NBER WORKING PAPER SERIES

THEORETICAL FOUNDATIONS OF BUFFER STOCK SAVING

Christopher Carroll

Working Paper 10867
http://www.nber.org/papers/w10867

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 2004

I would like to thank James Feigenbaum, Joseph Kaboski, Miles Kimball, and Misuzu Otsuka for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, and David Zervos for early guidance to the literature. The Mathematica programs that generate all the figures and numerical results from this paper are available on the author's website, http://www.econ.jhu.edu/people/ccarroll. The views expressed herein are those of the author(s) and not necessarily those of the National Bureau of Economic Research.

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NBER Working Paper No. 10867
October 2004
JEL No. D81, D91, E21

ABSTRACT

"Buffer-stock" versions of the dynamic stochastic optimizing model of saving are now standard in the consumption literature. This paper builds theoretical foundations for rigorous understanding of the main characteristics of buffer stock models, including the existence of a target level of wealth and the proposition that aggregate consumption growth equals aggregate income growth in a small open economy populated by buffer stock consumers.

Christopher Carroll
Department of Economics
Mergenthaler 440
Johns Hopkins University
Baltimore, MD 21218
and NBER
ccarroll@jhu.edu

1 Introduction

Following Friedman's (1957) introduction of the permanent income hypothesis, a fruitful period of formalization culminated in famous papers by Schectman and Escudero (1977) and Bewley (1977). However, despite powerful subsequent developments in recursive dynamic theory (codified in Stokey et. al. (1989)), surprisingly few theoretical results have ever been published about the commonly used version of the model with unbounded (constant relative risk aversion) utility, stochastic labor income, and no liquidity constraints.

Economists working in the area have nonetheless been able to use this model extensively because increasing computer power has allowed them to solve the problem numerically. Starting with Zeldes (1989), numerical solutions have now become the standard approach for serious quantitative consumption modeling.

But numerical methods have a 'black box' character. It is possible to use a consumption rule that emerges from a numerical solution algorithm without thoroughly understanding of the properties of that rule. Indeed, without foundational theory, it can even be difficult to be sure that numerical solutions are correct. And without theoretical underpinnings, the analyst often does not know the circumstances under which any given simulation result might change.

A good example is the finding that when consumers are both impatient and "prudent" there will be a target level of nonhuman wealth ('cash' for short) such that if actual cash exceeds the target, the consumer will spend freely and cash will fall (in expectation), while if actual cash is below the target the consumer will save and cash will rise. Carroll (1992; 1997) showed that target saving behavior can arise under plausible parameter values for both finite and infinite horizon models. Gourinchas and Parker (2002) estimate the model and conclude that the buffer-stock saving phase of life lasts from age 25 until around age 40-45; using the same model with different data Cagetti (2003) finds target saving behavior into the 50s for the median household. But none of these papers provides a formal explanation for why target saving behavior arises. In each case, target-saving behavior is simply observed in simulations under specific parameter values. The papers also draw a variety of other conclusions based on the numerical solutions, such as that the marginal propensity to consume appears to approach the perfect foresight MPC as cash gets large.

This paper provides the analytical foundations for these and other propositions that have emerged from the simulation literature. The paper pairs these theoretical results with illustrative simulation examples, providing an integrated framework for understanding buffer-stock saving behavior.[1]

The paper proceeds in three parts.

The first part states the maximization problem, demonstrates that the problem can

[1]The computer programs that generate these simulation results are available on the author's website.

be rewritten in terms of ratios to permanent labor income, and proves that the problem defines a contraction mapping with a limiting consumption function. It then shows that a related class of models (exemplified by Deaton (1991)) reflects a particular limit of the model examined here.

The next section demonstrates five key properties of buffer-stock saving models. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a specific simple analytical limit. Third, there exists a unique 'target' cash-on-hand-to-permanent-income ratio. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent labor income. Finally, the expected growth rate of consumption is declining in the level of cash. All of the first four propositions are proven generally; the last proposition is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

The final section examines properties of aggregate behavior in an economy populated by buffer-stock consumers. Szeidl (2002) has recently proven that an ergodic distribution of cash will exist in such an economy.[2] This section shows that even with a fixed aggregate interest rate that differs from the time preference rate, the economy converges to a balanced growth equilibrium in which the growth rate of consumption and cash tend toward the (exogenous) growth rate of permanent income. A similar proposition holds at the level of individual households.

2 The Problem

2.1 Setup

Consider a consumer solving an optimization problem from the current period t until the end of life at T defined by the objective

$$\max \ E_t \left[\sum_{s=t}^{T} \beta^{s-t} u(C_s) \right] \tag{1}$$

where $u(C)$ is a constant relative risk aversion utility function $u(C) = C^{1-\rho}/(1-\rho)$ for $\rho > 1$.[3] (We will ultimately be interested in the limit as the time until death $T-t$ approaches infinity, but we start with a finite horizon). Initial conditions are defined

[2]Szeidl's proof supplants the analysis in an earlier draft of this paper, which provided simulation evidence of ergodicity but no proof.

[3]The main results also hold for logarithmic utility which is the limit as $\rho \rightarrow 1$ but dealing with the logarithmic case is cumbersome and therefore omitted.

by a starting value of market resources M_t (cash) and an initial value of permanent noncapital income P_t. The consumer's circumstances evolve according to

$$\begin{aligned} A_t &= M_t - C_t, \\ M_{t+1} &= RA_t + Y_{t+1}, \\ Y_{t+1} &= P_{t+1}\xi_{t+1}, \\ P_{t+1} &= GP_t\Psi_{t+1}, \end{aligned} \tag{2}$$

where A_t indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $R = (1+r)$ between periods;[4] M_{t+1} is the sum of beginning-of-next-period resources RA_t and next-period noncapital income Y_{t+1}; actual noncapital income Y_{t+1} equals permanent noncapital income P_{t+1} multiplied by a mean-one iid transitory shock ξ_{t+1} (more generally, we assume that from the perspective of period t, all future transitory shocks satisfy $E_t[\tilde{\xi}_{t+n}] = 1 \ \forall\ n \geq 1$);[5] and permanent noncapital income in period $t+1$ is equal to its previous value, multiplied by a growth factor G, and modified by a mean-one truncated lognormal iid shock Ψ_{t+1}, $E_t[\tilde{\Psi}_{t+n}] = 1 \ \forall\ n \geq 1$ satisfying $\Psi \in [\underline{\Psi}, \bar{\Psi}]$ for $0 < \underline{\Psi} \leq 1 \leq \bar{\Psi} < \infty$ where $\underline{\Psi} = \bar{\Psi} = 1$ is the degenerate case with no permanent shocks.[6]

Following Carroll (1992), assume that in future periods $n \geq 1$ there is a small probability p that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } p > 0 \\ \Theta_{t+1}/q & \text{with probability } q \equiv (1-p) \end{cases} \tag{3}$$

where Θ_{t+n} is a mean-one random variable (guaranteeing $E_t[\tilde{\xi}_{t+n}] = 1$), and has a distribution satisfying $\Theta \in [\underline{\Theta}, \bar{\Theta}]$ where $0 < \underline{\Theta} \leq 1 \leq \bar{\Theta} < \infty$ (degenerately $\underline{\Theta} = \bar{\Theta} = 1$). Call the cumulative distribution functions F_Ψ and F_Θ (and F_ξ is derived trivially from (3) and F_Θ). Permanent income and cash start out strictly positive, $P_t \in (0, \infty)$ and $M_t \in (0, \infty)$, and the consumer cannot die in debt,

$$C_T \leq M_T. \tag{4}$$

[4] Allowing a stochastic interest factor is straightforward but adds little to the analysis.

[5] The notational convention is that stochastic variables have a $\sim$ over them when their expectation is being taken from the perspective of a period prior to their realization, but have no $\sim$ otherwise. Hence we write $P_{t+1} = GP_t\Psi_{t+1}$ but if we need the period-t expectation we write $E_t[\tilde{P}_{t+1}] = GP_tE_t[\tilde{\Psi}_{t+1}]$.

[6] The definition of permanent income here differs from Deaton's (1992) (which is often used in the macro literature), in which permanent income is the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to M_t as 'cash' rather than as wealth to avoid any confusion for those readers who might be accustomed to thinking of the discounted value of future labor income as a part of wealth.

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem questionable. However, it is easy to show that a model with a nonzero minimum value of ξ (because, for example, of unemployment insurance) can be redefined by capitalizing the present discounted value of perfectly certain income into current market assets,[7] transforming that model back into the model analyzed here. Also, the assumption that there is a positive point mass (as opposed to positive density) for the worst realization of the transitory shock is inessential, but simplifies and clarifies the proofs.

Combining the combinable transition equations, the recursive nature of the problem allows us to rewrite it more compactly in Bellman equation form,

$$\begin{aligned} V_t(M_t, P_t) &= \max_{C_t} \left\{ u(C_t) + \beta E_t \left[V_{t+1}(\tilde{M}_{t+1}, \tilde{P}_{t+1}) \right] \right\} && (5) \\ &\text{s.t.} \\ P_{t+1} &= G P_t \Psi_{t+1} && (6) \\ M_{t+1} &= R(M_t - C_t) + P_{t+1}\xi_{t+1}. && (7) \end{aligned}$$

This model differs from Bewley's (1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that $u(0)$ is well defined, or that $u'(0)$ is well defined and finite, so neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these formulations in that it permits permanent growth, and permanent shocks to income, which a large empirical literature finds to be quite substantial in micro data (MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004)) and which are almost certainly more consequential for utility than are transitory fluctuations. It differs from Deaton (1991) because liquidity constraints are absent; there are separate transitory and permanent shocks; and the transitory shocks here can occasionally cause income to reach zero. Finally, it differs from models found in Stokey et. al. (1989) because neither constraints nor bounds on utility or marginal utility are imposed.[8] Below it will become clear that the Deaton model can be thought of as a particular limit of this paper's model.

[7] So long as this PDV is a finite number and unemployment benefits are related to P_t; see the discussion in section 2.7.

[8] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (1986) and Clarida (1987). See Toche (2000) for an elegant analysis of a related but simpler continuous-time model.

2.2 The Perfect Foresight Benchmark

A useful benchmark is the solution to the corresponding perfect foresight model, which can be written as above with $p = 0$ and $\underline{\Theta} = \bar{\Theta} = \underline{\Psi} = \bar{\Psi} = 1$.

The dynamic budget constraint plus the can't-die-in-debt condition imply an exactly-holding intertemporal budget constraint

$$PDV_t(C) \quad = \quad M_t - P_t + PDV_t(P), \tag{8}$$

and with constant growth and interest factors

$$\begin{aligned} PDV_t(P) &= P_t + (G/R)P_t + (G/R)^2 P_t + ... + (G/R)^{T-t} P_t \quad (9)\\ &= \underbrace{\left(\frac{1-(G/R)^{T-t+1}}{1-(G/R)} \right) P_t}_{\equiv H_t} \quad (10) \end{aligned}$$

where H_t is 'human wealth,' the discounted value of future labor earnings.

The Euler equation implies

$$\begin{aligned} C_t^{-\rho} &= R\beta C_{t+1}^{-\rho} \quad (11)\\ (C_{t+1}/C_t) &= (R\beta)^{1/\rho} \quad (12) \end{aligned}$$

which can be used similarly to obtain

$$\begin{aligned} PDV_t(C) &= C_t \left(1 + R^{-1}(R\beta)^{1/\rho} + \left(R^{-1}(R\beta)^{1/\rho} \right)^2 + ... \right) \quad (13)\\ &= \left(\frac{1-(R^{-1}(R\beta)^{1/\rho})^{T-t+1}}{1-R^{-1}(R\beta)^{1/\rho}} \right) C_t \quad (14) \end{aligned}$$

and the IBC (8) therefore implies

$$C_t \quad = \quad \underbrace{\left(\frac{1-R^{-1}(R\beta)^{1/\rho}}{1-(R^{-1}(R\beta)^{1/\rho})^{T-t+1}} \right)}_{\equiv \underline{\kappa}_t} \overbrace{(M_t - P_t + H_t)}^{\equiv W_t} \tag{15}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume and W_t is total wealth, human and nonhuman.

We define the infinite horizon solution as the limit of the finite horizon solution as the horizon $T - t$ approaches infinity.[9] However, (10) makes plain that in order for $\lim_{n\to\infty} H_{T-n}$ to be finite, we must impose

$$G \quad < \quad R. \tag{16}$$

[9]This is not necessarily the same as the solution to a truly infinite horizon problem; se ignore this subtlety.

