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ABSTRACT

A firm's termination leads to bankruptcy costs. This may create an incentive for outside stakeholders or the firm's debtholders to bail out the firm as bankruptcy looms. Because of this implicit guarantee, firm shareholders have an incentive to increase volatility in order to exploit the implicit protection. However, if they increase volatility too much they may induce the guarantee-extending parties to "walk away". I derive the optimal risk management rule in such a framework and show that it allows high volatility choices, while net worth is high. However, risk limits tighten abruptly when the firm's net worth declines below an endogenously determined threshold. Hence, the model reproduces the qualitative features of existing risk management rules, and can account for phenomena such as "flight to quality".

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1 Introduction

In debating the charter for the Bank of England in 1840, Sir Robert Peel (the Prime Minister of Britain at the time) used the following words:

> While the charter is well-designed and while we are taking all precautions which legislation can prudently take against the recurrence of a monetary crisis, a crisis may occur despite of our precautions. If it does, and if it be necessary to assume grave responsibility for the purpose of meeting it, I dare say men will be found willing to assume such responsibility.

Sir Robert Peel's words are as relevant today as they were 168 years ago. As the Unites States is going through one of the worst financial crises of the last decades, and as its leaders contemplate a large bailout, it seems important to recall that the current crisis is not unique in its features. During the last few decades, the world has seen several financial crises (Asian crisis, Russian crisis etc.) that all shared a common theme: Periods of increased risk appetites, as typically evidenced by high leverage ratios, led financial institutions to the brink of bankruptcy. Bailouts and restructuring followed, sometimes undertaken by the government and sometimes resulting from negotiations between the parties directly involved in these institutions. At the same time, large liquidations of risky positions - sometimes referred to as "flight to quality" - exacerbated the initial negative shocks and led to prolonged periods of depressed asset valuations.

The subprime lending crisis that the United States is experiencing these days provides a reconfirmation of the general pattern: In the years 2004-2006 the quest for higher expected returns led financial institutions to increase their leverage and their lending to subprime borrowers. The expansion of leverage left little margin for error when house values declined and delinquencies increased. The initial reaction to the ensuing crisis consisted of private sector bailouts of the affected entities[1] followed by outright government bailouts once some of

[1] For instance, during the early stages of the recent subprime lending crisis the parent companies of hedge funds, structure investment vehicles, or originators of CDOs had to provide infusions of liquidity in order

the largest private entities were considered "too big to fail". At the same time, risky markets that attracted several participants between 2004-2006 (such as the market for collateralized debt obligations) were abandoned in a quite dramatic fashion in favor of simpler and safer investment forms. This abrupt change in the willingness to take risk led to large risk premia, illiquidity and deep discounts in several markets, which further exacerbated the crisis.

The commonality of the structure of financial crises suggests the possibility of an economic mechanism that can simultaneously explain their recurrent themes. Two phenomena seem to be of first order importance: a) the pattern of high initial risk taking followed by rapid reversals of risk appetite around the onset of a crisis and b) the prevalence of bailouts and restructuring during a crisis.

Pre-existing research has suggested that the first phenomenon may have a simple, almost mechanical explanation: A large body of research has argued[2] that it is the very nature of the risk management practices followed by financial institutions that makes them prone to risk appetite reversals. Indeed, existing risk management rules[3] allow high volatility choices in good times and automatically tighten the risk limits in response to declining market values. This tends to exacerbate the effects of negative shocks. Then why do such risk management rules exist in the first place ? This question is important both for positive as well as normative reasons.

The present paper proposes an answer to this question. It develops a model where risk management rules are derived as optimal responses to the adverse risk taking incentives created by bailouts. Additionally, the incentives to undertake a bailout are endogenously determined, making it possible to provide a joint explanation for the observed risk appetite reversals and the prevalence of bailouts.

to avoid abrupt liquidation of these entities. (For instance Bear Sterns had to bail out two of its hedge funds at the onset of the crisis). Such guarantees were sometimes explicit (through market value swaps) and sometimes implicit (due to reputational concerns of the investment banks). See JP Morgan "US Fixed Income Markets weekly" August 10, 2007 p. 66 for a discussion.

[2] See e.g. Basak and Shapiro (2001). Papers that are similar in spirit include Grossman and Zhou (1996), Basak (1995), Pavlova and Rigobon (2005), Gromb and Vayanos (2002).

[3] One such example is Value at Risk (VAR).

Specifically, in the baseline version of the model there are three agents: the firm's shareholders, its debtholders and a stakeholder (such as the parent company of the firm, an insurer that guarantees principal repayment to debtholders, junior claimants, potentially the government etc.). The stakeholder incurs a discrete cost or externality if the firm is terminated. The presence of this cost or externality makes the stakeholder willing to bail out the firm, by injecting funds, once bankruptcy looms. However, the stakeholder's guarantee to the shareholders is implicit and the benefit from the firm's continued presence is bounded. Hence, bailouts can occur only if the stakeholder finds it profitable to undertake them. (The paper also discusses a variant of the model, where there is no stakeholder and the bankruptcy cost is incurred directly by the debtholders, who may have an incentive to "forgive" some debt in order to avoid the discrete bankruptcy cost.)

In this baseline framework, the paper studies the shareholders' incentive to take risk. As one might expect, the presence of an implicit guarantee makes the shareholders inclined to raise the volatility of the projects that they undertake. However, high volatility choices could deter the stakeholder from bailing out the firm. This produces a tension. On the one hand, shareholders want to raise volatility, but not so much that the stakeholder will find it prohibitively costly to bail out the firm.

In reality, the tension produced by such conflicting goals leads to the adoption of regulations, self-regulations, covenants, laws etc. that I will refer to as "risk management rules" or commitments. Such rules place limits on the risks that firms can take and hence serve the purpose of reassuring the stakeholder.

A new aspect of the model is that rules, regulations and commitments are allowed to be imperfect, as they are likely to be in reality. The imperfection stems from the fact that future shareholders may choose to renege by paying a cost. This helps capture situations where firms can circumvent risk management rules by undertaking costly activities such as setting up offshore, off-balance sheet entities etc. The imperfection of commitment implies that the credibility of a risk management rule is not taken as given. Instead, adherence to the rule has to be dynamically consistent.

Within this framework, I analyze the optimal choice of a risk management rule and show that it has a particularly simple form: undertake projects with high risk levels when net worth (defined as assets minus liabilities) is sufficiently high and switch to projects with low risk levels when net worth falls below an endogenously determined threshold.

The intuition for this result is simple. An optimal risk management rule should induce the stakeholder to bail out the firm, in order to avoid the deadweight cost of bankruptcy. Simultaneously, it should provide future shareholders with high continuation values, in order to reduce the temptation to renege. The optimal risk management rule achieves both of these objectives. By tightening the risk limits when net worth is low, it becomes possible to allow projects with high volatilities when the firm's assets safely exceed its liabilities. By postponing the high volatilities for the times when net worth is high, the anticipated growth rate of shareholder value is maximized. This "backloading" effect is common in many dynamic contracting contexts.[4]

This paper belongs to the continuous time literature that analyzes capital structure via contingent claim methods. This literature was initiated by the seminal Merton (1974) paper. Duffie (2001) presents a textbook treatment.[5] This literature takes the cash flow and control rights of debt and equity claims as given and uses the risk neutral pricing approach of Cox and Ross (1976) in a continuous time framework to price claims on a firm (including implicit guarantees) by option valuation techniques. The present paper contributes to this literature by explicitly modeling the incentives of the shareholders to take risk and the incentives of the stakeholder to undertake a bailout.

Leland (1998) also models endogenous volatility choice. The present paper supports the results in Leland (1998), in that it shows analytically the optimality of simple Markovian "bang-bang" type volatility policies. However, the two papers have a different focus and consider different frictions and choices, so that the optimal volatility process takes a different form. Specifically, in Leland (1998) shareholders have an incentive to *increase* rather than

[4]See e.g. DeMarzo and Fishman (2007), DeMarzo and Sannikov (2006).

[5]A representative sample of papers in this voluminous literature includes Ronn and Verma (1986), Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996), Lucas and McDonald (2005), Constantinides, Donaldson, and Mehra (2002), Pennacchi and Lewis (1994).

decrease volatility as net worth declines and termination looms.[6] The reason is that in Leland (1998) there are no bailouts or debt renegotiations, so that the terminal nature of bankruptcy removes the incentives to mitigate risk that are present in this paper. Therefore, in Leland (1998) the incentives to mitigate risk result from the callability of debt, rather than the participation constraint of the stakeholder. For parsimony, and in order to illuminate the new insights of the present paper, I abstract from taxes, callability, and the endogenous choice of capital structure, so that the only reason to mitigate risk is the participation constraint of the stakeholder. In such a context the optimal rule is for firms to *increase* their volatility when their net worth is high and *reduce* it as they come close to bankruptcy, giving rise to a "flight to quality" phenomenon.

The model is also related to a literature in financial economics that studies how commonly observed risk management practices can lead to variations in institutional risk taking. See e.g. Grossman and Zhou (1996), Basak (1995) Pavlova and Rigobon (2005), Basak and Shapiro (2001), Gromb and Vayanos (2002). Taking these risk management approaches as *given*, previous literature has recognized their importance in limiting a firm's ability to absorb risk during times of crisis. The contribution of this paper is to understand *why* the prevailing risk management rules dictate risk limits that tighten as firm's net worth declines.

There exists a voluminous literature on debt, default and reorganization, that I will not attempt to summarize here.[7] This literature studies the allocation of control and cash flow rights, strategic default service, reorganization etc. There are several major differences between the present paper and that literature. First of all, for the most part the present paper studies the incentives to inject "new money" into a company, as opposed to splitting the existing cash flows. Second, and more importantly, the present paper focuses on the risk taking and risk management incentives of bailouts. By having an explicit intertemporal framework, it becomes possible to address the important issue of commitment and rules in the context of choosing optimal risk management policies.[8]

[6] This result holds irrespective of whether one assumes commitment or not.

[7] See Hart and Moore (1998) for a seminal contribution in this literature.

[8] See e.g. Leland (1994) on the issue of commitment. In the context of banking, Ritchken, Thompson, DeGennaro, and Li (1993) show that charter value can create risk management incentives. However, in their

As Leland (1998) points out, commitment (and the lack thereof) is a central issue behind the asset substitution problem of Jensen and Meckling (1976). Recent literature in economic theory and monetary economics has made advances in terms of making commitment an endogenous choice rather than imposing it as an assumption.[9] In the context of the asset substitution problem studied in this paper, section 5 introduces a new approach to modeling commitment and more generally regulations or self-regulations. This approach allows commitment that is somewhere between the two extremes of a) full commitment without any requirement of time consistency and b) full dynamic consistency which completely precludes firms from "tying their hands" through some form of covenant or regulation. The notion of commitment that is proposed in section 5 allows for the possibility of reneging at a cost. Therefore, commitment is not perfect, but it is not impossible either.

To endogeneize the extent of commitment, I let the involved parties choose both the risk management rule and how large will be the cost if the commitment is abandoned. Furthermore, I assume that higher costs of reneging (more stringent regulations or self-regulations) are associated with higher oppotunity costs associated with distortions, implementation, monitoring etc..

Surprisingly, it turns out that endogeneizing commitment in this way implies two results: a) Simple Markovian policies are optimal, since current decisionmakers have an incentive to choose a commitment, that limits future shareholders' temptation to renege. b) The qualitative nature of the optimal risk management rule (high volatility in high net worth states, low volatility in low net worth states) is not affected by whether the commitment is imposed as a regulation by the stakeholder or is voluntarily chosen by the shareholders; only the distribution of the rents from the bailout depends on the party that determines the volatility policy.

Motivated mostly by the Asian crisis, a literature in international economics considers the effects of bailouts and balance sheet effects for understanding crises in developing

simpler setup there are no commitment or strategic issues.

[9] See e.g. Caruana and Einav (2008) for game theoretic applications and Giannoni and Woodford (2002) for applications to monetary economics.

economies by taking a general equilibrium perspective.[10] Typically, this literature does not consider the incentives of the parties who undertake the bailout. The present paper takes a microeconomic approach in order to understand jointly the incentives of stakeholders and shareholders in a dynamic setting. The conclusions reached in this paper naturally complement the international finance literature, which is mostly concerned with general equilibrium issues. An application of the present model is that levered institutions in developed markets will rapidly decrease their risky positions in response to negative net worth shocks.[11] Such risk management practices could help explain contagion effects not only across countries but also across seemingly unrelated markets.[12]

Methodologically, the paper uses continuous time methods to analyze an intertemporal incentive problem. Continuous time methods allow a close and explicit characterization of the solution to dynamic incentive problems. However, the present paper differs with the dynamic contracting literature,[13] since the goal is not to study the optimal design of debt and equity or the dynamic evolution of a firm's capital structure, the allocation of cash flows etc.. The present paper takes the capital structure as given, and focuses exclusively on the incentives to take risk and the incentives to undertake bailouts within a dynamic framework.

The structure of the paper is as follows. Section 2 presents the setup of the basic model. In order to expedite the presentation of the main result, Section 3 restricts attention to Markovian policies and derives the optimal volatility policy in that class assuming the presence of full commitment. Section 4 presents several realistic extensions of the baseline model and a discussion of its real world implications. Section 5 introduces the notion of imperfect/costly commitment and shows that Markovian commitments are optimal even after allowing for general (potentially history dependent) commitment policies. Assuming imper-

[10] See e.g. Schneider and Tornell (2004) and references therein.

