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FIXED COSTS AND LONG-LIVED INVESTMENTS

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ABSTRACT

Neoclassical investment models predict that firms should make frequent, small adjustments to their capital stocks. Microeconomic evidence, however, shows just the opposite — firms make infrequent, large adjustments to their capital stocks. In response, researchers have developed models with fixed costs of adjustment to explain the data. While these models generate the observed firm-level investment behavior, it is not clear that the aggregate behavior of these models differs importantly from the aggregate behavior of neoclassical models. This is important since most of our existing understanding of investment is based on models without fixed costs. Moreover, models with fixed costs have non-degenerate, time-varying distributions of capital holdings across firms, making the models extremely difficult to analyze. This paper shows that, for sufficiently long-lived capital, (1) the cross-sectional distribution of capital holdings has virtually no bearing on the equilibrium and (2) the aggregate behavior of the fixed-cost model is virtually identical to that of the neoclassical model. The findings are due to a near infinite elasticity of investment timing for long-lived capital goods — a feature that fixed-cost models and neoclassical models share. The analysis shows that the so-called "irrelevance results" obtained in recent numerical studies of fixed-cost models are not parametric special cases but instead reflect fundamental properties of long-lived investments.

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I. Introduction

Conventional neoclassical investment models typically assume that capital adjustment costs rise smoothly with investment and thus predict that firms should make frequent, small adjustments to their capital stocks. Microeconomic evidence, however, shows that many firms make infrequent, large adjustments to their capital stocks. Motivated by the micro-evidence, researchers have developed investment models that feature fixed costs of adjustment. While these models generate the observed firm-level investment behavior, it is not clear whether the aggregate equilibrium behavior of such models differs significantly from the equilibrium behavior generated by more conventional investment models. Several prominent numerical studies of calibrated DSGE models with fixed adjustment costs suggest that there may be only minor differences between the two models. The cause of these "irrelevance results" is typically attributed to consumption smoothing forces present in general equilibrium settings. The irrelevance results have been contested by other researchers on the grounds that they hold only for some parameter values and are not a general feature of equilibrium models with fixed costs.

The aggregate behavior of models with fixed adjustment costs is important for several reasons. Much of our existing understanding of investment is based on neoclassical models that abstract from fixed adjustment costs. Because the earlier models contrast sharply with the microeconomic evidence, researchers are justifiably concerned that policy conclusions or econometric predictions based on these models may be misleading. On the other hand, if the aggregate behavior of the two modeling frameworks is similar, then the apparent failure of conventional neoclassical models at the micro level does not necessarily imply that we need to abandon neoclassical modeling techniques to analyze aggregate investment. Indeed, there may be reasons to prefer the neoclassical framework. Unlike neoclassical investment models, fixed-cost models are analytically very cumbersome. Models with fixed costs typically have non-degenerate distributions of capital across firms. The distribution is a time-varying object which enters the models as an additional state variable and makes the models extremely difficult to analyze, particularly in equilibrium settings.

This paper analyzes the approximate equilibrium behavior of an investment model where fixed costs matter at the microeconomic level. I focus on long-lived investment goods – investment goods with low rates of economic depreciation. The main insight of the paper is that, in the face of fixed adjustment costs, optimal investment behavior is characterized by an extremely high intertemporal elasticity of substitution for investment purchases. For sufficiently long-lived capital goods (goods with low rates of economic depreciation), the intertemporal elasticity of substitution is nearly infinite. This property has a number of implications.

First, for long-lived investment goods, the underlying distribution of capital holdings across firms has little bearing on the equilibrium. Because firms are willing to drastically change the timing of their investments, firms that are bunched up or spread out relative to the steady state distribution can simply delay or accelerate the timing of their investment purchases to avoid high prices or take advantage of low prices. Thus, the high intertemporal elasticity of investment timing effectively breaks the link between the distribution of firms' capital holdings and aggregate investment, thereby eliminating any role for the cross-sectional distribution at the aggregate level.

Second, the near infinite elasticity of investment timing is a property that the fixed-cost model shares with the neoclassical investment model. In an instructive limiting case in which the economic depreciation rate approaches zero, the equilibrium in the fixed-cost model corresponds exactly to the equilibrium in the neoclassical model. Thus, at the aggregate level, investment and investment prices, particularly for long-lived capital goods, can be accurately analyzed with traditional, neoclassical investment models. While the traditional models cannot match the behavior of the firms at the microeconomic level, they provide an easy, reliable guide to aggregate behavior, policy analysis and empirical predictions. This finding supports the recent "irrelevance results" in Thomas [2002] and Veracierto [2002]. Indeed, rather than being an artifact of a particular calibration, the irrelevance results reflect deep, fundamental properties of investment models with long-lived capital goods.

In contrast to the received wisdom of the literature, the source of the equivalence between neoclassical models and fixed-cost models is not consumption smoothing *per se*. Because both neoclassical and fixed-cost models have high intertemporal elasticities for the timing of investment, anything that causes the effective price of new capital goods to increase with aggregate investment will make the models difficult to distinguish with aggregate data. Thus, an increasing quadratic adjustment cost in a neoclassical framework and an upward-sloping supply curve in the fixed-cost model will result in the same equilibrium paths provided that the elasticity of the marginal cost of investment is the same in each case. Consumption smoothing in DSGE models is but one source of an increasing marginal cost of investment and is not the key cause of the equivalence between the models.

I supplement the approximate analytical results with numerical analysis. The analysis shows that the limiting approximations are highly accurate even for realistic depreciation rates away from the low-depreciation limit. Among other things, the numerical analysis shows that while there are substantial variations in the cross-sectional moments of the distribution, these moments provide little information on the evolution of future prices and investment.

The remainder of the paper is set out as follows: Section II presents background information and provides a brief overview of the related literature. Section III presents the basic model and analyzes the equilibrium in the low depreciation limit. Section IV presents a numerical analysis of the model and considers the quantitative performance of the limiting analysis in environments with realistic depreciation rates. Section V concludes.

II. Background and Related Literature

In micro data, plant level investment is characterized by long periods of relative inaction punctuated by episodes of high investment. Thus, rather than spreading investments over time, firms make large, infrequent adjustments to their capital stocks. Doms and Dunne [1998] show that, for U.S. manufacturing, most plants experience at least one year in which their capital stock rises by at least 50 percent. For many establishments, half of all plant-level investment spending over a 17-year horizon is concentrated in the three years surrounding the year with the plant's greatest investment. Cooper *et al*. [1999] show that each year, roughly 1 out of every 5 manufacturing plants experiences an "investment spike," which they define as an increase in plant-level capital of at least 20 percent. Aggregate variation in investment spikes accounts for the bulk of the variation in U.S. manufacturing investment. Gourio and Kashyap [2007] show that the aggregate variation in investment spikes is primarily driven by changes in the number of firms experiencing spikes rather than changes in the average size of spikes.[1] Taken as a whole, the evidence from the micro-data stands in stark contrast to the predictions of standard neoclassical investment models with convex adjustment costs (e.g., Abel [1982], Hayashi [1982] and Summers [1981]). Models with fixed costs rationalize the lumpy investment behavior seen in the data. To avoid paying the fixed cost, firms make infrequent, large changes to their capital stock.

Unlike earlier convex models, investment models with fixed costs are difficult to solve even in partial equilibrium settings and are often completely intractable in general equilibrium. Indeed, much of the recent literature focuses simply on numerically solving such models. The difficulty in solving these models arises because not all firms have the same capital stock. At any point in time, some firms have old, outdated capital and are likely to adjust in the near term while other firms have recently adjusted and will not purchase new capital for quite some time. The distribution of capital stocks changes whenever shocks or policies disturb the market. Thus, to solve the model, one must keep track of an endogenous, time-varying distribution of capital holdings across firms.

Because the position and dynamics of the distribution of capital holdings can influence the equilibrium, the distribution often plays a prominent role in the questions posed by the literature on fixed costs. For example, suppose there is an unusually large number of firms with relatively old capital. This situation might be thought of as "pent-up demand." In this case, one would expect to see a predictable surge of demand in the near term as these firms update their capital. Thus, investment prices would be high in the short-run and fall as time passes. The opposite scenario is also possible. If many firms recently adjusted, then there would be few firms that currently need new capital. This situation might be thought of as "capital overhang." In this case, investment demand and prices should be unusually low in the near term. Only later, when the other firms' capital depreciates sufficiently, will investment demand recover. Moreover,

[1] Doms and Dunne [1998] and Cooper *et al*. [1999] base their findings on data from the Longitudinal Research Database (LRD), which includes most U.S. manufacturing plants. Gourio and Kashyap [2007] use both LRD and Chilean data on manufacturing plants. (See also Fuentes and Gilchrist [2005] and Fuentes *et al.* [2006].) Like Cooper *et al.* [1999], Gourio and Kashyap define investment spikes to be increases in plant-level capital of 20 percent or more and show that variation in aggregate investment is associated with variation in aggregate investment spikes. (See also Cooper and Haltiwanger [2006].)

economic policies could have different effects in each case. In the pent-up demand case, a tax subsidy might have a considerable impact on investment since there are many firms close to the point at which they would invest. In the capital overhang case, because there are very few firms with low capital stocks, the same subsidy might have little effect.[2] In theory, each different configuration of the distribution could imply a different equilibrium outcome and have different policy implications.

The cross-sectional distribution of capital thus presents both a problem and an opportunity for researchers. Accounting for the equilibrium behavior of an endogenous distribution is computationally and analytically very difficult. The combination of an incredibly large state space (the distribution) and highly non-linear behavior on the part of firms makes fixed-cost models difficult to analyze even in numerical settings. At the same time, variations in the cross sectional distribution could have rich implications for the study of investment behavior and policy analysis.

While many researchers have analyzed models of investment with heterogeneous agents and fixed costs, most of the well known results in this area come from models of individual firms taking prices as given.[3] Caballero and Engel [1999] assume that all supply curves are perfectly elastic. This is tantamount to working in a partial equilibrium framework since, with perfectly flat supply curves, investment decisions of other firms have no influence on equilibrium prices. Adda and Cooper [2000] analyze a model of consumer durables with discrete replacement. Like Caballero and Engel, Adda and Cooper assume that prices, though stochastic, are independent of aggregate investment. In both cases, the complexity that arises from the distribution is suppressed.

Because obtaining analytical results for models with fixed costs in equilibrium settings is difficult, much of the progress in this area has been made with numerical studies of particular dynamic models.[4] Using numerical techniques, Thomas [2002] and

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[2] Adda and Cooper [2000] present a dynamic analysis of a French automobile scrapping subsidy with implications exactly in this spirit.

[3] See, among others, Abel and Eberly [1994], Bertola and Cabellero [1990], Cabellero [1993], Caballero and Leahy [1996], Caballero and Engel [1999], Cooper and Haltiwanger [1993], Cooper *et al*. [1999], Dixit and Pindyck [1994], and Eberly [1994]. For studies that confine attention to steady state analysis, see, Caplin and Spulber [1987], Hendel and Lizzeri [1999, 2002], House and Leahy [2004], House and Ozdenoren [2007], and Stolyarov [2002]. Caplin and Leahy [1991, 1997] make the simplifying assumption that firm-level investment demand does not react to endogenous changes in the distribution of firms.

[4] Typically, analytical results require strong assumptions to facilitate analysis. See Danziger [1999] for a closed-form analysis of a model with fixed costs. Gertler and Leahy [2006] adapt Danziger's approach to a

Veracierto [2002] find that calibrated DSGE models with fixed costs behave almost identically to conventional DSGE models that abstract from such micro-frictions. Thomas [2002] and Khan and Thomas [2007] attribute these "irrelevance results" to the consumption smoothing motives of the representative household in their models. Gourio and Kashyap [2007] and Bachmann, *et al.* [2008] have challenged these results on the grounds that they hold only for certain parameter values and are not general properties of models with fixed adjustment costs. While numerical analysis has advanced rapidly in recent years, numerical techniques are limited to solving and cataloging particular cases. Furthermore, the techniques required are still quite cumbersome and the underlying economic forces at play are often obscured. The main objective of this paper is to shed light on these forces.

III. MODEL

The basic structure of the model is inspired by the model in Caplin and Leahy [2004, 2006]. The model is in continuous time. The demand side of the model consists of a continuum of firms (measure one) that maximize their discounted profits net of investment costs. Firms discount the future at the discount rate r. Each firm owns a stock of capital k, which depreciates exponentially at the rate δ. Because I focus on long-lived investment goods, I assume the depreciation rate is less than five percent annually. Flow profits are $A(t)k(t)^{\alpha}$, where $0<\alpha<1$ and $A(t)$ is a shock to the profitability of capital. When a firm adjusts its capital stock, say from k to k', it incurs two costs. The first is a fixed cost of adjustment $F>0$, which is paid whenever investment at the firm is non-zero. The second is a cost per-unit of investment given by $p(t)\cdot[k'-k]$. To make matters simple, I assume that when a firm adjusts, it must adjust to a fixed level of capital $\bar{k}$. (The assumption of a constant reset level is innocuous. I discuss this assumption further later.) Thus, the firm's problem is simply to decide when

more conventional model of price rigidity. Caplin and Leahy [2006] assume that idiosyncratic depreciation shocks smooth out the distribution over time thus simplifying the solution. Feasible numerical approaches have only recently been made available. Krusell and Smith [1997, 1998] assume that expectations are based on a small number of moments of the distribution rather than on the entire distribution (see also Rios-Rull [1999]). Other approaches use additional heterogeneity to make the model differentiable so that linear methods can be used. For examples of the latter, see Dotsey, *et al.* [1999], Thomas [2002], Veracierto [2002], Khan and Thomas [2003] and King and Thomas [2006].

to adjust. If the firm doesn't adjust, its capital stock obeys $\dot{k} = -\delta k$. If the firm makes an adjustment at time *T*, it jumps from its current capital stock $k(T)$ to the reset level of capital $\bar{k}$ and incurs the adjustment cost $p(T)\cdot\left[\bar{k} - k(T)\right] + F$.

To focus attention on the demand side of the model, the supply side is intentionally kept as simple as possible. The flow supply of investment is governed by an investment supply curve $p(t) = z(t)\cdot S(I(t))$. $I(t)$ is the flow supply of aggregate investment, $p(t)$ is the prevailing market price of new investment goods and $z(t)$ is an investment supply shock. The supply curve is upward sloping ($S' > 0$) and $S\left(0\right) = 0$. Note that the model has no representative consumer and thus has no direct role for consumption smoothing as emphasized in the DSGE literature.

