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ABSTRACT

This paper shows how rational investors can have different degrees of optimism regarding the prospects of the economy, even if they share exactly the same information regarding all economic fundamentals. The key is that heterogeneity in expectations regarding endogenous outcomes can emerge as a purely self-fulfilling equilibrium property when investment choices are strategic complements. This in turn has interesting novel positive and normative implications for a wide class of models that feature such complementarities: (i) It can rationalize idiosyncratic investor sentiment. (ii) It can be the source of significant heterogeneity in real and financial investment choices, even in the absence of any heterogeneity in individual characteristics and despite the presence of a strong incentive to coordinate on the same course of action. (iii) It can sustain rich fluctuations in aggregate investment and asset prices, including fluctuations that are smoother than those often associated with multiple-equilibria models. (iv) It can capture the idea that investors learn slowly how to coordinate on a certain course of action. (v) It can boost welfare. (vi) It can render apparent coordination failures evidence of improved efficiency.

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1 Introduction

Going back to Keynes, many have argued that animal spirits, market sentiments, or other forms of *extrinsic* uncertainty can be the cause of aggregate fluctuations.¹ In this paper I argue that extrinsic uncertainty can be largely idiosyncratic and can therefore also be the source of heterogeneity in real and financial investment choices. In so doing, I propose a rational theory of *idiosyncratic investor sentiment*. I then explore some novel positive and normative implications.

I conduct this exercise within two closely related models. The first is a simple real investment game that abstracts from financial prices. The second is a variant that stylizes trading in financial markets. The common essential feature of the two models is that they allow for strategic complementarity in investment choices: an individual investor is more willing to invest when he expects others also to invest. Such a complementarity could originate in a variety of production, demand, thick-market, or credit-related externalities analyzed in prior work.

To deliver the central result of this paper in its sharpest form, I rule out any exogenous source of heterogeneity: all investors have identical preferences, face identical constraints, and share the same information about exogenous productivity and all other relevant economic fundamentals. These assumptions ensure that all investors would choose exactly the same level of investment if their choices had been strategically independent. One may expect this conclusion not to be affected by the presence of a complementarity in investment choices: if all investors find it optimal to make the same choice when they do not care about one another's choices, why should they do anything different when they only have a desire to align their choices with one another? Yet, there now exist equilibria in which identical investors make different investment choices.

The key to this apparent paradox is that individual investors may now face idiosyncratic extrinsic uncertainty about the aggregate level of investment. That is, if we take a snapshot of the economy at any given point, we will find different investors holding different expectations regarding endogenous economic outcomes, even though they hold identical expectations regarding all exogenous economic fundamentals. This idiosyncratic variation in “optimism” regarding the endogenous prospects of the economy requires neither any differences in information regarding fundamentals nor any deviation from Bayesian rationality; rather, it emerges as a self-fulfilling prophecy.

¹The role of extrinsic uncertainty has been formalized within two related but distinct classes of models: overlapping generations economies (e.g., Azariadis, 1981, Azariadis and Guesnerie, 1986) and models with complementarities (e.g., Benhabib and Farmer, 1984, Obstfeld, 1986; Chatterjee, Cooper and Ravikumar, 1993; Cooper and John, 1988, Kiyotaki and Moore, 1997; Matsuyama, 1991; Weill, 1989).

Formally, this is achieved by the introduction of “private sunspots”. Like the public sunspots used in previous work, the private sunspots considered in this paper are payoff-irrelevant random variables. But unlike public sunspots, private sunspots are only imperfectly correlated across agents and are privately observed by them. The equilibria that obtain with private sunspots are thus closely related to the correlated equilibria introduced in game theory by Aumann (1974, 1987): when there are endogenous prices, the equilibria considered in this paper are hybrids of correlated equilibria and rational-expectations equilibria.

As an example, one could imagine the agents measuring the brightness of the sun or the temperature outside their houses; idiosyncratic measurement error could then be a natural source of imperfect correlation. Alternatively, one could imagine the agents reading a newspaper in search of clues about what action other agents are likely to coordinate on; the choice of what newspaper to read, or the interpretation of what any given newspaper says, could then be somewhat idiosyncratic. However, one need not take these examples too literally. Rather, one should think of private sunspots as modeling devices that permit the construction of equilibria in which different investors have different degrees of optimism regarding the *endogenous* prospects of the economy.

One interpretation is that private sunspots rationalize idiosyncratic investor sentiment; another is that they capture, in a certain sense, idiosyncratic uncertainty regarding which equilibrium is played. Indeed, while the pertinent work has been criticized for assuming away the possibility that each individual agent may be uncertain what action other agents are trying to coordinate, private sunspots address this issue at its heart by generating such uncertainty as an integral feature of the equilibrium. But no matter what interpretation one gives to private sunspots, there is a number of novel positive and normative implications that they deliver.

On the *positive* front, I highlight that models with macroeconomic complementarities can generate significant heterogeneity in real and financial investment choices. Such heterogeneity can obtain even in the absence—or after controlling for—any heterogeneity in exogenous individual characteristics, but only to the extent that individual incentives depend strongly enough on forecasts of others’ choices. It is thus symptomatic of the “beauty-contest” character of financial and real investment emphasized by Keynes.

Furthermore, I show how introducing idiosyncratic extrinsic uncertainty can significantly enrich, not only the cross-sectional, but also the aggregate outcomes of these models. In the two models considered in this paper, with public sunspots aggregate investment and asset prices can only take

two extreme values (“high” and “low”); with private sunspots, instead, aggregate investment and asset prices can follow smooth stochastic processes spanning the entire interval between these two extreme values. Private sunspots can thus generate much smoother aggregate fluctuations than public sunspots, indeed fluctuations that are more reminiscent of unique-equilibria models.

On the *normative* front, I show that ignoring private sunspots may lead to erroneous welfare and policy conclusions. The models considered in this paper feature exactly two equilibria in the absence of sunspots: a “good” (Pareto-dominant) one in which everybody invests; and a “bad” one in which nobody invests. Adding public sunspots only randomizes among those two extreme levels of investment, achieving convex combinations of the welfare obtained in the two sunspot-less equilibria. Therefore, as long as one restricts attention to public sunspots, one can safely draw two conclusions: that the occurrence of an investment crash is *prima-facia* evidence of coordination failure; and that policy interventions that preclude this outcome (at no or small cost) are bound to improve welfare.

Neither conclusion is warranted once one allows for private sunspots. Suppose, in particular, that the aggregate level of investment in the “good” equilibrium is excessive relative to the first best. Then one can construct an equilibrium with private sunspots in which the economy fluctuates between states during which only a subset of the investors invest (“normal times”) and states during which nobody invests (“crashes”). Because the aggregate level of investment is now closer to the first-best level during normal times, this equilibrium can achieve higher welfare than the equilibrium where everybody invests. However, for certain individuals to have an incentive not to invest during normal times, it must be that these individuals believe that a crash will take place with sufficiently high probability, while many other individuals believe the opposite. But then note that, as long as agents are rational, such heterogeneity in beliefs is possible in equilibrium only if crashes *do* happen with positive probability.

Therefore, an occasional crash—what looks as apparent coordination failure—is actually boosting welfare by facilitating idiosyncratic uncertainty and thereby providing the necessary incentive that keeps investment from being excessive during normal times. It then also follows that well-intended policies that aim at preventing apparent coordination failures could actually reduce welfare by eliminating the aforementioned incentive.

Related literature. The literature on macroeconomic complementarities, coordination failures, and sunspots is voluminous. Key contributions include Azariadis (1981), Azariadis and Guesnerie (1986), Benhabib and Farmer (1984), Cass and Shell (1983), Chatterjee, Cooper and Ravikumar

(1993), Cooper and John (1988), Diamond and Dybvig (1983), Guesnerie and Woodford (1992), Howitt and McAfee (1992), Kiyotaki (1988), Kiyotaki and Moore (1997), Matsuyama (1991), Obstfeld (1986, 1996), and Woodford (1986, 1987, 1991).² None of these earlier works considers idiosyncratic extrinsic uncertainty. Although this paper uses only a highly stylized representative of this class of models, it clearly illustrates how the introduction of such uncertainty can enrich the cross-sectional and aggregate outcomes of these models, as well as their welfare implications.