[11] An illustrative example is the behaviour of Japanese banks during the Asian financial crisis. (See Kaminsky and Reinhart (2001).) It is interesting to note that one of the financing forms that most rapidly evaporated during the East Asian crisis was short term lending by international banks - especially Japanese and European banks - that provided the bulk of credit lending to these countries. The risk management practices of these banks coupled with capital adequacy requirements are viewed by many researchers as responsible for the prolonged capital flow reversal.

[12] See e.g. Calvo (1999), Caballero and Panageas (2005)

[13] See e.g. DeMarzo and Fishman (2007),DeMarzo and Sannikov (2006).

fect/costly commitment, section 6 establishes that the qualitative features of the optimal risk management rule are the same irrespective of whether the rule is determined by the shareholders or by the stakeholder through regulation. Finally, section 7 presents the implications of imperfect/costly commitment for times prior to the first bailout. Section 8 concludes. All proofs are relegated to the appendix.

2 Model

The baseline model makes a number of simplifying assumptions for expositional reasons. Several of the simplifying assumptions are relaxed in subsequent sections.

2.1 Lenders and the outside stakeholder

There are three types of agents in the baseline model: a continuum of competitive lenders, a continuum of anonymous shareholders and an outside stakeholder, who derives some benefit from the firm's continued existence.

The lenders hold a fixed liability of the firm in the amount L. This liability remains constant throughout time for simplicity. The firm also owns assets in the amount W_t, so that the firm's net worth at time t is $W_t - L$. The assets of the firm satisfy $W_0 > L$ at time 0.

The firm is a productive entity that can never fully eliminate the risks associated with its operation. However, it can choose to invest its assets in projects involving either high or low risk. Under the risk neutral measure[14] both high and low risk projects yield an expected return equal to the interest rate r per unit of time dt. However, projects involving high risk have instantaneous volatility σ_2, while less risky projects have a lower volatility $0 < \sigma_1 < \sigma_2$.[15] The firm can costlessly adjust the fraction that it invests in high and low risk

[14] Roughly speaking, pricing under the risk neutral measure means that the implied Arrow Debreu prices in the market are used to determine the value of the firm. For more details on the relationship between Arrow Debreu prices and risk neutral measures, see Duffie (2001).

[15] The fact that $\sigma_1 > 0$ implies that the nature of the firm's business is such, that it can never fully

projects. As a result, its assets follow a geometric Brownian Motion under the risk neutral measure

$$\frac{dW_t}{W_t} = rdt + \sigma_t dZ_t \tag{1}$$

where the drift $r > 0$ is the prevailing (real) interest rate in the economy, dZ_t is a standard Brownian motion, and σ_t presents the volatility of total assets. By constantly adjusting the fraction it invests in high risk and low risk projects, a firm can attain any level of $\sigma_t \in [\sigma_1, \sigma_2]$ for all $t \geq 0$.

To keep the analysis simple, in the baseline model the firm can pay no intermediate dividends to its shareholders until a random time τ, at which time it pays a liquidating dividend in the amount of $W_\tau - L$. Section 4 relaxes this assumption and considers intermediate dividends. The firm also pays a flow of rL to its lenders, up to the time of its liquidation.[16]

Liquidation occurs either exogenously or endogenously. Exogenous liquidation happens at a random exponentially distributed time τ with constant hazard $\lambda > 0$. This will facilitate the use of infinite horizon optimization techniques by making all solutions independent of time. In addition to this exogenous arrival of termination, lenders can terminate the firm prior to τ : By covenant, (or because lending is secured by the assets of the firm, or is short term) they can enforce liquidation if the assets of the firm fall below its liabilities, i.e. if $W_t < L$. This assumption and the associated simplicity of the bankruptcy trigger will expedite the presentation of the results without affecting the conclusions.

When the firm gets terminated by its lenders ("endogenous liquidation"), the outside stakeholder incurs a monetary cost B. Purely for simplicity, I assume that this cost does not arise if liquidation is exogenous.[17] The source of the cost B typically depends on whether

eliminate risk. For instance, if the firm is a mortgage granting institution, $\sigma_1 > 0$ implies that there is no perfectly safe mortgage. The idea that there is always some risk in a productive entity is a common assumption in production economies (see e.g. Cox, Ingersoll, and Ross (1985)).

[16] As Leland (1994), I assume that equity issuance can be used to finance the payments to the debtholders, as long as shareholder value is positive.

[17] This assumption can be easily relaxed without affecting any of the results.

the firm is non-financial or financial. For the first type of companies, the cost B could have political origins (e.g. the political cost associated with increased unemployment in a region). For financial companies, the cost B could be interpreted as a fire sale or bankruptcy cost due to rapid liquidation of the assets. In particular, assume that the outside stakeholder is an insurer to the lenders and has committed to incur any fire sale or bankruptcy costs in the event of a liquidation, so that debtholders do not experience any principal losses. In that case, if the firm's assets drop to L and the debtholders force liquidation of the firm, there will be bankruptcy and fire sale costs in the amount of B, that the stakeholder will have to incur.[18] A further interpretation of B as an externality occurs when a firm has claimants of different seniority that could be hit asymmetrically by bankruptcy costs.[19] B could also be the result of systemic risk, or it could have reputational origins. For instance, at the onset of the recent subprime lending crisis, several major investment banks bailed out structure investment vehicles or hedge funds they were sponsoring, so as to shield their claimholders from losing their invested capital.

Before proceeding, it should be noted that even though the assumption of a discrete bankruptcy cost or externality B is critical for the results, the assumption about the existence of an outside stakeholder isn't. Section 4.1 presents a variant of the basic model where it is the debtholders who incur the cost B rather than some outside stakeholder.

Whatever the reason for the cost or externality B, the outside stakeholder has the option of making transfers to the firm in order to keep its assets above L, and hence prevent liquidation by the lenders. In mathematical terms

$$dW_t = rW_t dt + \sigma_t W_t dZ_t + dG_t \tag{2}$$

[18] For instance, a standard practice of major investment banks was to provide their structured investment vehicles (SIV's) with a guarantee to purchase their short term paper at fixed rates, if the need presented itself. Economically this is identical to providing a guarantee to the debtholders of the fund. Similarly, the deposit insurance agency of a given country might have to incur such bankruptcy costs in order to protect the bank's lenders.

[19] As an example, consider a firm that has debt in the amount $L = L^S + L^J$, whrere L^S is senior debt and L^J is junior debt. If there are bankruptcy costs in the amount $B \leq L^J$ and senior debtholders can request liquidation once the firm's assets reach L, then they can impose an externality on junior debtholders by requesting liquidation.

where $dG_t \geq 0$ represents incremental transfers that can be used once $W_t = L$ in order to enforce $W_t \geq L$ for all t. Intuitively, one should think of these injections as follows: Each time the firm's assets W_t fall by an amount $\varepsilon > 0$ below L, the stakeholder transfers ε to the firm. Since the stakeholder has no incentive to make transfers to the firm beyond the ones that are absolutely necessary to ensure its existence, one can focus on the *minimal* process that is required to keep $W_t > L$. Karatzas and Shreve (1991) (p.210-211) show that the unique minimal process for G_t that will safeguard $W_t \geq L$ for all t is given by

$$\frac{\int_0^t dG_s}{L} = \max\left[0, \max_{0 \leq s \leq t}\left\{-\left(\log\left(W_0\right) - \log\left(L\right) + rs - \frac{1}{2}\int_0^s \sigma_u^2 du + \int_0^s \sigma_u dZ_u\right)\right\}\right]. \quad (3)$$

In the baseline model the stakeholder injects funds without receiving a share of the firm's dividends (or other form of security) in exchange for the transfers G_t. Section 4 enriches the model to allow for this realistic extension.

A key assumption of the model is that the stakeholder has a choice on whether to bail out the firm or not. In particular, once the assets of the firm become equal to its liabilities, the stakeholder can decide whether to make the transfers dG_t or to just let the lenders seize the assets and terminate the firm.

Defining τ^l to be the time of firm liquidation (be it exogenous or lender-induced), a sufficient condition for the stakeholder to always prefer to bail out the firm is that the net present value of the costs associated with keeping the firm alive is less than the benefit of doing so

$$E_t\left(\int_t^{\tau^l} e^{-r(s-t)} dG_s | W_t = L\right) \leq B, \quad (4)$$

where the process G_t is given by (3). The expectation is taken under the risk neutral measure, and so are all expectations in the rest of the paper. Since the firm controls the volatility process σ_t, it also influences the net present value of the transfers on the left hand side of this equation.

2.2 Shareholders

The volatility choices of the firm are determined by its shareholders. Therefore, I use the terms "the firm" and "the shareholders" interchangeably.

To determine the value of the firm to shareholders, observe that the total value of the firm is given by $W_t + P_t$, where P_t is the value of the implicit option that the stakeholder extends to the firm

$$P_t = E_t \left(\int_t^{\tau^l} e^{-r(s-t)} dG_s \right) \tag{5}$$

The total value of the firm is just equal to the sum of the claims that debtholders and shareholders hold. Letting V_t denote shareholder value and D_t denote debtholder value, one obtains $W_t + P_t = V_t + D_t$. Since debtholders can always induce liquidation once $W_t = L$, they hold effectively riskless debt. Accordingly $D_t = L$. Using this observation, shareholder value is

$$V(W_t) = W_t - L + P_t \tag{6}$$

Equation (6) has two implications. First, different volatility processes will affect shareholder value through their effect on the value of the guarantee P_t. Since $P_t \geq 0$, shareholders always have an incentive to induce the stakeholder to extend the guarantee once $W_t = L$.

To check intuition, it is also useful to confirm that the firm has an incentive to set high levels of volatility in order to exploit the guarantee provided by the stakeholder. To be more specific, ignoring temporarily the constraint (4) and assuming that the stakeholder unconditionally guarantees the perpetual continuation of the firm until the time of exogenous termination, the following result holds:

Lemma 1 *Assume that $\tau^l = \tau$ in expression (5). Assume furthermore that volatility is* *constant at the level $\overline{\sigma}$ for all $t \geq 0$ and define α as:*

$$\alpha\left(\overline{\sigma}\right) = \frac{-(r - \frac{1}{2}\overline{\sigma}^2) - \sqrt{\left(r - \frac{\overline{\sigma}^2}{2}\right)^2 + 2\overline{\sigma}^2\left(r + \lambda\right)}}{\overline{\sigma}^2} < 0 \tag{7}$$

Then, the value of the stakeholder guarantee is given by:

$$P\left(W_t; \overline{\sigma}\right) = \frac{L}{\left|\alpha\left(\overline{\sigma}\right)\right|}\left(\frac{W}{L}\right)^{\alpha(\overline{\sigma})} \tag{8}$$

It is also straightforward to show the following result

Lemma 2 *Assume that $\tau^l = \tau$ in expression (5). Then the volatility choice that maximizes P_t is given by $\sigma_t = \sigma_2$.*

In light of the above result, if the stakeholder extended an unconditional and perpetual guarantee to the firm, then the shareholder value maximizing choice of volatility would be to set σ_t equal to its upper bound σ_2 for all $t > 0$. This captures the standard asset substitution intuition of unconditional guarantees.

The above two Lemmas only apply if the guarantee is unconditional. The focus of this paper, however, is on guarantees that are implicit, i.e. guarantees that will only be extended if (4) is satisfied. In order to make the problem interesting, I make the following assumption:

Assumption 1

$$P\left(L; \sigma_1\right) = \frac{L}{\left|\alpha\left(\sigma_1\right)\right|} < B < \frac{L}{\left|\alpha\left(\sigma_2\right)\right|} = P\left(L; \sigma_2\right) \tag{9}$$

In light of Lemma 1 and equation (4), assumption 1 has two implications: a) to ensure that the firm has at least one feasible choice of volatility that will make it possible to satisfy the constraint (4) (namely by setting $\sigma_t = \sigma_1$) and b) to impose that setting volatility equal to the upper bound σ_2 for all $t > 0$ will violate the constraint (4).

In the baseline model, B is a time invariant constant that satisfies assumption 1. Previewing results, this will imply that optimal volatility choices will always induce the stakeholder to bail out the firm. Section 4.3 presents a simple example where B is time-varying and shows that termination can occur in equilibrium.

2.3 Commitment and risk management rules

The above discussion illustrates the tension that is at the core of this paper. On the one hand, shareholders would like to set high volatility levels in order to increase the value of the implicit guarantee. On the other hand, if volatility choices are too large, then it will become too expensive for the stakeholder to extend the guarantee.

To resolve this tension, I introduce commitment via some form of regulation that I will refer as a "risk management rule". Commitment serves the purpose of reassuring the stakeholder that the firm will not exploit the implicit protection.

To expedite the presentation of the key results, this section makes several simplifying assumptions: Specifically, once the firm is started shareholders have the ability to precommit costlessly and perfectly as to how future volatility will be determined. Furthermore, shareholders can only formulate Markovian commitments, i.e. the promised volatility choice depends exclusively on the state variable W_t. Finally, the risk management rule is determined in a shareholder-value maximizing way.

Section 5 relaxes all these simplifying assumptions by allowing arbitrary adapted policies (i.e. not necessarily Markovian policies). Furthermore, that section allows for the possibility of imperfect and costly commitment, in the sense that the risk management rule can be circumvented at a cost. Section 6 discusses the case where the stakeholder can impose the risk management rule on the shareholders via regulation or law. The main result of sections 5 and 6 is that the features of the optimal risk management rule are not altered by these extensions.

In light of the simplifying assumptions introduced in the present section, determining the optimal risk management rule amounts to solving the following problem.

