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DERIVATIVES

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**ABSTRACT**

We conduct a comprehensive analysis of unspanned stochastic volatility in commodity markets in general and the crude-oil market in particular. We present model-free results that strongly suggest the presence of unspanned stochastic volatility in the crude-oil market. We then develop a tractable model for pricing commodity derivatives in the presence of unspanned stochastic volatility. The model features correlations between innovations to futures prices and volatility, quasi-analytical prices of options on futures and futures curve dynamics in terms of a low-dimensional affine state vector. The model performs well when estimated on an extensive panel data set of crude-oil futures and options.

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# 1 Introduction

Volatility in commodity markets (as in most other asset markets) is obviously stochastic. Less obvious is the extent to which volatility risk can be hedged by trading in the commodities themselves or, more generally, their associated futures contracts; in other words, the extent to which volatility is spanned by the futures contracts. This question is of fundamental importance for the pricing, hedging and risk-management of commodity options. If, for a given commodity, volatility is unspanned by the futures contracts, then options on futures contracts are not redundant securities, and they cannot be fully hedged and risk-managed using only the futures contracts.

The purpose of this paper is to conduct a comprehensive analysis of unspanned stochastic volatility in commodity markets in general and the crude-oil market in particular. The paper makes a number of theoretical and empirical contributions to the literature. First, we present results that strongly suggest the presence of unspanned stochastic volatility in the crude-oil market. For different option maturities, we regress returns on at-the-money (ATM) option straddles and changes in implied volatilities – both reasonable proxies for changes in the true but unobservable volatility – on futures returns and find low  $R^2$ s, indicating that most volatility risk cannot be hedged by trading in the futures contracts. Furthermore, there is large common variation in regression residuals across option maturities, indicating that there is one dominant unspanned stochastic volatility factor. These results are model-free in the sense that no pricing model is used to derive the optimal hedge portfolios. Furthermore, the results hold true regardless of the number of options included in the analysis, the length of the sample and whether we run normal or “rolling” regressions.

Second, we develop a parsimonious and highly tractable model for pricing commodity derivatives in the presence of unspanned stochastic volatility. The model is specified directly under the risk-neutral probability measure. It is based on the Heath, Jarrow, and Morton (1992) (HJM) framework and takes the initial futures curve as given. Futures prices are driven by two factors, with one factor affecting the spot price of the commodity and another factor affecting the forward cost of carry curve. A third (square-root) factor drives the volatility of the futures prices; hence, options on futures contracts are driven by three factors. We allow for correlations between innovations to the three factors which implies that volatility may be partially spanned by the futures contracts. The model features quasi-analytical prices of European options on futures contracts based on transform techniques. By a suitable

parametrization of the shocks to the forward cost of carry curve, the dynamics of the futures curve can be described in terms of a four-dimensional affine state vector, which makes the model ideally suited for pricing derivatives by simulations.

Third, we estimate the model on an extensive panel data-set of crude-oil futures and options, involving a total of 9536 futures prices and 49001 option prices. Estimation is facilitated by parameterizing the market prices of risk, such that the state vector is also described by an affine diffusion under the actual probability measure. The estimation procedure is quasi-maximum likelihood in conjunction with the extended Kalman filter. We show that the model has an excellent fit to futures across different maturities and, generally, a good fit to options across both different moneyness categories and maturities. The model is able to match variations in implied volatilities across time, across the maturity dimension, and, with the exception of very short-term options, across moneyness (i.e. the implied volatility “smile”). The parameter estimates imply that volatility is mostly unspanned, which is consistent with our regression results.

Fourth, as an application of the model we investigate the relative pricing of plain-vanilla and calendar spread options – options on the price differential between two futures contracts with different maturities. We price ATM calendar spread options out-of-sample from our model estimated on the main data set and find, conditional on our model describing the true dynamics of the derivatives market, that ATM calendar spread options have been overvalued relative to plain-vanilla options on average. However, the pricing errors are also very volatile, which renders the results insignificant for all but a few options.

Our paper draws on the term structure literature. Using regression approaches similar to the one applied in this paper, Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2003) document the presence of unspanned stochastic volatility in the fixed-income market. The results for the crude-oil market are broadly consistent with the results for the bond markets analyzed in these papers.

Our model is related to the stochastic volatility HJM interest rate models in Casassus, Collin-Dufresne, and Goldstein (2005) and Trolle and Schwartz (2006). It appears to be the first stochastic volatility HJM-type model for pricing commodity derivatives. Previous HJM-type commodity models such as Cortazar and Schwartz (1994), Amin, Ng, and Pirrong (1995), Miltersen and Schwartz (1998), Clewlow and Strickland (1999) and Miltersen (2003) all assume

deterministic volatilities.<sup>1</sup>

An alternative approach for pricing commodity derivatives relies on specifying the (typically affine) dynamics of a limited set of state variables and deriving futures prices endogenously. Examples of this approach include Gibson and Schwartz (1990), Schwartz (1997), Hilliard and Reis (1998), Schwartz and Smith (2000), Richter and Sørensen (2002), Nielsen and Schwartz (2004) and Casassus and Collin-Dufresne (2005). Of these papers only Richter and Sørensen (2002) explicitly allow for stochastic volatility. The main drawback of these models is that volatility will almost invariably be completely spanned by the futures contracts; indeed, this is the case in the Richter and Sørensen (2002) model.<sup>2</sup>

The paper is organized as follows. Section 2 discusses the crude-oil derivatives data that we make extensive use of throughout the paper. Section 3 presents the evidence of unspanned stochastic volatility. Section 4 describes the model for pricing commodity derivatives. Section 5 discusses the estimation procedure and estimation results. Section 6 analyses the relative valuation of plain-vanilla and calendar spread options. Section 7 concludes. Three appendices contain proofs and additional information.

## 2 Overview of crude-oil derivatives data

In the paper we repeatedly use an extensive data set of crude-oil futures and options on futures trading on NYMEX.<sup>3</sup> The NYMEX crude-oil derivatives market is the world's largest and most liquid commodity derivatives market. The range of maturities that futures and options cover and the range of strike prices on the options are also greater than for other commodities. This makes it an ideal market for studying commodity derivatives pricing. The raw data set consists of daily data from January 2, 1990 until May 18, 2006 on settlement prices, open interest and daily volume for all available futures and options.<sup>4,5</sup>

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<sup>1</sup>Eydeland and Geman (1998) propose a stochastic volatility Heston (1993) model for pricing energy derivatives. However, they only model the evolution of the spot price, not the evolution of the entire futures curve.

<sup>2</sup>Collin-Dufresne and Goldstein (2002) derive the parameter restrictions necessary for volatility to be unspanned in affine term structure models. Similar conditions can be derived for affine commodity models.

<sup>3</sup>The New York Mercantile Exchange.

<sup>4</sup>The NYMEX light, sweet crude-oil futures contract trades in units of 1000 barrels. Prices are quoted as US dollars and cents per barrel.

<sup>5</sup>Note that all computations in the paper are based on settlement prices. Settlement prices for all contracts are determined by a "Settlement Price Committee" at the end of regular trading hours (currently 2.30 p.m.

The number and maximum maturity of the futures and options have increased significantly throughout the sample. On the first trading day in the sample, the number of futures and options with positive open interest (trading volume) were 17 (15) and 77 (52), respectively. The maximum maturity among the futures and options with positive open interest (trading volume) were 499 (470) and 164 (164) days, respectively. In contrast, on the last trading day in the sample, the number of futures and options with positive open interest (trading volume) were 45 (26) and 1435 (199), respectively, while the maximum maturity among the futures and options with positive open interest (trading volume) were 2372 (2372) and 2008 (1643) days, respectively.<sup>6</sup>

Liquidity has also increased. Open interest (daily volume) for the first future with more than 14 days to expiration has increased from 66,925 (45,177) to 273,746 (86,622) contracts. The combined open interest (daily volume) for the options on that future has increased from 92,083 (16,427) to 376,694 (32,820) contracts.

We make two observations regarding liquidity. First, open interest for futures tends to peak when expiration is a couple of weeks away, after which open interest declines sharply. Second, among futures and options with more than a couple of weeks to expiration, the first six monthly contracts tend to be very liquid. Beyond approximately six months, liquidity is concentrated in the contracts expiring in March, June, September and December. Beyond approximately one year, liquidity is concentrated in the contracts expiring in December. Due to these liquidity patterns, we screen the available futures and options contracts according to the following procedure: we discard all futures with 14 or less days to expiration. Among the remaining, we retain the first six monthly contracts. Beyond these, we choose the first two contracts with expiration in either March, June, September or December. Beyond these, we choose the next four contracts with expiration in December. This procedure leaves us with twelve generic futures contracts which we label M1, M2, M3, M4, M5, M6, Q1, Q2, Y1, Y2, Y3 and Y4.

Figure 1 displays the futures data. The run-up in crude-oil prices since 2002 is striking (EST) and represent a very accurate measure of the true market prices at the time of close. Settlement prices are widely scrutinized by all market participants since they are used for marking to market all account balances.

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<sup>6</sup>Futures expire on the third business day prior to the 25th calendar day of the month preceding the delivery month. If the 25th calendar day of the month is a non-business day, expiration is on the third business day prior to the business day preceding the 25th calendar day. Options expire three business days prior to the expiration of the underlying futures.

ing. Using the M1 futures contract as a proxy for the spot price, the Q2 futures contract is backwarddated 82.6 percent of the time and strongly backwarddated 66.3 percent of the time.<sup>7</sup>

Figure 2 displays the implied log-normal volatility for options on the first eight of the generic futures contracts.<sup>8</sup> It is evident that volatility is stochastic. In the next section we investigate to what extent stochastic volatility is unspanned.

### 3 Evidence of unspanned stochastic volatility

#### 3.1 Approach

If volatility is spanned by the futures contracts, it implies that changes in volatility can be hedged with a portfolio of futures contracts. Suppose we regress changes in volatility on the returns of futures contracts, the  $R^2$ s will indicate the extent to which volatility is spanned. However, this approach is not feasible since volatility is not directly observable. One alternative is to investigate how much of the variation in the prices of derivatives highly exposed to stochastic volatility (so-called “straddles”) can be explained by variation in the underlying futures prices. Another alternative is to investigate how much of the variation in log-normal implied volatilities (which is related to expectations under the risk-neutral measure of future volatility) can be explained by variation in the underlying futures prices. We use both approaches to investigate the extent to which volatility is spanned. Although they are related, they do have their relative strengths and weaknesses. Both approaches have previously been used by Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2003) to investigate unspanned stochastic volatility in the fixed income market.<sup>9</sup>

More specifically, we proceed in three steps. Suppose we have a set of contracts (futures and their associated options),  $i = 1, \dots, n$ . First, we factor analyze the covariance matrix of the

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<sup>7</sup>Let  $S(t)$  denote the time- $t$  spot price and  $F(t, T)$  [ $P(t, T)$ ] the time- $t$  price of a futures contract [zero-coupon bond] with maturity  $T - t$ . The futures contracts is backwarddated if  $S(t) - P(t, T)F(t, T) > 0$  and strongly backwarddated if  $S(t) - F(t, T) > 0$ . The numbers reported here are slightly lower than those reported in Litzenberger and Rabinowitz (1995) for an earlier sample.

<sup>8</sup>At each date we choose the options that are closest to ATM. The options are American and we invert their prices using the Barone-Adesi and Whaley (1987) formula. More details are given in Section 5.1.

<sup>9</sup>Yet a third alternative is to investigate how much of the variation in realized volatility, estimated from high-frequency data, can be explained by variations in the underlying futures prices. Andersen and Benzoni (2005) use this approach to investigate unspanned stochastic volatility in the fixed income market. However, our data set does not include high-frequency data for futures prices.

futures returns and retain the first three principal components,  $PC^{fut,1}$ ,  $PC^{fut,2}$  and  $PC^{fut,3}$ . These summarize virtually all of the information in the futures returns.

Second, for each futures contract  $i$ , we regress the return on the straddle that is closest to ATM on the first three principal components and squared principal components of the futures returns,<sup>10</sup>

$$r_t^{straddle,i} = \beta_0^i + \beta_1^i PC_t^{fut,1} + \beta_2^i PC_t^{fut,2} + \beta_3^i PC_t^{fut,3} + \beta_4^i \left( PC_t^{fut,1} \right)^2 + \beta_5^i \left( PC_t^{fut,2} \right)^2 + \beta_6^i \left( PC_t^{fut,3} \right)^2 + \epsilon_t^i. \quad (1)$$

A straddle consists of a put option and a call option with the same strike. The price of a near-ATM straddle has low sensitivity to variations in the price of the underlying futures contract (since “deltas” are close to zero for ATM straddles) but high sensitivity to variations in volatility (since “vegas” peak for ATM straddles). Consequently, the extent to which straddle returns can be explained by the principal components of the futures returns will indicate the extent to which volatility is spanned by the futures contracts.

For each futures contract  $i$  we also run regression (1) with the straddle return substituted by the change in implied log-normal volatility. The implied log-normal volatility is related to the average expected (under the risk-neutral measure) volatility over the life of the option. Hence, these regressions also indicate the extent to which volatility is spanned by the futures contracts.

Third, we factor analyze the covariance matrices of the  $n$  time-series of residuals from straddle return regressions and the implied volatility regressions. The principal components of the residuals are by construction independent of those of the futures returns. If there is unspanned stochastic volatility in the data, we should see large common variation in the residuals. If the residuals are simply due to noisy data, we should not find much common variation in the residuals.

The advantage of running the regressions with straddle returns is that the results are not conditional on a particular pricing model. The disadvantage is that even if volatility is completely unspanned by the futures contracts, the  $R^2$ s from the regressions will not necessarily be close to zero. The reason is that straddle returns are highly convex in the futures returns (near-ATM straddles have high “gammas”), and we include the squared principal components of the futures returns in the regressions. Hence, some of the variation in straddle returns will

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<sup>10</sup>We add the squared principal components to the set of independent variables in the regressions in an attempt to take account of non-linearities in the relation between straddle returns and futures returns.



be explained by futures returns even if none of the variation in volatility can be explained by the futures returns. In contrast, running the regressions with implied volatility should produce  $R^2$ s close to zero if volatility is completely unspanned. However, in this case the results will to some extent be conditional on the particular pricing model used for inverting implied volatilities from option prices.

A potential weakness of the procedure is that the parameters in the regressions are assumed constant over the entire sample. In reality we would expect the parameters to be time-varying. To take account of such time-variation, we also perform the analysis using a rolling window of 100 observations.<sup>11</sup> That is, for each window we extract the first three principal components of futures returns, run the regressions and factor analyze the residuals.

### 3.2 Data

We compute daily returns on the futures contracts. For each futures contract, we compute daily returns on the straddle that is closest to ATM. We only search among straddles with moneyness (strike divided by the price of the underlying futures contract) in the interval 0.95–1.05.<sup>12</sup> The implied log-normal volatilities are computed as the averages of the implied log-normal volatilities of the puts and calls that constitutes the straddles.<sup>13</sup>

An important issue is that our approach requires a full set of futures and straddle returns. The fact that the maturity range of futures and options has increased over the sample generates a tradeoff: as we include longer contracts in the analysis, we have fewer dates with a full set of futures and straddle returns but obtain more information about common variation in returns across different straddles. For this reason we perform the analysis with the following five different sets of contracts: M1–M4, M1–M6, M1–Q2, M1–Y2 and M1–Y4.