In so doing, the paper builds on Aumann's (1974, 1987) seminal work on correlated equilibria. Although the main contribution is to identify a set of positive and normative implications that have not been considered by prior applied work, a secondary contribution is to show how imperfect correlation can be accommodated within rational-expectations equilibria. The conceptual issue here is that equilibrium prices convey information about the underlying common components of the imperfect correlation devices that different agents observe, so that each agent's beliefs about these sunspots are endogenous to the strategies of other agents. This introduces a fixed-point element between beliefs and strategies that is absent in standard correlated equilibria.

Layout. The rest of the paper is organized as follows. Section 2 introduces the baseline model and revisits the set of equilibria with public sunspots. Section 3 introduces private sunspots and studies their positive implications. Section 4 studies a variant model that captures trading in financial markets. Section 5 turns to normative implications. Section 6 concludes.

2 The baseline model: a real investment game

The economy is populated by a measure-1 continuum of agents (investors), who are indexed by $i \in [0, 1]$, are endowed with one unit of wealth each, and decide how to allocate this wealth between a safe technology and a risky alternative. The safe technology delivers a return R no matter what, while the return of the risky technology depends on the aggregate level of investment in that technology. The payoff of investor i is given by

$$\pi_i = \Pi(k_i, K) \equiv (1 - k_i)R + k_i(R + A(K)) = R + A(K)k_i,$$

where k_i denotes investor i 's investment in the risky technology, K denotes the aggregate level of investment, and $A(K)$ the excess return of this technology relative to the safe one.

²See Benhabib and Farmer (1999) and Cooper (1999) for excellent reviews.

The key assumption needed for the positive results of this paper is that there exists a $\kappa \in (0, 1)$ such that $A(K) < 0$ for all $K < \kappa$ and $A(K) > 0$ for all $K > \kappa$. This assumption introduces strategic complementarity in investment choices and guarantees the existence of two Nash equilibria, one where all agents invest their entire wealth in the risky technology and another where all agents invest their entire wealth in the safe technology. To simplify the analysis, I henceforth normalize $R = 0$ and let $A(K) = -c < 0$ for $K < \kappa$ and $A(K) = b - c > 0$ for $K \geq \kappa$, where $b > c > 0$. One can then think of c as parameterizing the cost of investing in the risky technology, κ as the minimal level of aggregate investment for which the technology becomes profitable, and b as the gross benefit enjoyed in that event.³ I further assume that investment is indivisible: each investor can choose either $k_i = 1$ (which I henceforth call simply “invest”) or $k_i = 0$ (“don’t invest”), so that K is also the mass of agents investing.⁴

Model interpretation. This model can be interpreted as a highly stylized version of a variety of models considered in prior applied work. The core element is the presence of strategic complementarity in individual production, investment, or portfolio choices. Such complementarity could originate in a plethora of production, demand, or thick-market externalities, as well as in credit frictions.⁵ For the purposes of this paper, modeling the deeper foundations of such complementarity is not essential. What is essential is only that such complementarity opens the door to extrinsic uncertainty. Note, however, that the framework introduced so far abstracts from how prices (or other signals of aggregate activity) may limit idiosyncratic extrinsic uncertainty, a possibility that is evidently relevant for most applications of interest. I will deal with this issue in Section 4.

Public sunspots. As noted above, the model admits exactly two equilibria in the absence of sunspots. To see this, note that, in the absence of sunspots, the aggregate level of investment is deterministic,⁶ and the best response of investor i is simply

$$k_i = BR(K) \equiv \begin{cases} 1 & \text{if } K \geq \kappa \\ 0 & \text{if } K < \kappa \end{cases}$$

³All these parameters are common knowledge—there is no uncertainty about the economic fundamentals.

⁴Indivisibility has no bite here because agents are risk neutral.

⁵See, for example, Diamond (1976) for thick-market externalities; Kiyotaki (1988) and Woodford (1991) for aggregate demand externalities; Azariadis and Smith (1998), Kiyotaki and Moore (1997), and Matsuyama (2007) for complementarities due to credit frictions; Chatterjee, Cooper and Ravikumar (1993) for complementarities in business formation; Diamond and Dybvig (1983) and Obstfeld (1986, 1996) for coordinated bank runs and currency attacks; and Cooper (1999) for an excellent review of the role of complementarities in macroeconomics.

⁶Because there is a continuum of investors, this is true even if investors follow mixed strategies.

It follows that all investors necessarily make the same choice and there exist exactly two equilibria: one in which everybody invests ($k_i = K = 1$ for all i) and another in which nobody invests ($k_i = K = 0$ for all i).⁷ The one equilibrium is sustained by the self-fulfilling expectation that everybody will invest; the other by the self-fulfilling expectation that nobody will invest. In either case, investors face no uncertainty about what choices other investors are making and perfectly coordinate on the same course action.

Now let us introduce *public* sunspots. Before investors make their choices, they publicly observe a payoff-irrelevant random variable s , whose support is $\mathbb{S} \subseteq \mathbb{R}$ and whose cumulative distribution function (c.d.f.) is $F : \mathbb{S} \rightarrow [0, 1]$. Because the investors can now follow strategies that are contingent on s , the aggregate level of investment can be stochastic. However, because s is publicly observed, the investors continue to face no uncertainty about the equilibrium level of investment and continue to make identical choices. As a result, equilibria with public sunspots are merely lotteries over the two sunspot-less equilibria.

Proposition 1 *For any equilibrium with public sunspots, there exists a $p \in [0, 1]$ such that $K(s) = 1$ with probability p and $K(s) = 0$ with probability $1 - p$. Conversely, for any $p \in [0, 1]$, there exists an equilibrium in which $K(s) = 1$ with probability p and $K(s) = 0$ with probability $1 - p$.*

Beliefs and actions vary across equilibria, or across realizations of the public sunspot, but never in the cross-section of investors: in any given equilibrium and for any given realization of the sunspot, all investors share the same “sentiment” (i.e., the same belief about all endogenous outcomes), can perfectly forecast one another’s choices, and end up taking exactly the same action. The next section shows how none of these properties need to hold once we allow for private sunspots.

3 Private sunspots and idiosyncratic sentiment

I introduce private sunspots as follows. First, “Nature” draws a payoff-irrelevant random variable s that is *not* observed by any investor. The support of this variable is $\mathbb{S} \subseteq \mathbb{R}$ and its c.d.f. is $F : \mathbb{S} \rightarrow [0, 1]$. Then, each investor *privately* observes a payoff-irrelevant random variable m . Conditional on s , m is i.i.d. across investors, with support $\mathbb{M} \subseteq \mathbb{R}$ and c.d.f. $\Psi : \mathbb{M} \times \mathbb{S} \rightarrow [0, 1]$.

⁷When $A(\kappa) = 0$, there also exists a mixed-strategy equilibrium in each investor invests with probability κ ; aggregate investment is then κ and investors are indeed indifferent between investing and not investing so long as $A(\kappa) = 0$. I have ruled out this equilibrium by assuming $A(\kappa) \neq 0$. This, however, is not essential for any of the results.

These variables define what I call “private sunspots”: they are private signals of the underlying unobserved common sunspot s . I henceforth call $(\mathbb{S}, F, \mathbb{M}, \Psi)$ the “sunspot structure” and define an equilibrium as follows.

Definition 1 *An equilibrium with private sunspots consists of a sunspot structure $(\mathbb{S}, F, \mathbb{M}, \Psi)$ and a measurable strategy $k : \mathbb{M} \rightarrow \{0, 1\}$ such that*

$$k(m) \in \arg \max_{k \in \{0,1\}} \int_{\mathbb{S}} \Pi(k, K(s)) dP(s|m) \quad \forall m \in \mathbb{M},$$

with $K(s) = \int_{\mathbb{M}} k(m) d\Psi(m|s) \quad \forall s \in \mathbb{S}$, and with $P(s|m)$ denoting the c.d.f. of the posterior about s conditional on m (as implied by Bayes’ rule).

Note that the sunspot structure $(\mathbb{S}, F, \mathbb{M}, \Psi)$ is not part of the exogenous primitives of the environment. Rather, it is a modeling device that permits the construction of equilibria that sustain endogenous stochastic variation, not only in the aggregate, but also in the cross-section of agents. In the remainder of this section, I consider a specific Gaussian sunspot structure that best illustrates the novel positive properties equilibria with private sunspots can lead to.

