominantly allocate attention to the aggregate payoff shock, z_n. They use the information they observe to form a portfolio that covaries with z_n. In times when they learn that z_n will be high, they hold more risky assets whose returns are increasing in z_n. This positive covariance can be seen from equation (10) in which $\tilde{q}$ is increasing in $E_j[\tilde{f}]$ and from equation (9) in which $E_j[\tilde{f}]$ is increasing in η_j, which is further increasing in z_n. The positive covariances between the aggregate shock and funds' portfolio holdings in recessions, on the one hand, and between stock-specific shocks and the portfolio holdings in expansions, on the other hand, directly follow from optimal attention allocation decisions switching over the business cycle. As such, these covariances are the key moments that enable us to test the attention allocation predictions of the model. We define the empirical counterparts to these covariances in Section 2.

1.4 Portfolio Dispersion

Since many empirical studies on investment managers detect no outperformance, perhaps the most controversial implication of the attention reallocation result is that investment managers are processing information at all. The next four results speak directly to that implication. They do not identify changes in attention allocation, but they help to distinguish our theory from non-information-based alternative explanations for mutual fund performance patterns.

In recessions, as aggregate shocks become more volatile, the firm-specific shocks to assets' payoffs account for less of the variation, and the comovement in stock payoffs rises. Since asset payoffs comove more, the payoffs to all passive investment strategies that put fixed weights on assets also comove more. Dispersion across investor portfolios and portfolio returns would fall if investment strategies were passive. But when investment managers are processing information and actively investing based on that information, this prediction is reversed. To see why, consider a simple example in which there is no learning from prices.

A skilled agent is updating beliefs about a random variable $\tilde{f} \sim N(\mu, \Sigma)$, using a signal $\eta_j|\tilde{f} \sim N(\tilde{f}, \Sigma_\eta)$. Bayes' law says that the posterior mean is a weighted average of the prior mean μ and the signal, where each is weighted by their relative precision:

$$E[\tilde{f}|\eta_j] = \left(\Sigma^{-1} + \Sigma_\eta^{-1}\right)^{-1} \left(\Sigma^{-1}\mu + \Sigma_\eta^{-1}\eta_j\right) \tag{15}$$

If in recessions, aggregate shock variance σ_n rises, then the prior beliefs about asset payoffs become more uncertain: Σ rises and Σ^{-1} falls. This makes the weight on prior beliefs μ decrease and the weight on the signal η_j increase. The prior μ is common across agents, while the signal realization η_j is heterogeneous. When informed managers weigh their heterogeneous signals more, their resulting posterior beliefs become more different from each other and more different from the beliefs of uninformed managers or investors. More disagreement about asset payoffs results in more heterogeneous portfolios and portfolio returns. Since price signals are also common, the same result holds once they are incorporated. The feature of the model that underpins this result is the idiosyncratic component of signal noise. We could allow signal noise to be correlated across agents, as long as signals are not identical. Such idiosyncratic signal noise is inherent in the idea of rational inattention.

Thus, the model's second set of predictions are that in recessions, the cross-sectional dispersion in funds' investment strategies and returns should rise.

Proposition 3. *If $\bar{x}_i$ is sufficiently large then, an increase in variance σ_i weakly increases (a) the dispersion of fund portfolios, $\int E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]dj$, and (b) the dispersion of portfolio excess returns, $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$.*

This result takes into account that when variance of a shock changes, the equilibrium allocation of attention and equilibrium asset returns change as well. While this is a generic result for any risk i, the effect is particularly large for the aggregate risk because it affects every asset and therefore it is in abundant supply. This shows up in the proof as a high $\bar{x}_n$, which amplifies the effect of σ_n on portfolio and return dispersion.

Next, we consider the second effect of recessions: an increase in the price of risk. The following result shows that, when prices are sufficiently noisy, an increase in the price of risk increases the dispersion of portfolio returns.

Proposition 4. *If σ_x and $\bar{x}_n$ are sufficiently large, then an increase in risk aversion ρ increases the dispersion of portfolio excess returns, $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$.*

When risk aversion rises, skilled investors make smaller bets on their information. These smaller deviations from the market portfolio affect prices less and make prices less informative. The reduced precision of price information implies that agents weigh prices less and private signals more in their posterior beliefs. Just like priors, information in prices is common. Thus, weighing common signals less and heterogenous private signals more leads to higher dispersion in beliefs and therefore in portfolio returns as well.

This effect has to offset a counteracting force. Recall that the optimal portfolio for investor j takes the form $q = (1/\rho)\hat{\Sigma}_j^{-1}(f - pr)$. If ρ increases, investors scale down their risky asset positions and q falls. The increase in returns needs to increase dispersion enough to offset the decrease in dispersion coming from the effect of $1/\rho$ reducing q. The proof of the proposition in the Appendix shows that a sufficient condition for the first effect to dominate is that the elasticity of $V_1[\tilde{f} - \tilde{p}r]$ with respect to ρ be greater than 1, which requires a large enough asset supply variance. The high average supply of aggregate risk is what makes the n^{th} risk aggregate. In addition to this result, we can sign the effect of a change in risk aversion on the dispersion of *risk-adjusted* returns as well, with looser conditions on parameters that produce stronger equilibrium effects through aggregate attention reallocation. See Supplementary Appendix Section S.6 for a proof. In addition, our numerical example below confirms that portfolio dispersion increases in risk aversion, even in cases where our parameter restrictions are not satisfied.

1.5 Fund Performance

To measure performance, we want to measure the portfolio return, adjusted for risk. One risk adjustment that is both analytically tractable in our model and often used in empirical work is the certainty equivalent return, which is also an investor's objective (6). The following proposition shows that abnormal portfolio returns, defined as the fund's portfolio return, $\tilde{q}_j'(\tilde{f} - \tilde{p}r)$, minus the market return, $\bar{\tilde{q}}'(\tilde{f} - \tilde{p}r)$, for skilled funds exceeds that for unskilled funds and non-fund investors by more when volatility is higher.

Proposition 5. *If $\bar{x}_i$ is sufficiently large then, for each skilled investor j, an increase in the variance σ_i weakly increases the portfolio excess return, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$.*

Because aggregate risk factor payoffs are more uncertain in recessions (σ_n is higher), recessions are times when information is more valuable. The return effect is larger for the aggregate shock because it depends on how abundant the risk is ($\bar{x}_n$) and the aggregate shock is naturally the most abundant one.

Next, we consider the effect of an increase in the price of risk on performance.

Proposition 6. *If σ_x and $\bar{x}_n$ are sufficiently large then, for each skilled investor j, an increase in risk aversion ρ increases excess return, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$.*

The reason why a higher price of risk leads to higher performance is that information can resolve risk. Therefore, informed managers are compensated for risk that they do not bear because the information has resolved some of their uncertainty about random asset payoffs. When the price of risk rises, the value of being able to resolve this risk rises as well. Put differently, informed funds take larger positions in risky assets because they are less uncertain about their returns. These larger positions pay off more on average when the price of risk is high.

The role of the high σ_x and $\bar{x}_n$ is the same as in Proposition 4. And just like for Proposition 4, we can prove that risk-adjusted returns rise with looser parameter conditions. See Supplementary Appendix Section S.6. In addition, our numerical example confirms that when the price of risk increases, average performance of informed funds rises, for a wide range of parameter values.

Taken together, these results provide two reasons why skilled investors' advantage over unskilled investors increases in recessions. Of course, the model predicts that skilled investors should always outperform unskilled. In practice, this outperformance is difficult to detect. The model helps to guide the search for skill by explaining why one ought to focus on recessions as times when skill should be particularly salient.

Measuring Performance: Mapping skill into alpha The previous outperformance results were for abnormal fund returns, measured as the fund's return minus the market return. One other way to risk-adjust fund returns, which is common in the empirical literature, is to estimate a Capital Asset Pricing Model (CAPM) using each fund's returns and look at the fund's α, the intercept of the Security Market Line. This CAPM is estimated using only information that is in every investor's information set, which is the unconditional moments of asset returns. The following result shows that if one constructs such an unconditional CAPM from the fund returns in our model, the fund α captures information capacity K (skill) and rises in recessions.

Proposition 7. *If the net supply of idiosyncratic risk is small, then expected excess portfolio return of fund j is $E[R_j]-r = \alpha_j+\beta_j(E[r_m]-r)$, where $\alpha_j = \sum_i 1/\rho\left(var[\tilde{f}_i](\sigma_i^{-1} + K_{ij}) - 1\right)-\bar{\rho}_{ij}$.*

The model tells us that the CAPM alpha of a fund's return is increasing in its ability to process information about each type of risk. But the alpha also varies over the cycle as aggregate risk changes. In recessions, aggregate risk (σ_n) increases, which increases α_j. As in Hansen and Richard (1987), the unconditional CAPM correctly prices all portfolios constructed using only the common information set and assigns them zero alpha. But when skilled investors, who have a richer information set, construct portfolios, the portfolios will lie on a different mean-variance frontier and thus achieve a higher alpha.

1.6 Who Underperforms?

The requirement that markets clear implies that not all investors can be successful at investing in the right stock at the right time (stock-picking) or at timing the aggregate market fluctuations. In each period, someone must make poor stock-picking or market-timing decisions if someone else makes profitable decisions. We explain now why rational, unskilled investors underperform in equilibrium.

In recessions, unskilled investors have negative timing ability. When the aggregate state z_n is low, most skilled investors sell, pushing down asset prices, p, and making prior expected returns high. The high expected return (high $(\mu - pr)$) causes uninformed investors to demand more of the asset (equation (10)). The unskilled demand more because they cannot distinguish low prices that arise because of information from those that arise from positive asset supply shocks. Thus, unskilled investors' holdings covary negatively with aggregate payoffs, making their market timing measure negative. Since no investors learn about the aggregate shock in expansions, prices do not fall when unexpected aggregate shocks are negative and market timing is close to zero for both skilled and unskilled.

Likewise, unskilled investors exhibit negative stock-picking ability in expansions. When the stock-specific shock z_i is low, and some investors know this, they sell and depress the price of asset i. A low price raises the expected return $(\mu_i - p_i r)$. The high expected return induces unskilled investors to buy more of the asset. Since they buy more of assets that subsequently have negative asset-specific payoff shocks, these uninformed investors display negative stock-picking ability.

Note that when there is a positive aggregate supply shock, prices will be lower (Lemma 1), and assets will look more attractive to both uninformed and informed agents, all else equal. In that case, both informed and uninformed can trade in the same direction because of the additional asset supply.

1.7 Interaction Effects

The previous results describe the effects of aggregate risk and risk aversion separately. But there is also a subtle interaction between the two. Higher risk aversion amplifies the effect of aggregate risk on attention allocation, dispersion, and performance. The resulting testable prediction is that the effect of aggregate volatility on all three outcome variables should be greater in recessions, when the market price of risk is high. We derive these results in the Separate Appendix (Section S.7).

1.8 Endogenous Capacity Choice

So far, we have assumed that skilled investment managers choose how to allocate a fixed information-processing capacity, K. We now extend the model to allow for skilled managers to add capacity at a cost $\mathcal{C}(K)$. We model this cost as a utility penalty, akin to the disutility from labor in business-cycle models. Since there are no wealth effects in our setting, it would be equivalent to modeling a cost of capacity through the budget constraint. We draw three main conclusions. First, the proofs of Propositions 1 and 2 hold for any chosen level of capacity K, below an upper bound, no matter the functional form of $\mathcal{C}$. The other propositions also continue to hold because they only depend on the attention reallocation effects proven in Propositions 1 and 2. Endogenous capacity only has quantitative, not qualitative implications. Second, because the marginal utility of learning about the aggregate shock is increasing in its prior variance (Proposition 1), skilled managers choose to acquire higher capacity in recessions. This extensive-margin effect amplifies our dispersion and performance results. Third, the degree of amplification depends on the convexity of the cost function, $\mathcal{C}(K)$. The convexity determines how elastic equilibrium capacity choice is to the cyclical changes in the marginal benefit of learning. The Supplementary Appendix discusses numerical simulation results from an endogenous-K model; they are similar to our benchmark results.

2 Bringing the Model to Data

To test the model, we look at various measures of mutual fund investments in recessions and in non-recession periods. Of course, our model is not a dynamic one. It could be. A simple dynamic model would amount to a succession of static models that are either in the expansion or in the recession state. As we stated in the model setup, a recession state would be one in which aggregate risk and the price of risk are both high. Aggregate risk is captured by the variance parameter σ_a. We capture changes in the price of risk by varying risk aversion ρ. A variety of economic mechanisms can generate this kind of time-varying price of risk: external habits, heterogeneous labor income risk and limited commitment, borrowing constraints, or a concern for model misspecification (see Hansen (2013)). Since these mechanisms are too complex to embed in our model, we settle for varying a risk aversion parameter.

Propositions 1 and 2 teach us that both the increase aggregate shock variances and the increased price of risk prompt attention reallocation toward aggregate risk. Thus, the prediction is that in recessions, the average amount of attention devoted to aggregate shocks should increase and the average amount of attention devoted to stock-specific shocks should decrease. But of course, attention is not directly observable. Learning about a shock allows managers choose portfolio holdings that covary more with that shock. We see this in the portfolio first order condition (10). A manager who knows nothing about a shock cannot possibly hold a portfolio that covaries with the shock. It is not a feasible or measurable strategy. This covariance argument, combined with the reallocation results leads us to make the first testable prediction:

Prediction 1. *In recessions, portfolios should covary more with the aggregate component of payoffs. Conversely, in expansions, portfolio holdings should covary more with stock-specific payoff shocks.*

Because recessions are times of high aggregate risk and high risk prices, and both forces increase dispersion (Propositions 3 and 4), we make the next empirical prediction:

Prediction 2. *In recessions, the dispersion of fund portfolios should rise.*

Finally, both more aggregate risk and the higher price of risk cause skilled funds to generate higher returns (Propositions 5 and 6). The skill of these funds should be reflected in their portfolios' α, which increases in σ_n (Proposition 7). Since fund managers are skilled or unskilled, while other investors are only unskilled, an increase in the skill premium implies that the *average* mutual fund's excess return rises in recessions. Together, these findings lead us to make the following empirical prediction:

Prediction 3. *In recessions, the average fund should earn a higher excess return and have a higher alpha.*

Next, we introduce the empirical measures that we use in Section 3 to test each of these predictions.

