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DISTRIBUTION OF PRIZES IN A MATCH-PLAY TOURNAMENT WITH SINGLE ELIMINATIONS

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ABSTRACT

This paper begins to study the reward-incentive structure in sequential knock-out or elimination tournaments with matched, pairwise comparisons among players at each stage. The prize structure required to elicit constant expected quality of play in all matches throughout the tournament is characterized for competition among equally talented (or perfectly handicapped) players. The incentive maintaining prize structure is shown to concentrate extra weight on the top ranking prize, a phenomenon observed in most tournaments. More can be said. Prizes that maintain performance incentives at all stages award a constant increment for each match won up to the last stage; and an amount greater than this for the player who wins the final match. Players' incentives to perform in early rounds are propelled by the probability of achieving higher ranks and surviving to later stages where rewards are larger. These continuation options are played out in the final match, so it is only the difference between winning and losing prizes in the finals that controls incentives there.

Many athletic tournaments are structured in the manner analyzed here, but the general framework ultimately may have application to certain career games as well. More generally, a tournament structure may be viewed as a statistical, experimental design problem. The prize structure interacts with the design in providing incentives for the best players to survive to the finals and win the top prizes.

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THE DISTRIBUTION OF PRIZES IN A MATCH-PLAY TOURNAMENT WITH SINGLE ELIMINATIONS

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A number of recent papers have clarified the problem of incentives in simple one-shot games when players are paid on the basis of rank or relative performance [Lazear and Rosen (1981), Green and Stokey (1983), Nalebuff and Stiglitz (1983), O'Keeffe, Viscusi and Zeckhauser (1984), Holmstrom (1982)]. The chief result available so far is that rank/relative reward schemes can lead to efficient performance incentives, especially when precise measurement of individual outcomes is costly and when environmental factors (the "conditions of play") equally influence the connection between input and outcomes for all players. These models do not, however, yield many restrictions on what the resulting prize structure might look like. Yet prizes are observed to be heavily concentrated on the top ranks in most professional tournaments. See table 1.

This paper begins to investigate the structure of prizes in sequential elimination tournaments, where rewards are increasing in survival. Many athletic tournaments are organized in this way. Tennis immediately comes to mind (e.g., Wimbledon), but end of season playoffs in most team sports also follow this design (with entry conditioned on standings in the regular season round robin). In what follows, the rank-reward structure required to maintain constant expected performance throughout the tournament is characterized for games involving equally talented players. The incentive maintaining prize schedule is shown to be convex in rank order. More surprisingly, it is fully described in figure 1. Let W_R be the prize offered for achieving rank R in the tournament. Then constant incentives requires $W_R - W_{R+1} = \gamma$ for $R \geq 2$ and $W_1 - W_2 > \gamma$, where γ is a positive number. Reward increases linearly in rank order from (n + 1)st place through second place, but the first place prize takes a distinct jump, out of sync with and of a different character from the rest. "More convexity" than shown in figure 1 leads to increasing intensity or expected quality of play in the later stages of the tournament; and "less convexity" leads to declining expected quality of play as the tournament proceeds through its stages.

The economic interpretation of this result is interesting and appealing. The prize structure in figure 1 implies a fixed incremental reward of γ for advancing one more round from the beginning of the tournament up to the finals. However, a player's valuation of continuation in these stages exceeds the interrank reward difference (γ) because there is a nontrivial probability of surviving longer than one more round and winning an even larger prize. The value of continuation includes these higher order terms, and it is this value that sets a player's incentives to perform, not the interrank reward difference itself. But when a player has reached the finals, there is no possibility of further continuation. The incremental value of winning the finals is $W_1 - W_2$ alone. The difference between first and second place prize money is the sole instrument available for incentive maintenance in the finals, because there is no possibility of further advancement, as there is at earlier stages of the game. That $W_1 - W_2$ exceeds the other interrank reward differences therefore is fundamentally due to the no-tomorrow aspect of playing in the finals. Raising $W_1 - W_2$ above γ effectively extends the horizon of players reaching the finals, similar to the role of a pension in finitely repeated principal and agent problems [Becker and Stigler (1974)].

The next section describes the structure of the game and some notation. The strategies of players are set forth in section II. Sequential Nash strategies are adopted as the equilibrium concept. Since there is a natural end point, the method of solution is backward recursion, analyzing the finals and working one step at a time back to earlier stages of the game. The principal result on incentive maintaining prize structures is established in section III and is further discussed and qualified in section IV.

I. DESIGN OF THE GAME

The tournament begins with 2^n identical players in the initial round and proceeds sequentially in $\underline{n}$ distinct stages. Each stage is a set of paired matches with pairings randomly drawn among players eligible to enter that stage. Winners of these matches survive to the next round, where another pairing is drawn randomly, while losers are eliminated from all subsequent play. Thus half of all eligibles at any stage maintain eligibility and continue to the next, and the other half are eliminated at the end of their matches. See figure 2. All who lose eligibility at the same stage achieve the same rank in the tournament. No branching matches are offered for breaking ties (however, see section IV for some analysis of consolation matches). For example, the two losers of the semifinals achieve

3rd rank, the four losers of the quarter finals achieve 4th rank, etc. More generally, if s is the number of stages remaining to be played (equivalently, the maximum possible number of matches a currently eligible contestant can play in the remainder of the tournament), all losers of the next matches achieve rank s + 1 and all are awarded prize W_{s+1}, for $s = 0, 1, \ldots, n$. There are 2^{s-1} such players, for $s \geq 2$, so the amount paid to all players achieving (s+1)st place is $2^{s-1}W_{s+1}$. Of course there is a single winner. Prizes are designated in advance of the first round and are strictly increasing in rank: $W_{s+1} > W_{s+2}$, for all s. We study how the sequence $\{W_{s+1}\}$ affects the sequence of incentives to win at various stages of the game.

There are two features of these games that make it meaningless to specify input-output technologies and "marginal products" of contestants in the ordinary sense. First, competition is naturally relative because it involves face-to-face confrontation in most instances. Success depends on both offensive and defensive skills. Second, these games have an essential ordinal character because the calibration of point scores used to determine winners and losers has many arbitrary elements, much in common with the scores on a classroom test. For example, the nature of the game of tennis would be greatly affected by altering the height of the net, changing the size and composition of the court or adjusting the compression of the ball. The adopted standards and operating rules of the game have large effects on the productivity of various personal dimensions of talent, for example those affecting power, finesse and endurance[1]. For this reason point scales have little sense of cardinality. The best one can do for analysis is specify how players' actions affect the probability of winning.

The probability of winning a match is assumed to follow a Poisson process, a specification which has been used to great advantage in the recent liteature on patent races, from which it is borrowed [see especially Loury (1979); also Kamien and Schwartz (1972), Lee and Wilde (1979), and Reinganum (1982); Telser (1982) considers some sequential elements of these problems]. Let x index the intensity of effort expended by a player in some match. If player i is matched against player j the probability that i wins the match is assumed to follow the law

$$(1) \qquad p(x_i, x_j) = \frac{h(x_i)}{h(x_i) + h(x_j)}$$

with h(x) strictly increasing in x and h(0) = 0. A player increases the probability of winning the match by exerting greater effort given the effort of the opponent.

