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### **ABSTRACT**

We identify the relative importance of changes in the conditional variance of fundamentals (which we call “uncertainty”) and changes in risk aversion (“risk” for short) in the determination of the term structure, equity prices and risk premiums. Theoretically, we introduce persistent time-varying uncertainty about the fundamentals in an external habit model. The model matches the dynamics of dividend and consumption growth, including their volatility dynamics and many salient asset market phenomena. While the variation in dividend yields and the equity risk premium is primarily driven by risk, uncertainty plays a large role in the term structure and is the driver of counter-cyclical volatility of asset returns.

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# 1 Introduction

Without variation in discount rates, it is difficult to explain the behavior of aggregate stock prices within the confines of rational pricing models. An old literature, starting with Pindyck (1988), focused on changes in the variance of fundamentals as a source of price fluctuations, suggesting that increased variances would depress prices. Poterba and Summers (1986) argued that the persistence of return variances does not suffice to account for the volatility of observed stock returns, whereas Barsky (1989) was the first to focus attention on the fact that increased uncertainty may also affect riskless rates in equilibrium which may undermine the expected price effects. Abel (1988) examined the effects of changes in the riskiness of dividends on stock prices and risk premiums in a Lucas (1978) general equilibrium model, with the perhaps surprising result that increased riskiness only lowers asset prices when the coefficient of risk aversion is lower than one.

Changes in the conditional variance of fundamentals (either consumption growth or dividend growth) as a source of asset price fluctuations are making a comeback in the recent work of Bansal and Yaron (2004), Bansal, Khatchatrian and Yaron (2002), and Bansal and Lundblad (2002), which we discuss in more detail below. Nevertheless, most of the recent literature has not focused on changes in the variability of fundamentals as the main source of fluctuations in asset prices and risk premiums but on changes in risk aversion and risk preferences. The main catalyst here was the work of Campbell and Cochrane (1999), CC henceforth, who showed that a model with counter-cyclical risk aversion could account for a large equity premium, substantial variation in returns and price-dividend ratios and long-horizon predictability of returns. There have been a large number of extensions and elaborations of the CC framework (see e.g. Bekaert, Engstrom and Grenadier (2004), Brandt and Wang (2003), Buraschi and Jiltsov (2005), Menzly, Santos and Veronesi (2004), and Wachter (2004)) and a large number of articles trying to find an economic mechanism for changes in aggregate prices of risk (Chan and Kogan (2002), Lustig and Van Nieuwerburgh (2003), Santos and Veronesi (2000), Piazzesi, Schneider and Tuzel (2003), and Wei (2003)).

In this article, we try to identify the relative importance of changes in the conditional variance of fundamentals (which we call “uncertainty”) and changes in risk aversion (“risk” for short)<sup>2</sup>. We build

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<sup>2</sup>Hence, the term uncertainty is used in a different meaning than in the growing literature on Knightian uncertainty, see for instance Epstein and Schneider (2004). It is also consistent with a small literature in international finance which has focused on the effect of changes in uncertainty on exchange rates and currency risk premiums, see Hodrick (1989, 1990) and Bekaert (1996). The Hodrick (1989) paper provided the obvious inspiration for the title to this paper.

on the external habit model formulated in Bekaert, Engstrom and Grenadier (2004) which features stochastic risk aversion and introduce persistent time-varying uncertainty in the fundamentals. We explore the effects of both on price dividend ratios, equity risk premiums, the conditional variability of equity returns and the term structure, both theoretically and empirically. To differentiate time-varying uncertainty from stochastic risk aversion empirically, we use information on higher moments in dividend and consumption growth and the conditional relation between their volatility and a number of instruments.

The model is consistent with the empirical volatility dynamics of dividend and consumption growth and matches a large number of salient asset market features, including a large equity premium and low risk free rate and the volatilities of equity returns, dividend yields and interest rates. We find that variation in the equity premium is driven by both risk and uncertainty with risk aversion dominating. However, variation in asset prices (consol prices and dividend yields) is primarily due to changes in risk. These results arise because risk aversion acts primarily as a level factor in the term structure while uncertainty affects both the level and the slope of the real term structure and also governs the riskiness of the equity cash flow stream. Consequently, our work provides a new perspective on recent advances in asset pricing modelling. We confirm the importance of economic uncertainty as stressed by Bansal and Yaron (2004) and Kandel and Stambaugh (1990) but show that changes in risk are critical too. However, the main channel through which risk affects asset prices in our model is the term structure, a channel shut off in the original Campbell and Cochrane (1999) paper while stressed by the older partial equilibrium work of Barsky (1989).

The remainder of the article is organized as follows. The second section sets out the theoretical model and motivates the use of our state variables to model time-varying uncertainty of both dividend and consumption growth. In the third section, we derive closed-form solutions for price-dividend ratios and real and nominal bond prices as a function of the state variables and model parameters and examine some comparative statics results. We also demonstrate that two extant models, Abel (1988) and Wu (2001), severely restrict the relationship between uncertainty and equity prices and show why this is so. In the fourth section, we set out our empirical strategy. We use the General Method of Moments (Hansen (1982), GMM henceforth) to estimate the parameters of the model. The fifth section reports parameter estimates and discusses how well the model fits salient features of the data. The sixth section reports various variance decompositions and dissects how uncertainty and risk aversion affect asset prices. Section 7 concludes.

## 2 Theoretical Model

### 2.1 Fundamentals and Uncertainty

To model fundamentals and uncertainty, we begin with the specification of Abel (1988) but enrich the framework in a number of dimensions. Abel (1988) models log dividends as having a persistent conditional mean and persistent conditional variance and models the stochastic behavior of the conditional mean and the conditional coefficient of variation of dividends. Hence, he assumes that dividends are stationary. We modify this set up to allow for a unit root in the dividend process, as is customary in modern asset pricing, and model dividend growth as having a stochastic volatility process. In addition, we relax the assumption that dividends and consumption are identically equal. While consumption and dividends coincide in the original Lucas (1978) framework and many subsequent studies, recent papers have emphasized the importance of recognizing that consumption is financed by sources of income outside of the aggregate equity dividend stream, for example Santos and Veronesi (2005), and Bansal, Dittmar and Lundblad (2004). Our modeling choice for dividends and stochastic volatility is described by the following equations.

$$\begin{aligned}\Delta d_t &= \mu_d + \rho_{du}u_{t-1} + \sqrt{v_{t-1}}(\sigma_{dd}\varepsilon_t^d + \sigma_{dv}\varepsilon_t^v) \\ v_t &= \mu_v + \rho_{vv}v_{t-1} + \sigma_{vv}\sqrt{v_{t-1}}\varepsilon_t^v\end{aligned}\tag{1}$$

where  $d_t = \log(D_t)$  denotes log dividends,  $u_t$  is the demeaned and detrended log consumption-dividend ratio (described further below) and  $v_t$  represents “uncertainty,” and is proportional to the conditional volatility of the dividend growth process. All innovations in the model, including  $\varepsilon_t^d$  and  $\varepsilon_t^v$  follow independent  $N(0, 1)$  distributions. Consequently, covariances must be explicitly parameterized. With this specification, the conditional mean of dividend growth varies potentially with past values of the consumption-dividend ratio, which is expected to be a slowly moving stationary process. Uncertainty itself follows a square-root process and may be arbitrarily correlated with dividend growth through the  $\sigma_{dv}$  parameter. The sign of  $\sigma_{dv}$  is not a priori obvious. From a corporate finance perspective, an increase in the volatility of firm cash flows may increase the present value of the costs of financial distress but it may also make growth options more valuable (see Shin and Stulz (2000) for a recent survey). Because it is a latent factor,  $v_t$  can be scaled arbitrarily without empirical consequence and we therefore fix its unconditional mean at unity.

We model consumption as stochastically cointegrated with dividends, in a fashion similar to Bansal, Dittmar and Lundblad (2004), so that the consumption dividend ratio,  $u_t$ , becomes a relevant state variable. We model  $u_t$  symmetrically with dividend growth,

$$u_t = \mu_u + \rho_{uu}u_{t-1} + \sigma_{ud}(\Delta d_t - E_{t-1}[\Delta d_t]) + \sigma_{uu}\sqrt{v_{t-1}}\varepsilon_t^u. \quad (2)$$

By definition, consumption growth,  $\Delta c_t$ , is

$$\begin{aligned} \Delta c_t &= \delta + \Delta d_t + \Delta u_t \\ &= (\delta + \mu_u + \mu_d) + (\rho_{du} + \rho_{uu} - 1)u_{t-1} + (1 + \sigma_{ud})\sqrt{v_{t-1}}(\sigma_{dd}\varepsilon_t^d + \sigma_{dv}\varepsilon_t^v) + \sigma_{uu}\sqrt{v_{t-1}}\varepsilon_t^u. \end{aligned} \quad (3)$$

Note that  $\delta$  and  $\mu_u$  cannot be jointly identified. We proceed by setting the unconditional mean of  $u_t$  to zero and then identify  $\delta$  as the difference in means of consumption and dividend growth.<sup>3</sup> Consequently, the consumption growth specification accommodates arbitrary correlation between dividend and consumption growth, with heteroskedasticity driven by  $v_t$ . The conditional means of both consumption and dividend growth depend on the consumption-dividend ratio, which is an  $AR(1)$  process. Consequently, the reduced form model for dividend and consumption growth is an  $ARMA(1,1)$  which can accommodate either the standard nearly uncorrelated processes widely assumed in the literature, or the Bansal and Yaron (2004) specification where consumption and dividend growth have a long-run predictable component. Bansal and Yaron (2004) do not link the long run component to the consumption-dividend ratio as they do not assume consumption and dividends are cointegrated.<sup>4</sup>

Our specification raises two important questions. First, is there heteroskedasticity in consumption and dividend growth data? Second, can this heteroskedasticity be captured using our single latent variable specification? Perhaps surprisingly, there is substantial affirmative evidence regarding the first question, but to our knowledge none regarding the second question. Ferson and Merrick (1987), Whitelaw (2000) and Bekaert and Liu (2004) all demonstrate that consumption growth volatility varies through time. For our purposes, the analysis in Bansal, Khatchatrian and Yaron (2004) and Kandel and Stambaugh (1990) is most relevant. The former show that price-dividend

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<sup>3</sup>The presence of  $\delta$  means that  $u_t$  should be interpreted as the demeaned and detrended log consumption-dividend ratio.

<sup>4</sup>In a recent paper, Bansal, Gallant and Tauchen (2004) show that both a Campbell Cochrane (1999) and a Bansal-Yaron type model fit the data equally well.

ratios predict consumption growth volatility with a negative sign and that consumption growth volatility is persistent. Kandel and Stambaugh (1990) link consumption growth volatility to three state variables, the dividend yield, the AAA versus T-Bill spread, and the BBB versus AA spread. They find that, consistent with the Bansal et. al. (2004) results, dividend yields positively affect consumption growth volatility. In Table 1, we extend and modify this analysis. We estimate the following model by GMM,

$$x_{t+k}^2 = \nu_0 + \nu_1 r f_t + \nu_2 dp_t^f + \nu_3 spd_t + \varepsilon_{t+k} \quad (4)$$

where we alternatively model  $\Delta d_t^f$  and  $\Delta c_t^f$ , (filtered) dividend and consumption growth, as  $x_t$ . Because the observable fundamental series are filtered using a four-period moving average to eliminate seasonality, the prediction lag,  $k$ , is set at four quarters. Above,  $r f_t$  is the risk free rate,  $dp_t^f$  is the (also filtered) dividend yield, and  $spd_t$  is the nominal term spread. We defer a discussion of the data to Section 4 of the article. Suffice it to say that our analysis uses data starting in 1926 (but we lose one year when calculating lags), whereas the previous papers use post-war samples. We considered an alternative model where time-variation in the conditional means was removed and the conditional variance of the residuals was modelled as in Equation (4). Because consumption and dividend growth display little variation in the conditional mean, the results were quite similar for this case.

The results are reported in Table 1. Panel A focuses on univariate tests while Panel B reports multivariate tests. Wald tests in the multivariate specification very strongly reject the null of no time variation for the volatility of both consumption and dividend growth. Moreover, all three instruments are significant predictors of volatility in their own right: high interest rates are associated with low volatility, high term spreads are associated with high volatility as are high dividend yields. The results in Bansal et al (2004) and Kandel and Stambaugh (1990) regarding the dividend yield predicting economic uncertainty appear robust to the sample period and are also valid for dividend growth volatility.

Note that the coefficients on the instruments for the dividend growth volatility are 5-15 times as high as for the consumption growth equation. This suggests that one latent variable may capture the variation in both. We test this conjecture by estimating a restricted version of the model where the slope coefficients are proportional across the dividend and consumption equations. This restriction

is not rejected, with a p-value of 0.8958. We conclude that our use of a single latent factor for both fundamental consumption and dividend growth volatility is appropriate. The proportionality constant,  $\eta$ , is 0.0807, implying that the dividend slope coefficients are about 12 times larger than the consumption slope coefficients.

The last two lines of Panel B examine the cyclical pattern in the fundamentals' heteroskedasticity, demonstrating a strong counter-cyclical pattern. This is an important finding as it intimates that heteroskedasticity may be the driver of the counter-cyclical Sharpe ratios stressed by Campbell and Cochrane (1999) and interpreted as counter-cyclical risk aversion.

Table 1 (Panel A) also presents similar predictability results for excess equity returns. We will later use these results as a metric to judge whether our estimated model is consistent with the evidence for variation in the conditional volatility of returns. While the signs are the same as in the fundamentals' equations and the t-statistics are well over one, none of the coefficients are significantly different from zero at conventional significance levels.

## 2.2 Investor Preferences

Following CC, consider a complete markets economy as in Lucas (1978), but modify the preferences of the representative agent to have the form:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{(C_t - H_t)^{1-\gamma} - 1}{1-\gamma} \right], \quad (5)$$

where  $C_t$  is aggregate consumption and  $H_t$  is an exogenous "external habit stock" with  $C_t > H_t$ .

One motivation for an "external" habit stock is the "keeping up with the Joneses" framework of Abel (1990, 1999) where  $H_t$  represents past or current aggregate consumption. Small individual investors take  $H_t$  as given, and then evaluate their own utility relative to that benchmark.<sup>5</sup> In CC,  $H_t$  is taken as an exogenously modelled subsistence or habit level. In this situation, the local coefficient of relative risk aversion can be shown to be  $\gamma \frac{C_t}{C_t - H_t}$ , where  $\left( \frac{C_t - H_t}{C_t} \right)$  is defined as the surplus ratio. As the surplus ratio goes to zero, the consumer's risk aversion goes to infinity. In our model, we view the inverse of the surplus ratio as a preference shock, which we denote by  $Q_t$ . Thus, we have  $Q_t \equiv \frac{C_t}{C_t - H_t}$ , in which case risk aversion is now characterized by  $\gamma Q_t$ , and  $Q_t > 1$ . As  $Q_t$  changes over time, the representative consumer investor's "moodiness" changes, which led Bekaert,

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<sup>5</sup>For empirical analyses of habit formation models where habit depends on past consumption, see Heaton (1995) and Bekaert (1996).