2.1 Market-Timing and Stock-Picking Measures

We define a fund's fundamentals-based timing ability, $Ftiming$, as the covariance between its portfolio weights in deviation from the market portfolio weights, $w_i^j - w_i^m$, and the aggregate payoff shock, z_n, over a T-period horizon, averaged across assets:

$$Ftiming_t^j = \frac{1}{TN^j} \sum_{i=1}^{N^j} \sum_{\tau=0}^{T-1} (w_{it+\tau}^j - w_{it+\tau}^m)(b_i z_{n(t+\tau+1)}), \tag{16}$$

where N^j is the number of individual assets held by fund j. The portfolio weights are dated $t+\tau$ because they are chosen and thus known at $t+\tau$. The aggregate shock that affects the payoff of that portfolio is dated $t+\tau+1$ because that shock is not fully observed until one period later. Relative to the market, a fund with a high $Ftiming$ overweights assets that have high (low) sensitivity to the aggregate shock in anticipation of a positive (negative) aggregate shock realization and underweights assets with a low (high) sensitivity.

When skilled investment managers allocate attention to stock-specific payoff shocks, z_i, $i \in \{1, \ldots, n-1\}$, information about z_i allows them to choose portfolios that covary with z_i. Fundamentals-based stock-picking ability, $Fpicking$, measures the covariance of a fund's portfolio weights of each stock, relative to the market, with the stock-specific shock, z_i:

$$Fpicking_t^j = \frac{1}{N^j} \sum_{i=1}^{N^j} (w_{it}^j - w_{it}^m)(z_{it+1}). \tag{17}$$

How well can the manager choose portfolio weights in anticipation of future asset-specific payoff shocks is closely linked to her stock-picking ability.

$Ftiming$ and $Fpicking$ are closely related to commonly used market-timing and stock-picking variables, which measure how a fund's holdings of each asset, relative to the market, covary with the systematic and idiosyncratic components of the *stock return*. The key difference is that we measure how a portfolio covaries with aggregate and firm-specific *fundamentals*. We use the fundamentals-based measures because they correspond more closely to the idea in the model that funds are learning about fundamentals and using signals about those fundamentals to time the market and pick stocks. The returns-based picking and timing facts might be explained by managers who forecast sentiment, momentum, liquidity, etc. Also, since funds affect asset values, but do not directly affect earnings or production, the returns-based covariance can come from some reverse causality. The fundamentals-based results make it clear that the changing covariance between portfolios and returns comes from the covariance with one-quarter-ahead fundamentals. That offers a much clearer view of what information fund managers are collecting and processing. It also significantly narrows down the set of possible explanations consistent with the covariance facts.

2.2 Dispersion Measures

To connect the propositions to the data, we measure portfolio dispersion as the sum of squared deviations of fund j's portfolio weight in asset i at time t, w_{it}^{j}, from the average fund's portfolio weight in asset i at time t, w_{it}^{m}, summed over all assets held by fund j, N^j:

$$Portfolio\ Dispersion_t^j = \sum_{i=1}^{N^j} \left(w_{it}^{j} - w_{it}^{m}\right)^2 \tag{18}$$

This measure is similar to the portfolio concentration measure in Kacperczyk, Sialm, and Zheng (2005) and the active share measure in Cremers and Petajisto (2009). The average dispersion $\int Portfolio\ Dispersion_t^j dj$ is the same quantity as in Proposition 3, except that the number of shares q is replaced with portfolio weights w. In recessions, the portfolios of the informed managers differ more from each other and more from those of the uninformed investors. Part of this difference comes from a change in the composition of the risky asset portfolio and part comes from differences in the fraction of assets held in riskless securities. Fund j's portfolio weight w_{it}^{j} is a fraction of the fund's assets, including both risky and riskless, held in asset i. Thus, when one informed fund gets a bearish signal about the market, its w_{it}^{j} for all risky assets i falls. Dispersion can rise when funds take different positions in the risk-free asset, even if the fractional allocation among the risky assets remains identical.

The higher dispersion across funds' portfolio strategies translates into a higher cross-sectional dispersion in fund abnormal returns ($R^j - R^m$). To facilitate comparison with the data, we define the dispersion of variable X as $|X^j - \bar{X}|$, where $\bar{X}$ denotes the equally weighted cross-sectional average across all fund managers (excluding non-fund investors).

When funds get signals about the aggregate state z_n that are heterogenous, they take different directional bets on the market. Some funds tilt their portfolios to high-beta assets and other funds to low-beta assets, thus creating dispersion in fund betas. To look for evidence of this mechanism, we form a CAPM regression for fund j and test for an increase in the beta dispersion in recessions as well.

We measure outperformance by looking at abnormal fund returns, measured as the fund's return minus the market return, and several risk-adjusted returns. One way to do that risk adjustment is to estimate a CAPM for each fund's return and look at the fund α. Proposition 7 shows that the alpha of a CAPM regression of fund returns on market returns should capture a fund's total information capacity, or skill. As a robustness check, we also compute the α from models with multiple risk factors that are common in the empirical literature, with the proviso that these additional risk factors are not present in our model.

3 Evidence from Equity Mutual Funds

Our model studies attention allocation over the business cycle, and its consequences for investors' strategies. We now turn to a specific set of investors, active U.S. equity mutual fund managers, to test the predictions of the model. The richness of the data makes the mutual fund industry a great laboratory for these tests. In principle, similar tests could be conducted for hedge funds, real estate investment trusts, other professional investment managers, or even individual investors.

3.1 Data

Our sample builds upon several data sets. We begin with the Center for Research on Security Prices (CRSP) survivorship bias-free mutual fund database. The CRSP database provides comprehensive information about fund returns and a host of other fund characteristics, such as size (total net assets), age, expense ratio, turnover, and load. Given the nature of our tests and data availability, we focus on actively managed open-end U.S. equity mutual funds. We further merge the CRSP data with fund holdings data from Thomson Financial. The total number of funds in our merged sample is 3477.[9] We also use the CRSP/Compustat stock-level database, which is a source of information on individual stocks' returns, market capitalizations, book-to-market ratios, momentum, liquidity, and standardized unexpected earnings (SUE). The aggregate stock market return is the value-weighted average return of all stocks in the CRSP universe.

We use innovations in monthly seasonally adjusted industrial production, obtained from the Federal Reserve Statistical Release, as a proxy for aggregate shocks. We measure recessions using the definition of the National Bureau of Economic Research (NBER) business cycle dating committee. The start of the recession is the peak of economic activity and its end is the trough. Our aggregate sample spans 312 months of data from January 1980 until

[9]The unit of observation is a fund. In Supplementary Appendix S.10, we verify that our results are robust to using the manager as a unit of observation.

December 2005, among which 38 are NBER recession months (12%). We consider several alternative recession indicators and find our results to be robust.[10]

3.2 Motivating Fact: Aggregate Risk and Prices of Risk Rise in Recessions

At the outset, we present empirical evidence for the main assumption in our model: Recessions are periods in which individual *stocks* contain more aggregate risk and prices of risk are higher.

Table 1 shows that an average stock's aggregate risk increases substantially in recessions whereas the change in idiosyncratic risk is not statistically different from zero. The table uses monthly returns for all stocks in the CRSP universe. For each stock and each month, we estimate a CAPM equation based on a twelve-month rolling-window regression, delivering the stock's beta, β_t^i, and its residual standard deviation, $\sigma_{\varepsilon t}^i$. We define the aggregate risk of stock i in month t as $|\beta_t^i \sigma_t^m|$ and its idiosyncratic risk as $\sigma_{\varepsilon t}^i$, where σ_t^m is formed monthly as the realized volatility from daily return observations. Panel A reports the results from a time-series regression of the aggregate risk (columns 1 and 2), the idiosyncratic risk (columns 3 and 4), and the ratio of aggregate to idiosyncratic risk (columns 5 and 6), all averaged across stocks, on the NBER recession indicator variable. The aggregate risk is twenty percent higher in recessions than it is in expansions (8.04% versus 6.69% per month), an economically and statistically significant difference. In contrast, the stock's idiosyncratic risk is not statistically different in expansions and in recessions. As a result, the ratio of aggregate to idiosyncratic risk increases from 0.508 in expansions to 0.606 in recessions, and this cyclicality is driven exclusively by the numerator. The results are similar whether one controls for other aggregate risk factors (columns 2, 4, and 6) or not (columns 1, 3, and 5).

Panel B reports estimates from pooled (panel) regressions of each stock's aggregate risk (columns 1 and 2), idiosyncratic risk (columns 3 and 4), or the ratio of aggregate to idiosyncratic risk (columns 5 and 6) on the recession indicator variable, *Recession*, and additional stock-specific control variables including size, book-to-market ratio, and leverage. The panel results confirm the time-series findings.

A large literature in economics and finance presents evidence supporting the results in Table 1. First, Ang and Chen (2002), Ribeiro and Veronesi (2002), and Forbes and Rigobon (2002) document that stocks exhibit more comovement in recessions, consistent with stocks

[10] Results are omitted for brevity but are available from the authors upon request.

Table 1: **Individual Stocks Have More Aggregate Risk in Recessions**

For each stock i and month t, we estimate a CAPM equation based on twelve months of data (a twelve-month rolling-window regression). This estimation delivers the stock's beta, β_t^i, and its residual standard deviation, $\sigma_{\varepsilon t}^i$. We define stock i's aggregate risk in month t as $\left|\beta_t^i \sigma_t^m\right|$ and its idiosyncratic risk as $\sigma_{\varepsilon t}^i$, where σ_t^m is the realized volatility from daily market return observations. Panel A reports results from a time-series regression of the average stock's aggregate risk, $\frac{1}{N}\sum_{i=1}^N \left|\beta_t^i \sigma_t^m\right|$, in columns 1 and 2, of the average idiosyncratic risk, $\frac{1}{N}\sum_{i=1}^N \sigma_{\varepsilon t}^i$, in columns 3 and 4, and of the ratio of aggregate to average idiosyncratic risk, in columns 5 and 6, on *Recession*. *Recession* is an indicator variable equal to one for every month the economy is in a recession according to the NBER, and zero otherwise. In columns 2, 4, and 6 we include several aggregate control variables: the market excess return (MKTPREM), the return on the small-minus-big portfolio (SMB), the return on the high-minus-low book-to-market portfolio (HML), the return on the up-minus-down momentum portfolio (UMD). The data are monthly from 1980-2005 (309 months). Standard errors (in parentheses) are corrected for autocorrelation and heteroscedasticity. Panel B reports results of panel regressions of each stock's aggregate risk, $\left|\beta_t^i \sigma_t^m\right|$, in columns 1 and 2 and of its idiosyncratic risk, $\sigma_{\varepsilon t}^i$, in columns 3 and 4, and of the ratio of a stock's aggregate to idiosyncratic risk, in columns 5 and 6, on *Recession*. In Columns 2, 4, and 6 we include several firm-specific control variables: the log market capitalization of the stock, $log(Size)$, the ratio of book equity to market equity, $B-M$, the average return over the past year, $Momentum$, the stock's ratio of book debt to book debt plus book equity, *Leverage*, and an indicator variable, $NASDAQ$, equal to one if the stock is traded on NASDAQ. All control variables are lagged one month. The data are monthly and cover all stocks in the CRSP universe for 1980-2005. Standard errors (in parentheses) are clustered at the stock and time dimensions.

	(1)	(2)	(3)	(4)	(5)	(6)
	Aggregate Risk		Idiosyncratic Risk		Aggregate/Idiosyncratic Risk	
Panel A: Time-Series Regression						
Recession	1.348	1.308	0.058	0.016	0.098	0.097
	(0.693)	(0.678)	(1.018)	(1.016)	(0.027)	(0.027)
MKTPREM		-4.034		-1.865		-0.215
		(3.055)		(3.043)		(0.226)
SMB		8.110		12.045		0.167
		(3.780)		(4.923)		(0.199)
HML		0.292		9.664		-0.308
		(5.458)		(8.150)		(0.302)
UMD		-4.279		-1.112		-0.270
		(2.349)		(3.888)		(0.178)
Constant	6.694	6.748	13.229	13.196	0.508	0.513
	(0.204)	(0.212)	(0.286)	(0.276)	(0.013)	(0.014)
Observations	309	309	309	309	309	309
R-squared	6.85	9.70	0.10	3.33	8.58	10.52
Panel B: Pooled Regression						
Recession	1.203	1.419	0.064	0.510	0.096	0.104
	(0.242)	(0.238)	(0.493)	(0.580)	(0.021)	(0.024)
Log(Size)		-0.145		-1.544		0.043
		(0.021)		(0.037)		(0.002)
B-M Ratio		-0.934		-2.691		0.008
		(0.056)		(0.086)		(0.004)
Momentum		0.097		2.059		-0.040
		(0.101)		(0.177)		(0.005)
Leverage		-0.600		-1.006		-0.010
		(0.074)		(0.119)		(0.003)
NASDAQ		0.600		1.937		-0.043
		(0.075)		(0.105)		(0.005)
Constant	4.924	4.902	12.641	12.592	0.450	0.450
	(0.092)	(0.095)	(0.122)	(0.144)	(0.009)	(0.009)
Observations	1,312,216	1,312,216	1,312,216	1,312,216	1,312,216	1,312,216
R-squared	0.62	2.90	0.000	19.33	0.58	7.56

carrying higher systematic risk in recessions. In addition, Schwert (1989, 2011), Hamilton and Lin (1996), Campbell, Lettau, Malkiel, and Xu (2001), and Engle and Rangel (2008) show that aggregate stock market return volatility is much higher during periods of low economic activity. The evidence on the cyclicality of idiosyncratic risk is less unanimous. Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) find that the cross-sectional dispersion in firm earnings growth rises in recessions. Using a similar measure of stock-specific risk as ours, Campbell, Lettau, Malkiel, and Xu (2001) also report an increase in firm-level risk in recessions. Exploring the difference with the Campbell et al. (2001) results, Supplementary Appendix S.8 shows that the countercyclicality of idiosyncratic risk only holds for a value-weighted measure and only in the Campbell, Lettau, Malkiel, and Xu (2001) sample. For our sample as well as for a long sample, we find no significant differences between idiosyncratic risk in expansions and recessions. Finally, we reiterate that what matters for our theoretical results is that aggregate risk rises by more than idiosyncratic risk in recessions, a conclusion supported by the last two columns of Table 1.