Gaussian sunspots. Suppose s is drawn from a Normal distribution with mean $\mu_s \in \mathbb{R}$ and variance $\sigma_s^2 > 0$. The private signal observed by investor i is $m_i = s + \varepsilon_i$, where ε_i is Normal noise, i.i.d. across investors and independent of s , with variance $\sigma_\varepsilon^2 > 0$. One can then think of s as the “brightness of the sun” or the “average temperature in a city” and ε_i as idiosyncratic measurement error. The next proposition then constructs equilibria where an investor invests if and only if his private measurement of the brightness of the sun or the temperature is sufficiently high. In these equilibria, an investor’s private sunspot captures his idiosyncratic sentiment regarding the prospects of the economy: the higher m , the higher the investor’s expectation of the aggregate level of investment.

Proposition 2 *For any $(\mu_s, \sigma_s, \sigma_\varepsilon)$, there exists an equilibrium in which the following are true:*

- (i) *An investor invests when $m > m^*$ and not when $m < m^*$, for some $m^* \in \mathbb{R}$.*
- (ii) *The aggregate level of investment is stochastic, with full support on $(0, 1)$.*
- (iii) *The cross-sectional distribution of expectations regarding the aggregate level of investment, $\mathbb{E}[K|m]$, has full support on $(0, 1)$.*

Proof. Let Φ denote the c.d.f. of the standard Normal distribution. Suppose there exists an m^* such that an investor invests if and only if $m > m^*$. Aggregate investment is then given by

$$K(s) = \Pr(m \geq m^* | s) = \Phi\left(\frac{s - m^*}{\sigma_\varepsilon}\right), \quad (1)$$

and therefore $K(s) \geq \kappa$ if and only if $s \geq s^*$, where

$$s^* = m^* + \sigma_\varepsilon \Phi^{-1}(\kappa). \quad (2)$$

Because both the prior about s and the signal m are Gaussian, the posterior about s conditional on m is Normal with mean $\mathbb{E}[s|m] = \frac{\sigma_\varepsilon^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}m + \frac{\sigma_s^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}\mu_s - s^*$ and variance $\text{Var}[s|m] = (\sigma_s^{-2} + \sigma_\varepsilon^{-2})^{-1}$. It follows that the expected return from investing conditional on signal m is

$$\mathbb{E}[A(K(s))|m] = b \Pr(s \geq s^* | m) - c = b \Phi\left(\sqrt{\sigma_s^{-2} + \sigma_\varepsilon^{-2}} \left[\frac{\sigma_\varepsilon^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}m + \frac{\sigma_s^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}\mu_s - s^*\right]\right) - c.$$

Note that the latter is strictly increasing in m . For the proposed strategy to be part of an equilibrium, it is thus necessary and sufficient that m^* satisfies $\mathbb{E}[A|m^*] = 0$, or equivalently

$$\frac{\sigma_\varepsilon^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}m^* + \frac{\sigma_s^{-2}}{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}\mu_s - s^* = \frac{1}{\sqrt{\sigma_s^{-2} + \sigma_\varepsilon^{-2}}}\Phi^{-1}\left(\frac{c}{b}\right). \quad (3)$$

Substituting s^* from condition (2) into (3) and rearranging gives

$$m^* = \mu_s - \sigma_s \left\{ \frac{\sigma_s^2 + \sigma_\varepsilon^2}{\sigma_s \sigma_\varepsilon} \Phi^{-1}(\kappa) + \sqrt{1 + \frac{\sigma_s^2}{\sigma_\varepsilon^2}} \Phi^{-1}\left(\frac{c}{b}\right) \right\}, \quad (4)$$

which completes the proof of part (i). Part (ii) then follows from condition (1). Finally, part (iii) follows from part (ii) along with the fact that both the distribution of s conditional on m and that of m conditional on s have full supports. *QED*

Note that different investors hold different expectations about the distribution of the signals m in the population. In the equilibria constructed above, this means that different investors also hold different expectations about the mass of investors who have received $m \geq m^*$. The end result is different expectations about the aggregate level of investment, which in turn sustain different individual investment choices—a sharp difference from the case with public sunspots.

Because this heterogeneity in expectations and choices can not be traced to any heterogeneity in primitive characteristics (preferences, endowments, technologies, or payoff-relevant information), it can be interpreted as idiosyncratic variation in “sentiment” or “optimism”. This optimism is with regard to the endogenous prospects of the economy. It does not require any heterogeneity in expectations regarding the exogenous primitives of the environment, nor any deviation from Bayesian rationality. Rather, it is merely, and purely, a self-fulfilling equilibrium property.

Finally, note that, in the equilibria constructed above, the aggregate level of investment has full support on the $(0, 1)$ interval. In contrast, in the equilibria with no or only public sunspots, the aggregate level of investment could take only the extreme values 0 and 1. Therefore, private sunspots permit, not only endogenous heterogeneity in the cross-section of the population, but also a richer set of aggregate outcomes.

A simple dynamic extension. To better appreciate the aggregate implications of private sunspots, consider the following dynamic extension. There is an infinite number of periods. In each period t , each investor chooses whether to invest ($k_t = 1$) or not ($k_t = 0$). He then receives a contemporaneous payoff $\pi_t = A(K_t)k_t$, where K_t is the aggregate level of investment in period t and $A(K_t)$ is the net return to investment, with $A(K_t) = b - c > 0$ if $K_t \geq \kappa$ and $A(K_t) = -c < 0$ if $K_t < \kappa$. The investor’s intertemporal payoff is simply $\sum_{t=0}^{\infty} \beta^t \pi_t$, where $\beta \in (0, 1)$.

The sunspot structure is wherein the interesting dynamics enter. The unobserved sunspot in period t is given by $s_t = \rho s_{t-1} + u_t$, where $\rho \in (0, 1)$ is the auto-correlation in the sunspot and u_t is white noise, i.i.d. across time, with variance σ_u^2 . The private sunspot observed by an investor in period t is $m_t = s_t + \varepsilon_t$, where ε_t is white noise, i.i.d. across agents and time, with variance σ_ε^2 .

Now note that investors may learn over time about past realized sunspots by the observation of past aggregate investment and/or past payoffs. To maintain the analysis tractable, I ignore the learning through payoffs. I also assume that investors observe noisy private signals of past investment: each investor observes in period t a signal $x_t = \Phi^{-1}(K_{t-1}) + \xi_t$, where ξ_t is white noise, i.i.d. across agents and time, with variance σ_ξ^2 . These assumptions guarantee the existence of equilibria in which the information structure remains Gaussian.⁸

⁸The assumption that investors do not learn from their past payoffs is merely for convenience and can be justified as follows. Let the payoff of an investor be $\pi_t = z_t k_t$, where $z_t \equiv A(K_t) + \omega_t$ and where ω_t is white noise, i.i.d. across both time and agents, with variance σ_ω^2 . Suppose further that z_t is privately observed by the investor, independently of his choice of investment; this kills the value for experimentation that would have emerged if z_t was observed only when $k_t = 1$. Then, the observation of π_t conveys no more information than z_t , which by itself is a noisy private signal of K_t . Qualitatively, this is much alike the noisy private signal x_t that we have already introduced. The only difference is that the information contained z_t is not Gaussian, making the updating of beliefs intractable. However, letting

Indeed, as shown in the Appendix, we can find a sequence $\{m_t^*, \sigma_t\}_{t=0}^\infty$ and an equilibrium in which the following hold: (i) the entire sequence of private signals up to period t can be summarized in a sufficient statistic \tilde{m}_t , which is Normal, i.i.d. across investors, with mean s and variance σ_t^2 ; and (ii) an investor invests in period t if and only if $\tilde{m}_t \geq m_t^*$. Along this equilibrium, the sufficient statistic \tilde{m}_t and its variance σ_t can be constructed recursively as functions of $(\tilde{m}_{t-1}, \sigma_{t-1}; m_t, x_t)$. Moreover, as the history gets arbitrarily long, (m_t^*, σ_t) converges to some time-invariant (m^*, σ) . We thus obtain a stationary equilibrium along which aggregate investment is given by

$$K_t(s^t) = \Phi\left(\frac{s_t - m^*}{\sigma}\right).$$

Hence, up to a monotone transformation, aggregate investment follows a smooth AR(1) process.⁹

Note then that fictitious data generated by the present model would be virtually indistinguishable from fictitious data generated by a canonical unique-equilibrium model. This would not be the case if we had ignored private sunspots: with public sunspots, aggregate investment features discrete fluctuations (between 0 and 1), which would be more telling of multiple equilibria. We conclude that private sunspots can help generate very smooth aggregate fluctuations, making it difficult to identify fluctuations driven by sunspots from fluctuations driven by smooth changes in the underlying fundamentals.