A perfect-foresight equilibrium is a fixed point in prices. Taking the price path $p(t)$ and the productivity path $A(t)$ as given, firms make optimal investment decisions. The investment decisions imply a time path for aggregate investment

$$I\left(t\right) = \int_0^\infty f\left(s,t\right) i\left(s,t\right) ds, \tag{1}$$

where $f\left(t,s\right)$ is the date *t* measure of firms with capital of age *s* and $i(s,t)$ is optimal investment for a firm at date *t* that last adjusted *s* periods ago. Aggregate investment then implies a price path $p'(t) = z(t)\cdot S(I(t))$. Equilibrium requires $p'(t) = p(t)$.

Obtaining analytical results for equilibrium models with fixed costs is virtually impossible. Consequently, almost all results for these models come from numerical examples (the solution of which is also quite difficult). The difficulty in solving the model arises from the presence of the time-varying distribution *f* – an infinite-dimensional, endogenous state variable. The strategy I follow in this paper is to derive approximate analytical results by establishing some basic properties of firm behavior and incentives in the steady state. These properties can be used to analyze the approximate behavior of the system away from the steady state. I then use numerical techniques to confirm the approximate analytical results.

3.1 The Optimal Timing of Investment in the Steady State

In the steady state, the price level and the level of productivity are constant. I normalize both the steady state price (*p*) and steady state productivity (*A*) to be 1. Let $\bar{V}$ denote the steady state value of having $\bar{k}$ units of capital and behaving optimally. The optimization problem of a typical firm is to choose a time to adjust *T* to maximize

$$V(T) = \int_0^T e^{-rt}\left(e^{-\delta t}\bar{k}\right)^{\alpha} dt + e^{-rT}\left[\bar{V} - F\right] - e^{-rT}\left[\bar{k} - e^{-\delta T}\bar{k}\right]. \tag{2}$$

The first order condition for the optimal choice of *T* is

$$V_T(T) = \left(k(T)\right)^{\alpha} - r\left[\bar{V} - F - \bar{k}\right] - (r+\delta)k(T) = 0, \tag{3}$$

where $k(T) = e^{-\delta T}\bar{k}$. At the optimum, the loss the firm would incur by waiting a bit more (*dT*) is zero. The first term in V_T is the gain the firm would get by using its existing capital stock more. The second term reflects the fact that waiting delays the payoff $\bar{V} - F - \bar{k} > 0$. The last term shows that the firm also suffers by delaying the resale of its existing capital and because the capital stock deteriorates, reducing its resale value. At the optimum, all of these forces balance and the firm is indifferent between adjusting and waiting.

The second order condition shows what happens to the first-order costs and benefits as the firm delays or accelerates adjustment. The second order condition requires

$$\delta k(T)\left[\alpha k(T)^{\alpha-1} - (r+\delta)\right] > 0. \tag{4}$$

Condition (4) says that if it is optimal to adjust at time *T*, then the marginal product of capital when the firm adjusts $\alpha k(T)^{\alpha-1}$ must be strictly greater than the user cost of capital $r+\delta$. The difference between the marginal product and the user cost plays an important role in the analysis. I refer to this difference as the Jorgenson gap and denote it as $G(\delta,T) = \alpha k(T)^{\alpha-1} - (r+\delta)$.

While I do not allow the firm to choose its reset level of capital $\bar{k}$, I assume that $\bar{k}$ is optimal in the steady state. If the firm adjusts every *T* periods, and has a reset capital level $\bar{k}$, then I can write *V* as

$$V\left(\bar{k},T\right)=\frac{1}{1-e^{-rT}}\left\{\bar{k}^{\alpha}\frac{1-e^{-(r+\alpha\delta)T}}{r+\alpha\delta}-e^{-rT}\left[F+\bar{k}\left(1-e^{-\delta T}\right)\right]\right\}, \tag{5}$$

where I have used the fact that $\int_0^T e^{-rt}\left[e^{-\delta t}\bar{k}\right]^{\alpha}dt=\bar{k}^{\alpha}\left[1-e^{-(r+\alpha\delta)T}\right]\left[r+\alpha\delta\right]^{-1}$. If the firm could choose its reset capital stock, then $\bar{k}$ would solve $\max_{\bar{k}}\left\{V\left(\bar{k},T\right)-\bar{k}\right\}$. The first order condition for $\bar{k}$ would require

$$\alpha\left[\bar{k}\left(\delta,T\right)\right]^{\alpha-1}\left(\frac{1-e^{-(r+\alpha\delta)T}}{1-e^{-(r+\delta)T}}\right)\left(\frac{r+\delta}{r+\alpha\delta}\right)=r+\delta, \tag{6}$$

where I have written $\bar{k}\left(\delta,T\right)$ to reflect the dependence of the optimal reset level on the parameters δ and *T*. One can show that the marginal product of capital at $\bar{k}$ is less than the user cost $r+\delta$. For reference, I let k^J denote the capital stock at which the standard user cost relation holds, so $\alpha\left[k^J\right]^{\alpha-1}=r+\delta$. Thus, $\bar{k}$ exceeds the frictionless capital stock k^J, which in turn exceeds the capital stock at the optimal adjustment horizon $k\left(T\right)$. Note that $\lim_{T\to 0}\bar{k}\left(\delta,T\right)=k^J$, so as the horizon *T* gets shorter, the normal user cost relationship emerges. Figure 1 shows the relationship between $k\left(T\right)$, k^J, $\bar{k}\left(\delta,T\right)$ and the Jorgenson gap $G\left(\delta,T\right)$.

It is easy to show that the condition $V_T\left(\bar{k},T\right)=0$ implies condition (3). This first order condition gives the optimal *T* for any given $\bar{k}$ and any *F*. Alternatively, I can invert the first order condition to find a fixed cost $F\left(\delta,T\right)>0$ that rationalizes a given adjustment horizon *T* and a given $\bar{k}\left(\delta,T\right)$. I prefer to cast the problem in terms of adjustment horizons (*T*) rather than fixed costs (*F*) since firms' adjustment horizons are more easily observed than are their fixed costs. Thus, in what follows, I devote relatively little attention to the magnitude of the fixed costs themselves and instead focus on the length of time it takes firms to adjust. In Cooper *et al.* [1999], 18 percent of firms experience an investment spike each year implying an average adjustment horizon of roughly five to six years.

3.2 The Intertemporal Elasticity of Substitution.

In this section I demonstrate that firms in the fixed-cost model have very high intertemporal elasticities of substitution for the timing of investment purchases. This high intertemporal elasticity is the key observation that allows us to analyze the solution. It is also a property that fixed-cost models share with neoclassical investment models.

Consider the loss to the firm from adjusting early or late by an amount dT. The loss from this suboptimal behavior is $L(dT) = V(T) - V(T+dT)$, which, to a second order approximation, is

$$L(dT) \approx -\frac{1}{2}V''(T)(dT)^2 > 0.$$

Since T is optimal, we can use (3) and (4) to show that

$$\frac{L(dT)}{r\left[\bar{V} - F - \bar{k}\right]} \approx \frac{1}{2}\delta\alpha\left[\frac{G(\delta,T)}{G(\delta,T) + (1-\alpha)(r+\delta)}\right](dT)^2 < \frac{1}{2}\delta\alpha(dT)^2. \tag{7}$$

Equation (7) says that the loss relative to the annuity value of the firm's profits is less than $\delta\alpha(dT)^2/2$. To put this in quantitative terms, consider compensating the firm to invest one year in advance ($dT = 1$). If the annual depreciation rate were four percent ($\delta = .04$) and if $\alpha = .5$, then the left-hand side of (7) would need to be no greater than .01. That is, the firm would require only one percent of its annual flow profits in compensation for adjusting early (or late) by one year. Equation (7) also shows that the loss is related to the size of the Jorgenson gap $G(\delta,T)$. If $G(\delta,T)$ is small, then the loss is even less than $\delta\alpha(dT)^2/2$.

This finding – that losses from adjusting early or late even by large amounts are small relative to flow profits – provides our first glimpse into why the underlying distribution of firms has little influence on the aggregate behavior of investment. Figure 2 plots two distributions of firms' capital holdings in an environment in which firms adjust every 10 years in the steady state. The shaded rectangle represents the steady state distribution of capital holdings. The steady state distribution is uniform. There is an equal number of firms with capital of every age. The heavy dark line represents an extreme alternate distribution in which the firms are concentrated on only five capital vintages. Each vintage is owned by 1/5 of the firms and there are no other capital vintages. In fact, because the firms are so willing to retime their investments, this distribution is much closer to the steady state distribution than it appears. Suppose we modify the usual profit maximization requirement for equilibrium and instead require that firms only come within $\varepsilon > 0$ of maximum profits. This relaxed version of equilibrium is sometimes referred to as an ε-equilibrium (see Everett [1957]). With the parameter values above, adjusting early or late by one year costs the firm at most one percent of its annual flow profits. If $\varepsilon = (0.01)\left(r\left[\bar{V} - F - \bar{k}\right]\right)$, then the steady state price and investment paths $p(t) = \bar{p} = 1$ and $I(t) = \bar{I}$ for all *t* constitute an ε-equilibrium for both the steady state distribution and the extreme distribution. Even though the distribution looks starkly different from uniform, it is actually within ε of the steady state.

Returning to strictly optimal firm behavior, consider the change in payoffs from a small change in the purchase price of capital *dp*. In this case, the change in the payoff is simply $-dp\left[\bar{k} - k(T)\right]$. Putting this loss relative to the annuity value of profits gives

$$\frac{L(dp)}{r\left[\bar{V} - F - \bar{k}\right]} = \alpha\left[\frac{e^{\delta T} - 1}{G(\delta,T) + (1-\alpha)(r+\delta)}\right](dp), \tag{8}$$

which is positive if prices rise and negative if they fall. Using (7) we can solve for the price change required to make the firm indifferent between adjusting now and adjusting in one year. This price change is

$$dp \approx -\frac{1}{2}\delta\frac{G(\delta,T)}{e^{\delta T} - 1}(dT)^2 \approx -\frac{G(\delta,T)}{2T}(dT)^2. \tag{9}$$

Not surprisingly, the Jorgenson gap $G(\delta,T)$ again emerges as the central determining factor for how willing firms are to retime capital purchases in response to price changes.

At this point, it helps to get a sense of the magnitude of the gap. Recall that $G(\delta,T) = \alpha k(T)^{\alpha-1} - (r+\delta)$. If $\bar{k}$ is optimal, then $\bar{k}(\delta,T)$ satisfies equation (6) so that $k(T) = e^{-\delta T}\bar{k}(\delta,T)$ and we can solve directly for $G(\delta,T)$. To get a simple expression for $G(\delta,T)$, notice that (6) suggests that, for small *T*, the optimal reset level $\bar{k}(\delta,T)$ is not far from the frictionless level k^J. If we assume that $\bar{k}(\delta,T) \approx k^J$, then

$$G(\delta,T) \approx \alpha\left[e^{-\delta T}k^J\right]^{\alpha-1} - (r+\delta) \approx \delta(r+\delta)(1-\alpha)T\ , \tag{10}$$

where I have used $e^{(1-\alpha)\delta T} - 1 \approx (1-\alpha)\delta T$. For example, if $T = 10$, $\delta = .04$, $r = .02$, and $\alpha = .5$, then (10) suggests that $G(\delta, T) \approx .012$. Thus, the gap between the marginal product and the user cost when the firm adjusts is roughly 1 percentage point. Because the approximation above assumes that $\bar{k} = k^J$ rather than $\bar{k} = \bar{k}(\delta, T) > k^J$, the true gap is actually somewhat smaller than the approximation suggests. Figure 3 plots the exact $G(\delta, T)$ for several time horizons *T* and depreciation rates δ.

We can now use (9) and (10) to find the price change required to induce a firm to change its investment timing by an amount *dT*.

$$dp \approx -\frac{1}{2}\delta(r+\delta)(1-\alpha)(dT)^2 .$$

For long-lived investment goods, this price change is very small indeed. Mechanically, the magnitude of the price change is dominated by the presence of the term $\delta(r+\delta)$ which is very small for depreciation rates less than .05. For example, given the parameters above, the price change required to induce a firm to change its timing by one year is roughly $dp \approx -0.0006$ or 6 basis points (6/100ths of one percent). By retiming their investments, the firms will cause investment to be greatly smoothed out at the aggregate level. The price incentives necessary to achieve this smoothing are so small that they will not be detectable in data.

Of course, the quantitative evaluation above depends on parameters. Using a higher discount rate, or a lower curvature parameter α will increase the price response, though not by a large amount. Increasing *r* to .08 or reducing α to zero will both increase the price change to 12 basis points. (Gourio and Kashyap [2007] use a numerical model to analyze the effects of changing α.) Adding economic growth acts like a form of depreciation. With growth, the price change is $-(1/2)(\delta+g)(r+\delta+g)(1-\alpha)(dT)^2$. If $g = .02$, then the price response would again be 12 basis points.

Another way to see the same point is to compute a price path for which the firm is indifferent as to when to adjust. If we allow for a time-varying price $p(t)$ in (2), then the first-order condition for *T* is

$$k(T)^{\alpha} - r\left[\bar{V} - F - p(T)\bar{k}\right] - p(T)\left\{(r+\delta)k(T) - \frac{\dot{p}(T)}{p(T)}\left(k(T) - \bar{k}\right)\right\} = 0\,. \tag{11}$$

If the firm were indifferent between any adjustment horizon, then (11) must hold for all *T*. The solution to this differential equation satisfies

$$p(t)\left(1 - e^{-\delta t}\right) = \frac{\bar{V} - F}{\bar{k}} - \bar{k}^{\alpha-1}\frac{e^{-\alpha\delta t}}{r + \alpha\delta} + e^{-rt}C\,, \tag{12}$$

where C is an unknown constant.[5] Restricting the path to satisfy $p(T) = 1$ we have

$$p(t) \approx 1 - e^{-\delta T}\frac{1}{2}\delta(r+\delta)(1-\alpha)(t-T)^2.$$

If the gap $G(\delta, T)$ is small, the indifferent price path stays close to the steady state price $p = 1$. Clearly the price path that makes firms indifferent about when to adjust is more flat for long-lived capital than for short-lived capital.

Figure 4 plots several indifferent price paths for various depreciation rates δ assuming that firms adjust every 10 years in steady state. All of the indifferent price paths are quite flat. Even for $\delta = .2$, the firm requires only a 2 percent price cut to adjust two years early or late. For $\delta = .05$, a price reduction of only 20 basis points is enough to cause the firm to delay or accelerate investment by roughly two years. Again, the important thing to realize is that the indifferent price paths are very close to the steady state price. Put differently, in the steady state, while it is optimal to adjust at date *T*, the firm is willing to adjust at almost any date.