Our second assumption, that the price of risk rises in recessions, is supported in four ways. First is an empirical literature that documents the countercyclical nature of risk premia and Sharpe ratios on equity, bonds, options, and currencies.[11] Second, a large theoretical literature has developed models that generate such counter-cyclical market prices of risk (see Section 2). Third, Dew-Becker (2012) uses the structure of his model to construct an empirical proxy for risk aversion and shows it rises in recessions. Fourth, several papers show that aggregate risk aversion rises in recessions because of properties of aggregation.[12]

3.3 Testing Predictions 1 and 2: Time-Varying Skill

Turning to our main model predictions, we first test whether skilled investment managers reallocate their attention over the business cycle in a way that is consistent with measures of time-varying skill. To estimate time-varying skill, we need measures of $Ftiming$ and $Fpicking$ for each fund j in each month t. We proxy for the aggregate payoff shock with

[11] E.g., Fama and French (1989), Cochrane (2006), Ludvigson and Ng (2009), Lettau and Ludvigson (2010), Lustig, Roussanov, and Verdelhan (2014), and the references therein. A related fact consistent with counter-cyclical market prices of risk is high corporate bond yields in recessions despite only modestly higher default rates, see Chen (2010).

[12] See Dumas (1989), Chan and Kogan (2002), and Garleanu and Panageas (2015), among others. In these models, heterogeneous agents with the same preferences but different risk aversion parameters aggregate into a representative agent who has wealth-weighted functions of the individual agent's parameters. Because more risk-averse agents are more conservative, their relative wealth rises in recessions, making aggregate risk aversion counter-cyclical.

the innovation in log industrial production growth, estimated from an AR(1).[13] A time series of $Ftiming_t^j$ is obtained by computing the covariance of the innovations and each fund j's portfolio weights (as in equation (16)), using twelve-month rolling windows. Following equation (17), *Fpicking* is computed in each month t as a cross-sectional covariance across the assets between the fund's portfolio weights and firm-specific earnings shocks (SUE). We then estimate the following two equations using pooled (panel) regression model and calculating standard errors by clustering at the fund and time dimensions.

$$\begin{aligned} Fpicking_t^j &= a_0 + a_1 Recession_t + \mathbf{a_2 X_t^j} + \epsilon_t^j, & (19) \\ Ftiming_t^j &= a_3 + a_4 Recession_t + \mathbf{a_5 X_t^j} + \varepsilon_t^j, & (20) \end{aligned}$$

$Recession_t$ is an indicator variable equal to one if the economy in month t is in recession, as defined by the NBER, and zero otherwise. X is a vector of fund-specific control variables, including the fund age, the fund size, the average fund expense ratio, the turnover rate, the percentage flow of new funds, the fund load, the volatility of fund flows, and the fund style characteristics along the size, value, and momentum dimensions.

Our model predicts that *Ftiming* should be higher in recessions, which means that the coefficient of *Recession*, a_4, should be positive. Conversely, the fund's portfolio holdings and its returns covary more with subsequent firm-specific shocks in expansions. Therefore, our hypothesis is that *Fpicking* should fall in recessions, or that a_1 should be negative.

The parameter estimates appear in columns 1, 2, 4, and 5 of Table 2. Column 1 shows the results for a univariate regression model. In expansions, *Ftiming* is not different from zero, implying that funds' portfolios do not comove with future macroeconomic information in those periods. In recessions, *Ftiming* increases. The increase amounts to ten percent of a standard deviation of *Ftiming*. It is measured precisely, with a t-statistic of 3. To remedy the possibility of a bias in the coefficient due to omitted fund characteristics correlated with recession times, we turn to a multivariate regression. Our findings, in column 2, remain largely unaffected by the inclusion of the control variables. Columns 4 and 5 of Table 2 show that the average *Fpicking* across funds is positive in expansions and substantially lower in recessions. The effect is statistically significant at the 1% level. It is also economically significant: *Fpicking* decreases by approximately ten percent of one standard deviation. In sum, the data support both main predictions of the theory: Portfolio holdings are more sensitive

[13]Our results are robust to using industrial productions growth itself. Our results are also robust to measuring aggregate shocks to fundamentals as innovations in non-farm employment growth

Table 2: **Attention Allocation is Cyclical**

Dependent variables: Fund j's $Ftiming_t^j$ is defined in equation (16), where the rolling window T is 12 months and the aggregate shock a_{t+1} is the change in industrial production growth between t and $t+1$. A fund j's $Fpicking_t^j$ is defined as in equation (17), where s_{it+1} is the change in asset i's earnings growth between t and $t+1$. All are multiplied by 10,000 for readability. Independent variables: *Recession* is an indicator variable equal to one for every month the economy is in a recession according to the NBER, and zero otherwise. $Log(Age)$ is the natural logarithm of fund age in years. $Log(TNA)$ is the natural logarithm of a fund total net assets. *Expenses* is the fund expense ratio. *Turnover* is the fund turnover ratio. *Flow* is the percentage growth in a fund's new money. *Load* is the total fund load. $FlowVol$ is the volatility of fund flows, measures from the last twelve months of fund flows. The last three control variables measure the style of a fund along the size, value, and momentum dimensions, calculated from the scores of the stocks in their portfolio in that month. They are omitted for brevity. All control variables are demeaned. Flow and Turnover are winsorized at the 1% level. $PriceofRisk$ is an indicator variable for periods with high default spread. Default spread is defined as a difference in yields of Baa and Aaa-rated U.S. corporate bonds. Price of risk equals one if default spread is in the highest 10% of months in the sample. $Volatility$ is an indicator variable for periods of high volatile earnings. We calculate the twelve-month rolling-window standard deviation of the year-to-year log change in the earnings of S&P 500 index constituents; the earnings data are from Robert Shiller for 1926-2008. Volatility equals one if this standard deviation is in the highest 10% of months in the 1926-2008 sample. During 1985-2005, 12% of months are such high volatility months. The data are monthly and cover the period 1980 to 2005. Standard errors (in parentheses) are clustered by fund and time.

	(1)	(2)	(3)	(4)	(5)	(6)
		Ftiming			Fpicking	
Recession	0.011	0.012		-0.742	-0.680	
	(0.003)	(0.003)		(0.138)	(0.126)	
Price of Risk			0.019			-2.780
			(0.012)			(0.514)
Volatility			0.003			-0.440
			(0.003)			(0.123)
Log(Age)		-0.001	-0.001		0.447	0.040
		(0.001)	(0.001)		(0.061)	(0.144)
Log(TNA)		-0.001	-0.001		-0.130	-0.225
		(0.000)	(0.000)		(0.029)	(0.052)
Expenses		-0.208	-0.208		96.748	-90.819
		(0.219)	(0.219)		(11.200)	(21.241)
Turnover		-0.004	-0.004		-0.260	0.182
		(0.001)	(0.001)		(0.063)	(0.087)
Flow		-0.010	-0.010		0.637	1.305
		(0.011)	(0.011)		(0.652)	(0.526)
Load		0.006	0.006		-9.851	-9.876
		(0.022)	(0.023)		(1.951)	(5.322)
Flow Vol		-0.006	-0.004		6.684	3.931
		(0.017)	(0.017)		(1.042)	(1.164)
Constant	-0.001	0.000	-0.001	3.082	3.238	3.119
	(0.001)	(0.002)	(0.001)	(0.069)	(0.107)	(0.072)
Observations	221,488	221,488	221,488	165,029	165,029	165,029
R-squared	0.03	0.09	0.08	0.03	0.25	0.21

to aggregate shocks in recessions and more sensitive to firm-specific shocks in expansions.

These results differ from Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014) who measure timing and picking as covariances with returns, rather than covariances with fundamental payoffs. The return-based results in Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014) could, in principle, be explained by funds who forecast non-fundamental return drivers such as sentiment, momentum, liquidity, etc. That would be harder to reconcile with an information-processing theory like ours. In unreported results, we construct a measure of covariation of portfolio weights with innovations to the Baker and Wurgler (2006) sentiment index. We subsequently correlate this measure with the (return-based) timing measure, but find no relationship between the two quantities. In contrast, *Ftiming* shows a strong positive correlation with the return-based timing measure, highlighting that managers seem to adjust portfolio weights in anticipation of fundamental news.

Testing for Separate Effects of Volatility and Price of Risk. To identify a more nuanced prediction of the model, we can split the recession effect into that which comes from aggregate volatility and that which comes from an increased price of risk. Proposition 1 predicts that an increase in aggregate volatility alone should cause managers to reallocate attention to aggregate shocks. Furthermore, there should be an additional effect of recessions, after controlling for aggregate volatility, that comes from the increase in the price of risk (Proposition 2). To test for these two separate effects, we re-estimate the previous results with both an indicator for price of risk and an indicator for high aggregate payoff volatility. The price of risk indicator variable equals one in months with the highest level of default spread where default spread is defined as a difference in yields between BBB and AAA-rated bonds. The high-volatility indicator variable equals one in months with the highest volatility of aggregate earnings growth, where aggregate volatility is estimated from Shiller's S&P 500 earnings growth data.[14] We include both high price of risk and high aggregate payoff volatility indicators as explanatory variables in an empirical horse race.

Columns 3 and 6 of Table 2 show that both price of risk and volatility contribute to a lower *Fpicking* in expansions. For the *Ftiming* result, the price of risk effect is much stronger and drives out some of the volatility effect, while for the *Fpicking* result both price of risk and volatility contribute to a large degree. Clearly, there is an effect of recessions beyond the one coming through volatility. This is consistent with the predictions of our model, where recessions are characterized both by an increase in aggregate payoff volatility and an increase in the price of risk. In the Supplementary Appendix (Section S.9), we also explore a non-linear volatility specification and find the same pattern but somewhat stronger effects for the highest-volatility periods. Finally, when we interact volatility with a recession indicator and with an expansion indicator, we find the strongest effects of volatility in recessions. This is consistent with the model's prediction that the effect of aggregate risk

[14] We calculate the twelve-month rolling-window standard deviation of aggregate earnings growth. The volatility cutoff selects 6% of months. Of the high-volatility periods, 28% are recessions. Of all other periods (when high-volatility indicator is 0), 10.6% are recessions. Conversely, 14% of recessions are also high-volatility periods whereas only 4.8% of expansions are high-volatility periods.

(volatility) should be strong in recessions, when the price of risk is high (Section 1.7).

3.4 Testing Predictions 3 and 4: Dispersion

The second main prediction of the model states that heterogeneity in fund investment strategies and portfolio returns rises in recessions. To test this hypothesis, we estimate the following regression specification, using various return and investment heterogeneity measures, generically denoted as $Dispersion_t^j$, the dispersion of fund j at month t.

$$Dispersion_t^j = g_0 + g_1 Recession_t + \mathbf{g_2 X_t^j} + \epsilon_t^j, \tag{21}$$

The definitions of $Recession$ and other controls mirror those in regression (19). Our coefficient of interest is g_1.

The first dispersion measure we examine is $Portfolio\ Dispersion$, defined in equation (18). It measures a deviation of a fund's investment strategy from a passive market strategy, and hence, in equilibrium, from the strategies of other investors. The results in columns 1 and 2 of Table 3 indicate an increase in average $Portfolio\ Dispersion$ across funds in recessions. The increase is statistically significant at the 1% level. It is also economically significant: The value of portfolio dispersion in recessions goes up by about 15% of a standard deviation.

Since dispersion in fund strategies should generate dispersion in fund returns, we next look for evidence of higher return dispersion in recessions. To measure dispersion, we use the absolute deviation between fund j's return and the equally weighted cross-sectional average, $|return_t^j - \overline{return}_t|$, as the dependent variable in (21). Columns 5 and 6 of Table 3 show that return dispersion increases by 17% in recessions. Finally, portfolio and return dispersion in recessions should come from different directional bets on the market. This should show up as an increase in the dispersion of portfolio betas. Columns 3 and 4 show that the CAPM-beta dispersion increases by 36% in recessions, all consistent with the predictions of our model.

These findings are robust. Counter-cyclical dispersion in funds' portfolio strategies is also found in measures of fund style shifting and sectoral asset allocation. The dispersion in returns is also found for abnormal returns and fund alphas. Results are available on request.

Testing for Separate Effects of Volatility and Price of Risk. Propositions 3 and 4 tell us that return dispersion increases in recessions for two reasons. One is that the volatility of aggregate shocks increases and the other reason is that the price of risk increases.

Table 3: **Portfolio and Return Dispersion Rise in Recessions**

Dependent variables: Portfolio dispersion is the Herfindahl index of portfolio weights in stocks $i \in \{1, \cdots, N\}$ in deviation from the market portfolio weights $\sum_{i=1}^{N}(w_{it}^{j} - w_{it}^{m})^2 \times 100$. Return dispersion is $|return_t^j - \overline{return}_t|$, where $\overline{return}$ denotes the (equally weighted) cross-sectional average. The CAPM beta comes from twelve-month rolling-window regressions of fund-level excess returns on excess market returns (and returns on SMB, HML, and MOM). Beta dispersion is constructed analogously to return dispersion. The right-hand side variables, the sample period, and the standard error calculation are the same as in Table 2.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Portfolio Dispersion		Beta Dispersion		Return Dispersion		
Recession	0.204	0.118	0.083	0.088	0.316	0.380	
	(0.027)	(0.025)	(0.015)	(0.014)	(0.147)	(0.146)	
Price of Risk							0.188
							(0.088)
Volatility							0.637
							(0.201)
Log(Age)		0.210		-0.005		-0.121	-0.108
		(0.028)		(0.002)		(0.017)	(0.018)
Log(TNA)		-0.165		0.004		0.043	0.035
		(0.014)		(0.001)		(0.009)	(0.010)
Expenses		31.986		4.162		28.330	25.526
		(4.867)		(0.212)		(2.621)	(2.519)
Turnover		-0.113		0.013		0.090	0.076
		(0.026)		(0.001)		(0.013)	(0.015)
Flow		-0.230		-0.004		-0.230	-0.280
		(0.108)		(0.018)		(0.223)	(0.218)
Load		-1.658		-0.318		-4.071	-3.519
		(0.900)		(0.041)		(0.509)	(0.517)
Flow Vol		2.379		0.075		1.570	1.905
		(0.304)		(0.027)		(0.240)	(0.242)
Constant	1.525	1.524	0.228	0.228	1.904	1.899	1.843
	(0.024)	(0.022)	(0.006)	(0.006)	(0.084)	(0.077)	(0.078)
Observations	227,141	227,141	224,130	224,130	227,141	227,141	227,141
R-squared	0.10	4.80	1.35	8.10	0.19	7.00	7.83

We can disentangle these two effects by regressing return dispersion on volatility and price of risk simultaneously. The model would predict that volatility should be a significant determinant of dispersion and that after controlling for volatility, there should be some additional explanatory power of recessions that comes from the price of risk effect.

