NBER WORKING PAPER SERIES

OPTIMAL TAXATION IN THE PRESENCE OF BAILOUTS

Stavros Panageas

Working Paper 15405
http://www.nber.org/papers/w15405

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 2009

This paper was prepared for the April 2009 Carnegie Rochester Conference on Public Policy "Credit Market Turmoil: Implications for Policy". I would like to thank Andy Abel, Rui Albuquerque, John Heaton, and participants at the Carnegie Rochester conference for useful comments and discussion. The views expressed herein are those of the author(s) and do not necessarily reflect the views of the National Bureau of Economic Research.

Optimal taxation in the presence of bailouts
Stavros Panageas
NBER Working Paper No. 15405
October 2009
JEL No. E62,G28,H2,H21

ABSTRACT

The termination of a representative financial firm due to excessive leverage may lead to substantial bankruptcy costs. A government in the tradition of Ramsey (1927) may be inclined to provide transfers to the firm so as to prevent its liquidation and the associated deadweight costs. It is shown that the optimal taxation policy to finance such transfers exhibits countercyclicality and history dependence, even in a complete market. These results are in contrast with pre-existing literature on optimal fiscal policy, and are driven by the endogeneity of the transfer payments that are required to salvage the financial firm.

Stavros Panageas
University of Chicago
Booth School of Business
5807 South Woodlawn Avenue
Chicago, IL, 60637
and NBER
stavros.panageas@chicagobooth.edu

1 Introduction

During the recent financial crisis, several financial corporations received public-sector assistance. The usual reason given for such "bailouts", as they are commonly called in the popular press, is that massive liquidations would have severe adverse effects on the economy.

Predictably, such bailouts cause a lot of debate and questions as to their desirability. The present paper does not seek to resolve this debate. Instead, it studies a more focused question. Using a neoclassical framework in the tradition of Ramsey (1927), and simply assuming that private-sector contracting frictions induce governmental intervention to avoid bankruptcy-related deadweight costs, it studies optimal (distortionary) taxation to finance the transfer payments needed to salvage the financial firms. In the model a welfare-maximizing government adjusts tax rates, anticipating the effects of tax changes on equilibrium allocations, and ultimately consumer welfare. There are two types of (competitive) firms. Financial firms supply financial services using capital. Final goods firms produce consumption goods using labor and the services of financial firms as inputs. Financial firms are partly financed by debt. If the firm's assets fall short of its liabilities, then the financial firm must be liquidated. Liquidation leads to deadweight costs, which can induce the government to provide the financial firm with capital injections in order to prevent its liquidation and the associated deadweight losses.

In the baseline model, the transfer payments are financed by distortionary labor taxes similar to Lucas and Stokey (1983), and markets are (dynamically) complete. Because taxation is distortionary, Ricardian equivalence fails and the optimal welfare-maximizing way to raise taxes becomes a non-trivial problem.

A new feature of the model is that taxes are not raised to finance exogenous government expenditures, but rather to finance transfer payments from the government to the financial firm. Hence, government outlays do not drive a wedge between consumption and output; they merely reallocate existing consumption goods through taxes and transfers. Importantly, the net present value of the transfer payments is not exogenous to the model, but rather is determined in general equilibrium, since the value of the firm and the necessary transfer payments to prevent liquidation are endogenous to the model.

Within such a neoclassical framework optimal taxation balances two aspects.

On the one hand, the usual labor tax smoothing argument (see e.g. Barro (1979)) implies a constant tax rate, irrespective of current economic conditions. Indeed, a first result states that if one were to ignore the endogeneity of transfer payments, then the optimal tax rate within the model should be constant. This result is reminiscent of Lucas and Stokey (1983), who find that in the presence of complete markets and constant government expenditure (government expenditure is set to zero in the present model), optimal tax rates are constant.

On the other hand, inside the model countercyclical fiscal policy (raising tax rates in good times and lowering them in bad times) can help boost output, the demand for financial services and accordingly the value of the representative firm in bad times. In turn, increasing the firm's value in bad times can lower the amount of transfers that are required to salvage the firm, and can lead to a lower overall net present value of required (distortionary) taxes.

The main insight of the proposed neoclassical model is that the trade-off between smoothing tax distortions and reducing the net present value of transfers implies procyclical taxes. Specifically, optimal taxes are a non-decreasing function of total factor productivity. An additional result of theoretical interest is that optimal tax rates are history dependent *despite (dynamically) complete markets*; tax rates at some time t do not only depend on the productivity level at time t, but are also a non-decreasing function of past transfer payments. This result contrasts with Lucas and Stokey (1983), where tax rates are history independent, when government expenditure is Markovian.

The analysis is related to two strands of the literature. The first strand uses continuous time finance techniques to price government guarantees as contingent claims[1]. In this literature, the stochastic discount factor is taken as given, and the issue of how the government should raise the revenue to pay for the guarantees is not considered. By contrast, in the present framework the endogeneity of the stochastic discount factor and the optimal way to

[1]A representative sample of papers in this literature includes Merton (1978), Ronn and Verma (1986), Lucas and McDonald (2006), Panageas (2008), Pennacchi and Lewis (1994).

finance these guarantees are explicitly taken into account. The second strand of the literature studies optimal (labor-distortionary) taxation[2]. This literature is mostly considered with the optimal timing of taxes in the presence of exogenous expenditures. As mentioned above, the distinguishing feature of the present paper is that taxes are raised in order to finance endogenous transfer payments rather than *exogenous government expenditures*. Karantounias et al. (2008) also obtain history dependence of the optimal tax rate in a framework involving complete markets. However, their results are driven by fears of model mis-specification rather than endogenous transfers.

The paper is structured as follows. Section 2 describes the model and the problem of the government. Section 3 derives the optimal taxation policy. Section 4 considers additional forms of funding government transfers and derives the process of optimal debt holdings. Section 5 concludes. All proofs are in the appendix.

2 Model

2.1 Consumers, firms and assets

The representative consumer has preferences given by

$$E_0 \left\{ \int_0^\infty e^{-\rho t} \left[\log(c_t) + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right] dt \right\}, \tag{1}$$

where c_t is an adapted consumption process, and h_t denotes hours worked, $\rho > 0$ is the subjective discount rate, and $\phi > 0, \quad \phi \neq 1$ controls the elasticity of labor supply. The representative agent's endowment of hours is normalized to one. Specification (1) is attractive, since it allows hours to be stationary in the long run, while keeping the utility of consumption and leisure separable.

Firms in the economy are competitive and fall into two groups: Non-financial (NF) and financial (F). Within each group all firms are identical, so that one can speak of a

[2] A representative sample of papers in this literature includes Barro (1979),Lucas and Stokey (1983), Aiyagari et al. (2002), Angeletos (2002).

single representative financial and a single non-financial firm. The services produced by the representative financial firm are used by the non-financial firm as intermediate goods. Specifically, the non-financial firm is the sole producer of consumption goods and it utilizes the production function

$$Y_t = Z_t \left(F_t\right)^{1-\alpha} \left(h_t\right)^{\alpha}, \tag{2}$$

where F_t denotes the amount of financial services, h_t the hours worked, $\alpha \in (0,1)$ controls the labor share of production, and Z_t captures total factor productivity, which follows a geometric brownian motion

$$\frac{dZ_t}{Z_t} = \mu dt + \sigma dB_t. \tag{3}$$

The parameters $\mu > 0$ and $\sigma > 0$ control the drift and the volatility of the geometric brownian motion and satisfy $\mu - \frac{\sigma^2}{2} > 0$. The price of a unit of financial services is given by p_t. Accordingly, the optimization problem of the non-financial firm is given by

$$\max_{h_t, F_t} Y_t - w_t h_t - p_t F_t. \tag{4}$$

The financial firm employs a simple production technology. It owns K_t units of capital goods, and uses one unit of capital goods to produce one unit of financial services. To simplify matters, I follow Lucas (1978) and assume that K_t cannot be accumulated, nor does it depreciate, so that it remains constant.

Agents can frictionlessly trade a zero net supply instantaneously maturing riskless claim and a zero net supply[3] claim that pays Z_t as a dividend. Dynamic trading of these two securities in equilibrium[4] implies that financial markets are complete with respect to the filtration generated by B_t, and hence there exists a (unique) stochastic discount factor ξ_t.

Accordingly, the total value of the financial firm's capital stock, which I will denote as W_t, is given by the present value of the revenue that it produces

$$W_t = \left(E_t \int_t^{\infty} \frac{\xi_s}{\xi_t} p_s F_s ds \right). \tag{5}$$

[3]By introducing trading in a zero net supply claim paying the dividend Z_t, it is possible to ensure the dynamic completeness of markets.

[4]See e.g. Karatzas and Shreve (1998), Duffie (2001).

2.2 Default and Bailouts

The financial firm is levered; it owes debtholders an amount L_t. The coupon of this debt is variable and equals the rate of return on an instantaneously maturing risk-free bond.[5] The presence of debt introduces the possibility of default. The modeling of default follows Leland (1994). If a firm defaults, ownership of the entire capital stock is transferred to debtholders. To simplify the presentation, I assume that in the event of bankruptcy, the capital stock of the firm is sold by the debtholders to a newly formed set of financial firms that raise their funds by issuing equity to the representative households. However, this redeployment process is costly. Specifically, a fraction δ of the capital stock is lost in the process of redeployment, so that K_t jumps downward to $(1-\delta)\,K_t$.

To simplify the analysis further, I assume that debt includes a protective covenant that allows debtholders to liquidate the firm once

$$W_t \leq L_t \tag{7}$$

Equation (7) can be viewed as a constraint, that arises naturally in the presence of collateralized borrowing.

As in Leland (1998), loan modifications (principal write-downs, debt for equity swaps etc.) are not allowed as a mechanism to avoid impending bankruptcy and the associated deadweight losses. In the real world such modifications or write-downs are hard to implement because of various frictions, and that's why we observe bankruptcies in the first place[6].

[5] As Duffie (2001), and Karatzas and Shreve (1998) show, the stochastic discount factor in a brownian filtration model can be written as

$$\frac{d\xi_t}{\xi_t} = -r_t dt - dA_t - \kappa_t dB_t, \tag{6}$$

where (r_t) is the equilibrium interest rate, κ_t is the Sharpe ratio or price of risk and A_t is a continuous process of bounded variation. The rate of return of an instantaneously maturing risk-free bond is given by $r_t dt + dA_t$. (In many cases $dA_t = 0$ and the rate of return of an instantaneously maturing risk-free bond is simply $r_t dt$.)