Problem 1 *Let $\mathcal{M}$ denote the class of Markovian policies, i.e. policies of the form* $\sigma_t =$

$f(W_t)$ for some $f : [L, \infty) \to [\sigma_1, \sigma_2]$. Then for any W_t, choose σ so as to maximize

$$\max_{\sigma \in \mathcal{M}} P\left(W_t; \sigma\right) \tag{10}$$

subject to the constraint

$$P(L) \leq B. \tag{11}$$

In light of (6) maximizing $P\left(W_t; \sigma\right)$ is equivalent to maximizing shareholder value. Hence, the objective (10) is the familiar shareholder value maximization objective, while (11) simply re-states the stakeholder's participation constraint (4) taking into account that the Markovian nature of the volatility policies makes P_t also Markovian.

3 Solution

3.1 The set of feasible payoffs

The first step towards solving problem 1 is to characterize the set of payoff functions $P(W)$ that can be attained by $\sigma(W) \in \mathcal{M}$, while also satisfying (11). This is the purpose of the next Lemma.

Lemma 3 *Let the payoff function P be defined as in (5), and assume that it satisfies constraint (11). Then the following results hold for any $\sigma(W) \in \mathcal{M}$:*

1. *In the domain (L, ∞), P satisfies the ordinary differential equation*

$$\frac{\sigma^2(W)}{2} W^2 P_{WW} + r P_W W - (r + \lambda) P = 0 \tag{12}$$

2. *P is within the bounds $0 \leq P(W) \leq B$ for all $W \in [L, \infty)$. At $+\infty$ the function P satisfies $\lim_{W \to \infty} P(W) = 0$*

3. $P \in \mathcal{C}^1$ *and the derivatives of* P *satisfy* $P_W(L) = -1$, $P_W < 0, P_{WW} > 0$.

Lemma 3 states several properties of any feasible payoff function. The first property is a familiar Black-Scholes type differential equation. Heuristically, it can be derived by observing that P_t is a "claim" whose rate of appreciation in the domain $[L, \infty)$ is equal to the sum of the interest rate r and the hazard rate of termination λ

$$\frac{dE(P_t)}{dt} = (r + \lambda) P_t \tag{13}$$

Using Ito's Lemma, $\frac{dE(P_t)}{dt}$ can be expressed as $\frac{\sigma^2(W)}{2}W^2 P_{WW} + rP_W W$. Combining Ito's Lemma with (13) leads to (12).

Property 2 in Lemma 3 places upper and lower bounds on the set of feasible payoffs.[20] Property 3 has a somewhat more intricate proof, which is given in the appendix. It is however straightforward to give a heuristic intuition for $P_W(L) = -1$. Consider a situation where the assets of the firm fall below L by a small amount ε. Assuming participation compatibility, the stakeholder will intervene in order to restore the assets back to L by making a transfer of ε. Therefore, it is as if the "claim" P pays a "dividend" ε and the state variable W is reset back to L. Therefore $P(L - \varepsilon) = \varepsilon + P(L)$. Expanding the left hand side of this equation in a Taylor fashion around L gives $P(L) - \varepsilon P_W(L) = \varepsilon + P(L)$. Cancelling $P(L)$ from both sides and dividing by ε gives $P_W(L) = -1$.

[20] To see why P will always be between those two bounds, fix a $t \geq \tau_0$ and let τ^L be the first time (after t) such that $W_{\tau^L} = L$. Then

$$P(W_t) = E_t\left(e^{-(r+\lambda)(\tau^L - t)} P_{\tau^L}\right) \leq E_t\left(e^{-(r+\lambda)(\tau^L - t)} B\right) \leq B \tag{14}$$

The first equality in (14) follows from $dG_s = 0$ for all $s \in [t, \tau^L]$. The first inequality in (14) follows by constraint (11) and the second inequality follows since $e^{-(r+\lambda)(\tau^L - t)} \leq 1$.

3.2 The optimization problem as a regular optimal control problem

Note that $P(W_t)$ in the maximization problem 1 can be rewritten as

$$P(W_t) = P(L) + \int_L^\infty P'(x) 1\{x < W_t\} dx$$

$1\{x < W_t\}$ is an indicator function taking the value 1 if $x < W_t$ and 0 otherwise. Assuming that the participation constraint binds, $P(L) = B$.(Lemma 6 in section 5 verifies that this constraint optimally binds). Furthermore, using the characterization of all attainable payoffs from Lemma 3, one can rewrite the optimization problem 1 as a standard (deterministic) optimal control problem using (P, P') as state variables

$$\max_{\sigma(x)} \int_L^\infty P'(x) 1\{x < W_t\} dx \tag{15}$$

$$\begin{bmatrix} P' \\ P'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2(r+\lambda)}{\sigma^2}\frac{1}{x^2} & -\frac{2r}{\sigma^2}\frac{1}{x} \end{bmatrix} \begin{bmatrix} P \\ P' \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} P(L) \\ P'(L) \end{bmatrix} = \begin{bmatrix} B \\ -1 \end{bmatrix}, \lim_{x\to\infty} P(x) = 0 \tag{17}$$

Equation (16) is simply a transformation of the second order equation (12) into a system of two first order ordinary differential equations, while equation (17) gives the boundary conditions of the state variables (P, P') at L and ∞.

Letting π_1, π_2 denote the co-state variables for the two state variables (P, P'), the Hamiltonian for this optimal control problem is

$$H = 1\{x < W_t\} P'(x) + \pi_1 P'(x) + \pi_2 \frac{2}{\sigma^2}\left((r+\lambda) P(x)\frac{1}{x^2} - rP'(x)\frac{1}{x}\right) \tag{18}$$

The fact that $P'' > 0$ (by Lemma 3) implies that $\left((r+\lambda) P(x)\frac{1}{x^2} - rP'(x)\frac{1}{x}\right) > 0$ (by [12]).

Hence, maximizing H w.r.t. σ gives the optimal policy

$$\sigma^*(x) = \begin{cases} \sigma_1 \text{ if } \pi_2 > 0 \\ \sigma_2 \text{ if } \pi_2 < 0 \end{cases} \tag{19}$$

By standard optimal control theory, the co-state variables must satisfy:

$$\dot{\pi}_1 = -\frac{2(r+\lambda)}{[\sigma^*(x)]^2}\pi_2\frac{1}{x^2} \tag{20}$$

$$\dot{\pi}_2 = -(\pi_1 + 1\{x < W_t\}) + r\frac{2}{[\sigma^*(x)]^2}\pi_2\frac{1}{x} \tag{21}$$

The form of (19) suggests that the optimal policy will have a "bang-bang" form, with a switch at the point W^* where π_2 changes sign. Motivated by this observation, a reasonable conjecture is that the optimal policy is of the form

$$\sigma^*(x) = \begin{cases} \sigma_1 \text{ if } x < W^* \\ \sigma_2 \text{ if } x \geq W^* \end{cases} \tag{22}$$

for an appropriately chosen constant W^*. The next Lemma uses policy (22) to determine a closed form solution for $P(W_t;\sigma^*)$ taking an arbitrary $W^* \geq L$ as given. It then determines W^* in such a way as to satisfy the boundary condition $P(L;\sigma^*) = B$. In a final step, Proposition 1 verifies that (22) solves (19), and hence is optimal.

Lemma 4 *Take an arbitrary $W^* > L$ and suppose that shareholders adopt policy σ^* of equation (22). Then $P(W_t;\sigma^*)$ is given by*

$$\frac{P(W_t;\sigma^*)}{L} = \begin{cases} \left(\frac{W_t}{L}\right)^{\alpha_1^+} \dfrac{\frac{\alpha_2^- - \alpha_1^-}{\alpha_2^- - \alpha_1^+}\left(\frac{L}{W^*}\right)^{\alpha_1^+ - \alpha_1^-}}{\left[\alpha_1^- - \alpha_1^+ \frac{\alpha_2^- - \alpha_1^-}{\alpha_2^- - \alpha_1^+}\left(\frac{L}{W^*}\right)^{\alpha_1^+ - \alpha_1^-}\right]} - \left(\frac{W_t}{L}\right)^{\alpha_1^-} \dfrac{1}{\left[\alpha_1^- - \alpha_1^+ \frac{\alpha_2^- - \alpha_1^-}{\alpha_2^- - \alpha_1^+}\left(\frac{L}{W^*}\right)^{\alpha_1^+ - \alpha_1^-}\right]} & \text{if } L \leq W_t \leq W^* \\ \dfrac{\left[\frac{\alpha_2^- - \alpha_1^-}{\alpha_2^- - \alpha_1^+} - 1\right]\left(\frac{L}{W^*}\right)^{\alpha_2^- - \alpha_1^-}}{\left[\alpha_1^- - \alpha_1^+ \frac{\alpha_2^- - \alpha_1^-}{\alpha_2^- - \alpha_1^+}\left(\frac{L}{W^*}\right)^{\alpha_1^+ - \alpha_1^-}\right]}\left(\frac{W_t}{L}\right)^{\alpha_2^-} & \text{if } W_t > W^* \end{cases} \tag{23}$$

where

$$\alpha_1^{\pm} = \frac{-\left(r - \frac{\sigma_1^2}{2}\right) \pm \sqrt{\left(r - \frac{\sigma_1^2}{2}\right)^2 + 2\sigma_1^2\left(r + \lambda\right)}}{\sigma_1^2} \text{ and } \alpha_2^{\pm} = \frac{-\left(r - \frac{\sigma_2^2}{2}\right) \pm \sqrt{\left(r - \frac{\sigma_2^2}{2}\right)^2 + 2\sigma_2^2\left(r + \lambda\right)}}{\sigma_2^2} \tag{24}$$

Accordingly, $P(L;\sigma^*) = B$ *if and only if* W^* *is chosen as:*

$$W^* = L\left[\left(\frac{\alpha_2^- - \alpha_1^+}{\alpha_2^- - \alpha_1^-}\right)\frac{\left(1 + \frac{B}{L}\alpha_1^-\right)}{\left(1 + \frac{B}{L}\alpha_1^+\right)}\right]^{\frac{1}{\alpha_1^- - \alpha_1^+}} \tag{25}$$

Lemma 4 determines the appropriate value of W^*, that makes the policy of equation (22) satisfy (11). The appendix solves the differential equations (20), (21), and confirms that this policy also satisfies (19). It becomes then straightforward to establish the optimality of policy σ_t^* :

Proposition 1 *Let* σ_t^* *be defined as in (22) with* W^* *given by (25). Then*

$$P\left(W_t;\sigma_t^*\right) \geq P\left(W_t;\sigma_t\right)$$

for any volatility policy $\sigma_t \in \mathcal{M}$ and for any $W_t \geq L$. *The inequality becomes an equality if* $\sigma_t = \sigma_t^*$.

This proposition implies that the firm will always follow a simple policy: keep volatility at the lower bound σ_1 while $W_t \leq W^*$, and then switch to maximal volatility σ_2 if current assets W_t exceed W^*. The critical wealth level W^* is determined in such a way as to make the key constraint (11) hold as an equality.

Why is it optimal for the firm to lower, instead of raise volatility as its net worth declines ? To see intuitively why, let τ_0 be a time when $W_{\tau_0} = L$, fix a level $W_1 > L$ and let τ_1 be the first time after τ_0 such that $W_{\tau_1} = W_1$. Because of (4), the continuation value P_{τ_1} must satisfy the constraint

$$E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)}dG_s\right) + P_{\tau_1}E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right) \leq B,$$

which implies that

$$P_{\tau_1} \leq \frac{B - E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)}dG_s\right)}{E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right)} \leq \frac{B - \min_{\sigma_{s\in[\tau_0,\tau_1]}} E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)}dG_s\right)}{\min_{\sigma_{s\in[\tau_0,\tau_1]}} E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right)}. \quad (26)$$

The rightmost expression of equation (26) gives an upper bound to the continuation value that can be assigned at time τ_1. It is possible to show that the solution to the two minimization problems of equation (26) is obtained by setting $\sigma_{s\in[\tau_0,\tau_1]} = \sigma_1$. This is intuitive: By setting volatility at the lowest level between τ_0 and τ_1, it becomes possible to obtain the highest possible guarantee value at time τ_1, while still satisfying (4). Of course, after a certain point, the optimal risk management rule needs to switch to high levels of volatility in order to "deliver" on these high continuation values to the shareholders.

Proposition 1 is the main result of the paper. It illustrates that the optimal risk management rule is to set high volatilities when the firm is doing well (in the sense that it has a high net worth) and reduce volatility when the firm's net worth declines. The rest of the paper studies how general is this conclusion. The next section describes variations and extensions of the baseline model, that take into account the many forms that real-world bailouts take. Section 5 introduces costly and imperfect commitment and shows that Markovian policies are optimal. The last two sections develop implications of costly and imperfect commitment. Section 6 discusses cases, where the stakeholder has all the bargaining power and can impose the risk management rule via law or regulation, while Section 7 discusses implications of the model for times prior to the first bailout.