### 3.3 Results

We first discuss the results from the full regressions. Table 1 shows the explanatory power of the first three principal components of the futures returns. The first principal component explains the majority of the variation in futures returns. The importance of the second principal

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<sup>11</sup>We have tried window lengths of 50, 100 and 200 observations with little impact on the results.

<sup>12</sup>Furthermore, we only consider options that have open interest in excess of 100 contracts.

<sup>13</sup>The options are American and we invert their prices using the Barone-Adesi and Whaley (1987) formula. More details are given in Section 5.1.

component increases with the number of the contracts included in the analysis. The third principal component is not important in any of the cases.

Table 2 shows the adjusted  $R^2$ s from regressing straddles returns on the principal components (and squared principal components) of the straddle returns. The  $R^2$ s range from 6.6 percent to 62.5 percent, indicating that it is very difficult to hedge volatility risk using futures contracts.<sup>14</sup>

Table 3 shows the adjusted  $R^2$ s from regressing changes in implied volatilities on the principal components (and squared principal components) of the straddle returns. In this case, the  $R^2$ s range from 0.4 percent to 20.9 percent. This provides even stronger evidence of the inability of futures contracts to hedge volatility risk.

Table 4 displays the explanatory power of the first three principal components of the regression residuals. For the straddle return regressions, the first principal component explains between 50.8 percent and 79.8 percent of the variation in the residuals across maturities, while for the implied volatility regressions, it explains between 52.6 percent and 80.1 percent. Hence, there is large common variation in the residuals, which strongly indicates that the low  $R^2$ s from the regressions are primarily due to an unspanned stochastic volatility factor rather than noisy data.<sup>15</sup>

Similar results are obtained when we perform the analyses over a rolling window of 100 observations. The tables report average results (i.e. average explanatory power of the principal components and average  $R^2$ s from the regressions). The average explanatory power of the first three principal components of the futures returns is virtually the same as for the full regressions. As expected, the average regression  $R^2$ s are generally (but not always) higher than for the full regressions. The reason is that the regressions now are able to capture time variations in the relations between straddle returns and futures returns and between implied volatility changes and futures returns. Importantly, we still find evidence of unspanned stochastic volatility as the first principal component on average explains between 51.0 percent and 71.6 percent of the variation in the residuals from the straddle return regressions and between 55.5 percent

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<sup>14</sup>As discussed above, in these regressions the  $R^2$ s will not necessarily be close to zero even if volatility is completely unspanned. That the  $R^2$ s decrease with straddle maturity is consistent with the fact that “gammas” of ATM straddles also decrease with maturity.

<sup>15</sup>The correlation between the first principal component of the residuals from the straddle return regression and the first principal component of the residuals from the implied volatility regression is consistently above 0.95.

and 73.1 percent of the variation in the residuals from the implied volatility regressions.<sup>16</sup>

## 4 A model for commodity derivatives featuring unspanned stochastic volatility

Based on the evidence presented above, we now develop a parsimonious and highly tractable model for pricing commodity derivatives in the presence of unspanned stochastic volatility.

### 4.1 The model under the risk-neutral measure

Let  $S(t)$  denote the time- $t$  spot price of the commodity. The dynamics of  $S(t)$  is given by

$$\frac{dS(t)}{S(t)} = \delta(t)dt + \sigma_S \sqrt{v(t)} dW_1^Q(t), \quad (2)$$

where  $\delta(t)$  denotes the instantaneous spot cost of carry. Let  $y(t, T)$  denote the time- $t$  instantaneous forward cost of carry at time  $T$  with  $y(t, t) = \delta(t)$ . The dynamics of  $y(t, T)$  is given by

$$dy(t, T) = \mu_y(t, T)dt + \sigma_y(t, T) \sqrt{v(t)} dW_2^Q(t). \quad (3)$$

Finally,  $v(t)$  is assumed to follow a square-root process

$$dv(t) = \kappa(\theta - v(t))dt + \sigma_v \sqrt{v(t)} dW_3^Q(t). \quad (4)$$

$W_1^Q(t)$ ,  $W_2^Q(t)$  and  $W_3^Q(t)$  denote correlated Wiener processes under the risk-neutral measure, with  $\rho_{12}$ ,  $\rho_{13}$  and  $\rho_{23}$  denoting pairwise correlations. In particular, a negative correlation between shocks to the spot price and the cost of carry will have a mean-reverting effect on the spot price under the risk-neutral measure.

The forward cost of carry is given by the forward interest rate minus the forward convenience yield, and we could extend the model with separate processes for the forward interest rate and the forward convenience yield.<sup>17</sup> However, for pricing many commodity futures it

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<sup>16</sup>We have repeated all the analyses using correlation matrices rather than the covariance matrices. This yields virtually identical results, and the tables are therefore omitted.

<sup>17</sup>When we allow for stochastic interest rates, we should, strictly speaking, distinguish between forward and future convenience yields and forward and future cost of carry, see Miltersen and Schwartz (1998) for more on this issue.

appears sufficient to model the cost of carry as driven by one factor.<sup>18</sup> Furthermore, for pricing short-term or medium-term options on many commodity futures, ignoring stochastic interest rates results in negligible pricing errors, since the volatility of interest rates is typically orders of magnitudes smaller than the volatility of futures returns and the correlation between interest rates and futures returns tends to be very low.<sup>19</sup>

Let  $F(t, T)$  denote the time- $t$  price of a futures contract maturing at time  $T$ . By definition we have

$$F(t, T) \equiv S(t) \exp \left\{ \int_t^T y(t, u) du \right\}. \quad (5)$$

In the absence of arbitrage opportunities, the futures price process must be a martingale under the risk-neutral measure, see e.g. Duffie (2001). Applying Ito's Lemma to (5) and setting the drift to zero, it follows that the dynamics of  $F(t, T)$  is given by

$$\frac{dF(t, T)}{F(t, T)} = \sqrt{v(t)} \left( \sigma_S dW_1^Q(t) + \int_t^T \sigma_y(t, u) du dW_2^Q(t) \right). \quad (6)$$

Futures prices are driven by two risk factors,  $W_1^Q(t)$  and  $W_2^Q(t)$ . This should be adequate for pricing crude-oil futures, since Table 1 shows that two factors can capture virtually all variation in futures returns.

Volatility of futures prices depends on  $v(t)$ , which is driven by the risk factor  $W_3^Q(t)$ . Therefore, volatility risk and options on futures contracts cannot be completely hedged by trading in the futures contracts. Hence, the model features unspanned stochastic volatility consistent with the evidence presented in Section 3. To the extent that  $W_1^Q(t)$  and  $W_2^Q(t)$  are correlated with  $W_3^Q(t)$ , volatility risk is partly hedgeable. If  $\rho_{13}$  and  $\rho_{23}$  are both zero, volatility risk is completely unhedgeable. The model should perform well in terms of pricing options on crude-oil futures, since Table 4 shows that there is one dominant unspanned stochastic

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<sup>18</sup>See Schwartz (1997) for a comparison between the Gibson and Schwartz (1990) model (in which the spot cost of carry is driven by one factor) and a model that assumes stochastic spot interest rate and stochastic spot convenience yield (in which the spot cost of carry is driven by two factors). In terms of pricing crude-oil futures, the performance of the latter model is only marginally better than that of the former model.

<sup>19</sup>For instance, for the sample period January 2, 1990 to August 25, 2003, Casassus and Collin-Dufresne (2005) estimate the (instantaneous) crude-oil spot return volatility to 0.397 and the (instantaneous) spot interest rate volatility to 0.009. Furthermore, the (instantaneous) correlation between the spot return and the spot interest rate is estimated to 0.051 and is insignificant. Accounting explicitly for stochastic interest rates may become more important when pricing long-term options on futures contracts.

volatility factor. It is straightforward to extend the model to multiple unspanned stochastic volatility factors, if necessary.

From the requirement that the drift of the futures price process is zero, we obtain the following condition on the drift of the forward cost of carry process

**Proposition 1** *Absence of arbitrage implies that the drift term in (3) is given by*

$$\mu_y(t, T) = -v(t)\sigma_y(t, T) \left( \rho_{12}\sigma_S + \int_t^T \sigma_y(t, u)du \right). \quad (7)$$

**Proof:** See Appendix A. ■

This condition is analogous to the Heath, Jarrow, and Morton (1992) drift condition in forward rate term structure models.

## 4.2 An affine model for the dynamics of the futures curve

So far we have left  $\sigma_y(t, T)$  unspecified. We obtain a highly tractable model with the following time-homogeneous specification<sup>20,21</sup>

$$\sigma_y(t, T) = \alpha e^{-\gamma(T-t)}. \quad (8)$$

In the following proposition we show that  $y(t, T)$  is now an affine function of two state variables,  $x(t)$  and  $\phi(t)$ , where  $x(t)$  is stochastic while  $\phi(t)$  is an “auxiliary” locally non-stochastic state variable. Furthermore,  $x(t)$ ,  $\phi(t)$  and  $v(t)$  jointly constitute a three-dimensional affine state vector.

**Proposition 2** *The time- $t$  instantaneous forward cost of carry at time  $T$ ,  $y(t, T)$ , is given by*

$$y(t, T) = y(0, T) + A_x(T-t)x(t) + A_\phi(T-t)\phi(t), \quad (9)$$

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<sup>20</sup>In a preliminary version of the paper, we used the more general specification  $\sigma_y(t, T) = (\alpha_0 + \alpha_1(T-t))e^{-\gamma(T-t)}$ . This specification leads to a somewhat more complex, yet still tractable, model (see Trolle and Schwartz (2006) for an HJM-type stochastic volatility term structure model using this specification for the shocks to the forward rate curve). However,  $\alpha_1$  was estimated close to zero, and the pricing performance of the two models was almost indistinguishable. Therefore, in the interest of parsimony we will work with the simpler specification (8).

<sup>21</sup>With this specification,  $\sigma_S$ ,  $\alpha$ ,  $\theta$  and  $\sigma_v$  are not simultaneously identified, see e.g. the discussion of invariant affine transformations in Dai and Singleton (2000). In our empirical analysis we normalize  $\theta$  to one to achieve identification.

where

$$A_x(\tau) = \alpha e^{-\gamma\tau} \quad (10)$$

$$A_\phi(\tau) = \alpha e^{-2\gamma\tau} \quad (11)$$

and  $x(t)$  and  $\phi(t)$  evolve according to

$$dx(t) = \left( -\gamma x(t) - \left( \frac{\alpha}{\gamma} + \rho_{12}\sigma_S \right) v(t) \right) dt + \sqrt{v(t)} dW_2^Q(t) \quad (12)$$

$$d\phi(t) = \left( \frac{\alpha}{\gamma} v(t) - 2\gamma\phi(t) \right) dt \quad (13)$$

subject to  $x(0) = \phi(0) = 0$ .

**Proof:** See Appendix A. ■

It follows that the instantaneous spot cost of carry is given by

$$\delta(t) = y(0, t) + \alpha x(t) + \alpha\phi(t). \quad (14)$$

From (5) we have that  $F(t, T)$  is given by

$$F(t, T) = S(t) \frac{F(0, T)}{F(0, t)} \exp \{ B_x(T-t)x(t) + B_\phi(T-t)\phi(t) \} \quad (15)$$

where

$$B_x(\tau) = \frac{\alpha}{\gamma} (1 - e^{-\gamma\tau}) \quad (16)$$

$$B_\phi(\tau) = \frac{\alpha}{2\gamma} (1 - e^{-2\gamma\tau}), \quad (17)$$

while the dynamics of  $F(t, T)$  is given by

$$\frac{dF(t, T)}{F(t, T)} = \sqrt{v(t)} \left( \sigma_S dW_1^Q(t) + B_x(T-t) dW_2^Q(t) \right). \quad (18)$$

Futures prices now depend on three state variables ( $S(t)$ ,  $x(t)$  and  $\phi(t)$ ), but are still driven by two risk-factors, since  $\phi(t)$  is locally deterministic.<sup>22</sup>

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<sup>22</sup>To understand the model it may be helpful to note that the dynamics of the spot cost of carry is given by

$$d\delta(t) = \gamma(\theta(t) - \delta(t))dt + \alpha\sqrt{v(t)}dW_2^Q(t) \quad (19)$$

where

$$\theta(t) = \frac{1}{\gamma} \frac{\partial y(0, t)}{\partial t} + y(0, t) - \frac{1}{\gamma} \alpha \rho_{12} \sigma_S v(t) - \alpha\phi(t). \quad (20)$$

Therefore, the model can be seen as a stochastic volatility extension of the two factor Gibson and Schwartz (1990) model, where the first factor is the spot price and the second factor is a mean-reverting spot convenience yield or, equivalently, spot cost of carry (since they assume a constant spot interest rate). If we replace  $v(t)$  with a constant, we obtain the Gibson and Schwartz (1990) model fitted exactly to the initial futures curve.

It is convenient to use  $s(t) \equiv \log(S(t))$  instead of  $S(t)$  as a state vector. Using Ito's Lemma and substituting  $\delta(t)$  by (14), the dynamics of  $s(t)$  is given by

$$ds(t) = \left( y(0, t) + \alpha x(t) + \alpha \phi(t) - \frac{1}{2} \sigma_S^2 v(t) \right) dt + \sigma_S \sqrt{v(t)} dW_1^Q(t). \quad (21)$$

Now,  $s(t)$ ,  $x(t)$ ,  $\phi(t)$  and  $v(t)$  jointly constitute a four-dimensional affine state vector and the log of futures prices is an affine function of  $s(t)$ ,  $x(t)$  and  $\phi(t)$ ,

$$\log F(t, T) = \log F(0, T) - \log F(0, t) + s(t) + B_x(T - t)x(t) + B_\phi(T - t)\phi(t). \quad (22)$$

### 4.3 Pricing options on futures contracts

To price options on futures we follow Collin-Dufresne and Goldstein (2003), who extend the analysis in Duffie, Pan, and Singleton (2000) to HJM models, and introduce the transform

$$\psi(u, t, T_0, T_1) = E_t^Q \left[ e^{u \log(F(T_0, T_1))} \right]. \quad (23)$$

This transform has an exponentially affine solution as demonstrated in the following proposition

**Proposition 3** *The transform in (23) is given by*

$$\psi(u, t, T_0, T_1) = e^{M(T_0 - t) + N(T_0 - t)v(t) + u \log(F(t, T_1))}, \quad (24)$$

where  $M(\tau)$  and  $N(\tau)$  solve the following system of ODEs

$$\frac{dM(\tau)}{d\tau} = N(\tau)\kappa\theta \quad (25)$$

$$\begin{aligned} \frac{dN(\tau)}{d\tau} &= N(\tau) \left( -\kappa + u\sigma_v(\rho_{13}\sigma_S + \rho_{23}B_x(T_1 - T_0 + \tau)) \right) + \frac{1}{2}N(\tau)^2\sigma_v^2 \\ &\quad + \frac{1}{2}(u^2 - u)(\sigma_S^2 + B_x(T_1 - T_0 + \tau)^2 + 2\rho_{12}\sigma_S B_x(T_1 - T_0 + \tau)) \end{aligned} \quad (26)$$

subject to the boundary conditions  $M(0) = 0$  and  $N(0) = 0$ .