[6] As Leland (1998), p. 1219 argues "Repeated restructurings would always take place before default, and

Modeling these frictions is besides the scope of this paper, since it is inessential for the optimal taxation results that are derived in the next section. All that matters is that frictions in private contracting will lead to bankruptcy once $W_t < L_t$.

Since bankruptcy leads to deadweight costs, a benevolent government may have an incentive to intervene and "bail out" the financial firm. Bailouts are modeled in a way similar to Panageas (2008). Specifically, if the firm is threatened by imminent default, the government makes a transfer that allows the firm to pay down an amount $dL_t < 0$ of debt, so that[7]

$$W_t \geq L_{t^+} \equiv L_t + dL_t. \tag{8}$$

An implication of Karatzas and Shreve (1991) p. 210 is that there exists a unique minimal process dL_t that enforces (8) and it is given by

$$\int_0^t dL_t = \min\left(L_0, \min_{0 \leq s \leq t} W_s\right) - L_0. \tag{9}$$

Later, I show that if the parameter δ, which controls the deadweight costs of bankruptcy, is large enough, then an optimizing government will indeed provide the transfers implied by equation (9). I refer to such a situation as a "perpetual bailout". For what follows, I simply assume that the bailout is perpetual and proceed as if all agents anticipate that the government will provide the transfers implied by equation (9).

default would never occur. As default is not uncommon, this approach is not pursued." Leland's observation applies to a wide class of models, including the present one. If debt renegotiation were frictionless, there should never be any bankruptcies, since bankruptcy costs reduce the joint surplus of all claimholders. The fact that we do observe bankruptcies implies that various frictions that are not explicitly modeled here (e.g. hold up problems created by different seniority of the tranches, asymetric information about the long-run prospects of the firm etc.) can lead to failure of negotiations between the various claimants.

[7]This specification of the transfer and its use to reduce debt is stylized, but could be easily relaxed without changing the main insights of the analysis. For instance, replacing a fraction of the fair-market-rate loans to the private sector with below-market-rate loans to the government is equivalent to a government transfer whose value dL_t is equal to the difference in the economic value of the two loans.

2.3 Taxes to finance the bailout

The government needs to raise taxes in order to finance the transfers to the firm. Taxation is distortionary and the only source of funding for the government is labor income taxation. This simple assumption allows a comparison with the results in Lucas and Stokey (1983). Later, I enrich the model and consider what happens when the government can obtain a fraction of company stock, or a fraction of the firm's assets (or both) in exchange for providing the associated transfers.

Specifically, the government levies a (proportional) labor tax on workers. This tax raises a revenue given by $\tau_t w_t h_t$ at time t. For simplicity, there are no government expenditures and no initial debt. Hence taxes or raised only in order to finance the transfers to the firm. Given the above assumptions, and recalling that markets are complete and $dL_s < 0$, the government's intertemporal budget constraint is

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt = -E_0 \int_0^\infty \xi_t dL_t. \tag{10}$$

2.4 Formulation of the government's problem

Given a path of τ_t, a market equilibrium is defined as a tuple of adapted process for c_t, h_t, F_t, ξ_t, w_t and p_t, so that

1. Consumers maximize (1) over c_t, h_t subject to their intertemporal budget constraint

$$E_0 \int_0^\infty \xi_t c_t dt = E_0 \int_0^\infty \xi_t (1-\tau_t) w_t h_t dt + E_0 \int_0^\infty \xi_t p_t F_t dt - E_0 \int_0^\infty \xi_t dL_t. \tag{11}$$

2. Firms maximize (4) over h_t, F_t.

3. Goods markets, financial services markets, and labor markets clear, i.e. $c_t = Y_t$, $F_t = K_t$ and hours supplied by workers are equal to hours demanded by firms.

4. All asset markets clear.

Equation (11) has a natural interpretation. It states that the consumer's present value of consumption (left hand side of equation [11]) should equal the present value of after tax labor income (first term on the right hand side of [11]) plus the total value (sum of debt and equity) of the financial firm which is given by the last two terms on the right hand side of (11). Observe that the last term in (11) captures the increase in total firm value due to the government transfers.

Constructing an equilibrium for a given path τ_t is straightforward. Attaching a Lagrange multiplier ν to the intertemporal budget constraint (11) and maximizing over c_t, h_t leads to the pair of first order conditions

$$\begin{aligned} e^{-\rho t}\frac{1}{c_t} &= \nu\xi_t, & (12)\\ e^{-\rho t}\left(1-h_t\right)^{-\phi} &= \nu\xi_t(1-\tau_t)w_t. & (13) \end{aligned}$$

Combining (12), and (13) leads to

$$\left(1-h_t\right)^{-\phi} = (1-\tau_t)\frac{w_t}{c_t}. \tag{14}$$

Turning to firms, the first order conditions with respect to h_t and F_t yield

$$\begin{aligned} \alpha Y_t &= w_t h_t, & (15)\\ (1-\alpha)\,Y_t &= p_t F_t. & (16) \end{aligned}$$

These are familiar relationships for factor payments when the production function is of the Cobb-Douglas form. Multiplying both sides of (14) by h_t, using (15) and recognizing that in equilibrium $c_t = Y_t$ leads to

$$\left(1-h_t\right)^{-\phi} h_t = (1-\tau_t)\alpha. \tag{17}$$

Equation (17) implies a one-to-one mapping between a given tax rate and the hours worked in equilibrium. To capture this relationship, let

$$\tau\left(h_t\right) \equiv 1 - \frac{\left(1-h_t\right)^{-\phi} h_t}{\alpha} \tag{18}$$

denote the tax rate that is required to induce a given value of h_t. Straightforward calculations yield $\tau'(h_t) < 0, \tau''(h_t) < 0$.

Before proceeding with the formulation of the government's problem it is useful to use (12)-(17) in order to obtain the present value of taxes and bailout payments in a market equilibrium. Using (12) and (15) inside (10) and recognizing that in equilibrium $c_t = Y_t$ implies that the budget constraint of the government can be written as

$$\alpha E_0 \int_0^\infty e^{-\rho t} \tau(h_t)\, dt = -E_0 \int_0^\infty e^{-\rho t} \frac{1}{c_t} dL_t. \tag{19}$$

Furthermore, equations (5) together with (16), (12) and $c_t = Y_t$ yield

$$W_t = \left(\frac{1-\alpha}{\rho} \right) Y_t. \tag{20}$$

Equation (20) is a well known property of economies where the representative agent has logarithmic utility over consumption; in such economies the price to earnings ratio for a claim that pays a constant fraction $(1-\alpha)$ of aggregate consumption $(c_t = Y_t)$ is simply $\frac{1}{\rho}$. Defining

$$m_t \equiv \min_{0 \le s \le t} Y_s, \quad \text{and } \chi \equiv \frac{(1-\alpha)}{\rho}, \tag{21}$$

and using equations (20) and (9) leads to

$$dL_t = \begin{cases} 0 & \text{if } \chi m_t \ge L_0, \\ \chi dm_t & \text{otherwise.} \end{cases} \tag{22}$$

Equation (22) implies that dL_t changes only when $c_t = m_t$ and hence

$$\begin{aligned} -E_0 \int_0^\infty e^{-\rho t} \frac{1}{c_t} dL_t &= -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\{\chi m_t \le L_0\}} \frac{1}{m_t} dm_t \\ &= -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\left\{\log m_t \le \log\left(\frac{L_0}{\chi}\right)\right\}} d \log m_t, \end{aligned} \tag{23}$$

where the last equality obtains because m_t is a bounded variation process. Combining (19) and (23) leads to the following problem for a government that is assumed to be providing a perpetual bailout.

Problem 1 *Let $J(Z_0, K_0; h_t)$ denote the representative consumer's welfare*

$$U(Z_0, K_0; h_t) \equiv E_0 \left\{ \int_0^\infty e^{-\rho t} \left[\log(Z_t) + (1-\alpha)\log(K_0) + \alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right] dt \right\}. \tag{24}$$

Then the optimal taxation problem for a government is

$$\max_{h_t \in [0,1]} U(Z_0, K_0; h_t)$$

subject to

$$\alpha E_0 \int_0^\infty e^{-\rho t} \tau(h_t)\, dt = -\chi E_0 \int_0^\infty e^{-\rho t} 1_{\left\{\log m_t \leq \log\left(\frac{L_0}{\chi}\right)\right\}} d\log m_t, \tag{25}$$

where $\tau(h_t)$ is given by (18) and m_t is given by (21).

Equation (24) is simply a re-statement of equation (1), using $c_t = Y_t$, equation (2), and the fact that the supposition of a perpetual bailout implies $K_t = K_0$. Equation (25) is the government's intertemporal budget constraint, which follows from (23) and (19).

Before proceeding with the solution of problem 1, it is useful to make three remarks.

First, the one-to-one correspondence between hours worked and tax rates implies an equivalence between choosing τ_t and h_t. For convenience, it is easiest to have the government choose h_t rather than τ_t. Second, as is well known (see e.g. Ljungqvist and Sargent (2004), Chapter 15), the government's budget constraint (10) implies the consumer's budget constraint in a market equilibrium.[8] As a result, to check feasibility of an allocation, it suffices to ensure that the government's intertemporal budget constraint holds (equation [25]).[9] Third, problem 1 assumes that the government can commit to a sequence of taxes. However, as Lucas and Stokey (1983) show, this assumption can be relaxed if the government can issue contingent debt at all maturities.

[8] To see this add $E_0 \int_0^\infty \xi_t (1-\tau_t) w_t h_t dt +$ $E_0 \int_0^\infty \xi_t p_t F_t dt$ to both sides of (10) and use the fact that $w_t h_t + p_t F_t = Y_t = c_t$ on the left hand side of the resulting expression to obtain (11).

[9] Alternatively put, after solving for the optimal h_t^* in problem 1, it is always possible to find a market equilibrium that supports the resulting allocation. (In that equilibrium output and consumption are given by $c_t = Y_t = Z_t K_0^{1-\alpha} (h_t^*)^\alpha$, the taxes that yield the optimal allocation as a market equilibrium are given by $\tau(h_t^*)$ and the price processes ξ_t, w_t, p_t are given by equations (12), (14) and (16), evaluated at $F_t = K_0$.)