Two features of (1) are noteworthy. First, that the function h(x) is the same for all players embodies the assumption that all are equally talented. When two players exert the same effort (1) implies that the probability of either one winning is one-half -- entirely random outcomes. If players are not equally talented h(x) must be indexed by ability and the stronger player has a winning edge at equal effort levels (see section IV). Second, (1) neatly accomodates common environmental factors that influence the quality of play. Let the common factor multiply h(x). Then whether the commonality is match specific, stage specific or tournament specific, it factors out of the probability calculation in any match and at any stage of the game. It therefore has no effect on incentives.

Specification (1) has a racing game interpretation. Let τ be arrival time from the beginning of the match. Then $h_i = h(x_i)$ is the probability of "crossing the finish line" at τ given that player i has been racing up to τ. The unconditional duration density of finishing at τ exactly is $f_i(\tau) = h_i e^{-h_i \tau}$ and its CDF is $1 - e^{-h_i \tau}$. Expected finishing time for i is $1/h_i$. The player who arrives first is declared winner of the match: (1) gives the probability of this event. Expected finishing time in an (i,j) match is $(h_i + h_j)^{-1}$. While this interpretation is a bit strained in context, its spirit is maintained by identifying "shorter" expected match completion time with higher quality of play. In any case it is (1) that is the primative construct for this problem, not the particular route by which it is generated. Merely think of (1) and its counterpart for player j as symmetric functions where the values of the arguments (x_i, x_j) determine the expected quality of the match, and $p(x, x) = 1/2$ for $x > 0$.

II. STRATEGIES AND INCENTIVES

A player's decision of how much effort to expend in any match depends on a cost-benefit calculation. Greater effort at any stage increases the probability of achieving a higher rank and larger prize money, but involves additional cost. There are two complications. First, the anticipated value of advancing to a subsequent stage depends on future effort expenditures. In deciding how much to put out in the current match, a player must anticipate how he will behave should eligibility be maintained in more advanced stages of the tournament. This difficulty is common to all intertemporal decision problems and is solved by backward recursion. Second, the cost-benefit calculation for any player depends on anticipated actions of opponents in all possible future matches as well as in the current one. Given the sequential character of the game, this is best analyzed by adopting Nash noncooperative strategies as the equilibrium concept at each stage. I ignore time discounting between stages and, for now, assume players are risk neutral. It is also assumed that each match is independent: costs incurred in previous matches have no carry overs and do not affect either costs or the probability of success in subsequent matches.

Define V_s as the value of playing a match when s possible stages remain in the tournament, and define p_s by (1) as the probability of winning the match and maintaining eligibility into the next stage. Let c(x) be the cost of effort in any match and assume nondecreasing marginal cost: $c'(x) > 0$ and $c''(x) \geq 0$, and $c(0) = 0$. The value V_s consists of two components. One is the prize W_{s+1} awarded to players achieving (s+1)<u>st</u> place in the tournament if the match is lost and the player is eliminated, an event which occurs with probability $(1 - p_s)$. The other component is the value of achieving a rank superior to s+1 if the match is won. The value of winning is eligibility in the next stage, V_{s-1}, an event which occurs with probability p_s. Therefore,

$$(2) \qquad V_s = \max_{x_s} [p_s V_{s-1} + (1-p_s)W_{s+1} - c(x_s)]$$

where x_s is effort expended in the current match. I shall place sufficient structure on h(x) and c(x) to guarantee a unique equilbrium at each stage.[2]

Analysis begins with the <u>Finals</u>. Define $V_0 = W_1$ and substitute (1) into (2) for s = 1. The value of achieving the finals for player i in a match against player j is

(3) $$V_{1i} = \max_{x_i} \{[h_i/(h_i + h_j)](W_1 - W_2) + W_2 - c(x_i)\}$$

where h_i is shorthand for $h(x_i)$ and similarly for h_j. To avoid notational clutter it is understood that the x's in (3) refer to the final round and are not subscripted by s = 1. The best response of player i to the opponent's effort x_j is found by differentiating the bracketed expression in (3):

(4) $$\frac{h_j h_i'}{(h_i+h_j)^2} (W_1 - W_2) - c'(x_i) = 0$$

where $h_i' = h'(x_i)$. The second derivative is

(5) $$\Delta = c'(x_i)\{(h_i''/h_i') - 2p_i(h_i'/h_i) - c''(x_i)/c'(x_i)\} < 0$$

where $h_i'' = h''(x_i)$ and $p_i = h_i/(h_i+h_j)$. Δ must be negative at the solution for (4) to describe a local maximum of (3) given x_j. It proves convenient to express (5) slightly differently. Define the elasticities $\eta(x) = xh'(x)/h(x)$ and $\varepsilon(x) = xc'(x)/c(x)$. Then (5) becomes

(6) $$\Delta = c'(x_i)/x_i\{x_i h_i''/h_i' - 2p_i\eta(x_i) - \varepsilon(x_i)\} < 0.$$

Comparative statics on (4) yields:

(7) $$\partial x_i/\partial(W_1-W_2) = \frac{c'(x_i)/(W_1-W_2)}{-\Delta} > 0 \quad .$$

Second place money is assured if the player has reached the finals and players are contesting over the difference $(W_1 - W_2)$, so an increase in the differential reward to winning elicits greater intensity of effort by i, given x_j.

Differentiating (4) with respect to x_j yields

$$dx_i/dx_j = \frac{c'(x_i)(h_j'/h_j)(h_i - h_j)}{-\Delta} \tag{8}$$

which defines the slope of player i's best response function with respect to x_j. The response function $x_i = X(x_j)$ is increasing in x_j when $h_i > h_j$, or when $x_i > x_j$. It is decreasing when $x_j > x_i$ and has a turning point at $x_i = x_j$, as shown in figure 3. At smaller values of x_j, player i chooses x_i to have a winning edge over j. As x_j increases x_i responds positively to maintain a smaller winning edge. However, for x_j sufficiently large, x_i is chosen so that j has the winning edge: player j's effort is so great that it doesn't pay player i to compete on equal or better terms. As x_j increases further, player i puts out less and less effort. If $h'' < 0$ for all x then $\Delta < 0$ for all x because η and ε are both positive. Then $X(x_j)$ is continuous throughout its domain. However, if $h'' > 0$ the elasticity of $h'(x)$ must be sufficiently small for (6) to hold true at all, and even so it may fail for some values of x_j. If it does fail then $X(x_j)$ exhibits a point of discontinuity. Examination of (6) shows that failure is more likely when p_i is small (so x_j is large). At some value of x_j and beyond the opponent is putting forth so much effort that it is best for i to simply give up, to set $x_i = 0$ and benignly accept his fate as sure loser. In this paper attention is confined to pure strategy equilibria. This requires that if (6) fails it must do so beyond the turning point of $X(x_j)$. For this we require a strict upper bound on xh''/h'.[3]

The j-player's best reply function is the mirror image of that for the i-player. Therefore, if an equilibrium exists it is symmetric: $x^* = x_i = x_j > 0$, and (4) reduces to

$$h'(x^*)/h(x^*)[W_1 - W_2]/4 = c'(x^*). \tag{9}$$

Using the elasticity definitions above, (9) may be written equivalently as

$$\eta(x^*)/\varepsilon(x^*)[W_1 - W_2]/4 = c(x^*). \tag{10}$$

In equilibrium $p_i = p_j = 1/2$, both players have an equal chance of winning and the match is a "close call" in expected value. Therefore $V_{1i} = V_{1j} = V_1$ and

$$V_1 = (W_1 + W_2)/2 - c(x^*). \tag{11}$$

The prizes are assumed to be sufficiently large that $V_1 > 0$.