Column 7 of Table 3 shows that both the price of risk and the volatility effects are present in the data. Both are associated with a significant increase in the dispersion of returns. The volatility and price of risk fluctuations both have significant effects on portfolio dispersion, with the effect of volatility being somewhat larger. Similar results are found for the other dispersion measures. A non-linear volatility specification in the Supplementary Appendix shows that the effect of volatility on return dispersion is strongest in high-volatility periods. Both recession and high-volatility indicators are significant when a recession indicator is used instead of the price of risk as explanatory variable. Finally, the volatility effect on dispersion is significant both in recessions and expansions. But the fact that it is twice as strong in recessions supports the interaction effect predicted by the theory (Section 1.7).

3.5 Testing Predictions 5 and 6: Performance

The third prediction of our model is that recessions are times when information allows funds to earn higher average risk-adjusted returns. Empirical work by Moskowitz (2000), Kosowski (2011), Glode (2011), and de Souza and Lynch (2012) also documents such evidence. Their results are based on time-series analysis, while we account for differences in fund size, age, turnover, flows, loads, style and flow volatility, using the following regression specification:

$$Performance_t^j = c_0 + c_1 Recession_t + \mathbf{c_2 X_t^j} + \epsilon_t^j \tag{22}$$

where $Performance_t^j$ denotes fund j's performance in month t, measured as fund abnormal returns, or CAPM, three-factor, or four-factor alphas. All returns are net of management fees. The coefficient of interest is c_1.

Column 1 of Table 4 shows that the average fund's net return is statistically indistinguishable from zero in expansions. But the coefficient of *Recession* is 38bp per month, implying that the average mutual fund's abnormal return is 4.6% per year higher in recessions. This difference is highly statistically significant and increases further after we control for fund characteristics (column 2). Similar results (columns 3 and 4) obtain when we use the CAPM alpha as a measure of fund performance, except that the net alpha is now significantly negative in expansions. In recessions, the 34bp per month higher net alpha corresponds to 4% per year. When we use alphas from the three- and four-factor models, the recession return premium diminishes (columns 5-8). But in recessions, the four-factor alpha still represents a non-trivial 1% per year risk-adjusted excess return, 1.6% higher (significant at the 1% level) than the -0.6% recorded in expansions.

The advantage of this cross-sectional regression model is that it allows us to include fund-specific control variables. The disadvantage is that performance is measured using past twelve-month rolling-window regressions. Thus, a given observation can be classified as a recession when some or even all of the remaining eleven months of the window are expansions. To verify the robustness of our cross-sectional results, we also employ a time-series approach.[15] We explore alternative performance measures, such as *gross* fund returns,

[15] In each month, we form the equally weighted portfolio of funds and calculate its net return, in excess of

Table 4: **Fund Performance Improves in Recessions**

Dependent variables: *Abnormal Return* is the fund return minus the market return. The alphas come from twelve-month rolling-window regressions of fund-level excess returns on excess market returns for the CAPM alpha, additionally on the size (SMB) and the book-to-market (HML) factors for the three-factor alpha, and additionally on the momentum (UMD) factor for the four-factor alpha. The independent variables, the sample period, and the standard error calculations are the same as in Table 2.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Abnormal Return		CAPM Alpha		3-Factor Alpha		4-Factor Alpha		
Recession	0.384	0.433	0.339	0.399	0.043	0.062	0.108	0.131	
	(0.056)	(0.059)	(0.048)	(0.050)	(0.034)	(0.026)	(0.041)	(0.033)	
Price of Risk									0.052
									(0.031)
Volatility									0.149
									(0.061)
Log(Age)		-0.015		-0.032		-0.023		-0.035	-0.036
		(0.021)		(0.008)		(0.006)		(0.006)	(0.006)
Log(TNA)		0.023		0.040		0.018		0.019	0.010
		(0.013)		(0.004)		(0.003)		(0.003)	(0.003)
Expenses		-5.120		-0.929		-5.793		-5.970	-8.277
		(2.817)		(0.892)		(0.720)		(0.677)	(1.248)
Turnover		0.021		-0.054		-0.087		-0.076	-0.068
		(0.039)		(0.010)		(0.010)		(0.008)	(0.009)
Flow		2.127		2.308		1.510		1.386	1.544
		(0.672)		(0.172)		(0.096)		(0.096)	(0.056)
Load		-0.698		-0.810		-0.143		-0.371	-0.205
		(0.457)		(0.174)		(0.129)		(0.139)	(0.200)
Flow vol		-0.106		1.025		1.461		1.210	1.311
		(0.588)		(0.137)		(0.109)		(0.104)	(0.109)
Constant	-0.032	-0.036	-0.060	-0.065	-0.059	-0.061	-0.051	-0.053	-0.066
	(0.064)	(0.063)	(0.025)	(0.024)	(0.020)	(0.018)	(0.023)	(0.021)	(0.021)
Observations	224,130	224,130	224,130	224,130	224,130	224,130	224,130	224,130	224,130
R-squared	0.01	0.57	1.15	10.70	0.03	6.20	0.16	5.50	5.16

gross alphas, or the information ratio (the ratio of the CAPM alpha to the CAPM residual volatility). Finally, we find similar results when we lead alpha on the left-hand side by one month instead of using a contemporaneous alpha. All results point in the same direction: Outperformance increases in recessions.

Testing for Separate Effects of Volatility and Price of Risk. As before, two forces increase the performance of funds relative to non-funds in recessions: the increase in volatility and the increase in the price of risk (Propositions 5 and 6). Column 9 of Table 4 shows that the data are consistent with each force having a distinct effect on fund outperformance. We use the 4-factor alpha as the dependent variable for this exercise because we want to avoid

the risk-free rate. We then regress this time series of fund portfolio returns on *Recession* and common risk factors, adjusting standard errors for heteroscedasticity and autocorrelation.

conflating more risk taking in recessions with greater fund outperformance in recessions. When we regress each fund's 4-factor alpha on a price of risk indicator and a volatility measure, both have positive, significant coefficients. We also estimated the effect of price of risk and volatility on the other three measures of performance. The results are qualitatively similar but quantitatively stronger. A non-linear volatility specification shows that the effect of volatility on performance is strongest in high-volatility periods (Supplementary Appendix S.9). In a specification that adds the recession indicator both recession and high-volatility indicators retain significance. The volatility effect is only significant in recessions. These results suggest that fund outperformance in recessions is due mostly to the higher volatility of aggregate shocks and is due to a lesser extent to the increased price of risk. But the fact that both variables have a significant relationship with fund outperformance, dispersion, and attention, in the direction predicted by the theory offers solid support for the model. Furthermore, the fact that the volatility effect is four times as strong in recessions as in expansions is empirical support for the interaction effect between volatility and price of risk predicted by the model.

4 Conclusion

Do investment managers add value for their clients? The answer to this question matters for issues ranging from the discussion of market efficiency to practical portfolio advice for households. The large amount of randomness in financial asset returns makes it a difficult question to answer. The multi-billion investment management industry is first and foremost an information-processing business. We model investment managers not only as agents making optimal portfolio decisions, but also as human beings with finite mental capacity (attention), who optimally allocate that scarce capacity to process information at each point in time. Since the optimal attention allocation varies with the state of the economy, so do investment strategies and fund returns. As long as a subset of skilled investment managers can process information about future asset payoffs, the model predicts a higher covariance of portfolio holdings with aggregate asset payoff shocks, more cross-sectional dispersion in portfolio investment strategies and returns across funds, and a higher average outperformance in recessions. We observe these patterns in investments and returns of actively managed U.S. mutual funds. Hence, the data are consistent with a world in which some investment managers have skill.

On the technical side, our paper contributes a novel change-of variable technique to solve models with normal signals, but arbitrary signal structures. These tools can be used to generalize the information assumptions in related models of rational inattention, a generalization advocated by Sims (2006).

Beyond the mutual fund industry, a sizeable fraction of GDP currently comes from industries that produce and process information (consulting, business management, product design, marketing analysis, accounting, rating agencies, equity analysts, etc.). Ever increasing access to information has made the problem of how to best allocate a limited amount of information-processing capacity ever more relevant. While information choices have consequences for real outcomes, they are often poorly understood because they are difficult to measure. By predicting how information choices are linked to observable variables (such as the state of the economy) and by tying information choices to real outcomes (such as portfolio investment), we show how models of information choices can be brought to the data. This information-choice-based approach could be useful in examining other information-processing sectors of the economy.

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A Proofs

A.1 Useful notation, matrices and derivatives

All the following matrices are diagonal with ii entry given by:

1. Precision of the information prices convey about shock i: $(\Sigma_p^{-1})_{ii} = \frac{1}{\rho^2\sigma_x}(\bar{\Sigma}_\eta^{-1})_{ii}^2 = \frac{\bar{K}_i^2}{\rho^2\sigma_x} = \sigma_{ip}^{-1}$

2. Precision of posterior belief about shock i for an investor j is $\hat{\sigma}_{ij}^{-1}$, which is equivalent to

$$(\hat{\Sigma}_j^{-1})_{ii} = (\Sigma^{-1} + \Sigma_{\eta j}^{-1} + \Sigma_p^{-1})_{ii} = \sigma_i^{-1} + K_{ij} + \frac{\bar{K}_i^2}{\rho^2\sigma_x} = \hat{\sigma}_{ij}^{-1} \tag{23}$$

3. Average signal precision: $(\bar{\Sigma}_\eta^{-1})_{ii} = \bar{K}_i, \quad$ where $\quad \bar{K}_i \equiv \int K_{ij}dj$. Since we focus on symmetric equilibria and the fraction of skilled investors is χ, $\bar{K}_i = \chi K_{ij}$ for any skilled investor j.

4. Average posterior precision of shock i: $\bar{\sigma}_i^{-1} \equiv \sigma_i^{-1} + \bar{K}_i + \frac{\bar{K}_i^2}{\rho^2\sigma_x}$. The average variance is therefore $\bar{\Sigma}_{ii} = [\left(\sigma_i^{-1} + \bar{K}_i + \frac{\bar{K}_i^2}{\rho^2\sigma_x}\right)]^{-1} = \bar{\sigma}_i$, with derivatives:

$$\frac{\partial\bar{\sigma}_i}{\partial\sigma_i} = \left(\frac{\bar{\sigma}_i}{\sigma_i}\right)^2 > 0, \tag{24}$$

$$\frac{\partial\bar{\sigma}_i}{\partial\rho} = \frac{2}{\rho}\frac{\bar{\sigma}_i^2}{\sigma_{ip}} > 0. \tag{25}$$

5. Difference from average posterior beliefs: Recall that $\bar{\Sigma}_\eta^{-1} \equiv \int \Sigma_{\eta j}^{-1}dj$ is the average private signal precision and that $\bar{\Sigma}^{-1} \equiv \int \hat{\Sigma}_j^{-1}dj = \Sigma^{-1} + \Sigma_p^{-1} + \bar{\Sigma}_\eta^{-1}$ is the average posterior precision. Define Δ as the difference between the precision of an informed investor's posterior beliefs and the average posterior precision. Since the $\Sigma^{-1} + \Sigma_p^{-1}$ terms are equal for all investors, this quantity is also equal to the difference between the precision of an informed investor's private signals and the average private signal precision:

$$\Delta \equiv \hat{\Sigma}_j^{-1} - \bar{\Sigma}^{-1} = \Sigma_{\eta j}^{-1} - \bar{\Sigma}_\eta^{-1}. \tag{26}$$

   In symmetric information choice equilibria, $\Delta = (1-\chi)\Sigma_{\eta j}^{-1}$ for any skilled investor j.

6. Ex-ante mean and variance of returns: Using Lemma 1 and the coefficients given by (42), we can write the risk factor return as:

$$\begin{aligned} \tilde{f} - \tilde{p}r &= (I-B)z - Cx - A \\ &= \bar{\Sigma}\left[\Sigma^{-1}z + \rho\left(I + \frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\right)x\right] + \rho\bar{\Sigma}\bar{x}. \end{aligned}$$

   This expression is a constant plus a linear combination of two normal variables, which is also a normal variable. Therefore, we can write

$$\tilde{f} - \tilde{p}r = V^{1/2}u + w, \tag{27}$$

   where u is a standard normally distributed random variable $u \sim N(0, I)$, and w is a non-random vector measuring the ex-ante mean of excess returns

$$w \equiv \rho\bar{\Sigma}\bar{x}. \tag{28}$$

and V is the ex-ante variance matrix of excess returns:

$$\begin{aligned}
V &\equiv \bar{\Sigma}\left[\Sigma^{-1}+\rho^2\sigma_x\left(I+\frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\right)\left(I+\frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\right)'\right]\bar{\Sigma} \\
&= \bar{\Sigma}\left[\Sigma^{-1}+\rho^2\sigma_x\left(I+\frac{1}{\rho^2\sigma_x}(\bar{\Sigma}_\eta^{-1'}+\bar{\Sigma}_\eta^{-1})+\frac{1}{\rho^4\sigma_x^2}\bar{\Sigma}_\eta^{-1'}\bar{\Sigma}_\eta^{-1}\right)\right]\bar{\Sigma} \\
&= \bar{\Sigma}\left[\Sigma^{-1}+\rho^2\sigma_x I+(\bar{\Sigma}_\eta^{-1'}+\bar{\Sigma}_\eta^{-1})+\frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\bar{\Sigma}_\eta^{-1}\right]\bar{\Sigma} \\
&= \bar{\Sigma}\left[\rho^2\sigma_x I+\bar{\Sigma}_\eta^{-1'}+\Sigma^{-1}+\bar{\Sigma}_\eta^{-1}+\Sigma_p^{-1}\right]\bar{\Sigma} \\
&= \bar{\Sigma}\left[\rho^2\sigma_x I+\bar{\Sigma}_\eta^{-1'}+\bar{\Sigma}^{-1}\right]\bar{\Sigma}.
\end{aligned}$$

The first line uses $E[xx'] = \sigma_x I$ and $E[zz'] = \Sigma$, the fourth line uses (43) and the fifth line uses $\bar{\Sigma}^{-1} = \Sigma^{-1} + \Sigma_p^{-1} + \bar{\Sigma}_\eta^{-1}$.