“Learning to coordinate.” As another example of the rich dynamics that private sunspots can sustain, I now consider the following variant. Investors continue to receive the exogenous and endogenous private signals m_t and x_t considered above, but no the unobserved sunspot remains constant over time: $\rho = 1$ and $\sigma_u = 0$, so that $s_t = s$ for all t .

As before, we can find an equilibrium in which an investor invests in period t if and only if his sufficient statistic \tilde{m}_t exceeds some deterministic threshold m_t^* . For simplicity, suppose $\kappa = c/b = 1/2$, which gives $m_t^* = 0$ for all t . It follows that aggregate investment in period t is given by $K_t(s) = \Phi\left(\frac{s}{\sigma_t}\right)$. Because of the accumulation of new signals, σ_t is decreasing over time and converges to zero as $t \rightarrow \infty$. It follows that, whenever $s > 0$, $K_t(s)$ is bounded in $(1/2, 1)$ and increasing over time, asymptotically converging to 1; and whenever $s < 0$, $K_t(s)$ is bounded inside $(0, 1/2)$, and decreasing over time, asymptotically converging to 0.

$\sigma_\omega \rightarrow \infty$ avoids this problem by rendering the signal z_t uninformative. At the same time, because the expectation of ω_t is zero no matter σ_ω , investors continue to choose k_t so as to maximize their expectation of $A(K_t)k_t$. It follows that the error introduced by ignoring the information contained in payoffs vanishes as $\sigma_\omega \rightarrow \infty$.

⁹To be precise, $\Phi^{-1}(K_t)$ is a Gaussian AR(1).

Recall now that $K = 1$ and $K = 0$ represent the only two equilibria that are possible in the absence of private sunspots and that require all investors coordinating on the same course of action. We can thus interpret the dynamics that obtain here with private sunspots as situations where investors slowly learn on which action to coordinate: at any given date, some investors are making the “wrong” investment choice (i.e., do the opposite of what the majority does), but the fraction of investors who makes such a mistake falls over time and vanishes in the limit.

Also note that this form of learning can be either exogenous or endogenous: it can originate in either the signals m_t regarding the unobserved sunspot s or the signals x_t regarding past aggregate activity. We conclude that private sunspots can capture, not only the idea that agents may fail to perfectly coordinate on the same course of action, but also the possibility that agents slowly learn how to do so over time through the observation of one another’s actions.¹⁰

4 Private sunspots and financial markets

The preceding analysis has been conducted within a simple investment game that abstracted from market interactions. I now consider a variant model in which investors trade an asset within a competitive financial market. This exercise serves two purposes. First, it shows how the insights of the preceding analysis translate in the context of financial markets. Second, it shows how imperfect correlation can be accommodated within a rational-expectations-equilibrium framework, where prices partially reveal the unobserved common sunspot component that drives the correlation among the beliefs (the private sunspots) of different investors.

Model set-up. There is again a large number of risk-neutral investors, who now decide how much to trade of a certain financial asset. An individual’s investment in the asset is denoted by k_i and the aggregate investment by K . The price of the asset is denoted by p and its dividend by A . The later is assumed to increase with aggregate investment in the asset: $A = A(K)$. Once again, this is meant to capture, in a crude way, a variety of feedback effects identified in prior work.¹¹

¹⁰The form of social learning considered here is purely private, but one could easily extend the analysis to public signals about either s or past activity. A certain kind of public signals that is of special interest is prices; this brings us to the topic of the next section.

¹¹For example, Ozdenoren and Yuan (2008) argue that the higher the position of institutional investors in the stock of a particular company, the better the monitoring of the management of that company, and hence the higher return and the higher the demand for that stock; Subrahmanyam and Titman (2001) stress the role of complementarities among the customers, suppliers, or employees of a company; and many others emphasize feedback effects from stock prices to capital availability, and therefrom to firm profitability and back to stock prices.

Since investors are risk-neutral, their payoffs are simply given by

$$\pi_i = \Pi(k_i, K, p) \equiv [A(K) - p] k_i.$$

To rule out infinite positions, I assume that k_i is bounded in $[\underline{k}, \bar{k}]$, for some finite \underline{k} and \bar{k} . These bounds can be interpreted as the result of borrowing and short-selling constraints. (Allowing for risk aversion would be another natural, but less tractable, way to ensure that investors take finite positions.) Without any further loss of generality, let $\underline{k} = 0$ and $\bar{k} = 1$. Finally, the supply of the asset, which is denoted by Q , is assumed to be an increasing function of the price and of some unobserved supply shock: $Q = Q(p, u)$, where $u \in \mathbb{U} \subseteq \mathbb{R}$. The shock u can also be interpreted as the impact of “noise traders”; its sole role is to introduce noise in the price.

Private sunspots are introduced as before: nature first draws an unobserved common sunspot variable $s \in \mathbb{S}$ from some distribution F ; nature then sends each agent i a private signal $m_i \in \mathbb{M}$, which is drawn i.i.d. across agents from a conditional distribution Ψ . These variables are payoff-irrelevant and are independent of the supply shock u ; they are once again devices that introduce aggregate and idiosyncratic extrinsic uncertainty. What is novel here relative to the model of the previous section is that the price that clears the asset market may publicly reveal information about these sunspot variables. This motivates the following equilibrium definition, which introduces private sunspots within an otherwise-standard rational-expectations equilibrium concept.

Definition 2 *A rational-expectations equilibrium with private sunspots consists of a sunspot structure $(\mathbb{S}, F, \mathbb{M}, \Psi)$, a price function $P : \mathbb{S} \times \mathbb{R} \rightarrow \mathbb{R}$, an individual demand function $k : \mathbb{M} \times \mathbb{R} \rightarrow [\underline{k}, \bar{k}]$, and a belief (c.d.f.) $\mu : \mathbb{S} \times \mathbb{R} \times \mathbb{M} \times \mathbb{R} \rightarrow [0, 1]$, such that the following hold:*

- (i) *Beliefs are consistent with Bayes rule given the equilibrium price function.*
- (ii) *Given the beliefs and the price function, the demand function satisfies individual rationality:*

$$k(m, p) \in \arg \max_{k \in \{0, 1\}} \int_{\mathbb{S} \times \mathbb{U}} \Pi(k, K(s, P(s, u)), P(s, u)) d\mu(s, u | m, p) \quad \forall (m, p),$$

where $K(s, p) \equiv \int_{\mathbb{M}} k(m, p) d\Psi(m | s) \quad \forall s \in \mathbb{S}$.

- (iii) *Given the demand function, the price function satisfies market-clearing:*

$$K(s, P(s, u)) = Q(s, u) \quad \forall (s, u).$$

As in most rational-expectations models, the analysis is intractable without an “artful” choice of distributional assumptions and functional forms. I thus assume that all uncertainty is Gaussian: $u \sim N(0, \sigma_u^2)$, $s \sim N(\mu_s, \sigma_s^2)$, and $m_i = s + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ is i.i.d. across agents and independent of both s and u . I further impose the following functional forms for A and Q : $A(K) = 1$ if $K \geq \kappa$ and $A(K) = 0$ otherwise, for some scalar $\kappa \in (0, 1)$; and $Q(p, u) = \Phi(u + \lambda\Phi^{-1}(p))$, for some scalar $\lambda > 0$. This scalar parameterizes the price elasticity of the supply of the asset, while Φ denotes again the c.d.f. of the standardized Normal distribution.