Intuitively, finite human wealth requires that labor income grow at a rate less than the interest rate.

Similarly, if we start with any positive value of consumption, then in order for the PDV of consumption to be finite we must impose

$$(R\beta)^{1/\rho} \quad < \quad R. \tag{17}$$

Inspection of the formula for $\underline{\kappa}_t$ in (15) makes the reason for this restriction obvious: It is necessary to guarantee a positive marginal propensity to consume. This can be loosely thought of as imposing a maximum degree of 'patience,' in the sense that the consumer cannot be so pathologically patient as to wish to spend zero or a negative amount when $W_t > 0$. We will henceforth refer to this condition as 'nonpathological patience' or NPP for short.

2.3 Demonstration That the Problem Can Be Rewritten in Ratio Form

As written, the problem has two state variables, the level of permanent noncapital income P_t and the level of market resources M_t. We show now that for relative risk aversion $\rho > 1$, it is possible to normalize the model by P_t and thereby to reduce the effective number of state variables to one.[10,11] Specifically, defining lower-case variables as the upper-case variable normalized by P_t (e.g. $m_t = M_t/P_t$), assume that $V_{T+1} = 0$, and consider the problem in the second-to-last period of life,

$$\begin{aligned}
V_{T-1}(M_{T-1}, P_{T-1}) &= (1-\rho)^{-1} \max_{C_{T-1}} \left\{ C_{T-1}^{1-\rho} + \beta E_{T-1}[\tilde{M}_T^{1-\rho}] \right\} \\
&= (1-\rho)^{-1} \max_{c_{T-1}} \left\{ (P_{T-1}c_{T-1})^{1-\rho} + \beta E_{T-1}[(\tilde{P}_T \tilde{m}_T)^{1-\rho}] \right\} \\
&= (1-\rho)^{-1} P_{T-1}^{1-\rho} \left\{ \max_{c_{T-1}} \left(c_{T-1}^{1-\rho} + \beta E_{T-1}[(G\tilde{\Psi}_T \tilde{m}_T)^{1-\rho}] \right) \right\}.
\end{aligned}$$

Now define

$$\Gamma_t \quad \equiv \quad G\Psi_t \tag{18}$$

[10]The same normalization is possible in the logarithmic utility case; the derivation is omitted for brevity.

[11]This subsection reviews material that is well known in order to provide a notational and conceptual framework for subsequent novel material.

and consider the problem

$$
\begin{aligned}
v_t(m_t) &= \max_{c_t} \left\{ u(c_t) + \beta E_t[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}(\tilde{m}_{t+1})] \right\} && (19) \\
&\text{s.t.} \\
a_t &= m_t - c_t, && (20) \\
m_{t+1} &= \mathcal{R}_{t+1} a_t + \xi_{t+1} && (21)
\end{aligned}
$$

for

$$
\mathcal{R}_{t+1} \equiv (R/\Gamma_{t+1}). \qquad (22)
$$

If we specify $v_T(m_T) = m_T^{1-\rho}/(1-\rho)$ and define $v_{T-1}(m_{T-1})$ from (19) for $t = T-1$ we have that

$$
V_{T-1}(M_{T-1}, P_{T-1}) = P_{T-1}^{1-\rho} v_{T-1}(M_{T-1}/P_{T-1}). \qquad (23)
$$

Similar logic can be applied inductively to all earlier periods, which means that if we solve the normalized one-state-variable problem specified in (19)-(21) we will have solutions to the original problem from:

$$
\begin{aligned}
V_t(M_t, P_t) &= P_t^{1-\rho} v_t(M_t/P_t), && (24) \\
C_t(M_t, P_t) &= P_t c_t(M_t/P_t), && (25)
\end{aligned}
$$

and so on.

2.4 The Baseline Solution

Figure 1 depicts the successive consumption rules that apply in the last period of life ($c_T(m)$), the second-to-last period, and various earlier periods under the set of baseline parameter values listed in Table 1, which correspond to a standard calibration in the literature.

The 45 degree line is labelled as $c_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources. The figure shows the consumption rules are converging as the end of life recedes; the infinite-horizon consumption rule $c(m)$ is defined as

$$
c(m) = \lim_{n\to\infty} c_{T-n}(m). \qquad (26)
$$

2.5 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. Unfortunately, (19) cannot be proven to be a contraction

Table 1: Baseline Parameter Values

Description	Parameter	Value
Permanent Income Growth Factor	G	1.03
Interest Factor	R	1.04
Time Preference Factor	β	0.96
Coeff of Relative Risk Aversion	ρ	2
Probability of Zero Income	p	0.005
Std Dev of Log Permanent Shock	σ_ψ	0.1
Std Dev of Log Transitory Shock	σ_θ	0.1

mapping using the standard theorems in, say, Stokey et. al. (1989), because those theorems require marginal utility to be bounded over the space of possible values of m. For the problem specified here, the possibility (however unlikely) of an unbroken string of zero-income events for the remainder of life means that as m approaches zero c must approach zero (see the discussion in 2.5.2) and thus marginal utility is unbounded. Fortunately, Boyd (1990) provides a weighted contraction mapping theorem that can be used. To use Boyd's theorem we need

Definition 1 *Define $f \in \mathcal{C}(\mathbf{A},\mathbf{B})$ where $\mathcal{C}(\mathbf{A},\mathbf{B})$ is the space of continuous functions from $\mathbf{A}$ to $\mathbf{B}$. Suppose $\phi \in \mathcal{C}(\mathbf{A},\mathbf{B})$ with $\mathbf{B} \subset \mathbb{R}$ and $\phi > 0$. Then f is ϕ-bounded if the ϕ-norm of f,*

$$\|f\|_\phi = \sup_m \left[\frac{|f(m)|}{\phi(m)}\right], \tag{27}$$

is finite.

For $\mathcal{C}_\phi(\mathbf{A},\mathbf{B})$ defined as the set of functions in $\mathcal{C}(\mathbf{A},\mathbf{B})$ that are ϕ-bounded, Boyd (1990) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let $\mathcal{T} : \mathcal{C}_\phi(\mathbf{A},\mathbf{B}) \to \mathcal{C}(\mathbf{A},\mathbf{B})$ such that*

$$\begin{array}{ll} 1) & \mathcal{T} \text{ is non-decreasing, i.e. } w_a(m) \le w_b(m) \Rightarrow (\mathcal{T}w_a)(m) \le (\mathcal{T}w_b)(m) \\ 2) & \mathcal{T}(0) \in \mathcal{C}_\phi(\mathbf{A},\mathbf{B}) \\ 3) & \mathcal{T}(w+\gamma\phi) \le \mathcal{T}w + \gamma\varsigma\phi \text{ for some } \varsigma < 1 \text{ and all } \gamma > 0. \end{array} \tag{28}$$

Then $\mathcal{T}$ is a contraction with a unique fixed point.

For our problem, take $\mathbf{A}$ as $\mathbb{R}_{++}$, $\mathbf{B}$ as $\mathbb{R}$ and $\phi(m) = \eta + m + m^{1-\rho}$ where $\eta > 0$ is a real number whose specific value will be determined in the course of the proof. We introduce the mapping $\mathsf{T} : \mathcal{C}_\phi(\mathbf{A},\mathbf{B}) \to \mathcal{C}(\mathbf{A},\mathbf{B})$,

$$(\mathsf{T}w)(m_t) = \sup_{c_t \in [\underline{\kappa} m_t, \bar{\bar{\kappa}} m_t]} \left\{ u(c_t) + \beta E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} w(\tilde{m}_{t+1}) \right] \right\} \tag{29}$$

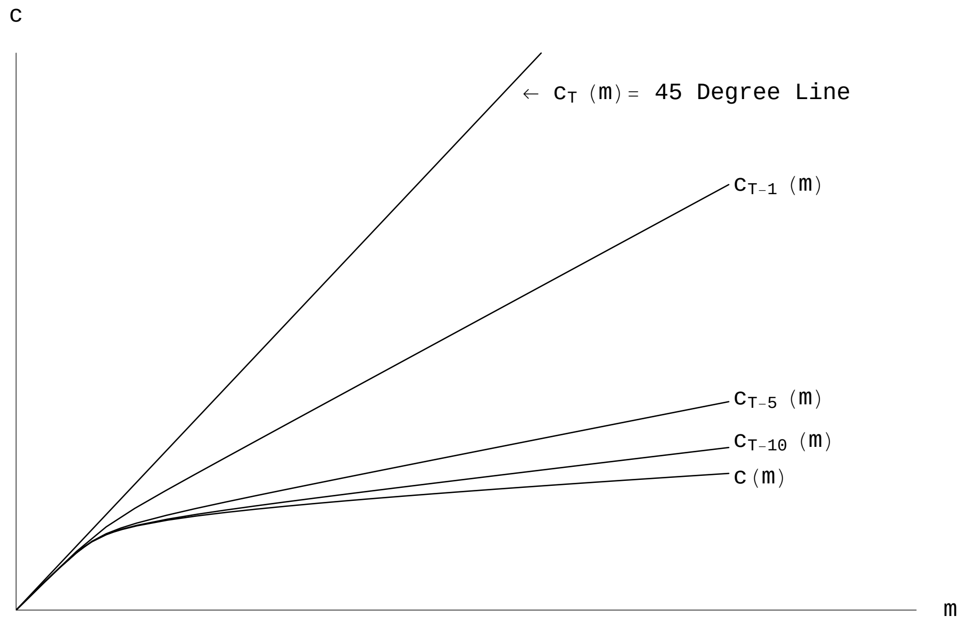

Figure 1: Convergence of the Consumption Rules

where m_{t+1} is defined as in (21) and

$$\bar{\bar{\kappa}} \equiv \left[1 + p^{1/\rho} R^{-1} (R\beta)^{1/\rho}\right]^{-1} \tag{30}$$

$$\underline{\kappa} \equiv \left[1 - R^{-1} (R\beta)^{1/\rho}\right] = \lim_{n\to\infty} \underline{\kappa}_{T-n} \tag{31}$$

where $\underline{\kappa}$ and $\bar{\bar{\kappa}}$ can be referred to respectively as the 'minimal' and the 'maximal' marginal propensies to consume (these terms will be justified later, along with the notation).

Our goal is to show that T satisfies the conditions that Boyd requires of his operator $\mathcal{T}$, if we impose two restrictions on parameter values.

The first restriction is the nonpathological patience requirement imposed for the perfect foresight model, (17). The second is

$$R\beta E[\tilde{\Gamma}^{-\rho}] < 1, \tag{32}$$

which is identical to the condition Deaton (1991) showed necessary for his model with constraints, and the two restrictions can be combined into a single expression,

$$R\beta \max\left[R^{-\rho}, E[\tilde{\Gamma}^{-\rho}]\right] < 1. \tag{33}$$

We discuss the meaning of these restrictions in detail below; essentially, they require the consumer to be suffiently impatient so that desired m does not head to infinity.

We are now in position to state the main theorem of the paper.

Theorem 1 *The mapping* T *is a contraction mapping if the restrictions on parameter values* (17) *and* (32) *are true. Furthermore,* $\mathsf{T}v_{t+1}(m) = v_t(m)$*, which means the mapping* T *generates the sequence of value functions defined in equation* (19).

2.5.1 Proof that T Is A Contraction Mapping

We must show that our operator T satisfies all of Boyd's conditions.

Boyd's operator $\mathcal{T}$ maps from $\mathcal{C}_\phi(\mathbf{A},\mathbf{B})$ to $\mathcal{C}(\mathbf{A},\mathbf{B})$. A preliminary requirement is therefore that $\mathsf{T}w$ be continuous for any $\phi-$bounded w, $\mathsf{T}w \in \mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (2003) for details.