One important detail remains. A global condition must be imposed to rule out $x_i = 0$ as a best reply to $x_j = x^*$ in order to guarantee existence of equilibrium (9). For this we require that $V_1 - W_2 > 0$. Otherwise the best reply to x_j^* is $x_i = 0$: Taking the sure loss is better than competing on equal terms. There cannot be a symmetric equilibrium at $x^* = 0$ because $h(0) = 0$; and one player guarantees a win by exerting vanishing small effort at vanishing small cost. But if one player can do this so can the other and the joint responses are driven back to $x^* > 0$ in figure 2.

Therefore if both players can do better by taking a sure loss to exerting x* in (10) there can be no equilibrium in the game. Substituting (10) into (11) and subtracting W_2 yields

$$(12) \qquad V_1 - W_2 = (1/2)(W_1 - W_2)[1 - \frac{\eta(x^*)}{2\varepsilon(x^*)}].$$

No player has incentives to default from x* defined by (9) or (10) if $V_1 - W_2 > 0$, or

$$(13) \qquad \eta(x)/\varepsilon(x) < 2$$

which will be assumed to hold true for all x.

The sense of condition (13) has independent interest and is related to the problem of arms races and proposals for limitations on political campaign expenditures. If the elasticity of response of effort is large relative to the elasticity of its cost then both players' efforts to win results in a negative sum game for which a stable equilibrium is not defined. It is not optimal for either player to default if the other one does, but at the locally stable equilibrium the costs of contesting have been escalated so far that both want to default. In athletic games this problem is controlled by a supreme authority which reviews standards of play from time to time and which limits rules changes and the introduction of new equipment that would otherwise lead to problems.[4]

Now that V_1 has been nailed down we may proceed to analyze the Semifinals. At s = 2 equation (2) becomes

$$(14) \qquad V_{2i} = \max_{x_{2i}} [p_{2i}(V_1 - W_3) + W_3 - c(x_{2i})].$$

Exactly the same line of argument as above establishes a unique nontrivial symmetric equilibrium for which $p_{2i} = p_{2j} = 1/2$, with $x_{2i} = x_{2j} = x_2^*$ determined by a condition analogous to (9)

$$h'(x_2^*)/h(x_2^*)[V_1 - W_3]/4 = c'(x_2^*). \tag{15}$$

Substituting $p_2 = 1/2$ and (11) into (14) we find

$$V_2 = \frac{W_1+W_2}{4} + \frac{W_3}{2} - \frac{c(x_1^*)}{2} - c(x_2^*) \ . \tag{16}$$

Notice that x_1^* depends on W_1 and W_2, from (9) and that V_1 also depends on W_1 and W_2 (see equation (11)). It follows from (15) that x_2^* depends on W_1, W_2 and W_3. It is in this way that the effects of the structure of prizes on incentives at each stage of the game may be studied. The fact that the equilibrium level of effort at any stage has no memory and is only forward looking simplifies the problem considerably, and is due to the assumption that effort at each stage has no spillovers to later stages.

Continuing in this manner, the solution at any stage is easily established. When s stages remain to be played all surviving players in the next match choose effort x_s^* to satisfy

$$h'(x_s^*)/h(x_s^*)[V_{s-1} - W_{s-1}]/4 = c'(x_s^*). \tag{17}$$

An induction argument shows that condition (13) is necessary for (17) to describe a global best response for each player at any stage. In equilibrium the probability of extinction in the next match is 1/2 at all stages of the game. The probability that any player is eliminated at the end of stage s and receives payoff W_{s+1} is $(1/2)^{n-s+1}$. At the start of play all players have the same chances in equilibrium of each rank because all are equally talented and the equilibrium is symmetric at any stage[5]. Substituting $p_s = 1/2$ into (2) and iterating yields

$$V_s = \frac{W_1+W_2}{2^s} + \frac{W_3}{2^{s-1}} + \frac{W_4}{2^{s-2}} + \ldots + \frac{W_{s+1}}{2} - c(x_s^*) - c(x_{s-1}^*)/2 - \ldots \tag{18}$$

$$- c(x_1^*)/2^{s-1}.$$

III. PRIZES THAT MAINTAIN INCENTIVES

A complete analysis of the optimal distribution of prizes requires precise specification of what services these games produce and how production affects willingness of consumers to pay to see them. The prize structure presumably evolves to produce the distribution in quality of play over stages that maximizes tournament profits, given supplies and talents of the players. Little can be said on this at present beyond the obvious that fan interest is stimulated by the closeness of matches and by higher general quality of play.

A less specific and at the same time less general question is asked here. How should the purse be divided to elicit the same intensity of play in every match at all stages of the game? What sequence $\{W_{s+1}\}$ guarantees $x_s^* = x^*$ for all s? This is a convenient benchmark because it roughly indicates how an increasing (or decreasing) sequence of effort and expected quality of play can be generated over the course of the tournament. I believe the answer, which is depicted in figure 1, is one of the reasons why prizes in real life tournaments are so heavily skewed toward the top ranks. Proof that figure 1 provides a complete answer to the question follows.

It is clear from (17) that the intensity of effort at each stage is determined by $V_{s-1} - W_{s+1}$, the incremental reward to winning the next match. Maintenance of incentives at each stage therefore requires that $V_{s-1} - W_{s+1}$ have the same value independent of s. An expression for $V_{s-1} - W_{s+1}$ is obtained by iterating (18) one step and subtracting W_{s+1}. After imposing the constraint that $x_s = x^*$ we obtain

$$(19) \qquad V_{s-1} - W_{s+1} = \frac{W_1+W_2}{2^{s-1}} + \frac{W_3}{2^{s-2}} + \ldots + \frac{W_s}{2} - W_{s+1}$$

$$- c(x^*)[1 - 2^{-(s-1)}]/(1/2)$$

Iterate (19) backward one step to obtain a similar expression for $V_s - W_{s+2}$. Equating the two expressions yields

$$(20) \qquad \frac{W_1+W_2}{2^{s-1}} + \frac{W_3}{2^{s-2}} + \ldots + \frac{W_s}{4} - (3/2)W_{s+1} - W_{s+2} = c(x^*)/2^{s-2}$$

as the condition that prizes must satisfy in order to guarantee effort x* at every stage of the game.

To eliminate c(x*) in (20) iterate it one step forward, multiply by 2 and subtract the result from (20) to obtain a difference equation,

$$(21) \qquad W_s - 2W_{s+1} + W_{s+2} = 0$$

which holds for all stages other than the finals. The characteristic equation of (21) has two unit roots, so the solution is linear in s. Since W_s must be declining in s in order for players to have incentives to continue playing, we have, for $s \geq 2$

$$W_s - W_{s+1} = \gamma \tag{22}$$

with $\gamma > 0$. Apart from a lump sum payment (possibly negative) at the beginning of the tournament, incentive maintenance implies that players receive a fixed reward γ for each match won in all stages up to the finals.[6]

However, the situation is slightly different for the finals. Comparing the finals with the semifinals, $x_1^* = x_2^* = x^*$ requires, from (9), (11), and (15)

$$W_1 - W_2 = V_1 - W_3 = (W_1 + W_2)/2 - c(x^*) - W_3 \tag{23}$$

or

$$W_1 - W_2 = 2(W_2 - W_3 - c(x^*)) = 2(\gamma - c(x^*)). \tag{24}$$

Substituting (24) into (10), x^* must satisfy

$$\eta(x^*)/2\varepsilon(x^*)[\gamma - c(x^*)] = c(x^*). \tag{25}$$

Solving for $c(x^*)$ in (25) and using (24) we have

$$W_1 - W_2 - (W_2 - W_3) = \gamma - 2c(x^*) \tag{26}$$

$$= \gamma[1-\eta(x^*)/2\varepsilon(x^*)]/[1+\eta(x^*)/2\varepsilon(x^*)],$$

so $W_1 - W_2$ exceeds γ whenever $\eta(x^*)/\varepsilon(x^*) < 2$, precisely condition (13) guaranteeing existence of a unique symmetric equilbrium at each stage. The tariff for winning the last match exceeds all the rest if incentives are to be maintained at x^* throughout the tournament.