3.3 Implications

In this section I consider the consequences of the near infinite elasticity of investment demand with respect to anticipated price changes in fixed-cost models. I also compare the fixed-cost model with a standard neoclassical investment model. The comparison reveals that the equilibrium behavior of both models can be reduced to a simple supply and demand system. The supply and demand analysis allows me to summarize how various shocks influence the equilibrium.

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[5] To derive this condition, I assume the reset value $\bar{V}$ is independent of the time of adjustment *T*.

Investment Demand. The analysis above shows that slight changes in prices cause firms to dramatically alter the timing of their investment decisions. For sufficiently long-lived investment projects and sufficiently patient firms (low δ and low *r*), the incentive to delay or accelerate investment in response to predictable price changes is nearly infinite. Thus, despite the apparent complexity of the fixed-cost model, characterizing its dynamic behavior is disarmingly simple. Investment demand is approximately summarized by a perfectly flat demand curve. The demand curve may shift, but these shifts are unpredictable. Specifically, the demand curve will shift in response to highly persistent innovations. If the shocks confronting the firm are short-lived, and thus have little impact on the long-run value of capital, the demand curve simply remains close to the steady state price. This continues to be true regardless of whether there are relatively many or relatively few firms near the steady state adjustment trigger. If there are many firms considering adjustment, the demand curve shifts to the right. If few firms are at near the adjustment margin, the demand curve shifts to the left. Because demand is nearly horizontal, the price is unaffected and the equilibrium quantity of investment is determined solely by the supply curve.

The reader may be struck that the implied price path is so close to a partial equilibrium framework (in which prices are assumed to be fixed), but yet aggregate investment is not influenced by the micro-level heterogeneity. After all, researchers have previously found that the distribution of capital has effects in partial equilibrium settings. Of course, the two results are perfectly consistent. Because firms are so sensitive to price changes, even a seemingly small departure from perfectly constant prices will cause dramatic changes in the equilibrium. This is why the early partial equilibrium models produce results that are in stark contrast to the more recent equilibrium models and is why partial equilibrium analyses of fixed cost models give such misleading results.

The near invariance of prices to transitory shocks in equilibrium is why the assumption of a constant reset capital stock ($\bar{k}$) has little bearing on the outcome. If prices changed over time then the firms would optimally vary the reset level. However, since prices are essentially constant, the optimal reset level is also essentially constant.

(Giving the firms the option to vary the reset capital level with prices makes them even more sensitive to price changes thus enhancing the results even further. [6])

<u>Comparison with Neoclassical Investment Models</u>. The extremely high intertemporal elasticity of investment demand is a feature that the fixed-cost model shares with neoclassical investment models and it is why the two models, though very different at the micro-level, are often indistinguishable at the aggregate level. In neoclassical settings, firms equate the marginal cost of investment with the marginal benefit of more capital. Let $q(t)$ be the marginal benefit of additional capital. In a neoclassical framework,

$$q(t)=\int_t^{\infty} e^{-(r+\delta)s} MP^k(s)\,ds\,, \tag{13}$$

where $MP^k(s)$ is the marginal product of capital at time *s*. For the optimal level of investment, $q(t)=p(t)$.

For low δ and low *r*, there cannot be large predictable movements in the shadow value of capital $q(t)$. With sufficiently long-lived capital and sufficiently short-lived shocks, one can safely approximate $q(t)$ with its steady state value $\overline{q}$.[7] The marginal value of capital $q(t)$ is a discounted sum of payoffs. If the firm is patient and depreciation is slow, the integral places substantial weight on terms in the distant future. Because transitory shocks influence only the first few terms in the integral, they have a negligible impact on the value of capital and thus $q(t)\approx\overline{q}$. By affecting most or all of the terms in the integral, more persistent, long-lasting shocks have a substantial effect on $q(t)$. Nevertheless, since the shadow value is largely determined by the future terms in (13), $q(t)$ will remain virtually constant in expected value.

For sufficiently long-lived capital goods, the investment demand curves implied by the neoclassical model and the fixed cost model are the same. Because it is nearly perfectly elastic with respect to predictable variations in prices, investment demand can

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[6] Using the numerical model in Section IV, I evaluated the relative performance of a model with a variable reset capital stock. The outcomes were essentially the same.

[7] Barsky *et al.* [2007] and House and Shapiro [2008] evaluate the accuracy of this approximation in neoclassical settings.

be approximately summarized by a flat line at the steady state price. As a result, in the low-depreciation limit, any differences in equilibrium outcomes reflect differences in the specification of supply. If supply is the same in the two models, then the equilibrium outcome in the fixed-cost model and the equilibrium outcome in the neoclassical model will be virtually identical. More precisely, in the low-depreciation limit, both models have identical reactions to transitory shocks. Neither transitory supply shocks nor transitory demand shocks cause perceptible changes in prices and only supply shocks cause changes in investment.[8]

The Distribution of Capital Holdings. The distribution of capital holdings features prominently in both the theoretical and empirical literature on fixed costs. In theory, the equilibrium should depend on the cross sectional distribution of capital holdings across firms. In empirical studies, researchers have tried to test whether observed variations in the distribution predict future movements in investment and prices. Using LRD data, Caballero *et al*. [1995] show that changes in the distribution of capital explain changes in the responsiveness of investment to shocks. Similar results are in Caballero and Engel [1999] who use BEA investment data and show that the distribution has predictive power for aggregate investment.

The analysis in this paper suggests that, for sufficiently long-lived investments, variations in the distribution of capital holdings across firms should have no independent influence on equilibrium investment or prices. In particular, if the only changes to the system are changes in the distribution, then equilibrium prices and investment should remain close to their steady state levels. Thus, the model and analysis so far indicate that, for long-lived investments, the coefficients on moments of the cross-sectional distribution in forecasting regressions should be close to zero and the increase in predictive power from adding additional moments should be negligible. That is, knowledge of the distribution should provide little to no information regarding the future behavior of investment or prices.

[8] This is reminiscent of recent work by Jonas Fisher who argues that investment supply shocks are primarily responsible for aggregate business cycle fluctuations. See Fisher [2006].

Of course, the analysis above is only approximate. Furthermore, the identical behavior of the neoclassical model and the fixed-cost model, and the irrelevance of the distribution are results that we should expect only in the low-depreciation limit. In the next section, I consider a numerical version of the model to assess the accuracy of the approximate solution and the limiting results. The numerical model shows that the approximation is accurate even for modest depreciation rates.

IV. NUMERICAL ANALYSIS AND APPLICATIONS

Based on the analysis in Section III and the discussion in Section 3.3, we should expect to observe the following in fixed-cost models for long-lived investments: (1) a temporary cost shock should have no noticeable effect on the price of new capital but should reduce equilibrium investment by the amount of the shock; (2) a temporary demand shock (modeled as a temporary increase in A) should have virtually no influence on prices or investment; (3) different initial distributions of capital should have no consequences for prices or investment; and (4) for sufficiently transitory shocks, the aggregate behavior of the fixed-cost model should be identical to the aggregate behavior of a conventional neoclassical investment model.

In this section I analyze a numerical version of the model in Section III. The numerical model confirms the limiting analysis of the preceding section and allows me to evaluate the accuracy of the approximation for realistic parameter values. I begin by sketching out the broad features of the numerical model. The details of the numerical solution are in the appendix.

4.1 Quantitative Model

The numerical model is cast in discrete time with time intervals of size Δ. There are $J+1$ possible capital stocks $\bar{k}, k_1, k_2, ..., k_J$ with $k_j = e^{-\Delta j}\bar{k}$. The lowest possible capital stock is k_J. Let $V_{j,t}$ be the value of having capital stock *j* at time *t* and let $\bar{V}_t$ be the value of having the reset level $\bar{k}$ at time *t*.

The numerical solution uses a method developed jointly by Robert King, Julia Thomas and Marcelo Veracierto.[9] The key simplifying assumption of the method is to assume that firms draw idiosyncratic fixed costs of adjustment each period. Thus, instead of facing the fixed cost F each period, firm i faces the stochastic fixed cost $\varepsilon_{i,t}$ where $\varepsilon_{i,t} \sim \Psi(\varepsilon)$, $E\left[\varepsilon_{i,t}\right] = F$, $\varepsilon_{i,t} \geq 0$ and $\varepsilon_{i,t}$ is *i.i.d.* across periods and across firms. For purposes of computation, I assume that $\varepsilon_{i,t}$ is a mixture of a log-normal random variable and a wide uniform. Given a time interval Δ, I construct the discount factor $\beta = e^{-r\Delta}$. I can then write the value for a firm with cost draw ε, capital stock $k = k_j$ at time t as

$$V_{j,t}(\varepsilon) = \max\left\{\Delta \cdot A_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right], \Delta \cdot A_t \bar{k}^{\alpha} + \beta E_t\left[v_{1,t+1}\right] - \varepsilon - p_t\left(\bar{k} - k_j\right)\right\}, \tag{14}$$

where

$$v_{j,t} = \int_0^{\infty} V_{j,t}(\varepsilon)\, d\Psi(\varepsilon). \tag{15}$$

The marginal firms with capital stock j have critical cost draw

$$\hat{\varepsilon}_{j,t} = \Delta \cdot A_t\left[\bar{k}^{\alpha} - k_j^{\alpha}\right] + \beta E_t\left[v_{1,t+1} - v_{j+1,t+1}\right] - p_t\left(\bar{k} - k_j\right). \tag{16}$$

The critical $\hat{\varepsilon}_{j,t}$ is differentiable. Firms with cost draws higher than $\hat{\varepsilon}_{j,t}$ choose not to adjust and firms with cost draws lower than $\hat{\varepsilon}_{j,t}$ adjust. If a firm with capital stock j chooses to adjust, its investment is $\bar{k} - k_j$. Aggregate investment I_t is the sum of individual firm-level investment.

To close the model, I assume an isoelastic supply curve

$$p_t = z_t \cdot \left(I_t / \bar{I}\right)^{\frac{1}{\xi}}. \tag{17}$$

Here ξ is the elasticity of investment supply, $\bar{I}$ is steady state investment and z_t is a cost shock with mean 1. The cost shock (z) and the productivity shock (A) are assumed to have simple autoregressive forms

$$A_{t+1} = \left(1 - \rho_A\right) + \rho_A A_t + \eta_{A,t+1}, \tag{18}$$

$$z_{t+1} = \left(1 - \rho_z\right) + \rho_z z_t + \eta_{z,t+1}. \tag{19}$$

[9] Dotsey, King and Wohlman [1999] used this technique to analyze a menu cost model. Thomas [2002] and Veracierto [2002] used it to analyze investment. King and Thomas [2006] analyze labor adjustment.

I choose parameter values for illustrative purposes only. The elasticity of supply ξ is set to 1. The autoregressive parameters ρ_z and ρ_A are set to imply a 6-month half-life of the shocks and, together with the variances of the innovations η_z and η_A, imply an unconditional variance of one percent for *z* and *A*.[10] I set the discount rate *r* to 2 percent annually. I set the parameter α to 0.35 and I set T to 10 so that firms adjust once every ten years in steady state. I choose a ten year adjustment horizon because it is both a plausible calibration and also a good number for expositional purposes. Since ten years is less frequent than the frequency of investment spikes in the Cooper *et al*. [1999] study (between five and six years), the fixed costs implied by the baseline calibration are greater than the data require. The baseline parameter values are summarized in Table 1.

Because it plays a central role in governing the system, I consider several different depreciation rates. By definition, long-lived capital goods have low depreciation rates. I consider a capital good to be long-lived if it has an annual depreciation rate less than five percent. Examples of such goods include manufacturing structures, commercial office buildings, electrical transmission and distribution apparatus, telecommunications structures, and so forth. Most structures are long-lived capital goods (structures typically have depreciation rates between two and four percent). Because structures make up roughly thirty percent of all non-residential investment, long-lived investments are not a trivial fraction of investment activity. The baseline annual depreciation rate δ is 5 percent.

4.2 Quantitative Analysis

I can now assess the accuracy of the analysis from Section III. I begin by considering temporary supply shocks and temporary demand shocks. I also consider the quantitative role of the distribution in governing the equilibrium and in forecasting future price movements.

[10] A 6-month half-life is fairly transitory compared with productivity shocks in the RBC literature. TFP shocks with quarterly autoregressive roots of 0.95 have half-lives of almost 3.5 years. The 6-month half-life is a common degree of persistence in the price rigidity literature. Allowing for more persistent shocks is straightforward. For sufficiently long-lived capital, the neoclassical model and the fixed costs model behave identically regardless of the persistence.

Supply Shocks. I consider a positive innovation of one percent to z_t in equation (17). This increases the cost of investment and shifts the supply schedule back. I consider five annual depreciation rates: 0.20, 0.10, 0.05, 0.02 and 0.01. Twenty percent depreciation is comparable to depreciation rates experienced by computers, software and some vehicles. Typical business equipment has a depreciation rate of roughly ten percent per year. The five, two and one percent depreciation rates correspond to depreciation rates of many structures (e.g., residential investment and business structures have depreciation rates of roughly two percent. For more on empirical depreciation rates, see Fraumeni [1997]).

Figure 5 shows the system's reaction to the temporary cost shock. The top panel shows the response of aggregate investment. The middle panel shows the response of the price level and the bottom panel shows the cost shock variable itself (the cost shock is the same for each depreciation rate). In the figure, as one would expect from the earlier analysis, the equilibrium price of new investment changes only slightly in response to the shock. For $\delta = 0.10$ and $\delta = 0.20$, the increase in prices on impact is roughly 12 basis points (0.12 percent). For lower depreciation rates the price change is even smaller. For example, for $\delta = 0.01$ and $\delta = 0.02$, the increase in prices is roughly 1 basis point (1/100th the size of the impulse). Since prices change only slightly, most of the adjustment to the shock occurs through changes in aggregate investment. For each depreciation rate, the drop in investment is almost 1.00 percent. For $\delta = 0.01$ and $\delta = 0.02$, the drop is 0.99 percent. For the higher depreciation rates, the drop in investment is roughly 0.9 percent. This behavior is exactly what the earlier analysis predicted. The approximation is better for low depreciation rates as the gap $G(\delta, T)$ approaches zero and the elasticity of demand approaches infinity.