Consider next condition 1). For this problem,

$$\begin{aligned} \mathsf{T}w_a \quad &\text{is} \quad \sup_{c_t\in[\underline{\kappa}m_t,\bar{\bar{\kappa}}m_t]} \left\{u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}w_a\left(\tilde{m}_{t+1}\right)\right]\right\} \\ \mathsf{T}w_b \quad &\text{is} \quad \sup_{c_t\in[\underline{\kappa}m_t,\bar{\bar{\kappa}}m_t]} \left\{u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}w_b\left(\tilde{m}_{t+1}\right)\right]\right\}, \end{aligned}$$

so $w_a \leq w_b$ implies $\mathsf{T}w_a \leq \mathsf{T}w_b$ by inspection.[12]

Consider condition 2), $\mathsf{T}(0) \in \mathcal{C}_\phi\left(\mathbf{A},\mathbf{B}\right)$. By definition $\mathsf{T}(0)$ is

$$\sup_{c_t\in[\underline{\kappa}m_t,\bar{\bar{\kappa}}m_t]} \left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right) + \beta 0\right\} \tag{34}$$

the solution to which is patently $u(\bar{\bar{\kappa}}m_t)$. Hence $\mathsf{T}(0)$ is ϕ-bounded for $\phi(m) = \eta + m + m^{1-\rho}$.[13]

Finally, we turn to Boyd's condition 3), $\mathsf{T}(w+\gamma\phi) \leq \mathsf{T}w + \gamma\varsigma\phi$. We begin by constructing the $\mathsf{T}(\phi)$ term. Expand the expectation $\beta E_t[\tilde{\Gamma}_{t+1}^{1-\rho}(\eta + \tilde{m}_{t+1} + \tilde{m}_{t+1}^{1-\rho})]$,

$$\beta E_t\left\{\tilde{\Gamma}_{t+1}^{1-\rho}\left[\eta + \tilde{\mathcal{R}}_{t+1}a_t + \tilde{\xi}_{t+1} + [\tilde{\mathcal{R}}_{t+1}a_t + \tilde{\xi}_{t+1}]^{1-\rho}\right]\right\}. \tag{35}$$

Now note that $a_t = (m_t - c_t) < m_t$ and $\xi_{t+1} < \overline{\Theta}$, so (35) is less than:

$$\begin{aligned} \beta E_t\left\{\tilde{\Gamma}_{t+1}^{1-\rho}\left[\eta + \tilde{\mathcal{R}}_{t+1}m_t + \overline{\Theta} + [\tilde{\mathcal{R}}_{t+1}a_t + \tilde{\xi}_{t+1}]^{1-\rho}\right]\right\} &= \\ R\beta E_t[\tilde{\Gamma}_{t+1}^{-\rho}]m_t + \beta E_t\left\{\tilde{\Gamma}_{t+1}^{1-\rho}\left[\eta + \overline{\Theta} + [\tilde{\mathcal{R}}_{t+1}a_t + \tilde{\xi}_{t+1}]^{1-\rho}\right]\right\}&. \end{aligned} \tag{36}$$

[12] Recall that $\tilde{m}_{t+1}$ is just a function of c_t and the stochastic shocks.

[13] Note that $\mathsf{T}(0)$ is not v_T; this maximization does not have an economic interpretation.

Now break up the second term in (36):

$$\beta E_t \left\{ \tilde{\Gamma}_{t+1}^{1-\rho} \left[\eta + \overline{\Theta} + [\tilde{\mathcal{R}}_{t+1} a_t + \tilde{\xi}_{t+1}]^{1-\rho} \right] \right\} = \tag{37}$$

$$\begin{aligned} & \beta E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} [\eta + \bar{\Theta}] \right] + \beta E_t \left[\left(R a_t + \tilde{\Gamma}_{t+1} \tilde{\xi}_{t+1} \right)^{1-\rho} \right] \\ = \; & (\eta + \bar{\Theta}) \beta G^{1-\rho} E_t [\tilde{\Psi}_{t+1}^{1-\rho}] + \beta \left\{ p (R a_t)^{1-\rho} + q E_t \left[\left(R a_t + \tilde{\Gamma}_{t+1} \tilde{\Theta}_{t+1} \right)^{1-\rho} \right] \right\} \\ < \; & (\eta + \bar{\Theta}) \beta G^{1-\rho} E_t [\tilde{\Psi}_{t+1}^{-\rho}] + \beta \left\{ p (R a_t)^{1-\rho} + q E_t \left[\left(\tilde{\Gamma}_{t+1} \underline{\Theta} \right)^{1-\rho} \right] \right\} \end{aligned} \tag{38}$$

where the last line follows from comparing the two components on each side of the inequality separately: For the second component, with $\rho > 1$, $\underline{\Theta}^{1-\rho} \geq E_t[(R a_t + \tilde{\Theta}_{t+1})^{1-\rho}]$, while for the first component, if Ψ were distributed as an untruncated lognormal we would have

$$\begin{aligned} E_t[\tilde{\Psi}_{t+1}^{1-\rho}] &= \exp(-(1-\rho)\sigma_\psi^2/2 + (1-\rho)^2 \sigma_\psi^2/2) & (39) \\ &= \exp(\rho \sigma_\psi^2/2 + \rho^2 \sigma_\psi^2/2) \exp(-\sigma_\psi^2/2 + \sigma_\psi^2/2 - 2\rho\sigma_\psi^2/2) & (40) \\ &= E_t[\tilde{\Psi}_{t+1}^{-\rho}] \exp(-2\rho\sigma_\psi^2/2) & (41) \\ &< E_t[\tilde{\Psi}_{t+1}^{-\rho}]. & (42) \end{aligned}$$

which means that if we pick wide enough truncation points $[\underline{\Psi}, \bar{\Psi}]$ we have $E_t[\tilde{\Psi}_{t+1}^{1-\rho}] < E_t[\tilde{\Psi}_{t+1}^{-\rho}]$.

Note that the impatience condition (32) implies that the first term on the RHS of (38) can be rewritten

$$(\eta + \bar{\Theta}) R \beta E_t [\tilde{\Gamma}_{t+1}^{-\rho}] (G/R) < (\eta + \bar{\Theta})(G/R). \tag{43}$$

This condition also implies that the first term on the RHS in (36) is less than m_t.

On the other hand, a_t satisfies $a_t = m_t - c_t \geq \underline{\underline{\lambda}} m_t$ for all t where $\underline{\underline{\lambda}} = (1 - \bar{\bar{\kappa}})$ is the minimal marginal propensity to save.[14] This means

$$a_t^{1-\rho} \leq \underline{\underline{\lambda}}^{1-\rho} m_t^{1-\rho}, \tag{44}$$

which implies that the term $(R a_t)^{1-\rho}$ in (38) is less than $R^{1-\rho} \underline{\underline{\lambda}}^{1-\rho} m_t^{1-\rho}$. Using this and (43), the RHS of (38) is less than

$$\begin{aligned} (\eta + \bar{\Theta})(G/R) + \beta \left\{ p (R \underline{\underline{\lambda}})^{1-\rho} m_t^{1-\rho} + q (G \underline{\Theta})^{1-\rho} E_t [\tilde{\Psi}_{t+1}^{1-\rho}] \right\} = \\ (\eta + \bar{\Theta})(G/R) + \beta p (R \underline{\underline{\lambda}})^{1-\rho} m_t^{1-\rho} + \beta q (G \underline{\Theta})^{1-\rho} E_t [\tilde{\Psi}_{t+1}^{1-\rho}]. \end{aligned} \tag{45}$$

[14] Mnemonic: λ is the Greek letter l and the amount $l$eft unspent.

Thus, combining all of these inequalities, we have that

$$\beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}\phi(\tilde{m}_{t+1})\right] < \tag{46}$$
$$R\beta E_t[\tilde{\Gamma}_{t+1}^{-\rho}]m_t + (\eta+\bar{\Theta})(G/R) + \beta p(R\underline{\underline{\lambda}})^{1-\rho}m_t^{1-\rho} + \beta q(G\underline{\Theta})^{1-\rho}E_t[\tilde{\Psi}_{t+1}^{1-\rho}].$$

Under the assumptions on parameters above, there exist $\eta > 0$ and $\varsigma \in (0,1)$ which satisfy

$$(\eta+\bar{\Theta})(G/R) + \beta q(G\underline{\Theta})^{1-\rho}E_t[\tilde{\Psi}_{t+1}^{1-\rho}] \;=\; \varsigma\eta \tag{47}$$
$$R\beta G^{-\rho}E_t[\tilde{\Psi}_{t+1}^{-\rho}] \;\leq\; \varsigma \tag{48}$$
$$\beta p(R\underline{\underline{\lambda}})^{1-\rho} \;\leq\; \varsigma. \tag{49}$$

Then we can obtain

$$\varsigma\phi(m_t) \;\geq\; \left(\eta+\overline{\Theta}\right)G^{1-\rho}\beta E_t[\tilde{\Psi}_{t+1}^{1-\rho}] + R\beta E_t[\tilde{\Gamma}_{t+1}^{-\rho}]m_t + \beta q\underline{\Theta}E_t[\tilde{\Gamma}_{t+1}^{1-\rho}] + \beta p(\underline{\underline{\lambda}}R)^{1-\rho}m_t^{1-\rho} \tag{50}$$

This means

$$\begin{aligned} & u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}(w(\tilde{m}_{t+1}) + \gamma\phi(\tilde{m}_{t+1}))\right] \qquad (51)\\ \leq\; & u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}w(\tilde{m}_{t+1})\right] + \gamma\varsigma\phi(m_t) \end{aligned}$$

so if we define

$$c_t^*(m_t) \;=\; \underset{c_t\in[\underline{\kappa}m_t,\bar{\bar{\kappa}}m_t]}{\arg\max}\left\{u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}\left\{w\left(\tilde{m}_{t+1}^*\right) + \gamma\phi\left(\tilde{m}_{t+1}^*\right)\right\}\right]\right\} \tag{52}$$

with m_{t+1}^* defined analogously to (21), then we obtain our final requirement:

$$\begin{aligned} \mathsf{T}(w+\gamma\phi) \;&=\; u(c_t^*) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}\left\{w\left(\tilde{m}_{t+1}^*\right) + \gamma\phi\left(\tilde{m}_{t+1}^*\right)\right\}\right] \\ &\leq\; u(c_t^*) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}w\left(\tilde{m}_{t+1}^*\right)\right] + \gamma\varsigma\phi(m_t) \\ &\leq\; \max_{c_t\in[\underline{\kappa}m_t,\bar{\bar{\kappa}}m_t]}\left\{u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}w(\tilde{m}_{t+1})\right] + \gamma\varsigma\phi(m_t)\right\} \\ &=\; \mathsf{T}(w) + \gamma\varsigma\phi \end{aligned}$$

where the second inequality holds because $E_t\left[\gamma\tilde{\Gamma}_{t+1}^{1-\rho}\phi(\tilde{m}_{t+1})\right] \leq \gamma\varsigma\phi(m_t)$ for all $m_t \in \mathbb{R}_{++}$ and m_{t+1} such that $c_t \in [\underline{\kappa}m_t, \bar{\bar{\kappa}}m_t]$.

Thus, the proof that T defines a contraction mapping is complete.

2.5.2 Existence of a Concave Consumption Function

We now show that the maximization problem (19) defines a sequence of continuously differentiable strictly increasing concave functions $\{c_T, c_{T-1}, ..., c_{T-k}\}$.

To do this, we need a definition. We will say that a function $n(z)$ is 'nice' if it satisfies

1. $n(z)$ is well-defined iff $z > 0$
2. $n(z)$ is strictly increasing
3. $n(z)$ is strictly concave
4. $n(z)$ is $\mathbf{C}^3$ (its first three derivatives exist)
5. $n(z) < 0$
6. $\lim_{z \downarrow 0} \; n(z) = -\infty$

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} n'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_s is nice by induction for all $s \leq T$ because $v_T(m) = u(m)$ and $u(m)$ is nice by inspection for our $u(c)$.

As a first step, define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}(\tilde{\mathcal{R}}_{t+1} a + \tilde{\xi}_{t+1}) \right]. \tag{53}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003)).

Next define $\underline{v}_t(m, c)$ as

$$\underline{v}_t(m, c) = u(c) + \mathfrak{v}_t(m - c) \tag{54}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (19) can be written as

$$v_t(m) \;\; = \;\; \max_{c} \; \underline{v}_t(m, c). \tag{55}$$

$\underline{v}_t$ is well-defined only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0} \underline{v}_t(m, c) = \lim_{c\uparrow m} \underline{v}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{v}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0} \frac{\partial \underline{v}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m} \frac{\partial \underline{v}_t(m,c)}{\partial c} = -\infty$. It follows that the $c_t(m)$ defined by

$$c_t(m) \;\; = \;\; \underset{0<c<m}{\arg\max} \; \underline{v}_t(m, c) \tag{56}$$

exists and is unique, and (19) has an internal solution that satisfies

$$u'(c_t(m)) = \mathfrak{v}_t'(m - c_t(m)). \tag{57}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $c_t(m)$ and $a_t(m) = m - c_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t(m)$ are three times continuously differentiable, using (57) we can conclude that $c_t(m)$ is continuously differentiable and

$$c_t'(m) \quad = \quad \frac{\mathfrak{v}_t''(a_t(m))}{u''(c_t(m)) + \mathfrak{v}_t''(a_t(m))}. \tag{58}$$

Similarly we can easily show that $c_t(m)$ is twice continuously differentiable (as is $a_t(m)$).[15] This implies that $v_t(m)$ is nice, since $v_t(m) = u(c_t(m)) + \mathfrak{v}_t(a_t(m))$.