As noted at the beginning of the paper, the jump in the top prize fundamentally is due to the fact that the finals is the last match to be played. Linearity of reward in previous stages appears paradoxical on these terms. The fact that the horizon draws closer as the game proceeds seems to require increasing incremental rank-rewards to maintain incentives. However, offsetting this is the fact that the probability of reaching the top increases with survival, so the shorter horizon is effectively discounted by a smaller amount. The two effects exactly cancel each other at every stage except the last.

The following example is instructive for illustrating the result and for showing how other prize structures affect incentives. Suppose $h(x) = x^\eta$ and $c(x) = x^\varepsilon$ where η and ε are positive constants. Defining $y_s = x_s^\varepsilon$, application of (17) and (18) gives the recursions

$$y_s = \left(\frac{\eta/\varepsilon}{4}\right)(V_{s-1} - W_{s+1}) \tag{27}$$

$$V_s = \beta V_{s-1} + (1 - \beta)W_{s+1} \tag{28}$$

where $\beta = (1 - \eta/2\varepsilon)/2$. The usual manipulations of (27) and (28) imply a difference equation for y_s:

(29) $$y_{s+1} = \beta y_s + (\frac{\eta/\varepsilon}{4})(W_{s+1} - W_{s+2})$$

which has solution

(30) $$y_s = x_s^\varepsilon = (\frac{\eta/\varepsilon}{4})[(W_s - W_{s+1}) + \beta(W_{s-1} - W_s) + \beta^2(W_{s-2} - W_{s-1}) + \ldots + \beta^{s-1}(W_1 - W_2)$$

It is discounted future interrank rewards that determines effort at each stage, where the discount rate β depends on the probability of winning ($= 1/2$) and the cost and arrival time distribution parameters η and ε. The discount rate is positive so long as $\eta/\varepsilon < 2$. Substituting (30) into (27) and iterating one step yields the solution for V_s:

$$V_s = (1 - \beta)[W_{s+1} + \beta W_s + \ldots + \beta^{s-1} W_2] + \beta^s W_1.$$

The value of eligibility depends on the level of rewards. For a given reward structure it is easy to show that V_s is declining in η/ε. The reason is that less effort is expended at any stage as η/ε falls, from (30).[7]

The achievement of a target effort level x^* requires, for each stage except the last, from (29)

$$W_{s+1} - W_{s+2} = (\frac{4}{\eta/\varepsilon})(1 - \beta)(x^*)^\varepsilon \qquad \text{for } s = 1, \ldots, n.$$

The spread between first and second place money must be larger than this: from (27) it must be

$$W_1 - W_2 = \left(\frac{4}{\eta/\varepsilon}\right)(x^*)^{\varepsilon}.$$

Consequently $(W_1 - W_2)/(W_s - W_{s+1}) = 1/(1 - \beta)$, which ranges between 1.0 and 2.0. It can be shown that a relative winning increment in excess of $1/(1 - \beta)$ results in monotonically increasing effort and quality of play as the game proceeds through its stages, and that a relative increment of less than $1/(1 - \beta)$ results in monotonically decreasing effort in later stages.

The incentive consequences of some other reward schemes follow from (30):

(i) Winner-Take-All

The reward structure specifies $W_1 > 0$ and $W_s = 0$ for $s \geq 2$. Here (29) implies $x_s^{\varepsilon} = \left(\frac{\eta/\varepsilon}{4}\right)W_1\beta^{s-1}$. The expected quality of play at each stage is larger than the previous stage. Effort rises more or less than geometrically across stages as $\varepsilon \lessgtr 1$. It rises geometrically if $\varepsilon = 1$.

(ii) Win, Place and Show Money Only.

In this case we find

$$x_s^{\varepsilon} = \left(\frac{\eta/\varepsilon}{4}\right)\beta^{s-2}[\beta(W_1 - W_2) + (W_2 - W_3)] \quad \text{for } s \geq 2$$

$$x_1^{\varepsilon} = \left(\frac{\eta/\varepsilon}{4}\right)(W_1 - W_2)$$

which yields an interstage quality of play pattern similar to winner-take-all in stages prior to the semi-finals. It remains true that $W_1 - W_2$ must be larger than $(W_2 - W_3)(1 - \beta)^{-1}$ for final round effort to be larger than effort in the semifinals.

(iii) Geometric Inverse Rank Rule

Suppose the prize ratio between adjacent ranks maintains a constant value $1 + \xi$, with $\xi > 0$. For example, if $\xi = 1$ then the Rth place reward is twice as large as the (R+1)st reward and the purse is split equally among all ranks. Table 1 is roughly of this form (except for W_1.) We find, for $\beta(1+\xi) < 1$

$$x_s^\varepsilon = \frac{(\frac{\eta/\varepsilon}{4})W_1\xi}{1-\beta(1+\xi)}[(1 + \xi)^{-s} - \beta^s],$$

for $\beta(1+\xi) > 1$

$$x_s^\varepsilon = \frac{(\frac{\eta/\varepsilon}{4})W_1\xi}{\beta(1+\xi)-1}[\beta^s - (1 + \xi)^{-s}],$$

and for $\beta(1+\xi) = 1$

$$x_s^\varepsilon = (\frac{\eta/\varepsilon}{4})\xi W_1[\frac{s}{(1+\xi)^s}].$$

In all parameter configurations effort is decreasing in s or increasing with survival. The intensity of play is largest in the finals and smallest in the first round.

IV. DISCUSSION AND EXTENSIONS

The result on incentive maintenance survives generalization to a broader class of preferences and win technologies.

Risk Aversion. The preference structure implicit in the problem above is strongly additive; linear in income and convex in effort. Suppose instead that preferences take the additive form $U(W) - \sum_s c(x_s)$, where $c(x)$ is as before and $U(W)$ is increasing, but not necessarily linear in W. Then the entire analysis goes through by merely replacing W_s with $U(W_s)$ wherever it appears. Incentive maintenance requires a constant difference in the utility of rewards $U(W_{s+1}) - U(W_{s+2})$ in all stages prior to the finals, but still requires a jump in the interrank difference in utiilty of winning the finals. Figure 1 applies so long as the ordinate is relabeled, with U(W) replacing W. If players are risk averse then $U''(W) < 0$ and the incentive maintenance prize structure requires strictly increasing incremental monetary rewards between adjacent ranks, with a much larger increment between first and second place. The prize structure is everywhere strictly convex in rank order, with greater concentration of the purse on the top prizes than appears in figure 1.