3 Optimal taxation

The key difficulty in solving problem 1 is the endogeneity of the cost of the bailout, which is reflected in the fact that output Y_t and as a result its running minimum m_t are affected by the choice of h_t.

To obtain intuition, it is useful to examine what would happen if the government behaved "naively" and optimized over h_t *as if* m_t were an exogenous process beyond its influence. In that case, the solution to problem 1 is straightforward. Letting λ denote the Lagrange multiplier on the government's budget constraint and maximizing

$$U\left(Z_0, K_0; h_t\right) + \lambda \left[\alpha E_0 \int_0^\infty e^{-\rho t} \tau\left(h_t\right) dt + \chi E_0 \int_0^\infty e^{-\rho t} 1_{\left\{\log m_t \leq \log\left(\frac{L_0}{\chi}\right)\right\}} d\log m_t\right],$$

amounts to a simple "point by point" maximization problem with solution

$$h^* = \arg\max_{h_t} \left[\alpha \log h_t + \frac{\left(1-h_t\right)^{1-\phi}}{1-\phi} + \lambda\alpha\tau\left(h_t\right)\right]. \tag{26}$$

By concavity of the objective in (26), there is a unique value h^* that maximizes (26). Furthermore, the one-to-one correspondence between h_t and τ_t (equation [17]) implies that the government would end up choosing a *constant* tax rate. This result is reminiscent of the well known labor tax smoothing results in the optimal taxation literature (see e.g. Ljungqvist and Sargent (2004) Chapter 15), and serves as an illustration of the labor tax smoothing forces that are present in the model.

However, matters are more complex, because a fully rational government takes into account that m_t (and hence the cost of the bailout) are both endogenous. This introduces two opposing forces: On the one, smoothing the distortions associated with taxation tends to favor a stable tax rate. On the other hand, lowering taxes in states where the financial firm is threatened with bankruptcy can boost demand for labor, and hence increase the marginal product of the financial firm and the value of its assets W_t. In turn this boost in value implies that the total cost of the bailout may be reduced by "stimulating" the economy through tax cuts.

The remainder of this section derives the optimal h_t^* (and hence the optimal τ_t^*) that solves problem 1. It is easiest to start by assuming that τ_0 and hence h_0 is given, so that $\log m_0 = \log Y_0 = \log Z_0 + (1-\alpha)\log K_0 + \alpha \log h_0$ is also given. It is also useful to assume that bailout payments haven't yet started at time 0

$$\log Y_0 = \log m_0 \geq \log\left(\frac{L_0}{\chi}\right). \tag{27}$$

Remark 1 in the appendix allows the government to freely choose τ_0 (and hence Y_0 and m_0) and shows that condition (27) is always satisfied for sufficiently high values of Z_0.[10]

Next, attaching a Lagrange multiplier $\lambda > 0$ to (25), defining

$$f(h_t) \equiv \alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} + \lambda\alpha\tau(h_t), \tag{28}$$

and observing that the first two terms inside the square brackets of (24) are exogenous, implies that maximizing U is equivalent to maximizing $\Phi(h_t)$, where

$$\Phi(h_t) \equiv E_0 \int_0^\infty e^{-\rho t} f(h_t)\, dt + \lambda\chi E_0 \int_0^\infty e^{-\rho t} 1_{\left\{\log m_t \leq \log\left(\frac{L_0}{\chi}\right)\right\}} d\log m_t. \tag{29}$$

To derive the optimal h_t, I first derive an upper bound to Φ over all processes h_t, and then show how to attain it with an appropriate choice of h_t.

To this end, I consider an arbitrary path of $\widetilde{h}_t$ and fix the associated process $\widetilde{m}_t$. By definition of $\widetilde{m}_t$,

$$\log Y_t = \log Z_t + (1-\alpha)\log K_0 + \alpha \log \widetilde{h}_t \geq \log \widetilde{m}_t. \tag{30}$$

Letting

$$g(Z_t, \widetilde{m}_t) \equiv \max_{h_t \text{ s.t. } \log Z_t + (1-\alpha)\log K_0 + \alpha\log h_t \geq \log \widetilde{m}_t} f(h_t), \tag{31}$$

and

$$J(\widetilde{m}_t) \equiv E_0 \int_0^\infty e^{-\rho t} g(Z_t, \widetilde{m}_t)\, dt + \lambda\chi E_0 \int_0^\infty e^{-\rho t} 1_{\left\{\log \widetilde{m}_t \leq \log\left(\frac{L_0}{\chi}\right)\right\}} d\log \widetilde{m}_t, \tag{32}$$

[10] If the initial conditions are such that $W_0 < L_0$, then the initial payment $L_0 - W_0$ must be added to the net present value of the guarantee. Even though this doesn't change the nature of the solution for optimal taxation, it does affect the cost-benefit evaluation of the bailout, that is described in section 3.3.

and using the definition of $g(Z_t, \widetilde{m}_t)$ implies that $g(Z_t, \widetilde{m}_t) \geq f\left(\widetilde{h}_t\right)$. Letting $D^{(m_0)}$ denote the set of decreasing adapted processes starting at m_0, and comparing (29) and (32) gives

$$\Phi\left(\widetilde{h}_t\right) \leq J(\widetilde{m}_t) \leq \max_{m_t \in D^{(m_0)}} J(m_t) \tag{33}$$

Hence, $\max_{m_t \in D^{(m_0)}} J(m_t)$ provides an upper bound to the feasible payoffs $\Phi\left(\widetilde{h}_t\right)$ for any admissible process $\widetilde{h}_t$. Furthermore, the optimal process h_t^* that attains this upper bound is given by

$$h_t^* = \arg \max_{h_t \text{ s.t. } \log Z_t + (1-\alpha)\log K_0 + \alpha \log h_t \geq \log m_t^*} f(h_t), \tag{34}$$

where $m_t^* = \arg\max_{m_t \in D^{(m_0)}} J(m_t)$. Letting

$$\overline{h} \equiv \arg\max_{h_t} f(h_t), \tag{35}$$

denote the unconstrained maximum of $f(h_t)$, and noting that $f(\cdot)$ is concave, the process h_t^* that solves (34) is simply given by

$$\log h_t^* = \max\left[\log \overline{h}, \frac{1}{\alpha}\left(\log m_t^* - \log Z_t - (1-\alpha)\log K_0\right)\right] \tag{36}$$

Equation (36) suggests a simple two-step strategy for solving the government's problem. First, determine the optimal m_t that solves $\max_{m_t \in D^{(m_0)}} J(m_t)$. In a second step, use equations (36) and (18) to determine the optimal process for hours and taxes given the optimal m_t^* from the first step. Intuitively, in the first step, the government determines a minimum level of output that it wants to achieve, while in the second step it determines the hours and the taxes that will implement that minimum level.

3.1 Determining m_t^*

To determine m_t^* one needs to solve the optimization problem

$$V(Z_0, m_0) = \max_{m_t \in D^{(m_0)}} J(m_t). \tag{37}$$

This maximization problem shares several similarities with problems of irreversible investment. The only difference is that irreversible investment-problems feature maximization over increasing rather than decreasing processes.

The next proposition derives the solution to problem (37).

Proposition 1 *Define*

$$\varphi_1 \equiv \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2}$$

Using the notation $\langle x\rangle^- \equiv \min[x, 0]$, *define*

$$y(u) \equiv \frac{1}{\alpha} u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \left\langle f'\left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}\right)\right\rangle^- + \lambda\chi\rho \tag{38}$$

and let β *be given by the solution to*

$$\inf_{\beta} \;\; s.t. \int_{\beta}^{\infty} \frac{y(u)}{(u)^{\varphi_1+1}} du = 0. \tag{39}$$

Equation (39) has a unique solution and the optimal m_t^* *that maximizes (37) is given by*

$$m_t^* = \min\left[\frac{L_0}{\chi}, \frac{1}{\beta}\left(\min_{0\le s\le t} Z_s\right)\right]. \tag{40}$$

Figure 1 illustrates the solution (40) for values of $m_t \le \frac{L_0}{\chi}$. The diagram is split into two regions that are referred to as the "inaction" region and the "forbidden" region. In the inaction region $Z_t > \beta m_t$ and hence $m_t = \frac{1}{\beta}\min_{0\le s\le t} Z_s < \frac{1}{\beta} Z_t$ or $Z_t > \min_{0\le s\le t} Z_s$. Accordingly, it is optimal to set $dm_t^* = 0$, i.e. take no action. By contrast, if Z_t becomes (instantaneously) smaller than βm_t, then it is optimal to immediately reduce m_t^* until the inequality $Z_t \ge \beta m_t^* = \min_{0\le s\le t} Z_s$ is restored.

3.2 Procyclical tax rates and history dependence

Using the definition of $\tau(h_t)$ (equation[18]), noting that $\tau'(h_t) < 0$ and using (36) leads to the following expression for the optimal tax rate

$$\tau\left(Z_t, \min_{0\le s\le t} Z_s\right) = \min\left\{\tau\left(\overline{h}\right), \tau\left[\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \left(\frac{Z_t}{\min_{0\le s\le t} Z_s}\right)^{-\frac{1}{\alpha}}\right]\right\}, \tag{41}$$

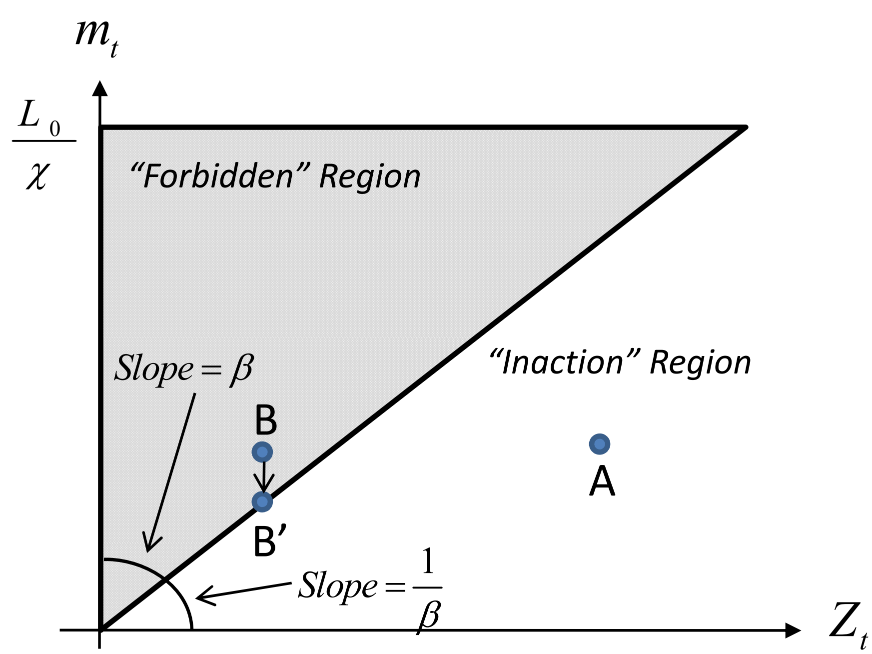

FIGURE 1: An illustration of the optimal policy m_t^*. At point A, $Z_t > \beta m_t^*$ and it is optimal to keep m_t^* unchanged. By contrast at point B it is optimal to instantaneously decrease m_t^* so as to move to point B' and restore the inequality $Z_t \geq \beta m_t^*$.

where m_t^* is given by (40) and $\overline{h}$ is given by (35). Equation (41) shows that the optimal tax rate can be expressed as the ratio of two state variables, namely Z_t and $\min_{0 \leq s \leq t} Z_s$.