Engstrom and Grenadier (2004) to label this a “moody investor economy.”

The marginal rate of substitution in this model determines the real pricing kernel, which we denote by  $M_t$ . Taking the ratio of marginal utilities of time  $t + 1$  and  $t$ , we obtain:

$$\begin{aligned} M_{t+1} &= \beta \frac{(C_{t+1}/C_t)^{-\gamma}}{(Q_{t+1}/Q_t)^{-\gamma}} \\ &= \beta \exp[-\gamma \Delta c_{t+1} + \gamma (q_{t+1} - q_t)], \end{aligned} \tag{6}$$

where  $q_t = \ln(Q_t)$ .

We proceed by assuming  $q_t$  follows an autoregressive square root process which is contemporaneously correlated with fundamentals, but also possesses its own innovation,

$$q_t = \mu_q + \rho_{qq} q_{t-1} + \sigma_{qc} (\Delta c_t - E_{t-1}[\Delta c_t]) + \sigma_{qq} \sqrt{q_{t-1}} \varepsilon_t^q \tag{7}$$

As with  $v_t$ ,  $q_t$  is a latent variable and can therefore be scaled arbitrarily without economic consequence; we therefore set its unconditional mean at unity. In our specification,  $Q_t$  is not forced to be perfectly negatively correlated with consumption growth as in Campbell and Cochrane (1999) and other interpretations of habit persistence. In this sense, our preference shock specification is closest in spirit to that of Brandt and Wang (2003) who also allow for  $Q_t$  to be correlated with other business-cycle factors. Only if  $\sigma_{qq} = 0$  and  $\sigma_{qc} < 0$  does a Campbell Cochrane like specification obtain where consumption growth and risk aversion shocks are perfectly negatively correlated. Consequently, we can test whether independent preference shocks are an important part of variation in risk aversion or whether its variation is dominated by shocks to fundamentals. Note that the covariance between  $q_t$  and consumption growth depends on  $v_t$  which is itself counter-cyclical. Hence, when  $\sigma_{qc} < 0$ , risk aversion and consumption are negatively correlated with the increase in risk aversion in recessions a positive function of the degree of fundamental uncertainty.

### 2.3 Inflation

When confronting consumption-based models with the data, real variables have to be translated into nominal terms. Furthermore, inflation may be important in realistically modeling the joint dynamics of equity returns, the short rate and the term spread. Therefore, we append the model

with a simple inflation process,

$$\pi_t = \mu_\pi + \rho_{\pi\pi}\pi_{t-1} + \kappa E_{t-1}[\Delta c_t] + \sigma_\pi \varepsilon_t^\pi \quad (8)$$

The impact of expected ‘real’ growth on inflation can be motivated by macroeconomic intuition, such as the Phillips curve (in which case we expect  $\kappa$  to be positive). Because there is no contemporaneous correlation between this inflation process and the real pricing kernel, the one-period short rate will not include an inflation risk premium. However, non-zero correlations between the pricing kernel and inflation may arise at longer horizons due to the impact of  $E_{t-1}[\Delta c_t]$  on the conditional mean of inflation. Note that expected real consumption growth varies only with  $u_t$ ; hence, the specification in Equation (8) is equivalent to one where  $\rho_{\pi u}u_{t-1}$  replaces  $\kappa E_{t-1}[\Delta c_t]$ .

To price nominal assets, we define the nominal pricing kernel,  $\widehat{m}_{t+1}$ , that is a simple transformation of the log real pricing kernel,  $m_{t+1}$ ,

$$\widehat{m}_{t+1} = m_{t+1} - \pi_{t+1}. \quad (9)$$

To summarize, our model has five state variables with dynamics described by the equations,

$$\begin{aligned} \Delta d_t &= \mu_d + \rho_{du}u_{t-1} + \sqrt{v_{t-1}}(\sigma_{dd}\varepsilon_t^d + \sigma_{dv}\varepsilon_t^v) \\ v_t &= \mu_v + \rho_{vv}v_{t-1} + \sigma_{vv}\sqrt{v_{t-1}}\varepsilon_t^v \\ u_t &= \rho_{uu}u_{t-1} + \sigma_{ud}(\Delta d_t - E_{t-1}[\Delta d_t]) + \sigma_{uu}\sqrt{v_{t-1}}\varepsilon_t^u \\ q_t &= \mu_q + \rho_{qq}q_{t-1} + \sigma_{qc}(\Delta c_t - E_{t-1}[\Delta c_t]) + \sigma_{qq}\sqrt{q_{t-1}}\varepsilon_t^q \\ \pi_t &= \mu_\pi + \rho_{\pi\pi}\pi_{t-1} + \rho_{\pi u}u_{t-1} + \sigma_{\pi\pi}\varepsilon_t^\pi \end{aligned} \quad (10)$$

with  $\Delta c_t = \delta + \Delta d_t + \Delta u_t$ .

As discussed above, the unconditional means of  $v_t$  and  $q_t$  are set equal to unity so that  $\mu_v$  and  $\mu_q$  are not free parameters. Finally, the real pricing kernel can be represented by the expression,

$$m_{t+1} = \ln(\beta) - \gamma(\delta + \Delta u_{t+1} + \Delta d_{t+1}) + \gamma \Delta q_{t+1} \quad (11)$$

We collect the 19 model parameters in the vector, ‘ $\Psi$ ,’

$$\Psi = \begin{bmatrix} \mu_d, \mu_\pi, \rho_{du}, \rho_{\pi\pi}, \rho_{\pi u}, \rho_{uu}, \rho_{vv}, \rho_{qq}, \dots \\ \sigma_{dd}, \sigma_{dv}, \sigma_{\pi\pi}, \sigma_{ud}, \sigma_{uu}, \sigma_{vv}, \sigma_{qc}, \sigma_{qq}, \delta, \beta, \gamma \end{bmatrix}'. \quad (12)$$

### 3 Asset Pricing

In this section, we present exact solutions for asset prices, and gain some intuition for how the model works. We then compare the behavior of our model to its predecessors in the literature, such as Abel (1988), Wu (2001) Bansal and Yaron (2004) and Campbell and Cochrane (1999). Our model represents a more elaborate framework than any of these. This is necessary because the scope of the current investigation is wider than that of former studies. As we will see shortly, this model is better able to match a wide variety of empirical features of the data which we believe is necessary to credibly discern the relative importance of uncertainty versus stochastic preferences in decomposing variation in asset prices and the equity premium. However, a drawback of this richness is that while we are able to readily calculate exact pricing formulas for stocks and bonds, these solutions are sufficiently complex and nonlinear that it is difficult, for instance, to trace pricing effects back to any single parameter’s value. Below, we provide as much intuition as possible.

The general pricing principle in this model is simple and follows the framework of Bekaert and Grenadier (2001). Assume an asset pays a real coupon stream  $K_{t+\tau}$ ,  $\tau = 1, 2 \dots T$ . We consider three assets: a real consol with  $K_{t+\tau} = 1$ ,  $T = \infty$ , a nominal consol with  $K_{t+\tau} = \Pi_{t,\tau}^{-1}$ ,  $T = \infty$ , (where  $\Pi_{t,\tau}$  represents cumulative gross inflation from  $t$  to  $\tau$ ) and equity with  $K_{t+\tau} = D_{t+\tau}$ ,  $T = \infty$ . The case of equity will be slightly more complex because dividends are non-stationary (see below). Then, the price-coupon ratio can be written as

$$PC_t = E_t \left\{ \sum_{n=1}^{n=T} \exp \left[ \sum_{j=1}^n (m_{t+j} + \Delta k_{t+j}) \right] \right\} \quad (13)$$

By induction, it is straight forward to show that

$$PC_t = \sum_{n=1}^{n=T} \exp (A_n + B_n \Delta d_t + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t) \quad (14)$$

with

$$X_n = f^X (A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1}, E_{n-1}, F_{n-1}, \Psi)$$

for  $X \in [A, B, C, D, E, F]$ . The exact form of these functions depends on the particular coupon stream as we now demonstrate. We proceed by first pricing real bonds (bonds that pay out 1 unit of the consumption good at a particular point in time), then nominal bonds and finally equity.

### 3.1 Real Term Structure

Consider the term structure of real zero coupon bonds. The well known recursive pricing relationship governing the term structure of these bond prices is

$$P_{n,t}^{rz} = E_t [M_{t+1} P_{n-1,t+1}^{rz}] \quad (15)$$

where  $P_{n,t}^{rz}$  is the price of a real zero coupon bond at time  $t$  with maturity at time  $(t+n)$ . The following proposition summarizes the solution for these bond prices. We solve the model for a slightly generalized (but notation saving) case where  $q_t = \mu_q + \rho_{qq}q_{t-1} + \sqrt{v_{t-1}} (\sigma_{qd}\varepsilon_t^d + \sigma_{qu}\varepsilon_t^u + \sigma_{qv}\varepsilon_t^v) + \sqrt{q_{t-1}}\sigma_{qq}\varepsilon_t^q$ . Our current model obtains when

$$\begin{aligned} \sigma_{qd} &= \sigma_{qc}\sigma_{dd}(1 + \sigma_{ud}) \\ \sigma_{qu} &= \sigma_{qc}\sigma_{uu} \\ \sigma_{qv} &= \sigma_{qc}\sigma_{dv}(1 + \sigma_{ud}). \end{aligned} \quad (16)$$

**Proposition 1** *For the economy described by Equations (10) and(11), the prices of real, risk free, zero coupon bonds are given by*

$$P_{n,t}^{rz} = \exp (A_n + B_n\Delta d_t + C_nu_t + D_n\pi_t + E_nv_t + F_nq_t) \quad (17)$$

where

$$\begin{aligned} A_n &= f^A (A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ B_n &= 0 \\ C_n &= f^C (A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ D_n &= 0 \\ E_n &= f^E (A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ F_n &= f^F (A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \end{aligned}$$

And the above functions are represented by

$$\begin{aligned}
f^A &= \ln \beta - \gamma \delta + A_{n-1} + (B_{n-1} - \gamma) \mu_d + E_{n-1} \mu_v + (F_{n-1} + \gamma) \mu_q \\
f^C &\equiv ((B_{n-1} - \gamma) \rho_{du} + C_{n-1} \rho_{uu} + \gamma (1 - \rho_{uu})) \\
f^E &\equiv E_{n-1} \rho_{vv} \\
&+ \frac{1}{2} ((B_{n-1} - \gamma) \sigma_{dd} + (C_{n-1} - \gamma) \sigma_{ud} \sigma_{dd} + (F_{n-1} + \gamma) \sigma_{qd})^2 \\
&+ \frac{1}{2} ((C_{n-1} - \gamma) \sigma_{uu} + (F_{n-1} + \gamma) \sigma_{qu})^2 \\
&+ \frac{1}{2} ((B_{n-1} - \gamma) \sigma_{dv} + (C_{n-1} - \gamma) \sigma_{ud} \sigma_{dv} + (F_{n-1} + \gamma) \sigma_{qv} + E_{n-1} \sigma_{vv})^2 \\
f^F &\equiv \left( E_{n-1} \rho_{qq} + \gamma (\rho_{qq} - 1) + \frac{1}{2} ((F_{n-1} + \gamma) \sigma_{qq})^2 \right)
\end{aligned}$$

and  $A_0 = B_0 = C_0 = E_0 = F_0 = 0$ . (Proof in Appendix).

We will examine the dynamics implied by this solution shortly, but first it is instructive to note the form of the price-coupon ratio of a hypothetical real consol (with constant real coupons) in the following proposition. This result is immediate once it is realized that the payoffs to such a consol are the sum of those of the above real bonds.

**Proposition 2** *Under the conditions set out in Proposition 1, the price-coupon ratio of a consol paying a constant real coupon is given by*

$$P_t^{rc} = \sum_{n=1}^{\infty} \exp(A_n + B_n \Delta d_t + C_n u_t + E_n v_t + F_n q_t) \quad (18)$$

Note that inflation has zero impact on real bond prices, but will, of course, affect the nominal term structure.

We now examine the impact of fundamentals on the real term structure of bond prices, starting with the consumption-dividend ratio, captured by the  $C_n$  term. The lagged consumption-dividend ratio enters the conditional mean of both dividend growth and itself. Either of these channels will in general impact future consumption growth given Equation (3). If, for example, the net effect of a high consumption-dividend ratio is higher expected future consumption growth, then this implies lower future marginal utility. All else equal, investors will desire to borrow from this happy future, but since bonds are assumed to be in zero net supply, interest rates must rise to offset the borrowing motive.

The volatility factor,  $v_t$ , has important term structure effects because it affects the volatility of both consumption growth and  $q_t$ . As such,  $v_t$  affects the volatility of the pricing kernel, thereby creating precautionary savings effects. In times of high uncertainty, investors desire to save more.

For equilibrium to obtain, interest rates must fall, raising bond prices. Note that the second, third and fourth lines of the  $E_n$  are positive: increased volatility unambiguously drives up bond prices. Thus the model features a classic ‘flight to quality’ effect. If we look at the ‘direct’ effect, we find that a unit change in  $v_t$  affects the bond price by:  $+\frac{1}{2}\gamma^2(\sigma_{qc}-1)^2\sigma_{cc}^2$  where  $\sigma_{cc}^2$  is defined as,

$$\sigma_{cc}^2 \equiv \left( (1 + \sigma_{ud})^2 (\sigma_{dd}^2 + \sigma_{dv}^2) + \sigma_{uu}^2 \right). \quad (19)$$

The risk aversion variable,  $q_t$ , affects bond prices through offsetting utility smoothing and precautionary savings channels. A high current realization of  $q_t$  leads to an expectation that future  $q_t$  will be relatively lower (due to stationarity), indicating a lower future marginal utility state. Smoothing motives again induce a desire to borrow from the future, forcing down bond prices in equilibrium. These effects are captured by the first two terms in the  $F_n$  equation. On the other hand, higher  $q_t$  also increases the volatility of the pricing kernel, which tends to increase the precautionary savings motive. This effect is governed by the third term in the expression for  $F_n$ . In sum, the direct effect (that is, excluding lagged functional coefficients) of a unit change in  $q_t$  on the consol price is  $\gamma(\rho_{qq}-1) + \frac{1}{2}(\gamma\sigma_{qq})^2$ .

It is instructive to gain some further insight into the determinants of the term structure in this model. Let us first focus on the real interest rate. While the rate is implicit in Proposition 1, it is also useful to derive it exploiting the log-normality of the model:

$$rrf_t = -E_t[m_{t+1}] - \frac{1}{2}V_t[m_{t+1}]. \quad (20)$$

The conditional mean of the pricing kernel economically represents consumption smoothing whereas the variance of the kernel represents precautionary savings effects. To make notation less cumbersome in terms of notation, let us reparameterize the consumption growth process as having conditional mean and variance

$$\begin{aligned} E_t[\Delta c_{t+1}] &= \delta + \mu_d + (\rho_{du} + \rho_{uu} - 1)u_t \equiv \mu_c + \rho_{cu}u_t \\ V_t[\Delta c_{t+1}] &= v_t\sigma_{cc}^2. \end{aligned} \quad (21)$$

Then the real rate simplifies to

$$rrf_t = -\ln(\beta) + \gamma(\mu_c - \mu_q) + \gamma\rho_{cu}u_t + \phi_{rq}q_t + \phi_{rv}v_t \quad (22)$$

with  $\phi_{rq} = \gamma(1 - \rho_{qq}) - \frac{1}{2}\gamma^2\sigma_{qq}^2$  and  $\phi_{rv} = -\frac{1}{2}\gamma^2(\sigma_{qc} - 1)^2\sigma_{cc}^2$ . Consequently, our model features a three-factor real interest rate model, with the consumption-dividend ratio, risk aversion, and uncertainty as the three factors. Changes in risk have an ambiguous effect on interest rates depending on whether the smoothing or precautionary savings effect dominates (the sign of  $\phi_{rq}$ ). If  $v_t$  is indeed counter cyclical, then variation in  $v_t$  will tend to make real rates pro-cyclical.