Finally, strict concavity of the consumption functions is shown by Carroll and Kimball (1996).

In intuitive terms, the reason $c_t(m) < m$ is that if the consumer spent all available resources, he would arrive in period $t+1$ with assets of zero, then might earn zero noncapital income for the rest of his life (an unbroken series of zero-income events is unlikely but possible). In such a case, the budget constraint and the can't-die-in-debt condition mean that the consumer would be forced to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything.

2.5.3 T generates $\{v_{T-n+1}(m)\}_{n=1}^{\infty}$

Here we show that our operator T produces the sequence of value functions defined in (19); that is, $\mathsf{T}v_{t+1} = v_t$. The only differences between v_t as defined in (19) and $\mathsf{T}v_{t+1}$ are 1) the restriction, for the T operator, that $c_t \in [\underline{\kappa} m_t, \bar{\bar{\kappa}} m_t]$, and, 2) the use of the sup operator in the definition of T as opposed to max for v_t. We show here that these differences do not matter.

The first step is to show that the lower bound for consumption is

$$c_t(m) \geq \underline{c}(m) \equiv \underline{\kappa} m \tag{59}$$

where $\underline{\kappa}$ is defined by (31).

To see that this holds, define $\vec{c}_t(m)$ as the solution to the normalized version of the perfect foresight finite horizon problem,

$$\vec{c}_t(m_t) \quad = \quad (m_t - 1 + h_t)\underline{\kappa}_t \tag{60}$$

and note from the definition of $\underline{\kappa}_t$, (15), that

$$\underline{\kappa}_{T-n-1} \quad < \quad \underline{\kappa}_{T-n} \tag{61}$$

so that $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa} < \underline{\kappa}_t$.

[15]See Appendix A.

Furthermore, Carroll and Kimball (1996) show, in finite horizon models of the class considered here, that the MPC for a consumer facing uncertainty is strictly greater than the MPC for the corresponding perfect foresight consumer:

$$c'_{T-n}(m) \quad > \quad \underline{\kappa}_{T-n}. \tag{62}$$

Combining these, we have

$$c'_t(m) \quad > \quad \underline{\kappa}_t > \underline{\kappa} \tag{63}$$

which justifies our earlier labeling of $\underline{\kappa}$ as the 'minimal' marginal propensity to consume. Since $\lim_{m\downarrow 0} c_t(m) = 0$ the fact that $\underline{\kappa}$ is the lower bound MPC implies

$$c_t(m) \quad > \quad \underline{\kappa}m \tag{64}$$

as required for all $t < T$ and $m > 0$.

The next step is to show that

$$c_t(m) \leq \bar{\bar{\kappa}}m \text{ for all } t \leq T-1 \tag{65}$$

where $\bar{\bar{\kappa}}$ is defined by (30). Begin by defining

$$e_t(m) \quad = \quad c_t(m)/m \tag{66}$$

$$\bar{\kappa}_t \quad = \quad \lim_{m\downarrow 0} e_t(m) > 0 \tag{67}$$

and note that this limit exists and is strictly positive because continuous differentiability and strict concavity of $c_t(m)$ along with $c_t(m) > 0$ imply that $e_t(m)$ is continuous, decreasing, and $0 < e_t(m) < 1$ for $t < T$.

The Euler equation says that

$$c_t(m_t)^{-\rho} \quad = \quad \beta R E_t\left[\left(c_{t+1}(\tilde{m}_{t+1})\tilde{\Gamma}_{t+1}\right)^{-\rho}\right] \tag{68}$$

$$(m_t e_t(m_t))^{-\rho} \quad = \quad \beta R E_t\left[\left(e_{t+1}(\tilde{m}_{t+1})\tilde{\Gamma}_{t+1}\tilde{m}_{t+1}\right)^{-\rho}\right] \tag{69}$$

$$e_t(m_t)^{-\rho} \quad = \quad \beta R E_t\left[\left(e_{t+1}(\tilde{m}_{t+1})\left(\frac{Ra_t(m_t)+\tilde{\Gamma}_{t+1}\tilde{\xi}_{t+1}}{m_t}\right)\right)^{-\rho}\right] \tag{70}$$

$$\begin{aligned} &= \quad q m_t^{\rho}\beta R \iint \left(e_{t+1}(\mathcal{R}_{t+1}a_t(m_t)+\Theta/q)\left(\mathcal{R}_{t+1}a_t(m_t)+\Theta/q\right)\right)^{-\rho} dF_\Psi dF_\Theta \\ &\quad +p\beta R^{1-\rho}\int\left(e_{t+1}(\mathcal{R}_{t+1}a_t(m_t))\frac{a_t(m_t)}{m_t}\right)^{-\rho} dF_\Psi \end{aligned} \tag{71}$$

but note that since $\lim_{m\downarrow 0} a_t(m) = 0$ the limit of the double integral in (71) tends toward bounds defined by $(e_{t+1}(\underline{\Theta}/q)\underline{\Theta}/q)^{-\rho}$ and $(e_{t+1}(\bar{\Theta}/q)\bar{\Theta}/q)^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by q goes to zero as m_t^ρ goes to zero. The integral in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. It follows that $\bar{\kappa}_t$ satisfies $(\bar{\kappa}_t)^{-\rho} = \beta p R^{1-\rho}(\bar{\kappa}_{t+1})^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. We can conclude that

$$\bar{\kappa}_t = (\beta p R)^{-\frac{1}{\rho}} R(1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \tag{72}$$

$$\underbrace{R^{-1}(\beta p R)^{1/\rho}}_{\equiv \underline{\lambda}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \tag{73}$$

which implies

$$(\underline{\lambda}\bar{\kappa}_t)^{-1} = (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \tag{74}$$

$$\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) = \underline{\lambda}\bar{\kappa}_{t+1}^{-1} \tag{75}$$

$$\bar{\kappa}_t^{-1} = 1 + \underline{\lambda}\bar{\kappa}_{t+1}^{-1}. \tag{76}$$

Then $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=1}^{\infty}$ is an increasing convergent sequence if

$$0 \leq \quad \underline{\lambda} \quad < 1 \tag{77}$$

$$0 \leq \quad p^{1/\rho}R^{-1}(R\beta)^{1/\rho} \quad < 1 \tag{78}$$

but since $0 \leq p \leq 1$ by the definition of probability, this is a weaker condition than $R^{-1}(R\beta)^{1/\rho} < 1$ which is the nonpathological patience condition (17) already imposed.

Since $\bar{\kappa}_T = 1$, from (76) we have that

$$\bar{\kappa}_{T-1} = \left[1 + R^{-1}(\beta p R)^{1/\rho}\right]^{-1} = \bar{\bar{\kappa}} \tag{79}$$

and since $\bar{\kappa}_{T-n}$ is a decreasing sequence we have $\bar{\kappa}_{T-n} < \bar{\bar{\kappa}}$ for $n > 1$, justifying our earlier labeling of $\bar{\bar{\kappa}}$ as the maximal MPC.

The foregoing analysis permits us to conclude that the solution to our original problem is identical to the solution to the problem

$$v_t(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\bar{\kappa}} m_t]} \left\{ u(c_t) + \beta E_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}(\tilde{m}_{t+1}) \right] \right\}. \tag{80}$$

The only difference between (80) and $\mathsf{T}v_{t+1}$ is the use of the max rather than the sup operator in (80). But the $u(c)$ and $E_{t+1}[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}(\tilde{\mathcal{R}}_{t+1}(m-c) + \tilde{\Gamma}_{t+1}\tilde{\xi}_{t+1})]$ functions are both continuous in c, and the sup of a sum of continuous functions over a bounded compact set is a max in that set, so we can replace the sup operator with a max operator without loss of generality. Thus, v_t is equivalent to $\mathsf{T}v_{t+1}$ as required.

2.5.4 Convergence of Consumption Functions

Application of Boyd's theorem demonstrates that the the v_{t-n} functions converge in a $\phi-$bounded space. What we are really interested in, however, is convergence of the consumption policy functions. The proof that the former implies the latter is uninteresting and is relegated to appendices B and C.

2.6 Liquidity Constraints as a Limit

This section demonstrates that a related problem commonly considered in the literature (e.g. by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the model considered here as the probability p of the zero-income event approaches zero.

Formally, suppose we change the description of the problem by making the following two assumptions:

$$
\begin{aligned}
p &= 0 && (81)\\
c_t &\leq m_t, && (82)
\end{aligned}
$$

and suppose we designate the solution to this consumer's problem $\grave{c}_t(m_t)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed p as $c_t(m_t;p)$ where we separate the arguments by a semicolon to distinguish between m_t, which is a state variable, and p, which is not. The proposition we wish to demonstrate is

$$\lim_{p\downarrow 0} c_t(m_t;p) = \grave{c}_t(m_t). \tag{83}$$

We will first examine the problem in period $T-1$, then argue that the key result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $R = \beta = G = 1$, and there are no permanent shocks, $\Psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $c_T(m_T)$ (in practice, this will be $c_T(m_T) = m_T$), consider the unrestrained optimization problem

$$\grave{a}^*_{T-1}(m) = \arg\max_a \left\{ u(m-a) + \beta \int_{\underline{\Theta}}^{\bar{\Theta}} v_T(a+\Theta)dF_\Theta \right\}. \tag{84}$$

As usual, the envelope theorem tells us that $v'_T(m) = u'(c_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined *a la* (53) as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\Theta}}^{\bar{\Theta}} u'(c_T(a+\Theta))dF_\Theta, \tag{85}$$

and the solution to (84) will satisfy

$$u'(m-a) = \beta\grave{\mathfrak{v}}'_{T-1}(a). \tag{86}$$

Defining $\grave{a}^*_{T-1}(m)$ as the function that solves (86), $\grave{a}^*_{T-1}(m)$ answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \geq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $p > 0$). The restrained consumer's actual asset position will be

$$\grave{a}_{T-1}(m) \;=\; \min[0, \grave{a}^*_{T-1}(m)], \tag{87}$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^{\#}_{T-1} \;=\; \left(\beta \grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho} \tag{88}$$

is the cusp value of m at which the constraint makes the transition from binding to not binding.

Analogously to (86), defining

$$\mathfrak{v}'_{T-1}(a;p) \;\equiv\; \left\{\beta \left[pa^{-\rho} + (1-p)\int_{\underline{\Theta}}^{\bar{\Theta}} \left(c_T(a + \Theta/(1-p))\right)^{-\rho} dF_{\Theta}\right]\right\}, \tag{89}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} \;=\; \beta \mathfrak{v}'_{T-1}(a;p) \tag{90}$$

with solution $a_{T-1}(m;p)$. Now note that for a fixed $a > 0$, $\lim_{p\downarrow 0} \mathfrak{v}'_{T-1}(a;p) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (86) and (90) are identical, this means that for such an m $\lim_{p\downarrow 0} a_{T-1}(m;p) = \grave{a}^*_{T-1}(m)$. That is, for any fixed value of $m > m^{\#}_{T-1}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $p \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^{\#}_{T-1}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in in (89) is $\lim_{a\downarrow 0} pa^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{p\downarrow 0} \mathfrak{v}'_{T-1}(a;p) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{a}^*_{T-1}(m) < 0$, and we showed earlier that

$\lim_{p\downarrow 0} a^{*}_{T-1}(m;p) = \grave{a}^{*}_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^{\#}_{T-1}$.

These arguments demonstrate that for any $m > 0$, $\lim_{p\downarrow 0} c_{T-1}(m;p) = \grave{c}_{T-1}(m)$ which is the period $T-1$ version of (83). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical in the limit, so (83) holds.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (30) for the maximal marginal propensity to consume satisfies

$$\lim_{p\downarrow 0} \bar{\kappa} \;=\; 1, \tag{91}$$

which makes sense because the marginal propsensity to consume for a constrained restrained consumer is one by the definition of 'constrained.'

2.7 Discussion of Conditions Required for Convergence

For the proof to hold we needed to impose two parametric conditions, (32) and (17). To understand these conditions, consider a comparison to the infinite horizon perfect foresight problem.