**Proof:** See Appendix A. ■

As in Duffie, Pan, and Singleton (2000) and Collin-Dufresne and Goldstein (2003) we can now price European options on a futures contract by applying the Fourier inversion theorem

**Proposition 4** *The time- $t$  price of a European put option expiring at time  $T_0$  with strike  $K$  on a futures contract expiring at time  $T_1$ ,  $\mathcal{P}(t, T_0, T_1, K)$ , is given by*

$$\begin{aligned}\mathcal{P}(t, T_0, T_1, K) &= E_t^Q \left[ e^{-\int_t^{T_0} r(s) ds} (K - F(T_0, T_1)) \mathbf{1}_{F(T_0, T_1) < K} \right] \\ &= P(t, T_0) (KG_{0,1}(\log(K)) - G_{1,1}(\log(K))),\end{aligned}\quad (27)$$

where  $P(t, T_0)$  denotes the time- $t$  price of a zero-coupon bond maturing at time  $T_0$ ,  $i = \sqrt{-1}$  and  $G_{a,b}(y)$  is defined as

$$G_{a,b}(y) = \frac{\psi(a, t, T_0, T_1)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im} [\psi(a + iub, t, T_0, T_1) e^{-iuy}]}{u} du. \quad (28)$$

**Proof:** See Appendix A. ■

This formula is exact when interest rates are uncorrelated with futures prices. To the extent that the correlation is low and the volatility of interest rates is significantly lower than the volatility of futures prices, the formula gives a very accurate approximation of the true price of short-term or medium-term options.<sup>23</sup>

#### 4.4 Market price of risk specification

For estimation we also need the dynamics of the state vector under the actual probability measure  $P$ , which are obtained by specifying the market prices of risk,  $\Lambda_i$ , that link the Wiener processes under  $Q$  and  $P$  through

$$dW_i^P(t) = dW_i^Q(t) - \Lambda_i(t) dt, \quad i = 1, 2, 3. \quad (29)$$

We apply the parsimonious “completely affine” specification, see e.g. Dai and Singleton (2000) which in our setting is given by

$$\Lambda_i(t) = \lambda_{iv} \sqrt{v(t)}, \quad i = 1, 2, 3. \quad (30)$$

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<sup>23</sup>To see this, note that in general we have

$$\begin{aligned}\mathcal{P}(t, T_0, T_1, K) &= P(t, T_0) E_t^Q [(K - F(T_0, T_1)) \mathbf{1}_{F(T_0, T_1) < K}] \\ &\quad + \text{Cov}_t^Q \left[ \exp \left\{ -\int_t^{T_0} r(s) ds \right\}, (K - F(T_0, T_1)) \mathbf{1}_{F(T_0, T_1) < K} \right].\end{aligned}$$

The covariance term is zero if  $r(t)$  and  $F(t, T_1)$  are uncorrelated, in which case the formula is exact. The covariance term is insignificant relative to the option price, provided that the correlation between  $r(t)$  and  $F(t, T_1)$  is low and the volatility of  $r(t)$  is low relative to the volatility of  $F(t, T_1)$ .



This specification preserves the affine structure of the state vector under the change of measure. The dynamics of  $s(t)$ ,  $x(t)$  and  $v(t)$  under  $P$  is now given by

$$ds(t) = \left( y(0, t) + \alpha x(t) + \alpha \phi(t) - \left( \frac{1}{2} \sigma_S^2 - \lambda_{1v} \sigma_S \right) v(t) \right) dt + \sigma_S \sqrt{v(t)} dW_1^Q(t) \quad (31)$$

$$dx(t) = \left( -\gamma x(t) - \left( \frac{\alpha}{\gamma} + \rho_{12} \sigma_S - \lambda_{2v} \right) v(t) \right) dt + \sqrt{v(t)} dW_2^Q(t) \quad (32)$$

$$dv(t) = (\kappa \theta - (\kappa - \lambda_{3v} \sigma_v) v(t)) dt + \sigma_v \sqrt{v(t)} dW_3^Q(t). \quad (33)$$

The dynamics of  $\phi(t)$  is given by (13) also under  $P$ , since it does not contain any stochastic terms.<sup>24</sup>

## 5 Model estimation

### 5.1 Data

We estimate the model on an extensive panel data set of futures and options on futures. We use weekly data to reduce the computational burden. In particular, we use data on Wednesdays.<sup>25</sup> We include the entire set of futures contracts M1–Y4 and options on the first eight futures contracts M1–Q2. We do not consider options on the remaining futures contracts Y1–Y4 for two reasons. Firstly, the quasi-analytical expression for option prices, that we develop in Section 4.3, does not take stochastic interest rates into account. While it appears that ignoring stochastic interest rates results in negligible pricing errors for short-term and medium-term options, it may result in non-negligible pricing errors for long-term options. Secondly, and more importantly, the main options in the data set are American, whereas our pricing formula

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<sup>24</sup>We have also estimated the model with the “extended affine” specification suggested by Cheredito, Filipovic, and Kimmel (2003) and Collin-Dufresne, Goldstein, and Jones (2003) and given by

$$\Lambda_i(t) = \frac{\lambda_{i0} + \lambda_{is}s(t) + \lambda_{ix}x(t) + \lambda_{iv}v(t)}{\sqrt{v(t)}}, \quad i = 1, 2 \quad (34)$$

$$\Lambda_3(t) = \frac{\lambda_{30} + \lambda_{3v}v(t)}{\sqrt{v(t)}}. \quad (35)$$

However, most of the additional parameters turn out to be very imprecisely estimated. Furthermore, the Feller restriction, which must be imposed to rule out arbitrage opportunities, is strongly binding under  $Q$ . This implies that although the “extended affine” specification gives the model more flexibility under the  $P$ -measure, it reduces the flexibility under the  $Q$ -measure which significantly affects the model’s pricing performance. For these reasons we only work with the “completely affine” specification.

<sup>25</sup>If a given Wednesday is not a business day, we choose the business day directly preceding the Wednesday.

is for European options.<sup>26</sup> For computational reasons, estimation is only feasible for European options, necessitating a conversion of American prices to European prices. This requires an approximation of the early exercise premium (described below), and since the size of the early exercise premium as a fraction of the total option price increases with option maturity, any errors in the early exercise approximation becomes more serious for longer term options. Considering options on the first eight generic futures contracts appears to strike a reasonable balance between including information from the maturity dimension of option prices, while limiting our exposure to the approximation errors associated with the American-to-European conversions and not explicitly modelling stochastic interest rates.

For each option maturity we consider eleven moneyness intervals 0.78–0.82, 0.82–0.86, 0.86–0.90, 0.90–0.94, 0.94–0.98, 0.98–1.02, 1.02–1.06, 1.06–1.10, 1.10–1.14, 1.14–1.18 and 1.18–1.22, where moneyness is defined as option strike divided by the price of the underlying futures contract. Among the options within a given moneyness interval we choose the one that is closest to mean of the interval.

The approach that we use for converting American prices to European prices is the following: we assume that the price of the underlying futures contract follows a geometric Brownian motion, in which case American options can be priced using the Barone-Adesi and Whaley (1987) formula. Inverting this formula for a given American option price yields an implied volatility, from which we can price the associated European option with the Black (1976) formula.<sup>27</sup> Naturally, this procedure is inherently inconsistent; the whole point of the paper is to investigate stochastic volatility, yet when approximating the early exercise premium we assume that the underlying future follows a geometric Brownian motion with constant volatility. Note, however, that the procedure implicitly takes variations in volatility into account, since the implied volatilities, although assumed constant, in reality vary over time and across moneyness. To minimize the effect of any errors in the early exercise approximation, we only consider options that are at- or out-of-the-money. More specifically, for moneyness less than 0.98 we only use put options, while for moneyness larger than 1.02 we only use call options.<sup>28,29</sup>

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<sup>26</sup>Recently, NYMEX has introduced European crude-oil options. However, the trading history is much shorter and the liquidity much lower than for the American options.

<sup>27</sup>We have also inverted the American prices using binomial or trinomial trees. This gives results that are very similar to using the Barone-Adesi and Whaley (1987) formula.

<sup>28</sup>Furthermore, we only consider options that have open interest in excess of 100 contracts and options with prices larger than 0.10 dollars. The reason for the latter is that prices are quoted with a precision of 0.01 dollars.

<sup>29</sup>Similar approaches for converting American options prices to European prices is used by e.g. Broadie,

The discount function  $P(t, T)$  in (27) is obtained by fitting a Nelson and Siegel (1987) curve each date to a LIBOR/swap curve consisting of the 1mth, 3mth, 6mth, 9mth and 12mth LIBOR rates and the 2yr swap rate.<sup>30</sup>

In total, the data set used for estimation consists of 9536 futures and 49001 options. There are 855 trading dates. On a given trading date, the number of futures range from 8 to 12 while the number of options range from 23 to 87.

## 5.2 Estimation approach

We estimate the model by quasi maximum-likelihood (QML) in conjunction with the extended Kalman filter.<sup>31</sup> To apply the Kalman filter we write the model in state space form, which consists of a measurement equation and a transition equation. The measurement equation describes the relationship between the prices of futures and options and the state variables, while the transition equation describes the discrete-time dynamics of the state variables.

Let  $z_t$  denote the data vector at time  $t$  and  $X_t$  the vector of state variables. The measurement equation is given by

$$z_t = h(X_t) + u_t, \quad u_t \sim \text{iid. } N(0, \Omega), \quad (37)$$

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Chernov, and Johannes (2004) for estimating equity derivatives models on S&P 500 futures options, Bikbov and Chernov (2004) for estimating term structure models on Eurodollar futures options and Richter and Sørensen (2002) for estimating a commodity derivatives model on soybean futures options.

<sup>30</sup>Let  $f(t, T)$  denote the time- $t$  instantaneous forward interest rate at time  $T$ . Nelson and Siegel (1987) parameterize the forward interest rate curve as  $f(t, T) = \beta_0 + \beta_1 e^{-\theta(T-t)} + \beta_2 \theta(T-t) e^{-\theta(T-t)}$  which yields the following expression for zero-coupon bond prices

$$P(t, T) = \exp \left\{ \beta_0(T-t) + (\beta_1 + \beta_2) \frac{1}{\theta} \left( 1 - e^{-\theta(T-t)} \right) + \beta_2 (T-t) e^{-\theta(T-t)} \right\} \quad (36)$$

from which we can price LIBOR and swap rates. The parameters  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$  and  $\theta$  are recalibrated on each observation date by minimizing the mean squared percentage differences between the observed LIBOR and swap rates on that date and those implied (36).

<sup>31</sup>Alternative estimation procedures include MCMC, Efficient Method of Moments (EMM) and Simulated Maximum likelihood (SML). Duffee and Stanton (2004) compare EMM, SML and QML/Kalman filter in the context of estimating affine term structure models. Their conclusion is that the latter procedure is preferable due to its better finite sample properties. Since the structure of our commodity model is similar to that of affine term structure models, we conjecture that a similar conclusion also holds in the present context. Computational considerations also speak in favor of the QML/Kalman filter approach, since the inclusion of options in the estimation makes even this otherwise simple procedure computationally intensive. Estimating the model with simulation based procedures such as MCMC, EMM or SML would be extremely time-consuming, if not impossible.

where  $h$  is the pricing function and  $u_t$  is a vector of iid. Gaussian measurement errors with covariance matrix  $\Omega$ .

Suppose at time  $t$  we observe  $m$  futures prices,  $F_{t,1}, \dots, F_{t,m}$ , and  $n$  option prices,  $\mathcal{P}_{t,1}, \dots, \mathcal{P}_{t,n}$ . We take the log of the futures prices and we divide the option prices by their Black (1976) “vegas”, the sensitivities of the option prices with respect to variations in log-normal volatilities. This is very similar to fitting the model to log-normal implied volatilities but is much faster, since computing implied volatilities requires a numerical inversion for each option, which would add an extra layer of complexity to the estimation procedure.<sup>32</sup> Hence,  $z_t$  is given by

$$z_t = (\log F_{t,1}, \dots, \log F_{t,m}, \mathcal{P}_{t,1}/\mathcal{V}_{t,1}, \dots, \mathcal{P}_{t,n}/\mathcal{V}_{t,n}), \quad (38)$$

where  $\mathcal{V}_{t,1}, \dots, \mathcal{V}_{t,n}$  denote the Black (1976) “vegas” for the  $n$  options, and  $X_t$  is given by

$$X_t = (s(t), x(t), \phi(t), v(t))'. \quad (39)$$

The dimension of the  $z_t$ -vector varies over time. This does not present a problem, however, since the Kalman filter easily handles missing observations.

The log of futures prices are affine functions of  $s(t)$ ,  $x(t)$  and  $\phi(t)$ , see (22). In the estimation we assume that the initial forward cost of carry curve is flat and equal to  $\varphi$ . In other words, we replace  $y(0, t)$  with  $\varphi$  in (31) and replace  $\log F(0, T) - \log F(0, t)$  with  $\varphi(T - t)$  in (22) and estimate  $\varphi$  as part of the estimation procedure. This reduces the model to a time-homogeneous model where  $\varphi$  is the long-run forward cost of carry.<sup>33</sup>

The option prices are non-linearly related to  $v(t)$  through (24) and (27). Since we price options based on the actual futures prices, option prices are independent of the  $s(t)$ ,  $x(t)$  and  $\phi(t)$  state variables. This has the advantage that an imperfect fit to the futures contracts does not get reflected in derivatives prices, which in turn should provide us with a cleaner estimate of the volatility process.

To reduce the number of parameters in  $\Omega$ , we make the conventional assumption that the measurement errors are cross-sectionally uncorrelated (that is,  $\Omega$  is diagonal). Furthermore,

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<sup>32</sup>To see this let  $\tilde{\mathcal{P}}_{t,i}$  and  $\hat{\mathcal{P}}_{t,i}$  denote the fitted and true price, respectively, for option  $i$  at time  $t$  and let  $\tilde{\sigma}_{t,i}$  and  $\hat{\sigma}_{t,i}$  denote the corresponding implied log-normal volatilities. Furthermore, let  $\hat{\mathcal{V}}_{t,i} \equiv \frac{\partial \mathcal{P}}{\partial \sigma} \Big|_{\sigma=\hat{\sigma}_{t,i}}$  denote the “vega” computed at the true option price. Then

$$\tilde{\mathcal{P}}_{t,i} \approx \hat{\mathcal{P}}_{t,i} + \hat{\mathcal{V}}_{t,i} (\tilde{\sigma}_{t,i} - \hat{\sigma}_{t,i}) \Leftrightarrow \tilde{\sigma}_{t,i} - \hat{\sigma}_{t,i} \approx \tilde{\mathcal{P}}_{t,i}/\hat{\mathcal{V}}_{t,i} - \hat{\mathcal{P}}_{t,i}/\hat{\mathcal{V}}_{t,i}.$$

<sup>33</sup>A similar approach is taken by de Jong and Santa-Clara (1999) and Trolle and Schwartz (2006) in their estimation of HJM-type term structure models.

we assume that one variance applies to all measurement errors for the log of futures prices, and that another variance applies to all measurement errors for scaled option prices.<sup>34</sup>

The transition equation describes the discrete-time dynamics of the state vector implied by the continuous-time processes (31), (32), (13) and (33)

$$X_{t+1} = \Phi(X_t) + w_{t+1}, \quad w_{t+1} \text{ iid.}, \quad E[w_{t+1}] = 0, \quad Cov[w_{t+1}] = Q(v_t). \quad (40)$$

Since  $X_t$  follows an affine diffusion, we have that  $\Phi(X_t) = \Phi_0 + \Phi_X X_t$  and  $Q(v_t) = Q_0 + Q_v v_t$ , where  $\Phi_0$ ,  $\Phi_X$ ,  $Q_0$  and  $Q_v$  are known in closed form and derived in Appendix B. The disturbance vector  $w_{t+1}$  is iid. but not Gaussian.