To obtain intuition for the term spread, let us consider a two period bond and exploit the log-normality of the model. We can decompose the spread into three components:

$$rrf_{2,t} - rrf_t = \frac{1}{2}E_t[rrf_{t+1} - rrf_t] + \frac{1}{2}Cov_t[m_{t+1}, rrf_{t+1}] - \frac{1}{4}Var_t[rrf_{t+1}]$$

The first term is the standard expectations hypothesis (EH) term, the second term represents the term premium and the third is a Jensen's inequality term (which we will ignore). Because of mean reversion, the effects of  $u_t$ ,  $v_t$ , and  $q_t$  on the first component will be opposite of their effects on the level of the short rate. For example, the coefficient on  $q_t$  in the EH term is  $\phi_{rq}(\rho_{qq} - 1)$ . Because preference shocks are positively correlated with marginal utility, the term premium effect of  $q_t$  will counter-balance the EH effect when  $\phi_{rq} > 0$ . In fact, it is straightforward to show that the coefficient on  $q_t$  for the term premium is  $\frac{1}{2}\gamma\phi_{rq}\sigma_{qq}^2$ .

Increased uncertainty depresses short rates and, consequently, the EH effect implies that uncertainty increases term spreads. The effect of  $v_t$  on the term premium is very complex because the correlation between  $q_t$  and the kernel is also driven by  $v_t$ . In fact, straightforward algebra shows that the coefficient on  $v_t$  is proportional to

$$(\sigma_{qc} - 1) [\sigma_{uu}^2 (\gamma\rho_{uc} + \phi_{rq}\sigma_{qc}) + (1 + \sigma_{ud}) (\gamma\rho_{uc}\sigma_{ud} + \phi_{rq}\sigma_{qc}(\sigma_{ud} + 1) (\sigma_{dd}^2 + \sigma_{dv}^2) - \phi_{rv}\sigma_{vv}\sigma_{dv})].$$

While the expression looks impossible to sign in general, it is at least conceivable the effect is positive. If that is the case, the EH and term premium effects reinforce one another.

### 3.2 Nominal Term Structure

We proceed as with the real term structure, keeping in mind that the appropriate recursion for the nominal term structure involves the nominal pricing kernel,  $\widehat{m}_t$  introduced in the previous section. The pricing relationship governing the nominal term structure of bond prices is therefore

$$P_{n,t}^z = E_t \left[ \widehat{M}_{t+1} P_{n-1,t+1}^z \right] \quad (23)$$

where  $P_{n,t}^z$  is the price of a nominal zero coupon bond at time  $t$  paying out a dollar at time  $(t+n)$ . The following proposition summarizes the solution for these bond prices.

**Proposition 3** *For the economy described by Equations (10) and (11), the time  $t$  price of a zero coupon bond with a risk free dollar payment at time  $t+n$  is given by*

$$P_{n,t}^z = \exp \left( \widetilde{A}_n + \widetilde{B}_n \Delta d_t + \widetilde{C}_n u_t + \widetilde{D}_n \pi_t + \widetilde{E}_n v_t + \widetilde{F}_n q_t \right) \quad (24)$$

where

$$\begin{aligned} \widetilde{A}_n &= f^A \left( \widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1} \right) + \left( \widetilde{D}_{n-1} - 1 \right) \mu_\pi + \frac{1}{2} \left( \widetilde{D}_{n-1} - 1 \right)^2 \sigma_{\pi\pi}^2 \\ \widetilde{B}_n &= 0 \\ \widetilde{C}_n &= f^C \left( \widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1} \right) + \left( \widetilde{D}_{n-1} - 1 \right) \rho_{\pi u} \\ \widetilde{D}_n &= \left( \widetilde{D}_{n-1} - 1 \right) \rho_{\pi\pi} \\ \widetilde{E}_n &= f^E \left( \widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1} \right) \\ \widetilde{F}_n &= f^F \left( \widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1} \right) \end{aligned}$$

where the functions  $f^X(\cdot)$  are given in Proposition 1 for  $X \in (A, B, C, E, F)$  and  $\widetilde{A}_0 = \widetilde{B}_0 = \widetilde{C}_0 = \widetilde{D}_0 = \widetilde{E}_0 = \widetilde{F}_0 = 0$ . (proof in Appendix.)

From Proposition 3, we can immediately glean the salient differences between the real and nominal term structures. First, the  $\widetilde{A}_n$  equation captures a drift effect from  $\mu_\pi$  - high unconditional inflation erodes the value of the prices of nominal bonds relative to their real counterparts. Additionally, a volatility effect, through  $\sigma_{\pi\pi}$ , is unambiguously positive, but is of second order importance.

Second, the effect of changes in inflation on the term structure is captured in the  $\widetilde{C}_n$  and  $\widetilde{D}_n$  terms. Assume  $\rho_{\pi\pi} > 0$ , the equation for  $\widetilde{D}_n$  implies higher inflation levels will further erode nominal bond prices, in line with economic intuition. Furthermore, because expected inflation is also affected by expected consumption growth through  $u_t$ , if inflation responds positively to higher real growth,

there will be a further relative erosion of nominal bond prices through  $\tilde{C}_n$ .

Because the conditional covariance between the real kernel and inflation is zero, the nominal short rate satisfies the Fisher hypothesis,

$$rf_t = rr f_t + \mu_\pi + \rho_{\pi\pi}\pi_t + \rho_{\pi u}u_t - \frac{1}{2}\sigma_{\pi\pi}^2 \quad (25)$$

The last term is the standard Jensen's inequality effect and the previous three terms represent expected inflation.

### 3.3 Equity Prices

In any present value model, under a no-bubble transversality condition, the equity price-dividend ratio is represented by the conditional expectation,

$$\frac{P_t}{D_t} = E_t \left[ \sum_{n=1}^{\infty} \exp \left( \sum_{j=1}^n (m_{t+j} + \Delta d_{t+j}) \right) \right] \quad (26)$$

where  $\frac{P_t}{D_t}$  is the price dividend ratio. This conditional expectation can also be solved in our framework as an exponential-affine function of the state vector, as is summarized in the following proposition.

**Proposition 4** *For the economy described by Equations (10) and(11), the price-dividend ratio of aggregate equity is given by*

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp \left( \hat{A}_n + \hat{B}_n \Delta d_t + \hat{C}_n u_t + \hat{E}_n v_t + \hat{F}_n q_t \right) \quad (27)$$

where

$$\begin{aligned}
\widehat{A}_n &= f^A \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \mu_d \\
\widehat{B}_n &= 0 \\
\widehat{C}_n &= f^C \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \rho_{du} \\
\widehat{E}_n &= f^E \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) \\
&\quad + \left( \frac{1}{2} \sigma_{dd}^2 + \sigma_{dd} \left( \left( \widehat{B}_{n-1} - \gamma \right) \sigma_{dd} + \left( \widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dd} + \left( \widehat{F}_{n-1} + \gamma \right) \sigma_{qd} \right) \right) \\
&\quad + \left( \frac{1}{2} \sigma_{dv}^2 + \sigma_{dv} \left( \left( \widehat{B}_{n-1} - \gamma \right) \sigma_{dv} + \left( \widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dv} + \left( \widehat{F}_{n-1} + \gamma \right) \sigma_{qv} + \widehat{E}_{n-1} \sigma_{vv} \right) \right) \\
\widehat{F}_n &= f^F \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right)
\end{aligned}$$

where the functions  $f^X(\cdot)$  are given in Proposition 1 for  $X \in (A, B, C, E, F)$  and  $A_0 = B_0 = C_0 = E_0 = F_0 = 0$ . (Proof in appendix)

It is clear upon examination of Propositions 1 and 4 that the price-coupon ratio of a real consol and the price-dividend ratio of an equity claim share many reactions to the state variables. This makes perfect intuitive sense. An equity claim may be viewed simply as a real consol with stochastic coupons. Of particular interest in this study is the *difference* in the effects of state variables on the two financial instruments.

Inspection of  $C_n$  and  $\widehat{C}_n$  illuminates an additional impact of a high realization of the consumption-dividend ratio,  $u_t$ , on the price-dividend ratio. This marginal effect depends positively on  $\rho_{du}$ . Feedback from  $u_t$  to the conditional mean of  $\Delta d_t$  may cause higher expected cash flows when  $u_t$  is high, increasing equity valuations.

Above, we established that higher uncertainty decreases interest rates and increases consol prices. Hence a first order effect of higher uncertainty is a positive ‘term structure’ effect. Two channels govern the differential impact of  $v_t$  on equity prices relative to consol prices. This is evident upon inspection of the expressions for  $E_n$  and  $\widehat{E}_n$ . Let us take them in turn. The two effects are governed by the volatility of future cash flows and the covariance between future cash flows and the pricing kernel. First, the terms,  $\frac{1}{2} \sigma_{dd}^2$  and  $\frac{1}{2} \sigma_{dv}^2$  arise from Jensen’s Inequality and tend towards an effect of higher cash flow volatility increasing equity prices relative to consol prices. While this may seem counterintuitive, it is simply an artifact of the log-normal structure of the model. The key terms for describing the riskiness of cash flows are represented by the second two lines in the expression for  $\widehat{E}_n$ . They arise from the conditional covariance between cash flow growth and the pricing kernel.

As in all modern rational asset pricing models, a negative covariance between the pricing kernel and cash flows induces a positive risk premium and depresses valuation. The ‘direct effect’ terms (those excluding lagged functional coefficients) can be signed. For the second two lines of the  $\widehat{E}_n$  line in Proposition 3, they are,

$$-\gamma(1 + \sigma_{ud})(1 - \sigma_{qc})(\sigma_{dd}^2 + \sigma_{dv}^2)$$

If the conditional covariance between consumption growth and dividend growth is positive,  $(1 + \sigma_{ud}) > 0$ , and consumption is negatively correlated with  $q_t$ ,  $\sigma_{qc} < 0$ , then the dividend stream is negatively correlated with the kernel and increases in  $v_t$  exacerbate this covariance risk. Consequently, uncertainty has two primary effects on stock valuation: a positive term structure effect and a potentially negative cash flow effect.

Interestingly, there is no marginal pricing difference in the effect of  $q_t$  on riskless versus risky coupon streams: the expressions for  $F_n$  and  $\widehat{F}_n$  are functionally identical. This is true by construction in this model because the preference variable,  $q_t$ , affects neither the conditional mean nor volatility of cash flow growth, nor the conditional covariance between the cash flow stream and the pricing kernel at any horizon. We purposefully excluded such relationships for two reasons. Economically, it does not seem reasonable for investor preferences to affect the productivity of the proverbial Lucas tree. Secondly, it would be empirically very hard to identify distinct effects of  $v_t$  and  $q_t$  without exactly these kinds of exclusion restrictions.

Finally, note that inflation has no role in determining equity prices for the same reason that it has no role in determining the real term structure. While such effects may be present in the data, we do not believe them to be of first order importance for the question at hand.

### 3.3.1 Relation to Previous Literature

It is useful at this point to reflect on the differences between these equity pricing results and those of two other papers which have considered the effects of uncertainty on equity prices. First, Abel (1988) creates an economy in which the effect of increased cash flow volatility on equity prices depends on a single parameter, the coefficient of relative risk aversion. That setup is vastly different from ours. Most importantly, Abel (1988) maintains that dividends themselves are stationary and so are prices (at least on a per-capita basis). Also, there is no distinction between consumption and dividends in his model, so that the covariance of cash flows with the pricing kernel and the volatility of the

pricing kernel are proportional. Finally, there is no preference shock. In the current framework, we can consider the effects of some of Abel's assumptions by simply shutting down the dynamics of the consumption dividend ratio ( $u_t = 0$ ) and stochastic risk aversion ( $q_t = 0$ ). However, we do not implement Abel's assumption that dividends and prices are stationary.

**Proposition 5** *For the economy described by Equations (10) and (11), and the additional assumption that the following parameters are zero,*

$$\mu_u, \mu_q, \rho_{du}, \rho_{uu}, \rho_{qq}, \sigma_{ud}, \sigma_{uu}, \sigma_{qc}, \sigma_{qq}$$

the equity price-dividend ratio is represented by

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(\overleftarrow{A}_n + \overleftarrow{E}_n v_t\right)$$

where

$$\begin{aligned} \overleftarrow{A}_n &= \ln \beta + \overleftarrow{A}_{n-1} + (B_{n-1} + 1 - \gamma) \mu_d + \overleftarrow{E}_{n-1} \mu_v \\ \overleftarrow{B}_n &= 0 \\ \overleftarrow{E}_n &= \overleftarrow{E}_{n-1} \rho_{vv} + \frac{1}{2} \left(\overleftarrow{E}_{n-1} + 1 - \gamma\right)^2 \sigma_{dd}^2 + \frac{1}{2} \left(\left(\overleftarrow{B}_{n-1} + 1 - \gamma\right) \sigma_{dv} + \overleftarrow{E}_{n-1} \sigma_{vv}\right)^2 \end{aligned}$$

with  $\overleftarrow{A}_0 = \overleftarrow{B}_0 = \overleftarrow{E}_0 = 0$  (Proof available upon request.)

The effect of volatility changes on the price dividend ratio is given by the  $\overleftarrow{E}_n$  coefficient. When volatility is positively autocorrelated,  $\rho_{vv} > 0$ ,  $\overleftarrow{E}_n > 0$  and increases in volatility always increase equity valuation, essentially because they depress the interest rate. In comparison to the differential effects of  $v_t$  in Proposition 3, only the Jensen's Inequality terms remain. There is no scope for  $v_t$  to alter the riskiness of the dividend stream beyond the real term structure effects because cash flows and the pricing kernel are proportional. Clearly, Abel's result is not robust to these different distributional assumptions and this simplified framework is too restrictive for our purposes.