Inspecting (41), and recalling that τ is a decreasing function, reveals that the tax rate is non-decreasing in Z_t, i.e. the tax rate is higher in times of higher productivity. Figure 2 illustrates the dependence of τ on Z_t. When $Z_t = \min_{0 \leq s \leq t} Z_s$, (i.e. at times when the government makes transfers to the private sector), the tax rate attains its lowest value. For values of Z_t that are to the right of $\min_{0 \leq s \leq t} Z_s$, the tax rate increases monotonically until the level $\tau\left(\overline{h}\right)$.

The lower panel of the figure depicts the associated behavior of Y_t. For values of Z_t such that τ is increasing in Z_t, output remains constant. Output starts increasing with Z_t only

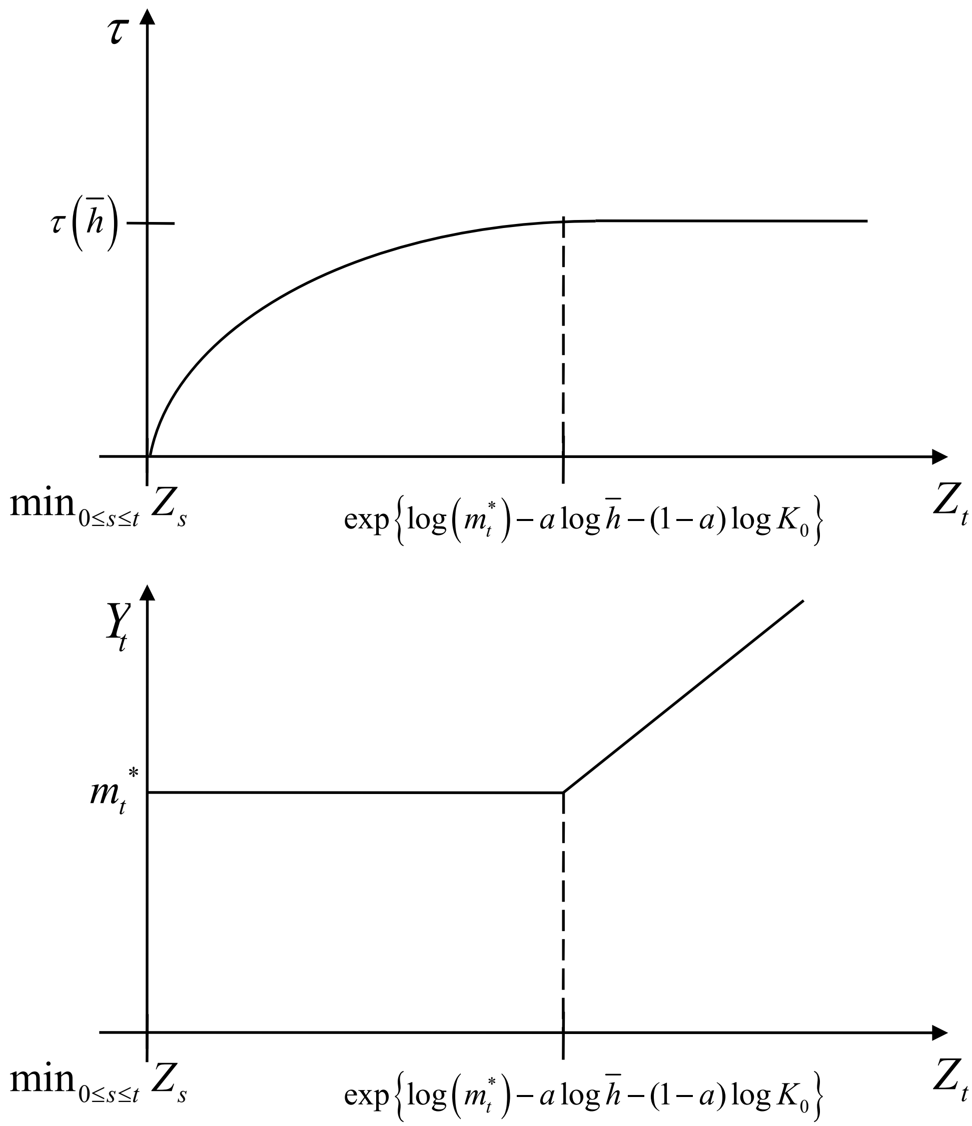

FIGURE 2: Procyclical taxes and output stabilization

once τ becomes constant. This behavior of Y_t is a manifestation of the two forces present in the model; on the one hand, the government would like to set a constant tax rate for standard labor tax smoothing reasons. On the other hand however, by making the tax rate procyclical, the government can stabilize output Y_t and accordingly ensure that the value of capital W_t does not fall further. This helps the government save transfers to the firm and reduce the overall cost of the bailout.

A further and somewhat surprising property of the optimal tax rate is its history dependence *despite dynamically complete markets*. Equations (40) and (41) imply that the tax rate is a non-increasing function of the running minimum of output and hence an increasing function of the total payments to the financial firm by time t (by equation [22]). Alternatively put, *fixing* a given level of productivity (Z_t) at time t, the model implies a positive relationship between total bailout payments *prior* to time t and tax rates at time t.

Equation (41) gives the optimal tax rate up to two constants $\overline{h}$ and β that depend on the Lagrange multiplier λ. To complete the determination of the optimal tax rate, the next Lemma shows how to compute λ.

Proposition 2 *The value of λ that enforces equation (25) is given as the solution to the equation*

$$V_\lambda \left(Z_0, \frac{L_0}{\chi}; \lambda \right) = 0. \tag{42}$$

An explicit expression for $V_\lambda \left(Z_0, \frac{L_0}{\chi}; \lambda \right)$ is contained in the proof of Proposition 2.

3.3 The optimality of transfer payments to the financial firm

Sofar the analysis has simply assumed that an optimizing benevolent government will choose to extend transfer payments to the financial firm. This subsection shows that this is indeed the case when δ is large enough.

Equations (36) and (40) imply that $h_t^* \in [\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}, \overline{h}]$. Importantly, neither β nor $\overline{h}$ depend on δ. Accordingly, at any point in time, the representative agent's welfare -assuming a perpetual bailout- is at least as large as

$$E_t \left\{ \int_t^\infty e^{-\rho(s-t)} \left[\log \left(Z_s \right) + (1-\alpha) \log \left(K_0 \right) + \psi_1 \right] ds \right\}, \tag{43}$$

where $\psi_1 = \min_{h_t \in [\beta^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}, \bar{h}]} \left(\alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right)$. Similarly, assuming that the government lets the financial firm fail at time t, the representative agent's welfare is bounded above by

$$E_t \left\{ \int_t^\infty e^{-\rho(s-t)} \left[\log(Z_s) + (1-\alpha) \log((1-\delta) K_0) + \psi_2 \right] ds \right\}, \tag{44}$$

where $\psi_2 = \max_{h_t} \left(\alpha \log h_t + \frac{(1-h_t)^{1-\phi}}{1-\phi} \right)$. Comparing (43) and (44) reveals that a *sufficient* condition for a perpetual bailout is

$$(1-\alpha) \log(1-\delta) + \psi_2 < \psi_1. \tag{45}$$

Since neither ψ_1 nor ψ_2 depend on δ, and $\log(1-\delta)$ can be made arbitrarily small as $\delta \to 1$, there always exist sufficiently large values of δ that make condition (45) hold.

It is important to stress that re-distribution concerns (which have been ignored sofar) may have a substantial impact on the welfare implications of a perpetual bailout. In a representative agent economy, the taxes raised through labor taxation get indirectly rebated back to the consumer in the form of an increased value of his total (financial and non-financial) wealth. In reality, a large fraction of the population has little or no financial wealth and has to rely on labor income alone to finance consumption. If the government cares mostly about these parts of the population, then the welfare calculations need to be modified. In these modified calculations, the benefit of a bailout would stem from the fact that the deadweight costs of bankruptcy reduce the capital stock and hence the wages of workers.

4 Additional forms of financing bailouts and the evolution of debt

Sofar, bailouts could only be financed with distortionary labor taxes. In reality, bailouts are at least partly financed by the beneficiaries of bailout payments. For instance, the government may provide cash injections in exchange for equity holdings in the underlying company.

Given the stylized nature of the model, which abstracts from capital accumulation, taxation of the existing capital stock in any shape or form is efficient, since it amounts to a non-distortionary, lump-sum tax. As is well understood in the literature, however, such forms of taxation can strain the government's commitment abilities, since typically they imply confiscatory capital taxation at time zero, accompanied by zero capital taxes in the long run in order to promote capital accumulation.

With this literature as a backdrop, it seems reasonable to place limits on the amount of tax that can be raised by taxing the existing capital stock within the model. For instance, such limitations can be motivated as resulting from pre-existing commitments to promote the capital accumulation that led to the initial capital stock K_0.

To be specific about the limits of capital taxation, this section continues to assume that the debtholders of the financial institution cannot be taxed. Besides the theoretical reasons laid out above, there are also some practical, institutional considerations, which make this assumption plausible. If the debtholders are thought of as deposit holders in depository institutions, then the presence of FDIC insurance along with the possibility to withdraw deposits on demand and invest them in tax-advantaged forms (municipal bonds, gold bars in safe deposit boxes etc.) would render deposit taxation practically very hard; in reality such taxation would induce the sort of bank run that governments try to prevent in the first place.