The Kalman filter is designed for linear Gaussian state space models. To apply the Kalman filter we therefore need to modify (37) and (40). We linearize the  $h$ -function in (37) and make the assumption that the disturbance term  $w_t$  in (40) is Gaussian. We can then apply the Kalman filter (now called the extended Kalman filter) to (37) and (40) and compute the likelihood function. The use of a Gaussian distribution to approximate the true distribution of  $w_{t+1}$  makes this a QML procedure.<sup>35,36</sup>

The log-likelihood function is maximized by initially using the Nelder-Mead algorithm and later switching to the gradient-based BFGS algorithm. The optimization is repeated with several different plausible initial parameter guesses to minimize the risk of not reaching the global optimum. The ODEs (25) and (26) are solved with a standard fourth-order Runge-Kutta algorithm, and the integral in (28) is evaluated with the Gauss-Legendre quadrature formula, using 20 integration points and truncating the integral at 400.

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<sup>34</sup>Note that the assumption of normally distributed additive measurement errors on the log of futures prices implies log-normally distributed multiplicative measurement errors on futures prices.

<sup>35</sup>For completeness the extended Kalman filter recursions are stated in Appendix B. Harvey (1989) and Hamilton (1994) are classic references.

<sup>36</sup>While QML estimation has been shown to be consistent in many settings, it is in fact not consistent in the present context due to the linearization of the  $h$ -function function, and the fact that the conditional covariance matrix  $Q$  in the recursions depends on the Kalman filter estimate  $\hat{v}_t$  rather than the true, but unobservable,  $v_t$ , see Duan and Simonato (1999) and Lund (1997). However, the Monte Carlo studies in these papers as well as Duffee and Stanton (2004) among others show the inconsistency problem to be of minor importance. This is particularly the case when the measurement equation is almost linear in the state variables, which is the case in our setting.

## 5.3 Results

### 5.3.1 Parameter estimates

Table 5 displays parameter estimates both for the entire sample 1990–2006 and for the two sub-samples 1990–1997 and 1998–2006.<sup>37</sup> The estimate of  $\sigma_S$  implies that the spot price volatility,  $\sigma_S\sqrt{v(t)}$ , equals 0.355 on average over the entire sample (0.306 and 0.414 over the two sub-samples), while the estimate of  $\alpha$  implies that the spot cost of carry volatility,  $\alpha\sqrt{v(t)}$ , equals 0.241 on average over the entire sample (0.302 and 0.312 over the two sub-samples). The correlation between innovations to the spot price and innovations to the cost of carry is estimated to be strongly negative -0.911 in the entire sample (-0.944 and -0.884 in the two sub-samples). This implies that the spot price will exhibit mean-reversion under the risk-neutral measure as found by Bessembinder et al. (1995). Our estimates are generally consistent with Schwartz (1997) and Casassus and Collin-Dufresne (2005), who also estimate commodity models on crude-oil futures data (but no options data). The estimates in Schwartz (1997) imply spot price volatility, spot cost of carry volatility and spot price–cost of carry correlation of 0.344, 0.372 and -0.915, respectively, while the corresponding numbers in Casassus and Collin-Dufresne (2005) are 0.397, 0.384 and -0.793.<sup>38</sup>

For the entire sample the volatility process is estimated to be moderately persistent under the risk-neutral measure with a half-life of 0.80 years. The persistence is estimated to be significantly lower in the second sub-sample with a half-life of only 0.37 years. This suggests that the 1990-1991 Gulf War I, which is by far the largest shock to the crude-oil derivatives market in our data set, has a significant impact on this estimate.

The volatility of volatility parameter  $\sigma_v$  is fairly constant across the three samples. It is often the case that stochastic volatility models fitted to equity option data lead to estimates of  $\sigma_v$  that are wildly inconsistent with the actual volatility of  $v(t)$ , see e.g. Bates (1996, 2000),

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<sup>37</sup>The asymptotic covariance matrix of the estimated parameters is computed from the outer-product of the first derivatives of the likelihood function. Theoretically, it would be more appropriate to compute the asymptotic covariance matrix from both the first and second derivatives of the likelihood function. In reality, however, the second derivatives of the likelihood function are somewhat numerically unstable.

<sup>38</sup>Schwartz (1997) and Casassus and Collin-Dufresne (2005) estimate Gaussian models that include a stochastic process for both the spot convenience yield and the spot interest rate. It is straightforward to compute the parameters of the implied spot cost of carry process. In the case of Schwartz (1997) we compare our results with those in his Table IX, column two. The sample period in Schwartz (1997) is January 2, 1990 to February 17, 1995 while the sample period in Casassus and Collin-Dufresne (2005) is January 2, 1990 to August 25, 2003.

Bakshi, Cao, and Chen (1997) and Broadie, Chernov, and Johannes (2004). When stochastic volatility models are estimated from panel data,  $\sigma_v$  is in principle identified from both the time-series dynamics of  $v(t)$  and the cross-section of option prices, where  $\sigma_v$  determines the curvature of the implied volatility “smile” by controlling the kurtosis of the futures return distribution. In reality, however,  $\sigma_v$  is primarily identified from the option prices, and there is no guarantee that the  $\sigma_v$ -estimate is consistent with the actual volatility of  $v(t)$ . A rough estimate of  $\sigma_v$  from the time-series dynamics of  $v(t)$  over the entire sample is given by  $\text{std} \left( \frac{v(t+\Delta)-v(t)}{\sqrt{v(t)\Delta}} \right) = 1.703$ ,<sup>39</sup> which is certainly of the same magnitude as the actual  $\sigma_v$ -estimate of 2.502.

The correlations between innovations to volatility and innovations to the spot price and the cost of carry are low with  $\rho_{13} = -0.105$  (-0.007 and -0.089 in the two sub-samples) and  $\rho_{23} = -0.001$  (-0.042 and -0.206 in the two sub-samples). Hence, the model implies that most of the shocks to volatility cannot be hedged by trading in futures contracts, consistent with the findings in Section 3. Similar to  $\sigma_v$ ,  $\rho_{31}$  and  $\rho_{32}$  are in principle identified from both the time-series dynamics of the state variables and the cross-section of option prices, where  $\rho_{31}$  and  $\rho_{32}$  determine the skewness of the implied volatility “smile” by controlling the skewness of the futures return distributions. However,  $\rho_{31}$  and  $\rho_{32}$  are also effectively identified from the option prices. Rough estimates of  $\rho_{31}$  and  $\rho_{32}$  from the time-series dynamics of the state variables over the entire sample are given by  $\text{corr} \left( \frac{s(t+\Delta)-s(t)}{\sqrt{v(t)\Delta}}, \frac{v(t+\Delta)-v(t)}{\sqrt{v(t)\Delta}} \right) = 0.013$  and  $\text{corr} \left( \frac{x(t+\Delta)-x(t)}{\sqrt{v(t)\Delta}}, \frac{v(t+\Delta)-v(t)}{\sqrt{v(t)\Delta}} \right) = -0.057$ , which are close to the actual estimates.

### 5.3.2 Model fit

Figure 3 displays the time-series of the root mean squared errors (RMSEs) and mean errors (MEs) of the futures contracts (where the errors are the percentage differences between actual and fitted prices) and option contracts (where the errors are the differences between actual and fitted implied log-normal volatilities) when the model is estimated on the entire data set. We have also highlighted the dates of four major shocks to the crude oil market. The Iraqi invasion of Kuwait on August 2, 1990, the beginning of the US-led liberation of Kuwait (“Operation Desert Storm”) on January 17, 1991, the September 11, 2001 terrorist attacks and the US-led invasion of Iraq on March 20, 2003. For both futures and options, the MEs are negligible which shows that the model tracks variations in the overall level of futures prices and implied volatilities almost perfectly. The RMSE of futures prices mostly fluctuates between 0.5 and

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<sup>39</sup>Since we use weekly data  $\Delta = \frac{1}{52}$ .

2.5 percent with an average of 1.23 percent. However, it reaches almost five percent at the beginning of “Operation Desert Storm”. The RMSE of the option implied volatilities mostly fluctuate between one to three percent with an average of 2.12 percent. It spikes around all four events that we have highlighted. The largest spike is again the beginning of “Operation Desert Storm”, where it reaches more than 20 percent.

The RMSE measure takes both variations and biases in the pricing errors into account. To see if the pricing errors of the individual futures and options deviate systematically from zero, Table 6 and Table 7 report the mean pricing errors and the associated  $t$ -statistics for all futures and options contracts. The model performs extremely well in terms of pricing the futures curve on average. It also generally performs well on average in terms of pricing options across the moneyness and the maturity dimensions. The main shortcoming of the model is in the pricing of deep in-the-money and out-of-the-money options on short term futures contracts, where the pricing errors are strongly significant. This is also evident from Figure 4, which displays the means of the actual and fitted implied volatility surfaces and implied “smile” surfaces. Comparing Panel A with Panel B and particularly Panel C with Panel D shows that while the model has a good fit to the average implied volatility “smile” for options on medium term and longer term futures contracts, it is not able to match the average implied volatility “smile” for options on short term futures contracts.<sup>40</sup> This is a well known deficiency of stochastic volatility models; as the option maturity goes to zero, the implied volatility “smile” flattens, whereas it tends to become more pronounced in the data. In the equity derivatives literature the solution has been to augment stochastic volatility models with jumps in the spot price process and possibly the volatility process, see Bates (1996, 2000), Bakshi, Cao, and Chen (1997) and Duffie, Pan, and Singleton (2000) among others. Our model can be extended along similar lines.<sup>41</sup>

Figure 5 shows the Kalman filtered state variables. Volatility (along with the spot price) increased dramatically in response to the Iraqi invasion of Kuwait and stayed elevated until the

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<sup>40</sup>Note that in Panel A and B we simply compute the average implied volatility in each moneyness–maturity category. Since the number of observations varies with moneyness and maturity care should be taken in comparing average implied volatilities across moneyness and maturity. In particular, Panel A and B appears to exaggerate the implied volatility “smile” compared to Panel C and D.

<sup>41</sup>Hilliard and Reis (1998) develop a model for pricing commodity derivatives where the spot price follows a jump-diffusion process. However, their model does not account for stochastic volatility, and they make no attempt to fit their model to actual option data.



beginning of “Operation Desert Storm”. Volatility also increased sharply on September 11, 2001 (contrary to the spot price which fell dramatically as the event was perceived as negative shock to the global economy and hence crude-oil demand). Finally, volatility increased in the run-up to Gulf War II but decreased sharply once the invasion of Iraq commenced.

## 6 The relative valuation of plain-vanilla and calendar spread options

A popular crude-oil derivative trading on the NYMEX exchange is the calendar spread option. Calendar spread options allow market participants to speculate on or hedge exposure to the slope of the futures curve. In this section we price calendar spread options out-of-sample to see if their prices are consistent with the plain-vanilla option market.

### 6.1 Pricing calendar spread options

The calendar spread option contract is a European style option on the price differential between two futures contracts with different maturities (the maturities can be spaced one, two, three, six and twelve months apart with the one-month and twelve month spreads being the most liquid).<sup>42</sup> At expiration the buyer of a put option contract, if choosing to exercise, will receive a short position in the shorter-dated futures contract and a long position in the longer-dated futures contract.<sup>43</sup> Assuming interest rates are uncorrelated with futures prices, the time- $t$  price of a European put option expiring at time  $T_0$  with strike  $K$  on the price differential between the futures contracts expiring at time  $T_1$  and  $T_2$  is

$$\mathcal{P}(t, T_0, T_1, T_2, K) = P(t, T_0) E_t^Q \left[ (K - (F(T_0, T_1) - F(T_0, T_2))) \mathbf{1}_{F(T_0, T_1) - F(T_0, T_2) < K} \right], \quad (41)$$

where  $P(t, T_0)$  denotes the time- $t$  price of a zero-coupon bond maturing at time  $T_0$ .<sup>44</sup>

In the particular case of  $K = 0$ , the calendar spread option reduces to an exchange option which can be priced in closed form using transform techniques along the lines of Section 4.3. This pricing formula is given in Appendix C. In the general case of  $K \neq 0$ , the calendar spread

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<sup>42</sup>NYMEX also lists options on “crack” spreads – the price differentials between crude-oil futures and either heating-oil futures or gasoline futures.

<sup>43</sup>Conversely, at expiration the buyer of a call option contract, if choosing to exercise, will receive a long position in the shorter-dated futures contract and a short position in the longer-dated futures contract.

<sup>44</sup>The option expires one business day before expiration of the shorter-dated futures contract.

option must be priced by Monte Carlo simulations. However, when  $K$  is close to zero, the efficiency of the Monte Carlo simulations can be greatly enhanced by using the corresponding calendar spread option with  $K = 0$  as a control variate.

## 6.2 Data

The data on calendar spread options covers the period June 10, 2002 to May 18, 2006. We consider the same weekly trading dates as in the Section 5. Furthermore, we restrict our attention to monthly calendar spreads and only consider the following five spreads: M1–M2, M2–M3, M3–M4, M4–M5, M5–M6, since the futures contracts that constitute these spreads are also part of the data set used in Section 5. For each spread we select the put and the call option with strike closest to the ATM strike.<sup>45</sup> This leaves us with 744 put options and 746 call options.

We price the options off the actual futures curves as in Section 5. This requires an estimate of the spot price and the forward cost of carry curve at each trading date. We use a parametrization of the forward cost of carry curve similar to the one proposed by Nelson and Siegel (1987) for forward interest rate curves.<sup>46</sup> This specification has the ability to fit a wide variety of futures curve shapes while ensuring a smooth forward cost of carry curve.

## 6.3 Results

To address the relative valuation of plain-vanilla and calendar spread options, we first estimate the model as in Section 5 over the period where calendar spread option data is available. We then price the calendar spread options out-of-sample by simulations.<sup>47</sup> Table 8 shows the mean pricing errors for the puts and the calls. On average, the model-implied prices of ATM short-

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<sup>45</sup>We require that the option strike deviates no more than 0.25 dollars from the ATM strike. Furthermore, we only consider options that have prices larger than 0.10 dollars, since prices are quoted with a precision of 0.01 dollars.

<sup>46</sup>Specifically, we parameterize the forward cost of carry curve as  $y(t, T) = \beta_0 + \beta_1 e^{-\theta(T-t)} + \beta_2 \theta (T-t) e^{-\theta(T-t)}$  which yields the following expression for futures prices

$$F(t, T) = S(t) \exp \left\{ \beta_0 (T-t) + (\beta_1 + \beta_2) \frac{1}{\theta} \left( 1 - e^{-\theta(T-t)} \right) + \beta_2 (T-t) e^{-\theta(T-t)} \right\}. \quad (42)$$

The spot price  $S(t)$  and the parameters  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$  and  $\theta$  are recalibrated on each observation date by minimizing the mean squared percentage differences between the actual futures prices and those those given by (42).

<sup>47</sup>Each calendar spread option is priced by simulations using 50,000 paths, antithetic variates and the corresponding zero-strike calendar spread option as control variate.

term calendar spread options are lower than the actual prices. For five of the ten options that we consider, the model-implied prices are more than ten percent lower than the actual prices, on average, and for the put option on the M1–M2 spread the model-implied price is more than 25 percent lower than the actual price, on average.<sup>48</sup> At the same time, on average, the model prices plain-vanilla ATM options almost without error.<sup>49</sup> This implies that, on average, ATM short-term calendar spread options have been overpriced relative to plain-vanilla ATM options. However, the pricing errors are also very volatile which renders the apparent mispricing insignificant except for put options on the M1–M2 and M2–M3 spreads. Figure 6 shows the time series of the pricing errors. The high volatility of the pricing errors is apparent.