The result is related to an "income effect." When U(W) is nonlinear the relevant marginal cost of effort is roughly the marginal rate of substitution between W and x, or $-c'(x)/U'(W)$. At the target level of effort $c'(x^*)$ is constant, but as a player continues and is guaranteed a higher and higher rank $U'(W)$ declines. The relevant marginal cost of effort effectively increases in each successive stage. Convexity of reward is required to overcome these wealth effects and maintain a player's interest in advancing to a later stage of the game.

Symmetric Win Technologies. The property $p(x_i,x_j) = p(x_j,x_i)$ and $p(x,x) = 1/2$ is crucial to the symmetric equilibrium resulting from specification (1). Notice that the proof of constancy of interrank rewards (or utility of rewards) for incentive maintenance in stages prior to the finals rests only on the fact that equilibrium is symmetric, with a survival probability of 1/2 at every stage of the game. Further, the jump in differential prize money between ranks one and two is due to the fact that the chance of continuing to higher ranks ceases in the finals. Hence the result applies to <u>any</u> symmetric probability specification resulting in a unique symmetric equilibrium, in which $p_s = 1/2$ for all s. It remains to be seen how much broader this class is compared with specification (1). One intersting possibility is $h(x_i,x_j)/[h(x_i,x_j) + h(x_j,x_i)]$ with $h_1 > 0$, $h_2 < 0$, and $h_{12} < 0$. Perhaps such a specification captures the direct confrontation nature of competition better than (1).

<u>Tie-Breakers and Consolation Matches</u>. It is basically the survival aspects of the game that lead to the result in figure 1, so any change in the game structure that preserves the "option value" of continuation results in an incentive maintenance prize schedule with similar features. For example, consider an alteration in the structure of figure 2 in which a branching consolation match is played among the losers of the semifinals, but no other tie-breakers are allowed. Let W_1 and W_2 be the prizes of the winner and loser in the finals as before; and let W_c and W_3 be the prizes of the winner and loser in the consolation match. Then the remaining notation in sections II and III remains intact.

Let us investigate the prize structure required to maintain effort at x* in all matches, including the consolation match. In the finals we know that incentives are determined by the spread $W_1 - W_2$. Similarly incentives in the consolation match are determined by $W_c - W_3$. Therefore equal effort requires $W_1 - W_2 = W_c - W_3 = k$ (say). Let V_f be the value of reaching the finals and let V_c be the value of the consolation match. Since $p = 1/2$ in equilibrium, we have

$$V_F = (W_1 + W_2)/2 - c(x^*) = k/2 + W_2 - c(x^*)$$

$$V_c = (W_c + W_3)/2 - c(x^*) = k/2 + W_3 - c(x^*).$$

In the semifinals incentives are provided by the difference $V_F - V_c = (W_2 - W_3)$. Therefore to guarantee effort x^* when $s = 2$ requires $W_2 - W_3 = k$. But $W_2 - W_3 = W_c - W_3$ implies $W_c = W_2$. The winner of the consolation match gets the same prize as the loser of the finals. This result is less surprising once one recognizes that these two players win the same number of matches in the overall tournament.

Given the constraint $x_s = x^*$, we have

$$V_2 = (1/2)(V_F - V_c) + V_c - c(x^*) = k + W_3 - 2c(x^*)$$

Incentives in the quarter finals are set by $V_2 - W_4 = k + W_3 - W_4 - 2c(x^*)$. Therefore $x_3 = x^*$ requires $V_2 - W_4 = k$, or $W_3 - W_4 = 2c(x^*)$. Using (10) to evaluate $c(x^*)$, we have $c(x^*) = (\eta/\varepsilon)k/4$, where η and ε are evaluated at x^*. Therefore $W_3 - W_4 = (\eta/2\varepsilon)k < k = W_1 - W_2 = W_c - W_3$ since $\eta/2\varepsilon$ is less than unity. One more step is necessary. For $s = 3$

$$V_3 = (1/2)(V_2 - W_4) + W_4 - c(x^*) = k/2 + W_4 - c(x^*).$$

Incentives at $s = 4$ are set by $V_3 - W_5$, so $x_4 = x^*$ requires
$V_3 - W_5 = k = k/2 + (W_4 - W_5) - c(x^*)$, or
$W_4 - W_5 = k/2 + c(x^*) = (k/2)(1 + \eta/2\varepsilon) < k$. All lower interrank differences equal $W_4 - W_5$ if x^* is maintained for all s.

We conclude that the differences $(W_1 - W_2) = (W_c - W_3)$ are larger than all the rest. Furthermore,
$(W_3 - W_4) - (W_4 - W_5) = (k/2)[(\eta/2\varepsilon) - 1] < 0$, which imparts a slight concavity to the incentive maintaining prize structure around rank 4. Otherwise, its general appearance resembles figure 1 if W_c and W_2 are both assigned $R = 2$. The schedule is a little more complicated if effort in the consolation match is constrained to be smaller than in the finals (and x_s is constrained to equal x^* in all other stages). Nevertheless, it resembles the previous case except $W_2 > W_c$ and $(W_1 - W_2) > (W_c - W_3)$. The jump at the top ranking prizes remains.

Equality between W_2 and W_c required for constant x_s in this example suggests that the incentive maintenence prize schedule in a complete tie-breaking structure awards a constant prize for each match won, irrespective of the stage or branch in which the win occurs. Such linearity arises because complete tie-breakers at every stage require every person to play the same number of matches in the overall tournament, and the design starts to resemble a round robin. Certainly a round robin design awards a constant prize for each match won in the problem analyzed here. Complete tie-breakers eliminate the survival-elimination elements which are crucial to players' incentives and strategies.[8] Extra concentration of the purse on the top ranking prize always is required for incentive purposes when tie-breakers are incomplete and confined to later stages of the game. The underlying logic also suggests that a qualitatively similar result applies if the tournament structure involves double (or more) eliminations.

Interstage Dependence. That effort/expenditure in any match is independent between stages implies a strong Markovian, strictly forward looking property of the solution that greatly simplifies analysis. The analytical problem is more complex if the path by which a contestant arrives at any stage affects either the productivity or cost of subsequent effort. For example, previous effort may increase subsequent productivity (or reduce subsequent costs) through a force of momentum or reinforcement, similar to a learning effect. Or current expenditure may deplete energy reserves and increase subsequent costs or reduce subsequent productivity through fatigue and "burnout."

The analytical issues raised by these forms of dependence are clear enough: The sequence $\{x_k\}_{s+1}^{n}$ conditions the functions $c(x)$ or $h(x_s)$. Define a state variable z_s as a function of the sequence $\{x_k\}_{s+1}^{n}$ of previous actions. Since z_s is given in round s and is an argument of $c(x)$ or p_s in (2), it follows that V_s is also a function of z_s. Therefore current x_s not only affects the probability of continuation. It also has a direct effect on the value of continuation. In the burnout case we have $\partial V_s/\partial z_s < 0$; whereas momentum implies $\partial V_s/\partial z_s > 0$. In contemplating action at s a player rationally takes account of its incremental direct effects on subsequent valuations (the derivatives above multiplied by dz_{s-1}/dx_s) as well as on the probability of continuing, with the realization that current and possible future opponents are doing the same thing.

A complete analysis of between stage spillovers is enormously complicated by the fact that a player's optimum strategy depends on the sequence of opponents' past actions as well as on his own and raises difficult issues of proving existence of equilibrium that are beyond the scope of this work. Even if a pure strategy equilibrium exists, it may be asymmetric. The best response functions may appear as in figure 4. For example, if fatigue is a factor and the j-player is working hard enough, the i-player may find it attractive to slack off in the current match, trading off a higher probability of elimination against the gain of starting the next match "fresh" and maintaining a possible winning edge in that (improbable) match. Nonuniqueness plays havoc with the backward recursion.