The sort of taxation that is allowed, however, is taxation of the equityholders. Specifically, this section allows for the possibility that part of the funds needed for the bailout may be raised by having the government obtain either a) a fraction $\pi_1 \geq 0$ of the equities of the financial firm and/or b) a share $\pi_2 \geq 0$ of the revenues of the representative financial firm. The motivation for considering equities and revenue fractions as two alternative forms for raising funds is based on proposals put forth during the crisis to fund part of the cost of bailouts by either diluting current shareholders, or by placing some of the troubled assets in the hands of the government (or in a government-sponsored "bad bank"). Since the goal of this section is mostly illustrative, I assume that π_1 and π_2 remain constant throughout time. Letting

$$P_t \equiv \left(\frac{1}{\xi_t} E_t \int_t^\infty \xi_s \left(1 - \pi_2\right) p_s F_s ds - L_t - \frac{1}{\xi_t} E_t \int_t^\infty \xi_s dL_s \right) \tag{46}$$

denote the total value of equity[11] at time at time t, I assume that

$$\pi_1 \xi_0 P_0 + \pi_2 E_0 \int_0^\infty \xi_t p_t F_t dt \leq -\zeta E_0 \int_0^\infty \xi_t dL_t, \ \zeta \in (0,1). \tag{47}$$

In equation (47), ζ is the fraction of the bailout that can be financed by taxation of the equity holders. Specification (47) is attractive, because it ensures that the value of equity is at least as large as $-(1-\zeta) E_0 \int_0^\infty \frac{\xi_t}{\xi_0} dL_t \geq 0$, and hence it implies allocations that are compatible with limited liability of equity. Moreover, values of ζ smaller than one can be motivated by the need to provide appropriate continuation incentives to managers and workers, who may be holding deferred compensation in the form of equities.

The government's modified budget constraint is given by

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt + \pi_1 \xi_0 P_0 + \pi_2 E_0 \int_0^\infty \xi_t p_t F_t dt = -E_0 \int_0^\infty \xi_t dL_t. \tag{48}$$

Combining (47) and (48) leads to

$$E_0 \int_0^\infty \xi_t \tau_t w_t h_t dt \geq -\left(1 - \zeta\right) E_0 \int_0^\infty \xi_t dL_t. \tag{49}$$

Since labor taxes are distortionary, (49) will hold with equality at the optimum. There are two obvious, yet important, implications of equation (49). First, obtaining a share of

[11] To see that this is the total value of equity, note that equity value is given as the difference between post-tax revenue and interest payments. By footnote 3 the payments to debtholders when debt is equal to L_t are given by $L_t \left(r_t dt + dA_t\right)$. Letting

$$P_t \equiv \frac{1}{\xi_t} \left(E_t \int_t^\infty \xi_s \left(1 - \pi_2\right) p_s F_s ds - E_t \int_t^\infty \xi_s L_s \left(r_s ds + dA_s\right) \right)$$

denote the total value of equity, using the fact that $L_t \left(r_t dt + dA_t\right) = -L_t \left(d\xi_t / \xi_t - \kappa_t \xi_t dB_t\right)$ (by footnote 3), integrating by parts and ignoring terms having expectation equal to zero leads to (46).

the dividends or the revenues of the financial firm is equivalent to limiting the cost of the guarantee and hence the distortions associated with labor taxation. And second, since π_1 and π_2 do not appear in (49), the choice between taxation of the profits of the financial firm or the underlying capital has no welfare consequences (since neither the maximization objective [1], nor the modified budget constraint [49] depend on π_1 or π_2.)

The second implication of (49) depends crucially on the assumption of dynamically complete markets. However, if one were to assume restrictions on the ability of the government to issue debt (say for commitment-to-repay issues), then one form of financing may become more preferable compared to the other. To illustrate this point, the next subsection considers the stochastic process of government debt under the two alternatives (obtaining a fraction of dividends or obtaining a fraction of revenues) and shows that in the presence of a debt ceiling, the latter form of financing may be preferable.

4.1 Evolution of debt and debt ceilings

Letting B_t denote the government's debt, one obtains

$$\begin{aligned} \xi_t B_t &= E_t \int_t^\infty \xi_s dL_s + E_t \int_t^\infty \xi_s \tau(h_s) w_s h_s ds \\ &+\pi_1 \left(E_t \int_t^\infty \xi_s (1-\pi_2) p_s F_s ds - \xi_t L_t - E_t \int_t^\infty \xi_s dL_s \right) + \pi_2 E_t \int_t^\infty \xi_s p_s F_s ds \end{aligned} \tag{50}$$

The next proposition shows that the maximal value of the debt to gdp ratio $b_t \equiv \frac{B_t}{Y_t}$ (across all $t > 0$) is smallest when $\pi_1 = 0$.

Proposition 3 *Assume that constraint (47) is satisfied with equality. Then the value of* π_1 *that minimizes* $\sup_{0 \leq s \leq t} b_s$ *is given by* $\pi_1 = 0$.

5 Conclusion

The present paper considered optimal fiscal policy, when distortionary taxes are used to finance bailouts. The key departure from pre-existing literature is that taxes are not used to finance exogenous *government expenditure*, but rather endogenous *transfer payments.*

Within such a framework, taxes are not only used to finance the bailout, but also to support real activity, raise the value of financial firms and hence reduce the (endogenously determined) net present value of the taxes required to finance the transfers. As a result, taxes turn out to be procyclical. This result is in contrast to a large literature (e.g. Barro (1979), Lucas and Stokey (1983)), that finds a constant, acyclical tax rate to be optimal, when government expenditures do not vary. Furthermore, tax rates are dependent on both current productivity and past transfer payments. This history dependence of the tax rate in a dynamically complete market is in contrast to Lucas and Stokey (1983), where the tax rate depends only on current government expenditure.

An extended version of the model considers additional forms of funding a bailout such as obtaining equity shares of the company and diluting shareholders or obtaining shares of the underlying capital stock. In a complete market it is only the total value of these claims that affects welfare, and not the choice between the two. However, the stochastic process for debt implied by the two alternatives is different. For instance, equity shares may lead to higher levels of the debt to gdp ratio, which may be unattractive if the government is subject to a debt ceiling.

A Appendix

Proof of Proposition 1. The first step towards proving proposition 1 is to show that one can focus attention to policies that set $m_0 = \frac{L_0}{\chi}$ (without loss of generality).

Lemma 1 $m_0^* \leq \frac{L_0}{\chi}$.

Proof of Lemma 1. Suppose otherwise and consider an optimal m_t^* such that $m_0^* > \frac{L_0}{\chi}$. Let τ^{L_0} denote the first time that $m^*_{\tau^{L_0}} = \frac{L_0}{\chi}$. Then, by Bellman's principle of optimality,

$$\begin{aligned} V(Z_0, m_0) &= E_0 \int_0^{\tau^{L_0}} e^{-\rho t} g(Z_t, m_t^*)\, dt + E e^{-\rho \tau^{L_0}} V\left(Z_{\tau^{L_0}}, \frac{L_0}{\chi}\right) \\ &\leq E_0 \int_0^{\tau^{L_0}} e^{-\rho t} g\left(Z_t, \frac{L_0}{\chi}\right) dt + E e^{-\rho \tau^{L_0}} V\left(Z_{\tau^{L_0}}, \frac{L_0}{\chi}\right) \\ &\leq V\left(Z_0, \frac{L_0}{\chi}\right), \end{aligned}$$

where the first inequality follows from $g_m < 0$. Since $V(Z_0, m_0) \leq V\left(Z_0, \frac{L_0}{\chi}\right)$, and it is always possible to decrease m_0 instantaneously, one may assume without loss of generality that $m_0^* = \frac{L_0}{\chi}$. ■

In light of Lemma 1, one can set $m_0 = \frac{L_0}{\chi}$ without loss of generality and accordingly $1_{\left\{\log m_t \leq \log\left(\frac{L_0}{\chi}\right)\right\}} = 1$. Letting

$$\eta(Z_t, m_t) \equiv g(Z_t, m_t) + \lambda \chi \rho \log m_t,$$

and applying integration by parts to (32) gives[12]

$$J\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) = \widehat{J}\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) - \lambda \chi \log\left(\frac{L_0}{\chi}\right), \tag{51}$$

[12]In applying integration by parts, note that

$$\lim_{T\to\infty} e^{-\rho T} E \log m_T = 0,$$

which follows from the fact that h_t is bounded and that $\lim_{T\to\infty} e^{-\rho T} E \min \log_{0\leq s\leq T} Z_T = 0$. (The fact that $\lim_{T\to\infty} e^{-\rho T} E \min \log_{0\leq s\leq T} Z_T = 0$ follows from the closed form expression for $\Pr\left(\min \log_{0\leq s\leq T} Z_s \geq x\right)$ in Corollary B.3.4. of Musiela and Rutkowski (1998) (p.470) after using integration by parts to compute $E e^{-\rho T} \min \log_{0\leq s\leq T} Z_s$ and sending T to infinity.)

where

$$\widehat{J}\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right) = E_0 \int_0^\infty e^{-\rho t} \eta\left(Z_t, m_t\right) dt.$$

Clearly, maximizing J is equivalent to maximizing $\widehat{J}$ (since $\lambda\chi \log\left(\frac{L_0}{\chi}\right)$ is a constant that does not depend on the choice of $m_{t>0}$). Before proceeding, it is useful to prove the following Lemma.

Lemma 2 *Let $\overline{Z}$ denote a value of Z that solves the equation*

$$\inf_{\overline{Z}} \ \ s.t. \int_{\overline{Z}}^\infty \frac{\eta_m\left(x, m_t\right)}{x^{\varphi_1+1}} dx = 0. \tag{52}$$

Such a value exists, is unique, and is increasing in m_t.