Both problems impose the nonpathological patience condition, (17). Surprisingly, however, our other restriction, (32), can actually be weaker than the second condition required for the perfect foresight problem, $G < R$. To see this, raise both sides of (32) to the $(1/\rho)$ power and extract the G term to obtain:

$$(R\beta)^{1/\rho}\left(E[\Psi^{-\rho}]\right)^{-1/\rho} \;<\; G \tag{92}$$

and note that if we turn off the permanent shocks $\underline{\Psi} = \bar{\Psi} = 1$ this reduces to

$$(R\beta)^{1/\rho} \;<\; G. \tag{93}$$

A particularly transparent case is $R = 1$ and $G \geq 1$. Human wealth is infinite so the perfect foresight infinite horizon model has no solution, but if $\beta < 1$ our model does have a solution.

To help interpret our condition, consider an infinite horizon perfect foresight consumer who does satisfy $G < R$ and who arrives in period t with beginning-of-period resources of zero, so that he has only human wealth. If initial permanent income is P_t then $W_t = H_t = P_t(R/(R-G))$ and the formula for consumption (15) becomes

$$C_t \;=\; \left(\frac{R-(R\beta)^{1/\rho}}{R-G}\right) P_t \tag{94}$$

so the condition $(R\beta)^{1/\rho} < G$ guarantees that consumption will exceed labor income of P_t. Thus, this consumer is 'impatient' in the sense of wanting to borrow against future labor income to finance current consumption.

The presence of permanent shocks tightens the restriction, since if Ψ is nondegenerate then $E[\Psi^{-\rho}]^{-1/\rho} > 1$. The interpretation of this effect is simple: The presence of uncertainty in permanent income increases the consumer's precautionary saving motive, and therefore increases the degree of patience; the condition requires that even after this boost to the saving motive, the consumer remains impatient in the relevant sense.

The simplest intuition for why our model has a solution when $G > R$ comes from the essential equivalence between the precautionary saving motive and liquidity constraints. Consider a version of the perfect foresight model with liquidity constraints. This model *does* have a well defined solution for $G > R$ because even a consumer with considerable current resources cannot spend an infinite amount (even if the PDV of future labor income is infinite) because that would violate the constraint. Furthermore, the amount the consumer is willing to spend today is limited by the knowledge that, because of impatience, they will be constrained at some point in the future.

The precautionary motive induced by the labor income risk can be thought of as being like a smoothed version of liquidity constraints. As cash declines toward zero, the size of the risk relative to the size of cash increases, which means that the relative variation in consumption increases, which means that the intensity of the precautionary motive increases. For a more rigorous and detailed treatment of the relationship between precautionary saving and liquidity constraints, see Carroll and Kimball (2001).

3 Analysis of the Converged Consumption Function

Figures 2 and 3a,b capture the main properties of the converged consumption rule.[16] Figure 2 shows the expected consumption growth factor $E_t[\tilde{C}_{t+1}/C_t]$ for a consumer using the converged consumption rule, while Figures 3a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

I will demonstrate five features of behavior captured, or suggested, by the figures. First, as $m_t \to \infty$ the expected consumption growth factor goes to $(R\beta)^{1/\rho}$, indicated by the lower bound in figure 2, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - R^{-1}(R\beta)^{1/\rho})$ (figure 3), the same as in the perfect foresight case. Second, as $m_t \to 0$ the consumption growth factor approaches ∞ (figure 2) and the MPC approaches $\bar{\kappa} = (1 - R^{-1}(R\beta p)^{1/\rho})$ (figure 3). Third (figure 2), there is a target

[16] These figures reflect the converged rule corresponding to the parameter values indicated in table 1.

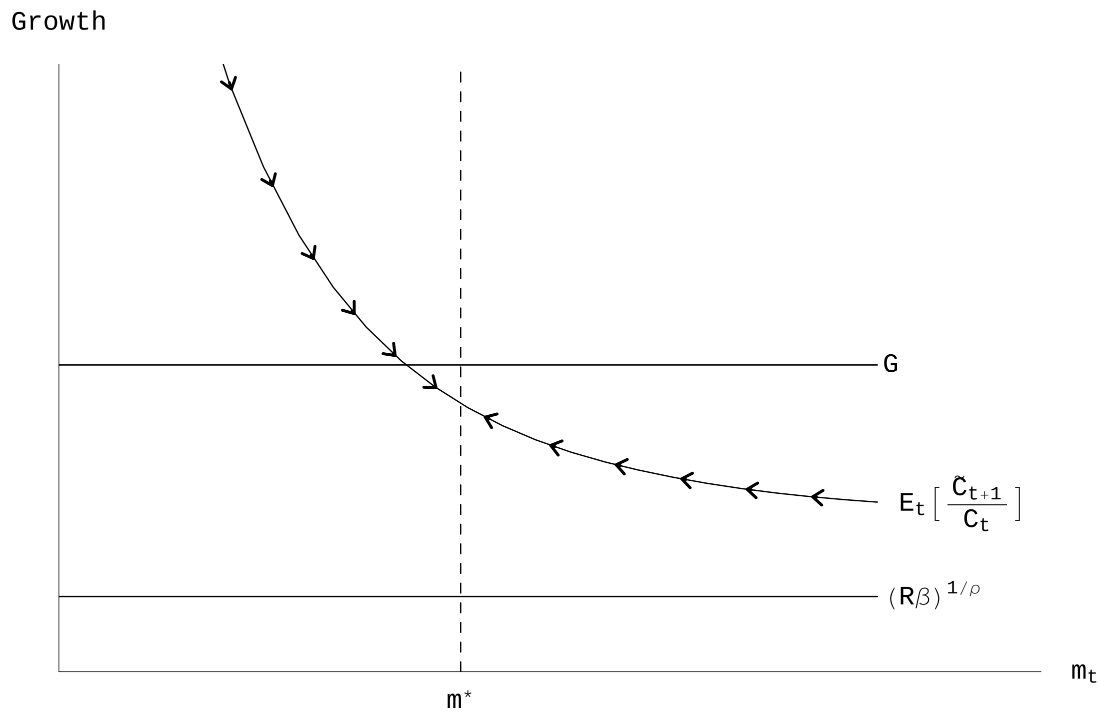

Figure 2: Target Saving, Expected Consumption Growth, and Permanent Income Growth

cash-on-hand-to-income ratio m^* such that if $m_t = m^*$ then $E_t[\tilde{m}_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $E_t[\tilde{C}_{t+1}/C_t]$ curve) that target ratio is stable in the sense that if $m_t < m^*$ then cash-on-hand will rise (in expectation), while if $m_t > m^*$ then above the target, it will fall (in expectation). Fourth (figure 2), at the target m, the expected growth rate of consumption is less than the expected growth rate of permanent labor income. The final proposition suggested by figure 2 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.

Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. From Carroll and Kimball (1996) we know that the finite-horizon consumption functions $c_{T-n}(m)$ are twice continuously differentiable and strictly concave, and we have shown above that these converge to a continuous function $c(m)$. It does not follow that $c(m)$ is twice continuously differentiable, but I will assume that it is.

3.1 Limits as $m_t \rightarrow \infty$

Recall our definition of

$$\underline{c}(m) \quad = \quad \underline{\kappa} m \tag{95}$$

in (59) which is the solution to an infinite-horizon problem with no labor income ($\xi_t = 0 \; \forall \; t$); clearly $\underline{c}(m) < c(m)$, since allowing the possibility of future labor income cannot reduce current consumption.

Assume that $G < R$ so that the infinite horizon perfect foresight solution,

$$\vec{c}(m) \quad = \quad (m - 1 + h)\underline{\kappa}, \tag{96}$$

exists. (We discuss the $G \geq R$ case below). This constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimball (1996) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

Thus, we can write

$$\underline{c}(m) < \quad c(m) \quad < \vec{c}(m) \tag{97}$$

$$1 < \quad c(m)/\underline{c}(m) \quad < \vec{c}(m)/\underline{c}(m). \tag{98}$$

But

$$\lim_{m \to \infty} \vec{c}(m)/\underline{c}(m) \quad = \quad \lim_{m \to \infty} (m - 1 + h)/m \tag{99}$$

$$= \quad 1. \tag{100}$$

Hence, as $m \rightarrow \infty, c(m)/\underline{c}(m) \rightarrow 1$, and the continuous differentiability and strict concavity of $c'(m)$ therefore imply

$$\lim_{m \to \infty} c'(m) = \underline{c}'(m) = \vec{c}'(m) = \underline{\kappa} \tag{101}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\vec{c}(m)$ or lower than $\underline{c}(m)$.

Figure 3 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \rightarrow \infty$:

$$\lim_{m_t \to \infty} E_t[\tilde{C}_{t+1}/C_t] \quad = \quad \lim_{m_t \to \infty} E_t[\tilde{\Gamma}_{t+1}\tilde{c}_{t+1}/c_t]. \tag{102}$$

But

$$\lim_{m_t \to \infty} \Gamma_{t+1}\underline{c}(m_{t+1})/\vec{c}(m_t) = \lim_{m_t \to \infty} \Gamma_{t+1}\vec{c}(m_{t+1})/\underline{c}(m_t) = \lim_{m_t \to \infty} \Gamma_{t+1} m_{t+1}/m_t, \tag{103}$$

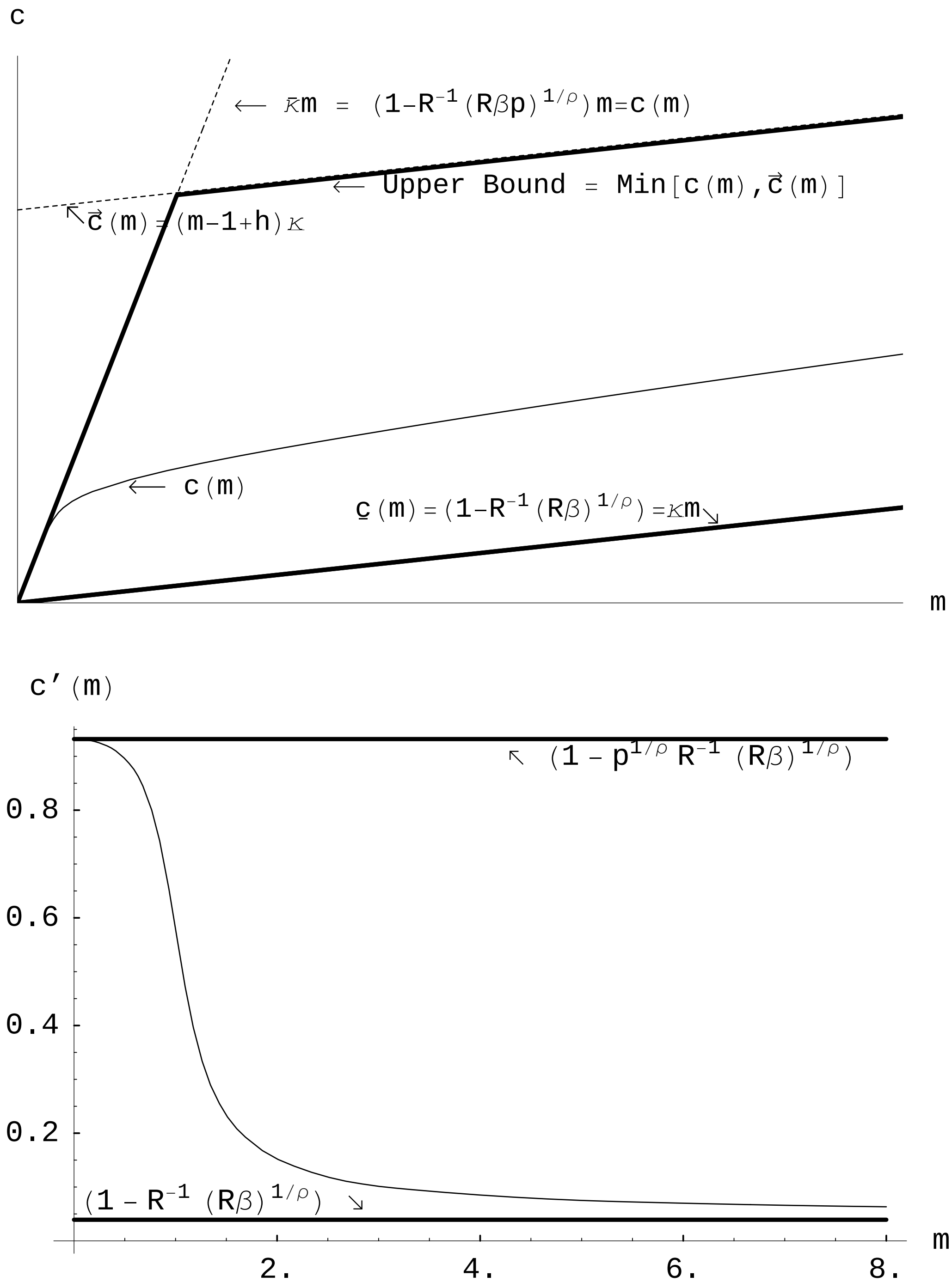

Figure 3: Bounds (in Dark) of $c(m)$ and $c'(m)$ as $m_t \to \infty$ and $m_t \to 0$

and

$$\lim_{m_t\to\infty} \Gamma_{t+1} m_{t+1}/m_t = \lim_{m_t\to\infty} \left(\frac{Ra(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t} \right) \tag{104}$$

$$= (R\beta)^{1/\rho} \tag{105}$$

because $\lim_{m_t\to\infty} a'(m) = R^{-1}(R\beta)^{1/\rho}$ and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (G\bar{\Psi}\bar{\Theta}/q)/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$(R\beta)^{1/\rho} \leq \lim_{m_t\to\infty} E_t[\tilde{C}_{t+1}/C_t] \leq (R\beta)^{1/\rho} \tag{106}$$

so as cash goes to infinity, consumption growth approaches its value in the perfect foresight model.