This variance matrix V is a diagonal matrix. Its diagonal elements are:

$$\begin{aligned}
V_{ii} &= (\bar{\Sigma}\left[\rho^2\sigma_x I+\bar{\Sigma}_\eta^{-1}+\bar{\Sigma}^{-1}\right]\bar{\Sigma})_{ii} \\
&= \bar{\sigma}_i[1+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i]. \qquad (29)
\end{aligned}$$

Diagonals of V have the following derivatives (using (24) and (25)):

$$\begin{aligned}
\frac{\partial V_{ii}}{\partial\sigma_i} &= \left(\frac{\bar{\sigma}_i}{\sigma_i}\right)^2\left(1+2(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i\right) > 0 \qquad (30) \\
\frac{\partial V_{ii}}{\partial\rho} &= 2\rho\sigma_x\bar{\sigma}_i^2\left[1+\frac{1}{\rho^2\sigma_x\sigma_{ip}}(1+2(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i)\right] > 0 \qquad (31)
\end{aligned}$$

7. The elasticity of V_{ii} with respect to ρ is

$$\begin{aligned}
\frac{\partial V_{ii}}{\partial\rho}\frac{\rho}{V_{ii}} &= 2\rho\sigma_x\bar{\sigma}_i^2\left[1+\frac{1}{\rho^2\sigma_x\sigma_{ip}}(1+2(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i)\right]\frac{\rho}{\bar{\sigma}_i[1+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i]} \\
&= \frac{2\rho^2\sigma_x\bar{\sigma}_i}{[1+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i]}\left[1+\frac{1}{\rho^2\sigma_x\sigma_{ip}}(1+2(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i)\right]
\end{aligned}$$

The second term is always larger than one. We look for a sufficient condition that also makes the first term larger than one:

$$\begin{aligned}
2\rho^2\sigma_x\bar{\sigma}_i &> 1+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i \\
\rho^2\sigma_x &> \bar{\sigma}_i^{-1}+\bar{K}_i \\
\rho^2\sigma_x &> \sigma_i^{-1}+2\bar{K}_i+\frac{\bar{K}_i^2}{\rho^2\sigma_x} \qquad (32)
\end{aligned}$$

Since the LHS is increasing in σ_x and the RHS is decreasing in σ_x, if σ_x is sufficiently high, the elasticity of V_{ii} with respect to ρ becomes larger than one.

A.2 Solving the Model

Step 1: Portfolio Choices From the FOC, the optimal portfolio of risk factors chosen by investor j is

$$\tilde{q}_j = \frac{1}{\rho}\hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r) \tag{33}$$

where $E_j[\tilde{f}]$ and $\hat{\Sigma}_j$ depend on the skill of the investor.

Next, we compute the risk factor portfolio of the average investor.

$$\begin{aligned}
\bar{\tilde{q}} \equiv \int \tilde{q}_j dj &= \frac{1}{\rho}\int \hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r)dj \\
&= \frac{1}{\rho}\left(\int \hat{\Sigma}_j^{-1}(\Gamma^{-1}\mu + E_j[z])dj - \bar{\Sigma}^{-1}\tilde{p}r\right) \\
&= \frac{1}{\rho}\left(\int \Sigma_{\eta j}^{-1}\eta_j dj + \Sigma_p^{-1}\eta_p + \bar{\Sigma}^{-1}(\Gamma^{-1}\mu - \tilde{p}r)\right) \\
&= \frac{1}{\rho}\left(\bar{\Sigma}_\eta^{-1}z + \Sigma_p^{-1}\eta_p + \bar{\Sigma}^{-1}(\Gamma^{-1}\mu - \tilde{p}r)\right), \qquad (34)
\end{aligned}$$

where the fourth equality uses the fact that average noise of private signals is zero. Using the portfolio expressions (33) and (34), we compute the difference between the portfolio of investor j and the average investor portfolio:

$$\begin{aligned}
\tilde{q}_j - \bar{\tilde{q}} &= \frac{1}{\rho}\left(\hat{\Sigma}_j^{-1}(E_j[\tilde{f}] - \tilde{p}r) - (\bar{\Sigma}_\eta^{-1} + \Sigma_p^{-1})z - \Sigma_p^{-1}\varepsilon_p - \bar{\Sigma}^{-1}(\Gamma^{-1}\mu - \tilde{p}r)\right) \\
&= \frac{1}{\rho}\left((\Sigma_{\eta j}^{-1}\eta_j + \Sigma_p^{-1}\eta_p) - \bar{\Sigma}_\eta^{-1}z - \Sigma_p^{-1}\eta_p + (\hat{\Sigma}_j^{-1} - \bar{\Sigma}^{-1})(\Gamma^{-1}\mu - \tilde{p}r)\right) \\
&= \frac{1}{\rho}\left((\Sigma_{\eta j}^{-1} - \bar{\Sigma}_\eta^{-1})z + \Sigma_{\eta j}^{-1}\varepsilon_j + (\hat{\Sigma}_j^{-1} - \bar{\Sigma}^{-1})(\Gamma^{-1}\mu - \tilde{p}r)\right) \\
&= \frac{1}{\rho}\left(\Delta(\tilde{f} - \tilde{p}r) + \Sigma_{\eta j}^{-1}\varepsilon_j)\right) \qquad (35) \\
&= \frac{1}{\rho}\left[\Sigma_{\eta j}^{-1}\varepsilon_j + \Delta(V^{1/2}u + w)\right], \qquad (36)
\end{aligned}$$

where the third equality uses $\eta_j = z + \varepsilon_j$, the fourth equality uses (26) and the definition $\tilde{f} = \Gamma^{-1}\mu + z$, and the last line uses (27).

Step 2: Clearing the asset market and computing expected excess return Lemma 1 describes the solution to the market-clearing problem and derives the coefficients A, B, and C in the pricing equation. The equilibrium price, along with the random signal realizations determines the time-2 expected return $(E_j[\tilde{f}] - \tilde{p}r)$. But at time 1, the equilibrium price and one's realized signals are not known. To compute period-1 utility, we need to know the time-1 expectation and variance of this time-2 expected return.

The time-2 expected excess return can be written as: $E_j[\tilde{f}] - \tilde{p}r = E_j[\tilde{f}] - \tilde{f} + \tilde{f} - \tilde{p}r$ and therefore its variance is:

$$V_1[E_j[\tilde{f}] - \tilde{p}r] = V_1[E_j[\tilde{f}] - \tilde{f}] + V_1[\tilde{f} - \tilde{p}r] + 2Cov_1[E_j[\tilde{f}] - \tilde{f}, \tilde{f} - \tilde{p}r]. \tag{37}$$

Combining (9) with the definitions $\eta_j = z + \varepsilon_j$ and $\eta_p = z + \varepsilon_p$, we can compute expectation errors:

$$\begin{aligned}
E_j[\tilde{f}] - \tilde{f} &= \hat{\Sigma}_j\left[(\Sigma_{\eta j}^{-1} + \Sigma_p^{-1} - \hat{\Sigma}_j^{-1})z + \Sigma_{\eta j}^{-1}\varepsilon_j + \Sigma_p^{-1}\varepsilon_p\right] \\
&= \hat{\Sigma}_j\left[-\Sigma^{-1}z + \Sigma_{\eta j}^{-1}\varepsilon_j + \Sigma_p^{-1}\varepsilon_p\right].
\end{aligned}$$

Since this is a sum of mean-zero variables, its expectation is $E_1[E_j[\tilde{f}]-\tilde{f}]=0$ and its variance is $V_1[E_j[\tilde{f}]-\tilde{f}]=\hat{\Sigma}_j\left[\Sigma^{-1}+\Sigma_{\eta j}^{-1}+\Sigma_p^{-1}\right]\hat{\Sigma}_j'=\hat{\Sigma}_j$.

From (27) we know that $V_1[\tilde{f}-\tilde{p}r]=V$. To compute the covariance term, we can use the definition $\tilde{f}=\Gamma^{-1}\mu+z$ and rearrange the definition of η_p to get $\tilde{p}r=B\eta_p+A$ and $\eta_p=z+\varepsilon_p$ to write

$$\begin{aligned}\tilde{f}-\tilde{p}r &= \Gamma^{-1}\mu+(I-B)z-A-B\varepsilon_p & (38)\\ &= \rho\bar{\Sigma}\bar{x}+\bar{\Sigma}\Sigma^{-1}z-(I-\bar{\Sigma}\Sigma^{-1})\varepsilon_p & (39)\end{aligned}$$

where the second line comes from substituting the coefficients A and B from Lemma 1. Since the constant $\rho\bar{\Sigma}\bar{x}$ does not affect the covariance, we can write

$$\begin{aligned}Cov_1[E_j[\tilde{f}]-\tilde{f},\tilde{f}-\tilde{p}r] &= Cov[-\hat{\Sigma}_j\Sigma^{-1}z+\hat{\Sigma}_j\Sigma_p^{-1}\varepsilon_p,\bar{\Sigma}\Sigma^{-1}z-(I-\bar{\Sigma}\Sigma^{-1})\varepsilon_p]\\ &= -\hat{\Sigma}_j\Sigma^{-1}\Sigma\bar{\Sigma}\Sigma^{-1}-\hat{\Sigma}_j\Sigma_p^{-1}\Sigma_p(I-\bar{\Sigma}\Sigma^{-1})]\\ &= -\hat{\Sigma}_j\bar{\Sigma}\Sigma^{-1}-\hat{\Sigma}_j(I-\bar{\Sigma}\Sigma^{-1}) \;=\; -\hat{\Sigma}_j\end{aligned}$$

Substituting the three variance and covariance terms into (37), we find that the variance of excess return is $V_1[E_j[\tilde{f}]-\tilde{p}r]=\hat{\Sigma}_j+V-2\hat{\Sigma}_j=V-\hat{\Sigma}_j$. Note that this is a diagonal matrix. Substituting the expressions (29) and (23) for the diagonal elements of V and $\hat{\Sigma}_j$ we have

$$V_1[E_j[\tilde{f}]-\tilde{p}r]=(V-\hat{\Sigma}_j)_{ii}=(\bar{\sigma}_i-\hat{\sigma}_i)+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i^2$$

In summary, the excess return is normally distributed as $E_j[\tilde{f}]-\tilde{p}r\sim\mathcal{N}(w,V-\hat{\Sigma}_j)$.

Step 3: Compute ex-ante expected utility Ex-ante expected utility for investor j is $U_{1j}=E_1\left[\rho E_j[W_j]-\frac{\rho^2}{2}V_j[W_j]\right]$. In period 2, the investor has chosen his portfolio and the price is in his information set, therefore the only random variable is z. We substitute the budget constraint in the optimal portfolio choice from (33) and take expectation and variance conditioning on $E_j[\tilde{f}]$ and $\hat{\Sigma}_j$ to obtain $U_{1j}=\rho rW_0+\frac{1}{2}E_1[(E_j[\tilde{f}]-\tilde{p}r)'\hat{\Sigma}_j(E_j[\tilde{f}]-\tilde{p}r)]$.

Define $m\equiv\hat{\Sigma}_j^{-1/2}(E_j[\tilde{f}]-\tilde{p}r)$ and note that $m\sim\mathcal{N}(\hat{\Sigma}_j^{-1/2}w,\hat{\Sigma}_j^{-1}V-I)$. The second term in the U_{ij} is equal to $E[m'm]$, which is the mean of a non-central Chi-square. Using the formula, if $m\sim\mathcal{N}(E[m],Var[m])$, then $E[m'm]=tr(Var[m])+E[m]'E[m]$, we get

$$U_{1j}=\rho rW_0+\frac{1}{2}tr(\hat{\Sigma}_j^{-1}V-I)+\frac{1}{2}w'\hat{\Sigma}_j^{-1}w.$$

Finally, we substitute the expressions for $\hat{\Sigma}_j^{-1}$ and w from (23) and (28):

$$\begin{aligned}U_{1j} &= \rho rW_0-\frac{N}{2}+\frac{1}{2}\sum_{i=1}^N\left(\sigma_i^{-1}+K_{ij}+\frac{\bar{K}_i^2}{\rho^2\sigma_x}\right)V_{ii}+\frac{\rho^2}{2}\sum_{i=1}^N\bar{x}_i^2\bar{\sigma}_i^2\left(\sigma_i^{-1}+K_{ij}+\frac{\bar{K}_i^2}{\rho^2\sigma_x}\right)\\ &= \frac{1}{2}\sum_{i=1}^N K_{ij}[V_{ii}+\rho^2\bar{x}_i^2\bar{\sigma}_i^2]+\rho rW_0-\frac{N}{2}+\frac{1}{2}\sum_{i=1}^N\left(\sigma_i^{-1}+\frac{\bar{K}_i^2}{\rho^2\sigma_x}\right)[V_{ii}+\rho^2\bar{x}_i^2\bar{\sigma}_i^2]\\ &= \frac{1}{2}\sum_{i=1}^N K_{ij}\lambda_i+constant & (40)\end{aligned}$$

$$\lambda_i=\bar{\sigma}_i[1+(\rho^2\sigma_x+\bar{K}_i)\bar{\sigma}_i]+\rho^2\bar{x}_i^2\bar{\sigma}_i^2 \quad (41)$$

where the weights λ_i are given by the variance of expected excess return V_{ii} from (29) plus a term that depends on the supply of the risk.