Proof of Lemma 2. Differentiating $\eta\left(Z_t, m_t\right)$ with respect to m_t gives

$$\eta_m\left(Z_t, m_t\right) = \frac{1}{\alpha} Z_t^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} m_t^{\frac{1}{\alpha}-1} \left\langle f'(m_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})\right\rangle^- + \lambda\chi\rho\frac{1}{m_t}.$$

Since $f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})$ is positive once $m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} < \overline{h}$, it follows that $\eta_m = \lambda\chi\rho\frac{1}{m_t} > 0$ whenever $x > \overline{h}^{-\alpha} K_0^{-(1-\alpha)} m_t$. Furthermore, since[13] $\lim_{h\to 1} f'(h) = -\infty$, it follows that $\lim_{x\to m_t K_0^{-(1-\alpha)}} \left[f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})\right] = -\infty$. Additionally, since f is concave, differentiating $\frac{1}{\alpha} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} m_t^{\frac{1}{\alpha}-1} f'(m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}})$ with respect to x implies that $\eta_m\left(x, m_t\right)$ is a non-decreasing function of x, and by the intermediate value theorem there exists a unique $\overline{x} \in \left(m_t K_0^{-(1-\alpha)}, +\infty\right)$ such that $\frac{\eta_m(\overline{x}, m_t)}{\overline{x}^{\varphi_1+1}} = 0$. Finally, $\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} > 0$ for all $x > \overline{x}$ and $\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} < 0$ for all $x < \overline{x}$. Furthermore, $\lim_{j\to m_t K_0^{-(1-\alpha)}} \int_j^{\overline{x}} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx = -\infty$. Accordingly, there exists a unique $\overline{Z}$ that solves equation (52). Inspection reveals that for $m_1 > m_2$, $\eta_m\left(x, m_1\right) < \eta_m\left(x, m_2\right)$ and hence $\overline{Z}\left(m_1\right) > \overline{Z}\left(m_2\right)$. (To see this, note that since η_m is

[13] A straightforward computation leads to

$$f'\left(h_t\right) = \frac{\alpha}{h_t} + \left(1-h_t\right)^{-\phi}\left[1 - \lambda - \phi\frac{h_t}{\left(1-h_t\right)}\right],$$

and since $\lim_{h_t\to 1}\left(1-h_t\right)^{-\phi} = +\infty$ and $\lim_{h_t\to 1}\left(-\phi\frac{h_t}{(1-h_t)}\right) = -\infty$, it follows that $\lim_{h_t\to 1} f'\left(h_t\right) = -\infty$.

decreasing[14], $\eta_{mm} < 0$.). An application of the implicit function theorem to (52) implies that

$$\frac{d\overline{Z}}{dm_t} = \frac{\int_{\overline{Z}}^{\infty} \frac{\eta_{mm}(x,m_t)}{x^{\varphi+1}} dx}{\overline{Z}^{-(\varphi_1+1)} \eta_m \left(\overline{Z}, m_t\right)} > 0,$$

since[15] $\eta_{mm} < 0$ and $\eta_m \left(\overline{Z}, m_t\right) < 0$ (note that $\overline{Z} < \overline{x}$ and hence $\eta_m \left(\overline{Z}, m_t\right) < 0$.) ■

The remainder of the proof proceeds via a verification argument; first I postulate a solution for the value function

$$\widehat{V}\left(Z_t, \frac{L_0}{\chi}\right) = \sup_{m_{t>0}} \widehat{J}\left(Z_0, \frac{L_0}{\chi}; m_{t>0}\right), \tag{53}$$

and then verify directly that $\widehat{V}\left(Z_t, \frac{L_0}{\chi}\right)$ is indeed the value function.

Specifically, let $\overline{Z}(m)$ denote the solution of equation (52), and let

$$\omega(x) \equiv \overline{Z}^{-1}(\cdot),$$

denote the inverse function of $\overline{Z}(\cdot)$. Observe that since $\overline{Z}(m)$ is increasing, $\omega(x)$ is also increasing. Next consider the following expression for the value function

$$\widehat{V}(Z_t, m_t) = \int_0^{\overline{Z}(m_t)} \mathcal{G}(Z_t, x)\, \eta(x, \omega(x))\, dx + \int_{\overline{Z}(m_t)}^{\infty} \mathcal{G}(Z_t, x)\, \eta(x, m_t)\, dx, \tag{54}$$

where $\mathcal{G}(Z_t, x)$ is defined as follows[16]

$$\mathcal{G}(Z_t, x) \equiv \begin{cases} \frac{2}{(\varphi_1-\varphi_2)\sigma^2} Z_t^{\varphi_2} x^{-\varphi_2-1} & \text{if } x \leq Z_t \\ \frac{2}{(\varphi_1-\varphi_2)\sigma^2} Z_t^{\varphi_1} x^{-\varphi_1-1} & \text{if } x > Z_t \end{cases} \tag{55}$$

and the constant φ_2 is defined as

$$\varphi_2 \equiv \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2}$$

---

[14] η_{mm} is defined everywhere except at the point where $m_t^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} = \overline{h}$. At that point one can arbitrarily set η_{mm} equal to either its left- or right-derivative (or any other value) without affecting the remainder of the proof.

[15] Note that the non-differentiability of η_m at a single point is irrelevant for the computation of the integral $\int_{\overline{Z}}^{\infty} \frac{\eta_{mm}(m_t,x)}{x^{\varphi_1+1}} dx$.

[16] In the literature $\mathcal{G}(Z_t, x)$ is known as the Green function, see e.g. Kobila (1993), or Øksendal (2003) Ch. 9.

The first step towards verifying that $\widehat{V}(Z_t, m_t)$ is the value function of (53) is contained in the following Lemma

Lemma 3 *$\widehat{V}(Z_t, m_t)$ satisfies the following differential inequality*

$$\sigma^2 Z_t^2 \widehat{V}_{ZZ} + \mu Z_t \widehat{V}_Z - \rho \widehat{V} + \eta(Z_t, m_t) \leq 0 \tag{56}$$

Proof of Lemma 3. By applying Ito's Lemma to (54) and using (55), it is simple to verify that[17]

$$\sigma^2 Z_t^2 \widehat{V}_{ZZ} + \mu Z_t \widehat{V}_Z - \rho \widehat{V} = - \begin{cases} \eta(Z_t, \omega(Z_t)) & \text{if } Z_t \leq \overline{Z}(m_t), \\ \eta(Z_t, m_t) & \text{if } Z_t > \overline{Z}(m_t). \end{cases} \tag{57}$$

When $Z_t \leq \overline{Z}(m_t)$ it follows that $\omega(Z_t) \leq m_t$ and hence

$$\eta(Z_t, m_t) - \eta(Z_t, \omega(Z_t)) = \int_{\omega(Z_t)}^{m_t} \eta_m(Z_t, u)\, du < 0, \tag{58}$$

since[18] $\eta_m(Z_t, \omega(Z_t)) < 0$ and η_m is declining in m. Combining (57) with (58) gives (56). ■

The second step of the verification argument is contained in the following statement

Lemma 4 *The derivative of $\widehat{V}$ with respect to m satisfies the following set of (in)equalities*

$$\widehat{V}_m(Z_t, m_t) = \begin{cases} = 0 & \text{if } Z_t < \overline{Z}(m_t) \\ \geq 0 & \text{if } Z_t \geq \overline{Z}(m_t) \end{cases}$$

Proof of Lemma 4. Differentiating (54) with respect to m_t, noting that $\eta\left(\overline{Z}(m_t), \omega\left(\overline{Z}(m_t)\right)\right)$ $= \eta\left(\overline{Z}(m_t), m_t\right)$ and using the definition of $\mathcal{G}(Z_t, x)$ in equation (55) implies that

$$\widehat{V}_m(Z_t, m_t) = \begin{cases} \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} Z_t^{\varphi_1} \int_{\overline{Z}(m_t)}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1 + 1}} dx & \text{if } Z_t < \overline{Z}(m_t), \\ \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} \left[Z_t^{\varphi_2} \int_{\overline{Z}(m_t)}^{Z_t} \frac{\eta_m(x, m_t)}{x^{\varphi_2 + 1}} dx + Z_t^{\varphi_1} \int_{Z_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1 + 1}} dx \right] & \text{if } Z_t \geq \overline{Z}(m_t). \end{cases}$$

[17]See Kobila (1993) for some technical details.

[18]As was shown above $\eta_m\left(\overline{Z}, m_t\right) < 0$, which implies that $\eta_m(Z_t, \omega(Z_t)) < 0$, since $Z_t = \overline{Z}(m_t)$ if and only if $m_t = \omega(Z_t)$.

In light of (52) it follows that $V_m(Z_t, m_t) = 0$ whenever $Z_t < \overline{Z}(m_t)$. Similarly, when $Z_t \geq \overline{Z}(m_t)$

$$\begin{aligned}\widehat{V}_m(Z_t, m_t) &= \frac{2Z_t^{\varphi_1}}{(\varphi_1 - \varphi_2)\sigma^2}\left[Z_t^{\varphi_2-\varphi_1}\int_{\overline{Z}(m_t)}^{Z_t}\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}}\frac{x^{\varphi_1+1}}{x^{\varphi_2+1}}dx + \int_{Z_t}^{\infty}\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}}dx\right] \\ &= \frac{2Z_t^{\varphi_1}}{(\varphi_1 - \varphi_2)\sigma^2}\left[\int_{\overline{Z}(m_t)}^{Z_t}\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}}\left(\frac{x}{Z_t}\right)^{\varphi_1-\varphi_2}dx + \int_{Z_t}^{\infty}\frac{\eta_m(x, m_t)}{x^{\varphi_1+1}}dx\right]\end{aligned}$$

Since $\int_{\overline{Z}(m_t)}^{Z_t}\frac{\eta_m(x,m_t)}{x^{\varphi_1+1}}dx + \int_{Z_t}^{\infty}\frac{\eta_m(x,m_t)}{x^{\varphi_1+1}}dx = 0$ by (52), and $\eta_m(x, m_t)$ is an increasing function of x it follows that $\int_{\overline{Z}(m_t)}^{Z_t}\frac{\eta_m(x,m_t)}{x^{\varphi_1+1}}dx < 0$. Additionally, since $\left(\frac{x}{Z_t}\right)^{\varphi_1-\varphi_2} < 1$ for all $x \in [\overline{Z}(m_t), Z_t)$, it follows that $\widehat{V}_m(Z_t, m_t) \geq 0$ for all $Z_t \geq \overline{Z}(m_t)$. ■

The rest of the verification argument follows similar steps to Kobila (1993), Proposition 6.1. To save space, I simply outline the argument and refer the reader to Kobila (1993) for additional technical details.