This argument applies equally well to the problem of the restrained consumer, because as m approaches infinity the constraint becomes irrelevant.

Of course, the constraint never becomes irrelevant if human wealth is infinite. We ruled out infinite human wealth at the beginning of this section by assuming $R > G$. We now consider the case where $R \leq G$.

Think first about the perfect foresight model with constraints. With infinite human wealth the constraint never becomes irrelevant, but the date at which it binds recedes arbitrarily far in the future as nonhuman approaches infinity. Between the present and that receding date, the optimal growth rate of consumption approaches $(R\beta)^{1/\rho}$, which in turn implies that the marginal propensity to consume approaches the same limit as before, $(1 - R^{-1}(R\beta)^{1/\rho})$. (For further discussion of the perfect foresight liquidity constrained case see Carroll and Kimball (2001)).

We argued earlier, however, that consumption of the perfect foresight liquidity constrained consumer is an upper bound to consumption of our consumers, which implies again that the marginal propensity to consume approaches $\underline{\kappa}$ as m approaches infinity; thus the above arguments continue to go through.

3.2 Limits as $m_t \to 0$

Now consider the limits of behavior as m_t gets arbitrarily small.

Equation (75) implies that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - R^{-1}(pR\beta)^{1/\rho}. \tag{107}$$

Defining $e(m) = c(m)/m$ as before we have

$$\lim_{m\downarrow 0} e(m) = (1 - p^{1/\rho} R^{-1}(R\beta)^{1/\rho}) = \bar{\kappa} \tag{108}$$

Now differentiate

$$
\begin{aligned}
e'(m) &= m^{-1}c'(m) - m^{-2}c(m) && (109)\\
me'(m) &= c'(m) - c(m)/m && (110)\\
c'(m) &= e(m) + me'(m) && (111)
\end{aligned}
$$

and using the continuous differentiability of the consumption function along with $0 < e(m) < 1$ we have

$$
\lim_{m \downarrow 0} c'(m) = \lim_{m \downarrow 0} e(m) = \bar{\kappa}. \tag{112}
$$

Figure 3 confirms that standard simulation methods obtain this limit for the MPC as m approaches zero.

For consumption growth, we have

$$
\begin{aligned}
\lim_{m_t \downarrow 0} E_t\left[\left(\frac{c(\tilde{m}_{t+1})}{c(m_t)}\right)\tilde{\Gamma}_{t+1}\right] &> \lim_{m_t \downarrow 0} E_t\left[\left(\frac{\underline{c}(\tilde{\mathcal{R}}_{t+1}a(m_t) + \tilde{\xi}_{t+1})}{\bar{\bar{\kappa}} m_t}\right)\tilde{\Gamma}_{t+1}\right]\\
&> \lim_{m_t \downarrow 0} E_t\left[\left(\frac{\underline{c}(\tilde{\xi}_{t+1})}{\bar{\bar{\kappa}} m_t}\right)\tilde{\Gamma}_{t+1}\right] && (113)\\
&= \infty && (114)
\end{aligned}
$$

where the last line follows because the minimum possible realization of ξ_{t+1} is $\underline{\Theta} > 0$ so the minimum possible value of expected next-period consumption is positive.

The same arguments establish $\lim_{m \downarrow 0} E_t[C_{t+1}/C_t] = \infty$ for the problem of the restrained consumer.

3.3 There exists exactly one target cash-on-hand ratio, which is stable

Define the target cash-on-hand-to-income ratio m^* as the value of m such that

$$
E_t[\tilde{m}_{t+1}/m_t] = 1 \text{ if } m_t = m^*. \tag{115}
$$

We prove existence by noting that $E_t[\tilde{m}_{t+1}/m_t]$ is continuous on $m_t > 0$, and takes on values both above and below 1.

Specifically, the same logic used in section 3.2 shows that $\lim_{m_t \downarrow 0} E_t[\tilde{m}_{t+1}/m_t] = \infty$.

The limit as m_t goes to infinity is

$$
\begin{aligned}
\lim_{m_t \to \infty} E_t[\tilde{m}_{t+1}/m_t] &= \lim_{m_t \to \infty} E_t\left[\frac{\tilde{\mathcal{R}}_{t+1}a(m_t) + \tilde{\xi}_{t+1}}{m_t}\right] && (116)\\
&= E_t[(R/\tilde{\Gamma}_{t+1})R^{-1}(R\beta)^{1/\rho}] && (117)\\
&= E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}] && (118)
\end{aligned}
$$

and appendix D shows that

$$E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}] \quad < \quad 1 \tag{119}$$

under our assumptions.

Stability means that in a local neighborhood of m^*, values of m_t above m^* will result in a smaller ratio of $E_t[\tilde{m}_{t+1}/m_t]$ than at m^*. That is, if $m_t > m^*$ then $E_t[m_{t+1}/m_t] < 1$. This will be true if

$$\left(\frac{d}{dm_t}\right) E_t[\tilde{m}_{t+1}/m_t] \quad < \quad 0 \tag{120}$$

in a neighborhood around $m_t = m^*$. But

$$\begin{aligned}\left(\frac{d}{dm_t}\right) E_t[\tilde{m}_{t+1}/m_t] &= E_t\left[\left(\frac{d}{dm_t}\right)\left[\tilde{\mathcal{R}}_{t+1}(1 - c(m_t)/m_t) + \tilde{\xi}_{t+1}/m_t\right]\right] \\ &= E_t\left[\frac{\tilde{\mathcal{R}}_{t+1}(c(m_t) - c'(m_t)m_t) - \tilde{\xi}_{t+1}}{m_t^2}\right]\end{aligned}$$

which will be negative if its numerator is negative. Define $\nu(m_t)$ as the expectation of the numerator,

$$\nu(m_t) = E_t[\tilde{\mathcal{R}}_{t+1}](c(m_t) - c'(m_t)m_t) - 1. \tag{121}$$

Now consider the definition of the target level of cash m^* such that $E_t[\tilde{m}_{t+1}] = m_t = m^*$:

$$\begin{aligned} E_t[\tilde{m}_{t+1}] &= E_t[\tilde{\mathcal{R}}_{t+1}(m_t - c_t) + \tilde{\xi}_{t+1}] && (122) \\ m^* &= E_t[\tilde{\mathcal{R}}_{t+1}](m^* - c(m^*)) + 1 && (123) \\ E_t[\tilde{\mathcal{R}}_{t+1}]c(m^*) &= 1 - (1 - E_t[\tilde{\mathcal{R}}_{t+1}])m^*. && (124) \end{aligned}$$

At the target, equation (121) is

$$\nu(m^*) \quad = \quad E_t[\tilde{\mathcal{R}}_{t+1}]c(m^*) - E_t[\tilde{\mathcal{R}}_{t+1}]c'(m^*)m^* - 1. \tag{125}$$

Substituting for the first term in this expression using (124) gives

$$\begin{aligned}
\nu(m^*) &= 1-(1-E_t[\tilde{\mathcal{R}}_{t+1}])m^* - E_t[\tilde{\mathcal{R}}_{t+1}]c'(m^*)m^* - 1 && (126)\\
&= m^*\Big(E_t[\tilde{\mathcal{R}}_{t+1}] - 1 - E_t[\tilde{\mathcal{R}}_{t+1}]c'(m^*)\Big) && (127)\\
&= m^*\Big(E_t[\tilde{\mathcal{R}}_{t+1}](1-c'(m^*)) - 1\Big) && (128)\\
&< m^*\Big(E_t[\tilde{\mathcal{R}}_{t+1}](1-(1-R^{-1}(R\beta)^{1/\rho})) - 1\Big) && (129)\\
&= m^*\Big(E_t[\tilde{\mathcal{R}}_{t+1}](R^{-1}(R\beta)^{1/\rho}) - 1\Big) && (130)\\
&= m^*\left(\underbrace{E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}]}_{<1 \text{ from } (119)} - 1\right) && (131)\\
&< 0. && (132)
\end{aligned}$$

We have now proven that some target m^* must exist, and that at any such m^* the solution is stable. Nothing we have said so far, however, rules out the possibility that there will be multiple values of m that satisfy the definition (115) of a target.

Multiple targets can be ruled out as follows. Suppose there exist multiple targets, indexed by i, so that the target with the smallest value is $m^{*,1}$. The argument just completed implies that since $E_t[\tilde{m}_{t+1}/m_t]$ is continuously differentiable there must exist some small ϵ such that $E_t[\tilde{m}_{t+1}/m_t] < 1$ for $m_t = m^{*,1}+\epsilon$. (Continuous differentiability of $E_t[\tilde{m}_{t+1}/m_t]$ follows from the continous differentiability of $c(m_t)$).

Now assume there exists a second value of m satisfying the definition of a target, $m^{*,2}$. Since $E_t[\tilde{m}_{t+1}/m_t]$ is continuous, it must be approaching 1 from below as $m_t \rightarrow m^{*,2}$, since by the intermediate value theorem it could not have gone above 1 between $m^{*,1}+\epsilon$ and $m^{*,2}$ without passing through 1, and by the definition of $m^{*,2}$ it cannot have passed through 1 before reaching $m^{*,2}$. But saying that $E_t[\tilde{m}_{t+1}/m_t]$ is approaching 1 from below as $m_t \rightarrow m^{*,2}$ implies that

$$\left(\frac{d}{dm_t}\right) E_t[\tilde{m}_{t+1}/m_t] \quad > \quad 0 \tag{133}$$

at $m_t = m^{*,2}$. However, we just showed above that precisely the opposite of equation (133) holds for any m that satisfies the definition of a target. Thus, assuming the existence of more than one target implies a contradiction, and so there must be only one target m^*.

The foregoing arguments rely on the continuous differentiability of $c(m)$, so the arguments do not directly go through for the restrained consumer's problem in which the existence of liquidity constraints can lead to discrete changes in the slope $c'(m)$ at particular values of m. But we can use the fact that the restrained model is the limit of the baseline model as $p \downarrow 0$ to conclude that there must be a unique target cash level in the restrained model.

If consumers are sufficiently impatient, however, the target level in the restrained model will be $m^* = E_t[\tilde{\xi}_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $a(1) = 0$, then the target level of m in the restrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of cash-on-hand will be greater than the expected level of income.

3.4 Expected consumption growth at target m is less than expected permanent income growth

In figure 2 the intersection of the target cash-on-hand ratio locus at m^* with the expected consumption growth curve lies below the intersection with the horizontal line representing the expected growth rate of permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $E_t[\tilde{m}_{t+1}] = m^* = m_t$ then

$$
\begin{aligned}
E_t\left[\frac{\tilde{\Gamma}_{t+1}c(\tilde{m}_{t+1})}{c(m_t)}\right] &< E_t\left[\left(\frac{\tilde{\Gamma}_{t+1}(c(m^*) + c'(m^*)(\tilde{m}_{t+1} - m^*))}{c(m^*)}\right)\right] && (134)\\
&= E_t\left[\tilde{\Gamma}_{t+1}\left(1 + \left(\frac{c'(m^*)}{c(m^*)}\right)(\tilde{m}_{t+1} - m^*)\right)\right] && (135)\\
&= G + \left(\frac{c'(m^*)}{c(m^*)}\right) E_t\left[\tilde{\Gamma}_{t+1}(\tilde{m}_{t+1} - m^*)\right] && (136)\\
&= G + \left(\frac{c'(m^*)}{c(m^*)}\right)\left[E_t[\Gamma_{t+1}]\underbrace{E_t[\tilde{m}_{t+1} - m^*]}_{=0} + \text{cov}_t(\tilde{\Gamma}_{t+1}, \tilde{m}_{t+1})\right] && (137)
\end{aligned}
$$

and since $m_{t+1} = (R/\Gamma_{t+1})a(m^*) + \xi_{t+1}$ and $a(m^*) > 0$ it is clear that $\text{cov}_t(\tilde{\Gamma}_{t+1}, \tilde{m}_{t+1}) < 0$ which implies that the entire term added to G in (137) is negative, as required.