Demand Shocks. Figure 6 shows the response to a temporary one percent increase in productivity A_t. Since supply is unchanged and since the elasticity of supply is 1.00, the reactions of prices and investment are identical. As predicted, the changes in prices and investment are small. For $\delta = 0.20$ and $\delta = 0.10$, prices and investment rise by roughly 15 basis points and 8 basis points, respectively. For $\delta = 0.01$ and $\delta = 0.02$, the increases are 3 basis points and 2.5 basis points. Because the shock is transitory, the value of long-lived capital goods is essentially unaffected.[11]

A Non-uniform Initial Distribution. I now consider the equilibrium when the system begins with an out-of-steady-state distribution.[12] The initial distribution I consider has an unusually large number of firms with capital that is five years old. To make the illustration stark, the initial density of firms with capital between 4.5 and 5.5 years old is twice the density elsewhere. The distribution is shown in the top panel of Figure 7. The steady state distribution is shown for comparison. Because the out-of-steady-state distribution has twice as many firms with five-year-old capital, one would anticipate that, in roughly five years, prices and investment would rise sharply as these firms approach the adjustment trigger. If firms could not change the timing of investment at all, then prices and aggregate investment would rise by 100 percent.

The middle panel of Figure 7 shows the equilibrium path of aggregate investment. Since the supply curve is stable and the elasticity of supply is 1.00, investment and prices are identical. The initial distribution has some influence on the equilibrium. The conventional supply and demand prediction that prices and investment should rise as the mass of firms adjusts is present in the figure but is quantitatively negligible relative to the magnitude of the distributional change. Instead of an increase of 100 percent, investment rises by only one-half of one percent for capital with a ten percent depreciation rate. For capital with a five percent depreciation rate, the equilibrium increase in investment is only 20 basis points. The reason the distribution exerts such little influence on the equilibrium is the high intertemporal elasticity for the timing of investment combined with a slight increase in prices.

The middle panel presented results for an elasticity of investment supply of 1.00. Some estimates of investment supply elasticities are substantially higher than this (see for example House and Shapiro [2008]). If the intertemporal elasticity of investment demand were literally infinite (as it is in the low-depreciation limit), then the form of the

[11] The near-infinite elasticity of investment demand implies that after-tax prices are constant for temporary shocks. A temporary investment subsidy increases pre-tax prices by the amount of the subsidy. House and Shapiro [2008] use this property to estimate ξ following the 2002 bonus depreciation provisions. While their analysis uses a neoclassical model, the estimates are valid in a model with fixed-costs of adjustment.

[12] This thought experiment is inspired by Gourio and Kashyap [2007] who consider a similar out-of-equilibrium experiment in their numerical model.

supply curve would not matter at all. Since the elasticity of investment demand is actually finite, higher supply elasticities will temper the price changes and allow the distribution to play a slightly greater role. The bottom panel of Figure 7 considers five different supply elasticities. Each line corresponds to a different value of ξ. The depreciation rate is set to its baseline value $\delta = 0.05$. It is remarkable how little influence the supply elasticity has on the equilibrium. Even with $\xi = 20$, the maximum change in aggregate investment is only 2 percent. Compared to the exogenous 100 percent increase in firms with five-year-old capital, this is very small. Only for an elasticity of 100 does aggregate investment react noticeably, and even then by less than 10 percent.

Using the Distribution to Forecast Prices. Another way to quantify the importance of the distribution of capital holdings it to ask whether it can be used to forecast prices. Firms care about the distribution only because it contains information about future prices. If the distribution is important in forecasting prices then the R^2 of a forecasting equation should be higher if we include information about the distribution. The analysis in Section III suggests that, because the distribution of capital holdings has only minor bearing on the equilibrium, the improvement in forecasts of future prices (and investment) should be negligible.

Consider forecasting equations of the form

$$p_{t+h} = \beta_0 + \beta_p p_t + \beta_z z_t + \beta_A A_t + \sum_m \beta_m M_t^m + e_{t+h}\,, \tag{20}$$

where h is the forecast horizon, the variables M_t^m are a set of moments of the date t distribution and e is a reduced-form error. Although any set of moments is admissible, I consider the number of firms in each fifth of the capital space at time t. Specifically, at time t, M_t^m is

$$M_t^m = \sum_{a=(m-1)\frac{T}{5}}^{m\frac{T}{5}} f_t(a)\,, \tag{21}$$

for $m = 1,...,5$ and where $f_t(a)$ is the number of firms with capital of age a at time t. In the steady state, $M^m = 1/5$ for all m.

To assess the predictive value of these moments, I use the numerical model to calculate the asymptotic values of the coefficients in (20).[13] I consider the baseline parameter values and uncorrelated, i.i.d., supply and demand shocks η_z and η_A. The shocks are normally distributed with variances that, together with ρ_A and ρ_z, imply z_t and A_t have unconditional variances of one percent.

Table 2 reports the standard deviations of investment I_t, price p_t, the supply and demand variables z_t and A_t and the moments $M_t^1,...,M_t^5$ from the numerical model. Table 3 reports estimated coefficients for the forecasting equation (20) for horizons $h = 1$, 2, 4 and 8 quarters. The most important forecasting variable is the price itself. The model implies that prices are very close to a random walk. Thus, while the shocks have only small impacts on prices, the effects are long-lasting. Notice that the coefficients on the moments, while small, are not zero. The distribution is a true state variable so it is relevant for forecasting prices. However, the gain in forecast accuracy measured by the change in R^2 as we add more and more moments is negligible (M_t^5 is not included because it is an exact linear combination of the other moments). To a first approximation, it is reasonable for investors to simply ignore the distribution when forming expectations about future prices.

These findings are governed to some extent by parameter values. Table 4 reports results for several different parameter values. Each row reports the R^2 for particular forecast horizons and specifications (i.e., how many moments are included). The columns consider different parameter values. Column (0) reports results for the baseline specification. Columns (1) – (9) consider models with baseline parameter values except for the parameter listed in the column heading. Columns (1) – (3) consider depreciation rates $\delta = .02$, .10, and .50; (4) – (6) report results for supply elasticities $\xi = 5$, 10, and 100; (7) – (9) consider curvature parameters $\alpha = .50$, .15, and .05. Column (10) reports results from a "myopic" model with $\delta = .50$, $r = .50$, $\alpha = .10$ and $\xi = 5$.

As in the baseline case, the performance of the forecasting equations is for the most part unchanged as we include additional moments of the distribution. There are

[13] The results reported below come from a simulation of 100,000 years of quarterly observations.

exceptions. For high depreciation rates (e.g. $\delta = .50$) and for high supply elasticities (e.g. $\xi = 100$), the moments of the distribution matter somewhat. The distribution also matters more for distant forecast horizons. This is not surprising since the average adjustment horizon in the model is T = 10. In the myopic calibration, the distribution matters at almost every horizon.[14]

4.3 Comparison with Neoclassical Investment Models

Because both the fixed-cost model and the neoclassical investment model have extremely high elasticities of substitution for the timing of investment, the models should be difficult to distinguish using aggregate data alone. In this section, I solve a standard neoclassical investment model and compare the equilibrium outcomes with a similarly calibrated fixed-cost model.

Figure 8 presents simulated data from both the neoclassical investment model and the fixed-cost model. The neoclassical model is a standard discrete time investment model with flow production function $A_t k_t^{\alpha}$. The supply curve for both models is given by (17). The parameters of both models are set to the baseline values in Table 1. Both models are subjected to exactly the same sequence of shocks.

The upper panels in Figure 8 show results for aggregate investment, while the lower panels show results for prices. The panels on the left show simulated time series. The thin black line is the fixed-cost model while the thick grey line is the neoclassical model. While the time paths for aggregate investment are essentially identical, there are noticeable differences in the price series. The middle panels show the impulse response to a cost shock like the one considered in Figure 5. The response of aggregate investment is identical while the price response displays small differences. The panels on the right show scatterplots of 500 years of quarterly data. Each dot represents a data point from the neoclassical model and the corresponding data point from the fixed-cost model.

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[14] Careful readers will note that the R^2 in Krusell and Smith [1998] are much closer to 1.00 than those reported here. While the model they study is different, the main cause of the difference is that Krusell and Smith approximate the contemporaneous pricing function $p_t(\{M_t^m\})$ and forecast future prices with an approximate transition function for the moments themselves $Q:\{M_t^m\} \rightarrow \{M_{t+1}^m\}$, while I form the price forecasts directly. Regressing current prices p_t on current states z_t, A_t and moments $\{M_t^m\}$, gives an R^2 close to 1.00 (essentially regardless of the number of moments included). I thank Gianluca Violante for particularly helpful comments on this point.

Again, the investment data are virtually the same (all the observations are on the 45 degree line) while the price data display a noticeable difference. This pattern is robust to wide variations in the parameters.

Why is aggregate investment so similar across the two models while prices are not? Two points are worth emphasizing. First, the analysis in Section III said that the near-infinite elasticity of investment timing would eliminate price fluctuations in equilibrium. That result, however, was only approximate in nature. While the elasticity is very high, it is not infinite. The fact that we observe price changes in the simulation reflects this approximation error. Since observed changes in prices arise from imperfections in the approximation, we cannot use the approximations to argue that the price paths should be identical.

Second, price changes reflect changes in the equilibrium value of capital. In both cases, the price is tied to the long-run demand for capital. Unlike the demand for investment, which can be characterized by a (nearly) flat demand curve, the long-run demand for *capital* is downward sloping and the shape of this demand curve depends on the details of the model. In the neoclassical model, the price reflects the discounted marginal product of capital. In the fixed-cost model, the price reflects the discounted *average* product of capital. Thus, while we can ignore the details of the demand side of the model when we analyze investment, we cannot ignore these details when we analyze the long-run demand for capital. Since the price depends on the demand for capital, it is not surprising that we observe different equilibrium price paths.[15]

4.4 Relation to DSGE Models.

Most of the other well-known work in this area considers the numerical evaluation of calibrated DSGE models. Superficially, the investment supply and demand framework analyzed here—the supply side in particular—may seem fundamentally different from the DSGE models. Here I briefly consider the relationship of my model to DSGE models. I pay particular attention to the analog of the supply curve in the GE models.

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[15] Caplin and Leahy [2004] use a model with fixed costs and permanent shocks to establish a mapping between the parameters of the fixed-cost model and the neoclassical model such that the equilibria are the same. Their analysis requires a key assumption to keep the distribution of capital close to the steady state.

Consider the following conventional one-good general equilibrium model. A representative agent maximizes utility $E_t \sum_{j=0}^{\infty} \beta^j \left\{ u\left(C_{t+j}\right) - v\left(N_{t+j}\right) \right\}$ subject to $F\left(K_t, N_t\right) = C_t + I_t$ and $K_{t+1} = K_t\left(1-\delta\right) + I_t$. Here C is consumption, N is labor, $F(\cdot)$ is the production function, I is investment and K is the capital stock.

The marginal cost of an additional unit of investment at date t is the marginal utility of consumption $u'\left(C_t\right)$. The elasticity of the marginal utility of consumption with respect to investment is the analog of the elasticity of supply in the framework in Section III. That is, $\xi = \left(\partial I / \partial\left[u'(C)\right]\right) \times \left(u'/I\right)$. Near the steady state

$$\xi = \left(\frac{v''}{v'}N - \frac{F''}{F'}N\right)^{-1} \frac{F'N}{I} - \left(\frac{u''}{u'}C\right)^{-1} \frac{C}{I}. \tag{22}$$

F' and F'' are the first and second derivatives of the production function F with respect to labor N. The elasticity of supply will be high if the curvature terms $C \cdot \left(u''/u'\right)$, $N \cdot \left(v''/v'\right)$ and $N \cdot \left(F''/F'\right)$ are small or if the ratios $\left(F'N\right)/I$ and C/I are large.

If investment is small relative to total labor product and total consumption, then large percent changes in investment do not entail large percent changes in either consumption or labor and thus the percent increase in marginal cost is limited. Likewise, if the curvature terms are low then there can be large swings in consumption and labor without large changes in the marginal utility of consumption, the marginal disutility of labor, or the marginal product of labor. Consider the standard example: $u\left(C\right) = C^{1-\frac{1}{\sigma}}$, $v\left(N\right) = N^{1+\frac{1}{\eta}}$ and $F\left(K, N\right) = K^{\gamma} N^{\theta}$; σ is the intertemporal elasticity of substitution for consumption, η is the Frisch labor supply elasticity, and θ is the elasticity of output with respect to labor. In this case,

$$\xi = \frac{\eta}{1+\eta\left(1-\theta\right)} \theta \frac{Y}{I} + \sigma \frac{C}{I}. \tag{23}$$

A reasonable calibration might be $\sigma = 0.2$, $\eta = 1/2$, and $\theta = 2/3$. If we assume that the investment to GDP ratio is roughly 0.15 then $Y/I \approx 6.6$ and $C/I \approx 5.6$. In this case, the elasticity implied by (23) is roughly 3. If $\sigma = \eta = 1$, then the implied elasticity is 8.9.

Based on the analysis above, these calibrations will not permit micro-level heterogeneity to play a noticeable role in the equilibrium.

Assuming that the one-good DSGE framework is correct, one could estimate the parameters and calculate the implied elasticity. In her original paper, Thomas [2002] calibrates her model so that $\sigma = 1$ and $\eta = \infty$, which would seem to be indulging in a parameterization that gives micro-level heterogeneity a good chance to play a role. The investment to GDP ratio implied by her model is 0.20. The implied elasticity in her model is roughly 14. The baseline calibration in Bachmann *et al*. [2008] has $\sigma = 1$, $\eta = \infty$ and an investment to GDP ratio of 0.145 implying $\xi \approx 20$. They also consider $\sigma = 10$, which gives $\xi \approx 73$. Alternatively, one could estimate the elasticity directly. House and Shapiro [2008] use variation in tax rates to estimate ξ. Their estimates suggest that ξ is between 6 and 13.

4.5 Discussion.

In this section I explain why the near-infinite intertemporal elasticity of investment demand is the source of the "irrelevance results" found by the earlier literature. I also briefly discuss credit constraints, idiosyncratic shocks, and the cyclicality of aggregate investment.

<u>The Source of the Irrelevance Results</u>. The analysis in this paper strengthens and extends the "irrelevance results" obtained in many earlier papers on fixed costs in equilibrium settings. Prominent examples include Thomas [2002], Veracierto [2002], Gourio and Kashyap [2007], and Khan and Thomas [2003, 2008]. Broadly speaking, these papers show that the distribution of capital holdings matters if prices are held constant, but the distributional effects on aggregate investment vanish in general equilibrium.