4 Extensions and Discussion

4.1 Absence of a stakeholder and debt forgiveness

It seems reasonable to ask if the model's predictions carry through even in cases where no stakeholder is present. To answer this question, this subsection discusses a simple model with only shareholders and debtholders. It is possible to show in a simple and stylized variant of the previous model that optimal principal writedowns can produce effects that are similar to bailouts.[21]

Specifically, this subsection keeps all the assumptions made sofar with the exceptions that 1) there is no stakeholder and hence $dG_s = 0$, 2) endogenous liquidation makes the firm's assets instantaneously decline by a fraction $b < 1$, i.e. $W_{\tau^+} = (1-b)\, W_{\tau^-}$, and most importantly 3) whenever $W_\tau < L_\tau$ the debtholders either have the option to declare the firm bankrupt and obtain its assets that are worth $(1-b)\, W_\tau$, or they can forgive debt $\Delta L_\tau < 0$ so as to ensure that $L_\tau + \Delta L_\tau = W_\tau$. Under these assumptions, if a firm has survived by time t, it means that L_t is given by

$$L_t = \min\{L_0, \min_{0\le s\le t} W_s\}.$$

A *sufficient* condition to induce debtholders to always prefer to write down principal is that for every time τ such that $W_\tau = L_\tau$, the value of a bankrupt firm is less than the anticipated present value of interest and principal payments. Mathematically,

$$(1-b)\, W_\tau \le E_\tau \int_\tau^{\tau^L} e^{-r(s-\tau)}(rL_s)ds + E_\tau\left[e^{-r(\tau^L-\tau)} L_{\tau^L}\right] \text{ for all } \tau : W_\tau = L_\tau \tag{27}$$

Applying integration by parts on the right hand side of (27), using the fact that $W_\tau = L_\tau$,

[21] For a rich and tractable model that studies allocation of cash flows and strategic debt service within a dynamic valuation framework, see Anderson and Sundaresan (1996). Sundaresan and Wang (2007) study strategic debt service in the presence of real options.

and simplifying gives the simpler condition:

$$-E_{\tau}\int_{\tau}^{\tau^{L}} e^{-r(s-\tau)}dL_s \leq bL_{\tau}. \tag{28}$$

Notice that (28) has a form that is quite similar to (4) except that dG_s is replaced by $-dL_s$ (recall that $dL_s < 0$) and the right hand side is proportional to L_{τ}.

Assuming that equation (28) will always be satisfied, shareholder value is given by

$$V(W_t, L_t) = W_t - E_t\int_t^{\tau^L} e^{-r(s-t)}(rL_s)ds + E_t\left[e^{-r(\tau^L - t)}\left(W_{\tau^L} - L_{\tau^L}\right)\right] = W_t - L_t + \widehat{P}(W_t, L_t)$$

where $\widehat{P}(W_t, L_t)$ is defined as

$$\widehat{P}(W_t, L_t) \equiv -E_t\int_t^{\tau^L} e^{-r(s-t)}dL_s. \tag{29}$$

Assuming that one can constrain attention to volatility policies that set σ_t as a function of the asset to liability ratio $w_t \equiv \frac{W_t}{L_t}$, then the following analog of Lemma 3 obtains.

Lemma 5 *Let the payoff function $\widehat{P}$ be defined as in (29), and assume that it satisfies (28). Then the following results hold for any $\sigma(w_t)$*

1. *There exists a function $p(w_t)$ such that $\widehat{P}(W_t, L_t) = L_t p(w_t)$. In the domain $(1, \infty)$, $p(w_t)$ satisfies the ordinary differential equation*

$$\frac{\sigma^2(w)}{2}w^2 p_{ww} + rp_w w - (r+\lambda)p = 0$$

2. *p is within the bounds $0 \leq p(w) \leq b$ for all $w \in [1, \infty)$. At $+\infty$ the function p satisfies $\lim_{w\to\infty} p(w) = 0$.*

3. *$p \in \mathcal{C}^1$ and the derivatives of p satisfy $p_w(1) = -1 + p(1)$, $p_w < 0, p_{ww} > 0$.*

The proof of this Lemma is practically identical to the proof of Lemma 3 and is therefore omitted.[22]

Given the similarity between the characteristics of feasible payoffs of Lemma 3 and Lemma 5, the volatility policy that maximizes $p(w_t)$ has a similar form. Indeed, assuming that

$$\frac{1}{1-\alpha_1^-} \leq b \leq \frac{1}{1-\alpha_2^-},$$

and repeating the same logic of section 3, one obtains that the optimal policy is given by

$$\sigma^*(x) = \begin{cases} \sigma_1 \text{ if } x \in \left[1, \left\{\left(\frac{\alpha_2^- - \alpha_1^+}{\alpha_2^- - \alpha_1^-}\right)\left(\frac{1+b\alpha_1^- - b}{1+b\alpha_1^+ - b}\right)\right\}^{\frac{1}{\alpha_1^- - \alpha_1^+}}\right] \\ \sigma_2 \text{ if } x \geq \left\{\left(\frac{\alpha_2^- - \alpha_1^+}{\alpha_2^- - \alpha_1^-}\right)\left(\frac{1+b\alpha_1^- - b}{1+b\alpha_1^+ - b}\right)\right\}^{\frac{1}{\alpha_1^- - \alpha_1^+}} \end{cases}$$

Notice that the optimal policy has the same familiar form as before: choose high volatilities when the asset to liability ratio is high and switch to low volatilities when the asset to liability ratio is low.

In summary, this section showed that the assumption of a stakeholder is not crucial for the key results. All that is required is a discrete drop in the value of assets upon bankruptcy. Of course in reality debt renegotiations are much more complex than outright principal writedowns. In most cases the existing debt is exchanged for some other claim (for instance some debt with alternative terms, equity etc.) The next subsection discusses such cases.

[22] The only part that requires some explanation is the fact $p_w = -1 + p(1)$. This follows from the fact that when W_t declines below L_t by a small ε we obtain $\widehat{P}(L_t - \varepsilon, L_t) = \varepsilon + \widehat{P}(L_t - \varepsilon, L_t - \varepsilon)$. (This follows because once W_t becomes smaller than L by $\varepsilon > 0$, the assets W_t don't change but L_t is reset downward by ε.) Taking a first order Taylor expansion leads to $\widehat{P}(L_t - \varepsilon, L_t) = \widehat{P}(L_t - \varepsilon, L_t - \varepsilon) + \varepsilon \widehat{P}_L(L_t - \varepsilon, L_t - \varepsilon) + o(\varepsilon)$ where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Combining the last two equations gives $\varepsilon \widehat{P}_L(L_t - \varepsilon, L_t - \varepsilon) + o(\varepsilon) = \varepsilon$. Dividing by ε and sending $\varepsilon \to 0$ leads to $\widehat{P}_L(L_t, L_t) = 1$. Since $\widehat{P}(W_t, L_t) = L_t p(w_t)$, and $w_t = \frac{W_t}{L_t}$ it follows at once that $\widehat{P}_L(L_t, L_t) = p(1) - p'(1)$ and therefore $p_w(1) = -1 + p(1)$. Since any optimal policy should make the constraint (28) binding, it follows that $p(1) = b$ and $p_w(1) = -1 + b$.

4.2 Bailouts and Security Issuance

In the baseline model bailouts have taken the form of direct transfers. In reality, the stakeholder undertaking the bailout often obtains some form of security in exchange for injecting funds. To have a simple example, return to the baseline model and suppose that the first time that $W_t = L$, the firm gives the stakeholder a claim to a share $x < 1$ of the firm's liquidating dividends in exchange for receiving the transfer process G_t as described by equation (3).

Since the shareholder value of the firm is given by (6), and $W_t = L$, the total value of the firm (including the newly issued shares) is P_t. By obtaining a fraction x of the firm's shares, the cost of the bailout to the shareholder is reduced to $(1-x)\,P_t$ and hence the constraint (4) becomes

$$(1-x)\,E_t\left(\int_t^{\tau^l} e^{-r(s-t)}dG_s|W_t = L\right) \leq B \tag{30}$$

More importantly, even though the value of the firm to the original shareholders is now xP_t, the same volatility commitments that maximize P_t will maximize xP_t, and all the analysis of the paper can be repeated after replacing the constraint (4) with (30).

An alternative way to think about equation (30) is as follows: Through the bailouts, the stakeholder "pays" an amount P_t (this is the net present value of the transfers) to purchase shares that are worth $(1-x)\,P_t$. Hence, she effectively purchases over-priced shares, which results in a net transfer to the original shareholders. Whether the transfer takes the form of outright cash injections, or purchases of "over-priced" shares has no material consequence for the analysis.

It should also be clear that the above argument does not depend on the type of claim that stakeholders obtain. As long as a) the firm survives once $W_t = L$, b) the shareholder retains some non-zero cash flow rights on the liquidating dividend, and c) the debtholder gets repaid capital and interest in all states of the world, then the stakeholder must be purchasing overpriced securities.

Even though the argument of this subsection was developed in the context of the baseline model, it carries over also to the case of principal writedowns (section 4.1). In particular, the conclusions of this section also apply to the setups, where the debtholders write down principal in exchange for obtaining equity - a common outcome of renegotiations to avoid bankruptcy.

Finally, a practical implication of allowing security issuance during a bailout is that the guarantee-extending party can even gain from a bailout *ex-post*, even though *ex-ante* the bailout is not profitable.

4.3 Temporary Externalities

Realistically, it is sometimes observed that firms wind down after being bailed out. The present framework allows for this, if one assumes that externalities are temporary. In reality, one motivation for bailouts is that there is a temporary shortage of liquidity in the market that could produce large termination costs B. With the passage of time however, and with active search for buyers for the firm's assets, these costs could be eliminated.

To model such a situation in the simplest possible way, suppose that after bailing out a firm (i.e. the first time that $W_t = L$), the stakeholder can start a project to reduce the termination cost B (such as searching for an appropriate buyer for the firm's assets). This project has a constant hazard rate of success β. However, if the project succeeds, it can reduce the externality from B down to 0.[23]

This modification of the setup can be easily accommodated by changing equation (12) to

$$\frac{\sigma^2(W)}{2} W^2 P_{WW} + r P_W W - (r + \lambda + \beta) P = 0. \tag{31}$$

Equation (31) asserts that once the stakeholder starts her efforts to eliminate the externality, the overall hazard rate of guarantee termination is $\lambda + \beta$. This stylized setup shows

[23] The results also go through when B drops to a level smaller than $\frac{L}{|\alpha(\sigma_1)|}$, because then assumption 1 is violated.

that if bailouts are associated with active efforts by the stakeholder to reduce the externality produced by the firm, then one should expect that bailouts will be temporary phenomena.

4.4 Bailouts through mergers and acquisitions

Especially in government sponsored bailouts, there is pressure for the government to not bailout the existing shareholders for reasons of "fairness". However, in many such cases the government may still try to salvage the company by finding a buyer, who acquires the company with all of its assets and its liabilities. In such a situation the government and the buyer typically enter negotiations. The joint surplus that the government and the buyer can obtain is B, i.e. the cost of liquidating the firm. In these negotiations the government may agree (implicitly or explicitly) to provide transfers to the buyer in order to incentivize him or her to acquire the company[24] and the buyer will have an incentive to give a volatility promise that satisfies the constraint (4). From this point on, one can simply repeat the analysis of the baseline version of the model. Furthermore, if political redistribution concerns imply that the amount of time that the new shareholders can enjoy the government's protection is exponentially distributed with parameter β, then the model from the perspective of the new buyer becomes formally equivalent to the model of temporary externalities of the subsection 4.3.[25]

In summary, the model's qualitative predictions do not depend on whether it is the existing shareholders or the new buyers who enter the volatility management commitment. As long as there is a surplus to be had, negotiations between the government and the acquiror will give the acquiror the incentives to enter a volatility commitment. (See also section 6 for the case where the government has some bargaining power.)

4.5 Intermediate Dividends

[24] For instance in the recent bailout of Bear Stearns by JP Morgan, the government agreed to provide a muti-billion dollar credit line to JP Morgan in order to provide incentives for the acquisition.

[25] One can extend the model in a straightforward way to allow for repeated purchases of the company by newly arriving buyers, once the "grace" period of past buyers has elapsed and the company is about to become bankrupt again.

Finally, the assumption that the firm only distributes a terminal dividend can easily be relaxed. Specifically, assume that the firm distributes a constant fraction δ of its assets as dividends to shareholders. In this case, equation (2) becomes

$$dW_t = (r - \delta) W_t dt + \sigma_t W_t dZ_t + dG_t.$$

The expression for shareholder value, however, is still given by (6). The reason is that the total value of the firm is still given by $W_t + P_t$ and the claim of the debtholders is still L. The only subtlety introduced by intermediate dividends is that equation (12) becomes

$$\frac{\sigma^2(W)}{2} W^2 P_{WW} + (r - \delta) P_W W - (r + \lambda) P = 0.$$

Accordingly, letting $\widetilde{r} = r - \delta$ and $\widetilde{\lambda} = \lambda + \delta$, all the expressions obtained in Lemma 4 and Proposition 1 continue to apply in the presence of intermediate dividends with $\widetilde{r}$ replacing r and $\widetilde{\lambda}$ replacing λ.[26]

5 Imperfect and Costly commitment

So far, the paper has only considered Markovian policies. A voluminous literature restricts attention to Markovian policies on a priori informational grounds.[27] However, a more important reason why the analysis sofar has focused attention on such policies, is that Markovian policies turn out to be optimal in the space of all (possibly history-dependent) policies as long as commitment is imperfect and costly rather than perfect and costless.

To introduce the notion of imperfect and costly commitment, let τ_0 be the first time that

[26] A more complex issue, that is beyond the scope of this paper, arises if dividends are assumed to be endogenous. Then the shareholders have an incentive to distribute excessive dividends in order to maximize the value of the guarantee. In that case the optimal commitment to the stakeholder needs to include provisions not only about volatility choice, but about dividend distribution as well.

[27] See e.g. Chapter 13 in the textbook of Fudenberg and Tirole (1991) for a list of papers that restrict player's actions to be Markovian. In the case where the players are firms, this restriction to "memory-less" strategies is routinely motivated by the fact that firms -unlike individuals- are run by continuously changing managers who may not have full access to the past history. However, they do know the current state variables.