Wu (2001) develops a model wherein increases in volatility unambiguously depress the price-dividend ratio. The key difference between his model and ours is that Wu models the interest rate as exogenous and constant. To recover something like Wu's results in our framework requires making the real interest rate process exogenous and maintaining the volatility process of Equation (10). Assume for example that we introduce a stochastic process  $x_t$  and modify the specification of

the dividend growth process to be:

$$\begin{aligned}\Delta d_t &= \frac{\ln \beta}{\gamma} + \frac{1}{\gamma} x_{t-1} + \frac{\gamma}{2} v_{t-1} + \sqrt{v_{t-1}} \varepsilon_t^d \\ x_t &= \mu_x + \rho_{xx} x_{t-1} + \sigma_x \varepsilon_t^x + \sigma_{xv} \sqrt{v_{t-1}} \varepsilon_t^v + \sigma_{xd} \sqrt{v_{t-1}} \varepsilon_t^d\end{aligned}\tag{28}$$

It is easily verified that under these specifications and the additional assumptions of Proposition 5,  $x_t$  is equal to the one-period real risk free rate. The solution for the price-dividend ratio in this economy is described in the following proposition

**Proposition 6** *For the economy described in Proposition 5. with the dividend process modified as in Equations (28) the equity price-dividend ratio can be expressed as*

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(\vec{A}_n + \vec{B}_n \Delta d_t + \vec{G}_n x_t + \vec{E}_n v_t\right)$$

where

$$\begin{aligned}\vec{A}_n &= \vec{A}_{n-1} + \left(1 + \vec{B}_{n-1}\right) \frac{\ln \beta}{\gamma} + \vec{G}_{n-1} \mu_x + \vec{E}_{n-1} \mu_v + \frac{1}{2} \left(\vec{G}_{n-1} \sigma_x\right)^2 \\ \vec{B}_n &= 0 \\ \vec{G}_n &= \left(-1 + \frac{1}{\gamma} + \vec{G}_{n-1} \rho_{xx}\right) \\ \vec{E}_n &= -\frac{\gamma^2}{2} + \frac{\gamma}{2} + \vec{E}_{n-1} \rho_{vv} + \frac{1}{2} (-\gamma + 1)^2 + \frac{1}{2} \left(\vec{G}_{n-1} \sigma_{xv} + \vec{E}_{n-1} \sigma_v\right)^2\end{aligned}$$

with  $\vec{A}_0 = \vec{B}_0 = \vec{E}_0 = \vec{G}_0 = 0$ . (Proof available upon request.)

By considering the expression for  $\vec{E}_n$ , we can see that the direct effect of an increase in  $v_t$  is  $\frac{1}{2}\gamma(1-\gamma)$ . Therefore, only when  $\gamma > 1$  will an increase in volatility depress the price-dividend ratio, but this ignores equilibrium term structure effects. In the context of an endogenous term structure model therefore, Wu's results are not readily generalizable.

Bansal and Yaron (2004) assume that the conditional volatility of consumption growth follows an AR(1) process proportional to that of dividend growth. By assuming Epstein and Zin (1989) preferences, they separate the intertemporal elasticity of substitution (IES) from pure risk aversion. They find that an increase in volatility lowers price-dividend ratios when the IES and risk aversion are larger than unity.

### 3.4 Sharpe Ratios

Campbell and Cochrane (1999) point out that in a lognormal model the maximum attainable Sharpe ratio of any asset is an increasing function of the conditional variance of the log real pricing kernel.

In our model, this is given by,

$$V_t(m_{t+1}) = \gamma^2 \sigma_{qq}^2 q_t + \gamma^2 (\sigma_{qc} - 1)^2 \sigma_{cc}^2 v_t$$

The Sharpe ratio is increasing in preference shocks and uncertainty. Thus, counter-cyclical variation in  $v_t$  may imply counter-cyclical Sharpe ratios. The effect of  $v_t$  on the Sharpe ratio is larger if risk aversion is itself negatively correlated with consumption growth. In Campbell and Cochrane (1999), the kernel variance is a positive function of  $q_t$  only.

## 4 Empirical Implementation

In this section, we describe how we bring the model to the data. We proceed by describing our data and estimation strategy.

### 4.1 Data

We measure all variables at the quarterly frequency and our base sample period extends from 1927:1 to 2004:3.

#### 4.1.1 Equity Market

We used the CRSP quarterly data files from 1926-2003 to create stock market variables. Our stock return measure is the standard CRSP value-weighted return index. To compute excess equity returns,  $r_t^x$ , we subtract the 90-day continuously compounded T-Bill yield earned over the same period (see next subsection for a description of bond market data). For the dividend yield and dividend growth, our methods differ slightly from the most common constructions in the literature. For the dividend yield, we proceed by first calculating a (highly seasonal) quarterly dividend yield series as,

$$DP_{t+1} = \left( \frac{P_{t+1}}{P_t} \right)^{-1} \left( \frac{P_{t+1} + D_{t+1}}{P_t} - \frac{P_{t+1}}{P_t} \right)$$

where  $\frac{P_{t+1}+D_{t+1}}{P_t}$  and  $\frac{P_{t+1}}{P_t}$  are available directly from the CRSP dataset as the value weighted stock return series including and excluding dividends respectively. We then use the four-period moving average of  $\ln(1 + DP_t)$  as our observable series,

$$dp_t^f = \frac{1}{4} [\ln(1 + DP_t) + \ln(1 + DP_{t-1}) + \ln(1 + DP_{t-2}) + \ln(1 + DP_{t-3})].$$

This measure of the dividend yield differs from the more standard technique of summing dividends over the course of the past four quarters and simply scaling by the current price. We prefer our filter because it represents a linear transformation of the underlying data for which we can account explicitly when bringing the model to the data. As a practical matter, the properties of our filtered series and the more standard measure are very similar with close means and volatilities and an unconditional correlation between the two of approximately 0.95 (results available upon request). For dividend growth, we first calculate quarterly dividend growth,

$$\Delta d_{t+1} = \ln \left[ \frac{DP_{t+1}}{DP_t} \frac{P_{t+1}}{P_t} \right]$$

Then, to eliminate seasonality, we use the four-period moving average as the observation series,

$$\Delta d_t^f = \frac{1}{4} (\Delta d_t + \Delta d_{t-1} + \Delta d_{t-2} + \Delta d_{t-3}). \quad (29)$$

Because the above moving average filters for dividends require four lags, our sample is shortened, effectively beginning in 1927.

#### 4.1.2 Bond Market and Inflation

We use standard Ibbotson data (from the SBBI Yearbook) for Treasury market and inflation series for the period 1927-2003. The short rate,  $rf_t$  is the (continuously compounded) 90-day T-Bill rate. The log yield spread,  $spd_t$ , is the average log yield for long term government bonds (maturity greater than ten years) less the short rate. Note that the timing convention of these yields is such that they are dated when they enter the econometrician's data set. For instance, the 90-day T-Bill return earned over January-March 1990 is dated as December 1989, as it entered the data set at the end of that month. Inflation,  $\pi_t$ , is the continuously compounded end of quarter change in the CPI as reported by Ibbotson.

### 4.1.3 Consumption

To avoid the look-ahead bias inherent in standard seasonally adjusted data, we obtained nominal non-seasonally adjusted (NSA) aggregate non-durable and service consumption data from the website of the Bureau of Economic Analysis (BEA) of the United States Department of Commerce for the period 1946-2004. We denote the continuously compounded growth rate of the sum of non-durable and service consumption series as  $\Delta c_t$ . From 1929-1946, consumption data from the BEA is available only at the annual frequency. For these years, we use repeated values equal to one-fourth of the compounded annual growth rate. Because this methodology has obvious drawbacks, we repeated all our analysis using an alternate consumption interpolation procedure which presumed the consumption-dividend ratio, rather than consumption growth was constant over the year. Results using this alternate method are very similar to those reported. Finally, for 1927-1929, no consumption data is available from the BEA. For these years, we obtain the growth rate for real per-capita aggregate consumption from the website of Robert Shiller at [www.yale.edu](http://www.yale.edu), and computed aggregate nominal consumption growth rates using the inflation data described above in addition to historical population growth data from the United States Bureau of the Census. Then, repeated values of the annual growth rate are used as quarterly observations. The raw consumption growth data was deflated with the inflation series described above. Due to the strong seasonality of consumption data and to mitigate the near term look-ahead bias of the repeated value methodology used for converting annual growth rates to the quarterly frequency, we use the four-period moving average of  $\Delta c_t$  as our observation series,

$$\Delta c_t^f = \frac{1}{4} (\Delta c_t + \Delta c_{t-1} + \Delta c_{t-2} + \Delta c_{t-3}). \quad (30)$$

## 4.2 Estimation and Testing Procedure

We now discuss the GMM methodology we use to estimate the model parameters.

### 4.2.1 Parameter Estimation

Our economy has five state variables, which we collect in the vector  $Y_t = [\Delta d_t, v_t, u_t, q_t, \pi_t]$ . While  $u_t, \Delta d_t$  and  $\pi_t$  are directly linked to the data,  $v_t$  and  $q_t$  are latent variables. We are interested in the implications of the model for seven variables: filtered dividend and consumption growth,  $\Delta d_t^f$

and  $\Delta c_t^f$ , inflation,  $\pi_t$ , the short rate,  $r_t^f$ , the term spread,  $spd_t$ , the dividend yield,  $dp_t$ , and log excess equity returns,  $rx_t$ . For all these variables we use the data described above. The first three variables are (essentially) observable state variables; the last four are endogenous asset prices and returns. We collect all the observables in the vector  $W_t$ .

The relation between term structure variables and state variables is affine, but the relationship between the dividend yield and excess equity returns and the state variables is non-linear. In the Computational Appendix, we linearize this relationship and show that the approximation is quite accurate. Note that this approach is very different from the popular Campbell-Shiller (1988) and Campbell (1990) linearization method, which linearizes the return expression itself before taking the linearized return equation through a present value model. We first find the correct solution for the price-dividend ratio and linearize the resulting equilibrium.

Conditional on the linearization, the following property of  $W_t$  obtains,

$$W_t = \mu^w(\Psi) + \Gamma^w(\Psi) Y_t^c \quad (31)$$

where  $Y_t^c$  is the companion form of  $Y_t$  containing five lags and the coefficients superscripted with ‘ $w$ ’ are nonlinear functions of the model parameters,  $\Psi$ . Because  $Y_t$  follows a linear process with square-root volatility dynamics, unconditional moments of  $Y_t$  are available analytically as functions of the underlying parameter vector,  $\Psi$ . Let  $X(W_t)$  be a vector valued function of  $W_t$ . For the current purpose,  $X(\cdot)$  will be comprised of first, second, third and fourth order monomials, unconditional expectations of which are uncentered moments of  $W_t$ . Using Equation (31), we can also derive the analytic solutions for uncentered moments of  $W_t$  as functions of  $\Psi$ . Specifically,

$$E[X(W_t)] = f(\Psi) \quad (32)$$

where  $f(\cdot)$  is also a vector valued function (subsequent appendices provide the exact formulae)<sup>6</sup>.

This immediately suggests a simple GMM based estimation strategy. The GMM moment conditions

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<sup>6</sup>In practice, we simulate the unconditional moments of order three and four during estimation. While analytic solutions are available for these moments, they are extremely computationally expensive to calculate at each iteration of the estimation process. For these moments, we simulate the system for roughly 30,000 periods (100 simulations per observation) and take unconditional moments of the simulated data as the analytic moments implied by the model without error. Due to the high number of simulations per observation, we do not correct the standard errors of the parameter estimates for the simulation sampling variability. To check that this is a reasonable strategy, we perform a one-time simulation at a much higher rate (1000 simulations / observation) at the conclusion of estimation. We check that the identified parameters produce a value for the objective function close to that obtained with the lower simulation rate used in estimation.

are,

$$g_T(W_t; \Psi_0) = \frac{1}{T} \sum_{t=1}^T X(W_t) - f(\Psi_0). \quad (33)$$

Moreover, the additive separability of data and parameters in Equation (33) suggests a ‘fixed’ optimal GMM weighting matrix free from any particular parameter vector and based on the data alone. Specifically, the optimal GMM weighting matrix is the inverse of the spectral density at frequency zero of  $g_T(W_t; \Psi_0)$ , which we denote as  $S(W_T)$ .

To reduce the number of parameters estimated in calculating the optimal GMM weighting matrix, we exploit the structure implied by the model. Under the model, we can project  $X(W_t)$  onto the vector of state variables  $Y_t^c$ , which stacks the contemporaneous five state variables and a number of lags,

$$X(W_t) = \widehat{B}Y_t^c + \widehat{\varepsilon}_t$$

where  $\widehat{B}$  and  $\widehat{\varepsilon}_t$  are calculated using a standard linear projection of  $X(W_t)$  onto  $Y_t^c$ . We assume the covariance matrix of the residuals,  $\widehat{D}$ , is diagonal and estimated it using the residuals,  $\widehat{\varepsilon}_t$ , of the projection. The projection implies

$$\widehat{S}(W_T) = \widehat{B}\widehat{S}(Y_T^c)\widehat{B}' + \widehat{D}$$

where  $\widehat{S}(Y_T^c)$  is the spectral density at frequency zero of  $Y_T^c$ . To estimate  $\widehat{S}(Y_T^c)$ , we use a standard pre-whitening technique as in Andrews and Monahan (2004). Because  $Y_t^c$  contains two unobservable variables,  $v_t$  and  $q_t$ , we use instead the vector  $Y_t^p = [\Delta d_t^f, \pi_t, \Delta c_t^f, r f_t, dp_t^f]'$  and one lag of  $Y_t^p$  to span  $Y_t^c$ <sup>7</sup>.

To estimate the system, we minimize the standard GMM objective function,

$$J(W_T; \widehat{\Psi}) = g_T(\widehat{\Psi}) \left( \widehat{S}(W_T) \right)^{-1} g_{1T}(\widehat{\Psi})' \quad (34)$$

in a one-step GMM procedure.

Because the system is nonlinear in the parameters, we took precautionary measures to assure that a global minimum has indeed been found. First, over 100 starting values for the parameter vector are chosen at random from within the parameter space. From each of these starting values,

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<sup>7</sup>In reality, the moving average filters of  $\Delta d_t^f$ ,  $\Delta c_t^f$  and  $dp_t^f$  would require using 3 lags, but the dimensionality of that system is too large.

we conduct preliminary minimizations. We discard the runs for which estimation fails to converge, for instance, because the maximum number of iterations is exceeded, but retain converged parameter values as ‘candidate’ estimates. Next, each of these candidate parameter estimates is taken as a new starting point and minimization is repeated. This process is repeated for several rounds until a global minimizer has been identified as the parameter vector yielding the lowest value of the objective function. In this process, the use of a fixed weighting matrix is critical. Indeed, in the presence of a parameter-dependent weighting matrix, this search process would not be well defined. Finally, the parameter estimates producing the global minimum are confirmed by starting the minimization routine at small perturbations around the parameter estimate, and verifying that the routine returns to the global minimum.

### 4.3 Moment Conditions

We use a total of 34 moment conditions to estimate the model parameters. These moments are explicitly listed in Table 2. They can be ordered into 6 groups. The first set is simply the unconditional means of the  $W_t$  variables; the second group includes the second uncentered moments of the state variables. In combination with the first moments above, these moments ensure that we are matching the unconditional volatilities of all the variables of interest. The third set of moments is aimed at identifying the autocorrelation of the fundamental processes. Because of the moving average filter applied to dividend and consumption growth, it is only reasonable to look at the fourth order autocorrelations. Because our specification implies complicated ARMA behavior for inflation dynamics, we attempt to fit both the first and fourth order autocorrelation of this series. The fourth set of moments concerns contemporaneous cross moments of fundamentals with asset prices and returns. As was pointed out by Cochrane and Hansen (1995), the correlation among fundamentals and asset prices implied by standard implementations of the consumption CAPM model can be much too high. We also include cross moments between inflation, the short rate, and consumption growth to help identify the  $\rho_{\pi u}$  parameter in the inflation equation and a potential inflation risk premium.