The main contribution of this paper is to reveal the source of the irrelevance results. My numerical results echo the earlier findings in a more general setting, showing that specific reasons for an upward sloping investment supply schedule are not essential. In contrast, most of the earlier research attributes the irrelevance results to consumption smoothing motives. For example, in her 2002 paper Thomas writes (p. 510) that the "households' preference for smooth consumption profiles restrains shifts in investment demand. This dampening force plays the predominant role in equilibrium investment determination and produces the invariance result." The consumption smoothing incentive results in "pronounced differences in interest rates" (Thomas, p. 527) and "procyclical real wage(s)" (Thomas, p. 530). Thus, "equilibrium price movements smooth the economy's response to such a degree that distributional effects are eliminated" (Thomas, p. 511).[16]

This reasoning is, however, incomplete. While price movements are a "dampening" force which "restrains" investment demand, the earlier work did not find that the partial equilibrium results were dampened; they were eliminated entirely. Moreover, the price movements at the heart of the explanation are barely detectable in the earlier papers (see for instance Figures 5 and 6 in this paper, or Figures 4.a and 4.b in Thomas [2002]). The missing link in the explanation is the near-infinite intertemporal elasticity of investment demand. Extreme price sensitivity implies that price movements are virtually eliminated in equilibrium. While the distribution may change, it has no bearing on aggregate investment because firms are willing to delay or accelerate the timing of their investments by seemingly large amounts in response to small price changes. The precise source of price movements is irrelevant. Consumption smoothing and variations in interest rates play no role in the result. In my model there is no representative consumer and interest rates are assumed constant.

<u>Idiosyncratic Shocks, Credit Constraints and One-Hoss Shays.</u> The model intentionally abstracts from a variety of complicating features to expose the mechanisms at work. Undoubtedly many firms base their investment decisions on factors other than price. For example, many firms face binding credit constraints when making investment decisions. Idiosyncratic demand or supply considerations also surely play an important role in determining investment timing. Some firms simply have to replace capital because of an unforeseen event like a fire or a flood which might cause their existing capital stock to fail suddenly – a "one-hoss shay" depreciation process.

[16] This reasoning permeates the literature. In their most recent paper, Khan and Thomas [2008] write that "movements in relative prices […] eliminate the implications of plant-level nonconvexities for aggregate dynamics." The effects disappear because "procyclical […] real wages and interest rates substantially dampen the changes in plants' target capital stocks […]" (p. 429).

Surprisingly, as long as some firms are free to re-time their investments as the model here assumes, complications like those mentioned above leave the basic results intact. Firms that are free to re-time their investments effectively arbitrage away predictable movements in investment prices. The other firms invest as their circumstances dictate (due to credit conditions or other firm-specific factors). The presence of firms that can freely change the timing of their investments will imply that the expected price of investment goods remains nearly constant and thus, again, aggregate investment is determined by investment supply alone.[17]

The Cyclicality of Aggregate Investment. Aggregate investment is highly cyclical. Figures 5 and 6 suggest that most variation in aggregate investment comes from investment supply shocks rather than demand shocks. This is not a necessary feature of investment in fixed-cost models. The reason demand shocks play such a small role here is that they are temporary. Permanent (or very long-lasting) changes to the productivity of capital will shift investment demand and cause sharp changes in both investment and prices. In addition, the supply shocks need not be investment-specific. A general technology shock, like that in conventional RBC models, increases both supply and demand for investment (i.e., a one-percent productivity shock would increase A and z simultaneously by one percent).

As seen in the data, most of the fluctuations in aggregate investment in the model are associated with changes in the number of firms making adjustments. There is some variation in investment for each firm that adjusts, since, by delaying or accelerating the timing of investment, the firm influences the size of its capital purchases $\bar{k} - k(T)$. However, since depreciation is slow, the implied variation in $k(T)$ is quite small.

Breaking the Irrelevance Result. The reason for the irrelevance results is the extraordinarily high elasticity of investment demand. A modification that would break the irrelevance result must interfere directly with this elasticity. Unfortunately, this elasticity cannot be traced to a single free parameter. Instead, the near-infinite elasticity

[17] Khan and Thomas [2003] numerically demonstrate that idiosyncratic shocks have little impact on the equilibrium in a DSGE model with fixed investment adjustment costs.

arises naturally in models of long-lived investments. If there is a parameter in the model that governs this elasticity it is $\delta(r+\delta)$ – a combination of parameters that are not subject to much dispute. For the distribution of capital to play a role at the aggregate level, there must be some additional friction that tempers the firms' willingness (or their ability) to delay or accelerate investment. These frictions could take the form of planning costs which inhibit the firm's ability to adjust the timing of their investments, or information costs which prevent the firms from reacting to small price changes. In any case, a mechanism that succeeds in breaking the irrelevance result must not only make demand less elastic, it must do so for virtually all firms in the economy. If some firms are free to adjust the timing of their investments, the irrelevance result will emerge.

V. CONCLUSION

The study of investment is of central importance to understanding business cycles and economic activity. The drive to base aggregate theories on solid micro-foundations as well as the desire to match firm-level investment patterns has led to the development of complex models of investment behavior at the firm level. Investment models featuring fixed costs of adjustment are attractive because they imply that investment at the plant-level will be infrequent, as seen in micro data sets. In this paper, I have analyzed the approximate equilibrium behavior of a dynamic investment model with fixed adjustment costs. The analysis shows that for sufficiently long-lived capital goods, the elasticity of intertemporal substitution for the timing of investment is extremely high. As the depreciation rate approaches zero, this elasticity approaches infinity. The near-infinite elasticity of intertemporal substitution eliminates virtually any role for microeconomic heterogeneity in governing investment demand. This high elasticity of intertemporal substitution is a property that conventional neoclassical models of investment demand and models with fixed costs have in common. Thus, even though simple neoclassical investment models are starkly at odds with the micro data, they capture virtually all of the relevant aggregate investment dynamics embodied in models with fixed investment adjustment costs. This finding is highly robust and explains why researchers working in the DSGE tradition have found little role for fixed costs in numerical trials. Because the differences between the two models are small for plausible depreciation rates and vanish in the low-depreciation limit, conventional models offer an easy and accurate vehicle for economic analysis of investment decisions at the aggregate level.

Earlier numerical studies demonstrated that in many cases, DSGE models with fixed costs have aggregate dynamics that are virtually the same as the aggregate dynamics of conventional DSGE models that ignore fixed costs at the micro-level. My analysis makes two basic contributions to this research area. First, it shows why the irrelevance results occur in the numerical studies. The similarity at the aggregate level is not caused by consumption smoothing in a general equilibrium setting. Rather it is due to the extreme willingness on the part of firms to adjust the timing of investment to take advantage of predictable price changes. The firms are so willing to retime their investments that in equilibrium there can be no such price changes. With prices pinned down, aggregate investment is simply determined by the supply curve. Consumption smoothing is just one example of a source of increasing marginal costs of investment. Decreasing returns to scale in the production of capital goods, upward sloping labor supply curves or rising input costs of any sort will all cause marginal costs to rise and eliminate any meaningful role for fixed costs in governing aggregate investment.

Second, the analysis suggests mechanisms that could undo the apparent irrelevance of fixed costs. Specifically, anything which causes firms to resist changing the timing of their investments can potentially revive the potency of the distribution as a state variable and thus make the fixed cost model and the neoclassical model behave differently. These frictions must apply to many, if not all, firms. Frictions such as idiosyncratic shocks or credit constraints which constrain some firms but not others will leave the equilibrium largely unchanged as the other firms freely re-time their investments and eliminate price movements through arbitrage. In the absence of frictions that inhibit firms' freedom or incentive to change the timing of investments, fixed cost models and neoclassical models will display virtually identical aggregate behavior.

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Appendix

This appendix presents the numerical model analyzed in "Fixed Costs and Long-Lived Investments" by C.L.House.

<u>NUMERICAL MODEL:</u>

Here I present details on the numerical model used to analyze the behavior of the system away from the low-depreciation limit. The numerical solution follows the approach advanced by Dotsey, King and Wohlman [1999], Thomas [2002], Veracierto [2002], and Khan and Thomas [2003], King and Thomas [2006].

The numerical model is in discrete time. The size of each time interval is Δ. The possible capital stocks are given by a list of length $J+1$ so that k_j is the capital stock for a firm that last adjusted j periods ago. Then, $k_{j+1} = k_j e^{-\delta\Delta}$ and $k_0 = \bar{k}$. The minimum possible capital stock is k_J. Let $V_{j,t}$ be the value of having k_j at the beginning of period t and let $\bar{V}_t$ be the value of having $\bar{k}$ at the beginning of period t. These values are time-dependent because prices and other endogenous variables fluctuate over time.

The key aspect of the numerical approach is the use of idiosynchratic fixed costs rather than the single fixed cost F. Each firm i is presented with a fixed cost at time t given by $\varepsilon_{i,t}$. The fixed costs are i.i.d. across firms and over time. The fixed cost is assumed to have positive support (i.e., ε_{it} takes values in $[0,\infty)$) and to have mean F. I assume the stochastic fixed cost has a density function $\psi(\varepsilon)$ and associated distribution $\Psi(\varepsilon)$. For purposes of computation, I take ε to be a mixture of a log normally distributed variable ε^{LN} and a wide uniform ε^{U}.

The log normal random variable obeys $\ln\left(\varepsilon^{LN}\right) \sim \Phi(\mu,\sigma)$ where Φ is a Gaussian distribution with mean μ and variance σ^2. Because I require $E[\varepsilon_{it}] = F$, for any σ, the parameter μ must satisfy

$$\mu = \ln(F) - \frac{1}{2}\sigma^2 \tag{1}$$

which follows from a well-known property of log-normal distributions. Thus, once F is given, the log normal distribution has only a single free parameter: σ. The wide uniform variable has density $\frac{1}{2F}$ centered around F. The final composite random variable is $\varepsilon = \omega\varepsilon^{LN} + (1-\omega)\varepsilon^{U}$ with $\omega \in (0,1)$. Thus the expected value of ε is F. The density of ε is the weighted sum of the two densities:

$$\psi(\varepsilon) = \omega\left[\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\frac{1}{\sigma}\exp\left\{-\frac{1}{2}\left(\frac{\ln\varepsilon-\mu}{\sigma}\right)^2\right\}\right] + \frac{(1-\omega)}{2F}$$

and (using properties of the log-normal distribution), the c.d.f. of ε is

$$\Psi(\varepsilon) = \frac{\omega}{2}\left[1+\operatorname{erf}\left(\frac{\ln\varepsilon-\mu}{\sigma\sqrt{2}}\right)\right] + \frac{(1-\omega)}{2F}\varepsilon$$

where erf is the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left\{-s^2\right\} ds$. In the numerical setup below, I also require the truncated expectation $\int_0^b \varepsilon\psi(\varepsilon)d\varepsilon$. This expectation is

$$\int_0^b \varepsilon\psi(\varepsilon)d\varepsilon = \omega\frac{1}{2}\cdot\exp\left\{\mu+\frac{1}{2}\sigma^2\right\}\cdot\left[1+\operatorname{erf}\left(\frac{\xi^b-\sigma}{\sqrt{2}}\right)\right] + \frac{(1-\omega)}{4F}b^2$$

where $\xi^b = \frac{\ln b - \mu}{\sigma}$.

Since the $V_{j,t}$'s are the values of having k_j at the beginning of period t, we can write

$$V_{j,t}(\varepsilon) = \max\left\{\Delta\cdot z_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right], \Delta\cdot z_t\bar{k}^{\alpha} + \beta E_t\left[v_{1,t+1}\right] - \varepsilon - p_t\left(\bar{k}-k_j\right)\right\}$$

where $v_{j,t} = \int V_{j,t}(\varepsilon)\psi(\varepsilon)d\varepsilon$ is the expected value of being in state j at time t prior to the realization of the stochastic fixed cost ε. For the lowest capital stock k_J I assume that the firm must adjust and pays F with certainty. Thus, the expected value of entering the last grid point is

$$v_{J,t} = V_{J,t} = \Delta\cdot z_t\bar{k}^{\alpha} + \beta E_t\left[v_{1,t+1}\right] - F - p_t\left(\bar{k}-k_J\right) \tag{2}$$

Define $\hat{\varepsilon}_{j,t}$ as the critical draw for the fixed cost for firms in position j at time t that makes them just indifferent between adjusting and not:

$$\hat{\varepsilon}_{j,t} = \Delta \cdot z_t \left[\bar{k}^{\alpha} - k_j^{\alpha}\right] + \beta E_t \left[v_{1,t+1} - v_{j+1,t+1}\right] - p_t \left[\bar{k} - k_j\right] \tag{3}$$

Note, if $\varepsilon < \hat{\varepsilon}_{j,t}$ then I adjust. Thus, we can write $v_{j,t}$ as

$$\begin{aligned}
v_{j,t} &= \int V_{j,t}(\varepsilon)\,\psi(\varepsilon)\,d\varepsilon \\
&= \int_0^{\hat{\varepsilon}_{j,t}} \left[\Delta \cdot z_t \bar{k}^{\alpha} + \beta E_t\left[v_{1,t+1}\right] - \varepsilon - p_t\left(\bar{k} - k_j\right)\right]\psi(\varepsilon)\,d\varepsilon + \left[1 - \Psi\left(\hat{\varepsilon}_{j,t}\right)\right]\left(\Delta \cdot z_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right]\right) \\
&= \Psi\left(\hat{\varepsilon}_{j,t}\right)\left(\Delta \cdot z_t \bar{k}^{\alpha} + \beta E_t\left[v_{1,t+1}\right] - p_t\left(\bar{k} - k_j\right)\right) - \int_0^{\hat{\varepsilon}_{j,t}} \varepsilon\psi(\varepsilon)\,d\varepsilon + \left[1 - \Psi\left(\hat{\varepsilon}_{j,t}\right)\right]\left(\Delta \cdot z_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right]\right) \\
&= \Psi\left(\hat{\varepsilon}_{j,t}\right)\left(\Delta \cdot z_t\left[\bar{k}^{\alpha} - k_j^{\alpha}\right] + \beta E_t\left[v_{1,t+1} - v_{j+1,t+1}\right] - p_t\left(\bar{k} - k_j\right)\right) - \int_0^{\hat{\varepsilon}_{j,t}} \varepsilon\psi(\varepsilon)\,d\varepsilon + \left(\Delta \cdot z_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right]\right) \\
&= \Psi\left(\hat{\varepsilon}_{j,t}\right)\hat{\varepsilon}_{j,t} - \int_0^{\hat{\varepsilon}_{j,t}} \varepsilon\psi(\varepsilon)\,d\varepsilon + \Delta \cdot z_t k_j^{\alpha} + \beta E_t\left[v_{j+1,t+1}\right]
\end{aligned}$$

Thus, we have

$$v_{j,t} = \Psi\left(\hat{\varepsilon}_{j,t}\right)\hat{\varepsilon}_{j,t} - \int_0^{\hat{\varepsilon}_{j,t}} \varepsilon\psi(\varepsilon)\,d\varepsilon + \Delta \cdot z_t k_j^{\alpha} + \beta v_{j+1,t+1} \tag{4}$$

Finally, the supply curve for new investment is

$$p_t = \bar{p}\left(\frac{I_t}{\bar{I}}\right)^{\frac{1}{\xi}}$$

where I_t is total (aggregate) investment at time t and $\xi > 0$ is the elasticity of supply.