$W_{\tau_0} = L$, so that the stakeholder needs to form a view as to how the firm will set volatility in the future. Importantly, from this point onward shareholders will be able to choose arbitrary adapted volatility policies.

In reality, commitment is likely to be imperfect and costly. To model these notions, I assume that any risk management rule can be circumvented (i.e. abandoned) by future shareholders at a cost $I > 0$. For instance, in a world of imperfect accounting, shareholders can pay a cost and create legal entities or invest in off-balance sheet items that make it hard to assess the value of the firm's assets, thus allowing them to take extra risk.

To endogeneize the extent of commitment, I assume that the parties currently involved in the formulation of a risk management rule can choose ex-ante the penalty I that future shareholders will have to pay if they choose to deviate. In most real-world examples such a choice is achieved by restricting the firm's ability to invest in certain instruments. As already noted, such restrictions can be circumvented -say by creating new legal entities. However, by making the restrictions more stringent, the cost of circumventing them is likely to become higher. In that sense, by making the restrictions more stringent, the current shareholders and stakeholders can choose how costly it will be to circumvent the risk management rule.

Clearly, the higher I, the more credible any risk management rule will become. However, it also seems plausible that rules that impose a high I may have other unintended distortions: For instance, by making it very costly for a firm to engage in certain types of transactions (such as derivatives, off-balance sheet and off-shore transactions) it also becomes hard for the firm to use these instruments for tax-planning or risk sharing. To capture the idea that more "draconian" commitment devices lead to more distortions, I assume that if future shareholders' cost of reneging is set to some level I, then current shareholders will incur a monetary deadweight cost equal to kI, where $k \in (0, 1)$. This is the "implementation cost" associated with the risk management rule.

A novel implication of this setup is that since commitment is costly, both the rules and the cost of reneging are jointly and endogenously determined. Different choices of I between 0 and infinity span the spectrum between no commitment and limitless commitment.

The definitions that follow formalize the notions described above. In this section, shareholders have all the bargaining power and hence can choose both the risk management rule and the penalties associated with deviation (self-regulation). The next section discusses the opposite case, whereby the stakeholders can impose both the risk management rule and the penalties via law or regulation.

In the following definitions, the notation $\sigma_{s>t}$ refers to the volatility process that is adopted after time t. Importantly, $\sigma_{s>t}$ can be an arbitrary adapted process (i.e. not necessarily Markovian).

Definition 1 *Let τ^L be any time such that $W_{\tau^L} = L$. Then a volatility process σ_s is* participation compatible *if*

$$P\left(L; \sigma_{s \geq \tau^L}\right) \leq B \tag{32}$$

Clearly, a stakeholder will never agree to bail out a firm at time τ^L unless constraint (32) is satisfied. The next definition captures the idea of commitment credibility.

Definition 2 *Let τ^0 be the time at which the commitment is entered. Fix a $t > \tau_0$ and let χ be the first time after t, such that $W_\chi = L$. For a given level of I, a volatility process $\sigma_{s \geq \tau^0}$ is* credible *if for all t and W_t*

$$P\left(W_t; \sigma_{s \geq t}\right) \geq \sup_{\sigma_{s \in (t,\chi)}} E e^{-(r+\lambda)(\chi - t)} \left[\sup_{\sigma_{s \geq \chi}} P(L; \sigma_{s \geq \chi}) - kI\right] - I \tag{33}$$

and $\sigma_{s \geq \chi}$ is participation compatible.

Definition 2 captures the simple notion that the value of the guarantee under commitment and/or the cost I should always be large enough, so that future shareholders will not find it optimal to pay the cost I and then reset the volatility from that point on. The term inside square brackets is the value that the shareholder can obtain by re-entering a new promise at some future time $\chi > t$, at which time the shareholders and the stakeholder will have to contemplate a new bailout. The term $\sup_{\sigma_{s\in(t,\chi)}} Ee^{-(r+\lambda)(\chi-t)}$ captures the idea that shareholders can choose volatility freely between t and χ, if they choose to renege.

Definition 2 implies that for any given volatility process $\sigma_{s\geq t}$, there exists a minimal amount $\widehat{I}(\sigma_{s\geq t})$ that will make that volatility process credible.[28]

With all these definitions in hand, it is now possible to give the definition of an optimal volatility process.

Definition 3 *Let* $\widehat{I}(\sigma) = \min I \in [0,\infty)$ *such that* $\sigma_{s\geq\tau_0}$ *is credible. A volatility process* $\sigma^*_{s\geq\tau^0}$ *is optimal if it is participation compatible, and*

$$P\left(L;\sigma^*_{s\geq\tau^0}\right) - k\widehat{I}\left(\sigma^*_{s\geq\tau^0}\right) \geq P\left(L;\sigma_{s\geq\tau^0}\right) - k\widehat{I}\left(\sigma_{s\geq\tau^0}\right) \tag{34}$$

for any other participation compatible $\sigma_{s\geq\tau^0}$.

According to this definition, a process $\sigma^*_{s\geq\tau^0}$ is optimal if it is participation compatible and maximizes the value of the implicit guarantee net of the costs that are required to ensure its credibility.

It will be useful at this stage to make a conjecture, that is verified later. Let τ^L denote any time at which $W_{\tau^L} = L$. Then

$$P(L;\sigma^*_{s\geq\tau^L}) = B \tag{35}$$

In particular the conjecture (35) applies to time τ^0.

Given conjecture (35), the search for the optimal commitment $\sigma^*_{s\geq\tau^0}$ amounts to minimizing $\widehat{I}(\sigma_{s\geq\tau^0})$ over all policies that satisfy (35). Intuitively, shareholders would like to put the stakeholder against her participation constraint, while keeping the implementation cost $k\widehat{I}(\sigma_{s\geq\tau^0})$ as low as possible. An additional implication of (35) is that equation (33)

[28] Since the volatilities are bounded, both the left hand side and the second term on the right hand side are bounded.

becomes

$$P\left(W_t;\sigma_{s\geq t}\right)\geq\left[B-kI\right]\sup_{\sigma_{s\in(t,\chi)}}Ee^{-(r+\lambda)(\chi-t)}-I=\left[B-kI\right]\left(\frac{W_t}{L}\right)^{\alpha(\sigma_2)}-I \tag{36}$$

The rightmost equality of (36) asserts that once shareholders renege, they will set volatility to the highest possible level, until they have to negotiate with the stakeholder again.[29] Given this constant volatility choice, the expression $Ee^{-(r+\lambda)(\chi-t)}$ has a simple closed form expression[30] given by $(W_t/L)^{\alpha(\sigma_2)}$.

Re-arranging equation (36) and recognizing that $\widehat{I}$ will have to be determined so that (36) holds at all times and for all levels of W_t gives

$$\widehat{I}\left(\sigma_{s\geq\tau^0}\right)=\max_{W_t\geq L}\frac{B\left(\frac{W_t}{L}\right)^{\alpha(\sigma_2)}-\inf_{t\geq\tau_0}P\left(W_t;\sigma_{s\geq t}\right)}{1+k\left(\frac{W_t}{L}\right)^{\alpha(\sigma_2)}} \tag{37}$$

Under conjecture (35), the aim of the shareholders is to choose a volatility process that will minimize $\widehat{I}$, while satisfying (35).

Equation (37) implies that $\widehat{I}$ is a decreasing function of the continuation values $P\left(W_t;\sigma_{s\geq t}\right)$. Between two participation compatible commitments, the one that implies higher continuation values at each point in time, will be preferred since shareholders in the future will be less tempted to renege. Accordingly, $\widehat{I}$ will be lower.

An important implication of (37) is that shareholders at two times t_1 and t_2 such that $W_{t_1}=W_{t_2}$ are treated symmetrically; equation (37) implies that it is only $\min(P_{t_1},P_{t_2})$ that matters for the determination of $\widehat{I}\left(\sigma_{s\geq\tau^0}\right)$, whether $t_1<t_2$ or $t_2>t_1$. This time invariance makes Markovian policies (that set by definition $P_{t_1}=P_{t_2}$) optimal. Formally, this is shown in the next proposition.

Proposition 2 *For* any *(potentially non-Markovian)* participation compatible *policy* σ, *there exists a lower bound* $\widehat{I}^*$, *such that* $\widehat{I}^*\leq\widehat{I}(\sigma)$. *Finally, for the Markovian policy* σ^*

[29] The proof of this fact follows the same steps as Lemma 2 and is omitted.

[30] See Øksendal (2003), p. 217.

of equation (22), one obtains $\widehat{I}^* = \widehat{I}(\sigma^*)$.

Proposition 2 shows that the policy of equation (22) is optimal, since it attains the lower bound $\widehat{I}^*$, while satisfying the constraint (35).[31][32]

The last step is to verify the conjecture (35).

Lemma 6 *It is always optimal for constraint (32) to hold as an equality.*

6 Allocation of bargaining power

In several realistic situations risk management rules are imposed by the stakeholder via regulation. Assuming costly commitment, this section shows that the distinction between regulation and self-regulation affects the rents of the two parties, but not the qualitative features of the optimal rule.

To be precise, suppose that the shareholders of the firm have some outside option when $W_\tau = L$. Such an outside option could be the result of legal difficulties in enforcing absolute priority, or more simply it could result from some scarce expertise that the shareholders can use elsewhere if the firm gets liquidated. Whatever the source, suppose that the monetary value of this outside option is $\widetilde{v} \in (P(L;\sigma_1), B)$. Since shareholders can always "walk away"

[31] An implication of Proposition 2 is that there exists a minimum amount of $\widehat{I}^*$, that needs to be paid in order to insure the credibility of any participation compatible policy. A corollary of Proposition 2 that can be proven in a similar way, is that it is impossible to find any participation compatible strategy unless the firm pays at least $\widehat{I}^*$. From a practical perspective this shows that the conclusions of the model are robust to how exactly one models credibility. Specifically, definition 2 requires that a commitment be credible at *all* states and dates. This assumption may seem too strong at first. However, the above argument implies that participation compatibility *alone and by itself* places a lower bound on I. The reason is intuitive. If I is not large enough, then the stakeholder anticipates that the shareholders will renege, which in turn increases the stakeholder's cost and makes it harder to satisfy the constraint (32). Hence, as long as one requires the rather unobjectionable property of participation compatibility, the proposed policy of this paper will be optimal.

[32] Proposition 2 does *not* assert that σ^* is the unique policy that attains the lower bound $\widehat{I}^*$. However, it does assert that no other participation compatible (potentially non-Markovian) policy can improve on the markovian policy σ^*.

with $\widetilde{v}$, it must be the case that

$$P(L;\sigma) - k\widehat{I}(\sigma) \geq \widetilde{v} \tag{38}$$

Now suppose that the stakeholder can determine the risk management rule and the associated punishments. Since the stakeholder is trying to minimize the value of the implicit guarantee, the stakeholder has an incentive to impose a risk management rule that will make equation (38) hold as an equality. However, as long as the assumptions of section 5 still hold, and shareholders could deviate from the prescribed policy at some cost I, then the optimal policy that minimizes $P(L;\sigma)$ is the one that minimizes the implementation cost $k\widehat{I}(\sigma)$. To see this, note that since (38) has to hold as an equality it follows that $P(L;\sigma) = \widetilde{v} + k\widehat{I}(\sigma)$ for any policy and hence $\min_\sigma P(L;\sigma) = \widetilde{v} + k \min_\sigma \widehat{I}(\sigma)$. (It should also be noted here, that it doesn't matter if the implementation cost $k\widehat{I}(\sigma)$ is "levied" on the shareholders or the stakeholder, since it simply reduces the joint surplus.)

From this point on, the entire analysis of the paper is applicable, with the only exception that the binding constraint $P(L) = \widetilde{v} + k\widehat{I}(\sigma)$ replaces the binding constraint (35). Since the optimal policy has always the same qualitative form irrespective of the value of $P(L)$, it follows that the optimal risk management rule remains qualitatively intact: choose low values of σ_1 when W_t is lower than some threshold, and choose σ_2 when W_t is above that threshold. However, the exact magnitude of the threshold, and hence the distribution of rents, does depend on whether it is the stakeholder or the shareholders that choose the optimal risk management rule.

In summary, the qualitative features of the optimal rule are invariant with respect to the party that chooses the rule. Be it through regulation or self-regulation, an optimal rule should "tempt" future shareholders as little as possible. By choosing a rule that postpones the high volatilities for times when net worth is high, the anticipated growth rate of the value of the guarantee is maximized. In turn, this reduces the temptation of future shareholders to circumvent the risk management rule, which results in smaller required punishments $\widehat{I}(\sigma)$ and smaller distortions $k\widehat{I}(\sigma)$.

7 The implications of the model for $t < \tau_0$

In section 5, the discussion focused on risk management rules that are imposed at τ_0, the time at which the firm may be threatened with bankruptcy. However, risk management rules are likely to take effect already at time $t < \tau_0$.

To give an example suppose that at time 0 shareholders have the ability to start a firm by making an equity contribution equal to $W_0 - L$. If started, the firm will produce a total gain equal to v_0 to the shareholder and U_0 to the stakeholder. For instance, if the stakeholder is the government, the new firm will produce a positive externality because it will pay taxes, reduce unemployment due to job market frictions etc. In a private sector context the new firm could be a joint venture between an established firm ("the stakeholder") and a smaller firm with scarce abilities ("the shareholders") that will lead to synergies. In either case, the termination of the newly created firm could lead to an externality B for the stakeholder as assumed throughout.