Take any arbitrary process $m_t \in D^{(L_0/\chi)}$. Applying Ito's Lemma to $\widehat{V}$ gives

$$\begin{aligned}E_0\left(e^{-\rho T}\widehat{V}(Z_T, m_T)\right) - \widehat{V}(Z_0, m_0) &= E_0\int_0^T e^{-\rho t}\left(\sigma^2 Z_t^2\widehat{V}_{ZZ} + \mu Z_t\widehat{V}_Z - \rho\widehat{V}\right)dt \\ &\quad +E_0\int_0^T e^{-\rho t}\sigma Z_t\widehat{V}_Z dB_s \\ &\quad +E_0\int_0^T e^{-\rho t}\widehat{V}_m dm_t.\end{aligned}$$

Letting $T \to \infty$, and using arguments similar to Kobila (1993), one obtains the limit

$$\begin{aligned}\widehat{V}(Z_0, m_0) &= -E_0\int_0^{\infty} e^{-\rho t}\left(\sigma^2 Z_t^2\widehat{V}_{ZZ} + \mu Z_t\widehat{V}_Z - \rho\widehat{V}\right)dt - E_0\int_0^{\infty} e^{-\rho t}\widehat{V}_m dm_t \\ &\geq E_0\int_0^{\infty} e^{-\rho t}\eta(Z_t, m_t)\,dt, \end{aligned} \tag{59}$$

where the second line follows from equation (56) and the fact that $E_0\int_0^{\infty} e^{-\rho t}\widehat{V}_m dm_t < 0$ (since $\widehat{V}_m \geq 0$, and m_t is decreasing). Since $m_t \in D^{(L_0/\chi)}$ was arbitrary, equation (59) implies that $\widehat{V}(Z_0, m_0)$ provides an upper bound to any attainable payoff. Additionally, by the Skorohod equation (Karatzas and Shreve (1991) p. 210) this upper bound is attained for the process

$$m_t^* = \min\left[\frac{L_0}{\chi}, \omega\left(\min_{0\leq s\leq t} Z_s\right)\right] \tag{60}$$

of equation (40). The next Lemma allows one to obtain an explicit expression for $\omega(\cdot) = \overline{Z}^{-1}(\cdot)$.

Lemma 5 *Let $y(u)$ be given by (38) and β by (39). Then the value $\overline{Z}(m_t)$ that solves (52) is given by $\overline{Z}(m_t) = \beta m_t$.*

Proof of Lemma 5. By Lemma 2, equation (52) has a unique solution $\overline{Z}(m_t)$. Accordingly, in order to prove the statement of Lemma 5, it suffices to show that

$$\int_{\beta m_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx = 0 \text{ for all } m_t. \tag{61}$$

To this end, observe first that $\eta_m(x, m_t) = \frac{1}{m_t} y\left(\frac{x}{m_t}\right)$. Next, let $u = \frac{x}{m_t}$ and apply a change of variables to (39) to obtain

$$0 = \int_{\beta m_t}^{\infty} \frac{y\left(\frac{x}{m_t}\right)}{\left(\frac{x}{m_t}\right)^{\varphi_1+1}} \frac{1}{m_t} dx = m_t^{\varphi_1+1}\left(\int_{\beta m_t}^{\infty} \frac{\eta_m(x, m_t)}{x^{\varphi_1+1}} dx\right).$$

This proves (61). ∎

Combining (60), Lemma 5 and the definition of $\omega(\cdot)$ yields (40). This concludes the proof of Proposition 1. ∎

Remark 1 *The purpose of this remark is to show that $\sup_{\lambda\in[0,\infty)} \beta(\lambda) < \infty$. To see this, note that when $\lambda = 0$, inspection of (38) and (39) reveals that $\beta = \overline{h}^{-\alpha} K_0^{1-\alpha}$. Additionally, for any β*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \int_{\beta}^{\infty} \frac{y(u)}{(u)^{\varphi_1+1}} du = \int_{\beta}^{\infty} \frac{u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}} \tau'\left(u^{-\frac{1}{\alpha}} K_0^{-\frac{1-\alpha}{\alpha}}\right) + \chi\rho}{(u)^{\varphi_1+1}} du. \tag{62}$$

By an argument similar to Lemma 2, there exists a finite value β, such that the right hand side of equation (62) is zero. Since β is a continuous function of λ, $\sup_{\lambda\in[0,\infty)} \beta(\lambda)$ is finite. This implies that if $Z_0 \geq \frac{L_0}{\chi}\left[\sup_{\lambda\in[0,\infty)} \beta(\lambda)\right]$, then one can guarantee that $m_0 < \frac{L_0}{\chi}$ is not optimal. It is useful to note, that even though $Z_0 \geq \frac{L_0}{\chi}\left[\sup_{\lambda\in[0,\infty)} \beta(\lambda)\right]$ is a sufficient condition for $m_0 \geq \frac{L_0}{\chi}$, it is not necessary.

Proof of Proposition 2. By equation (51) one obtains $J_\lambda\left(Z_0, \frac{L_0}{\chi}; m_{t>0}, \lambda\right) = \widehat{J}_\lambda\left(Z_0, \frac{L_0}{\chi}; m_{t>0}, \lambda\right) - \chi \log\left(\frac{L_0}{\chi}\right)$ for any process $m_{t>0}$. In particular for $m_{t>0} = m^*_{t>0}$,

$$V_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = \widehat{V}_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right) - \chi \log\left(\frac{L_0}{\chi}\right). \tag{63}$$

The next Lemma establishes two properties of $\widehat{V}_\lambda(Z_0, m_t; \lambda)$.

Lemma 6 *Let $B(Z_t, m^*_t; \lambda) \equiv \widehat{V}_\lambda(Z_t, m^*_t; \lambda)$ and $h^*_t = h^*(Z_t, m^*_t; \lambda)$ denote the hours that maximize (34), assuming that $m_t = m^*_t$. Then, the function $B(Z_t, m^*_t; \lambda)$ satisfies the following properties*

$$\sigma^2 Z_t^2 B_{ZZ} + \mu Z_t B_Z - \rho B = -\alpha\tau\left(h\left(Z_t, m^*_t\right)\right) - \chi\rho \log m^*_t, \tag{64}$$

$$B_m\left(\beta m^*_t, m^*_t\right) = 0. \tag{65}$$

Proof of Lemma 6. By (54) and Proposition 1, $\widehat{V}(Z_t, m^*_t; \lambda) = \int_0^{\beta m^*_t} \mathcal{G}(Z_t, x)\, \eta\left(x, \frac{x}{\beta}\right) dx$ $+ \int_{\beta m^*_t}^{\infty} \mathcal{G}(Z_t, x)\, \eta(x, m^*_t)\, dx$. Differentiating this expression with respect to λ and using the definition of $\widehat{V}_\lambda(Z_t, m^*_t; \lambda) = B(Z_t, m^*_t; \lambda)$ gives

$$\begin{aligned} B\left(Z_t, m^*_t; \lambda\right) \;=\; & \int_0^{\beta m^*_t} \mathcal{G}(Z_t, x)\, \eta_\lambda\left(x, \frac{x}{\beta}\right) dx + \int_{\beta m^*_t}^{\infty} \mathcal{G}(Z_t, x)\, \eta_\lambda(x, m^*_t)\, dx \\ & - \frac{\frac{d\beta}{d\lambda}}{\beta} \int_0^{\beta m^*_t} \mathcal{G}(Z_t, x)\, \eta_m\left(x, \frac{x}{\beta}\right) \frac{x}{\beta} dx. \end{aligned} \tag{66}$$

Since $Z_t \geq \beta m^*_t$, the definition of $\mathcal{G}(Z_t, x)$, along with an application of Ito's Lemma to (66) leads to $\sigma^2 Z_t^2 B_{ZZ} + \mu Z_t B_Z - \rho B = -\eta_\lambda(Z_t, m^*_t)$. Using the definition of $\eta(x, m^*_t)$ and applying the envelope theorem to compute g_λ yields $\eta_\lambda(Z_t, m^*_t) = \alpha\tau\left(h\left(Z_t, m^*_t; \lambda\right)\right)$ $+\chi\rho \log m^*_t$. This proves (64). To prove (65) note that when $Z_t = \beta m^*_t$,

$$B_m\left(\beta m^*_t, m^*_t\right) = \int_{\beta m^*_t}^{\infty} \mathcal{G}\left(\beta m^*_t, x\right) \eta_{\lambda m}(x, m^*_t)\, dx - \frac{d\beta}{d\lambda} \mathcal{G}\left(\beta m^*_t, \beta m^*_t\right) \eta_m\left(\beta m^*_t, m^*_t\right) m^*_t. \tag{67}$$

Substituting in for $\mathcal{G}(\beta m^*_t, x)$ gives

$$\int_{\beta m^*_t}^{\infty} \mathcal{G}\left(\beta m^*_t, x\right) \eta_{\lambda m}(x, m^*_t)\, dx = \frac{2\left(\beta m^*_t\right)^{\varphi_1}}{\left(\varphi_1 - \varphi_2\right)\sigma^2} \int_{\beta m^*_t}^{\infty} \frac{\eta_{\lambda m}(x, m^*_t)}{x^{\varphi_1 + 1}} dx$$

Since $\eta_m(x, m_t^*) = \frac{1}{m_t^*} y\left(\frac{x}{m_t^*}\right)$ it follows that $\eta_{\lambda m}(x, m_t^*) = \frac{1}{m_t^*} y_\lambda\left(\frac{x}{m_t^*}\right)$. Hence,

$$\begin{aligned}
\int_{\beta m_t^*}^{\infty} \mathcal{G}(\beta m_t^*, x)\, \eta_{\lambda m}(x, m_t^*)\, dx &= \frac{2(\beta m_t^*)^{\varphi_1}}{(\varphi_1 - \varphi_2)\sigma^2} \int_{\beta m_t^*}^{\infty} \frac{y_\lambda\left(\frac{x}{m_t^*}\right)}{x^{\varphi_1+1}} \frac{1}{m_t^*} dx \\
&= \frac{2}{(\varphi_1 - \varphi_2)\sigma^2} \frac{\beta^{\varphi_1}}{m_t^*} \int_{\beta}^{\infty} \frac{y_\lambda(u)}{u^{\varphi_1+1}} du. \qquad (68)
\end{aligned}$$

An application of the implicit function theorem to (39) yields $\frac{d\beta}{d\lambda} = \left(\frac{\beta^{\varphi_1+1}}{y(\beta)}\right)\left(\int_\beta^\infty \frac{y_\lambda(u)}{(u)^{\varphi_1+1}} du\right)$. Substituting this expression for $\frac{d\beta}{d\lambda}$, along with (68) into (67), and recalling the definition of $\mathcal{G}(\beta m_t^*, x)$ and that $\eta_m(x, m_t^*)\, m_t^* = y\left(\frac{x}{m_t^*}\right)$ gives (65). ■

Applying Ito's Lemma to $B(Z_t, m_t^*; \lambda)$, and ignoring terms with expectation equal to zero[19] leads to

$$\begin{aligned}
B(Z_t, m_t^*; \lambda) &= -E_t \int_t^\infty e^{-\rho(t-s)} \left[\sigma^2 Z_s^2 B_{ZZ} + \mu Z_s B_Z - \rho B\right] ds + E_t \int_t^\infty e^{-\rho(s-t)} V_m dm_s^* \\
&= E_t \int_t^\infty e^{-\rho(t-s)} \left[\alpha\tau\left(h(Z_s, m_s^*)\right) + \chi\rho \log m_s^*\right] ds. \qquad (69)
\end{aligned}$$

Furthermore, integration by parts implies that

$$E_t \int_t^\infty e^{-\rho(t-s)} \chi\rho \log m_s^* ds - \chi \log m_t^* = \chi E_t \int_t^\infty e^{-\rho(t-s)} d\log m_s^*. \qquad (70)$$

By Lemma 1, $m_0^* = \frac{L_0}{\chi}$. Using (63), and evaluating (69), (70) at $t = 0$ leads to $V_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right)$ $= E_0 \int_0^\infty e^{-\rho s} \alpha\tau\left(h(Z_s, m_s^*)\right) ds + \chi E_t \int_t^\infty e^{-\rho(t-s)} d\log m_s^*$. This implies that equation (25) is satisfied if and only if $V_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = 0$.