<u>Steady State:</u>

I normalize the supply curve so that in the steady state $p_t = p = 1$. There is then the question of how one can solve for the steady state values v_j, $\hat{\varepsilon}_j$. It is tempting to use the solution from the non-stochastic model in Section III of the text to find $\bar{V}$ however this is not correct. The presence of the stochastic fixed costs (rather than the pure fixed cost F) makes the value of being at $\bar{k}$ higher than otherwise because the firm has the option to adjust early to take advantage of a low fixed cost or to adjust late and avoid a high fixed cost.

To find the steady state of the modified model I follow the procedure outlined below:

1. Pick parameters r α δ σ ω J and T. Set μ from equation (1). Set the step size Δ. The discount factor is $\beta = e^{-r\Delta}$.

2. Set $\bar{k}$ at the non-stochastic level from equation (6) in the text. Construct the grid $k_1 = \bar{k}e^{-\delta\Delta}$, $k_2 = \bar{k}e^{-\delta 2\Delta}$, ... $k_j = \bar{k}e^{-\delta j\Delta}$.

3. Set v_1 (Note for the initial guess of v_1, I appeal to the non-stochastic setting in the text in which case $\bar{V} \approx \Delta \cdot \bar{k}^{\alpha} + \beta v_1$. The initial setting of v_1 is therefore $v_1 \approx \beta^{-1}\left(\bar{V} - \Delta \cdot \bar{k}^{\alpha}\right)$).

4. Equation (2) gives the steady state $v_J = V_J$ as

$$v_J = V_J = \Delta \cdot \bar{k}^{\alpha} + \beta v_1 - F - \left(\bar{k} - k_J\right).$$

5. Equation (3) then implies $\hat{\varepsilon}_{J-1}$

$$\hat{\varepsilon}_{J-1} = \Delta \cdot \left[\bar{k}^{\alpha} - k_{J-1}^{\alpha}\right] + \beta\left[v_1 - v_J\right] - \left[\bar{k} - k_{J-1}\right].$$

6. I then calculate v_{J-1} via quadrature using equation (4)

$$v_{J-1} = \Psi\left(\hat{\varepsilon}_{J-1}\right)\hat{\varepsilon}_{J-1} - \int_0^{\hat{\varepsilon}_{J-1}} \varepsilon\psi(\varepsilon)\,d\varepsilon + \Delta \cdot k_{J-1}^{\alpha} + \beta v_J$$

7. Then given v_{j+1} we can calculate $\hat{\varepsilon}_j$ with equation (3)

$$\hat{\varepsilon}_j = \Delta \cdot \left[\bar{k}^{\alpha} - k_j^{\alpha}\right] + \beta \left[v_1 - v_{j+1}\right] - \left[\bar{k} - k_j\right]$$

   and v_j with (4)

$$v_j = \Psi\left(\hat{\varepsilon}_j\right) \hat{\varepsilon}_j - \int_0^{\hat{\varepsilon}_j} \varepsilon \psi\left(\varepsilon\right) d\varepsilon + \Delta \cdot k_j^{\alpha} + \beta v_{j+1}$$

8. I repeat step (7) until I arrive at an implied v_1 say v_1'. If my initial guess $v_1 = v_1'$ then I have a set of steady state values and cutoffs. If not, I update v_1 and repeat from step 3. The steady state cutoffs $\hat{\varepsilon}_j$ values imply adjustment probabilities $\Psi_j = \Psi\left(\hat{\varepsilon}_j\right)$ for each grid point $j = 1, 2, ...J-1$ and I set $\Psi_J = 1$ since they must adjust at this point.

EQUILIBRIUM:

Let $f_{j,t}$ be the number (i.e., fraction) of investors at grid point j. Total investment at any date t is then

$$I_t = \sum_{j=1}^{J} \Psi_{j,t} \cdot f_{j,t} \cdot \left(\bar{k} - k_j\right)$$

The total number of firms is fixed $\sum_{j=1}^{J} f_{j,t} = 1$. Note, the numbers of firms at each grid point evolve according to

$$f_{j,t} = f_{j-1,t-1}\left(1 - \Psi_{j-1,t-1}\right)$$

for $2 \leq j \leq J$. For $j = 1$, we have

$$f_{1,t} = \sum_{j=1}^{J} \Psi_{j,t-1} \cdot f_{j,t-1}$$

so that all of the firms that adjusted last period arrive at gridpoint 1 the following period. To find the steady state values for f_j I use

$$f_j = \left(1 - \Psi_{j-1}\right) f_{j-1} = \left(1 - \Psi_{j-1}\right)\left(1 - \Psi_{j-2}\right) f_{j-2} = ... = f_1 \prod_{m=1}^{j-1} \left(1 - \Psi_{j-m}\right)$$

for all j between 2 and J. Then, to find f_1, I use

$$\sum_{j=1}^{J} f_j = f_1 + f_1\left(1 - \Psi_1\right) + f_1\left(1 - \Psi_1\right)\left(1 - \Psi_2\right) + ... = f_1\left[1 + \left(1 - \Psi_1\right) + \left(1 - \Psi_1\right)\left(1 - \Psi_2\right) + ...\right] = 1$$

So that

$$f_1 = \left[1 + \sum_{j=2}^{J} \left\{\prod_{m=1}^{j-1} \left(1 - \Psi_{j-m}\right)\right\}\right]^{-1}$$

The following auxiliary parameters are used in the numerical model: $\sigma = 0.0025$, $\omega = 0.99$, $\Delta = 1/4$ and $J = 80$. The model is linearized and solved with the Anderson-Moore (AIM) algorithm.

TABLE 1. BASELINE PARAMETERS

Parameter	Baseline Value
Discount rate, annual (r)	0.02
Curvature of profit function (α)	0.35
Steady state adjustment horizon (T) (years)	10.00
Elasticity of aggregate investment supply (ξ)	1.00
Half-life of demand shock (years)	0.50
Half-life of supply shock (years)	0.50

TABLE 2: STATISTICAL PROPERTIES OF SIMULATED DATA

Standard Deviations

I_t	p_t	z_t	A_t	M_t^1	M_t^2	M_t^3	M_t^4	M_t^5
0.961	0.127	1.000	1.000	0.179	0.174	0.165	0.156	0.494

Correlation Matrix

I_t	p_t	z_t	A_t	M_t^1	M_t^2	M_t^3	M_t^4	M_t^5
1.000	-0.266	-0.993	0.059	0.406	-0.067	-0.081	-0.066	-0.075
-0.266	1.000	0.382	0.426	-0.773	-0.680	-0.569	-0.456	0.853
-0.993	0.382	1.000	-0.002	-0.487	-0.022	0.006	0.005	0.180
0.059	0.426	-0.002	1.000	0.033	0.007	0.002	0.003	-0.016
0.406	-0.773	-0.487	0.033	1.000	0.482	0.328	0.318	-0.742
-0.067	-0.680	-0.022	0.007	0.482	1.000	0.452	0.294	-0.770
-0.081	-0.569	0.006	0.002	0.328	0.452	1.000	0.417	-0.744
-0.066	-0.456	0.005	0.003	0.318	0.294	0.417	1.000	-0.673
-0.075	0.853	0.180	-0.016	-0.742	-0.770	-0.744	-0.673	1.000

Note: The table shows the standard deviations and correlation coefficients for simulated variables: Investment, prices, supply parameters, productivity parameters, and moments M_t^j. The moments are described in the text. The data are simulated from a version of the model with $\delta = .05$. The supply and demand shocks are normally distributed and independent. Their variances are set to imply a 1 percent unconditional standard deviation in the long run. The estimated coefficients come from a simulated data set of 100,000 years of quarterly observations.

TABLE 3: FORECASTING EQUATIONS FOR INVESTMENT PRICES

Forecast Horizon	Forecast Coefficients								
	β_0	β_p	β_z	β_A	β_1	β_2	β_3	β_4	R^2
1 quarter	0.000	0.981	0.001	-0.016	n.a.	n.a.	n.a.	n.a.	0.877
	0.001	0.978	0.001	-0.016	-0.002	n.a.	n.a.	n.a.	0.877
	0.002	0.972	0.001	-0.015	-0.004	-0.003	n.a.	n.a.	0.877
	0.005	0.951	0.001	-0.014	-0.010	-0.008	-0.005	n.a.	0.877
	0.008	0.937	0.002	-0.013	-0.015	-0.011	-0.008	-0.002	0.877
2 quarters	0.000	0.961	0.001	-0.027	n.a.	n.a.	n.a.	n.a.	0.804
	0.001	0.957	0.001	-0.026	-0.004	n.a.	n.a.	n.a.	0.804
	0.003	0.943	0.002	-0.026	-0.007	-0.007	n.a.	n.a.	0.804
	0.010	0.904	0.002	-0.023	-0.020	-0.016	-0.010	n.a.	0.804
	0.013	0.890	0.003	-0.023	-0.024	-0.019	-0.013	-0.002	0.804
1 year	0.000	0.924	0.002	-0.039	n.a.	n.a.	n.a.	n.a.	0.719
	0.002	0.914	0.002	-0.038	-0.008	n.a.	n.a.	n.a.	0.719
	0.007	0.886	0.003	-0.037	-0.015	-0.014	n.a.	n.a.	0.719
	0.019	0.814	0.004	-0.033	-0.038	-0.030	-0.018	n.a.	0.719
	0.022	0.799	0.005	-0.032	-0.043	-0.034	-0.021	-0.002	0.719
2 years	0.000	0.850	0.004	-0.046	n.a.	n.a.	n.a.	n.a.	0.612
	0.004	0.828	0.003	-0.044	-0.018	n.a.	n.a.	n.a.	0.612
	0.013	0.775	0.004	-0.041	-0.032	-0.025	n.a.	n.a.	0.612
	0.032	0.668	0.007	-0.035	-0.067	-0.050	-0.028	n.a.	0.612
	0.030	0.680	0.006	-0.036	-0.063	-0.047	-0.026	0.001	0.612

Note: The table shows the estimated coefficients for reduced-form forecasting equations of the form $p_{t+h} = \beta_0 + \beta_p p_t + \beta_z z_t + \beta_A A_t + \sum_m \beta_m M_t^m + e_{t+h}$ where M_t^m are moments of the cross-sectional distribution of capital holdings at date *t*. The moments are described in the text. The data are simulated from a version of the model with $\delta = .05$. The supply and demand shocks are normally distributed and independent. Their variances are set to imply a 1 percent unconditional standard deviation in the long run. The estimated coefficients come from a simulated data set of 100,000 years of quarterly observations.

TABLE 4: PRICE FORECASTS (R^2), SENSITIVITY ANALYSIS

Forecast Horizon	Moments	Model (0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
		Baseline	$\delta=.02$	$\delta=.10$	$\delta=.50$	$\xi=5$	$\xi=10$	$\xi=100$	$\alpha=.50$	$\alpha=.15$	$\alpha=.05$	Myopic
1 quarter	0	0.880	0.923	0.837	0.692	0.927	0.920	0.849	0.867	0.905	0.910	0.624
	1	0.880	0.923	0.837	0.692	0.927	0.920	0.849	0.867	0.905	0.910	0.624
	2	0.880	0.923	0.838	0.692	0.927	0.920	0.849	0.867	0.905	0.910	0.624
	3	0.880	0.923	0.838	0.692	0.927	0.920	0.849	0.867	0.905	0.910	0.626
	4	0.880	0.923	0.838	0.692	0.927	0.920	0.849	0.867	0.905	0.910	0.632
6 months	0	0.809	0.878	0.740	0.524	0.865	0.847	0.708	0.790	0.843	0.849	0.449
	1	0.809	0.878	0.740	0.524	0.865	0.847	0.708	0.790	0.843	0.849	0.449
	2	0.809	0.878	0.740	0.525	0.865	0.847	0.708	0.790	0.843	0.849	0.451
	3	0.809	0.878	0.740	0.525	0.866	0.847	0.709	0.790	0.843	0.849	0.457
	4	0.809	0.878	0.740	0.525	0.866	0.847	0.710	0.790	0.843	0.849	0.475
1 year	0	0.726	0.826	0.625	0.358	0.759	0.715	0.473	0.703	0.762	0.765	0.259
	1	0.726	0.826	0.625	0.358	0.759	0.715	0.473	0.703	0.762	0.765	0.261
	2	0.726	0.826	0.625	0.359	0.759	0.715	0.474	0.703	0.762	0.765	0.269
	3	0.726	0.826	0.626	0.361	0.760	0.716	0.477	0.703	0.762	0.766	0.296
	4	0.726	0.826	0.626	0.361	0.760	0.716	0.478	0.703	0.762	0.766	0.321
2 years	0	0.622	0.761	0.478	0.189	0.578	0.497	0.191	0.598	0.647	0.640	0.067
	1	0.622	0.761	0.479	0.191	0.578	0.498	0.192	0.598	0.647	0.641	0.078
	2	0.622	0.761	0.480	0.196	0.579	0.499	0.197	0.598	0.647	0.641	0.108
	3	0.622	0.761	0.480	0.200	0.580	0.500	0.204	0.598	0.647	0.641	0.164
	4	0.622	0.761	0.480	0.200	0.580	0.500	0.204	0.598	0.647	0.641	0.165

Note: The table shows the R^2 for different forecasting equations, model specifications and forecast horizons. Forecast equations are of the form $p_{t+h}=\beta_0+\beta_p p_t+\beta_z z_t+\beta_A A_t+\sum_m \beta_m M_t^m+e_{t+h}$ where M_t^m are moments as described in the text. Column 1 is the baseline calibration. Columns 2 – 10 consider alternate calibrations. Parameter changes are described in the column heading. Parameters not listed are kept at baseline values. Column 11 (Myopic) gives results for $\delta=.50$, $r=.50$, $\alpha=.10$ and $\xi=5$. Supply and demand shocks are normally distributed and independent with variances set to imply a 1 percent unconditional standard deviation. Statistics come from a simulation of 100,000 years of quarterly observations.