Next, the fifth set of moments identifies higher order moments of dividend growth. This is crucial to ensure that the dynamics of  $v_t$  are identified by, and consistent with, the volatility predictability of the fundamental variables in the data. Moreover, this helps fit their skewness and kurtosis.

Note that there are  $34 - 19 = 15$  over-identifying restrictions and that we can use the standard

$J$ -test to test the fit of the model.

## 5 Estimation Results

This section describes the estimation results of the structural model, and characterizes the fit of the model with the data.

### 5.1 GMM Parameter Estimates

Table 2 reports the results of the above described estimation procedure. We start with dividend growth dynamics. First,  $u_t$  significantly forecasts dividend growth. Second, the conditional volatility of dividend growth,  $v_t$ , is highly persistent with an autocorrelation coefficient of 0.9795 and itself has significant volatility ( $\sigma_{vv}$ , is estimated as 0.3288 with a standard error of 0.0785). This confirms that dividend growth volatility varies through time. Further, the conditional covariance of dividend growth and  $v_t$  is positive and economically large:  $\sigma_{dv}$  is estimated at 0.0413 with a standard error of 0.0130.

The results for the consumption-dividend ratio are in line with expectations. First, it is very persistent, with an autocorrelation coefficient of 0.9826 (standard error 0.0071). Second, the contemporaneous correlation of  $u_t$  with  $\Delta d_t$  is sharply negative as indicated by the coefficient  $\sigma_{ud}$  which is estimated at  $-0.9226$ . In light of Equation (3), this helps to match the low volatility of consumption growth. However, because  $(1 + \sigma_{ud})$  is estimated to be greater than zero, dividend and consumption growth are positively correlated, as is true in the data. Finally, the own volatility parameter for the consumption dividend ratio is 0.0127 with a standard error of just 0.0007, ensuring that the correlation of dividend and consumption growth is not unrealistically high.

The dynamics of the stochastic preference process,  $q_t$ , are presented next. It is estimated to be quite persistent, with an autocorrelation coefficient of 0.9787 (standard error 0.0096) and it has significant independent volatility as indicated by the estimated value of  $\sigma_{qq}$  of 0.1753 (standard error 0.0934). Of great importance is the contemporaneous correlation parameter between  $q_t$  and consumption growth,  $\sigma_{qc}$ . While  $\sigma_{qc}$  is negative, it is not statistically different from zero. This indicates that risk is indeed moving countercyclically, in line with its interpretation as risk aversion under a habit persistence model such as that of Campbell and Cochrane (1999) (further discussion below). What is different in our model is that the correlation between consumption growth and risk

aversion<sup>8</sup> is  $-0.37$  instead of  $-1.00$  in Campbell and Cochrane. The impatience parameter  $\ln(\beta)$  is negative as expected and the  $\gamma$  parameter (which is not the same as risk aversion in this model) is positive, but not significantly different from zero. The wedge between mean dividend growth and consumption growth,  $\delta$ , is both positive and significantly different from zero.

Finally, we present inflation dynamics. As expected, past inflation positively affects expected inflation with a coefficient of  $0.2404$  (standard error  $0.1407$ ) and there is negative and significant predictability running from the consumption-dividend ratio to inflation.

## 5.2 Model Moments Versus Sample Data

Table 2 also presents the standard test of the overidentifying restrictions. The overidentification test fails to reject, with a p-value of  $0.6234$ . However, there are a large number of moments being fit and in such cases, the standard GMM overidentification tests are known to have low power in finite samples. Therefore, we examine the fit of the model with respect to specific moments in Tables 3 and 4.

Table 3 focuses on linear moments of the variables of interest: mean, volatilities and autocorrelations. The model fits the data exceedingly well with respect to the unconditional means of all seven of the endogenous variables. This includes generating a realistic low mean for the nominal risk free rate of about  $1\%$  and a realistic equity premium of about  $1.2\%$  (all quarterly rates). The volatilities of the endogenous variables are also well matched to the data. The implied volatilities of both the financial variables and fundamental series are within one standard error of the data moment. Finally, the model is broadly consistent with the autocorrelation of the endogenous series. The (fourth) autocorrelation of filtered consumption growth is somewhat too low relative to the data. However, in unreported results we verified that the complete autocorrelograms of dividend and consumption growth implied by the model are consistent with the data. The model fails to generate sufficient persistence in the term spread but this is the only moment not within a two standard bound around the data moment. However, it is within a  $2.05$  standard error bound!

As explored below, the time varying volatility of dividend growth is an important driver of equity returns and volatility, and it is therefore important to verify that the model implied nonlinearities in fundamentals are consistent with the data. In Table 4, we determine whether the estimated model

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<sup>8</sup>More specifically, the conditional correlation between  $\Delta c_{t+1}$  and  $q_{t+1}$  when  $v_t$  and  $q_t$  are at their unconditional mean of unity.

is consistent with the reduced form evidence presented in Table 1, and we investigate skewness and kurtosis of fundamentals and returns. In Panel A, we find that the volatility dynamics for fundamentals are quite well matched. The model produces the correct sign in forecasting dividend and consumption growth volatility with respect to the short rate and the spread; only the volatility dynamics with respect to the dividend yield are of the wrong sign. However, for return volatility, all the predictors have the right sign, including the dividend yield.

Panel B focuses on multivariate regressions. This is a very tough test of the model as it implicitly requires the model to also fit the correlation among the three instruments. Nevertheless, for consumption growth volatility the model gets all the signs right and every coefficient is within two standard errors of the data coefficient. The model also produces a fantastic fit with respect to time-variation in return predictability. However, the fit with respect to dividend growth volatility is not as stellar with two of three signs missed.

Panel C focuses on skewness and kurtosis. The model implied kurtosis of filtered dividend growth is consistent with that found in the data and the model produces a bit too much kurtosis in consumption growth rates. Equity return kurtosis is somewhat too low relative to the data, but almost within a 2 standard error bound. The model produces realistic skewness numbers for all three series. We conclude that the nonlinearities in the fundamentals implied by the model are reasonably consistent with the data.

## 6 Risk, Uncertainty and Asset Prices

In this section, we explore the dominant sources of time variation in equity prices (dividend yields), equity returns, the term structure, expected equity returns and the conditional volatility of equity returns. We also investigate the mechanisms leading to our findings.

Tables 5 and 6 contain the core results in the paper. Table 5 reports basic properties of some critical unobserved variables, including  $v_t$  and  $q_t$ . Table 6 reports variance decompositions with standard errors for several endogenous variables of interest and essentially summarizes the response of the endogenous variables to each of the state variables. Rather than discussing these tables in turn, we organize our discussion around the different variables of interest using information from the two tables.

## 6.1 Uncertainty and Risk

Table 5, Panel A presents properties of unobservable variables under the estimated model. First, note that ‘uncertainty,’  $v_t$ , which is proportional to the conditional volatility of dividend growth is quite volatile relative to its mean and is extremely persistent. These properties reflect the identifying information in the characteristics of the dividend yield, short rate and spread as well as the higher moments of fundamentals. Similarly,  $q_t$  has significant volatility and autocorrelation. Because local risk aversion,  $RA_t$ , in this model is given by  $\gamma \exp(q_t)$ , we can examine its properties directly. The median level of risk aversion in the model is 2.52, a level which would be considered perfectly reasonable by most financial economists. However, risk aversion is positively skewed and has large volatility so that risk aversion is occasionally extremely high in this model.

Panel B of Table 5 presents results for means of the above endogenous variables conditional on whether the economy is in a state of expansion or recession. For this exercise, recession is defined as one quarter of negative consumption growth. Both  $v_t$  and  $q_t$  (and hence local risk aversion) are strongly counter-cyclical.

## 6.2 Uncertainty, Risk and the Term Structure

Panel A of Table 5 also displays the properties of the real interest rate and the real term spread. The average real rate is 17 basis points (68 annualized) and the real interest rate has a standard deviation of around 90 basis points. The real term spread has a mean of 38 basis points, a volatility of only 28 basis points and is about as persistent as the real short rate. In Panel B, we see that real rates are pro-cyclical and spreads are counter-cyclical.

Panel C of Table 5 shows that uncertainty tends to depress real interest rates, while positive risk aversion shocks tend to increase them. In the theoretical section, we derived that the effect of  $q_t$  on real interest rates is ambiguous depending on whether the consumption smoothing or precautionary savings effect dominates. At our parameter values, the consumption smoothing effect dominates. The effect of  $v_t$  is entirely through the volatility of the pricing kernel and represents a precautionary savings motive. Hence, the correlation between real rates and  $q_t$  is actually positive, while the correlation between real rates and  $v_t$  is negative. Overall, real rates are pro-cyclical because  $v_t$  is strongly counter-cyclical.

The real term spread displays a positive correlation with both  $v_t$  and  $q_t$ , but for different reasons.

As we discussed in the theoretical section, if the expectations hypothesis were to hold, mean reversion would imply that the effect of either variable on the spread would be the opposite sign of its effect on the interest rate level. Figure 1 decomposes the exposures of both the real interest rate and the spread to  $v_t$  and  $q_t$  into an expectations hypothesis part and a term premium part and does so for various maturities (to 40 quarters). The exposure to  $v_t$  is negative and weakens with horizon leading to a positive EH effect. Because  $v_t$  has little effect on the term premium, the spread effect remains positive. Hence, when uncertainty increases, the term structure steepens and vice versa.

Figure 1 also shows why  $q_t$  has a positive effect on the real term spread, despite the EH effect being negative. Yields at long maturities feature a term premium that is strongly positively correlated with  $q_t$ . In the theoretical section, we derived that the sign of the term premium only depends on  $\phi_{rq}$  which is positive: because higher risk aversion increases interest rates (and lowers bond prices) at a time when marginal utility is high, bonds are risky.

In Table 6, we report the variance decompositions. While three factors ( $u_t$ ,  $v_t$ , and  $q_t$ ) affect the real term structure,  $v_t$  accounts for the bulk of its variation. An important reason for this fact is that  $v_t$  is simply more variable than  $q_t$ . The most interesting aspect of the results here is that  $q_t$  contributes little to the variability of the spread, so that  $q_t$  is mostly a level factor not a spread factor, whereas uncertainty is both a level and a spread factor. When we consider a real consol, we find that  $q_t$  dominates its variation. Because consol prices reflect primarily longer term yields, they are primarily driven by the most persistent level factor, which is  $q_t$ , through its effect on the term premium.

For the nominal term structure, inflation becomes an important additional state variable accounting for about 12% of the variation in the nominal interest rates. However, inflation is an even more important spread factor accounting for about 31% of the spread's variability. What may be surprising is that the relative importance of  $q_t$  increases going from the real to nominal term structure. The reason is the rather strong positive correlation between inflation and  $v_t$ , which arises from the negative relation between inflation and the consumption dividend ratio, that ends up counterbalancing the negative effect of  $v_t$  on real interest rates.

### 6.3 Uncertainty, Risk, and Equity Prices

Here we start with the variance decompositions for dividend yields and equity returns in Table 6. For the dividend yield,  $q_t$  dominates as a source of variation. The contribution of  $q_t$  to variation

in the dividend yield is almost 90%. To see why, recall first that  $q_t$  only affects the dividend yield through its effect on the term structure of real interest rates (see Proposition 4). Under the parameters presented in Table 2, the impact of  $q_t$  on real interest rates is positive at every horizon and therefore it is positive for the dividend yield as well. Formally, under the parameters of Table 2,  $\widehat{F}_n$  in Proposition 3 is negative at all horizons.

Next, consider the effect of  $v_t$  on the dividend yield. Uncertainty has a ‘real consol effect’ and a ‘cash-flow risk premium’ effect which offset each other. We already know that  $v_t$  creates a strong precautionary savings motive, which decreases interest rates. All else equal, this will serve to increase price-dividend ratios and decrease dividend yields. However,  $v_t$  also governs the covariance of dividend growth with the real kernel. This risk premium effect may be positive or negative, but intuitively the dividend stream will represent a risky claim to the extent that dividend growth covaries negatively with the kernel. For instance, if dividend growth is low in states of the world where marginal utility is high, then the equity claim is risky. In this case, we would expect high  $v_t$  to exacerbate this riskiness and depress equity prices when it is high, increasing dividend yields. As we discussed in section 5.5,  $\sigma_{qc}$  contributes to this negative covariance. On balance, these countervailing effects of  $v_t$  on dividend yield largely cancel out, so that the net effect of  $v_t$  on dividend yields is small. This shows up in the variance decomposition of the dividend yield. On balance,  $q_t$  is responsible for the overwhelming majority of dividend yield variation, and is highly positively correlated with it. The negative effect of  $u_t$  arises from its strong negative covariance with dividend growth.

Looking back to panel C in Table 5, while increases in  $q_t$  have the expected depressing effect on equity prices (a positive correlation with dividend yields), increases in  $v_t$  do not. This contradicts the findings in Wu (2001) and Bansal and Yaron (2004) but is consistent with early work by Barsky (1988) and Naik (1994). Because the relation is only weakly negative, there may be instances where our model will generate a classic “flight to quality” effect with uncertainty lowering interest rates, driving up bond prices and depressing equity prices.

Next notice the determinants of realized equity returns in Table 6. First, over 30% of the variation in excess returns is driven by dividend growth and dividend growth is positively correlated with excess returns. This is not surprising in light of the fact that dividend growth enters the definition of stock returns directly and dividend growth has almost half as much variation as returns themselves. The other primary driver of stock returns is  $q_t$ . This is a compound statistic which

includes the effect of current and lagged  $q_t$ . In fact, the contemporaneous effect of  $q_t$  on returns is negative (see Table 5) as increases in  $q_t$  depress stock valuations. However, the lagged effect of  $q_t$  on returns is positive because, all else equal, lower lagged prices imply higher current returns.

## 6.4 Uncertainty, Risk and the Equity Premium

We again go back to Table 5 to investigate the properties of the conditional equity premium,  $E_t[rx_{t+1}]$ . The premium is quite persistent, with an autocorrelation coefficient of 0.9789. In Panel B, we also find that expected equity returns are higher in recessions which is consistent with counter-cyclical risk aversion. Panel C shows that both  $v_t$  and  $q_t$  are positively correlated with the equity premium. The risk premium in any model will be negatively correlated with the covariance between the pricing kernel and returns. We already discussed how uncertainty is negatively correlated with cash flows and this dominates the small positive correlation with price-dividend ratios. The effect of  $q_t$  comes mostly through the capital gain part of the return: increases in  $q_t$  both raise marginal utility and lower prices making stocks risky. Table 6 shows that the point estimate for the share of the equity premium variation due to  $v_t$  is about 17% but with a standard error of 13%, with the remainder due to  $q_t$ .