If the shareholders have an outside option with value equal to $\widetilde{v}_0$, (such as investing abroad) then the firm will only get started if

$$W_0 - L + v_0 + P\left(W_0\right) \geq W_0 - L + \widetilde{v}_0. \tag{39}$$

The left hand side of equation (39) is the value of the firm to shareholders once started, while the right hand side is the value of the outside option. To make the situation interesting, suppose that the stakeholder enjoys no benefit if the firm does not get created and that $v_0 < \widetilde{v}_0$ and $U_0 \geq \widetilde{v}_0 - v_0$. In such a situation, there is an incentive for the stakeholder to extend an implicit guarantee such that

$$\widetilde{v}_0 - v_0 \leq P_0 = E_0\left(\int_0^{\tau^l} e^{-r(s-t)} dG_s\right) \leq U_0. \tag{40}$$

Moreover, the risk management rule should also satisfy (32) since the stakeholder has the possibility of "walking away" in the future. The inequality (40) leaves room for many possible values of P_0. For instance, if shareholders have all the bargaining power then

$$P_0 = E_0 \left(\int_0^{\tau^l} e^{-r(s-t)} dG_s \right) = U_0, \tag{41}$$

while $P_0 = \widetilde{v}_0 - v_0$ when the stakeholder has all the bargaining power. I will focus on the first case, since the latter one can be handled analogously. Assuming that the shareholders (or the stakeholder) still have to incur costs $k\widehat{I}(\sigma)$ for any risk management rule σ that is adopted at time 0, then the policy that satisfies (41) as an equality and minimizes $\widehat{I}(\sigma)$ is given as follows.

Lemma 7 *Let $P^*(W) \equiv P(W;\sigma^*)$ be the value of the guarantee that is defined in Lemma 4 and let τ_0 be the first time that $W_{\tau_0} = L$. Letting W_0 denote the assets of the firm at $t = 0$, and assuming that*[33] *$U_0 \geq P^*(W_0)$, there exists a level of assets $\widehat{W}^* \leq W^*$, such that the optimal risk management rule prescribes the policy of equation (22) for $t \geq \tau_0$, and the following policy for $t < \tau_0$:*

$$\sigma^*(x) = \begin{cases} \sigma_1 \text{ if } x < \widehat{W}^* \\ \sigma_2 \text{ if } x \geq \widehat{W}^* \end{cases}$$

8 Conclusion

This paper presented a model, whereby a firm is bailed out so as to avoid costs associated with bankruptcy. The optimal actions for the stakeholder, the firm and the lenders were derived endogenously. Even though the presence of an implicit guarantee increases the shareholders' incentives to take risk, it also makes it more and more costly for the stakeholder to continue

[33] The assumption $U_0 \geq P^*(W_0)$ is made purely to simplify the proof and save space. A full discussion of the optimal policy for arbitrary values of U_0 is beyond the illustrative scope of this Lemma. An empirical advantage of this assumption is that a firm that hasn't received bailouts in the past will have less tight risk limits than firms that have received bailouts in the past, since $\widehat{W}^* \leq W^*$.

providing the implicit protection.[34]

The optimal risk management rule is to increase volatility when the firm's net worth is high and reduce volatility when its net worth declines. This policy reduces future shareholders' temptation to renege, when assets are safely above liabilities. At the same time, it keeps the stakeholder at her participation constraint.

The predictions of the model seem to be qualitatively in line with existing risk management practices that tighten risk limits in response to declining net worth. Therefore, the model provides a potential justification for existing risk management rules, and is consistent with empirical phenomena such as flight to quality.

[34] For a paper that shows that moral hazard is attenuated in an infinite horizon setting, see e.g. Panageas and Westerfield (2005).

A Appendix

Proof of Lemma 1. Let $\tau^{\overline{W}}$ denote the first passage time to some $\overline{W} > W_t > L$, defined as $\tau^{\overline{W}} = \inf_{s \geq t}\{s : W_s \geq \overline{W}\}$. Consider the price of a guarantee that is terminated at either the exogenous liquidation time τ^l or $\tau^{\overline{W}}$, whichever comes first

$$P^{(\overline{W})}\left(W_t; \overline{\sigma}\right) = E_t\left(\int_t^{\tau^{\overline{W}} \wedge \tau^l} e^{-r(s-t)} dG_s\right) \tag{42}$$

It is easiest to price this claim first and then take the limit as $\overline{W} \to \infty$ in order to arrive at (8). One can use standard results to express $P^{(\overline{W})}$ as

$$P^{(\overline{W})}\left(W_t; \overline{\sigma}\right) = E_t\left(\int_t^{\tau^{\overline{W}}} e^{-(r+\lambda)(s-t)} dG_s\right) \tag{43}$$

In order to construct $P^{(\overline{W})}$ it is simplest to start by searching for a function that satisfies the following properties

$$\begin{aligned} \frac{\overline{\sigma}^2 W_s^2}{2} P_{WW}^{(\overline{W})} + r P_W^{(\overline{W})} W_s - (r+\lambda) P^{(\overline{W})} &= 0 && (44) \\ P_W^{(\overline{W})}(L) &= -1 && (45) \\ P^{(\overline{W})}(\overline{W}) &= 0 && (46) \\ P_W^{(\overline{W})} &< \infty \text{ for all } W \in \left[L, \overline{W}\right] && (47) \end{aligned}$$

Finding a $P_W^{(\overline{W})}$ that satisfies (44), along with the boundary conditions (45) and (46) is straightforward. With α given by (7) and α^+ defined as

$$\alpha^+ = \frac{-(r - \frac{1}{2}\overline{\sigma}^2) + \sqrt{\left(r - \frac{\overline{\sigma}^2}{2}\right)^2 + 2\overline{\sigma}^2\left(r+\lambda\right)}}{\overline{\sigma}^2} > 0,$$

the general solution to (44) is:

$$P^{(\overline{W})}(W_t) = C_1 W_t^{\alpha} + C_2 W_t^{\alpha^+}$$

where C_1, C_2 are arbitrary constants. One needs to determine the constants C_1, C_2 so that (45) and (46) hold. Carrying out this computation, yields the following unique solution to (44), that satisfies (45), (46) and (47):

$$P^{(\overline{W})}(W_t) = \frac{\frac{L}{\alpha}\left(\frac{\overline{W}}{L}\right)^{\alpha} W_t^{\alpha^+} - \frac{L}{\alpha}\left(\frac{\overline{W}}{L}\right)^{\alpha} \overline{W}^{\alpha^+-\alpha} W_t^{\alpha}}{\overline{W}^{\alpha^+} - \frac{\alpha^+}{\alpha} L^{\alpha^+}\left(\frac{\overline{W}}{L}\right)^{\alpha}} \tag{48}$$

It is now straightforward to verify that (48) is the solution to (43). Applying Ito's Lemma to $P^{(\overline{W})}$ and taking expectations yields

$$\begin{aligned} 0 \;=\; & P^{(\overline{W})}(W_t) \\ & -E_t\left[e^{-(r+\lambda)(\tau^{\overline{W}}-t)} P^{(\overline{W})}(\overline{W})\right] + \\ & +E_t\left[\int_t^{\tau^{\overline{W}}} e^{-(r+\lambda)(s-t)}\left(\frac{\overline{\sigma}^2 W_s^2}{2} P_{WW}^{(\overline{W})} + r P_W^{(\overline{W})} W_s - (r+\lambda) P^{(\overline{W})}\right) ds\right] \\ & +E_t\left[\int_t^{\tau^{\overline{W}}} e^{-(r+\lambda)(s-t)} \overline{\sigma} P_W^{(\overline{W})} W_s dB_s\right] \\ & +E_t\left[\int_t^{\tau^{\overline{W}}} e^{-(r+\lambda)(s-t)} P_W^{(\overline{W})}(L) dG_s\right] \end{aligned} \tag{49}$$

The second line in (49) is zero because of (46). The third line is zero because of (44). The fourth line is zero because $\overline{\sigma} P_W^{(\overline{W})} W_s$ is bounded for all $W \in \left[L, \overline{W}\right]$ by (47). Hence (49) reduces to

$$P^{(\overline{W})}(W_t) = -E_t\left[\int_t^{\tau^{\overline{W}}} e^{-(r+\lambda)(s-t)} P_W^{(\overline{W})}(L) dG_s\right] \tag{50}$$

Combining (45) and (50) leads to (43).

To conclude the proof, let $\overline{W} \to \infty$ in equation (48) and apply the monotone convergence theorem to obtain $\lim_{\overline{W}\to\infty} P^{(\overline{W})}(W_t) = \ P\left(W_t\right) = -\frac{L}{\alpha}\left(\frac{W_t}{L}\right)^{\alpha}$. ■

Proof of Lemma 2. By Lemma 1, $P^{(\sigma_2)} = P(W;\sigma_2) = -\frac{L}{\alpha}\left(\frac{W_t}{L}\right)^{\alpha}$ is convex in W, because $\alpha(\sigma_2) < 0$. Hence $P^{(\sigma_2)}$ satisfies the Hamilton Jacobi Bellman equation

$$\max_{\sigma\in[\sigma_1,\sigma_2]}\left\{\frac{\sigma^2}{2} W^2 P_{WW}^{(\sigma_2)}\right\} + r W P_W^{(\sigma_2)} - (r+\lambda)\, P^{(\sigma_2)} = 0. \tag{51}$$

The boundary conditions at L and at $+\infty$ are the same as in Lemma 1. Given the continuous differentiability of $P^{(\sigma_2)}$, a classical verification theorem along the lines of Fleming and Soner (1993) implies that setting $\sigma_t = \sigma_2$ is optimal. ■

Proof of Lemma 3. To show result 1, let $\mathcal{U}$ be any domain of the form: (L, W_2) for arbitrarily large W_2 such that $W_t < W_2 < \infty$. Consider now any stopping time $\tau^{\mathcal{U}}$ before W_t exits the domain $\mathcal{U}$. Then, by the definition of P and for any volatility process $\overline{\sigma}_t$:

$$e^{-(r+\lambda)t} P\left(W_t\right) = E_t\left[e^{-(r+\lambda)\tau^{\mathcal{U}}} P\left(W_{\tau^{\mathcal{U}}}\right)\right]$$

This local martingale property of $e^{-(r+\lambda)t}P\left(W_t\right)$ in the domain $\mathcal{U}$ implies that (12) holds and that $P \in C^1$ (for details see Øksendal (2003), Chapter 9). The first part of the proof of result 2 is contained in the text (see equation [14]). To see why $\lim_{W\to\infty} P(W) = 0$, define $\tau^L = \inf_{s\geq t}\{s : W_s = L\}$ and note that for arbitrary $x > t$:

$$\begin{aligned}
P(W_t) &= E\left(e^{-(r+\lambda)(\tau^L-t)} E\left(\int_{\tau^L}^{\tau} e^{-r(s-\tau^L)} dG_s | W_{\tau^L} = L\right)\right) \leq E\left(e^{-(r+\lambda)(\tau^L-t)} B\right) = \\
&= \Pr\left(\tau^L < x\right) E\left(e^{-(r+\lambda)\left(\tau^L-t\right)} B | \tau^L < x\right) + \Pr\left(\tau^L \geq x\right) e^{-(r+\lambda)(x-t)} E\left(e^{-(r+\lambda)(\tau^L-x)} B | \tau^L \geq x\right) \\
&\leq B\left[\Pr\left(\tau^L < x\right) + \Pr\left(\tau^L \geq x\right) e^{-(r+\lambda)(x-t)}\right] \qquad (52)
\end{aligned}$$

Now, fix an arbitrary $\varepsilon > 0$ and choose large x such that $e^{-(r+\lambda)(x-t)} = \frac{\varepsilon}{2B}$. The properties of Brownian motion imply that there always exists W_t large enough such that $\Pr\left(\tau^L < x\right) < \frac{\varepsilon}{2B}$. In light of (52), this then implies that $P(W_t) < \varepsilon$. Since ε can be chosen arbitrarily small, the result follows.

Assertion 3 contains three specific statements. The first statement is that $P_W(L) = -1$. To see why this is so, take any $\overline{W}$ and define $\overline{\tau} = \inf\{s \geq t : W_s \geq \overline{W}\}$. Applying Ito's Lemma to P gives:

$$\begin{aligned}
e^{-(r+\lambda)(T\wedge\overline{\tau}-t)} P\left(W_{T\wedge\overline{\tau}}\right) &= P(W_t) + \int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} \left(\frac{\sigma^2(W_s)}{2} W_s^2 P_{WW} + rP_W W_s - (r+\lambda)P\right) ds \\
&+ \int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} P_W \sigma(W_s) W_s dB_s \\
&+ \int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} P_W(L) dG_s
\end{aligned}$$

Taking expectations on both sides and using equation (12) leads to:

$$\begin{aligned} P(W_t) &= -E_t\left(\int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} P_W(L)dG_s\right) \\ &\quad -E_t\left(\int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} P_W\sigma(W_s)W_s dB_s\right) \\ &\quad +E_t\left[e^{-(r+\lambda)(T\wedge\overline{\tau}-t)} P\left(W_{T\wedge\overline{\tau}}\right)\right] \end{aligned} \tag{53}$$

Since $P(W_t)$ represents the payoff of strategy $\sigma(W)$ it follows that:

$$P(W_t) = E_t\left(\int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} dG_s\right) + E_t\left[e^{-(r+\lambda)(T\wedge\overline{\tau}-t)} P\left(W_{T\wedge\overline{\tau}}\right)\right] \tag{54}$$

for any stopping time $\overline{\tau}$. Combining (54) and (53), it follows that:

$$E_t\left(\int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)}[1+P_W(L)]dG_s\right) = -E_t\left(\int_t^{T\wedge\overline{\tau}} e^{-(r+\lambda)(s-t)} P_W\sigma(W_s)W_s dB_s\right) \tag{55}$$

As the differential equation (12) has a classical solution[35], P_W is a continuous and hence bounded function in the closed interval $\left[L, \overline{W}\right]$.Therefore, $P_W\sigma(W_s)W_s$ is bounded in $\left[L, \overline{W}\right]$. Hence the integrand on the right hand side of equation (55) is a martingale. Therefore, the right hand side of equation (55) is 0, and hence so must be the left side. This can only be the case if $P_W(L) = -1$.