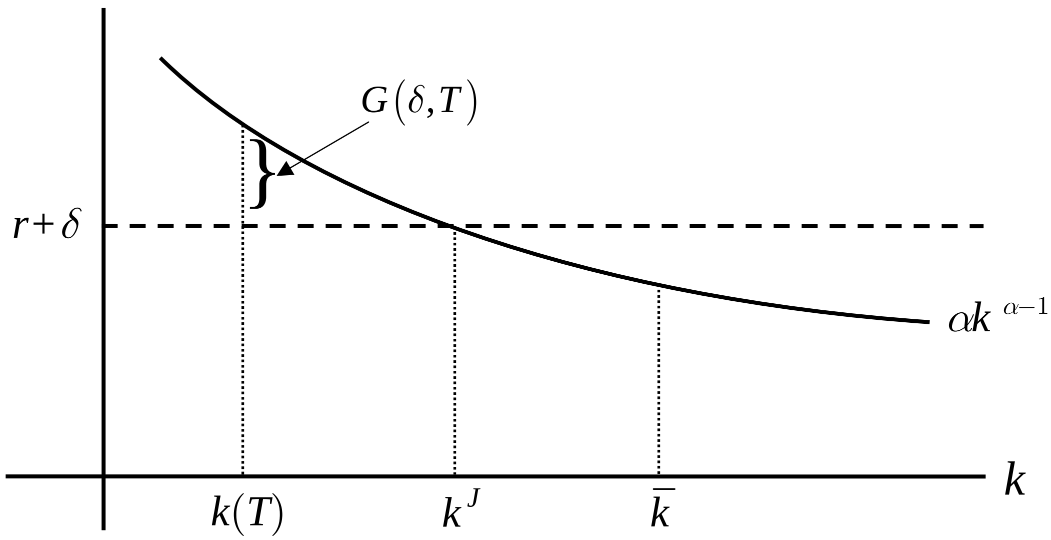


FIGURE 1: OPTIMAL BEHAVIOR IN THE STEADY STATE AND THE JORGENSON GAP $G(\delta,T)$

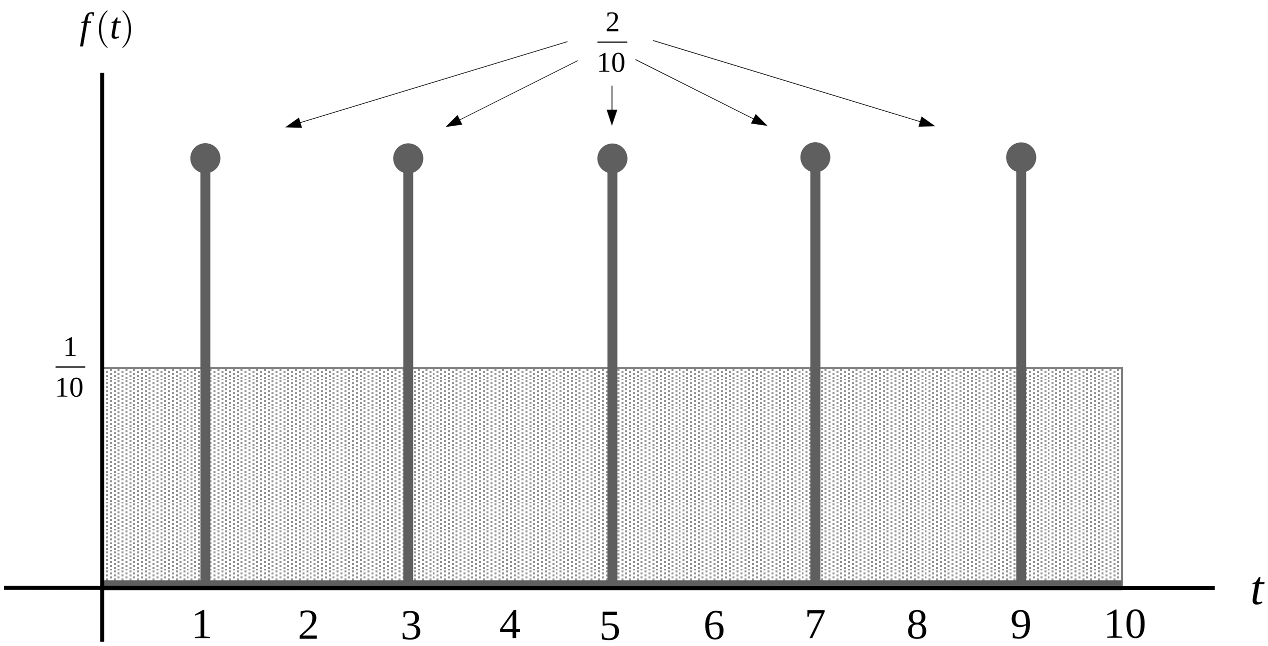
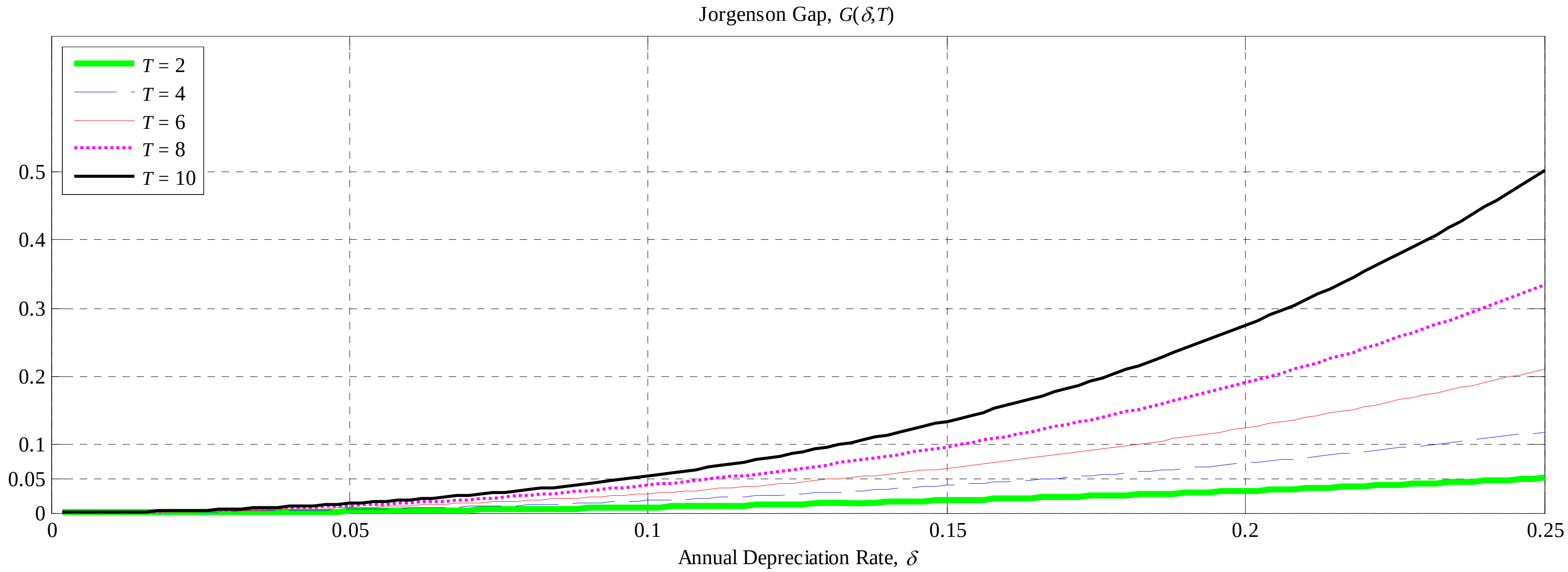


FIGURE 2: TWO DISTRIBUTIONS OF CAPITAL HOLDINGS.

The shaded rectangle represents the uniform steady state distribution. In this case, there is an even number of firms with capital t years old for $t \in (0,10)$. The heavy grey line represents an extreme alternative distribution. There are mass points of firms with 1-year-old capital, 3-year-old capital, etc. Each mass point has 2/10 of the firms. There are no firms with capital of any other age.

FIGURE 3: THE JORGENSON GAP



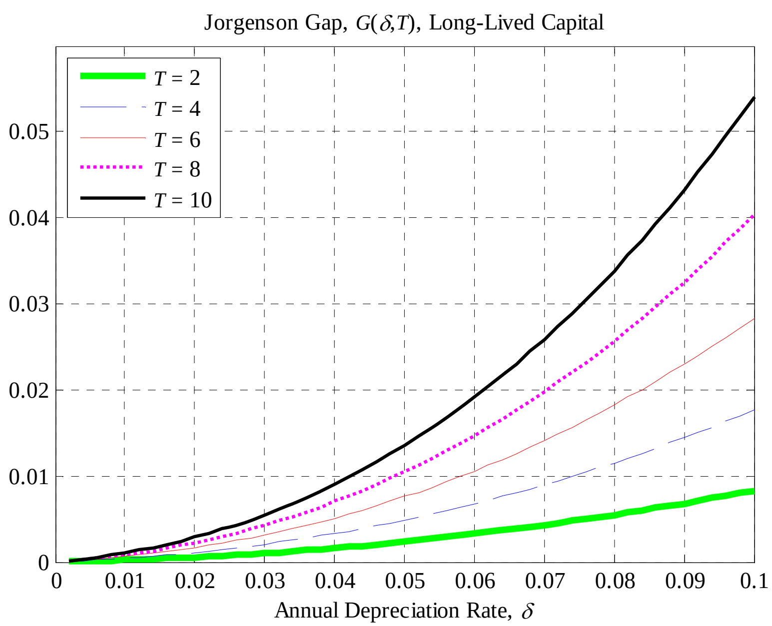
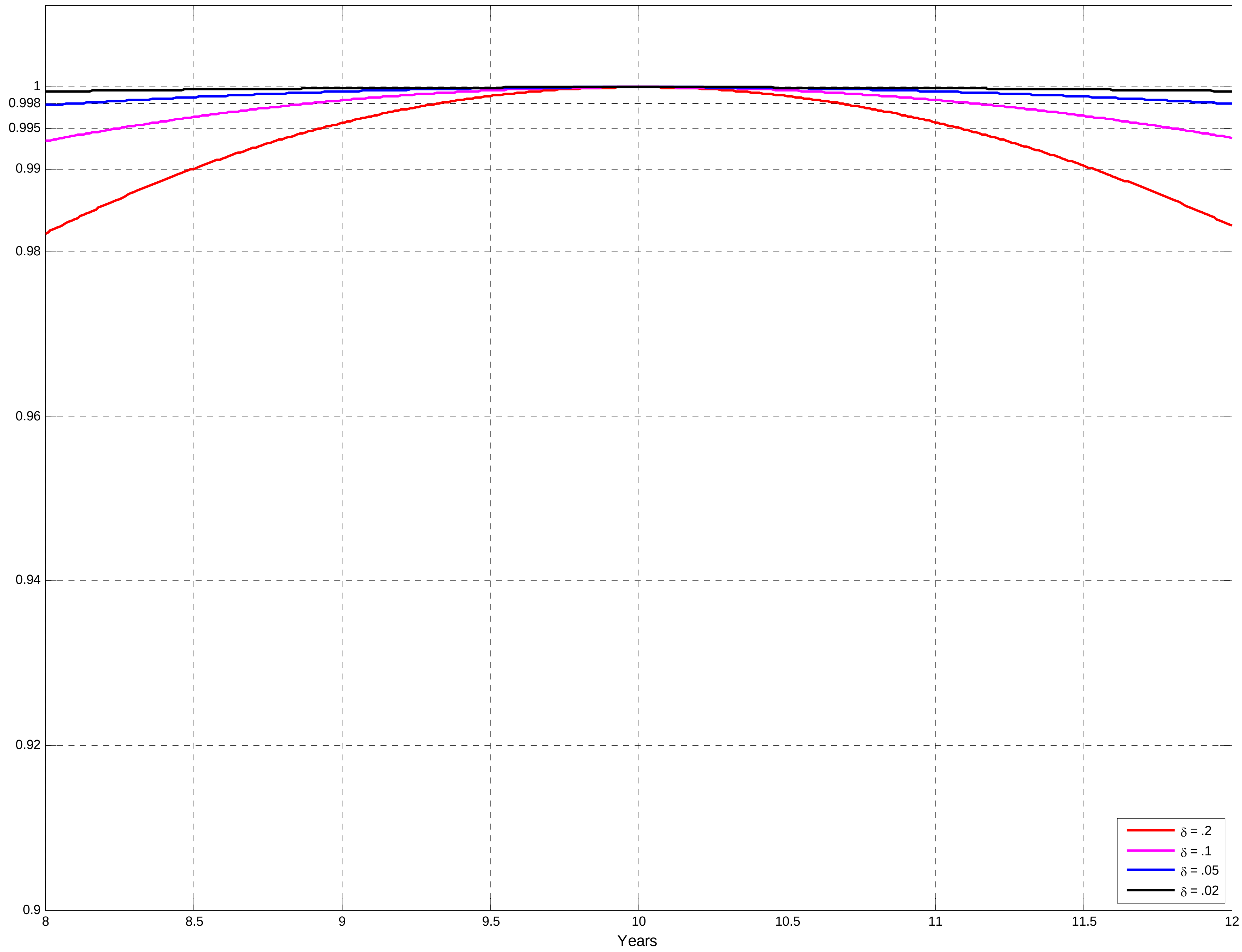


FIGURE 4: INDIFFERENT PRICE PATHS



Notes: The lines plot price paths $p(t)$ for which the firms are indifferent as to when they adjust their capital stock. The paths were made under the assumption that the reset value $\bar{V}$ was constant. The steady state price level is 1.00.