The fact that both the dividend yield and expected equity returns are primarily driven by  $q_t$  suggest that the dividend yield may be a strong predictor of equity returns in this model. Table 7 shows that this is indeed the case, with a regression of future returns on dividend yields generating a 1.53 coefficient. We also compare the model coefficients with the corresponding statistics in the data. It turns out that the predictability of equity returns during our sample period is rather weak. Table 7 reports univariate coefficients linking equity returns to short rates, dividend yields and spreads. The sign of the coefficients matches well-known stylized facts but none of the coefficients are significantly different from zero. The model produces coefficients within two standard errors of these data coefficients but this is of course a rather weak test. While it is theoretically possible to generate a negative link between current short rates and the equity premium which is observed empirically, our model fails to do so at the estimated parameters. We also report the results of a multivariate regression on the aforementioned instruments. The model here gets all the signs right and is always within two standard errors of the data coefficients. More generally, the ratio,  $VAR(E_t[rx_{t+1}])/VAR(rx_{t+1})$ , from Tables 3 and 5, implies a quarterly  $R^2$  of less than one percent, so the model does not generate much short term predictability of equity returns consistent with

recent evidence. There is a large debate on whether predictability increases with the horizon. In our model, the variance ratio discussed above for 10 year returns equals about 12% (not reported).

While we have studied the conditional equity premium, it remains useful to reflect on the success of the model in matching the unconditional equity premium. To make the model's success complete, it also matches the low risk free rate while keeping the correlation between fundamentals (dividend and consumption growth) and returns low. In fact, the correlation between dividend growth and equity returns is 0.28 in the data and 0.33 in the model. For consumption growth, the numbers are 0.07 and 0.11 respectively. Consequently, this model performs in general better than the Campbell and Cochrane (1999) model, which had trouble with the fundamentals-return correlation. The success of our model is primarily driven by the added flexibility offered by an additional state variable. When we set  $\sigma_{qq} = 0$ , and re-estimate the model, the model is strongly rejected and we fail to match the high equity premium and the low risk free rate. Consequently, while we have formulated a consumption-based asset pricing model that successfully matches many salient asset pricing phenomena, the presence of preference shocks *not* correlated fundamental shocks are essential to its success.

## 6.5 Uncertainty, Risk, Equity Return Volatility and Sharpe Ratios

To conclude, we investigate the properties of the conditional variance of equity returns, the equity Sharpe ratio and the maximum attainable Sharpe ratio available in the economy discussed in Section 2. We begin with the numbers in Panel A of Table 5. The conditional variance of excess equity returns has a mean of 0.0092, a standard deviation of 0.0070 and an autocorrelation of 0.9794 at the quarterly frequency. The final two columns of Table 5 report results for the conditional Sharpe ratio of equity and the maximum attainable Sharpe ratio available in the economy discussed in Section 2. The mean equity Sharpe ratio attains approximately three quarters of the maximum attainable value. Both Sharpe ratios are strongly persistent and possess significant time variation driven by  $v_t$  and  $q_t$ . These Sharpe ratios are quarterly, and so their magnitude is roughly half of annualized values.

The conditional variance of equity returns is counter-cyclical. Interestingly, the increase in expected equity returns during recessions is not as large as the increase in the expected variance which contributes to the equity Sharpe ratio being not counter-cyclical. The maximum Sharpe ratio does display counter-cyclical behavior.

Moving to Table 6, not surprisingly, the conditional volatility of equity returns is largely governed largely by  $v_t$ , which accounts for 75% of its variation with a standard error of only 32%. Here,  $q_t$  contributes 25% to the total volatility variation.

## 7 Conclusion

This paper has attempted to sort out the relative importance of two competing hypotheses for the sources of the magnitude and variation of asset prices. First, one literature has explored the role of cash flow volatility dynamics as a determinant of equity premiums both in the time series and cross section. Recent work in this area includes Wu (2000), Bansal and Yaron (2003), Bansal, Khatchatrian and Yaron (2002), and Bansal and Lundblad (2004). A quite separate literature has explored shocks to investors preferences as drivers of equity prices. Prominent papers in this area include Campbell and Cochrane (1999), Abel (1990, 1999), and a large number of elaborations such as Wachter (2004), Bekaert, Engstrom and Grenadier (2004), Brandt and Wang (2003), Menzly, Santos and Veronesi (2004), Wei (2004), and Lustig and van Nieuwerburgh (2004). With some exceptions, the focus has been on equities.

We design a theoretical model and empirical strategy which are capable of accommodating both explanations, and then implement an optimal GMM estimation to determine the relative importance of each story. We stress that from a theoretical perspective, it is important to consider the term structure effects on equity prices, a point prominent in the work of Abel (1988) and Barsky (1989). We conclude that both the conditional volatility of cash flow growth and time varying risk aversion emerge as important factors driving variation in the term structure, dividend yields, and equity risk premium and the conditional volatility of returns. Not surprisingly, uncertainty is more important for volatility whereas risk aversion is more important for dividend yields and the risk premium.

Our work is indirectly related to two other important literatures. First, there is a large literature on the conditional CAPM which predicts a linear, positive relation between expected excess returns on the market and the conditional variance of the market. Since the seminal work of French, Schwert and Stambaugh (1987), the literature has struggled with the identification of the price of risk, which is often negative in empirical applications (see Scruggs (2003)). Of course, in our model, there are multiple sources of time variation in risk premiums and both the price of risk and the quantity of risk varying through time. Upon estimation of our structural model, we identify a

strong positive contemporaneous correlation between expected equity returns and their conditional volatility. However, this relationship varies through time and contains a cyclical component (see Table 5).

Second, the volatility feedback literature has provided a link between the phenomenon of asymmetric volatility (or the leverage effect, the conditional return volatility and price shocks are negatively correlated) and risk premiums. It suggests that prices can fall precipitously on negative news as the conditional volatility increases and hence induces higher risk premiums (when the price of risk is positive). Hence, the literature primarily builds on the conditional CAPM literature (see Campbell and Hentschel (1992) and Bekaert and Wu (2000)). Wu (2001) sets up a present value model in which the variance of dividend growth follows a stochastic volatility process and shows under what conditions the volatility feedback effect occurs. There are two reasons why Wu's (2001) conclusions may not be generally valid. First, he ignores equilibrium considerations—that is the discount rate is not tied to preferences. Tauchen (2005) also shows how the presence of feedback may depend on preference parameters. Second, he assumes a constant interest rate. Within our set up, we can re-examine the validity of an endogenous volatility feedback effect. We intend to explore the implications of our model for these two literatures in the near future.

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# A Proof of Propositions

## A.1 Proof of Proposition 1

The well known recursive pricing relationship governing the term structure of these bond prices is

$$P_{n,t}^{rz} = E_t [M_{t+1} P_{n-1,t+1}^{rz}] \quad (35)$$

where  $P_{n,t}^{rz}$  is the price of a real zero coupon bond at time  $t$  with maturity at time  $(t+n)$ . The following proposition summarizes the solution for these bond prices. We solve the model for a slightly generalized (but notation saving) case where the  $q_t = \mu_q + \rho_{qq}q_{t-1} + \sqrt{v_{t-1}}(\sigma_{qd}\varepsilon_t^d + \sigma_{qu}\varepsilon_t^u + \sigma_{qv}\varepsilon_t^v) + \sqrt{q_{t-1}}\sigma_{qq}\varepsilon_t^q$ . Our current model obtains when

$$\begin{aligned} \sigma_{qd} &= \sigma_{qc}\sigma_{dd}(1 + \sigma_{ud}) \\ \sigma_{qu} &= \sigma_{qc}\sigma_{uu} \\ \sigma_{qv} &= \sigma_{qc}\sigma_{dv}(1 + \sigma_{ud}). \end{aligned} \quad (36)$$

Suppose the prices of real, risk free, zero coupon bonds are given by

$$P_{n,t}^{rz} = \exp(A_n + B_n\Delta d_t + C_nu_t + D_n\pi_t + E_nv_t + F_nq_t) \quad (37)$$

where

$$\begin{aligned} A_n &= f^A(A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ B_n &= 0 \\ C_n &= f^C(A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ D_n &= 0 \\ E_n &= f^E(A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \\ F_n &= f^F(A_{n-1}, B_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi) \end{aligned}$$

Then we have

$$\begin{aligned} &\exp(A_n + B_n\Delta d_t + C_nu_t + D_n\pi_t + E_nv_t + F_nq_t) \\ &= E_t\{\exp(m_{t+1} + A_{n-1} + B_{n-1}\Delta d_{t+1} + C_{n-1}u_{t+1} + D_{n-1}\pi_{t+1} + E_{n-1}v_{t+1} + F_{n-1}q_{t+1})\} \\ &= E_t\{\exp(\ln(\beta) - \gamma(\delta + \Delta u_{t+1} + \Delta d_{t+1}) + \gamma\Delta q_{t+1} \\ &\quad + A_{n-1} + B_{n-1}\Delta d_{t+1} + C_{n-1}u_{t+1} + D_{n-1}\pi_{t+1} + E_{n-1}v_{t+1} + F_{n-1}q_{t+1})\} \end{aligned}$$

Equating the coefficient on the two side of the equation, we get:

$$\begin{aligned}
f^A &= \ln \beta - \gamma \delta + A_{n-1} + (B_{n-1} - \gamma) \mu_d + E_{n-1} \mu_v + (F_{n-1} + \gamma) \mu_q \\
f^C &\equiv ((B_{n-1} - \gamma) \rho_{du} + C_{n-1} \rho_{uu} + \gamma (1 - \rho_{uu})) \\
f^E &\equiv E_{n-1} \rho_{vv} \\
&+ \frac{1}{2} ((B_{n-1} - \gamma) \sigma_{dd} + (C_{n-1} - \gamma) \sigma_{ud} \sigma_{dd} + (F_{n-1} + \gamma) \sigma_{qd})^2 \\
&+ \frac{1}{2} ((C_{n-1} - \gamma) \sigma_{uu} + (F_{n-1} + \gamma) \sigma_{qu})^2 \\
&+ \frac{1}{2} ((B_{n-1} - \gamma) \sigma_{dv} + (C_{n-1} - \gamma) \sigma_{ud} \sigma_{dv} + (F_{n-1} + \gamma) \sigma_{qv} + E_{n-1} \sigma_{vv})^2 \\
f^F &\equiv \left( E_{n-1} \rho_{qq} + \gamma (\rho_{qq} - 1) + \frac{1}{2} ((F_{n-1} + \gamma) \sigma_{qq})^2 \right)
\end{aligned}$$

Proof for Proposition 3 follows the same strategy as above.

## A.2 Proof of Proposition 4

Let  $P_t$  and  $D_t$  be the time- $t$  ex-div stock price and dividend.

Guess

$$J_{n,t} \triangleq E_t \exp \left[ \sum_{j=1}^n (m_{t+j} + \Delta d_{t+j}) \right] = \exp \left( \hat{A}_n + \hat{B}_n \Delta d_t + \hat{C}_n u_t + \hat{E}_n v_t + \hat{F}_n q_t \right)$$

Then

$$\begin{aligned}
J_{n,t} &= E_t \left[ \exp(m_{t+1} + \Delta d_{t+1}) E_{t+1} \sum_{j=1}^{n-1} \exp(m_{t+1+j} + \Delta d_{t+1+j}) \right] \\
&= E_t [\exp(m_{t+1} + \Delta d_{t+1}) J_{n-1,t+1}]
\end{aligned}$$

$$\begin{aligned}
&\exp \left( \hat{A}_n + \hat{B}_n \Delta d_t + \hat{C}_n u_t + \hat{E}_n v_t + \hat{F}_n q_t \right) \\
&= E_t \{ \exp[\ln(\beta) - \gamma(\delta + \Delta u_{t+1} + \Delta d_{t+1}) + \gamma \Delta q_{t+1} + \Delta d_{t+1} \\
&\quad + \hat{A}_{n-1} + \hat{B}_{n-1} \Delta d_{t+1} + \hat{C}_{n-1} u_{t+1} + \hat{E}_{n-1} v_{t+1} + \hat{F}_{n-1} q_{t+1}] \}
\end{aligned}$$

Using the property of lognormality distribution and equating coefficients on both side of the equation gives us:

$$\begin{aligned}
\widehat{A}_n &= f^A \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \mu_d \\
\widehat{B}_n &= 0 \\
\widehat{C}_n &= f^C \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \rho_{du} \\
\widehat{E}_n &= f^E \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) \\
&\quad + \frac{1}{2} \left( \sigma_{dd}^2 + \sigma_{dd} \left( \left( \widehat{B}_{n-1} - \gamma \right) \sigma_{dd} + \left( \widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dd} + \left( \widehat{F}_{n-1} + \gamma \right) \sigma_{qd} \right) \right) \\
&\quad + \frac{1}{2} \left( \sigma_{dv}^2 + \sigma_{dv} \left( \left( \widehat{B}_{n-1} - \gamma \right) \sigma_{dv} + \left( \widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dv} + \left( \widehat{F}_{n-1} + \gamma \right) \sigma_{qv} + \widehat{E}_{n-1} \sigma_{vv} \right) \right) \\
\widehat{F}_n &= f^F \left( \widehat{A}_{n-1}, \widehat{B}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right)
\end{aligned}$$

where the functions  $f^X(\cdot)$  are given in Proposition 1 for  $X \in (A, B, C, E, F)$  and  $A_0 = B_0 = C_0 = E_0 = F_0 = 0$ .

For the purposes of estimation the coefficient sequences are calculated out 200 years. If the resulting calculated value for  $PD_t$  has not converged, then the sequences are extended another 100 years until either the  $PD_t$  value converges, or becomes greater than 1000 in magnitude.

## B Log Linear Approximation of Equity Prices

In the estimation, we use a linear approximation to the price-dividend ratio. From Equation (??), we see that the price dividend ratio is given by

$$\begin{aligned}\frac{P_t}{D_t} &= \sum_{n=1}^{\infty} q_{n,t}^0 \\ &= \sum_{n=1}^{\infty} \exp(b_n^0 + b'_n Y_t)\end{aligned}\quad (38)$$

and the coefficient sequences,  $\{b_n^0\}_{n=1}^{\infty}$  and  $\{b'_n\}_{n=1}^{\infty}$ , are given above. We seek to approximate the log price-dividend ratio using a first order Taylor approximation of  $Y_t$  about  $\bar{Y}$ , the unconditional mean of  $Y_t$ . Let

$$\bar{q}_n^0 = \exp(b_n^0 + b'_n \bar{Y}) \quad (39)$$

and note that

$$\frac{\partial}{\partial Y_t} \left( \sum_{n=1}^{\infty} q_{n,t}^0 \right) = \sum_{n=1}^{\infty} \frac{\partial}{\partial Y_t} q_{n,t}^0 = \sum_{n=1}^{\infty} q_{n,t}^0 \cdot b'_n \quad (40)$$

Approximating,

$$\begin{aligned}pd_t &\simeq \ln \left( \sum_{n=1}^{\infty} \bar{q}_n^0 \right) + \frac{1}{\sum_{n=1}^{\infty} \bar{q}_n^0} \left( \sum_{n=1}^{\infty} \bar{q}_n^0 \cdot b'_n \right) (Y_t - \bar{Y}) \\ &= d_0 + d' Y_t\end{aligned}\quad (41)$$

where  $d_0$  and  $d'$  are implicitly defined. Similarly,

$$\begin{aligned}gpd_t &\equiv \ln \left( 1 + \frac{P_t}{D_t} \right) \simeq \ln \left( 1 + \sum_{n=1}^{\infty} \bar{q}_n^0 \right) + \frac{1}{1 + \sum_{n=1}^{\infty} \bar{q}_n^0} \left( \sum_{n=1}^{\infty} \bar{q}_n^0 \cdot b'_n \right) (Y_t - \bar{Y}) \\ &= h_0 + h' Y_t\end{aligned}\quad (42)$$

where  $h_0$  and  $h'$  are implicitly defined. Note also that the dividend yield measure used in this study can be expressed as follows

$$dp_t \equiv \ln \left( 1 + \frac{D_t}{P_t} \right) = gpd_t - pd_t \quad (43)$$

so that it is also linear in the state vector under these approximations. Also, log excess equity returns can be represented follows. Using the definition of excess equity returns,

$$\begin{aligned}rx_{t+1} &= -rf_t - pd_t + gd_{t+1} + \pi_{t+1} + gpd_{t+1} \\ &\sim (h_0 - d_0) + (e'_d + e'_\pi + h') Y_{t+1} + (-e'_{rf} - d') Y_t \\ &= r_0 + r'_1 Y_{t+1} + r'_2 Y_t\end{aligned}\quad (44)$$

where  $r_0$ ,  $r'_1$  and  $r'_2$  are implicitly defined.