Equilibrium analysis. The next proposition establishes the existence of rational-expectations equilibria in which investors’ demand functions are decreasing in the price and increasing in their private sunspots. As a result, the aggregate demand for the asset is increasing in s . Along with the fact that supply is increasing in u , this ensures that the equilibrium price is increasing in both s and u . Because the supply shock u is unobserved (recall, this shock captures more generally any noise in prices), the price is only a noise indicator of the underlying common sunspot component s . This ensures that, although investors do learn something about one another’s investment choices from the observed price, they continue to face some residual idiosyncratic uncertainty regarding one another’s investment choices, and hence about the eventual dividend of the asset. As a result, these equilibria feature different investors finding it strictly optimal to make different portfolio choices, even though they all share the same preferences, constraints, and beliefs regarding any exogenous component of asset returns—heterogeneity in portfolio choices originates merely in self-fulfilling heterogeneity in beliefs regarding the endogenous component of asset returns.

Proposition 3 *For any (σ_u, λ) , there exists a rational-expectations with private sunspots in which the following are true:*

(i) *An investor’s equilibrium demand for the asset is given by*

$$k(m, p) = \begin{cases} 1 & \text{if } m \geq m^*(p) \\ 0 & \text{otherwise} \end{cases}$$

where $m^(p)$ is a continuous increasing function of p . By implication, the aggregate demand for the asset, $K(s, p)$, is continuously increasing in s and continuously decreasing in p .*

(ii) *The equilibrium price is given by $p = P(s, u)$, where P is a continuously increasing function of s and a continuously decreasing function of u .*

Proof. Consider a sunspot structure such that $\lambda\sigma_\varepsilon(\sigma_s^{-2} + \sigma_\varepsilon^{-2} + \sigma_\varepsilon^{-2}\sigma_u^{-2})^{-1/2}\sigma_\varepsilon^{-2}\sigma_u^{-2} > 1$. Next, suppose there exists an $m^*(p)$ such that an investor invests if and only if $m > m^*(p)$. Given the proposed strategy, aggregate demand is given by

$$K(s, p) = \Phi\left(\frac{s - m^*(p)}{\sigma_\varepsilon}\right).$$

Market clearing imposes $K(s) = Q(p, u)$. Equivalently, p must satisfy $m^*(p) + \sigma_\varepsilon\lambda\Phi^{-1}(p) = s - \sigma_\varepsilon u$, for all (s, u) . Since the function m^* is common knowledge in equilibrium (and so are $\sigma_\varepsilon, \lambda$, and Φ), the observation of p is informationally equivalent to the observation of the signal

$$z(p) \equiv m^*(p) + \sigma_\varepsilon\lambda\Phi^{-1}(p) = s + n, \quad (5)$$

where $n \equiv -\sigma_\varepsilon u$ is Normal noise with variance $\sigma_n^2 = \sigma_\varepsilon^2\sigma_u^2$. Because the prior about s , the private signal m , and the public signal z are all Gaussian, the posterior about s conditional on m and p is also Gaussian, with mean

$$\mathbb{E}[s|m, p] = \frac{\sigma_s^{-2}}{\sigma_{post}^{-2}}\mu_s + \frac{\sigma_\varepsilon^{-2}}{\sigma_{post}^{-2}}m + \frac{\sigma_n^{-2}}{\sigma_{post}^{-2}}z(p) \quad (6)$$

and variance $Var[s|m, p] = \sigma_{post}^2$, where $\sigma_{post} \equiv (\sigma_s^{-2} + \sigma_\varepsilon^{-2} + \sigma_n^{-2})^{-1/2}$. It follows that the expected dividend conditional on signal m is

$$\begin{aligned} \mathbb{E}[A|m, p] &= \Pr[K(s, p) \geq \kappa|m, p] = \Pr[s \geq m^*(p) + \sigma_\varepsilon\Phi^{-1}(\kappa) | m, p] \\ &= \Phi\left(\frac{1}{\sigma_{post}}(\mathbb{E}[s|m, p] - m^*(p) - \sigma_\varepsilon\Phi^{-1}(\kappa))\right) \end{aligned}$$

By (6), the latter is increasing in m . It follows that an investor finds it optimal to invest if and only if and only if $m \geq m^{**}(p)$, where $m^{**}(p)$ is the unique solution to

$$\Phi\left(\frac{1}{\sigma_{post}}(\mathbb{E}[s|m^{**}(p), p] - m^*(p) - \sigma_\varepsilon\Phi^{-1}(\kappa))\right) = p$$

In any equilibrium, $m^{**}(p) = m^*(p)$. Along with (6), this gives a unique solution for $m^*(p)$:

$$m^*(p) = \mu_s + \sigma_\varepsilon\Phi^{-1}(\kappa) + [\lambda\sigma_\varepsilon\sigma_{post}\sigma_n^{-2} - 1]\sigma_s^2\sigma_{post}^{-1}\Phi^{-1}(p). \quad (7)$$

We conclude that there exists a unique equilibrium demand function and is given by

$$k(m, p) = \begin{cases} 1 & \text{if } m \geq m^*(p) \\ 0 & \text{otherwise} \end{cases}$$

with $m^*(p)$ as in (7). By assumption, $\lambda\sigma_\varepsilon\sigma_{post}\sigma_n^{-2} > 1$, which guarantees that $m^*(p)$ is a continuously increasing in p and hence the equilibrium demand for the asset is continuously decreasing in p . Along with the fact that the supply of the asset is continuously increasing in p , this also guarantees that there exists a unique equilibrium price function, $p = P(s, u)$. The latter is found by substituting $m^*(p)$ from (7) into (5) and solving for p . Doing so gives

$$p = P(s, u) \equiv \Phi \left(\frac{s - \sigma_\varepsilon u - \mu_s - \sigma_\varepsilon \Phi^{-1}(\kappa)}{[\lambda\sigma_\varepsilon\sigma_{post}(\sigma_n^{-2} + \sigma_s^2) - 1] \sigma_s^2 \sigma_{post}^{-1}} \right),$$

which is continuously increasing in s and continuously decreasing in u . *QED*

In the equilibria constructed above, the aggregate demand for the asset is globally decreasing in its price and therefore intersects only once with supply. Moreover, these equilibria feature only smooth fluctuations in asset prices. This is unlike the backward-bending demand functions, multiple demand-supply intersections, and discrete price changes (crashes) featured in Angeletos and Werning (2006), Barlevy and Veronesi (2003), or Ozdenoren and Yuan (2008). Therefore, an outsider could, once again, fail to detect any obvious symptoms of multiplicity and could fail to identify this model from a smoother, unique-equilibrium model of the financial market.

5 Private sunspots and efficiency

In the baseline model of Sections 2 and 3, the best sunspot-less equilibrium (the one in which $K = 1$) coincides with the first-best allocation. This, however, need not be the case in general. Investment booms may sometimes be excessive, leading to inefficient bubbles, crowding out of other productive activities, or having adverse price effects. For any of these reasons, the sunspot-less equilibrium with high investment ($K = 1$) may feature inefficiently high investment, even if it is the best among all equilibria with no (or only public) sunspots.

In a certain sense, this is precisely the case in the financial-market model of Section 4. In that model, a proper welfare analysis is complicated by the fact that I have assumed an exogenous supply

of the asset: I have not modeled the “noise traders” that lie behind this supply. We can nevertheless bypass this complication by focusing on the welfare of the investors that have been modeled—think of the latter as domestic agents and the ones behind the supply as “unloved” foreigners. Note then that higher aggregate investment implies a higher price at which the asset can be acquired. As a result, although domestic investors are better off in the equilibrium in which $K = 1$ than the one in which $K = 0$, they would have been even better off if they could somehow coordinate on some $K \in (\kappa, 1)$, for they would have then guaranteed the same rate of return at a lower price.

Whenever there are such inefficiencies, it is natural to think about Pigou-like policies that correct these inefficiencies and implement the first-best allocation as an equilibrium (although not necessarily the unique one). Suppose, though, that such policies are unavailable, too costly, or far from perfect, for reasons that are beyond the scope of this paper. I will now show how private sunspots, unlike public sunspots, can then improve welfare.

Towards this goal, consider the following variant of the baseline model. The net return to investment is now given by

$$A(K) = \begin{cases} 1 - c - hK & \text{if } K \geq \kappa \\ -c - hK & \text{if } K < \kappa \end{cases} \quad (8)$$

where $\kappa \in (0, \frac{1}{2}]$ and $h \geq 0$.¹² The baseline model is nested with $h = 0$. Allowing $h > 0$ introduces a congestion effect: a negative externality similar to the pecuniary externality featured in Section 4 or, more generally, a source of inefficiency in the best sunspot-less equilibrium.