The proof that $P_W < 0$ proceeds by contradiction. Assume otherwise. In particular assume that there exist a $W^{***} > L$ such that $P_W\left(W^{***}\right) > 0$. Since $P_W(L) = -1$ and the differential equation (12) has a continuous first derivative, there must be a point $\widehat{W} > L$ such that $P_W(\widehat{W}) = 0$. Since equation (12) holds at $\widehat{W}$, one obtains $(\sigma^2(\widehat{W})/2)\widehat{W}^2 P_{WW}\left(\widehat{W}\right) = (r+\lambda)P\left(\widehat{W}\right) > 0$, since $P > 0$. Hence $P_{WW}\left(\widehat{W}\right) > 0$. Therefore, at $\widehat{W}$ the function P must have a local minimum. Since $P > 0$ for all $W \geq L$ and $\lim_{W\to\infty} P(W) = 0$, the function P must also have a local maximum at some point $\widetilde{W} > \widehat{W}$, so that $P_W\left(\widetilde{W}\right) = 0$, and $P_{WW}\left(\widetilde{W}\right) < 0$. But this is impossible, by equation (12), since at $\widetilde{W}$ it would have to be the case that $(\sigma^2(\widetilde{W})/2)\widetilde{W}^2 P_{WW}\left(\widetilde{W}\right) = (r+\lambda)P\left(\widetilde{W}\right) > 0$, which is a contradiction to $P_{WW}\left(\widetilde{W}\right) < 0$. Hence it must be the case that $P_W(W) \leq 0$ for all W. Given that $P_W \leq 0$ it is now straightforward to use (12) to establish that $P_{WW} = \frac{2}{\sigma^2(W)W^2}\left[-rP_W W + (r+\lambda)P\right] > 0$. In turn, $P_{WW} > 0$ implies that P_W is increasing

[35] See Øksendal (2003), Chapter 9

throughout. Moreover it can never cross 0. Hence it must be bounded between $P_W(L) = -1$ and 0 as was asserted above. ■

Proof of Lemma 4. A detailed proof of this Lemma would replicate the same steps as Lemma 1. To save space, I only give a sketch. Applying the same logic as in Lemma 1, P should satisfy:

$$0 = \begin{cases} \frac{\sigma_2^2 W^2}{2} P_{WW} + rP_W W - (r+\lambda)P \text{ if } W > W^* \geq L \\ \frac{\sigma_1^2 W^2}{2} P_{WW} + rP_W W - (r+\lambda)P \text{ if } L \leq W \leq W^* \end{cases}$$

The general solution to this equation is

$$P(W) = \begin{cases} C_{21} W^{\alpha_2^+} + C_{22} W^{\alpha_2^-} \text{ if } W > W^* \geq L \\ C_{11} W^{\alpha_1^+} + C_{12} W^{\alpha_1^-} \text{ if } L \leq W \leq W^* \end{cases}$$

where the constants $\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-$ are defined in (24). In order to be able to replicate the same steps as in Lemma 1, $P(W)$ must be continuous and continuously differentiable[36] at W^*. This implies:

$$C_{21} (W^*)^{\alpha_2^+} + C_{22} (W^*)^{\alpha_2^-} = C_{11} (W^*)^{\alpha_1^+} + C_{12} (W^*)^{\alpha_1^-} \quad (56)$$

$$\alpha_2^+ C_{21} (W^*)^{\alpha_2^+ - 1} + \alpha_2^- C_{22} (W^*)^{\alpha_2^- - 1} = \alpha_1^+ C_{11} (W^*)^{\alpha_1^+ - 1} + \alpha_1^- C_{12} (W^*)^{\alpha_1^- - 1} \quad (57)$$

To enforce $\lim_{W \to \infty} P(W) = 0$, it is also necessary to impose $C_{21} = 0$. Finally, the condition $P_W(L) = -1$ implies:

$$\alpha_1^+ C_{11} (L)^{\alpha_1^+ - 1} + \alpha_1^- C_{12} (L)^{\alpha_1^- - 1} = -1 \quad (58)$$

Solving for C_{11}, C_{12}, C_{22} from equations (56), (57), (58) leads to (23). Equation (25) follows immediately by setting $P(L) = B$ and solving for W^*. ■

Proof of Proposition 1. The first step towards proving Proposition 1 is to establish the existence of a solution to the system of equations (20) and (21), satisfying $\pi_2(W^*) = 0$ and

$$\pi_2(x) \begin{cases} \geq 0 \text{ if } x < W^* \\ \leq 0 \text{ if } x > W^* \end{cases} \quad (59)$$

[36] In particular, these conditions will make it possible to apply Ito's Lemma as in Lemma 1.

with at least one of the two inequalities being strict for some values x. Furthermore, to provide sufficient conditions for the optimality of policy (22), the following properties will also be required:

$$\lim_{x\to\infty} \quad |\pi_1(x)| \quad < \quad \infty \tag{60}$$

$$\lim_{x\to\infty} \quad |\pi_2(x)| \quad < \quad \infty \tag{61}$$

The next Lemma constructs an explicit continuous solution to π_1, π_2 that satisfies (20), (21), (59), (60), (61) and $\pi_2(W^*) = 0$.

Lemma 8 *Let W^* be given by (25). Then, there exist continuous functions π_1 and π_2 that solve the pair of differential equations (20), (21) and satisfy $\pi_2(W^*) = 0$, (59), (60), (61).*

Proof of Lemma 8. The proof proceeds by explicitly constructing two functions that satisfy all the stated properties. Assume first that $W > W^*$. By the form of the conjectured optimal policy, one needs to distinguish 3 sub-regions for x :

(a) $L \leq x < W^*$

(b) $W^* \leq x \leq W$

(c) $x > W$

Define the four constants $\beta_1^+, \beta_1^-, \beta_2^+, \beta_2^-$ as

$$\beta_1^{\pm} = \frac{-\left(\frac{\sigma_1^2}{2} - r\right) \pm \sqrt{\left(\frac{\sigma_1^2}{2} - r\right)^2 + 2\sigma_1^2 (r+\lambda)}}{\sigma_1^2} \text{ and } \beta_2^{\pm} = \frac{-\left(\frac{\sigma_2^2}{2} - r\right) \pm \sqrt{\left(\frac{\sigma_2^2}{2} - r\right)^2 + 2\sigma_2^2 (r+\lambda)}}{\sigma_2^2}$$

In light of the conjectured optimal policy, in region (a) the differential equation (20), (21) has the general solution:

$$\begin{aligned} \pi_1(x) &= D_{11} x^{\beta_1^+} + D_{21} x^{\beta_1^-} - 1 \\ \pi_2(x) &= -\frac{\sigma_1^2 \beta_1^+}{2(r+\lambda)} D_{11} x^{\beta_1^+ + 1} - \frac{\sigma_1^2 \beta_1^-}{2(r+\lambda)} D_{21} x^{\beta_1^- + 1} \end{aligned}$$

for appropriate constants D_{11}, D_{21}. Similarly, in region (b) the general solution is:

$$\begin{aligned}
\pi_1(x) &= D_{12}x^{\beta_2^+} + D_{22}x^{\beta_2^-} - 1 \\
\pi_2(x) &= -\frac{\sigma_2^2\beta_2^+}{2(r+\lambda)}D_{12}x^{\beta_2^+ + 1} - \frac{\sigma_2^2\beta_2^-}{2(r+\lambda)}D_{22}x^{\beta_2^- + 1}
\end{aligned}$$

and in region (c):

$$\begin{aligned}
\pi_1(x) &= D_{13}x^{\beta_2^+} + D_{23}x^{\beta_2^-} \\
\pi_2(x) &= -\frac{\sigma_2^2\beta_2^+}{2(r+\lambda)}D_{13}x^{\beta_2^+ + 1} - \frac{\sigma_2^2\beta_2^-}{2(r+\lambda)}D_{23}x^{\beta_2^- + 1}
\end{aligned}$$

It remains to determine the 6 constants in the above equations in order to obtain the solution to π_1, π_2. Starting with region (c), it is clear that (60), (61) can only hold if $D_{13} = 0$, since $\beta_2^+ > 0$. To ensure continuity of $\pi_1(x), \pi_2(x)$ at point W, the constants D_{23}, D_{12}, D_{22} need to satisfy (after some straightforward cancellations):

$$D_{12}W^{\beta_2^+} + (D_{22} - D_{23})\,W^{\beta_2^-} = 1 \tag{62}$$

$$-\beta_2^+ D_{12}W^{\beta_2^+ + 1} - \beta_2^-\,(D_{22} - D_{23})\,W^{\beta_2^- + 1} = 0 \tag{63}$$

Similarly, continuity of $\pi_1(x), \pi_2(x)$ at W^* implies that

$$\begin{aligned}
D_{11}\,(W^*)^{\beta_1^+} + D_{21}\,(W^*)^{\beta_1^-} &= D_{12}\,(W^*)^{\beta_2^+} + D_{22}\,(W^*)^{\beta_2^-} \\
-\beta_1^+ D_{11}\,(W^*)^{\beta_1^+ + 1} - \beta_1^- D_{21}\,(W^*)^{\beta_1^- + 1} &= -\left(\frac{\sigma_2}{\sigma_1}\right)^2 \left[\beta_2^+ D_{12}\,(W^*)^{\beta_2^+ + 1} + \beta_2^- D_{22}\,(W^*)^{\beta_2^- + 1}\right]
\end{aligned}$$

Finally, to ensure $\pi_2(W^*) = 0$ it must also be the case that

$$-\beta_1^+ D_{11}\,(W^*)^{\beta_1^+ + 1} - \beta_1^- D_{21}\,(W^*)^{\beta_1^- + 1} = 0 \tag{64}$$

Solving this system of equations leads to the following solution for π_1, π_2 :

(a) $L \leq x < W^*$

$$\begin{aligned}
\pi_1(x) &= \left(\frac{W^*}{W}\right)^{\beta_2^+} \frac{1}{\left(\beta_1^+ - \beta_1^-\right)} \left(\frac{x}{W^*}\right)^{\beta_1^-} \left[\beta_1^+ - \beta_1^- \left(\frac{x}{W^*}\right)^{\beta_1^+ - \beta_1^-}\right] - 1 \\
\pi_2(x) &= -\frac{\sigma_1^2}{2\,(r+\lambda)} \left(\frac{W^*}{W}\right)^{\beta_2^+} \frac{\beta_1^- \beta_1^+}{\left(\beta_1^+ - \beta_1^-\right)} \left(\frac{x}{W^*}\right)^{\beta_1^-} \left[1 - \left(\frac{x}{W^*}\right)^{\beta_1^+ - \beta_1^-}\right] x
\end{aligned}$$

(b) $W^* \leq x \leq W$

$$\begin{aligned}
\pi_1(x) &= \frac{1}{\left(\beta_2^+ - \beta_2^-\right)} \left(\frac{x}{W}\right)^{\beta_2^-} \left[\beta_2^+ \left(\frac{W^*}{W}\right)^{\beta_2^+ - \beta_2^-} - \beta_2^- \left(\frac{x}{W}\right)^{\beta_2^+ - \beta_2^-}\right] - 1 \\
\pi_2(x) &= -\frac{\sigma_2^2}{2\,(r+\lambda)} \frac{\beta_2^+ \beta_2^-}{\left(\beta_2^+ - \beta_2^-\right)} \left(\frac{x}{W}\right)^{\beta_2^-} \left[\left(\frac{W^*}{W}\right)^{\beta_2^+ - \beta_2^-} - \left(\frac{x}{W}\right)^{\beta_2^+ - \beta_2^-}\right] x
\end{aligned}$$

(c) $x > W$

$$\begin{aligned}
\pi_1(x) &= \frac{\beta_2^+}{\left(\beta_2^+ - \beta_2^-\right)} \left(\frac{W^*}{W}\right)^{\beta_2^-} \left[\left(\frac{W^*}{W}\right)^{\beta_2^+ - \beta_2^-} - 1\right] \left(\frac{x}{W^*}\right)^{\beta_2^-} \\
\pi_2(x) &= -\frac{\sigma_2^2}{2\,(r+\lambda)} \frac{\beta_2^+ \beta_2^-}{\left(\beta_2^+ - \beta_2^-\right)} \left(\frac{W^*}{W}\right)^{\beta_2^-} \left[\left(\frac{W^*}{W}\right)^{\beta_2^+ - \beta_2^-} - 1\right] \left(\frac{x}{W^*}\right)^{\beta_2^-} x
\end{aligned}$$

By construction, $\pi_1(x), \pi_2(x)$ are continuous and satisfy $\pi_2(W^*) = 0$, (60), (61). It remains to verify that this solution also satisfies (59). This follows from $\beta_2^+ > 0, \beta_2^- < 0$ and also $\beta_1^+ > 0, \beta_1^- < 0$. The proof for $W < W^*$ follows similar steps and is therefore omitted. ■