### B.1 Accuracy of the Equity Approximation

To assess the accuracy of the log linear approximation of the price dividend ratio, the following experiment was conducted. For the model and point estimates reported in Table 2, a simulation was run for 10,000 periods. In each period, the 'exact' price dividend ratio and log dividend yield

were calculated in addition to their approximate counterparts derived in the previous subsection. The resulting series for exact and approximate dividend yields and excess stock returns compare as follows (quarterly rates).

	appx $dp_t$	exact $dp_t$	appx $r_t^x$	exact $r_t^x$
mean	0.0099	0.0100	0.0118	0.0119
std. dev.	0.0032	0.0034	0.0945	0.0891
correlation	0.9948		0.9853	

## C Analytic Moments of $Y_t$ and $W_t$

Recall that the data generating process for  $Y_t$  is given by,

$$\begin{aligned} Y_t &= \mu + AY_{t-1} + (\Sigma_F F_{t-1} + \Sigma_H) \varepsilon_t \\ F_t &= \text{sqr}t(\text{diag}(\phi + \Phi Y_t)) \end{aligned} \quad (45)$$

It is straightforward to show that the uncentered first, second, and first autocovariance moments of  $Y_t$  are given by,

$$\begin{aligned} \overline{Y_t} &= (I_k - A)^{-1} \mu \\ \text{vec}(\overline{Y_t Y_t'}) &= (I_{k^2} - A \otimes A)^{-1} \cdot \text{vec}(\mu \mu' + \mu \overline{Y_t'} A' + A \overline{Y_t} \mu' + \Sigma_F \overline{F_t^2} \Sigma_F' + \Sigma_H \Sigma_H') \\ \text{vec}(\overline{Y_t Y_{t-1}'}) &= (I_{k^2} - A \otimes A)^{-1} \cdot \text{vec}(\mu \mu' + \mu \overline{Y_t'} A' + A \overline{Y_t} \mu' + A (\Sigma_F \overline{F_t^2} \Sigma_F' + \Sigma_H \Sigma_H')) \end{aligned} \quad (46)$$

where overbars denote unconditional means and  $\overline{F_t^2} = \text{diag}(\phi + \Phi \overline{Y_t})$ .

Now consider the unconditional moments of a n-vector of observable variables  $W_t$  which obey the condition

$$W_t = \mu^w + \Gamma^w Y_{t-1} + (\Sigma_F^w F_{t-1} + \Sigma_H^w) \varepsilon_t \quad (47)$$

where  $\mu^w$  is an n-vector and  $\Sigma_F^w$ ,  $\Sigma_H^w$  and  $\Gamma^w$  are  $(n \times k)$  matrices. It is straightforward to show that the uncentered first, second, and first autocovariance moments of  $W_t$  are given by,

$$\begin{aligned} \overline{W_t} &= \mu^w + \Gamma^w \overline{Y_t} \\ \overline{W_t W_t'} &= \mu^w \mu^{w'} + \mu^w \overline{Y_t'} \Gamma^{w'} + \Gamma^w \overline{Y_t} \mu^{w'} + \Gamma^w \overline{Y_t Y_t'} \Gamma^{w'} + \Sigma_F^w \overline{F_t^2} \Sigma_F^{w'} + \Sigma_H^w \Sigma_H^{w'} \\ \overline{W_t W_{t-1}'} &= \mu^w \mu^{w'} + \mu^w \overline{Y_t'} \Gamma^{w'} + \Gamma^w \overline{Y_t} \mu^{w'} + \Gamma^w \overline{Y_t Y_{t-1}'} \Gamma^{w'} + \Gamma^w (\Sigma_F^w \overline{F_t^2} \Sigma_F^{w'} + \Sigma_H^w \Sigma_H^{w'}) \end{aligned} \quad (48)$$

It remains to demonstrate that the observable series used in estimation obey Equation (47). This is trivially true for elements of  $W_t$  which are also elements of  $Y_t$  such as  $\Delta d_t$ ,  $\Delta c_t$ ,  $\pi_t$ . Using Equations (25), (44) and (41), it is apparent that  $r_t^f$ ,  $dp_t$  and  $r_t^x$  satisfy Equation (47) as well.

Table 1: Heteroskedasticity in Fundamentals

Panel A: Univariate Regressions

	$(\Delta d_t^f)^2$	$(\Delta c_t^f)^2$	$(rx_t)^2$
$v_1$	-0.0779 (0.0260)	-0.0062 (0.0016)	-0.4620 (0.2740)
$v_2$	0.1112 (0.0420)	0.0106 (0.0039)	1.8843 (1.1279)
$v_3$	0.0984 (0.0485)	0.0080 (0.0034)	0.7759 (0.6435)

Panel B: Multivariate Regressions

	$(\Delta d_t^f)^2$	$(\Delta c_t^f)^2$	<i>restricted</i>	
$v_1$	-0.0621 (0.0214)	-0.0046 (0.0014)	-0.0593 (0.0205)	
$v_2$	0.0737 (0.0364)	0.0079 (0.0036)	0.0824 (0.0304)	
$v_3$	0.0438 (0.0334)	0.0040 (0.0031)	0.0437 (0.0296)	
$\eta$			0.0807 (0.0329)	
<i>tests</i>				
$v_1, v_2, v_3$	10.50	22.54	<i>Jstat</i>	0.2196
( <i>pval</i> )	(0.0148)	(< 0.0001)	( <i>pval</i> )	(0.8958)
<i>second moment change during recession</i>				
	+45%	+33%		+38%

## Table 1: Notes

The symbols  $\Delta d_t^f$ ,  $\Delta c_t^f$ , and  $rx_t$  refer to log filtered dividend and consumption growth and log excess equity returns. The table presents regressions of squared future values of these variables onto a set of three instruments: the log yield on a 90 day T-bill,  $rf_t$ , the filtered log dividend yield,  $dp_t^f$ , and the log yield spread,  $spd_t$ . The regressions are of the form:

$$x_{t+k}^2 = \nu_0 + \nu_1 rf_t + \nu_2 dp_t^f + \nu_3 spd_t + \varepsilon_{t+k}$$

The dependent variables are  $(\Delta d_t^f)^2$ ,  $(\Delta c_t^f)^2$  and  $(rx_t)^2$ . For consumption and dividend growth,  $k = 4$ , and 4 Newey West lags are used in the GMM estimation, but for returns,  $k = 1$  and no Newey West lags are used (standard errors are reported in parentheses throughout). In Panel A, the regression is univariate; only one instrument is used per regression. Panel B reports the full multivariate regressions for  $x_{t+k}^2 = (\Delta d_{t+k}^f)^2$  and  $x_{t+k}^2 = (\Delta c_{t+k}^f)^2$  in the first two columns. The first line below the column reports a joint test of the null of no predictability of the (uncentered) second moment with the p-value in parentheses. The test statistic is distributed as  $\chi^2(3)$ . The third column presents results for a restricted specification where the slope coefficients are proportional across the  $(\Delta d_t^f)^2$  and  $(\Delta c_t^f)^2$  equations such that  $\eta \nu_1^{div} = \nu_1^{cons}$ , etc. A likelihood ratio test is presented at the bottom of the third column which tests this restriction (p-value in parentheses). The final row of Panel B reports the percentage change in fitted squared variation during NBER defined recessions. Data are quarterly US aggregates from 1927:1-2004:3.

Table 2: Dynamic Risk and Uncertainty Model Estimation

Parameter Estimates			
$E[\Delta d]$	$\rho_{du}$		$\sigma_{dd}$
0.0039	0.0214		0.0411
(0.0011)	(0.0082)		(0.0116)
			$\sigma_{dv}$
			0.0413
			(0.0130)
$E[v_t]$	$\rho_{vv}$		$\sigma_{vv}$
1.0000	0.9795		0.3288
(fixed)	(0.0096)		(0.0785)
	$\rho_{uu}$		$\sigma_{ud}$
	0.9826		-0.9226
	(0.0071)		(0.0233)
			$\sigma_{uu}$
			0.0127
			(0.0007)
$E[q_t]$	$\rho_{qq}$		$\sigma_{qc}$
1.0000	0.9787		-5.2211
(fixed)	(0.0096)		(4.5222)
			$\sigma_{qq}$
			0.1753
			(0.0934)
$\ln(\beta)$	$\gamma$	$\delta$	
-0.0168	1.1576	0.0047	
(0.0042)	(0.7645)	(0.0011)	
$E[\pi_t]$	$\rho_{\pi\pi}$	$\rho_{\pi u}$	$\sigma_{\pi\pi}$
0.0081	0.2404	-0.0203	0.0086
(0.0010)	(0.1407)	(0.0073)	(0.0017)
Overidentification Test			
$J(15)$	12.7262		
(pval)	(0.6234)		

Table 2: Notes

The model is defined by the equations

$$\begin{aligned}
\Delta d_t &= \mu_d + \rho_{du} u_{t-1} + \sqrt{v_{t-1}} (\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v) \\
v_t &= \mu_v + \rho_{vv} v_{t-1} + \sigma_{vv} \sqrt{v_{t-1}} \varepsilon_t^v \\
u_t &= \mu_u + \rho_{uu} u_{t-1} + \sigma_{ud} (\Delta d_t - E_{t-1} [\Delta d_t]) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u \\
q_t &= \mu_q + \rho_{qq} q_{t-1} + \sigma_{qc} (\Delta c_t - E_{t-1} [\Delta c_t]) + \sigma_{qq} \sqrt{q_{t-1}} \varepsilon_t^q \\
\pi_t &= \mu_\pi + \rho_{\pi\pi} \pi_{t-1} + \rho_{\pi u} u_{t-1} + \sigma_\pi \varepsilon_t^\pi \\
\Delta c_t &= \delta + \Delta d_t + \Delta u_t \\
&= (\delta + \mu_d) + (\rho_{ud} + \rho_{uu} - 1) u_t + (1 + \sigma_{ud}) \sqrt{v_{t-1}} (\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u \\
m_{t+1} &= \ln(\beta) - \gamma \Delta c_{t+1} + \gamma \Delta q_{t+1}
\end{aligned}$$

The moments used to estimate the model are

$$\begin{aligned}
&E \left[ \Delta d_t^f, \Delta c_t^f, \pi_t, r f_t, dp_t^f, spd_t, r_t^{ex} \right] \quad (7) \\
&E \left[ \left( \Delta d_t^f \right)^2, \left( \Delta c_t^f \right)^2, \left( \pi_t \right)^2, \left( r f_t \right)^2, \left( dp_t^f \right)^2, \left( spd_t \right)^2, \left( r_t^{ex} \right)^2 \right] \quad (7) \\
&E \left[ \left( \pi_t \pi_{t-1} \right), \left( \pi_t \pi_{t-4} \right), \left( \Delta d_t^f \Delta d_{t-4}^f \right), \left( \Delta c_t^f \Delta c_{t-4}^f \right) \right] \quad (4) \\
&E \left[ \left( \Delta d_t^f \Delta c_t^f \right), \left( \Delta d_t^f r f_t \right), \left( \Delta d_t^f dp_t \right), \left( \Delta d_t^f spd_t \right) \right] \quad (4) \\
&E \left[ \left( \Delta c_t^f r f_t \right), \left( \Delta c_t^f dp_t \right), \left( \Delta c_t^f spd_t \right) \right] \quad (3) \\
&E \left[ \left( \pi_t \Delta c_t^f \right), \left( \pi_t r f_t \right) \right] \quad (2) \\
&E \left[ \left( \Delta d_t^f \right)^2 \otimes \left( r f_{t-4}, dp_{t-4}^f, spd_{t-4} \right) \right] \quad (3) \\
&E \left[ \left( \Delta d_t^f \right)^3, \left( \Delta d_t^f \right)^4, \left( \Delta c_t^f \right)^3, \left( \Delta c_t^f \right)^4 \right] \quad (4)
\end{aligned}$$

The model is estimated by GMM. Data are quarterly US aggregates from 1927:1-2004:3.  $\Delta d_t^f$ ,  $\Delta c_t^f$ ,  $\pi_t$ ,  $r f_t$ ,  $dp_t^f$ ,  $spd_t$ , and  $r x_t$  refer to filtered log dividend growth, filtered log consumption growth, log inflation, the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, and log excess equity returns (with respect to the 90 day T-bill). See text for data construction and estimation details.

Table 3: The Fit of the Model: Linear Moments

Simulated observable moments							
	$\Delta d_t^f$	$\pi_t$	$\Delta c_t^f$	$r f_t$	$dp_t^f$	$spd_t$	$rx_t$
mean	[0.0038] 0.0026 (0.0029)	[0.0084] 0.0077 (0.0013)	[0.0085] 0.0080 (0.0008)	[0.0097] 0.0094 (0.0010)	[0.0096] 0.0099 (0.0004)	[0.0038] 0.0040 (0.0004)	[0.0121] 0.0141 (0.0062)
std.dev.	[0.0291] 0.0308 (0.0034)	[0.0121] 0.0130 (0.0015)	[0.0068] 0.0075 (0.0009)	[0.0074] 0.0078 (0.0007)	[0.0035] 0.0035 (0.0003)	[0.0033] 0.0032 (0.0002)	[0.0967] 0.1085 (0.0126)
autocorr	[-0.0275]* 0.0699 (0.0995)	[0.5837] 0.6016 (0.0802)	[0.0233]* 0.2460 (0.2008)	[0.9170] 0.9582 (0.0356)	[0.9429]* 0.9347 (0.1751)	[0.6840] 0.8107 (0.0618)	[-0.0071] -0.0446 (0.1004)

Simulated moments, in square brackets, are calculated by simulating the system for 100,000 periods using the point estimates from Table 2 and calculating sample moments of the simulated data. Autocorrelations are all at one lag except for series denote with an asterisk (\*): dividend growth, consumption growth and the dividend price ratio, which are calculated at 4 lags. The second and third numbers for each entry are the sample moments and corresponding standard errors (in parentheses) computed using GMM with 4 Newey West lags. Data are quarterly US aggregates from 1927:1-2004:3.  $\Delta d_t^f$ ,  $\pi_t$ ,  $\Delta c_t^f$ ,  $r f_t$ ,  $dp_t^f$ ,  $spd_t$ , and  $rx_t$ , refer to filtered log dividend growth, log inflation, filtered log consumption growth, the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, log excess equity returns (with respect to the 90 day T-bill). See text for data construction details.