Proposition 4 *Suppose $h \in (\frac{1-c}{2}, 1-c)$.*

- (i) There exist only two sunspot-less equilibria, one with $K = 1$ and another with $K = 0$.*
- (ii) The equilibrium in which $K = 1$ achieves higher welfare (ex-ante utility) than the equilibrium in which $K = 0$, as well as than any equilibrium with public sunspots.*
- (iii) The first-best level of aggregate investment is $K^* \in [\kappa, 1)$.*

Proof. Part (i) follows from the fact that $A(K) < 0$ for all $K \in [0, \kappa)$ and, as long as $h < 1 - c$, $A(K) > 0$ for all $K \in [\kappa, 1]$. Now let $w(K)$ denote welfare (ex-ante utility) when the fraction of agents investing is K : $w(K) \equiv K\Pi(1, K) + (1 - K)\Pi(0, K) = KA(K)$. For part (ii), note that $w(1) = 1 - c - h$ and $w(0) = 0$, so that the result follows again from the assumption $h < 1 - c$. Finally, for part (iii), note that $w(K)$ is continuous, strictly decreasing, and strictly concave for

¹²Letting $b = 1$ is merely a normalization, while $\kappa \leq \frac{1}{2}$ simplifies a step in the proof of Proposition 6.

$K < \kappa$; it has an upward jump at $K = \kappa$ (at which point it is right- but not left-continuous); and thereafter it is again continuous and strictly concave, but possibly non-monotonic. In particular, for $K > \kappa$, $w'(K) = 1 - c - 2hK$, so that the first-best level of investment is given by

$$K^* \equiv \arg \max_K w(K) = \begin{cases} \kappa & \text{if } 1 - c - 2h\kappa \leq 0 \\ \frac{1-c}{2h} \in (\kappa, 1) & \text{if } 1 - c - 2h < 0 < 1 - c - 2h\kappa \\ 1 & \text{if } 1 - c - 2h \geq 0 \end{cases} \quad (9)$$

Therefore, $K^* < 1$ if and only if $h > \frac{1-c}{2}$. *QED*

The key result here is that, as long as the congestion effect is not too high ($h < 1 - c$), there continue to exist exactly two equilibria in the absence of sunspots; but, as long as the congestion effect is not too low ($h > \frac{1-c}{2}$), neither equilibrium is first-best efficient. That public sunspots can not improve upon those two equilibria is clear: public sunspots only attain convex combinations of the welfare levels attained by the two sunspot-less equilibria and they are thus dominated by the equilibrium in which $K = 1$. This, however, is not true once we allow for private sunspots.

Proposition 5 *Whenever $h \in (1 - \sqrt{c}, 1 - c)$, there exist equilibria with private sunspots that sustain strictly higher welfare than any of the equilibria with no or only public sunspots.*

This result can be established with a specific example. Here, I go one step further by characterizing the *best* possible equilibrium with private sunspots. This permits me to identify what equilibrium properties are necessary for efficiency when one allows for private sunspots—and then to contrast them with those that one identifies if one restricts attention to public sunspots.

Proposition 6 *Suppose $h \in (1 - \sqrt{c}, 1 - c)$, allow for private sunspots, and consider the set of equilibria that maximize welfare. There exists a unique pair (q^*, p^*) , with $K^* < q^* < 1$ and $0 < p^* < 1$, such that all these equilibria are characterized by the following properties:*

(i) $K(s) = q^*$ with probability p^* and $K(s) = 0$ with probability $1 - p^*$; that is, the economy fluctuates between “normal times”, events during which aggregate investment is positive, and “crashes”, events during which investment collapses to zero.

(ii) q^* and p^* decrease with c or h ; that is, the probability of a crash increases, and the level of investment in normal times decreases, as fundamentals get worse.

Proof. By the revelation principle, any equilibrium with private sunspots can be represented by a c.d.f. $F : [0, 1] \rightarrow [0, 1]$ such as the following hold: first, “Nature” draws q from F ; next, a “mediator” sends private messages that say “invest” to a fraction q of the population, while it sends private messages that say “don’t invest” to the remaining fraction $1 - q$; finally, investors find it individually rational to follow the action recommended in their respective messages.¹³ We can thus identify the best equilibria by studying the distributions F that maximize welfare (ex-ante utility) subject to the relevant incentive-compatibility constraints.

Take any F . Let μ_1 (resp., μ_0) be the c.d.f. of the posterior about q for an investor who receives the message “invest” (resp., “don’t invest”). By Bayes’ rule,

$$\mu_1(q) = \frac{\int_0^q q' dF(q')}{\int_0^1 q' dF(q')} \quad \text{and} \quad \mu_0(q) = \frac{\int_0^q (1 - q') dF(q')}{\int_0^1 (1 - q') dF(q')} \quad (10)$$

For the recommended actions to be incentive-compatible, the expected net return from investing must be positive conditional on the message “invest” and negative conditional on the message “don’t invest”: $\int_0^1 A(q) d\mu_1(q) \geq 0$ and $\int_0^1 A(q) d\mu_0(q) \leq 0$. Using (10), these constraints reduce to

$$W(F) \equiv \int_0^1 w(q) dF(q) \geq 0 \quad \text{and} \quad R(F) \equiv \int_0^1 r(q) dF(q) \leq 0,$$

where $w(q) \equiv qA(q)$ and $r(q) \equiv (1 - q)A(q)$. For any F that satisfies these constraints, welfare (ex-ante utility) is given as follows:

$$\mathbb{E}\pi = \int_0^1 [q\Pi(1, q) + (1 - q)\Pi(0, q)] dF(q) = \int_0^1 w(q) dF(q) = W(F).$$

The best equilibria are thus identified by maximizing $W(F)$ subject to $W(F) \geq 0 \geq R(F)$. Clearly, the set of F that satisfy these constraints is non-empty and the constraint $W(F) \geq 0$ does not bind at the optimum. The remainder of the proof thus characterizes the functions F that maximize $W(F)$ among the set of non-decreasing functions $F : [0, 1] \rightarrow [0, 1]$ that satisfy $R(F) \leq 0$.

I first show that any solution to this problem assigns zero measure to $q \in (0, K^*)$. Towards a contradiction, take any F that violates this property and construct a variation \tilde{F} by letting $\tilde{F}(q) = \lim_{q \rightarrow \kappa^-} F(q)$ for $q \in [0, K^*)$ and $\tilde{F}(q) = F(q)$ for $q \geq K^*$; \tilde{F} is thus constructed from F by reassigning to $q = 0$ all the mass that F assigns to $q \in [0, \kappa)$ and to $q = K^*$ all the mass that

¹³Restricting attention to pure strategies is immaterial because of the continuum of agents.

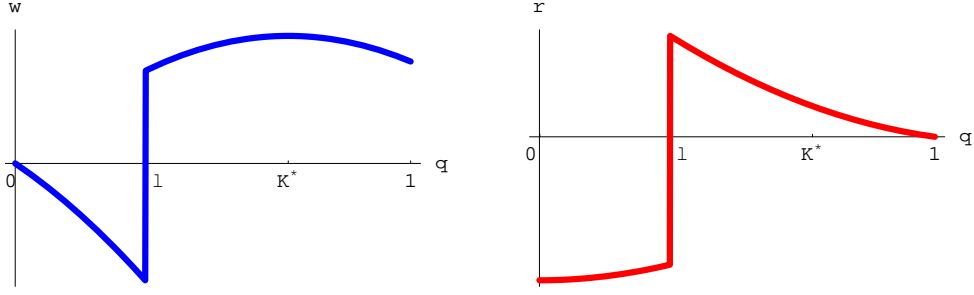


Figure 1: The functions $w(q) \equiv qA(q)$ and $r(q) \equiv (1 - q)A(q)$

F assigns to $q \in [\kappa, K^*]$, while not affecting the mass assigned to $q > K^*$. As illustrated in the left panel of Figure 1, the function $w(q) \equiv qA(q)$ is continuous and strictly decreasing in q for $q < \kappa$; it has an upward jump at $q = \kappa$; and thereafter it is again continuous and strictly concave, reaching its maximum at $q = K^* \in [\kappa, 1)$. It follows that $w(0) \geq w(q)$ for all $q \in [0, \kappa)$ and $w(K^*) \geq w(q)$ for all $q \in [\kappa, 1]$. By implication, the variation \tilde{F} improves welfare:

$$W(\tilde{F}) - W(F) = \int_0^\kappa [w(0) - w(q)] dF(q) + \int_\kappa^{K^*} [w(K^*) - w(q)] dF(q) > 0.$$