Proof of Proposition 1 continued. Given the existence of an appropriate pair of co-state variables π_1, π_2 it is now possible to verify optimality by using a standard sufficiency theorem of optimal control (see e.g. Leonard and Van Long (1992), p. 289). To save space, I omit a proof of the sufficiency theorem and make it available on request. ■

Proof of Proposition 2. Let Π be the set of all participation compatible policies σ. The first step in order to obtain $\widehat{I}^*$ is to find a function $g(W_t)$ such that

$$g(W_t) \geq \inf_{t \geq \tau_0} P\left(W_t; \sigma_{s \geq t}\right) \text{ for all } \sigma \in \Pi \tag{65}$$

Constructing such an upper bound is straightforward. First, fix a level $W_1 > L$ and let τ_1 be the first time after τ_0 such that $W_t = W_1$. An upper bound to $\inf_{t\geq\tau_0} P\left(W_1; \sigma_{s\geq t}\right)$ is given by the highest possible value P_{τ_1} that can be assigned by any participation compatible policy. Two observations are useful in order to determine P_{τ_1}. The first observation is that P_{τ_1} must satisfy the constraint

$$E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)} dG_s\right) + P_{\tau_1} E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right) \leq B,$$

which can be rewritten as

$$P_{\tau_1} \leq \frac{B - E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)} dG_s\right)}{E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right)} \tag{66}$$

Using an argument similar to the proof of Lemma 2, one can show that[37] setting $\sigma_s = \sigma_1$ for all $s \in [\tau_0, \tau_1]$ will minimize both $E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)} dG_s\right)$ and $E\left(e^{-(r+\lambda)(\tau_1-\tau_0)}\right)$, and hence will maximize the right hand side of (66). This is intuitive. In order to have the highest possible flexibility to promise a high level of P_{τ_1}, one needs to set volatility prior to τ_1 as low as possible. More importantly, this simple observation suggests that it is possible to find an explicit expression for the right hand side of equation (66). In particular, let $u(W)$ be the solution to the differential equation $\frac{\sigma_1^2}{2} u_{WW} W^2 + r u_W W - (r+\lambda) u = 0$, subject to the boundary conditions $u(L) = B$, and $u_W(L) = -1$. There is a unique solution to this equation which is given by

$$u(W) = L\left[\frac{\alpha_1^+ \frac{B}{L} + 1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W}{L}\right)^{\alpha_1^-} - \frac{\alpha_1^- \frac{B}{L} + 1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W}{L}\right)^{\alpha_1^+}\right]. \tag{67}$$

By arguments similar to the ones used in the proof of Lemma 3, the function $u(W)$ satisfies:

$$u(W_1) = \frac{B - E_{\tau_0}\left(\int_{\tau_0}^{\tau_1} e^{-(r+\lambda)(s-\tau_0)} dG_s | \sigma_{s\in[\tau_0,\tau_1]} = \sigma_1\right)}{E_{\tau_0}\left(e^{-(r+\lambda)(\tau_1-\tau_0)} | \sigma_{s\in[\tau_0,\tau_1]} = \sigma_1\right)} \tag{68}$$

and hence it gives a closed form expression for the right hand side of (66).

Letting χ be the first time after τ_1 such that $W_\chi = L$, a second observation about P_{τ_1} is that

[37] The proof of this fact follows steps similar to Lemma 2 and is ommited. It is available upon request.

it is bounded above by

$$P_{\tau_1} \leq \max_{\sigma} E\left(e^{-(r+\lambda)(\chi-\tau_1)}\right) B = E\left(e^{-(r+\lambda)(\chi-\tau_0)} | \sigma_{s\in[\tau_1,\chi]} = \sigma_2\right) B = B\left(\frac{W_1}{L}\right)^{\alpha_2^-}. \tag{69}$$

This bound is intuitive. It states that even if volatility is set at its maximum between times τ_1 and χ, the continuation value after that point cannot be larger than B.

The above observations, together with the fact that W_1 is arbitrary, imply that the function $g(W) \equiv \min\left[u(W), B\left(\frac{W}{L}\right)^{\alpha_2^-}\right]$ satisfies the equation (65). In turn, this implies that for any participation compatible policy

$$\widehat{I}(\sigma) \geq \widehat{I}^* \equiv \max_{W>L}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - g(W)}{1+k\left(\frac{W}{L}\right)^{\alpha(\sigma_2)}}\right] = \max_{W>L}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - g(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right], \tag{70}$$

where the rightmost equality follows from $\alpha(\sigma_2) = \alpha_2^-$. It will be useful to establish a few properties of the expression inside the square brackets of (70). To this end define

$$n(W) = \frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - g(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}$$

By its definition $g(L) = B,$ and hence $n(L) = 0$. Also, the definition of $g(W)$ implies that $n(W) \geq 0$. Moreover, $u_W(L) = -1$ and assumption (9) implies that

$$\frac{d\left[B\left(\frac{W}{L}\right)^{\alpha_2^-}\right]}{dW_{|W_t=L}} = \alpha_2^- \frac{B}{L} > -1.$$

These two last facts can be used to show that $n_W(L) > 0$, and hence $n > 0$ in a neighborhood of L.

Finally, by assumption (9), $\alpha_1^- \frac{B}{L} + 1 < 0$. Hence, $u(W) \to \infty$ and[38] $u_W \to 0$ as $W \to \infty$. By contrast $B\left(\frac{W}{L}\right)^{\alpha_2^-} \to 0$ as $W \to \infty$ and the derivative of $B\left(\frac{W}{L}\right)^{\alpha_2^-}$ is always negative. Hence there always exists a value W^u, such that $g(w) = B\left(\frac{w}{L}\right)^{\alpha_2^-}$ for all $w \geq W^u$. Therefore $n(w) = 0$ for all $w \geq W^u$. Since the function n starts at 0 when $W = L$, and becomes 0 for all $W \geq W^u$, and is positive and continuous for $W \in [L, W^u]$, it must attain a maximum at some point W^{**} between

[38] Since $\alpha_1^+ > 1$.

L and W^u.

To compute this maximum it is easiest to take the log of $n(W)$, differentiate with respect to W and set the resulting expression equal to 0 to obtain

$$\frac{\alpha_2^- \frac{B}{L}\left(\frac{W^{**}}{L}\right)^{\alpha_2^- -1} - \left[\alpha_1^- \frac{\alpha_1^+ \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{**}}{L}\right)^{\alpha_1^- -1} - \alpha_1^+ \frac{\alpha_1^- \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{**}}{L}\right)^{\alpha_1^+ -1}\right]}{\frac{B}{L}\left(\frac{W^{**}}{L}\right)^{\alpha_2^-} - \left[\frac{\alpha_1^+ \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{**}}{L}\right)^{\alpha_1^-} - \frac{\alpha_1^- \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{**}}{L}\right)^{\alpha_1^+}\right]} = \frac{\alpha_2^- k\left(\frac{W^{**}}{L}\right)^{\alpha_2^- -1}}{1+k\left(\frac{W^{**}}{L}\right)^{\alpha_2^-}} \quad (71)$$

Straightforward, but tedious algebra can be used to show that this equation has a unique root.[39]

Having obtained $\widehat{I}^*$ as a lower bound on $\widehat{I}(\sigma)$, it is now possible to verify the optimality of the policy σ^* of equation (22), by showing that $\widehat{I}(\sigma^*) = \widehat{I}^*$. As a first step towards showing this, I use the quantity W^* as defined in equation (25) and show that $W^* < W^{**}$. After some manipulations one can verify that

$$\frac{\alpha_2^- \frac{B}{L}\left(\frac{W^{*}}{L}\right)^{\alpha_2^- -1} - \left[\alpha_1^- \frac{\alpha_1^+ \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{*}}{L}\right)^{\alpha_1^- -1} - \alpha_1^+ \frac{\alpha_1^- \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{*}}{L}\right)^{\alpha_1^+ -1}\right]}{\frac{B}{L}\left(\frac{W^{*}}{L}\right)^{\alpha_2^-} - \left[\frac{\alpha_1^+ \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{*}}{L}\right)^{\alpha_1^-} - \frac{\alpha_1^- \frac{B}{L}+1}{\alpha_1^+ - \alpha_1^-}\left(\frac{W^{*}}{L}\right)^{\alpha_1^+}\right]} = \alpha_2^- \left(\frac{W^*}{L}\right)^{-1} < \frac{\alpha_2^- k\left(\frac{W^{*}}{L}\right)^{\alpha_2^- -1}}{1+k\left(\frac{W^{*}}{L}\right)^{\alpha_2^-}} \quad (72)$$

where the equality follows from (56)-(57) and the inequality follows from $\alpha_2^- < 0$, $k < 1$, $\frac{W^*}{L} > 1$. Combining (71) and (72) shows that $n_W(W^*) < 0$. Hence it must be the case that $W^{**} < W^*$.

Since the functions $P(W)$ of equation (23) and $u(W)$ coincide between L and W^*, and $W^{**} < W^*$, it follows that

$$\widehat{I}^* = \max_{L<W}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - g(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right] = \max_{L<W<W^*}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - u(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right] = \max_{L<W<W^*}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-} - P(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right]. \quad (73)$$

The first equation in (73) is the definition of $\widehat{I}^*$, the second and third equations follow from the fact that $u(W) = P(W) < B(W/L)^{\alpha_2^-}$ for all $W \in (L, W^*]$. The final step of the proof is to verify

[39] Details are available upon request.

that

$$\max_{L\leq W\leq W^*}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-}-P(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right]=\max_{L<W}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-}-P(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right]=\widehat{I}(\sigma^*). \tag{74}$$

This follows from the fact that $\left[B\left(\frac{W}{L}\right)^{\alpha_2^-}-P(W)\right]/\left[1+k\left(\frac{W}{L}\right)^{\alpha_2^-}\right]$ is a declining function of W for $W>W^*$, and hence

$$\max_{W>W^*}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-}-P(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right]=\frac{B\left(\frac{W^*}{L}\right)^{\alpha_2^-}-P(W^*)}{1+k\left(\frac{W^*}{L}\right)^{\alpha_2^-}}\leq\max_{L\leq W\leq W^*}\left[\frac{B\left(\frac{W}{L}\right)^{\alpha_2^-}-P(W)}{1+k\left(\frac{W}{L}\right)^{\alpha_2^-}}\right] \tag{75}$$

Using (75), it follows that (74) holds. Finally combining (74) and (73) implies that $\widehat{I}^*=\widehat{I}(\sigma^*)$. ■

Proof of Lemma 6. Lemma 2 has established that for any level of B, the policy σ^* of equation (22) is optimal, in the sense that it attains the lower bound $\widehat{I}^*$ (which also depends on B). Since the optimal policy σ^* assigns the same value $P(L)=B$ every time that $W_t=L$, it suffices to check that is is optimal to set $P_{\tau_0}=B$. To verify this, note that the shareholders' value if they set $P_{\tau_0}=B$, is given by $V=B-k\widehat{I}^*$. Differentiating V with respect to B, combining (70) with (67) and using the envelope theorem shows that

$$V_B=1-\left[\frac{k\left(\frac{W^{**}}{L}\right)^{\alpha_2^-}}{1+k\left(\frac{W^{**}}{L}\right)^{\alpha_2^-}}\right]\frac{\alpha_1^+\left[1-\left(\frac{W^{**}}{L}\right)^{\alpha_1^--\alpha_2^-}\right]-\alpha_1^-\left[1-\left(\frac{W^{**}}{L}\right)^{\alpha_1^+-\alpha_2^-}\right]}{\alpha_1^+-\alpha_1^-} \tag{76}$$

The second term on the right hand side of (76) is smaller than 1, because $1-\left(\frac{W^{**}}{L}\right)^{\alpha_1^--\alpha_2^-}<1$ (because $\alpha_1^-<\alpha_2^-<0$) and $1-\left(\frac{W^{**}}{L}\right)^{\alpha_1^+-\alpha_2^-}<0$. Hence $V_B>0$, and hence it is optimal to set $P(L)=B$. ■

Proof of Lemma 7. The proof of this fact is rather straightforward. To save space, I only give a sketch. First observe that the definition (37) implies that extending the duration of a risk management rule from $[\tau_0,\infty)$ to $[0,\infty)$ can never lower $\widehat{I}(\sigma)$, since $\inf_{t\geq\tau_0}P\left(W_t;\sigma_{s\geq t}\right)\geq\inf_{t\geq0}P\left(W_t;\sigma_{s\geq t}\right)$. Furthermore, proposition 2 implies that any participation compatible commitment will involve $\widehat{I}(\sigma_{s\geq0})\geq\widehat{I}^*$. Therefore, it suffices to show that $\widehat{I}(\sigma_{s\geq0})=\widehat{I}^*$. Since the policy

σ^* is identical to the policy of equation (22) for $t \geq \tau_0$, it is sufficient to show that $P_{t\in[0,\tau_0]}(W_t) \geq P_{t\geq\tau_0}(W_t)$. In turn this will automatically be the case if $\widehat{W}^* \leq W^*$. Following a strategy similar to Lemma 4, the value $P(W_t)$ for $t < \tau_0,$ satisfies (12), $P(L) = B$, and $\lim_{W\to\infty} P(W) = 0$. Furthermore, $\widehat{W}^*$ needs to be determined so that $P(W_0) = U_0$. Since $U_0 \geq P(W_0, \sigma^*_{s\geq\tau_0})$, it follows that $\widehat{W}^* \leq W^*$. ■

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