Step 4: Information choices The attention allocation problem maximizes ex-ante utility in (40) subject to the information capacity and no-forgetting constraints:

$$\begin{aligned} \max_{\{K_{ij}\}_{i=1}^{N}} \quad & \frac{1}{2}\sum_{i=1}^{N} K_{ij}\lambda_i + \text{constant} \\ \text{subject to} \quad & \sum_{i=1}^{N} K_{ij} \leq K \quad \text{and} \quad K_{ij} \geq 0 \quad \forall i \end{aligned}$$

Observe that λ_i depends only on parameters and on aggregate average precisions. Since each investor has zero mass within a continuum of investors, he takes λ_i as given. Since the constant is irrelevant, the optimal choice maximizes a weighted sum of attention allocations, where the weights are given by λ_i (equation (14)), subject to a constraint on an un-weighted sum. This is not a concave objective, so a first-order approach will not deliver a solution. A simple variational argument reveals that allocating all capacity to the risk(s) with the highest λ_i achieves the maximum utility. For a formal proof of this result, see Van Nieuwerburgh and Veldkamp (2010). Thus, the solution is given by: $K_{ij} = K$ if $\lambda_i = \max_k \lambda_k$, and $K_{ij} = 0$, otherwise. There may be multiple risks i that achieve the same maximum value of λ_i. In that case, the manager is indifferent about how to allocate attention between those risks. We focus on symmetric equilibria.

A.3 Proofs

Proof of Lemma 1

Proof. Following Admati (1985), we know that the equilibrium price takes the following form $\tilde{pr} = A + Bz + Cx$ where

$$\begin{aligned} A &= \Gamma^{-1}\mu - \rho\bar{\Sigma}\bar{x} \\ B &= I - \bar{\Sigma}\Sigma^{-1} \\ C &= -\rho\bar{\Sigma}\left(I + \frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\right) \end{aligned} \tag{42}$$

and therefore the price is given by $\tilde{pr} = \Gamma^{-1}\mu + \bar{\Sigma}\left[(\bar{\Sigma}^{-1} - \Sigma^{-1})z - \rho(\bar{x} + x) - \frac{1}{\rho\sigma_x}\bar{\Sigma}_\eta^{-1'}x\right]$. Furthermore, the precision of the public signal is

$$\Sigma_p^{-1} \equiv \left(\sigma_x B^{-1}CC'B^{-1'}\right)^{-1} = \frac{1}{\rho^2\sigma_x}\bar{\Sigma}_\eta^{-1'}\bar{\Sigma}_\eta^{-1} \tag{43}$$

□

Proof of Lemma 2

Proof. To show: If $\bar{x}_i$ is sufficiently large $\forall i$ and $\sum_i \sum_j K_{ij} \geq \underline{K}$, then there exist risks l and l' such that $\lambda_l = \lambda_{l'}$.

Suppose not. Then, there would be a unique maximum λ_i in the set of $\{\lambda_l\}_{l=1}^N$, no matter how large K is. Since there is a unique maximum, the solution above dictates that all information capacity is used to study this risk: $K_{ij} = K$ for all skilled investors j. Thus, $\bar{K}_i$ becomes arbitrarily large.

However, the value of learning about risk i falls as the aggregate capacity devoted to studying it increases: $\partial\lambda_i/\partial\bar{K}_i < 0$. We show this next. The solution for λ_i is given by (41). It is clearly increasing in $\bar{K}_i$ directly. But there is also an indirect negative effect through $\bar{\sigma}_i$. Recall that by Bayes' Law, the average posterior precision $\bar{\sigma}_i^{-1} = \sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i$. Thus, $\frac{\partial \bar{\sigma}_i}{\partial \bar{K}_i} < 0$. To sign the net effect, it is helpful to rewrite λ_i as

$$\lambda_i = \bar{\sigma}_i^2 \left[\bar{\sigma}_i^{-1} + \rho^2(\sigma_x + \bar{x}_i^2) + \bar{K}_i\right]$$

Substituting in $\bar{\sigma}_i^{-1} = \sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i$, we get

$$\lambda_i = \frac{\sigma_i^{-1} + \sigma_{pi}^{-1} + \rho^2(\sigma_x + 2\bar{x}_i^2) + \bar{K}_i}{(\sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i)^2}$$

Finally, the partial derivative with respect to $\bar{K}_i$ is

$$\frac{\partial \lambda_i}{\partial \bar{K}_i} = \frac{2(\sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i) - 2(\sigma_i^{-1} + \sigma_{pi}^{-1} + \rho^2(\sigma_x + 2\bar{x}_i^2) + \bar{K}_i)}{(\sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i)^3}$$

$$= \frac{-2\rho^2(\sigma_x + 2\bar{x}_i^2) - 2\bar{K}_i)}{(\sigma_i^{-1} + \sigma_{pi}^{-1} + \bar{K}_i)^3} < 0$$

Since the numerator is all terms that can only be negative and the denominator is a sum of precisions, that can only be positive, the sign is negative. This proves that λ_i is decreasing in $\bar{K}_i$.

Furthermore, as the supply of the risk factor $\bar{x}_i$ becomes large, $\partial\lambda_i/\partial\bar{K}_i$ becomes an arbitrarily large negative number. Thus, for a sufficiently large $\bar{x}_i$, there exists a $\underline{K}$ such that if $\bar{K}_i = \underline{K}$, then $\lambda_i < \lambda_{i'}$ for some other risk i'. But then, λ_i is not a unique maximum in the set of $\{\lambda_l\}_{l=1}^N$, which is a contradiction. □

Proof of Proposition 1 *For each skilled investor j, the optimal attention allocation for risk i (K_{ij}) is weakly increasing in its variance σ_i.*

Proof. The information choice problem is not a concave optimization problem. Therefore, a first-order approach is not valid. Instead we need to consider each of the various possible corner solutions, one by one. Let j denote an informed investor. From step 4 of the model solution, we know that when there is a unique maximum λ_i the optimal information choice is $K_{lj} = K$ if $\lambda_l = \max_i \lambda_i$, and $K_{lj} = 0$, otherwise. If multiple risks achieve the same maximum λ_l then all attention will be allocated amongst those risks. Therefore, there are three cases to consider.

Case 1: λ_l is the unique maximum λ_i. Holding attention allocations constant, a marginal increase in σ_l will cause λ_l to increase:

$$\frac{\partial \lambda_l}{\partial \sigma_l} = \left[1 + 2\bar{\sigma}_i(\rho^2(\sigma_x + \bar{x}_l^2) + \bar{K}_l)\right]\left(\frac{\bar{\sigma}_l}{\sigma_l}\right)^2 > 0.$$

The marginal increase in σ_l will not effect $\lambda_{l'}$ for $l' \neq l$ (see equations (14) and (23)). It follows that after the increase in σ_l, λ_l will still be the unique maximum λ_i. Therefore, in the new equilibrium, attention allocation is unchanged.

Case 2: Prior to the increase in σ_l, multiple risks – including risk l – attain the maximum λ_i. Let $\mathcal{I}_M$ be the set of such risks. If σ_l marginally increases and we held attention allocations fixed, then λ_l would be the unique maximum λ_i. If λ_l is the unique maximum, then K_{lj} should increase and $K_{l'j}$ for $l' \in \mathcal{I}_M \setminus l$ should decrease. However, using equations (14) and (23) we can show that an increase in K_{lj} would decrease λ_l:

$$\frac{\partial \lambda_l}{\partial \bar{K}_l} = -2\bar{\sigma}_l^2\Big\{\frac{\bar{K}_l}{\rho^2\sigma_x} + \bar{\sigma}_l\big[\rho^2(\sigma_x + \bar{x}_l^2) + \bar{K}_l\big]\Big(1 + \frac{2\bar{K}_l}{\rho^2\sigma_x}\Big)\Big\} < 0, \tag{44}$$

and since $\bar{K}_l = \chi K_{lj}$, $\partial\lambda_l/\partial K_{lj} < 0$. This effect works to partially offset the initial increase in λ_l. In the rest of the proof that follows, we construct the new equilibrium attention allocation, following an initial increase in λ_l and show that even though the attention reallocation works to reduce λ_l, the net effect is a larger $\bar{K}_l$.

This solution to this type of convex problem is referred to as a "waterfilling" solution in the information theory literature. (See textbook by (Cover and Thomas 1991).) To construct a new equilibrium, we reallocate attention from risk $l' \in \mathcal{I}_M \setminus l$ to risk l (increasing $\bar{K}_l$, decreasing $\bar{K}_{l'}$). This decreases λ_l and increases $\lambda_{l'}$. We continue to reallocate attention from all risks $l' \in \mathcal{I}_M \setminus l$ to risk l in such a way that $\lambda_{l'} = \lambda_{l''}$ for all $l', l'' \in \mathcal{I}_M \setminus l$ is maintained. We do this until either (i) all attention has been allocated to risk l or (ii) $\lambda_l = \lambda_{l'}$ for all $l' \in \mathcal{I}_M \setminus l$. Note that in the new equilibrium λ_l will be larger than before and the new equilibrium $\bar{K}_i$ will be larger than before.

Case 3: Prior to the increase in σ_l, $\lambda_l < \lambda_{l'}$ for some $l' \neq l$. Because λ_l is a continuous function of σ_l, a marginal increase in σ_l, will only change λ_l marginally. Because λ_l is discretely less than λ_l', the ranking of the λ_i's will not change and the new equilibrium will maintain the same attention allocation.

In cases one and three K_{lj} does not change in response to a marginal increase in σ_l. In case two K_{lj} is strictly increasing in σ_l. Therefore, K_{lj} is weakly increasing in σ_l. □

Proof of Proposition 2 If $\bar{x}_i$ is sufficiently large then, for each skilled investor j, the optimal attention allocation for risk i (K_{ij}) is weakly increasing in risk aversion ρ.

Proof. Let j denote an informed investor. Differentiating (41), we see that the partial derivative of λ_i with respect to ρ is

$$\frac{\partial \lambda_i}{\partial \rho} = 2\bar{\sigma}_i^2 \Big[\rho(\sigma_x + \bar{x}_i^2) + \frac{\bar{K}_i^2}{\rho^3 \sigma_x}\Big(1 + 2\bar{\sigma}_i[\rho^2(\sigma_x + \bar{x}_i^2) + \bar{K}_i]\Big)\Big] > 0. \tag{45}$$

The remaining task is to determine how the change in the marginal value of all signals λ_i, $\forall i$ affects the attention allocation $\bar{K}_i$, $\forall i$. There are again three cases to consider.

Case 1: Prior to the increase in ρ, there is a unique maximum λ_i. Holding $\bar{K}_i$ fixed, λ_i is continuous in ρ, so a marginal change in ρ cannot change the rankings of the λ_i's. Therefore, it is an equilibrium to maintain the same K_{ij} for all i.

Case 2: Let $\mathcal{I}_M$ be the set of risks which attain the maximum λ_l. In the previous proof, we showed that an increase in λ_l increases $\bar{K}_l$ if $\lambda_l \in \mathcal{I}_M$. The same equilibrium assignment argument demonstrates that $\bar{K}_l$ will increase after the change in ρ if $\partial\lambda_l/\partial\rho \geq \partial\lambda_{l'}/\partial\rho$ for all $l' \in \mathcal{I}_M \setminus l$.

From equation (45) we see that, $\partial\lambda_i/\partial\rho$ is strictly increasing in $\bar{x}_i$, finite-valued and not bounded above. Therefore, there exists $\bar{x}^*$ such that $\partial\lambda_i/\partial\rho > \partial\lambda_{i'}/\partial\rho$, $\forall i'$ if $\bar{x}_i > \bar{x}^*$. It follows that $\bar{K}_i$, and therefore K_{ij}, is weakly increasing in ρ if $\bar{x}_i > \bar{x}^*$.

Case 3: Prior to the increase in ρ, $\lambda_i < \max_j \lambda_j$. Since λ_i is not part of the maximal set, $\bar{K}_i = 0$ before the increase in ρ. But λ_i is continuous in ρ, so a marginal change in ρ cannot cause $\lambda_i \geq \max_j \lambda_j$ to hold. Since λ_i is not part of the maximal set, $\bar{K}_i = 0$ after the increase in ρ. Thus, $\bar{K}_i$ does not change.

In all three cases, K_{ij}, is weakly increasing in ρ if $\bar{x}_i > \bar{x}^*$. □

Derivation of excess returns and their dispersion We begin by calculating the portfolio excess return. Note that the return of the portfolio expressed in terms of assets is equal to the return expressed in risk factors:

$$(q_j - \bar{q})'(f - pr) = (q_j - \bar{q})'\Gamma^{-1}(\Gamma f - \Gamma pr) = (\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r) \tag{46}$$

Substitute (27) and (36) into (46) to get

$$
\begin{aligned}
E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)] &= \frac{1}{\rho}E\left[\left(\Sigma_{\eta j}^{-1}\varepsilon_j + \Delta(V^{1/2}u + w)\right)'(V^{1/2}u + w)\right] \\
&= \frac{1}{\rho}E\Big[\epsilon_j'\Sigma_{\eta j}^{-1}w + \epsilon_j'\Sigma_{\eta j}^{-1}V^{1/2}u + 2w'\Delta V^{1/2}u + w'\Delta w + u'V^{1/2}\Delta V^{1/2}u\Big] \\
&= \frac{1}{\rho}E\Big[w'\Delta w + u'V^{1/2}\Delta V^{1/2}u\Big] \\
&= \frac{1}{\rho}\Big[\rho^2\bar{x}'\bar{\Sigma}\Delta\bar{\Sigma}\bar{x} + Tr\Big(V^{1/2}\Delta V^{1/2}E(uu')\Big)\Big] \\
&= \rho Tr(\bar{x}'\bar{\Sigma}\Delta\bar{\Sigma}\bar{x}) + \frac{1}{\rho}Tr(\Delta V) \qquad (47)
\end{aligned}
$$

where the third equality comes from the fact that w is a constant and ε_j and u are mean zero and uncorrelated.