One interpretation of the results is that there are temporary mis-pricing between plain-vanilla and calendar spread options that a dynamic trading strategy might be able to exploit. Note, however, that the liquidity of the calendar spread option market is significantly lower than the liquidity in the plain-vanilla option market. Another possibility is that an extension of the model with a more flexible correlation structure might be consistent with the observed variation in the relative valuations of plain-vanilla and calendar spread options.<sup>50</sup>

## 7 Conclusion

We have analyzed unspanned stochastic volatility in commodity markets in general and the crude-oil market in particular. We first present model-free regression-based results which strongly indicate that in the crude-oil market it is very difficult to hedge volatility risk with crude-oil futures contracts. That is, volatility appears mostly unspanned by the futures contracts. These results are model-free in the sense that they are not conditional on a particular pricing model.

We then develop a parsimonious model for pricing commodity derivatives in the presence of unspanned stochastic volatility. The model takes the initial futures curve as given. Futures prices depend on two factors. The volatility of futures prices depend on a third factor implying

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<sup>48</sup>The median pricing errors (not reported) are generally slightly lower than the mean pricing errors so the results are not driven by a few observations with extreme mis-valuation.

<sup>49</sup>The mean pricing errors for plain-vanilla ATM options (ATM here refers to those options with moneyness in the interval 0.98–1.02) range from -0.14 percent to 2.14 percent.

<sup>50</sup>Calendar spread options are sensitive to variations in correlations between the futures contracts. In our model the correlations are constant due to the assumption of a single volatility factor. An extension of the model with multiple volatility factors would feature a time-varying correlation structure.

that options on futures contracts depend on three factors. The model features correlations between innovations to futures prices and volatility; therefore, volatility may be partially spanned by the futures contracts. The model is highly tractable as it has quasi-analytical prices of European options on futures. Furthermore, the dynamics of the futures curve can be described in terms of a four-dimensional affine state vector, which makes the model suitable for estimation and for pricing derivatives by simulations. Estimating the model on an extensive panel data-set of crude-oil futures and options, we show that it has an excellent fit to futures across different maturities and generally a good fit to options across both different moneyness categories and maturities.

As an application of the model, we price ATM calendar spread options out-of-sample from the model estimated on the main data set. Conditional on the model being true, ATM calendar spread options have been overvalued relative to plain-vanilla options on average, although the mis-pricing is insignificant for all but a few options.

The model can be extended along several dimensions. For instance, we might include separate processes for the forward interest rate and the forward convenience yield as well as multiple (partially) unspanned stochastic volatility factors. Perhaps more importantly, the model can be extended with jumps in the spot price and possibly volatility. This would increase the ability of the model to match the implied volatility “smile” for very short-term options.

In the empirical parts of the paper we have used data from the crude-oil derivatives market since this is the most important and most liquid commodity derivatives market in the world. It is likely that markets related to crude-oil such as gasoline and heating-oil as well natural gas also exhibit unspanned stochastic volatility. An interesting question is whether unspanned stochastic volatility is an important feature of commodity derivatives markets less related to crude-oil, such as metals and agricultural products. We leave these issues for future research.

## Appendix A: Proofs

### Proof of Proposition 1

We introduce the process

$$Y(t) = \int_t^T y(t, u) du, \quad (43)$$

the dynamics of which is given by

$$dY(t) = \left( -\delta(t) + \int_t^T \mu_y(t, u) du \right) dt + \sqrt{v(t)} \int_t^T \sigma_y(t, u) du dW_2(t). \quad (44)$$

$F(t, T)$  is given by

$$F(t, T) = S(t)e^{Y(t)} \quad (45)$$

which follows

$$\begin{aligned} \frac{dF(t, T)}{F(t, T)} &= \frac{dS(t)}{S(t)} + dY(t) + \frac{1}{2}(dY(t))^2 + \frac{dS(t)}{S(t)}dY(t) \\ &= \left( \int_t^T \mu_y(t, u) du + \left( \frac{1}{2} \left( \int_t^T \sigma_y(t, u) du \right)^2 + \rho_{12}\sigma_S \int_t^T \sigma_y(t, u) du \right) v(t) \right) dt \\ &\quad + \sqrt{v(t)} \left( \sigma_S dW_1(t) + \int_t^T \sigma_y(t, u) du dW_2(t) \right). \end{aligned} \quad (46)$$

In the absence of arbitrage, the drift must equal zero. Imposing this condition and differentiating w.r.t.  $T$  yields (7).

### Proof of Proposition 2

With  $\sigma_y(t, T)$  given as (8),  $\mu_y(t, T)$  is given by (7) as

$$\mu_y(t, T) = v(t) \left( \frac{\alpha^2}{\gamma} e^{-2\gamma(T-t)} - \left( \frac{\alpha^2}{\gamma} + \rho_{12}\sigma_S \right) e^{-\gamma(T-t)} \right). \quad (47)$$

Integrating (3) and using that  $e^{-\gamma(T-u)} = e^{-\gamma(T-t)}e^{-\gamma(t-u)}$ , we obtain

$$\begin{aligned} y(t, T) &= y(0, T) + \int_0^t \mu_y(u, T) du + \int_0^t \sigma_y(u, T) \sqrt{v(u)} dW(u) \\ &= y(0, T) + \alpha e^{-\gamma(T-t)} x(t) + \alpha e^{-2\gamma(T-t)} \phi(t), \end{aligned} \quad (48)$$

where

$$x(t) = - \int_0^t v(u) \left( \frac{\alpha}{\gamma} + \rho_{12}\sigma_S \right) e^{-\gamma(t-u)} du + \int_0^t e^{-\gamma(t-u)} \sqrt{v(u)} dW(u) \quad (49)$$

$$\phi(t) = \int_0^t v(u) \frac{\alpha}{\gamma} e^{-2\gamma(t-u)} du. \quad (50)$$

Applying Ito's Lemma to these expressions gives the dynamics of  $x(t)$  and  $\phi(t)$  stated in (12) and (13).

### Proof of Proposition 3

The proof is similar to those in Duffie, Pan, and Singleton (2000) and Collin-Dufresne and Goldstein (2003). We can rewrite (23) as

$$\begin{aligned}\psi(u, t, T_0, T_1) &= E_t^Q \left[ E_{T_0}^Q \left[ e^{u \log(F(T_0, T_1))} \right] \right] \\ &= E_t^Q [\psi(u, T_0, T_0, T_1)].\end{aligned}\quad (51)$$

Therefore, the proof consists of showing that the process  $\eta(t) \equiv \psi(u, t, T_0, T_1)$  is a martingale under  $Q$ . To this end we conjecture that  $\psi(u, t, T_0, T_1)$  is of the form (24). Applying Ito's Lemma to  $\eta(t)$  and setting  $\tau = T_0 - t$ , we obtain

$$\begin{aligned}\frac{d\eta(t)}{\eta(t)} &= \left( -\frac{dM(\tau)}{d\tau} - \frac{dN(\tau)}{d\tau}v(t) \right) dt + N(\tau)dv(t) + u\frac{dF(t, T_1)}{F(t, T_1)} \\ &\quad + \frac{1}{2}N(\tau)^2(dv(t))^2 + \frac{1}{2}(u^2 - u) \left( \frac{dF(t, T_1)}{F(t, T_1)} \right)^2 + N(\tau)udv(t)\frac{dF(t, T_1)}{F(t, T_1)}.\end{aligned}\quad (52)$$

For  $\eta(t)$  to be a martingale, it must hold that

$$\begin{aligned}0 &= \frac{1}{dt}E_t^Q \left[ \frac{d\eta(t)}{\eta(t)} \right] \\ &= -\frac{dM(\tau)}{d\tau} - \frac{dN(\tau)}{d\tau}v(t) + N(\tau)\kappa(\theta - v(t)) + \frac{1}{2}N(\tau)^2\sigma_v^2v(t) \\ &\quad + \frac{1}{2}(u^2 - u)(\sigma_S^2 + B_x(T_1 - t)^2 + 2\rho_{12}\sigma_S B_x(T_1 - t))v(t) \\ &\quad + N(\tau)u\sigma_v(\rho_{13}\sigma_S + \rho_{23}B_x(T_1 - t))v(t) \\ &= -\frac{dM(\tau)}{d\tau} + N(\tau)\kappa\theta \\ &\quad \left( -\frac{dN(\tau)}{d\tau} + N(\tau)(-\kappa + u\sigma_v(\rho_{13}\sigma_S + \rho_{23}B_x(T_1 - t))) + \frac{1}{2}N(\tau)^2\sigma_v^2 \right. \\ &\quad \left. + \frac{1}{2}(u^2 - u)(\sigma_S^2 + B_x(T_1 - t)^2 + 2\rho_{12}\sigma_S B_x(T_1 - t)) \right)v(t).\end{aligned}\quad (53)$$

Hence,  $\eta(t)$  is a martingale, provided that  $M(\tau)$  and  $N(\tau)$  satisfy (25) and (26). Furthermore, we have that

$$\psi(u, T_0, T_0, T_1) = e^{u \log(F(T_0, T_1))}, \quad (54)$$

which is true, provided that  $M(0) = 0$  and  $N(0) = 0$ .

### Proof of Proposition 4

Again, we follow Duffie, Pan, and Singleton (2000) and Collin-Dufresne and Goldstein (2003). Assuming  $r(t)$  and  $F(t, T_1)$  are uncorrelated, the time- $t$  price a European put option expiring at time  $T_0$  with strike  $K$  on a futures contract expiring at time  $T_1$ ,  $\mathcal{P}(t, T_0, T_1, K)$ , is given by

$$\begin{aligned} \mathcal{P}(t, T_0, T_1, K) &= E_t^Q \left[ e^{-\int_t^{T_0} r(s) ds} (K - F(T_0, T_1)) \mathbf{1}_{F(T_0, T_1) < K} \right] \\ &= P(t, T_0) \left( K E_t^Q \left[ \mathbf{1}_{\log(F(T_0, T_1)) < \log(K)} \right] - E_t^Q \left[ e^{\log(F(T_0, T_1))} \mathbf{1}_{\log(F(T_0, T_1)) < \log(K)} \right] \right) \\ &= P(t, T_0) (K G_{0,1}(\log(K)) - G_{1,1}(\log(K))), \end{aligned} \quad (55)$$

where

$$G_{a,b}(y) = E_t^Q \left[ e^{a \log(F(T_0, T_1))} \mathbf{1}_{b \log(F(T_0, T_1)) < y} \right]. \quad (56)$$

To evaluate  $G_{a,b}(y)$ , note that its Fourier transform is given by

$$\begin{aligned} \mathcal{G}_{a,b}(y) &= \int_{\mathbb{R}} e^{iuy} dG_{a,b}(y) \\ &= E_t^Q \left[ e^{(a+iub)\log(F(T_0, T_1))} \right] \\ &= \psi(a + iub, t, T_0, T_1), \end{aligned} \quad (57)$$

where  $i = \sqrt{-1}$ . Applying the Fourier inversion theorem we have

$$G_{a,b}(y) = \frac{\psi(a, t, T_0, T_1)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im} [\psi(a + iub, t, T_0, T_1) e^{-iuy}]}{u} du. \quad (58)$$

## Appendix B: Estimation by the extended Kalman filter

### The extended Kalman filter

Let  $\hat{X}_t = E_t[X_t]$  and  $\hat{X}_{t|t-1} = E_{t-1}[X_t]$  denote expectations of  $X_t$  (respectively including and excluding  $z_t$ ), and let  $P_t$  and  $P_{t|t-1}$  denote the corresponding estimation error covariance matrices. Linearizing the  $h$ -function in (37) around  $\hat{X}_{t|t-1}$  we obtain

$$z_t = (h(\hat{X}_{t|t-1}) - H_t' \hat{X}_{t|t-1}) + H_t' X_t + u_t, \quad u_t \sim \text{iid. } N(0, S\Omega), \quad (59)$$

where

$$H_t' = \left. \frac{\delta h(X_t)}{\delta X_t'} \right|_{X_t = \hat{X}_{t|t-1}}. \quad (60)$$

Assuming  $w_t$  in (40) is Gaussian, we obtain

$$X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \quad w_{t+1} \sim \text{iid. } N(0, Q_t). \quad (61)$$

The Kalman filter applied to (59) and (61) yields

$$\hat{X}_{t+1|t} = \Phi_0 + \Phi_X \hat{X}_t \quad (62)$$

$$P_{t+1|t} = \Phi_X P_t \Phi_X' + Q_t \quad (63)$$

and

$$\hat{X}_{t+1} = \hat{X}_{t+1|t} + P_{t+1|t} H_t' F_t^{-1} \epsilon_t \quad (64)$$

$$P_{t+1} = P_{t+1|t} - P_{t+1|t} H_t' F_t^{-1} H_t P_{t+1|t}, \quad (65)$$

where

$$\epsilon_t = z_{t+1} - h(\hat{X}_{t+1|t}) \quad (66)$$

$$F_t = H_t P_{t+1|t} H_t' + S \Omega. \quad (67)$$

The log-likelihood function is constructed from (66) and (67):

$$\log L = -\frac{1}{2} \log 2\pi \sum_{i=1}^T N_t - \frac{1}{2} \sum_{i=1}^T \log |F_t| - \frac{1}{2} \sum_{i=1}^T \epsilon_t' F_t^{-1} \epsilon_t, \quad (68)$$

where  $T$  is the number of observation dates and  $N_t$  is the dimension of  $z_t$ . We follow standard practice in the literature and initialize the Kalman filter at the unconditional values of  $\hat{X}_{t|t-1}$  and  $P_{t|t-1}$ .