3.5 Expected consumption growth is a declining function of m_t (or is it?)

Figure 2 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$
\begin{aligned}
\Upsilon &= C_{t+1}/C_t && (138)\\
&= \Gamma_{t+1}c(\mathcal{R}_{t+1}a(m_t) + \xi_{t+1})/c(m_t) && (139)
\end{aligned}
$$

and the proposition in which we are interested is

$$(d/dm_t)E[\Upsilon] \quad < \quad 0. \tag{140}$$

Now define

$$\begin{aligned} \Upsilon' &\equiv d\Upsilon/dm_t & (141) \\ &= \Gamma_{t+1}\left(\frac{c'(m_{t+1})\mathcal{R}_{t+1}a'(m_t)c(m_t) - c(m_{t+1})c'(m_t)}{c(m_t)^2}\right) & (142) \end{aligned}$$

and note that we can differentiate through the expectations operator so (140) is equivalent to

$$E[\Upsilon'] \quad < \quad 0. \tag{143}$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $c'_{t+1} \equiv c'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = R$ equation (142) can be rewritten

$$\begin{aligned} c_t\Upsilon' &= c'_{t+1}a'_tR - c'_t\Gamma_{t+1}c_{t+1}/c_t & (144) \\ &= c'_{t+1}a'_tR - c'_t\Upsilon & (145) \end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned} 1 &= R\beta E[\Upsilon^{-\rho}] & (146) \\ 0 &= E[\Upsilon^{-\rho-1}\Upsilon'] & (147) \\ &= E[\Upsilon^{-\rho-1}]E[\Upsilon'] + \text{cov}(\Upsilon^{-\rho-1}, \Upsilon') & (148) \\ E[\Upsilon'] &= -\text{cov}(\Upsilon^{-\rho-1}, \Upsilon')/E[\Upsilon^{-\rho-1}] & (149) \end{aligned}$$

but since $\Upsilon > 0$ we can see from (149) that (143) is equivalent to

$$\text{cov}(\Upsilon^{-\rho-1}, \Upsilon') \quad > \quad 0 \tag{150}$$

which, using (145), will be true if

$$\text{cov}(\Upsilon^{-\rho-1}, c'_{t+1}a'_tR - c'_t\Upsilon) \quad > \quad 0 \tag{151}$$

which in turn will be true if both

$$\text{cov}(\Upsilon^{-\rho-1}, c'_{t+1}) \quad > \quad 0 \tag{152}$$

and

$$\text{cov}(\Upsilon^{-\rho-1}, \Upsilon) \quad < \quad 0. \tag{153}$$

The latter proposition is obviously true. The former will be true if

$$\text{cov}\left((G\Psi_{t+1}c(m_{t+1}))^{-\rho-1}, c'(m_{t+1})\right) \;\; > \;\; 0$$

where recall that $m_{t+1} = (R/G\Psi_{t+1})a_t + \xi_{t+1}$.

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of Ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $G\Psi_{t+1}$, so the effect of a higher Ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term.

I have constructed an example in which this perverse effect dominates. However, extreme assumptions were required ($p < 0.000000001$, very small transitory shocks) and the region in which $\Upsilon' > 0$ was tiny. In practice, for plausible parametric choices, $E[\Upsilon'] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy (we will continue to take the aggregate interest rate as exogenous and constant). It is also possible, though more difficult, to solve a closed-economy version of the model where the interest rate is endogenous; see Carroll (2000) for an example.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above.

4.1 Convergence of the Cross-Section Distribution

A recent paper by Szeidl (2002) proves that such a population will be characterized by an ergodic (invariant) distribution of m which induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$. (Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper).

The proof of convergence does not yield any sense of how quickly convergence occurs, which in principle depends on all of the parameters of the model as well as the initial conditions. To build intuition, Figure 4 supplies an example in which a population begins with a particularly simple distribution that is far from the ergodic one:

$$m_{i,1} \quad = \quad \xi_{i,1}, \tag{154}$$

which would characterize a population in which all assets had been wiped out immediately before the receipt of period 1's labor income. The figure plots the distributions of a (for technical reasons, this is slightly better than plotting m) at the ends of 1, 4, 10, and 40 periods.

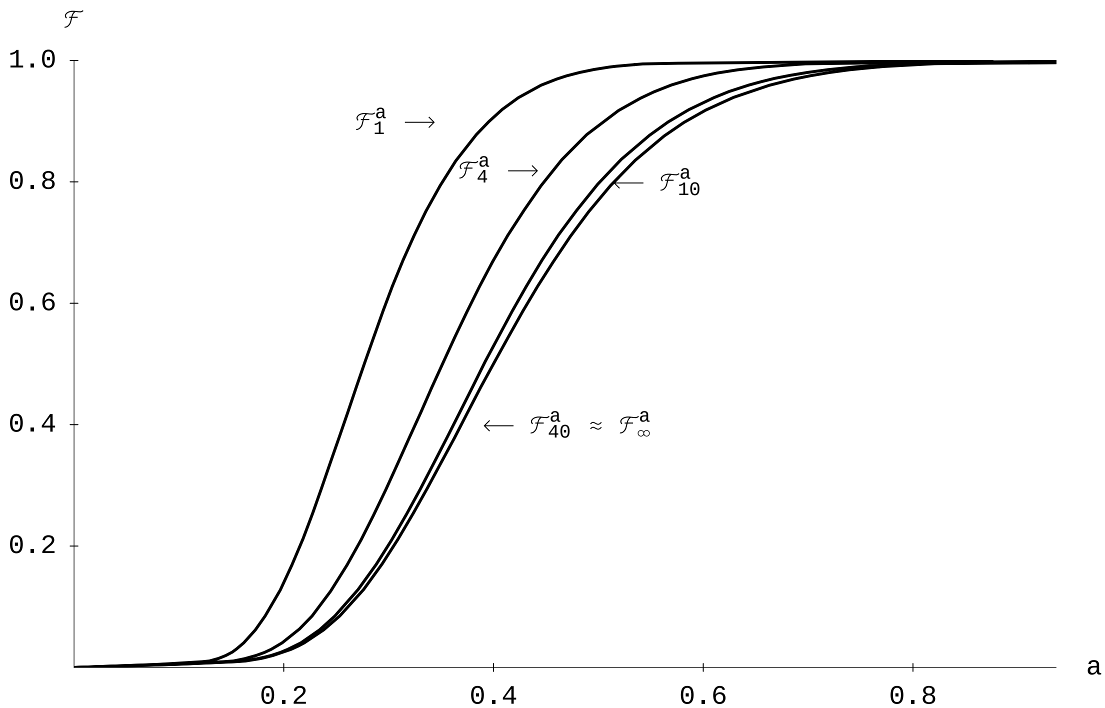

Figure 4: Convergence of $\mathcal{F}^a$ to Invariant Distribution

As the figure indicates, under these parameter values convergence of the CDF to the invariant distribution has largely been accomplished within 10 periods. By 40 periods, the distribution is indistinguishable from the invariant distribution.

4.2 Consumption and Income Growth at the Household Level

It is useful to define the operator $\mathcal{E}_t\{\}$ which yields the mean value of its argument in the population, as distinct from the expectations operator represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}
\mathcal{E}\left\{\Delta \log C_{i,t+1}\right\} &= \mathcal{E}\left\{\log c_{i,t+1}P_{i,t+1} - \log c_{i,t}P_{i,t}\right\} \\
&= \mathcal{E}\left\{\log P_{i,t+1} - \log P_{i,t} + \log c_{i,t+1} - \log c_{i,t}\right\} \\
&= \mathcal{E}\left\{\log P_{i,t+1} - \log P_{i,t}\right\} + \mathcal{E}\left\{\log c_{i,t+1} - \log c_{i,t}\right\} \\
&= g - \sigma_{\psi}^2/2, \qquad (155)
\end{aligned}$$

where the last equality follows because the invariance of $\mathcal{F}^c$ means that $\mathcal{E}\left\{\log c_{i,t+1}\right\} = \mathcal{E}\left\{\log c_{i,t}\right\}$.

Papers in the simulation literature have observed an approximate equivalence between the average growth rates of idiosyncratic consumption and permanent income, but formal proof was not possible until Szeidl's proof of ergodicity.

4.3 Growth Rates of Aggregate Income and Consumption

Attanasio and Weber (1995) point out that nonlinearities in consumption models make it important to distinguish between the growth rate of average consumption and the average growth rate of consumption. We have just examined the average growth rate; it is now time to examine the growth rate of the average.

Let bold varibles designate the average and aggregate values of variables. The growth factor for aggregate income is given by:

$$\begin{aligned}
\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathcal{E}\left\{\xi_{i,t+1}G\Psi_{i,t+1}P_{i,t}\right\}/\mathcal{E}\left\{P_{i,t}\xi_{i,t}\right\} \qquad (156)\\
&= G\mathcal{E}\left\{P_{i,t}\right\}/\mathcal{E}\left\{P_{i,t}\right\} \qquad (157)\\
&= G \qquad (158)
\end{aligned}$$

because of the independence assumptions we have made about ξ and Ψ.

Aggregate assets are:

$$\begin{aligned}
\mathbf{A}_t &= \mathcal{E}\left\{a_{i,t}P_{i,t}\right\} \\
&= \mathbf{a}\mathbf{P}_t + \mathrm{cov}(a_{i,t}, P_{i,t}) \qquad (159)
\end{aligned}$$

where we are assuming that a in period t was distributed according to the invariant distribution which justifies the lack of a time subscript on $\mathbf{a}$. Since permanent income grows at mean rate G while the distribution of a is invariant, if we normalize $\mathbf{P}_t$ to one we will similarly have for any period $n \geq 1$

$$\mathbf{A}_{t+1} = \mathbf{a}G^n + \mathrm{cov}(a_{i,t+n}, P_{i,t+n}). \qquad (160)$$

Unfortunately, the proof of the invariance of $\mathcal{F}^a$ does not yield the required information about how the covariance between a and P evolves.

We can show the desired result if there are no permanent shocks. Suppose the population starts in period t with an arbitrary value for $\text{cov}(a_{i,t}, P_{i,t})$. Then if we define

$$\hat{a}(\hat{m}) = \int_{\mathbf{m}}^{\hat{m}} a'(z+1)dz \tag{161}$$

we can write

$$a_{i,t+1} = a(\mathcal{R}\mathbf{a}+1) + \hat{a}(\mathcal{R}a_{i,t} + \xi_{i,t+1} - 1) \tag{162}$$

so

$$\begin{aligned} \text{cov}(a_{i,t+1}, P_{i,t+1}) &= \text{cov}\left(\hat{a}(\mathcal{R}a_{i,t} + (\xi_{i,t+1} - 1)), GP_{i,t}\right) & (163) \\ &= \text{cov}\left(\hat{a}(\mathcal{R}a_{i,t}), P_{i,t}\right) & (164) \end{aligned}$$

which holds because $\xi_{i,t+1}$ is a mean one variable independent of $P_{i,t}$. But since $R^{-1}(pR\beta)^{1/\rho} < \hat{a}'(m) < R^{-1}(R\beta)^{1/\rho}$

$$\text{cov}((pR\beta)^{1/\rho}a_{i,t}, P_{i,t}) < \text{cov}(a_{i,t+1}, P_{i,t+1}) < \text{cov}((R\beta)^{1/\rho}a_{i,t}, P_{i,t}) \tag{165}$$

and for the version of the model with no permanent shocks we know that $(R\beta)^{1/\rho} < G$, which implies

$$\text{cov}(a_{i,t+1}, P_{i,t+1}) < G\text{cov}(a_{i,t}, P_{i,t}). \tag{166}$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $\mathbf{a}G^n$ term which is growing steadily at rate G). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = G$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{i,t+1}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\bar{\mathcal{R}} = \mathcal{E}\{\mathcal{R}_{i,t+1}\}$ and consider a second order Taylor expansion of $\hat{a}(\mathcal{R}_{i,t+1}a_{i,t})$ around $\bar{\mathcal{R}}a_{i,t}$,

$$\hat{a}_{i,t+1} \approx \hat{a}(\bar{\mathcal{R}}a_{i,t}) + \hat{a}'(\bar{\mathcal{R}}a_{i,t})(\mathcal{R}_{i,t+1} - \bar{\mathcal{R}})a_{i,t} + \hat{a}''(\bar{\mathcal{R}}a_{i,t})\left(\frac{(\mathcal{R}_{i,t+1}a_{i,t} - \bar{\mathcal{R}}a_{i,t})^2}{2}\right).$$

The problem comes from the $\hat{a}''$ term. The concavity of the consumption function implies convexity of the $\hat{a}$ function, so this term is strictly positive but we have no theory to place bounds on its size as we do for $\hat{a}'$. Intuitively, a large positive shock to permanent income will produce a low ratio of assets to permanent income, which will be associated with a low marginal propensity to save (a high MPC).