Next, as illustrated in the right panel of Figure 1, $r(q) \equiv (1 - q)A(q)$ is continuous and strictly increasing in q for $q \in [0, \kappa)$; it has an upward jump at $q = \kappa$; and it is continuous and strictly decreasing in q for $q \in [\kappa, 1]$. It follows that the variation \tilde{F} relaxes incentive compatibility:

$$R(\tilde{F}) - R(F) = \int_0^\kappa [r(0) - r(q)] dF(q) + \int_\kappa^{K^*} [r(K^*) - r(q)] dF(q) < 0.$$

Since the variation \tilde{F} is both feasible and welfare-improving, no F that assigns positive measure to $q \in (0, K^*)$ can be optimal. We conclude that, for any optimal F , there exists a scalar $p \in [0, 1]$ and a c.d.f. $G : [K^*, 1] \rightarrow [0, 1]$ such that $F(q) = 1 - p$ for $q < K^*$ and $F(q) = (1 - p) + pG(q)$ for $q \geq K^*$. That is, $1 - p$ is the mass assigned to $q = 0$, p is the mass assigned to $q \geq K^*$, and G is the distribution of q conditional on $q \geq K^*$. It then also follows that

$$W(F) = (1 - p)w(0) + p \int_{K^*}^1 w(q) dG(q) \quad \text{and} \quad R(F) = (1 - p)r(0) + p \int_{K^*}^1 r(q) dG(q).$$

¹⁴That $r(q)$ is increasing for all $q \in [0, \kappa)$ is guaranteed by the assumption that $\kappa \leq 1/2$ and holds more generally as long as $c + h(1 - 2\kappa) \geq 0$.

Consider now the subproblem of choosing G for given $p \in (0, 1]$. This is the same as maximizing $\tilde{W}(G)$ subject to $\tilde{R}(G) \leq b(p)$, where $\tilde{W}(G) \equiv \int_{K^*}^1 w(q) dG(q)$, $\tilde{R}(G) \equiv \int_{K^*}^1 r(q) dG(q)$, and $b(p) \equiv -(1-p)r(0)/p$. Because this is a convex optimization problem, there exists a Lagrange multiplier $\hat{\lambda}_p \geq 0$ such that the optimal G solves $\max_G \int_{K^*}^1 [w(q) - \hat{\lambda}_p r(q)] dG(q)$. But now note that, for any $\hat{\lambda}_p \geq 0$, the function $w(q) - \hat{\lambda}_p r(q)$ is continuous and strictly concave in q over $[K^*, 1]$, and therefore there exists a unique \hat{q}_p such that $\hat{q}_p = \arg \max_{q \in [K^*, 1]} [w(q) - \hat{\lambda}_p r(q)]$, which in turn implies that the optimal G assigns all measure to $q = \hat{q}_p$. We can thus identify any optimal F with a pair $(\hat{p}, \hat{q}) \in [0, 1] \times [K^*, 1]$ that maximizes $\hat{W}(\hat{p}, \hat{q}) \equiv (1 - \hat{p})w(0) + \hat{p}w(\hat{q})$ subject to

$$\hat{R}(\hat{p}, \hat{q}) \equiv (1 - \hat{p})r(0) + \hat{p}r(\hat{q}) \leq 0. \quad (11)$$

Note that the constraint (11) must bind: if it did not, the optimum would be $(\hat{p}, \hat{q}) = (1, K^*)$, but then (11) would be violated, since $r(K^*) = (1 - K^*)A(K^*) > 0$. Thus, let $\lambda^* > 0$ be the Lagrange multiplier associated with (11). Using the fact that $w(0) = 0$ and $r(0) = A(0) = -c$, the first-order conditions for \hat{q} and \hat{p} reduce to the following:

$$w'(q^*) - \lambda r'(q^*) \begin{cases} \leq 0 & \text{if } q^* = K^* \\ = 0 & \text{if } q^* \in (K^*, 1) \\ \geq 0 & \text{if } q^* = 1 \end{cases} \quad w(q^*) - \lambda[r(q^*) + c] \begin{cases} \leq 0 & \text{if } p^* = 0 \\ = 0 & \text{if } p^* \in (0, 1) \\ \geq 0 & \text{if } p^* = 1 \end{cases} \quad (12)$$

Recall that $r'(q) < 0$ for all $q \in [K^*, 1]$. Together with $w'(K^*) = 0$ and $\lambda^* > 0$, this rules out $q^* = K^*$. If $p^* = 1$, (11) implies $q^* = 1$. But then the left part of (12) gives $w(1) - \lambda^*[r(1) + c] \geq 0$, or equivalently $\lambda^* \leq (1 - c - h)/c$, while the right part of (12) gives $w'(1) - \lambda^*r'(1) \geq 0$, or equivalently $\lambda^* \geq -(1 - c - 2h)/(1 - c - h)$. Hence, $p^* = 1$ is possible only if $q^* = 1$ and $-(1 - c - 2h)/(1 - c - h) \leq (1 - c - h)/c$; the latter in turn holds if and only if $c \leq (1 - h)^2$. If instead $p^* < 1$, then (11) gives $r(q^*) = c(1 - p^*)/p^* > 0$, which guarantees that $q^* < 1$ and, along with (12), gives the following unique non-negative solution (q^*, p^*, λ^*) :

$$q^* = \frac{1 - \sqrt{c}}{h}, \quad p^* = \frac{h\sqrt{c}}{h - (1 - \sqrt{c})^2}, \quad \lambda^* = \frac{(1 - \sqrt{c})^2}{h - (1 - \sqrt{c})^2}. \quad (13)$$

Note then that this solution satisfies $q^* < 1$ and $p^* < 1$ if and only if $c > (1 - h)^2$, or equivalently $h > 1 - \sqrt{c}$; if instead $h \leq 1 - \sqrt{c}$, the optimum is attained with $q^* = 1$ and $p^* = 1$. *QED*

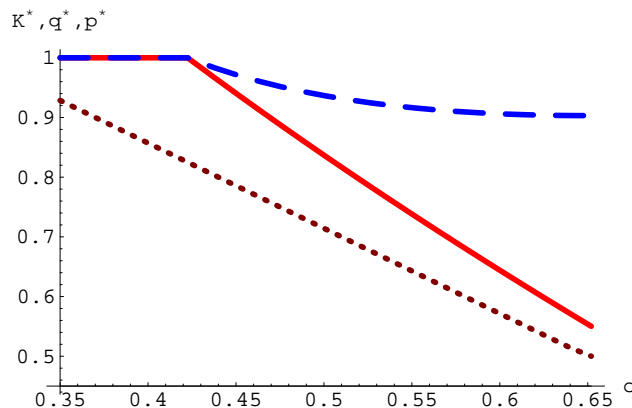


Figure 2: Comparative statics of best private-sunspot equilibrium.

This result establishes that the best equilibrium with private sunspots has the economy alternating between “normal times”, i.e., states during which a large fraction of the population invests, and “crashes”, i.e., states during which nobody invests. From the perspective of the pertinent literature, this seems quite paradoxical: the occurrence of a crash is considered prima-facia evidence of a coordinate failure, for the best equilibrium with no or only public sunspots would never feature a crash. The key to this apparent paradox is the incentive effect that the possibility of a crash has during normal times. In particular, the fact that many but not all investors invest during normal times contributes towards higher welfare than in the best sunspot-less equilibrium: the level of investment during normal times is now closer to the first-best level. However, for certain individuals to have an incentive not to invest during normal times, it must be that these individuals believe that a crash will take place with sufficiently high probability, while many other individuals believe the opposite. In turn, such heterogeneity in beliefs is possible in equilibrium only if crashes *do* happen with positive probability, which explains the result.