### The transition density

The dynamics of the state vector under the actual measure can be written as

$$dX(t) = (\Psi - \mathcal{K}X(t))dt + \sqrt{v(t)}\Sigma dW^P(t), \quad (69)$$

where  $X_t = (s(t), x(t), \phi(t), v(t))'$ ,  $W^P(t) = (W_1^P(t), W_2^P(t), W_3^P(t))'$  and

$$\Psi = \begin{pmatrix} \varphi \\ 0 \\ 0 \\ \kappa\theta \end{pmatrix}, \mathcal{K} = \begin{pmatrix} -1 & -\alpha & -\alpha & \frac{1}{2}\sigma_S^2 - \lambda_{1v}\sigma_S \\ 0 & -\gamma & 0 & \frac{\alpha}{\gamma} + \rho_{12}\sigma_S - \lambda_{2v} \\ 0 & 0 & 2\gamma & -\frac{\alpha}{\gamma} \\ 0 & 0 & 0 & \kappa - \lambda_{3v}\sigma_v \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_v \end{pmatrix}. \quad (70)$$



Applying Ito's Lemma to  $e^{\mathcal{K}t}X(t)$ , we obtain

$$\begin{aligned} d(e^{\mathcal{K}t}X(t)) &= e^{\mathcal{K}t}\mathcal{K}X(t)dt + e^{\mathcal{K}t}dX(t) \\ &= e^{\mathcal{K}t}\Psi dt + e^{\mathcal{K}t}\sqrt{v(t)}\Sigma dW^P(t). \end{aligned} \quad (71)$$

It follows that  $X(s)$ ,  $s > t$  is given as

$$X(s) = e^{-\mathcal{K}(s-t)}X(t) + \int_t^s e^{-\mathcal{K}(s-u)}\Psi du + \int_t^s e^{-\mathcal{K}(s-u)}\sqrt{v(u)}\Sigma dW^P(u). \quad (72)$$

The conditional mean of  $X(s)$ , given time- $t$  information, is given by

$$E_t[X(s)] = \int_t^s e^{-\mathcal{K}(s-u)}\Psi du + e^{-\mathcal{K}(s-t)}X(t), \quad (73)$$

and the conditional covariance matrix of  $X(s)$ , given time- $t$  information, is given by

$$\begin{aligned} Cov_t[X(s)] &= E_t \left[ \left( \int_t^s e^{-\mathcal{K}(s-u)}\sqrt{v(u)}\Sigma dW^P(u) \right) \left( \int_t^s e^{-\mathcal{K}(s-u)}\sqrt{v(u)}\Sigma dW^P(u) \right)' \right] \\ &= \int_t^s E_t[v(u)] e^{-\mathcal{K}(s-u)}\Sigma\rho\Sigma' e^{-\mathcal{K}'(s-u)} du \\ &= \int_t^s \left( 1 - e^{-\kappa_v^P(u-t)} \right) \theta_v^P e^{-\mathcal{K}(s-u)}\Sigma\rho\Sigma' e^{-\mathcal{K}'(s-u)} du \\ &\quad + \int_t^s e^{-\kappa_v^P(u-t)} e^{-\mathcal{K}(s-u)}\Sigma\rho\Sigma' e^{-\mathcal{K}'(s-u)} du v(t), \end{aligned} \quad (74)$$

where  $\rho$  denotes the correlation matrix for the Wiener processes, i.e.  $dW^P(t)dW^P(t)' = \rho dt$ .  $\Phi_0$ ,  $\Phi_X$ ,  $Q_0$  and  $Q_v$  can be inferred from (73) and (74).

## Appendix C: Pricing zero-strike calendar spread options

To price a zero-strike European calendar spread option, we introduce the transform

$$\psi(u_1, u_2, t, T_0, T_1, T_2) = E_t^Q \left[ e^{u_1 \log(F(T_0, T_1)) + u_2 \log(F(T_0, T_2))} \right], \quad (75)$$

which has an exponentially affine solution given in the following proposition

**Proposition 5** *The transform in (75) is given by*

$$\psi(u_1, u_2, t, T_0, T_1, T_2) = e^{M(T_0-t) + N(T_0-t)v(t) + u_1 \log(F(t, T_1)) + u_2 \log(F(t, T_2))}, \quad (76)$$

where  $M(\tau)$  and  $N(\tau)$  solve the following system of ODEs

$$\frac{dM(\tau)}{d\tau} = N(\tau)\kappa\theta \quad (77)$$

$$\begin{aligned} \frac{dN(\tau)}{d\tau} = & N(\tau) (-\kappa + u_1\sigma_v(\rho_{13}\sigma_S + \rho_{23}B_x(T_1 - T_0 + \tau)) + u_2\sigma_v(\rho_{13}\sigma_S + \rho_{23}B_x(T_2 - T_0 + \tau))) \\ & + \frac{1}{2}N(\tau)^2\sigma_v^2 + \frac{1}{2}(u_1^2 - u_1)(\sigma_S^2 + B_x(T_1 - T_0 + \tau)^2 + 2\rho_{12}\sigma_S B_x(T_1 - T_0 + \tau)) \\ & + \frac{1}{2}(u_2^2 - u_2)(\sigma_S^2 + B_x(T_2 - T_0 + \tau)^2 + 2\rho_{12}\sigma_S B_x(T_2 - T_0 + \tau)) \\ & + u_1u_2(\sigma_S^2 + B_x(T_1 - T_0 + \tau)B_x(T_2 - T_0 + \tau)) \\ & + \rho_{12}\sigma_S(B_x(T_1 - T_0 + \tau) + B_x(T_2 - T_0 + \tau)), \end{aligned} \quad (78)$$

subject to the boundary conditions  $M(0) = 0$  and  $N(0) = 0$ .

**Proof:** The proof is similar to that of Proposition 3 and therefore omitted.<sup>51</sup> ■

By the Fourier inversion theorem we can now price a zero-strike European calendar spread option.

**Proposition 6** *The time- $t$  price of a European calendar spread put option expiring at time  $T_0$  with strike  $K = 0$  on the price differential between the futures contracts expiring at time  $T_1$  and  $T_2$ ,  $\mathcal{P}(t, T_0, T_1, T_2, K = 0)$ , is given by*

$$\begin{aligned} \mathcal{P}(t, T_0, T_1, T_2, K = 0) &= E_t^Q \left[ e^{-\int_t^{T_0} r(s)ds} (0 - (F(T_0, T_1) - F(T_0, T_2))) \mathbf{1}_{F(T_0, T_1) - F(T_0, T_2) < 0} \right] \\ &= P(t, T_0)(G_{0,1,1,-1}(0) - G_{1,0,1,-1}(0)), \end{aligned} \quad (79)$$

where  $P(t, T_0)$  denotes the time- $t$  price of a zero-coupon bond maturing at time  $T_0$ ,  $i = \sqrt{-1}$  and  $G_{a_1, a_2, b_1, b_2}(y)$  is defined as

$$G_{a_1, a_2, b_1, b_2}(y) = \frac{\psi(a_1, a_2, t, T_0, T_1, T_2)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im} [\psi(a_1 + iub_1, a_2 + iub_2, t, T_0, T_1, T_2)e^{-iuy}]}{u} du. \quad (80)$$

**Proof:** The proof is similar to that of Proposition 4 and therefore omitted.<sup>52</sup> ■

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<sup>51</sup>The proof is available on our web-sites.

<sup>52</sup>The proof is available on our web-sites. This formula is exact when interest rates are uncorrelated with futures prices. To the extent that the correlation is low and the volatility of interest rates is low relative to the volatility of futures spreads, the formula gives a very accurate approximation of the true price of short-term or medium-term options.

	# obs	<i>Entire sample</i>			<i>Rolling sample</i>		
		$PC^{fut,1}$	$PC^{fut,2}$	$PC^{fut,3}$	$PC^{fut,1}$	$PC^{fut,2}$	$PC^{fut,3}$
M1 – M4	3686	98.69	1.08	0.21	99.05 (0.10)	0.85 (0.08)	0.09 (0.01)
M1 – M6	2664	98.69	1.17	0.11	98.72 (0.09)	1.16 (0.09)	0.10 (0.01)
M1 – Q2	1479	97.85	1.94	0.16	97.87 (0.18)	1.93 (0.17)	0.15 (0.01)
M1 – Y2	1055	96.01	3.51	0.34	96.06 (0.40)	3.48 (0.36)	0.32 (0.04)
M1 – Y4	162	96.77	2.94	0.20	97.15 (0.20)	2.59 (0.19)	0.19 (0.01)

Notes: The principal components are obtained by factor analyzing the covariance matrix of the futures returns. The table reports the percentage of futures return variation explained by the first three principal components. We consider five different sets of futures contracts. “Entire sample” denotes that the analysis is performed over the entire sample. “Rolling sample” denotes that the analysis is performed on a rolling window of 100 observations. The latter case yields time-series of the explanatory power of the first three principal components, and the table reports the means of these time-series with the standard deviations of the means in parentheses (these standard deviations are computed using Newey and West (1987) standard errors with 20 lags).

Table 1: Common variation in futures returns

		Underlying futures contract											
		M1	M2	M3	M4	M5	M6	Q1	Q2	Y1	Y2	Y3	Y4
		<i>Entire sample</i>											
	M1 – M4	15.35	14.72	14.42	13.39	—	—	—	—	—	—	—	—
	M1 – M6	39.36	39.52	29.89	24.30	22.66	20.72	—	—	—	—	—	—
	M1 – Q2	45.46	41.65	33.47	26.28	22.45	19.15	14.19	12.27	—	—	—	—
	M1 – Y2	43.55	41.30	33.35	26.84	26.31	21.58	21.21	15.65	16.56	6.61	—	—
	M1 – Y4	62.49	61.38	58.53	54.27	53.62	51.74	50.40	43.49	44.65	28.19	34.55	37.56
		<i>Rolling sample</i>											
	M1 – M4	49.29 (2.06)	47.17 (1.94)	36.83 (1.91)	33.05 (1.91)	—	—	—	—	—	—	—	—
	M1 – M6	53.16 (2.64)	51.21 (2.21)	38.24 (2.43)	35.37 (2.36)	34.85 (2.19)	30.15 (2.56)	—	—	—	—	—	—
	M1 – Q2	58.63 (2.32)	53.82 (2.37)	44.54 (2.48)	39.01 (2.98)	36.52 (2.74)	29.67 (3.49)	27.21 (3.29)	23.65 (3.51)	—	—	—	—
	M1 – Y2	59.45 (2.82)	55.94 (2.47)	45.39 (3.19)	40.34 (3.81)	39.72 (2.63)	33.35 (3.57)	31.16 (3.22)	27.10 (3.67)	28.05 (2.39)	16.69 (2.76)	—	—
	M1 – Y4	66.14 (0.40)	64.71 (0.13)	62.22 (1.09)	56.92 (3.00)	56.32 (2.95)	58.08 (0.61)	55.43 (1.16)	49.59 (0.45)	44.96 (0.76)	25.14 (1.29)	29.03 (0.46)	24.78 (2.03)

Notes: The adjusted  $R^2$ s from regressing straddle returns on the first three principal components and squared principal components of the futures returns. We consider five different sets of futures contracts. “Entire sample” denotes that the analysis is performed over the entire sample. “Rolling sample” denotes that the analysis is performed on a rolling window of 100 observations. The latter case yields time-series of adjusted  $R^2$ s, and the table reports the means of these time-series with the standard derivations of the means in parentheses (these standard deviations are computed using Newey and West (1987) standard errors with 20 lags).

Table 2:  $R^2$ s from straddle return regressions

	Underlying futures contract											
	M1	M2	M3	M4	M5	M6	Q1	Q2	Y1	Y2	Y3	Y4
	<i>Entire sample</i>											
M1 – M4	20.88	16.89	13.86	12.89	—	—	—	—	—	—	—	—
M1 – M6	14.95	12.81	8.15	5.23	4.80	3.45	—	—	—	—	—	—
M1 – Q2	12.08	8.76	6.64	2.98	3.26	2.79	1.67	0.99	—	—	—	—
M1 – Y2	11.29	9.90	7.18	5.85	6.23	3.21	4.37	2.35	3.23	0.92	—	—
M1 – Y4	20.41	19.83	18.42	12.84	15.48	16.47	14.81	15.06	12.12	0.42	0.85	0.94
	<i>Rolling sample</i>											
M1 – M4	19.43 (1.66)	22.50 (1.63)	17.33 (1.62)	16.05 (1.44)	—	—	—	—	—	—	—	—
M1 – M6	18.95 (1.63)	20.75 (1.52)	14.33 (1.19)	15.10 (1.22)	13.21 (1.19)	9.79 (1.17)	—	—	—	—	—	—
M1 – Q2	20.02 (1.63)	20.24 (1.32)	16.22 (1.30)	16.14 (1.41)	12.55 (1.18)	9.26 (1.27)	8.52 (1.34)	7.46 (1.64)	—	—	—	—
M1 – Y2	20.62 (2.22)	22.97 (1.98)	17.32 (2.27)	15.94 (1.99)	13.84 (2.17)	8.31 (1.17)	7.86 (1.48)	7.33 (1.79)	9.05 (1.73)	4.05 (2.11)	—	—
M1 – Y4	27.02 (2.91)	23.46 (2.23)	23.91 (2.22)	17.24 (0.76)	19.50 (2.13)	25.35 (1.08)	21.92 (0.95)	22.89 (1.86)	9.62 (1.94)	−6.40 (1.26)	−6.24 (0.39)	−12.58 (0.70)

Notes: The adjusted  $R^2$ s from regressing changes in implied log-normal volatilities on the first three principal components and squared principal components of the futures returns. We consider five different sets of futures contracts. “Entire sample” denotes that the analysis is performed over the entire sample. “Rolling sample” denotes that the analysis is performed on a rolling window of 100 observations. The latter case yields time-series of adjusted  $R^2$ s, and the table reports the means of these time-series with the standard deviations of the means in parentheses (these standard deviations are computed using Newey and West (1987) standard errors with 20 lags).

Table 3:  $R^2$ s from implied volatility regressions

	# obs	<i>Entire sample</i>			<i>Rolling sample</i>		
		$PC^{res,1}$	$PC^{res,2}$	$PC^{res,3}$	$PC^{res,1}$	$PC^{res,2}$	$PC^{res,3}$
<i>Straddle return regressions</i>							
M1 – M4	3686	79.79	12.18	4.61	71.60 (0.69)	17.31 (0.41)	6.79 (0.29)
M1 – M6	2664	64.94	16.47	6.37	62.23 (1.15)	18.29 (0.57)	8.09 (0.34)
M1 – Q2	1479	53.63	17.93	7.76	54.81 (2.28)	18.46 (0.80)	8.87 (0.49)
M1 – Y2	1055	50.81	14.44	9.09	51.00 (2.64)	17.00 (0.94)	9.71 (0.55)
M1 – Y4	162	56.19	18.61	11.63	55.58 (0.33)	20.50 (0.27)	12.11 (0.19)
<i>Implied volatility regressions</i>							
M1 – M4	3686	80.13	12.28	4.75	73.05 (0.76)	15.80 (0.39)	6.91 (0.29)
M1 – M6	2664	64.75	15.23	7.26	63.56 (1.27)	16.48 (0.47)	8.41 (0.38)
M1 – Q2	1479	54.97	15.97	8.07	57.09 (2.27)	16.97 (0.74)	8.65 (0.43)
M1 – Y2	1055	52.62	13.83	7.01	55.51 (2.57)	15.08 (0.87)	8.51 (0.49)
M1 – Y4	162	66.31	14.42	8.03	66.84 (0.99)	14.36 (0.91)	8.44 (0.15)

Notes: The principal components are obtained by factor analyzing the covariance matrix of the residuals from both the straddle return regressions and the implied volatility regressions. The table reports the percentage of the variation in residuals explained by the first three principal components. We consider five different sets of futures contracts. “Entire sample” denotes that the analysis is performed over the entire sample. “Rolling sample” denotes that the analysis is performed on a rolling window of 100 observations. The latter case yields time-series of the explanatory power of the first three principal components, and the table reports the means of these time-series with the standard deviations of the means in parentheses (these standard deviations are computed using Newey and West (1987) standard errors with 20 lags).