To get return dispersion, we substitute (27) and (36) into (46), then square the excess return and take the expectation:

$$
E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2] = E\left[\left(\frac{1}{\rho}[\Sigma_{\eta}^{-1}\varepsilon_j + \Delta V^{1/2}u + \Delta w]'(w + V^{1/2}u)\right)^2\right]
$$

Using the fact that for any random variable x, $V(x) = E(x^2) - E^2(x)$, the dispersion of funds' portfolio returns is equal to:

$$
\begin{aligned}
E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2] &= \frac{1}{\rho^2}V\Big([\Sigma_{\eta j}^{-1}\varepsilon_j + \Delta V^{1/2}u + \Delta w]'(V^{1/2}u + w)\Big) \\
&\quad + \frac{1}{\rho^2}\Big(E[\Sigma_{\eta j}^{-1}\varepsilon_j + \Delta V^{1/2}u + \Delta w]'(V^{1/2}u + w)\Big)^2
\end{aligned}
$$

We compute each term separately.

$$
\begin{aligned}
V(\cdot) &= V\Big[\varepsilon_j'\Sigma_{\eta j}^{-1}w + \varepsilon_j'\Sigma_{\eta j}^{-1}V^{1/2}u + 2w'\Delta V^{1/2}u + w'\Delta w + u'V^{1/2}\Delta V^{1/2}u\Big] \\
&= w'\Sigma_{\eta j}^{-1}w + 0 + 4w'\Delta V\Delta w + 0 + 2Tr(\Delta V\Delta V) \\
&= \rho^2 Tr(\bar{x}'\bar{\Sigma}\Sigma_{\eta j}^{-1}\bar{\Sigma}\bar{x}) + 4\rho^2 Tr(\bar{x}'\bar{\Sigma}\Delta V\Delta\bar{\Sigma}\bar{x}) + 2Tr(\Delta V\Delta V) \\
E(\cdot)^2 &= (w'\Delta w + Tr(\Delta V))^2 = \big(\rho^2\bar{x}'\bar{\Sigma}\Delta\bar{\Sigma}\bar{x} + Tr(\Delta V)\big)^2
\end{aligned}
$$

where the last lines uses the definition of w from (28). Next, we use the definition of Δ and the focus on symmetric information acquisition equilibria to get $\Delta = (1-\chi)K_j$ for any informed investor j. For an uninformed investor, the expression is the same, except that the $(1-\chi)$ terms are replaced with $-\chi$. Substituting in the squared expectation and variance, we have that for any informed investor j:

$$
\begin{aligned}
E[((\tilde{q}_j - \bar{qz})'(\tilde{f} - \tilde{p}r))^2] &= Tr(\bar{x}'\bar{\Sigma}\Sigma_{\eta j}^{-1}\bar{\Sigma}\bar{x}) + 4(1-\chi)Tr(\bar{x}'\bar{\Sigma}K_j V\Delta\bar{\Sigma}\bar{x}) \\
&\quad + \frac{2}{\rho^2}(1-\chi)^2 Tr(\Delta K_j V K_j V) + \frac{(1-\chi)^2}{\rho^2}\big(\rho^2\bar{x}'\bar{\Sigma}K_j\bar{\Sigma}\bar{x} + Tr(K_j V)\big)^2 \\
&= \sum_{i=1}^{n}\bar{x}_i^2\bar{\sigma}_i^2 K_{ij}\big(1 + 4(1-\chi)^2 K_{ij}V_{ii}\big) + \frac{2}{\rho^2}(1-\chi)^2 K_{ij}^2 V_{ii}^2 \\
&\quad + (1-\chi)^2\left(\sum_{i=1}^{n}\rho\bar{x}_i^2\bar{\sigma}_i^2 K_{ij} + \frac{1}{\rho}K_{ij}V_{ii}\right)^2 \qquad (48)
\end{aligned}
$$

The last line uses the fact that all square matrices are diagonal and that the trace is the sum of the diagonal elements.

Proof of Proposition 3 We prove part (a) and then part(b).

Proposition 3(a) If $\bar{x}_i$ is sufficiently large then an increase in variance σ_i weakly increases the dispersion of fund portfolios, $\int E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]dj$.

Proof. We prove the proposition by proving that for any given investor j, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]$ increases. Thus, the integral over j increases as well.

From (35), we know that $\tilde{q}_j - \bar{\tilde{q}}_j = \frac{1}{\rho}(\Delta(\tilde{f} - \tilde{p}r) + \Sigma_{\eta j}^{-1}\varepsilon_j))$ where Δ and Σ_η are diagonal matrices with diagonal elements $\Delta_{ii} = K_{ij} - \bar{K}_i$ and $(\Sigma_{\eta j}^{-1})_{ii} = K_{ij}$. Using these elements, we can write

$$E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})] = \frac{1}{\rho^2} E\left[\sum_{l=1}^{n}\Big((K_{lj} - \bar{K}_l)(\tilde{f}_l - \tilde{p}_l r) + K_{lj}\varepsilon_{lj}\Big)^2\right].$$

Recall that the expected return is $\tilde{f} - \tilde{p}r = V^{1/2}u + w$, with $u \sim N(0,1)$ and $w \equiv \rho\bar{\Sigma}\bar{x}$. Since $E[\varepsilon_{ij}^2] = K_{ij}^{-1}$, ε_{lj} is uncorrelated with $(\tilde{f}_l - \tilde{p}_l r)$, $u_l \sim N(0,1)$, and in equilibrium $\sum_l K_{lj} = K$, we get

$$\begin{aligned} E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})] &= \frac{1}{\rho^2} E\left[\sum_l (K_{lj} - \bar{K}_l)^2 (V_{ll}^{1/2}u_l + w_l)^2\right] + \frac{1}{\rho^2}K \\ &= \frac{1}{\rho^2}\sum_l (K_{lj} - \bar{K}_l)^2\Big(V_{ll} + \rho^2\bar{\sigma}_l^2\bar{x}_l^2\Big) + \frac{1}{\rho^2}K. \end{aligned} \tag{49}$$

To assess the effect of an increase in σ_i we consider two cases. The first case is when there is no change in attention allocation after a marginal increase in σ_i and the second case is when there is a change in attention allocation. The first case occurs if all attention or no attention is allocated to risk i before the change in σ_i, otherwise the second case occurs (this is explained in the proof of Proposition 1).

Case 1 (includes also what was previously called case 3): A marginal increase in σ_i will only change two variables on the right-hand side of equation (49), V_{ii} and $\bar{\sigma}_i$. In equations (24) and (30), we showed that that both $\bar{\sigma}_i$ and V_{ii} are strictly increasing in σ_i. If all attention or no attention is allocated to risk i before the increase in σ_i, then $K_{ij} - \bar{K}_i > 0$ or $K_{ij} - \bar{K}_i = 0$, respectively. Therefore, when all attention is allocated to risk i before the change in σ_i, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]$ strictly increases in σ_i. When no attention is allocated to risk i $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]$ is constant in σ_i.

Case 2: From Proposition 1 we know that a marginal increase in σ_i will cause $K_{ij} - \bar{K}_i$ to increase and $K_{lj} - \bar{K}_l$ to decrease for all risks $l \in \mathcal{I}_M \setminus i$. The other variables that will change in equation (49) are V_{ll} and $\bar{\sigma}_l$ for all risks $l \in \mathcal{I}_M$. If $\bar{x}_i$ is sufficiently large then the sign of the effect of σ_i on $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]$ will be determined by its effect on $\bar{\sigma}_i$. We will now show that, when $\bar{x}_i$ is sufficiently large, $\bar{\sigma}_i$ is increasing in σ_i, even after accounting for the reallocation of attention, so $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{q}_j - \bar{\tilde{q}})]$ is increasing in σ_i. We will prove this by contradiction.

Suppose that $\bar{\sigma}_i$ decreases when σ_i increases. Recall that

$$\lambda_i = \bar{\sigma}_i[1 + (\rho^2\sigma_x + \bar{K}_i)\bar{\sigma}_i] + \rho^2\bar{x}_i^2\bar{\sigma}_i^2.$$

Therefore, if $\bar{x}_i$ is sufficiently large and $\bar{\sigma}_i$ decreases, λ_i decreases. But, we know from Proposition 1 that if $K_{ij} > 0$ and σ_i increases, then λ_i increases. Therefore, $\bar{\sigma}_i$ must increase in σ_i.

Combining cases, if $\bar{x}_i$ is sufficiently large, dispersion weakly increases in σ_i. $\square$

Proposition 3(b) *Prove:* If $\bar{x}_i$ is sufficiently large then an increase in variance σ_i weakly increases the dispersion of portfolio excess returns, $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$.

Proof. As before, we prove that the integral increases by proving that the expectation increases for every investor j, and we consider three cases: The first case is when all attention is allocated to risk i before the change in σ_i. The second case is where some, but not all, attention is allocated to risk i. In the third case, no attention is allocated to risk i

Case 1: All attention is allocated to risk i. Since $\lambda_i > \lambda_l$, $\forall l \neq i$, a marginal change in σ_i will change λ's continuously and will not reverse the inequality. Thus λ_i will still be the unique maximum and attention will not change. The only variables on the right-hand side of equation (48) that will change when σ_i increases are V_{ii} and $\bar{\sigma}_i$. Both will increase strictly. Both are multiplied by quantities and parameters that are always non-negative. Thus, dispersion increases strictly.

Case 2: For an informed investor some, but not all, attention is allocated to risk i before the change in σ_i. To prove that expression (48) increases in σ_i we use the same method that we used for this case in the proof of Proposition 3(a). If $\bar{x}_i$ is sufficiently large, then the positive effect of the increase in K_{ij} and $\bar{\sigma}_i$ will outweigh any negative effect of the decrease in K_{lj} and $\bar{\sigma}_l$ for $l \neq i$. We established in the proof of Proposition 3(a) that an increase in σ_i causes $\bar{\sigma}_i$ to increase. Therefore, if $\bar{x}_i$ is sufficiently large, $E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]$ is increasing in σ_i.

Case 3: When no attention is allocated to risk i ($K_{ij} = 0$), dispersion is constant in σ_i because all the σ_i and V_{ii} terms are multiplied by K_{ij}.

For an uninformed investor, dispersion is the same, except that the $(1-\chi)^2$ terms are replace with $(-\chi)^2$ terms. Since both are non-negative, the same arguments hold for any uninformed investor j. Since $E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]$ weakly increases in σ_i for every investor j, the integral $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$ weakly increases as well. □

Proof of Proposition 4

If σ_x and $\bar{x}_n$ are sufficiently large, then an increase in risk aversion ρ increases the dispersion of portfolio excess returns, $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$.

Proof. Dispersion of excess returns is given in (48). We first work through the direct effect of ρ on $\bar{\sigma}_l$ and V_{ll} and then turn to the indirect effect that works through attention allocation K. Both $\bar{\sigma}_l$ and V_{ll} are increasing in ρ, as shown in (25) and (31). Both are multiplied by parameters and variables that are always non-negative. Therefore, the only terms of (48) whose derivative we need to work out to sign are the ones with V_{ii}/ρ or V_{ii}^2/ρ^2.

$$\frac{\partial}{\partial \rho}\frac{V_{ll}}{\rho} = \frac{1}{\rho}\left[\frac{\partial V_{ll}}{\partial \rho} - \frac{V_{ll}}{\rho}\right]$$

This expression is positive if the elasticity of V_{ll} with respect to ρ is larger than one for all l, which is ensured if σ_x is sufficiently large, i.e. it satisfies (32). Thus, the direct effect of risk aversion is to increase $E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]$ for each investor j and therefore increase $\int E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2]dj$ as well.

The total derivative is the sum of the partial derivative and the indirect effect that comes from reallocation of attention: $d/d\rho = \partial/\partial\rho + (\partial/\partial K_j)(\partial K_j/\partial\rho)$. The previous part of the proof signed the first term. This second part signs the second term.

From (41), note that $\partial\lambda_i/\partial\bar{x}_i = 2\rho^2\bar{\sigma}_i^2\bar{x}_i$. This is positive and increasing in $\bar{x}_i$. For any values of $\rho^2\bar{\sigma}_i^2$, there is an $\bar{x}_i$ sufficiently large that $\lambda_i > \lambda_j$, $\forall j \neq i$. Specifically for the supply of aggregate risk, if $\bar{x}_n$ is sufficiently large then $\lambda_n > \lambda_j$, $\forall j \neq n$ and thus $K_{nj} = K$, for all informed investors j. At this corner solution, where $\lambda_n > \lambda_j$, with strict inequality $\forall j \neq n$, a marginal change in ρ will not change the inequality because λ_i is continuous in ρ. Thus, after a marginal change in ρ, it is still true that $K_{nj} = K$, for all informed investors j. Because attention allocation is unchanged by a marginal change in ρ, the direct effect and the total effect are identical. Next, consider lower levels of $\bar{x}_n$ where a marginal increase in ρ does change the attention allocation. Since dispersion is continuously differentiable in K_j, and is strictly increasing in ρ for a given capacity allocation, there exists a ball of parameters such that $\partial K_{ij}/\partial\rho > 0$ for some risk $i \neq n$ and $d/d\rho\, E[((\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r))^2] > 0$. □

Proof of Proposition 5 *If $\bar{x}_i$ is sufficiently large then an increase in the variance σ_i weakly increases the portfolio excess return of an informed fund, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$.*

Proof. Writing the trace terms in equation (47) as sums and using the definition of Δ yields:

$$E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)] = \frac{1}{\rho}\Big[\sum_l (K_{lj} - \bar{K}_l)(V_{ll} + (\rho\bar{\sigma}_l\bar{x}_l)^2)\Big]. \tag{50}$$

To determine the effect of an increase in σ_i on this expression we consider two cases. The first case is when there is no change in attention allocation after the increase in σ_i and the second one is when there is a change in attention allocation. Recall (from the proof of Proposition 1) that the first case occurs if all attention or no attention is allocated to risk i before the change in σ_i, otherwise the second case occurs.

Case 1: As discussed in the proof of Proposition 3(a), the only variables that will change on the right-hand side of equation (50) when σ_i increases are V_{ii} and $\bar{\sigma}_i$. Both will increase. If no attention is allocated to risk i before the change in σ_i then $K_{ij} - \bar{K}_i = 0$, and the change in σ_i has no effect on $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$. If all attention is allocated to risk i then $K_{ij} - \bar{K}_i > 0$. Thus, the increase in σ_i causes $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$ to increase strictly.