To further illustrate the economics behind the determination of the best equilibrium, Figure 2 considers its comparative statics with respect to c ; the comparative statics with respect to h are similar. The dashed line gives p^* , while the solid line gives q^* . For comparison, the dotted lined gives K^* , the first-best level of investment. Note that $q^* > K^*$ always, that $p^* < 1$ and $q^* < 1$ as soon as $1 - \sqrt{c} < h$, and that thereafter both p^* and q^* decrease with c . In words, as the fundamentals worsen, so that the first-best level of investment falls, the equilibrium level of investment during normal times also falls, and the probability of a crash increases. This is true even though the best sunspot-less equilibrium is invariant with the fundamentals.

6 Conclusion

The pertinent literature on macroeconomic complementarities and endogenous fluctuations has focused on *aggregate* extrinsic uncertainty. In this paper I introduced *idiosyncratic* extrinsic uncertainty and showed how this can lead to novel positive and normative implications.

In one sense, the private sunspots considered in this paper capture the idea that agents face uncertainty about which equilibrium is played: each individual does not know what is the action upon which other agents are trying to coordinate.¹⁵ In another sense, they capture idiosyncratic variation in investor sentiment: different agents hold different expectations regarding the *endogenous* prospects of the economy. These possibilities were absent from previous work: in equilibria with public sunspots, all agents share the same beliefs about endogenous outcomes, face no uncertainty about what other investors are doing, and play the same action.

The heterogeneity of beliefs regarding endogenous economic outcomes obtained in this paper does not require any heterogeneity in information about exogenous economic fundamentals, nor any deviation from Bayesian rationality. Rather, it obtains as a self-fulfilling equilibrium property. Furthermore, it sustains significant heterogeneity in choices even in the absence of any heterogeneity in primitive characteristics. It can thus show up as significant “residual” variation in any econometric exercise that attempts to explain the observed heterogeneity in investment or portfolio choices on the basis of heterogeneity in individual characteristics such as wealth, risk aversion, or information regarding economic fundamentals. At the same time, it can sustain richer and smoother aggregate fluctuations, possibly making it easier for this class of models to match aggregate data and harder for an econometrician to detect the underlying multiplicity of equilibria.

Another intriguing possibility is that, with private sunspots, social learning can regard endogenous coordination rather than exogenous fundamentals. In particular, asset prices or data about aggregate activity may facilitate better predictability of the endogenous prospects of the economy and better coordination among agents, even if there is nothing to be learned from them regarding the exogenous economic fundamentals.¹⁶

¹⁵In this respect, the paper also relates to the recent work on global games (e.g., Morris and Shin, 2001; Angeletos and Werning, 2006). This literature introduces heterogeneous information regarding the fundamentals and studies how the resulting strategic uncertainty affects equilibrium outcomes, in certain cases selecting a unique equilibrium. In contrast, private sunspots accommodate strategic uncertainty without obstructing equilibrium multiplicity. An interesting question is how the two sources of strategic uncertainty may interact if one introduces private sunspots in global games with multiple equilibria.

¹⁶The role of prices in facilitating coordination has also been emphasized in Angeletos and Werning (2006); in their context, however, this is only because prices serve as a public signal regarding underlying unobserved fundamentals.

Finally, private sunspots unrest the conventional wisdom regarding the normative properties of environments with coordination problems. In certain cases, occasional investment crashes may be necessary for facilitating idiosyncratic uncertainty and thereby improving efficiency during normal times. When this is the case, what looks ex post as a coordination failure is actually contributing towards higher ex-ante welfare; and policies aimed at preventing such apparent coordination failures may backfire by eliminating a social mechanism that improves efficiency during normal times.

All these insights are evidently relevant for a wide class of models in macroeconomics that feature complementarities. However, the present paper delivered these insights only within two highly stylized representatives of this class of models. Embedding the analysis within richer micro-founded models remains an open direction for future research.

Appendix: dynamics and learning

Consider the dynamic extension of Section 3. The sunspot s_t follows an AR(1): $s_{t+1} = \rho s_t + u_t$, with $u_t \sim \mathcal{N}(0, \sigma_u)$, $s_1 \sim \mathcal{N}(0, \sigma_s)$, and $\sigma_s = \frac{\sigma_u}{1-\rho}$. The private signals are given by $m_t = s_t + \varepsilon_t$ and $x_t = \Phi^{-1}(K_t) + \xi_t$, with $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon)$ and $\xi_t \sim \mathcal{N}(0, \sigma_\xi)$. Let $\alpha_u \equiv \sigma_u^{-2}$, $\alpha_\xi \equiv \sigma_\xi^{-2}$, $\alpha_\varepsilon \equiv \sigma_\varepsilon^{-2}$.

Proposition 7 *There exists a sequence $\{m_t^*, \sigma_t\}_{t=0}^\infty$ and an equilibrium such that (i) the private information of an investor at t with respect to s_t is summarized in a sufficient statistic \tilde{m}_t that is Normal with mean s_t and variance σ_t^2 , and (ii) an investor invests at t if and only if $\tilde{m}_t \geq m_t^*$.*

Proof. The proof is by induction. The result trivially holds at $t = 1$, since the first period coincides with the static benchmark. Thus suppose the result holds at $t \geq 1$. Then, $K_t(s^t) = \Phi\left(\frac{s_t - m_t^*}{\sigma_t}\right)$ and $x_{t+1} = \frac{1}{\sqrt{\beta_t}}(s_t - m_t^*) + \xi_{t+1}$, where $\beta_t \equiv \sigma_t^{-2}$, so that x_{t+1} is effectively a Gaussian signal about s_t with precision $\beta_t \alpha_\xi$. It follows that the private information regarding s_{t+1} can be summarized in a sufficient statistic \tilde{m}_{t+1} that is Normal with mean s_{t+1} and variance $\sigma_{t+1}^2 \equiv \beta_{t+1}^{-1}$, where

$$\tilde{m}_{t+1} = \frac{\rho^2 \alpha_u (1 + \alpha_\xi) \beta_t}{\beta_{t+1} (\alpha_u + (1 + \alpha_\xi) \beta_t)} \left\{ \frac{1}{1 + \alpha_\xi} \tilde{m}_t + \frac{\alpha_\xi}{1 + \alpha_\xi} \left(m_t^* - \frac{1}{\sqrt{\beta_t}} x_t \right) \right\} + \frac{\alpha_\varepsilon}{\beta_{t+1}} m_{t+1} \quad (14)$$

$$\beta_{t+1} = \Gamma(\beta_t) \equiv \frac{\rho^2 \alpha_u (1 + \alpha_\xi) \beta_t}{\alpha_u + (1 + \alpha_\xi) \beta_t} + \alpha_\varepsilon. \quad (15)$$

But then, by a similar argument as in Proposition 2, it is indeed a continuation equilibrium that an investor invests in period $t + 1$ if and only if $\tilde{m}_{t+1} \geq m_{t+1}^*$, where

$$m_{t+1}^* = -\sigma_s \left\{ \frac{\sigma_s^2 + \sigma_{t+1}^2}{\sigma_s \sigma_{t+1}} \Phi^{-1}(\kappa) + \sqrt{1 + \frac{\sigma_s^2}{\sigma_{t+1}^2}} \Phi^{-1}\left(\frac{c}{b}\right) \right\}, \quad (16)$$

which proves that the result holds at $t + 1$ and completes the induction argument. ■

Now note that, for any $\rho \in (0, 1)$ and any finite $(\alpha_u, \alpha_\xi, \alpha_\varepsilon)$, the function $\Gamma(\beta)$ is strictly increasing and strictly concave, with $\Gamma(0) > 0$. It follows that (i) there exists a unique $\beta > 0$ such that $\beta = \Gamma(\beta)$ and (ii) the sequence $\{\beta_t\}_{t=0}^\infty$ converges to this fixed point for any initial $\beta_0 > 0$. This proves the claim that, as the history becomes infinitely long, both m_t^* and σ_t converge.

Finally, consider the variant with learning over a constant underlying sunspot ($s_t = s$ for all t). This is nested with $\rho = 1$ and $\alpha_u = \infty$, in which case (15) reduces to $\beta_{t+1} = \Gamma(\beta_t) = (1 + \alpha_\xi)\beta_t + \alpha_\varepsilon$. It is then immediate that σ_t decreases monotonically over time and asymptotes to 0 as $t \rightarrow \infty$. Finally, letting $\kappa = c/b = 1/2$ into (16) gives $m_t^* = 0$ for all t , as claimed in the main text.

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