Table 4: Common variation in regression residuals

	1990–2006	1990–1997	1998–2006
$\kappa$	0.8638 (0.0249)	1.0813 (0.0397)	1.8728 (0.0290)
$\sigma_v$	2.5017 (0.0434)	2.7276 (0.0614)	2.4388 (0.0303)
$\alpha$	0.1751 (0.0058)	0.2405 (0.0096)	0.2794 (0.0070)
$\gamma$	0.7917 (0.0026)	1.3248 (0.0073)	0.8163 (0.0028)
$\sigma_S$	0.2576 (0.0044)	0.2440 (0.0053)	0.3705 (0.0032)
$\rho_{12}$	−0.9112 (0.0057)	−0.9438 (0.0053)	−0.8839 (0.0059)
$\rho_{13}$	−0.1050 (0.0036)	−0.0066 (0.0062)	−0.0893 (0.0043)
$\rho_{23}$	−0.0005 (0.0223)	−0.0427 (0.0307)	−0.2064 (0.0211)
$\varphi$	0.0056 (0.0002)	0.0179 (0.0002)	−0.0019 (0.0003)
$\lambda_{1v}$	0.3723 (0.1969)	0.2245 (0.2763)	0.7228 (0.4025)
$\lambda_{2v}$	−0.2383 (0.1899)	−0.3439 (0.2647)	−0.3753 (0.4044)
$\lambda_{3v}$	0.1882 (0.0753)	0.1818 (0.1119)	0.1866 (0.1413)
$\sigma_{futures}$	0.0153 (0.0001)	0.0109 (0.0001)	0.0153 (0.0001)
$\sigma_{options}$	0.0271 (0.0000)	0.0300 (0.0000)	0.0235 (0.0000)
Log-likelihood	139855.7	52925.4	83105.9

Notes: Maximum-likelihood estimates with outer-product standard errors in parentheses.  $\sigma_{futures}$  denotes the standard deviation of log futures price measurement errors and  $\sigma_{options}$  denotes the standard deviation of scaled option measurement errors.  $\theta$  has been normalized to one.

Table 5: Parameter estimates

Futures contract	Pricing error
M1	-0.07 (-0.25)
M2	-0.13 (-0.99)
M3	-0.11** (-2.04)
M4	-0.04 (-0.73)
M5	0.03 (0.36)
M6	0.11 (0.90)
Q1	0.22 (1.38)
Q2	0.29 (1.51)
Y1	0.06 (0.33)
Y2	-0.46*** (-4.38)
Y3	-0.25 (-1.23)
Y4	0.49 (1.54)

Notes: The table reports the mean percentage pricing errors for the futures contracts when the model is estimated on the entire data set. The pricing errors are defined as the differences between the fitted and actual prices divided by the actual prices. *T*-statistics corrected for serial correlation up to 12 lags are in parentheses. Each statistic is computed using a maximum of 855 weekly observations from January 3, 1990 to May 17, 2006. \*, \*\* and \*\*\* denote significance at the ten, five and one percent level, respectively.

Table 6: Summary statistics of pricing errors for futures contracts



Moneyness	Contract							
	M1	M2	M3	M4	M5	M6	Q1	Q2
0.78–0.82	−3.04** (−2.18)	−1.50*** (−3.41)	−0.65** (−2.25)	−0.19 (−0.62)	−0.17 (−0.54)	−0.23 (−0.64)	−0.32 (−0.73)	−1.23** (−1.97)
0.82–0.86	−1.80** (−2.19)	−1.16*** (−3.10)	−0.42* (−1.88)	−0.03 (−0.14)	0.05 (0.19)	−0.05 (−0.16)	−0.08 (−0.20)	−0.72 (−1.25)
0.86–0.90	−1.09* (−1.84)	−0.80*** (−2.88)	−0.17 (−1.05)	0.14 (0.98)	0.17 (0.93)	0.03 (0.13)	−0.11 (−0.35)	−0.40 (−0.75)
0.90–0.94	−0.49 (−1.05)	−0.38* (−1.69)	−0.02 (−0.19)	0.25** (2.24)	0.20 (1.15)	−0.01 (−0.03)	−0.05 (−0.17)	−0.46 (−0.93)
0.94–0.98	0.40 (1.08)	−0.06 (−0.29)	0.12 (1.22)	0.16* (1.77)	0.18 (1.01)	−0.04 (−0.19)	−0.10 (−0.33)	−0.51 (−1.12)
0.98–1.02	1.16*** (3.42)	0.04 (0.16)	0.13 (1.10)	0.17* (1.85)	0.16 (1.07)	−0.01 (−0.05)	−0.04 (−0.12)	−0.45 (−1.03)
1.02–1.06	0.50 (1.38)	−0.16 (−0.64)	0.01 (0.09)	−0.04 (−0.36)	0.06 (0.41)	−0.09 (−0.43)	−0.19 (−0.66)	−0.56 (−1.20)
1.06–1.10	−0.19 (−0.41)	−0.44 (−1.57)	−0.18 (−1.12)	−0.09 (−0.76)	0.01 (0.06)	−0.16 (−0.76)	−0.08 (−0.26)	−0.54 (−1.08)
1.10–1.14	−0.95* (−1.67)	−0.77** (−2.26)	−0.33* (−1.73)	−0.10 (−0.79)	0.02 (0.11)	0.11 (0.44)	0.04 (0.14)	−0.17 (−0.31)
1.14–1.18	−1.71** (−2.19)	−1.15** (−2.54)	−0.49** (−2.11)	−0.35* (−1.93)	0.07 (0.39)	0.20 (0.82)	0.34 (1.02)	0.31 (0.56)
1.18–1.22	−3.16*** (−2.92)	−1.51*** (−2.64)	−0.88*** (−2.84)	−0.46** (−2.10)	−0.02 (−0.12)	0.17 (0.65)	0.36 (1.13)	0.42 (0.86)

Notes: The table reports the mean pricing errors for the options within each moneyness–maturity category when the model is estimated on the entire data set. The pricing errors are defined as the differences between fitted and actual log-normal implied volatilities.  $T$ -statistics corrected for serial correlation up to 12 lags are in parentheses. Each statistic is computed using a maximum of 855 weekly observations from January 3, 1990 to May 17, 2006. \*, \*\* and \*\*\* denote significance at the ten, five and one percent level, respectively.

Table 7: Summary statistics of pricing errors for options on futures contracts

Calendar spread	Put	Call
M1–M2	–25.41*** (–4.71)	–7.12 (–1.33)
M2–M3	–13.32** (–2.04)	–10.18* (–1.70)
M3–M4	–13.01* (–1.67)	–10.93* (–1.79)
M4–M5	–6.11 (–0.64)	–9.16 (–1.14)
M5–M6	–0.59 (–0.06)	–1.87 (–0.18)

Notes: The table reports the mean percentage pricing errors for ATM calendar spread options. The pricing errors are defined as the differences between the fitted and actual prices divided by the actual prices.  $T$ -statistics corrected for serial correlation up to 12 lags are in parentheses. The calendar spread options are priced out-of-sample from the model estimated on the main option data set using weekly data from June 6, 2002 to May 17, 2006. Each calendar spread option is priced by simulations using 50,000 paths, antithetic variates and the corresponding zero-strike calendar spread option as control variate. The number of observations range from 101 (for put options on the M5–M6 spread) to 197 (for put options on the M1–M2 spread). \*, \*\* and \*\*\* denote significance at the ten, five and one percent level, respectively.

Table 8: Summary statistics of out-of-sample pricing errors for ATM calendar spread options

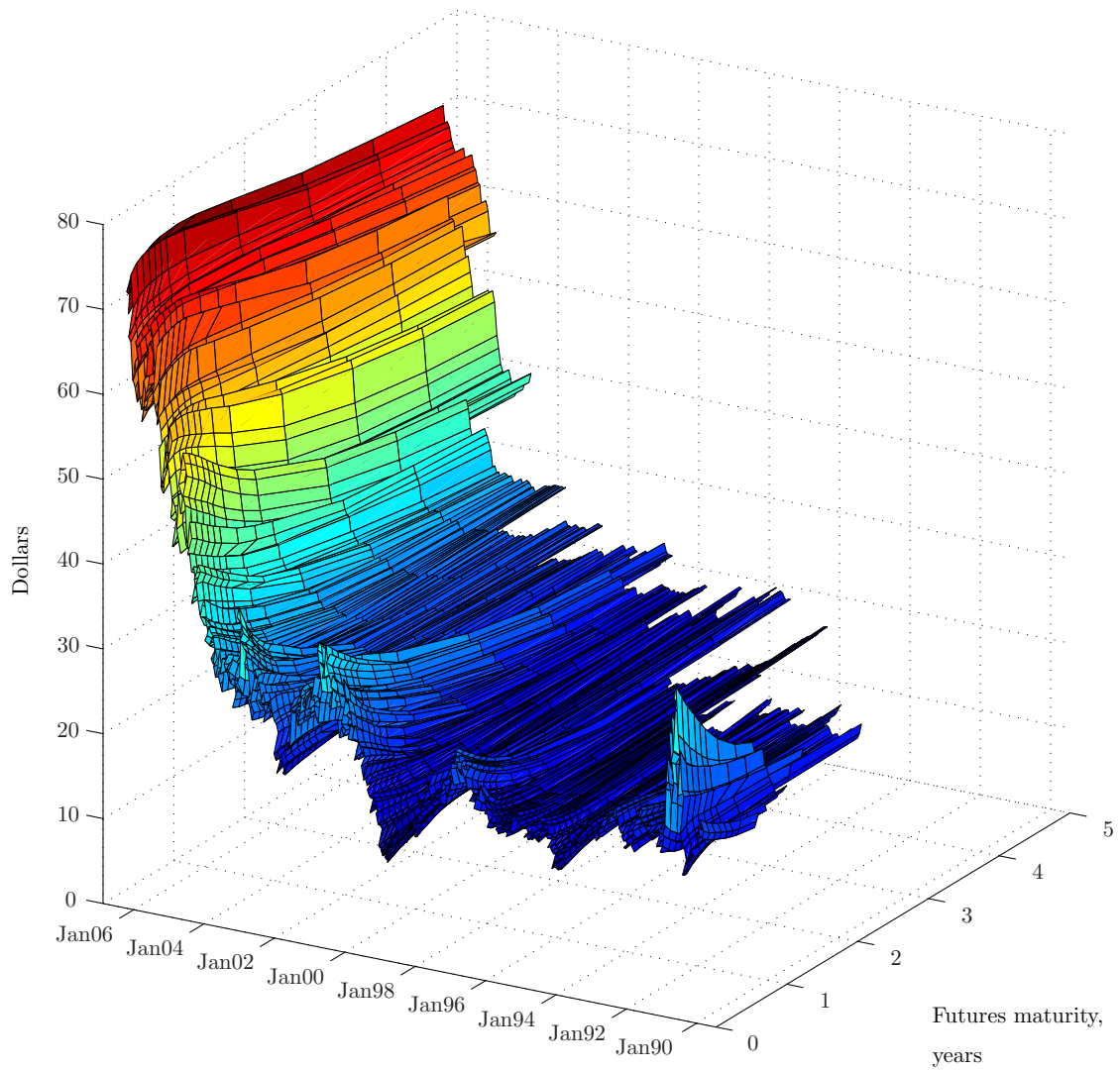


Figure 1: Prices of futures contracts

Prices of M1, M2, M3, M4, M5, M6, Q1, Q2, Y1, Y2, Y3 and Y4 futures contracts. Along the time-dimension there are 855 weekly observations from January 3, 1990 to May 17, 2006.

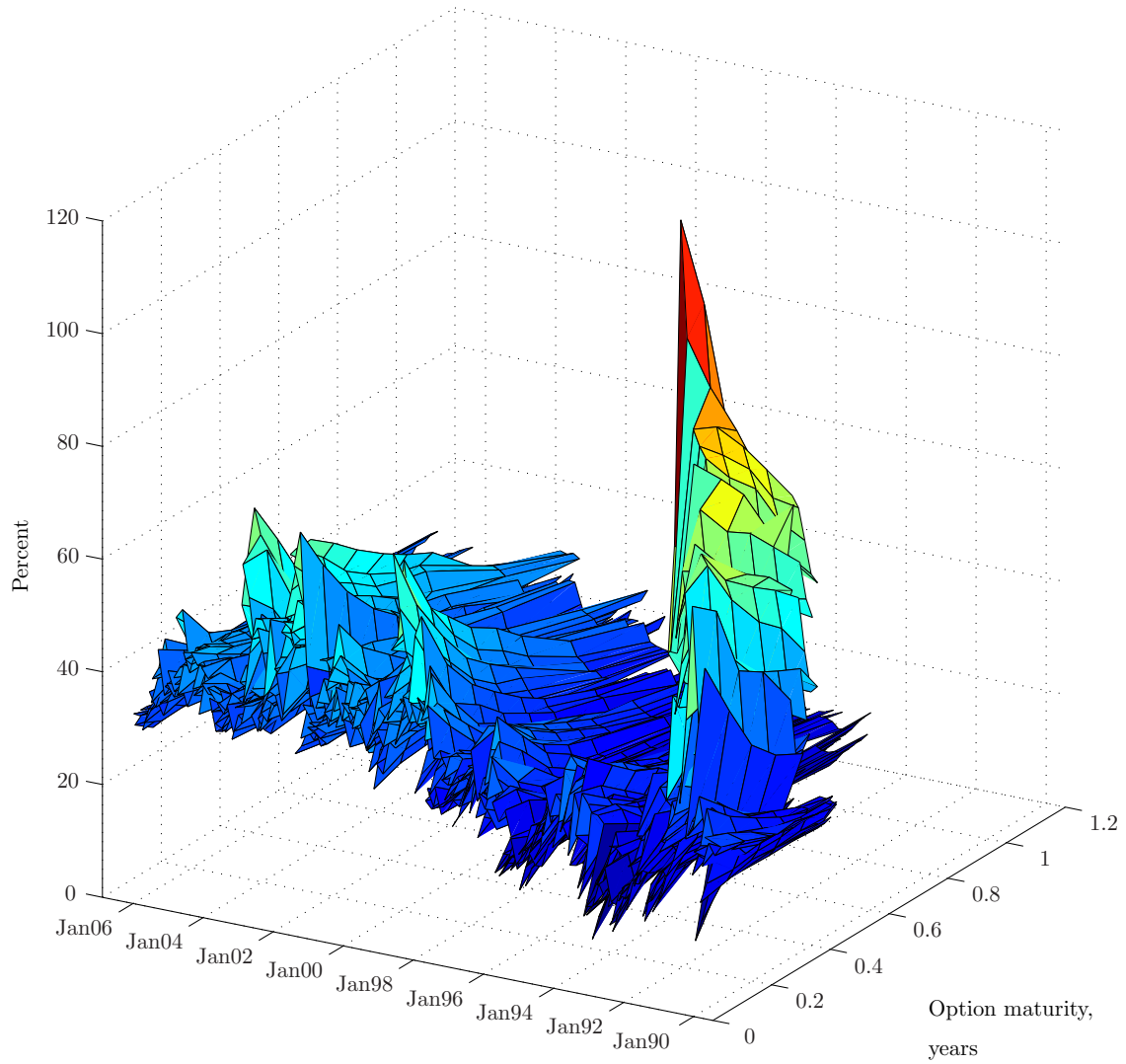


Figure 2: Implied log-normal volatility of futures options

Implied log-normal volatility of options on the M1, M2, M3, M4, M5, M6, Q1 and Q2 futures contracts. Implied volatilities are computed from option prices by inverting the Barone-Adesi and Whaley (1987) formula. Along the time-dimension there are 855 weekly observations from January 3, 1990 to May 17, 2006.