It is clear intuitively how these effects alter strategies at the symmetric equilibrium.[9] Let the spillover be confined to one round only. Then the state variable is $z_{s-1} = x_s$, so in choosing x_s in (4) or (9) an additional term in $p_s \partial V_{s-1}/\partial x_s$ appears on the left hand side. This term is positive in the case of momentum, so effort in the earlier stages tends to be larger than indicated above. It is negative when fatigue is important, so early round effort tends to be smaller than indicated above: Players tend to hold back effort and coast in the earliest rounds, saving energy for later stages, should they reach them, where the stakes are larger. In the first case the prize structure has to be more concentrated on the top to insure a constant interstage intensity of play. In the case of fatigue it must be less concentrated on the top to discourage early round coasting and maintain a constant effort level across stages.[10]

Another more interesting form of interstage dependence arises when players differ in talent.[11] Then the prize structure affects survival probabilities and the natural selection of players by talent through various stages of the game. These selection effects interact in an important way with incentives.

The Poisson specification offers an attractive parameterization in terms of proportional hazards. Index talent by I and write $h_I(x) = \alpha_I h(x)$. An I-player is stronger than a J-player if $\alpha_I > \alpha_J$ because I has a winning edge of $\alpha_I/(\alpha_I + \alpha_J)$ over J if both exert equal effort. This problem actually is technically less demanding than the case discussed above because the backward recursion methodology applies directly. Here V_{s-1} in (2) is replaced by its expectation as of stage $\underline{s}$, and the expected value of continuation depends on the distribution of talents of players still alive at s. In choosing a current strategy each player rationally contemplates the identities of probable future opponents, which in turn depend on the win probabilities of players in other matches, the conditional talent distribution surviving the previous stage and the pairing (or seeding) rules of the game. These interactions provide an interstage linkage that is absent in the problem addressed here.

The complexity of this more general problem arises from the fact that the expected value E_sV_{s-1} for any player in (2) depends on what players are doing in other matches at the same stage. This intermatch dependence means that strategies are not determined on a strictly pairwise basis, as they are when players are equally talented. Rather, the effort decision depends on the decisions of players in all other matches as well as on the decision of the specific opponent. Therefore, it is necessary to study the sequence of simultaneous 2^s player games through the stages. While the mechanics of this are conceptually straightforward, analytical solutions are impossible to obtain. Results must be obtained from computer simulation, which awaits future work. It remains to be seen how the horizon effect identified here affects the distribution of play intensity throughout the tournament and how the prize and seeding structure help assure that the best player wins.

Perhaps study of these highly structured and simple environments ultimately will illuminate a much bigger set of problems of incentive and selection in the labor market more generally. If so, this work may be a little less frivolous than appears on the surface.

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FOOTNOTES

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[1]An esteemed economist and (less esteemed) golfer is said to have proposed expanding the diameter of the cup by a factor of three. The Royal and Ancient society has not yet acted on this proposal.

[2]I do not mean to deny that nonmonetary considerations such as pride, self-esteem, and the desire for fame do not influence actions. Winning a prestigious tournament has great value apart from direct prize money in the form of endoresements and future appearance money, for reasons discussed in Rosen (1981). Many prestigious tournaments offer smaller purses, probably for this reason.

[3]A related discussion appears in Lazear and Rosen (1981). Nalebuff and Stiglitz (1983) consider equilibria involving mixed strategies.

[4]The precise rules of play and procedures used to determine winners affect the functional forms of $c(x)$ and $h(x)$. For viable games the rules and calibration of scores must be set so that (13) holds. An Authority is needed because equipment manufacturers and individual players have strong private incentievs to create a winning edge by introducing new styles, techniques of play and complementary capital. Many of these changes are beneficial and improve the social value of the game. However, those that greatly escalate the collective costs of all players relative to value for the group as a whole are prohibited. O'Keeffe _et al_. (1984) provide a different and interesting discussion of rules in terms of monitoring.

[5]Consider a tour of length T over a season, in which the tournament is repeated T times among the same 2^n players. It is conceptually straightforward to work out the season (or partial season) multinomial distribution of earnings among players.

[6]The purse must be large enough to support $V_s > 0$ for all s. For a given purse it is obvious that feasible x* is bounded from above for this condition to hold. Another bound is implied by contestants' outside opportunities, but is ignored here.

[7]However, effort has little effect on outcomes when η/ε is small. Such games generate little spectator interest and offer small prizes.

[8]That the structure of the game interacts with the incentive effects of prizes is further illustrated by the example in the appendix. The sequencing of that game differs from this one.

[9]Again, it is conceivable that the rules and standards of play are devised to eliminate the asymmetric equilibria in figure 4. Notice that the occurrence of turning points in the response functions of figure 3 at $x_i = x_j$ rules out asymmetric equilibria in the problem above.

[10]Stage dependence without memory is easily analyzed by allowing s to shift the cost or hazard functions. The analysis above is only slightly modified. For example write $c(x_s)f(s)$. $f'(s) > 0$ implies reinforcement because marginal cost declines in later stages. $f'(s) < 0$ implies fatigue. The reader is invited to rework the example in section III with this cost function to verify the statements in the text.

[11]Lazear and Rosen (1981) show that a larger prize spread is required to self-screen less talented players. However, in most games entry is based on direct performance indicators, not only on self-selection.

Appendix

Instead of sequential eliminations with paired comparisons, think of an outright race. n players start, all racing against each other. The first player to cross the finish line is declared overall winner and achieves the highest rank. The remaining n - 1 players continue racing, resetting their effort to take account of the fact that there is one less player to compete against. The first among these n - 1 to hit the finish line achieves the second highest rank. Then the n - 2 remaining players vie for third place, again resetting their efforts to account for the lesser number of players and the prize money remaining, etc.

This problem also has a recursive structure. Let V_t be the value of racing when t players remain. Then.

$$V_t = \max_{x_t} \{p_t W_{n-t+1} + (1 - p_t)V_{t-1} - c(x_t)\} \tag{A.1}$$

where p_t is the conditional probability of winning. Assuming Poisson arrivals and equally talented players

$$p_{ti} = \frac{h(X_i)}{\sum_j h(X_j)} \tag{A.2}$$

for player i, where the index of summation for j is over all remaining players including himself. p_t is independent of past action, by assumption. The Nash solution at each stage is symmetric and shares many of the features noted above. The only substantial difference is an adjustment for the change in the number of opponents at each stage.

Analyzing this game in the same manner as the example in section III yields recursions ($\varepsilon = 1$ is assumed for simplicity)

$$V_t = \beta_t W_{n-t+1} + (1 - \beta_t)V_{t-1} \tag{A.3}$$

$$x_t = \frac{(t-1)\eta}{t^2} [W_{n-t+1} - V_{t-1}] \tag{A.4}$$

with

$$\beta_t = \frac{1}{t} (1 - \frac{t-1}{t} \eta). \tag{A.5}$$

Note that in the text, the equivalent of "t" in (A.3)-(A.5) is 2, since there is exactly one opponent at each stage. In fact, substituting $t = 2$ into (A.5) yields the expression for β used in section III. However, here the number of opponents is changing as the game proceeds so the equilibrium conditional probability of winning at each stage is $1/t$ rather than $1/2$. β_t incorporates this effect.