Case 2: As argued in the proof of 3(a), if $\bar{x}_i$ is sufficiently large then we can assess the effect of an increase in σ_i on $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$ by only considering the effect on $K_{ij} - \bar{K}_i$, V_{ii} and $\bar{\sigma}_i$. As proved in Proposition 3(a), $K_{ij} - \bar{K}_i$ is increasing in σ_i and, if $\bar{x}_i$ is large enough, $\bar{\sigma}_i$ is increasing in σ_i. Therefore, it follows from equation (50) that if $\bar{x}_i$ is sufficiently large then $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$ is increasing in σ_i. □

Proof of Proposition 6 *If σ_x and $\bar{x}_n$ are sufficiently large, then an increase in risk aversion ρ increases expected excess return, $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$.*

Proof. Taking a partial derivative of (47) with respect to ρ, we get:

$$\begin{aligned}\frac{\partial E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]}{\partial \rho} &= Tr(\bar{x}'\bar{\Sigma}\Delta\bar{\Sigma}\bar{x}) + 2\rho Tr\Big(\bar{x}'\Delta\left[\frac{\partial\bar{\Sigma}}{\partial\rho}\right]\bar{\Sigma}\bar{x}\Big) - \frac{1}{\rho^2}Tr(\Delta V) + \frac{1}{\rho}Tr\Big(\Delta\left[\frac{\partial V}{\partial\rho}\right]\Big)\\ &= Tr(\bar{x}'\bar{\Sigma}\Delta\bar{\Sigma}\bar{x}) + 2\rho Tr\Big(\bar{x}'\Delta\left[\frac{\partial\bar{\Sigma}}{\partial\rho}\right]\bar{\Sigma}\bar{x}\Big) + \frac{1}{\rho}\left[Tr\Big(\Delta\left[\frac{\partial V}{\partial\rho} - \frac{V}{\rho}\right]\Big)\right].\end{aligned}$$

Since (25) tells us that $\partial\bar{\Sigma}_{ii}/\partial\rho \geq 0, \forall i$, a sufficient condition for this expression to be positive is $\frac{\partial V}{\partial\rho} - \frac{V}{\rho} > 0$, which is equivalent to the elasticity of V_{ii} with respect to ρ larger than one for each i. This holds if σ_x is sufficiently large, i.e. it satisfies (32).

The total derivative is the sum of the partial derivative and the indirect effect that comes from reallocation of attention: $d/d\rho = \partial/\partial\rho + (\partial/\partial K_j)(\partial K_j/\partial\rho)$. The previous part of the proof signed the first term. This second part signs the second term. Note that capacity allocation K_j enters through Δ.

From (41), note that $\partial\lambda_i/\partial\bar{x}_i = 2\rho^2\bar{\sigma}_i^2\bar{x}_i$. This is positive increasing in $\bar{x}_i$. For any values of $\rho^2\bar{\sigma}_i^2$, there is an $\bar{x}_i$ sufficiently large that $\lambda_i > \lambda_j, \forall j \neq i$. Specifically for the supply of aggregate risk, if $\bar{x}_n$ is sufficiently large then $\lambda_n > \lambda_j, \forall j \neq n$ and thus $K_{nj} = K$, for all informed investors j. At this corner solution, where $\lambda_n > \lambda_j$, with strict inequality $\forall j \neq n$, a marginal change in ρ will not change the inequality because λ_i is continuous in ρ. Thus, after a marginal change in ρ, it is still true that $K_{nj} = K$, for all informed investors j. Because K is unchanged by a marginal change in ρ, the direct effect and the total effect are identical. Next, consider lower levels of $\bar{x}_n$ where a marginal increase in ρ does change K. Since expected return is continuously differentiable in K_j, and is strictly increasing in ρ for a given capacity allocation, there exists a ball of parameters such that $\partial K_{ij}/\partial\rho > 0$ for some risk i and $E[(\tilde{q}_j - \bar{\tilde{q}})'(\tilde{f} - \tilde{p}r)]$ is still increasing in risk aversion. □

Proof of Proposition 7 *If the net supply of idiosyncratic risk is small, then expected excess portfolio return of fund j is $E[R_j] - r = \alpha_j + \beta_j(E[r_m] - r)$, where $\alpha_j = \sum_i 1/\rho\Big(var[\tilde{f}_i](\sigma_i^{-1} + K_{ij}) - 1\Big) - \bar{\rho}_{ij}$.*

Proof. Define the weight that fund j puts on asset i as

$$\omega_{ij} \equiv \frac{q_{ij} p_i}{\sum_k q_{kj} p_k} = \frac{q_{ij} p_i}{W_0},$$

let $\omega_j \equiv [\omega_{1j} \dots \omega_{nj}]'$ and define $R_j \equiv \omega_j' R$, where R is the vector of all risky asset returns, $[r_1, r_2, \dots, r_n]'$. The unconditional expected value of fund j's excess return R_j is

$$E[\omega_j'(R - r)] = \sum_i E[\omega_{ij}(r_i - r)]$$

Next, we substitute in the following definitions. Let R be a vector of returns with ith entry $R_i \equiv f_i/p_i$ and $\omega_{ij} = p_i q_{ij}/W_0$ be portfolio weight of investor j on asset i, where W_0 is initial wealth and by the budget constraint $W_0 = \sum_i p_i q_{ij}$.

$$\begin{aligned}
E[\omega_j'(R - r)] &= \sum_i E[\frac{1}{W_0} p_i q_{ij} (\frac{f_i}{p_i} - r)] \\
&= \frac{1}{W_0} \sum_i E[q_{ij}(f_i - p_i r)] \\
&= \frac{1}{W_0} \sum_i E[q_{ij}] E[(f_i - p_i r)] + cov[q_{ij}, (f_i - p_i r)]
\end{aligned}$$

where the last line follows from the definition of a covariance.

First, we work out the sum of the covariances. In matrix notation, this sum is $\sum_i cov[q_{ij}, (f_i - p_i r)] = Tr(Cov(q_j, (f - pr)))$. This covariance is slightly different from the unconditional covariance we worked out to solve the model, because this is a covariance conditional on the signals and price in fund j's interim information set. This is the term that will distinguish skilled funds, whose portfolios covary with payoffs, from unskilled ones. Since $f = \Gamma \tilde{f}$, $q_j = (\Gamma')^{-1} \tilde{q}_j$, and $(\Gamma')^{-1} = (\Gamma^{-1})'$ we can express this covariance in terms of risk quantities and payoffs as $Tr(Cov((\Gamma^{-1})' \tilde{q}_j, \Gamma(\tilde{f} - \tilde{p} r))) = \Gamma^{-1} \Gamma Tr(Cov(\tilde{q}_j, \tilde{f} - \tilde{p} r))$. Canceling the Γ terms and rewriting this as a sum, we obtain $\sum_i cov[\tilde{q}_{ij}, (\tilde{f}_i - \tilde{p}_i r)]$. Recall from the portfolio first-order condition that $\tilde{q}_{ij} = \frac{1}{\rho} \hat{\sigma}_{ij}^{-1}(E_j[\tilde{f}_i] - \tilde{p}_i r)$. Thus,

$$cov[\tilde{q}_{ij}, (\tilde{f}_i - \tilde{p}_i r)] = 1/(\rho \hat{\sigma}_i) var[E_j[\tilde{f}_i]].$$

By the law of total variance, the unconditional variance of a posterior belief $var[E_j[\tilde{f}_i]]$ is the variance of the prior σ_i minus the posterior variance $\hat{\sigma}_i$.

$$cov[\tilde{q}_{ij}, (\tilde{f}_i - \tilde{p}_i r)] = 1/(\rho \hat{\sigma}_{ij})(\sigma_i - \hat{\sigma}_{ij}).$$

By Bayes' law, this posterior variance is $\hat{\sigma}_{ij} = 1/(\sigma_i^{-1} + K_{ij})$. Substituting this in we get

$$cov[\tilde{q}_{ij}, (\tilde{f}_i - \tilde{p}_i r)] = \frac{1}{\rho} \left(\sigma_i (\sigma_i^{-1} + K_i) - 1 \right)$$

Since $\rho > 0$ and the variance term is positive, this covariance is increasing in signal precision K_i.

Next, we work out the product of the expectations $E[q_{ij}] E[(f_i - p_i r)]$ and rewrite it in a CAPM representation.

$$\begin{aligned}
E[q_{ij}] E[(f_i - p_i r)] &= E[q_{ij}] E[p_i (R_i - r)] \\
&= E[q_{ij}] \left(E[p_i] E[R_i - r] + cov(p_i, R_i) \right) \\
&= E[q_{ij}] E[p_i] \beta_i (E[r_m] - r) + E[q_{ij}] cov(p_i, R_i)
\end{aligned}$$

where the last line holds approximately if the relative supply of aggregate risk is large, and thus $R_i = \beta_i r_m$. Using the definitions $\hat{\omega}_{ij} \equiv E[q_{ij}]E[p_i]/W_0$ and $\bar{\rho}_{ij} \equiv -E[q_{ij}]cov(p_i, R_i)/W_0$, we can write

$$\frac{1}{W_0}E[q_{ij}]E[(f_i - p_i r)] = \hat{\omega}_j \beta_i (E[r_m] - r) - \bar{\rho}_{ij}$$

Note that since $R_i = f_i/p_i$, $cov(f_i, R_i) < 0$, the $\bar{\rho}_{ij}$ terms are positive for positive expected portfolio holdings.

Putting the two pieces together,

$$R_j = \sum_i \hat{\omega}_{ij}\beta_i(E[r_m] - r) - \bar{\rho}_{ij} + \frac{1}{\rho}\left(var[\tilde{f}_i](\sigma_i^{-1} + K_i) - 1\right)$$

$$R_j = \alpha_j + \beta_j(E[r_m] - r)$$

where $\alpha_j = \sum_i 1/\rho \left(var[\tilde{f}_i](\sigma_i^{-1} + K_i) - 1\right) - \bar{\rho}_{ij}$ and $\beta_j = \sum_i \hat{\omega}_{ij}\beta_i$. $\square$

B Model with a General Signal Covariance Structure

For the purposes of this mutual fund theory, we assumed a particular risk factor structure and assumed the signals are the payoffs of these risk factors, plus independent noise. However, our methodology can solve a much more general class of models. We show here how to transform any problem with an arbitrary asset and signal covariance structure into an equivalent problem of independent signals about independent risk factors.

Model: Suppose there are N assets with a random $N \times 1$ vector of payoffs $f \sim N(\mu, \Sigma)$. Each agent j receives a signal vector η_j about linear combinations of these asset payoffs plus noise:

$$\eta_j = \psi f + e_j \tag{51}$$

where ψ is invertible and the $N \times 1$ vector of signal noise $e_j \sim N(0, \Sigma_e)$. Σ_e and Σ need not be diagonal, but must be positive-definite and invertible. As before, portfolio choices q maximize (3) subject to (4) and information choices maximize (6) subject to (7) and (8).

Solution: Begin by using a Cholesky decomposition to transform signals so that each signal is about an independent payoff event. Consider the transformed signal

$$\Sigma^{-1/2}\psi^{-1}\eta_j = \Sigma^{-1/2} f + \Sigma^{-1/2}\psi^{-1} e_j \tag{52}$$

Note that $var(\Sigma^{-1/2} f) = I$. Thus, each signal in the signal vector $\Sigma^{-1/2}\psi^{-1}\eta_j$ is about an independent random event – an entry of $\Sigma^{-1/2} f$, albeit with correlated signal error.

Next, use an eigen-decomposition to make the signal noise independent. The variance-covariance matrix of the transformed signal above is $\Sigma^{-1/2}\psi^{-1}\Sigma_e\psi^{-1\prime}\Sigma^{-1/2\prime} = GLG'$ where G is the eigenvector matrix and L is the diagonal matrix of eigenvalues of the variance-covariance matrix. Next, let $\tilde{\eta_j} \equiv G'\Sigma^{-1/2}\psi^{-1}\eta_j$, let $\tilde{f} \equiv G'\Sigma^{-1/2} f$ and let $\tilde{e}_j \equiv G'\Sigma^{-1/2}\psi^{-1} e_j$. Then, premultiplying each term in (52) by G' yields

$$\tilde{\eta_j} = \tilde{f} + \tilde{e}_j \quad \text{s.t.} \quad var(\tilde{f}) = I \;\text{ and }\; var(\tilde{e}_j) = L \;\text{ (diagonal).} \tag{53}$$

The new signal $\tilde{\eta_j}$ is simply a linear combination of observed signals η_j. Each new signal (element of $\tilde{\eta_j}$) is about an independent event – an element of the vector $\tilde{f}$ and each has independent signal noise. To see that, note that $var(\tilde{f}) = G'\Sigma^{-1/2}\Sigma\Sigma^{-1/2}G$. Cancelling inverse matrices yields $G'G$. Since eigenvector matrices are idempotent, $G'G = I$. Thus, $var(\tilde{f}) = I$. Furthermore, $var(\tilde{e_j}) = G'\Sigma^{-1/2}\psi^{-1}\Sigma_e\psi^{-1\prime}\Sigma^{-1/2\prime}G$. Substituting in the definition of the eigen-decomposition GLG', we get $var(\tilde{e_j}) = G'GLG'G$. Since $G'G = I$, $var(\tilde{e_j}) = L$. Since L is an eigenvalue matrix, it is diagonal.

Given this set of independent signals about independent risk factors (combinations of assets with payoffs $\tilde{f}$), we can follow the steps above to solve our portfolio choice problem: Choose quantities $\tilde{q}$ of each risk to hold. Compute expected utility from that portfolio problem for a given signal precision matrix Σ_e^{-1}. Then choose precisions, diagonal entries of the L^{-1} matrix, to maximize the expected continuation utility, as outlined in the steps below. Finally, map the solutions of the risk-factor problem q^*, L^* back to quantities and precision in the underlying problem: $q = \Sigma^{-1/2}G\tilde{q}^*$ and $\Sigma_e = \psi\Sigma^{1/2}GL^*G'\Sigma^{1/2}\psi'$.