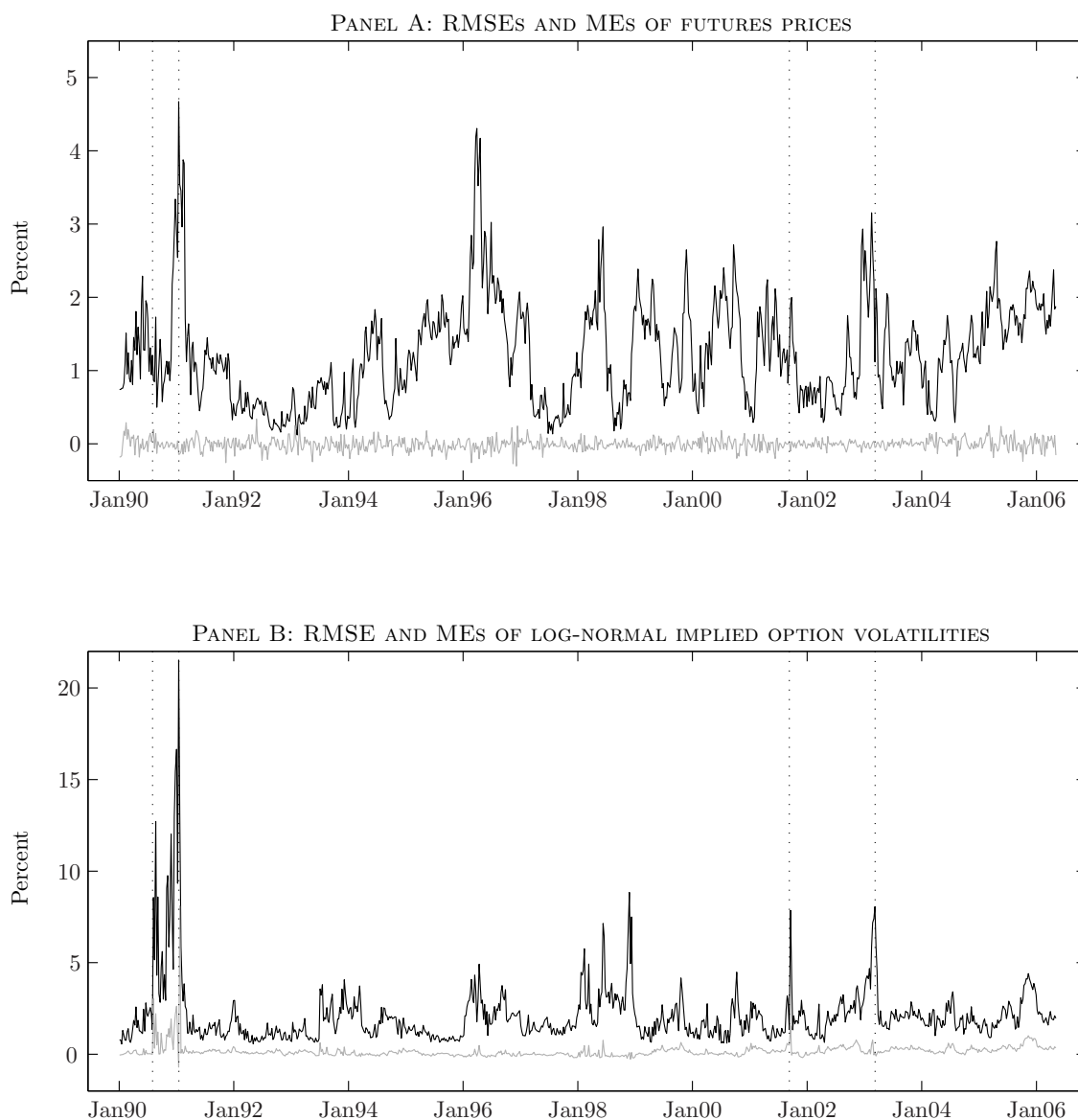


Figure 3: Time-series of RMSEs and MEs of futures and options

Panel A shows time-series of root-mean-squared-errors (RMSEs) and mean errors (MEs) of the percentage differences between actual and fitted futures prices when the model is estimated on the entire data set. Panel B shows time-series of RMSEs and MEs of the differences between actual and fitted implied option volatilities. — denotes RMSEs and — denotes MEs. The vertical dotted lines mark the Iraqi invasion of Kuwait on August 2, 1990, the beginning of the US-led liberation of Kuwait (“Operation Desert Storm”) on January 17, 1991, the September 11, 2001 terrorist attacks and the US-led invasion of Iraq on March 20, 2003, respectively. The number of futures at a given date varies between 8 and 12. The number of options at a given date varies between 23 and 87. Each time-series consists of 855 weekly observations from January 3, 1990 to May 17, 2006.

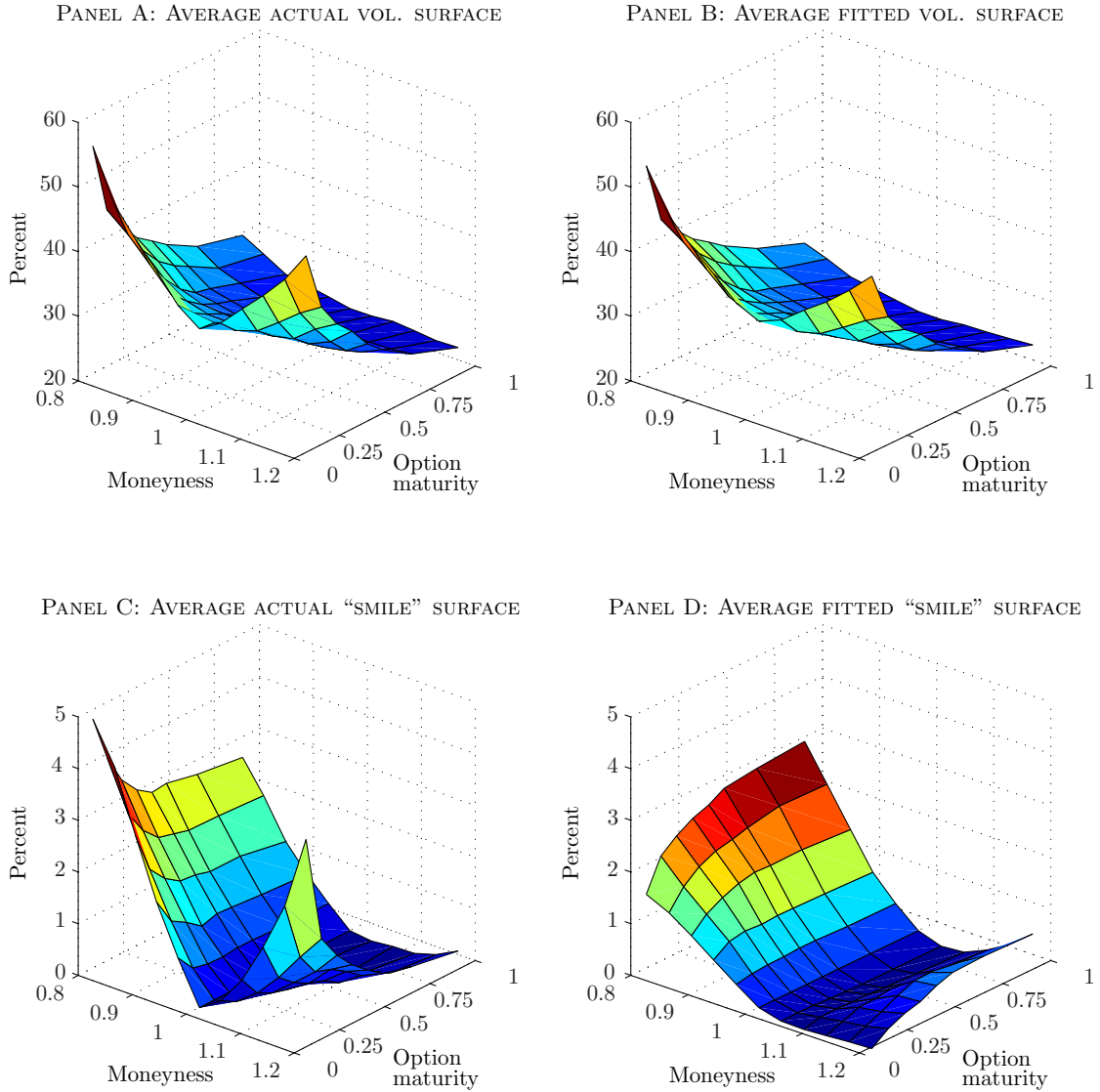


Figure 4: Average log-normal implied volatility and “smile” surfaces

Panel A and B show averages of the actual and fitted log-normal implied volatilities. Panel C and D show averages of the actual and fitted log-normal implied volatility “smiles”. For a given option maturity the “smile” is the difference between the implied volatilities across moneyness and the implied volatility of the ATM option. Moneyness is defined as option strike divided by the price of the underlying futures contract.

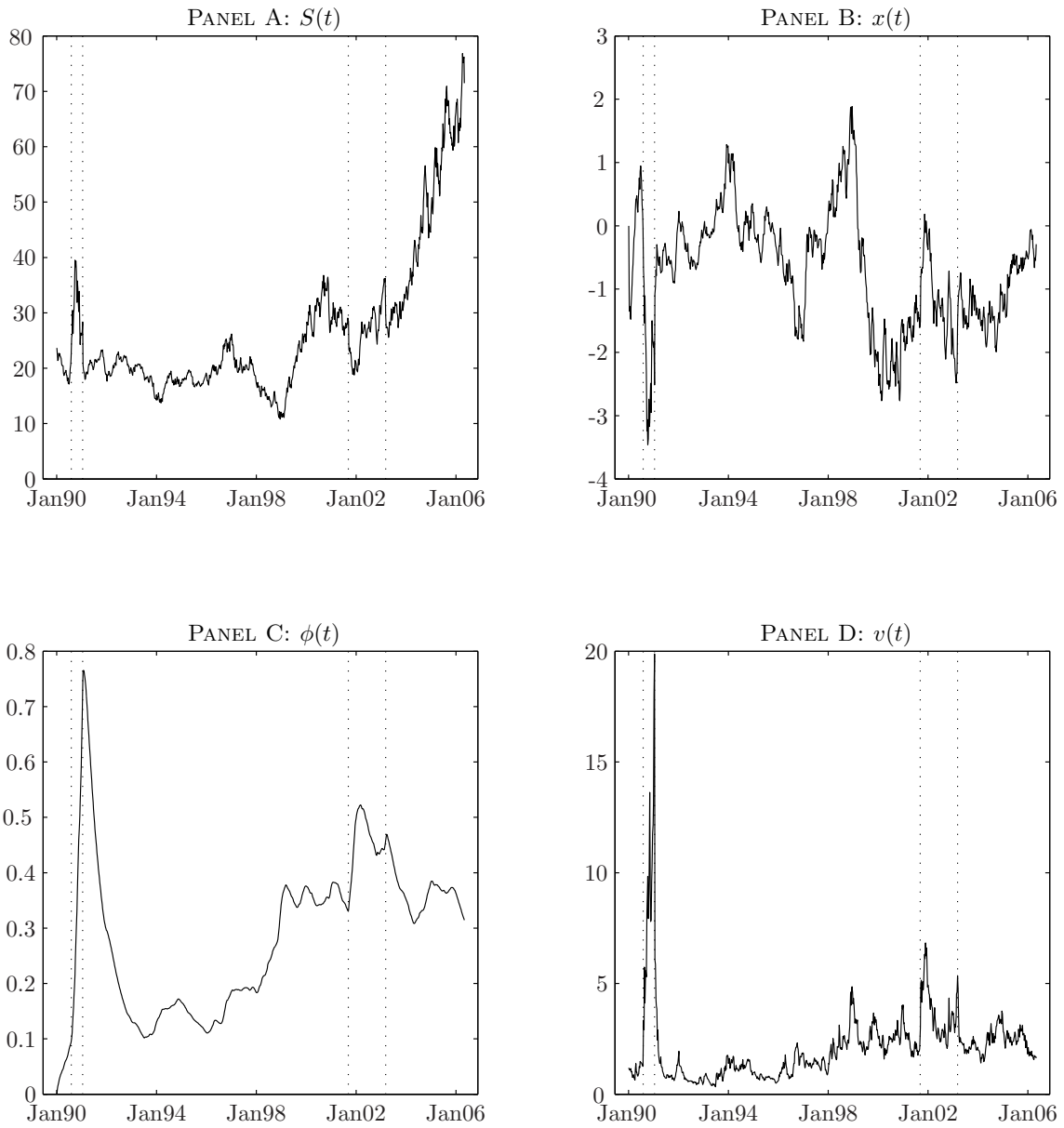


Figure 5: Time-series of Kalman-filtered state variables

The Kalman-filtered state variables when the model is estimated on the entire data set. The vertical dotted lines mark the Iraqi invasion of Kuwait on August 2, 1990, the beginning of the US-led liberation of Kuwait (“Operation Desert Storm”) on January 17, 1991, the September 11, 2001 terrorist attacks and the US-led invasion of Iraq on March 20, 2003, respectively. Each time-series consists of 855 weekly observations from January 3, 1990 to May 17, 2006.

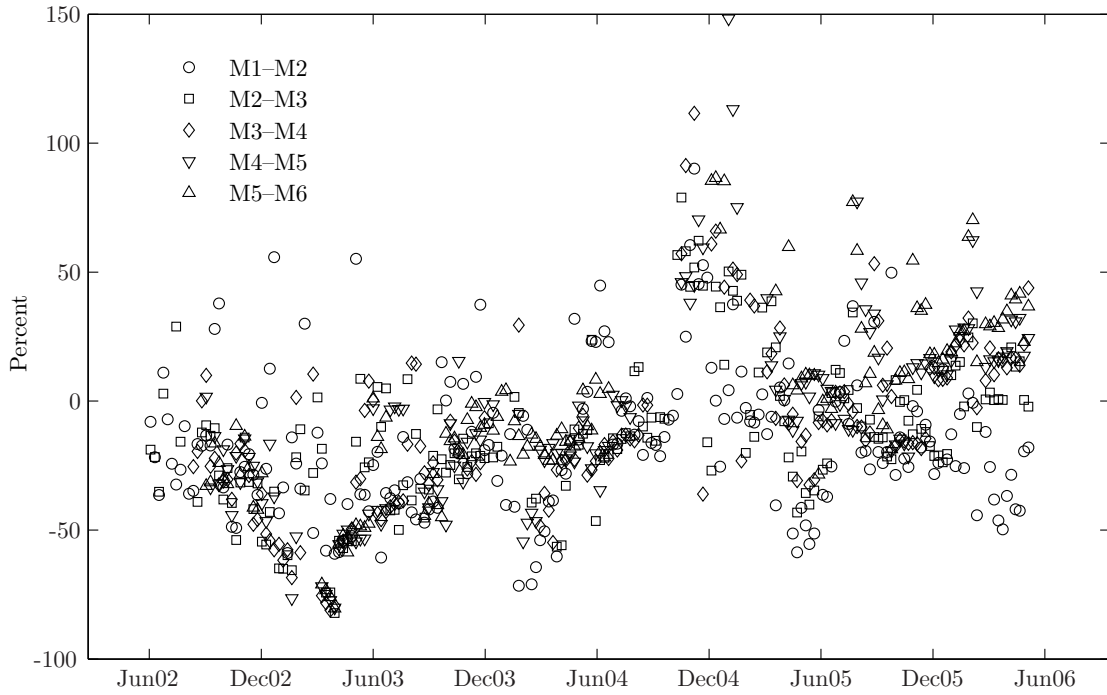


Figure 6: Time series of the pricing errors for ATM calendar spread options

Time-series of pricing errors for ATM options on the M1–M2, M2–M3, M3–M4, M4–M5 and M5–M6 calendar spreads. We display averages of ATM put and ATM call option pricing errors. The pricing errors are defined as the differences between the fitted and actual prices divided by the actual prices. The calendar spread options are priced out-sample from the model estimated on the main option data set using weekly data from June 6, 2002 to May 17, 2006. Each calendar spread option is priced by simulations using 50,000 paths, antithetic variates and the corresponding zero-strike calendar spread option as control variate.



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