Defining $V_1 = W_n$, the complete solution for x_t is

$$x_t = \frac{t-1}{t^2} \eta\{(W_{n-t+1} - W_{n-t+2}) + (1 - \beta_2)(W_{n-t+2} - W_{n-t+3}) \tag{A.6}$$

$$+ \ldots [(1 - \beta_{t-1})(1 - \beta_{t-2}) \ldots (1 - \beta_2)](W_{n-1} - W_n)\}.$$

The ordering of events is reversed from the text (since survival signals a smaller reward rather than a larger one), but the solution has similar features after taking account of the stage-varying discount factor.

Time varying discounts and the presence of the factor $(t-1)/t^2$ in A.6 make it more difficult to find the incentive maintaining prize structure because the schedule contains no linear segments. Some experimentation shows that it has both concave and convex portions, and the prize money need not pile up on the top ranks. This example is designed to show that the tournament design influences the incentive maintenance schedule, but it otherwise has very limited interest due to the assumed strong Markovian property that the probability of winning at any stage is independent of how far one has traveled in the past. Putting memory into this game leads to the same problems as were identified in section IV. This process is therefore better suited to tournaments with a natural survival-sequential structure, such as in the text.

TABLE 1

Men's Tennis: 1984 On-Site Prize Money Distribution Formula, Volvo Grand Prix Circuit[a]

Rank	Percent of Purse[b]			
	Grand Slam Events[c]		Other Grand Prix[d]	
	Singles (128 Draw)	Doubles (64 Draw)	Singles (64 Draw)	Doubles (32 Draw)
1	19.23	27.27	20.51	27.27
2	9.62	13.64	10.26	11.36
3-4	4.81	6.82	5.64	5.91
5-8	2.44	2.95	3.08	3.18
9-16	1.41	1.36	1.92	2.10
17-32	.77	.68	1.03	1.25
33-64	.45	.40	.43	
65-128	.22			

Notes

[a]Covers 80 international single elimination events. On-site money does not include contributions to end-of-season bonus pools. 62.5 percent of the $2.4M singles pool goes to the top 4 season ranked players and 64.2 percent of the $.6M doubles pool goes to the top 4 teams.

[b]Total tournament on-site purse split 78 percent for singles, 22 percent for doubles. Figures refer to shares of singles and doubles components of the total respectively. Each person in a tied rank receives the share indicated. Weighted shares may not sum to 100 due to rounding.

[c]French Open, Wimbledon, U.S. Open and Australian Open. Draw refers to number of players or teams. 96 draw singles events are slightly more concentrated on top ranks.

[d]On-site total purse of $25,000 or more.

Source: Official 1984 Professional Tennis Yearbook of the Men's International Professional Tennis Council. New York, 1984.