Table 4: The Fit of the Model: Nonlinear Moments

Panel A: Univariate Heteroskedasticity				Panel B: Multivariate Heteroskedasticity			
	$(\Delta d_t^f)^2$	$(\Delta c_t^f)^2$	$(rx_t)^2$		$(\Delta d_t^f)^2$	$(\Delta c_t^f)^2$	$(rx_t)^2$
$v_1$	[-0.0758]	[-0.0044]	[-0.1637]	$v_1$	[0.0362]	[-0.0035]	[-0.1051]
	-0.0779	-0.0062	-0.4620		-0.0621	-0.0046	-0.1245
	(0.0260)	(0.0016)	(0.2740)		(0.0214)	(0.0014)	(0.1388)
$v_2$	[-0.0371]	[-0.0028]	[0.8337]	$v_2$	[-0.1183]	[0.0021]	[0.8795]
	0.1112	0.0106	1.8843		0.0737	0.0079	1.8492
	(0.0420)	(0.0039)	(1.1279)		(0.0364)	(0.0036)	(1.1297)
$v_3$	[0.2290]	[0.0114]	[1.7147]	$v_3$	[0.2839]	[0.0070]	[1.5086]
	0.0984	0.0080	0.7759		0.0438	0.0040	0.7679
	(0.0485)	(0.0034)	(0.6435)		(0.0334)	(0.0031)	(0.6754)

Panel C: Skewness and Kurtosis			
	$\Delta d_t^f$	$\Delta c_t^f$	$rx_t$
skew	[-0.2250]	[-0.4574]	[0.1494]
	-0.3287	-0.7537	0.1254
	(0.6339)	(0.4450)	(0.7228)
kurt	[10.0250]	[10.1726]	[5.4295]
	7.9671	6.4593	9.7118
	(1.3668)	(0.9673)	(2.0755)

Panels A and B repeat the regression models for squared series of Table 1 and also reports analogous simulated statistics generated by the model estimated in Table 2. Panel C reports unconditional skewness and kurtosis for the variables in each column. In each panel, the simulated moments (100,000 simulations) are reported in square brackets and the corresponding data statistics and standard errors are reported below, with the standard errors in parentheses.

In panels A and B, the regressions are of the form:

$$x_{t+k}^2 = \nu_0 + \nu_1 r f_t + \nu_2 dp^f + \nu_3 spd_t + \varepsilon_{t+k}$$

The dependent variables are  $(\Delta d_t^f)^2$ ,  $(\Delta c_t^f)^2$ , and  $(rx_t)^2$ . For consumption and dividend growth,  $k = 4$  and 4 Newey West lags are used in the GMM estimation, but for returns,  $k = 1$  and no Newey West lags are used. In Panel A, the regression is univariate; that is only one instrument is used per regression. Panel B reports the full multivariate regressions.

Table 5: Dynamic Properties of Risk, Uncertainty and Asset Prices

## Panel A: Unconditional

Simulated unobservable univariate moments									
	$v_t$	$q_t$	$RA_t$	$rrf_t$	$rspd_t$	$E_t [rx_{t+1}]$	$V_t [rx_{t+1}]$	$S_t$	$MaxS_t$
mean	1.0090	1.0097	7.06	0.0017	0.0038	0.0121	0.0092	0.1396	0.2075
median	0.3611	0.7784	2.52	0.0037	0.0034	0.0103	0.0070	0.1320	0.2095
std.dev.	1.6063	0.9215	36.34	0.0093	0.0028	0.0075	0.0083	0.1265	0.0491
autocorr	0.9788	0.9784	0.9212	0.9784	0.9777	0.9789	0.9794	0.5384	0.9653

## Panel B: Cyclicalilty of Means

Simulated unobservable univariate means									
	$v_t$	$q_t$	$RA_t$	$rrf_t$	$rspd_t$	$E_t [rx_{t+1}]$	$V_t [rx_{t+1}]$	$S_t$	$MaxS_t$
Expansion	0.8665	0.9893	6.73	0.0024	0.0035	0.0117	0.0085	0.1406	0.2053
Recession	2.7195	1.2544	10.96	-0.0064	0.0076	0.0171	0.0183	0.1283	0.2349

Panel C: Correlations with  $v_t$  and  $q_t$ 

Simulated correlations between $v_t$ , $q_t$ and observables							
	$rrf_t$	$rspd_t$	$rf_t$	$dp_t$	$rx_t$	$E_t [rx_{t+1}]$	$V_t [rx_{t+1}]$
$v_t$	-0.9232	0.9562	-0.5163	-0.1835	0.1470	0.3428	0.8799
$q_t$	0.4687	0.1756	0.5375	0.9215	-0.1071	0.8943	0.3758

Simulated moments are calculated by simulating the system for 100,000 periods using the point estimates from Table 2 for a number of variables including:  $v_t$ , dividend growth volatility,  $q_t$ , the log inverse consumption surplus ratio,  $RA_t$ , local risk aversion which is  $\gamma \exp(q_t)$ . The variables  $rrf_t$  and  $rspd_t$  represent the real short rate and real term spread respectively, and  $E_t [rx_{t+1}]$  and  $V_t [rx_{t+1}]$  denote the conditional mean and conditional variance of excess stock returns.  $S_t$  denotes the conditional Sharpe ratio for equity.  $MaxS_t$  denotes the maximum attainable Sharpe ratio for any asset in the economy which is given by the quantity,  $[\exp(V_t(m_{t+1})) - 1]^{1/2}$ .

In Panel B, means of simulated data conditional on a binary recession/expansion variable are presented. Recessions are defined in the simulated data as periods of negative real consumption growth. Recessions represent approximately 8% of all observations in the simulated data.

In Panel C, the simulated unconditional correlations among  $v_t$ ,  $q_t$  and other endogenous variables are reported.

Table 6: Variance Decompositions

Fraction of variance due to variation in each state element					
	$\Delta d_t$	$\pi_t$	$u_t$	$v_t$	$q_t$
$rrf_t$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.0999] ⟨0.1154⟩	[0.7239] ⟨0.1472⟩	[0.1761] ⟨0.0698⟩
$rspd_t$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.0752] ⟨0.01010⟩	[0.8653] ⟨0.0943⟩	[0.0596] ⟨0.0510⟩
$cp_t^{rcons}$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.0502] ⟨0.0630⟩	[0.2299] ⟨0.1041⟩	[0.7199] ⟨0.1296⟩
$rf_t$	[0.0000] ⟨0.0000⟩	[0.1230] ⟨0.0765⟩	[0.0904] ⟨0.2222⟩	[0.5010] ⟨0.1216⟩	[0.2856] ⟨0.1796⟩
$spd_t$	[0.0000] ⟨0.0000⟩	[0.3148] ⟨0.3407⟩	[0.0035] ⟨0.0599⟩	[0.6019] ⟨0.3413⟩	[0.0797] ⟨0.0745⟩
$dp_t^f$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.0655] ⟨0.0901⟩	[0.0544] ⟨0.0798⟩	[0.8801] ⟨0.0627⟩
$rx_t$	[0.3605] ⟨0.0733⟩	[0.0091] ⟨0.0036⟩	[-0.1593] ⟨0.0401⟩	[0.1640] ⟨0.0895⟩	[0.6257] ⟨0.1397⟩
$E_t [rx_{t+1}]$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[-0.0167] ⟨0.0146⟩	[0.1665] ⟨0.1281⟩	[0.8502] ⟨0.1182⟩
$V_t [rx_{t+1}]$	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.0000] ⟨0.0000⟩	[0.8029] ⟨0.2229⟩	[0.1971] ⟨0.2229⟩

The symbols,  $rrf_t$ ,  $rspd_t$ , and  $cp_t^{rcons}$  refer to the theoretical real short rate, real term spread, and the coupon-price ratio of a real consol. The table reports the fraction of variation of selected variables due to variation in elements of the state vector.

The variable in each row can be expressed as a linear combination of the current state and lagged vector. Generally, under the model in Table 2, for the row variables,  $x_t$ ,

$$x_t = \mu + \Gamma' Y_t^c$$

where  $Y_t^c$  is the ‘companion form’ of the  $N$ -vector,  $Y_t$ ; that is,  $Y_t^c$  is comprised of ‘stacked’ current and lagged values of  $Y_t$ .  $\mu$  and  $\Gamma$  are constant vectors implied by the model and parameter estimates of Table 2. Let  $Var(Y_t^c)$  be the variance covariance matrix of  $Y_t^c$ . Based on  $\mu$  and  $\Gamma$ , the proportion of the variation of each row variable attributed to the  $n^{th}$  element of the state vector is calculated as

$$\frac{\Gamma' Var(Y_t^c) \Gamma^{(n)}}{\Gamma' Var(Y_t^c) \Gamma}$$

where  $\Gamma^{(n)}$  is a column vector such that  $\{\Gamma^{(n)}\}_i = \{\Gamma\}_i$  for  $i = n, N + n, \dots$  and zero elsewhere. Standard errors are reported below in angle brackets and are calculated from the variance covariance matrix of the parameters in Table 2 using the  $\Delta$ -method.

Table 7: Model Implied Reduced Form Return Predictability

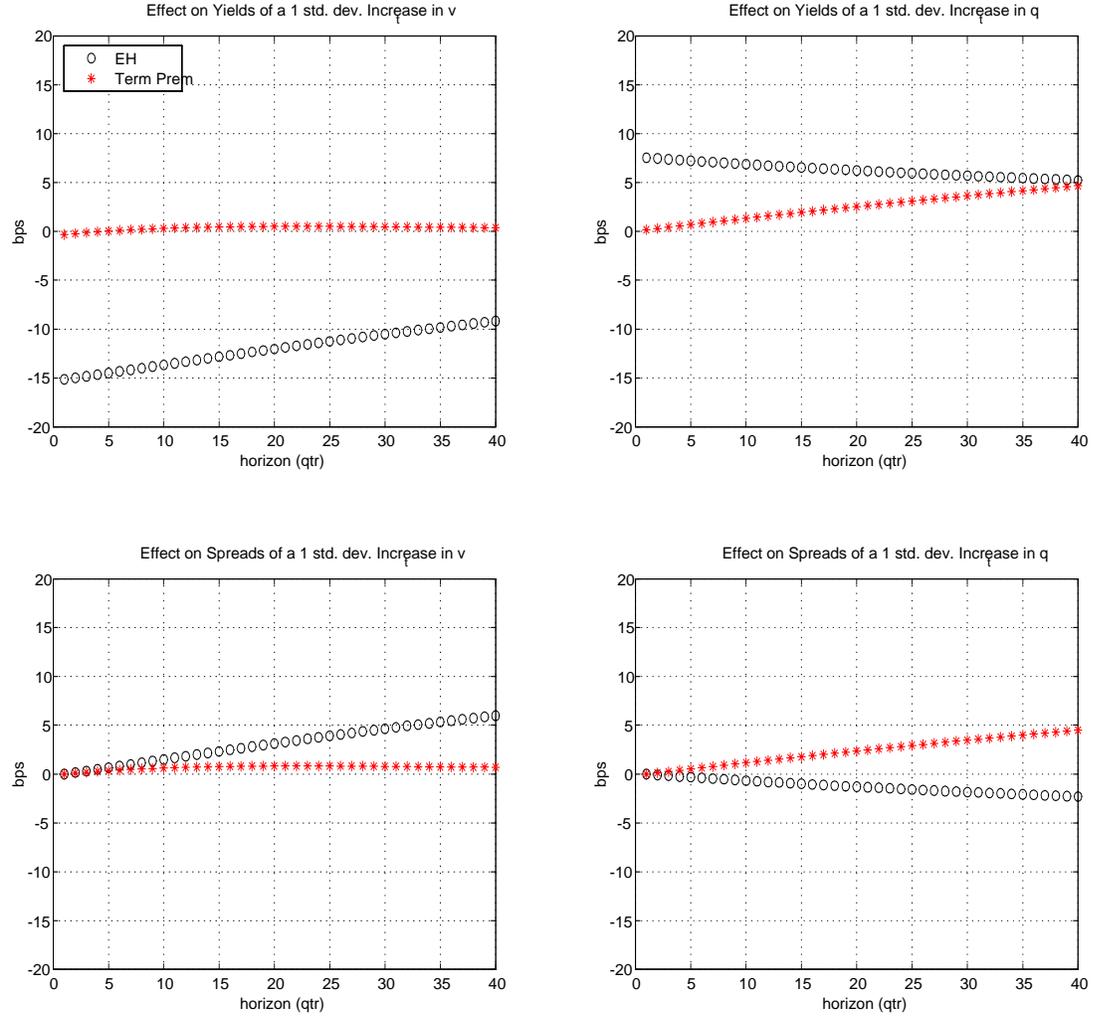
<u>Excess Returns</u>		
<u>Parameter estimates</u>		
	multivariate	univariate
$\beta_0$	[-0.0037] -0.0358 (0.0256)	
$\beta_1$	[-0.3097] -0.1669 (0.7464)	[0.2192] -1.1651 (0.7839)
$\beta_2$	[2.0695] 3.7980 (2.0231)	[1.5770] 3.7260 (1.9952)
$\beta_3$	[0.7668] 3.5376 (1.8900)	[1.2389] 3.4728 (1.8728)

The predictability model for excess returns is defined as,

$$rx_{t+1} = \beta_0 + \beta_1 rf_t + \beta_2 dp_t^f + \beta_3 spd_t + \varepsilon_{t+1}$$

and is estimated by GMM. Data are quarterly US aggregates from 1927:1-2004:3. The symbols  $rf_t$ ,  $dp_t^f$ ,  $spd_t$ , and  $rx_t$  refer to the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, and log excess equity returns (with respect to the 90 day T-bill). Simulated moments, in square brackets, are calculated by simulating the model for 100,000 periods using the point estimates from Table 2 and estimating the above model on the simulated data. The second and third numbers for each entry are the sample moments and corresponding standard errors (in parentheses).

Figure 1: Term Structure Determinants



Under the model of Table 2, real risk free yields of horizon,  $h$ , have solutions of the form,

$$rr f_{h,t} = a_h + A'_h Y_t$$

where the coefficients above are functions of the ‘deep’ model parameters. This figure shows the effect on these yields and the associated spreads (relative to the 1 period yield) of 1 standard deviation changes in the latent factors,  $v_t$  and  $q_t$  using the point estimates in Table 2. At horizons greater than 1, these effects can be further decomposed into parts corresponding to the expectations hypothesis (EH), and term premiums, which are drawn in blue and red bars respectively.