This concludes the proof of Proposition 2 ■

Remark 2 *Equation (42) always has a solution as long as Z_0 is large enough. To see this, notice first that $V_\lambda < 0$ when $\lambda = 0$ since in that case $\tau\left(h(Z_t, m_t^*; \lambda = 0)\right) = 0$, while $-\chi E_t \int_t^\infty e^{-\rho(t-s)} d\log m_s^* > 0$. Also, fixing any $\lambda > 0$, inspection of (66) and (55) reveals that*

$$\lim_{Z_0 \to \infty} \widehat{V}_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = \frac{1}{\rho}\left[\alpha\tau\left(\overline{h}; \lambda\right) + \chi\rho \log\left(\frac{L_0}{\chi}\right)\right]$$

[19] Panageas (2008) contains a more elaborate proof of the next expression. The reader is referred to that paper for technical details.

By equation (63) it follows that $\lim_{Z_0\to\infty} V_\lambda\left(Z_0, \frac{L_0}{\chi}; \lambda\right) = \frac{1}{\rho}\alpha\tau\left(\overline{h};\lambda\right) > 0$. *Since* $B\left(Z_t, m_t^*; \lambda\right)$ *is continuous in* λ, *equation (42) always has a positive root for large enough* Z_0.

Proof of Proposition 3. Reading (47) with equality and using (12), (16) and $c_t = Y_t$ leads to

$$\left[\pi_1\left(1-\pi_2\right)+\pi_2\right]\frac{1-\alpha}{\rho} = -\left(\zeta-\pi_1\right)E_0\int_0^\infty e^{-\rho s}\frac{1}{c_s}dL_s + \pi_1\frac{L_0}{c_0}. \tag{71}$$

Re-arranging (50) gives

$$\begin{aligned}\xi_t B_t &= \left(1-\pi_1\right)E_t\int_t^\infty \xi_s dL_s - \pi_1\xi_t L_t + E_t\int_t^\infty \xi_s\tau\left(h_s\right)w_s h_s ds \\ &\quad +\left[\pi_1\left(1-\pi_2\right)+\pi_2\right]E_t\int_t^\infty \xi_s p_s F_s ds\end{aligned} \tag{72}$$

Using (12), (15), $c_t = Y_t$ and substituting into (50) implies that the debt to gdp ratio $b_t = \frac{B_t}{Y_t} = \frac{B_t}{c_t}$ can be expressed as

$$\begin{aligned}b_t &= \left(1-\pi_1\right)E_t\int_t^\infty e^{-\rho(s-t)}\frac{1}{c_s}dL_s - \pi_1\left(\frac{L_t}{c_t}\right) + \alpha E_t\int_t^\infty e^{-\rho(s-t)}\tau\left(h_s\right)ds \\ &\quad +\left[\pi_1\left(1-\pi_2\right)+\pi_2\right]\frac{1-\alpha}{\rho}\end{aligned} \tag{73}$$

$$\begin{aligned}&= \left(1-\pi_1\right)E_t\int_t^\infty e^{-\rho(s-t)}\frac{1}{c_s}dL_s - \pi_1\left(\frac{L_t}{c_t}\right) + \alpha E_t\int_t^\infty e^{-\rho(s-t)}\tau\left(h_s\right)ds \\ &\quad -\left(\zeta-\pi_1\right)E_0\int_0^\infty e^{-\rho s}\frac{1}{c_s}dL_s + \pi_1\frac{L_0}{c_0},\end{aligned}$$

where the second equation follows from (71). Re-arranging, one obtains

$$\begin{aligned}b_t &= \left(1-\pi_1\right)E_t\int_t^\infty e^{-\rho(s-t)}\frac{1}{c_s}dL_s - \zeta E_0\int_0^\infty \frac{1}{c_s}dL_s + \alpha E_t\int_t^\infty e^{-\rho(s-t)}\tau\left(h_s\right)ds \\ &\quad +\pi_1\left[\frac{L_0}{c_0}-\frac{L_t}{c_t}\right] + \pi_1 E_0\int_0^\infty e^{-\rho s}\frac{1}{c_s}dL_s \\ &\le -\zeta E_0\int_0^\infty \frac{1}{c_s}dL_s + \alpha\frac{1}{\rho}\tau\left(\overline{h}\right) + \pi_1\left(\frac{L_0}{c_0} + E_0\int_0^\infty e^{-\rho s}\frac{1}{c_s}dL_s\right),\end{aligned} \tag{74}$$

where the last inequality follows from i) $\left(1-\pi_1\right)E_t\int_t^\infty e^{-\rho(s-t)}\frac{1}{c_s}dL_s < 0$, ii) $\tau\left(h_s\right) \le \tau\left(\overline{h}\right)$ and iii) $\frac{L_t}{c_t} > 0$. (Moreover, the upper bound (74) is approached arbitrarily closely as $Z_t \to$

∞.). To conclude the proof, note that $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$. To see this, note that methods similar to Panageas (2008), can be used to explicitly compute the value of the guarantee $E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s$ as $\frac{\chi}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2}$. Using this fact, one obtains

$$\begin{aligned} \frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s &= \frac{L_0}{m_0^*} \left(\frac{m_0^*}{c_0} \right) + \frac{\chi}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2} \\ &= \chi \left[\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_0^\alpha} + \frac{1}{\varphi_2} \left(\frac{Z_0}{\beta m_0^*} \right)^{\varphi_2} \right], \end{aligned}$$

where $h_{\min}$ denotes the lower bound on hours worked. By (36), $\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_0^\alpha}$ is equal to 1, as long as Z_0 is below some cutoff $\overline{Z}_0$ and then becomes equal to $\frac{\beta m_0^* h_{\min}^\alpha}{Z_0 h_{\max}^\alpha}$ for value larger than $\overline{Z}_0$. Since $|\varphi_2| > 1$, it follows that $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$, irrespective of Z_0.

Since $\frac{L_0}{c_0} + E_0 \int_0^\infty e^{-\rho s} \frac{1}{c_s} dL_s > 0$, equation (74) implies that $\sup_{0 \leq s \leq t} b_s$ is minimal when $\pi_1 = 0$. ■

References

Aiyagari, R., T. J. Sargent, A. Marcet, and Y. Seppala (2002). Optimal taxation without state-contingent debt. *Journal of Political Economy 110*(6), p. 1220 – 1254.

Angeletos, G.-M. (2002). Fiscal policy with noncontingent debt and the optimal maturity structure. *Quarterly Journal of Economics 117*(3), p 1105 – 1131.

Barro, R. J. (1979). On the determination of the public debt. *Journal of Political Economy 87*(5), p. 940 – 971.

Duffie, D. (2001). *Dynamic asset pricing theory.* Princeton and Oxford: Princeton University Press.

Karantounias, A., L. P. Hansen, and T. J. Sargent (2008). Ramsey taxation and fear of misspecification. mimeo. New York University.

Karatzas, I. and S. E. Shreve (1991). *Brownian motion and stochastic calculus*, Volume 113. Springer-Verlag.

Karatzas, I. and S. E. Shreve (1998). *Methods of mathematical finance*, Volume 39. Springer-Verlag.

Kobila, T. (1993). A class of solvable stochastic investment problems involving singular controls. *Stochastics and Stochastics Reports 43*, 29 – 63.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance 49*(4), 1213–1252.

Leland, H. E. (1998). Agency costs, risk management, and capital structure. *Journal of Finance 53*(4), 1213 – 1243.

Ljungqvist, L. and T. J. Sargent (2004). *Recursive macroeconomic theory.* MIT Press, Cambridge MA and London, England, Second Edition.

Lucas, D. and R. L. McDonald (2006). An options-based approach to evaluating the risk of fannie mae and freddie mac. *Journal of Monetary Economics 53*(1), 155 – 176.

Lucas, R. E.-J. (1978). Asset prices in an exchange economy. *Econometrica 46*(6), 1429 – 1445.

Lucas, R. E.-J. and N. L. Stokey (1983). Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics 12*(1), 55 – 93.

Merton, R. C. (1978). On the cost of deposit insurance when there are surveillance costs. *Journal of Business 51*(3), 439 – 452.

Musiela, M. and M. Rutkowski (1998). *Martingale methods in financial modelling*, Volume 36. Springer-Verlag.

Øksendal, B. (2003). *Stochastic differential equations*. Springer-Verlag.

Panageas, S. (2008). Bailouts, the incentive to manage risk, and financial crises. Forthcoming, Journal of Financial Economics.

Pennacchi, G. and C. Lewis (1994). The value of pension benefit guarantee corporation insurance. *Journal of Money, Credit and Banking 26*(3), 735–53.

Ramsey, F. P. (1927). A contribution of the theory of taxation. *Economic Journal*, 47 – 61.

Ronn, E. I. and A. K. Verma (1986). Pricing risk-adjusted deposit insurance: An option-based model. *Journal of Finance 41*(4), 871–895.