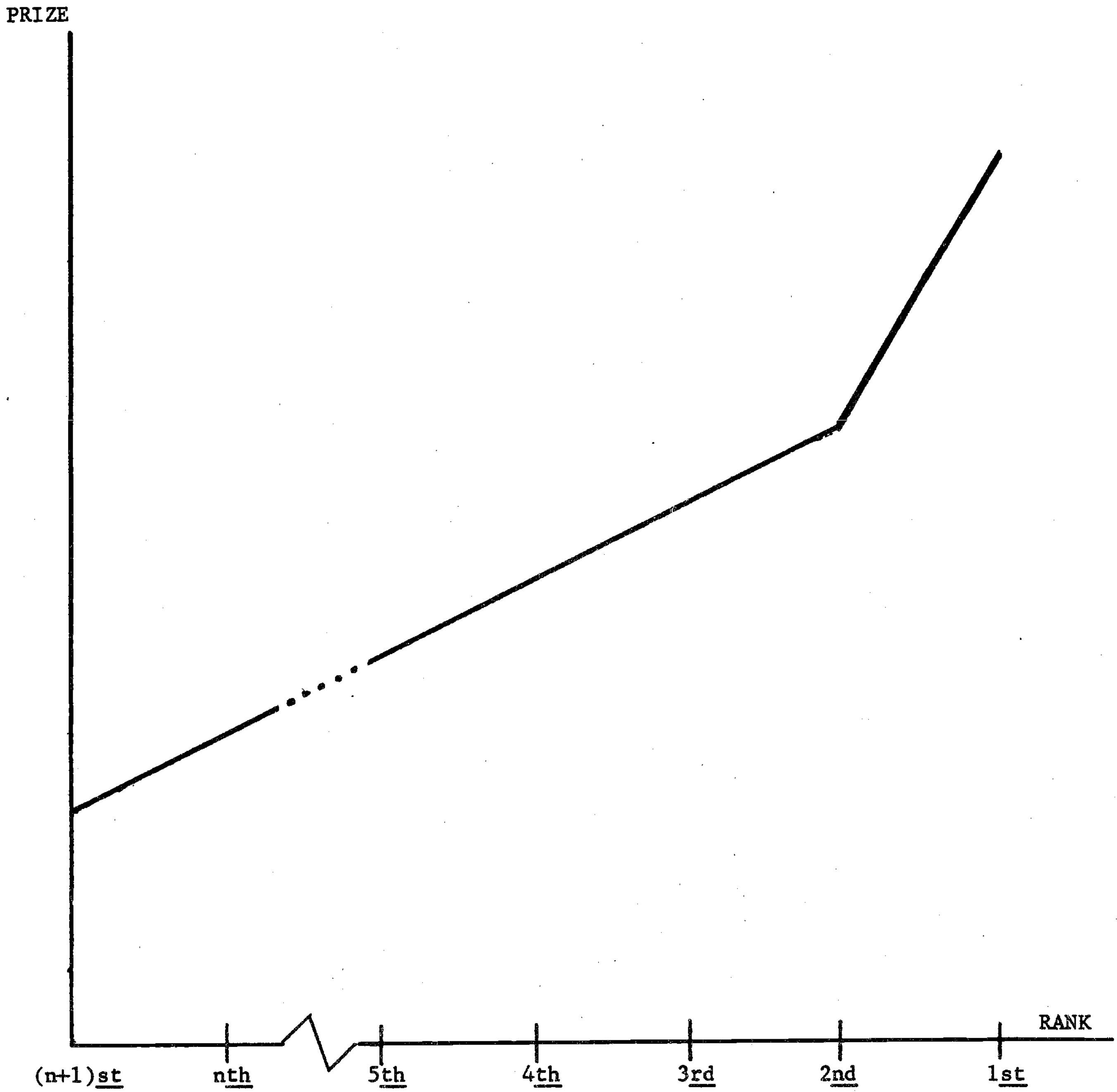

Figure 1: Incentive Maintaining Prize Structure

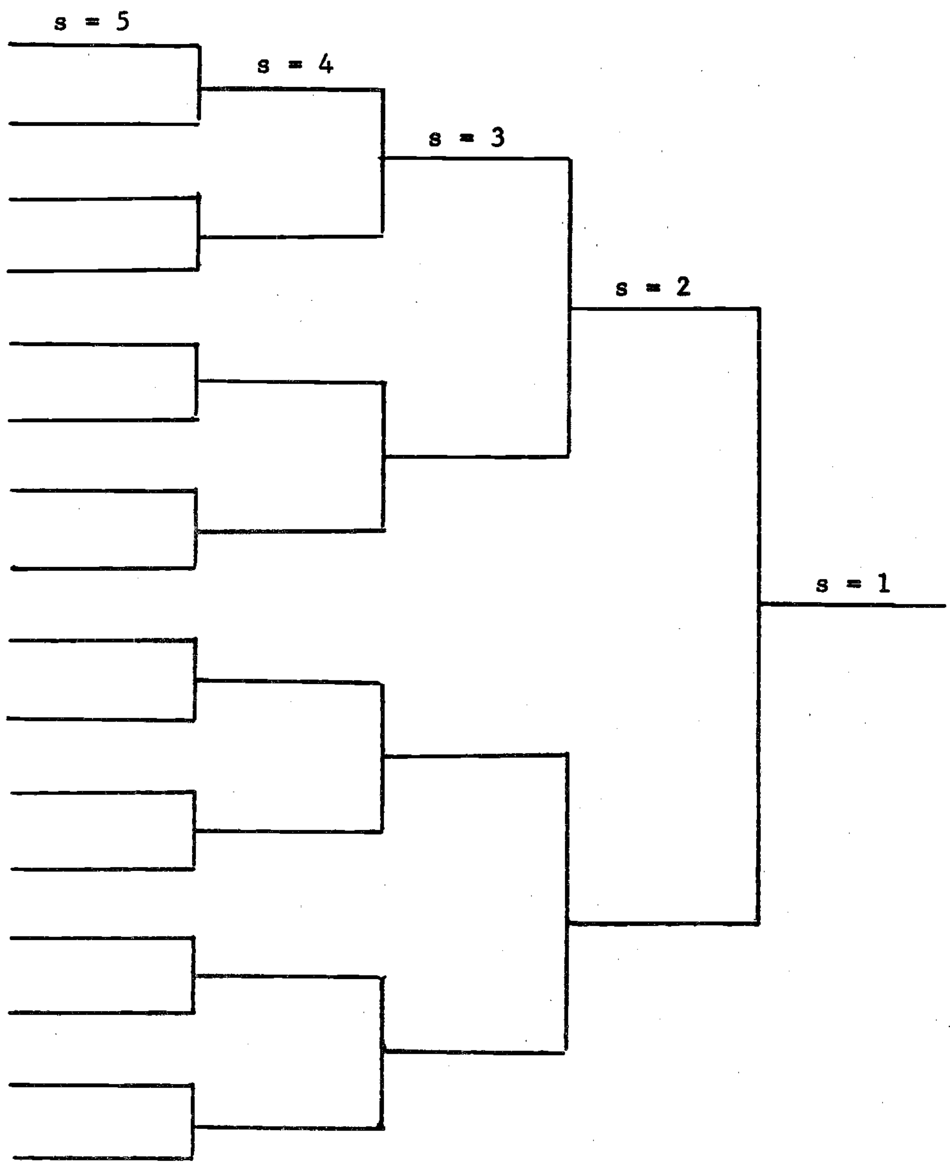

Figure 2: Tournament Design

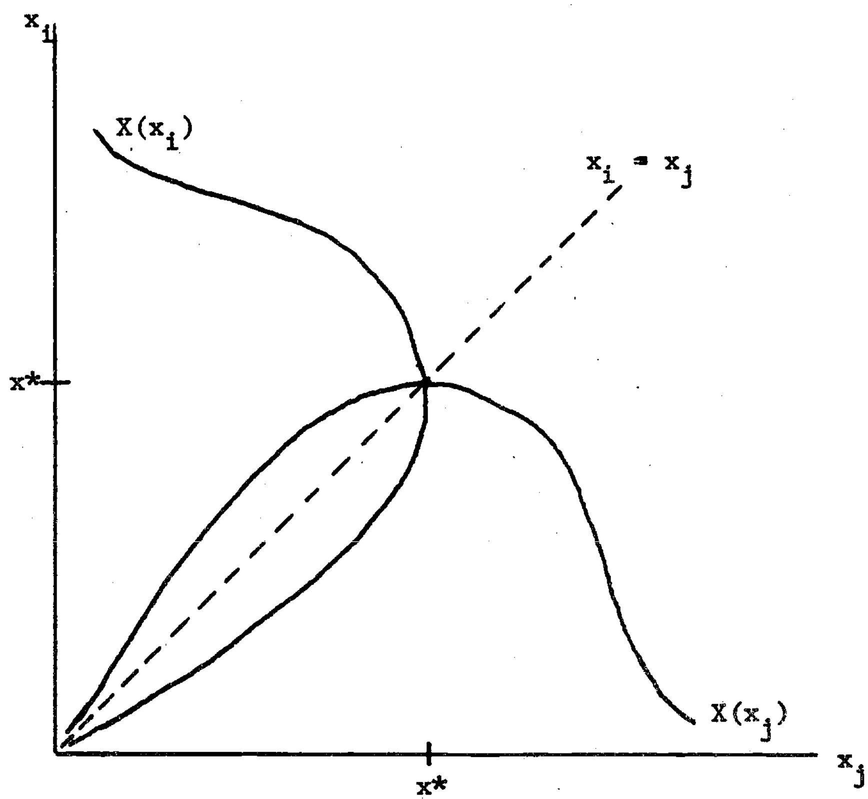

Figure 3: Best Reply Functions

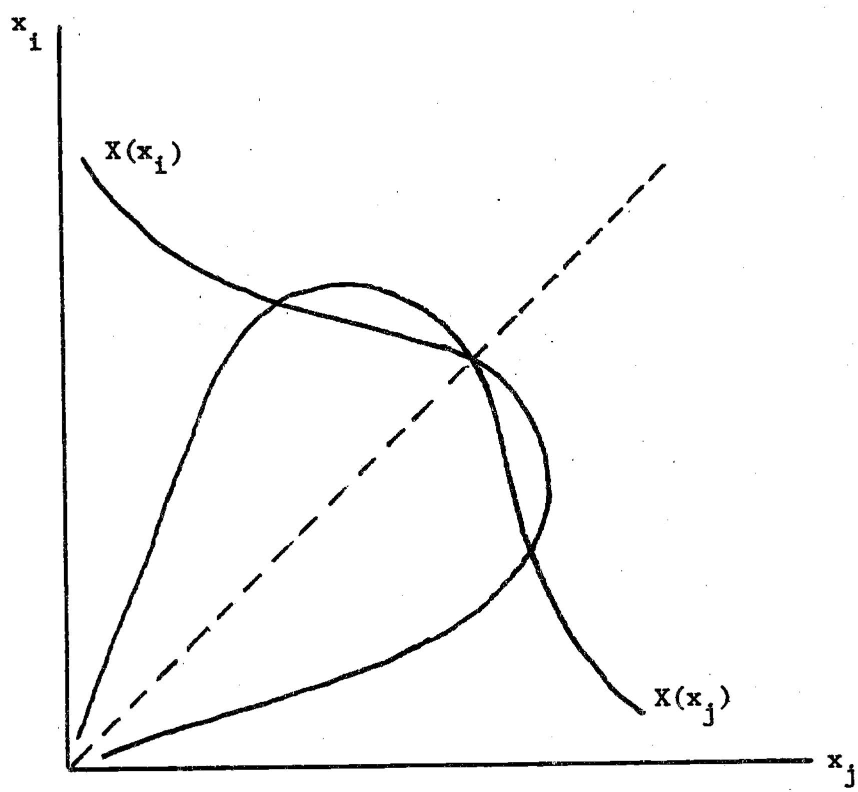

Figure 4: Asymmetric Equilibria