FIGURE 5: EQUILIBRIUM RESPONSE TO A SUPPLY SHOCK

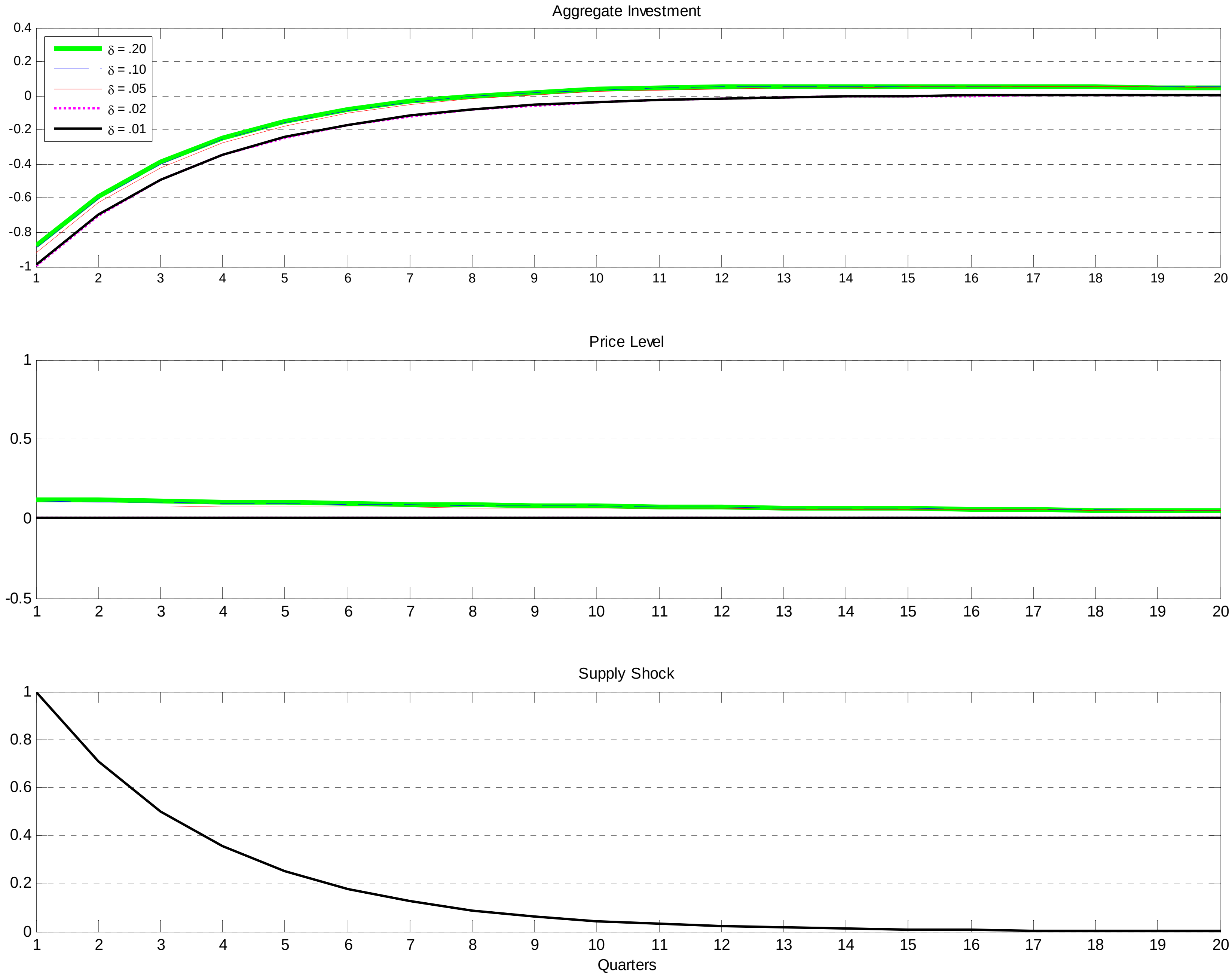


FIGURE 6: EQUILIBRIUM RESPONSE TO A DEMAND SHOCK

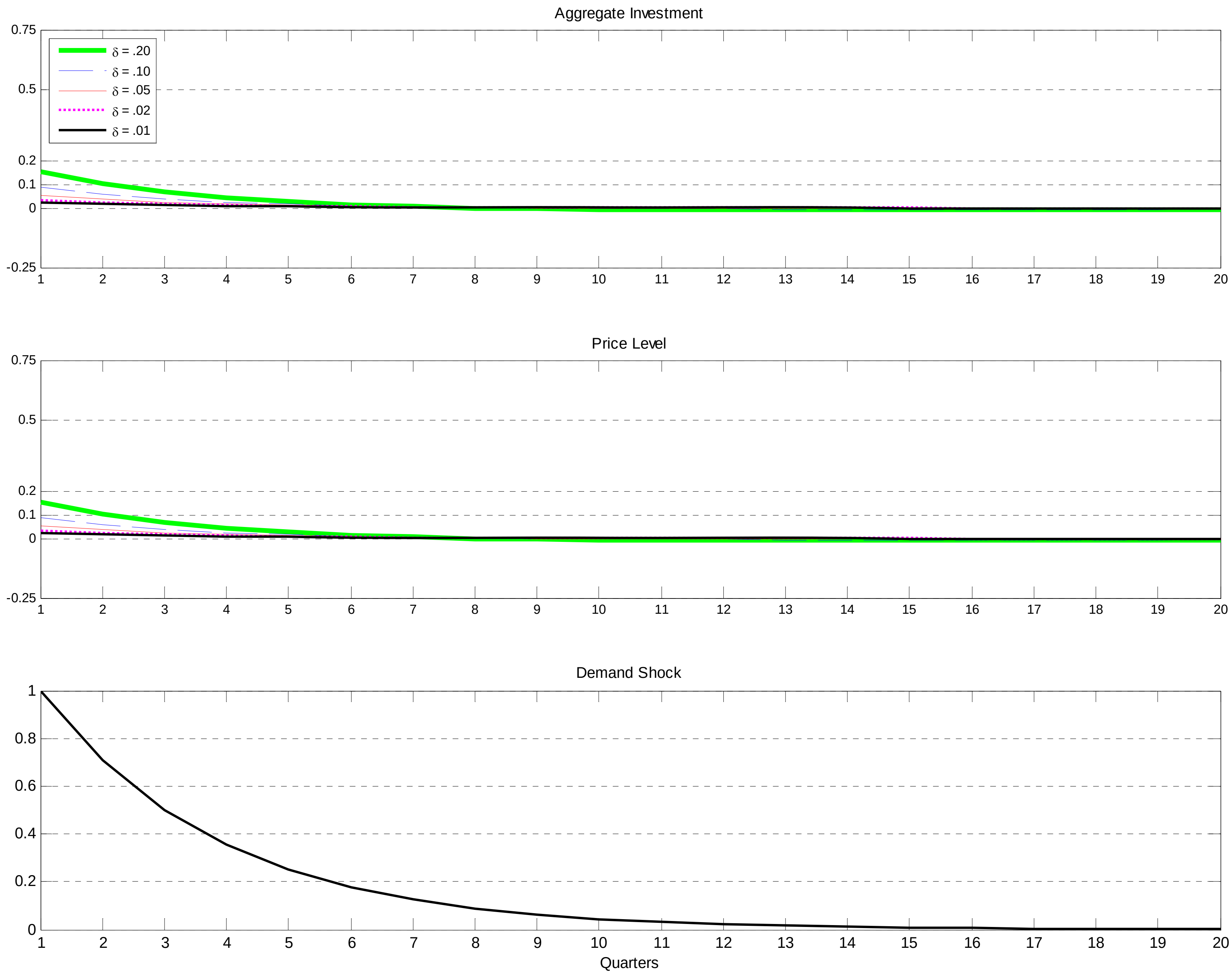


FIGURE 7: EQUILIBRIUM FROM AN OUT-OF-STEADY-STATE DISTRIBUTION

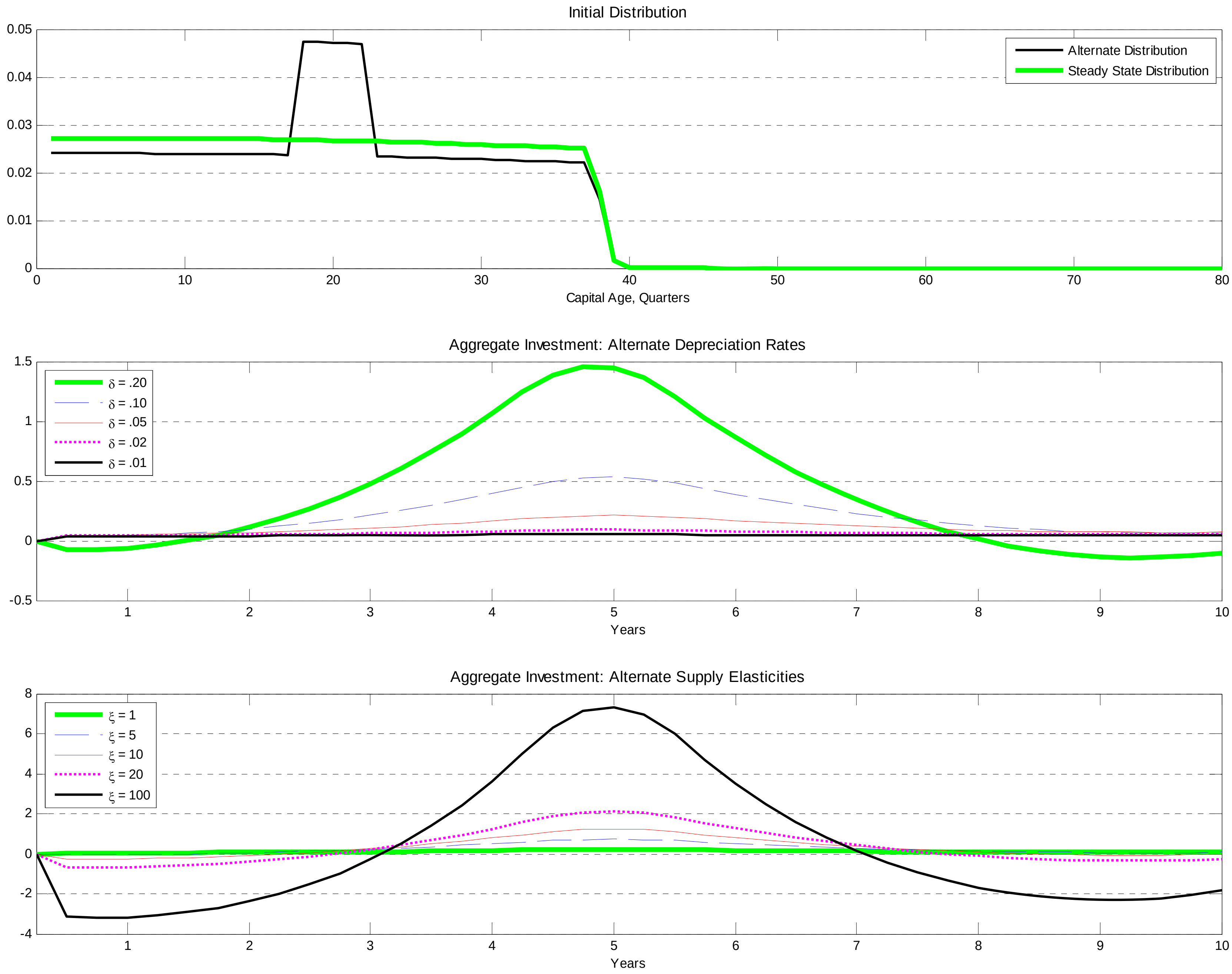
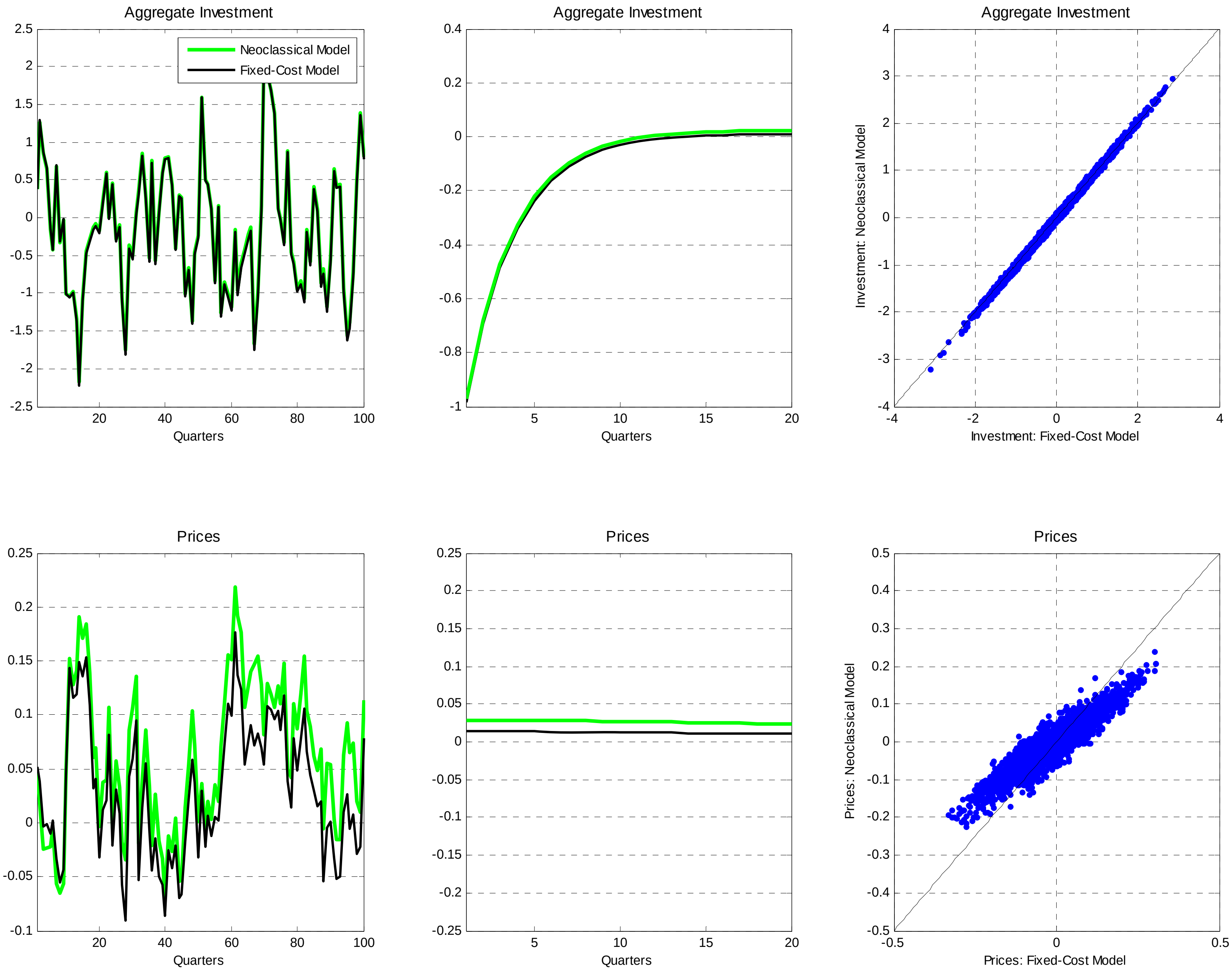


FIGURE 8: COMPARING THE FIXED-COST MODEL WITH THE NEOCLASSICAL MODEL, BASELINE PARAMETER VALUES



Notes: The parameter values for the fixed-cost model are given in Table 1. The neoclassical model is described in the text. Parameter values are identical to those in the fixed cost model. The scatter-plot shows 500 years of simulated data. Both models experienced identical shocks.