It is possible that methods like those developed by Szeidl (2002) might be able to establish the long run properties of the covariance term, and thus verify or refute the proposition that the economy heads towards a balanced growth path from any starting position. In the absence of such a proof, we must rely on simulation evidence. In practice, a wide range of simulations finds that the influence of the covariance term is modest. An example is given in Figure 5, which plots $\mathbf{C}_{t+1}/\mathbf{C}_t$ for the economy for which the CDFs were depicted in 4.

As an experiment, after the 40 periods of simulation that generated CDFs plotted in 4, we reset the level of permanent income to be identical for all consumers ('the revolution'):

$$P_{i,41} \quad = \quad G\mathbf{P}_{40} \tag{167}$$

and we redistribute cash among consumers in such a way as to leave each consumer with the same value of $m_{i,41}$ that they would have had in the absence of the revolution. The purpose of this experiment is to wipe out the inherited covariance between P and m in order to gauge the importance of the covariance effect and the dynamic effects of that term on aggregate growth.

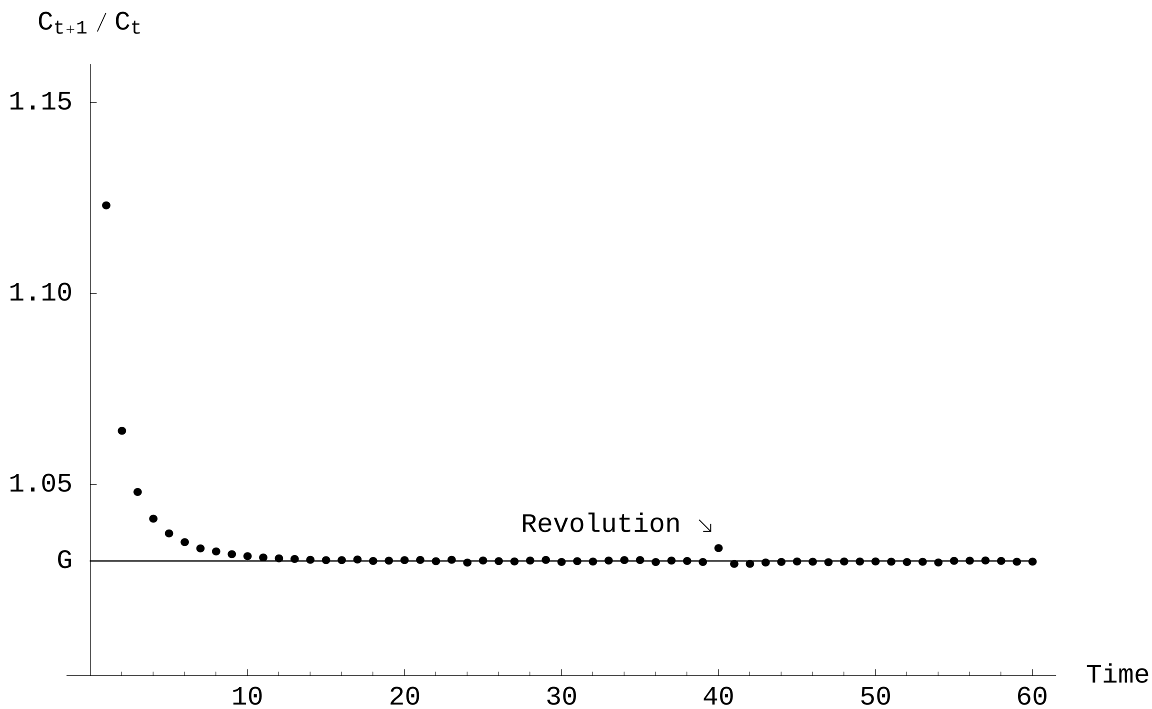

Figure 5: Consumption Growth in Simulated Economy with $G = 1.03$

The effect of the revolution on consumption growth is small, and dissipates almost immediately. This simulation and others suggest that the practical effects of the covariance between m and P are likely to be negligible, so that we should expect that in buffer stock economies aggregate consumption growth will be very close to aggregate permanent income growth.

5 Conclusions

This paper provides theoretical foundations necessary to prove many characteristics of buffer stock saving models that have heretofore been observed in simulations but not proven. The main results apply either to a model without liquidity constraints, or to the model with constraints (e.g. Deaton (1991)) considered as a limiting case. Perhaps the most important such proposition is the existence of a target cash-to-income ratio toward which actual cash will tend.

Another contribution of the paper is that it provides a set of tools for simulation analysis (available on the author's web page) that confirm and illustrate the theoretical propositions. These programs demonstrate how the incorporation of the theoretical results can make numerical solution algorithms more efficient and simpler. The simulation programs also provide quantification of the qualitative properties derived analytically (what is the target level of the buffer stock? what is the population-average marginal propensity to consume? etc.). Much previous work in the consumption literature has been either purely theoretical or purely simulation-based. This paper aims to bridge that gap.

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Appendices

A $c_t(m)$ Is Twice Continuously Differentiable

First we show that $c_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv x + dx$. Since $u'\left(c_t(y)\right) - u'\left(c_t(m)\right) = \mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))$ and $\frac{a_t(y)-a_t(m)}{dm} = 1 - \frac{c_t(y)-c_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)} = \\ \left(\frac{u'\left(c_t(y)\right) - u'\left(c_t(m)\right)}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)}\right) \frac{c_t(y) - c_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm \to +0} \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} < 0$ and $\lim_{dm \to 0} \frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)} \leq 0$ are satisfied. Then $\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)} < 0$ for sufficiently small dx. Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)}}{\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}_t'(a_t(y)) - \mathfrak{v}_t'(a_t(m))}{a_t(y) - a_t(m)}}$$

This implies that the right-derivative, $c_t'^{+}(m)$ is well-defined and

$$c_t'^{+}(m) = \frac{\mathfrak{v}_t''(a_t(m))}{u''(c_t(m)) + \mathfrak{v}_t''(a_t(m))}$$

Similarly we can show that $c_t'^{+}(m) = c_t'^{-}(m)$, which means $c_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $c_t'(m)$ exists and is continuous. $c_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $c_t(m)$ is $\mathbf{C}^1$ and $u''(c_t(m)) + \mathfrak{v}_t''\left(a_t(m)\right) < 0$. $c_t''(m)$ is given by

$$c_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'u'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right]^2} \tag{168}$$

Since $\mathfrak{v}_t''(a_t(m))$ is continuous, $c_t''(m)$ is also continuous.

B Convergence of v_t in Euclidian Space

Boyd's theorem shows that T defines a contraction mapping in a ϕ-bounded space. We now show that T also defines a contraction mapping in Euclidian space.

Since $v^*(m) = \mathsf{T}v^*(m)$,

$$\left\|v_{T-n+1}(m) - v^*(m)\right\|_\phi \leq \varsigma^{n-1}\left\|v_T(m) - v^*(m)\right\|_\phi \tag{169}$$

On the other hand, $v_T - v^* \in \mathcal{C}_\phi(\mathbf{A}, \mathbf{B})$ and $\kappa = \|v_T(m) - v^*(m)\|_\phi < \infty$ because v_T and v^* are in $\mathcal{C}_\phi(\mathbf{A}, \mathbf{B})$. It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa\varsigma^{n-1}|\phi(m)| \tag{170}$$

Then we obtain

$$\lim_{n\to\infty} v_{T-n+1}(m) = v^*(m). \tag{171}$$

Since $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $v_{T-1}(m) \leq \frac{(\bar{\bar{\kappa}}m)^{1-\rho}}{1-\rho} < v_T(m)$. On the other hand, $v_{T-1} \leq v_T$ means $\mathsf{T}v_{T-1} \leq \mathsf{T}v_T$, in other words, $v_{T-2}(m) \leq v_{T-1}(m)$. Inductively one gets $v_{T-n}(m) \geq v_{T-n-1}(m)$. This means that $\{v_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence.

C Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$.

We start by showing that

$$c(m) = \underset{c_t\in[\underline{\kappa}m,\bar{\bar{\kappa}}m]}{\arg\max}\left\{u(c_t) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v(\tilde{m}_{t+1})\right]\right\} \tag{172}$$

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\check{c} = (c_1 + c_2)/2$. c_i satisfies

$$\mathsf{T}v(m) = u(c_i) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v(\tilde{m}_{t+1}(m, c_i))\right] \tag{173}$$

where $m_{t+1}(m, c) = \mathcal{R}_{t+1}(m-c)+\xi_{t+1}$ and $i = 1, 2$. $\mathsf{T}v$ is concave for concave v. Since the space of continuous and concave functions is closed, v is also concave and satisfies

$$\frac{1}{2}\sum_{i=1,2} E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v(\tilde{m}_{t+1}(m, c_i))\right] \leq E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v(\tilde{m}_{t+1}(m, \check{c}))\right]. \tag{174}$$

On the other hand, $\frac{1}{2}\{u(c_1) + u(c_2)\} < u(\check{c})$. Then one gets

$$\mathsf{T}v(m) < u(\check{c}) + \beta E_t\left[\tilde{\Gamma}_{t+1}^{1-\rho}v(\tilde{m}_{t+1}(m, \check{c}))\right] \tag{175}$$

Since $\check{c}$ is a feasible choice for c_j, $v(\check{c})$ is not a maximum, which contradicts the definition.

Using uniqueness of $c(m)$ we can now show

$$\lim_{n\to\infty} c_{T-n+1}(m) = c(m). \tag{176}$$

Suppose this does not hold for some $m = m^*$. In this case, $\{c_{T-n+1}(m^*)\}_{n=1}^{\infty}$ has a subsequence $\left\{c_{T-n(i)}(m^*)\right\}_{i=1}^{\infty}$ that satisfies $\lim_{i\to\infty} c_{T-n(i)}(m^*) = c^*$ and $c^* \neq c(m^*)$. Now define $c^*_{T-n+1} = c_{T-n+1}(m^*)$. $c^* > 0$ because $\lim_{i\to\infty} v_{T-n(i)+1}(m^*) \leq \lim_{i\to\infty} u(c^*_{T-n(i)})$. Because $a(m^*) > 0$ and $\Psi \in [\underline{\Psi}, \bar{\Psi}]$ there exist $\{\underline{m}^*_+, \bar{m}^*_+\}$ satisfying $0 < \underline{m}^*_+ < \bar{m}^*_+$ and $m_{T-n+1}(m^*, c^*_{T-n+1}) \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. It follows that $\lim_{n\to\infty} v_{T-n+1}(m) = v(m)$ and the convergence is uniform on $m \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. (Uniform convergence is obtained from Dini's theorem.[17]) Hence for any $\delta > 0$, there exists an n_1 such that

$$\beta E_{T-n}\left[\tilde{\Gamma}^{1-\rho}_{T-n+1}\left|v_{T-n+1}(\tilde{m}_{T-n+1}(m^*, c^*_{T-n+1})) - v(\tilde{m}_{T-n+1}(m^*, c^*_{T-n+1}))\right|\right] < \delta$$

for all $n \geq n_1$. It follows that if we define

$$w(m^*, z) = u(z) + \beta E_{T-n}\left[\tilde{\Gamma}^{1-\rho}_{T-n+1} v(m_{T-n+1}(m^*, z))\right] \tag{177}$$

then $v_{T-n}(m^*)$ satisfies

$$\lim_{n\to\infty}\left|v_{T-n}(m^*) - w(m^*, c^*_{T-n+1})\right| = 0. \tag{178}$$

On the other hand, there exists an $i_1 \in \mathbb{N}$ such that

$$\left|v(m_{T-n(i)}(m^*, c^*_{T-n(i)})) - v(m_{T-n(i)}(m^*, c^*))\right| \leq \delta \text{ for all } i \geq i_1 \tag{179}$$

because v is uniformly continuous on $[\underline{m}^*_+, \bar{m}^*_+]$. $\lim_{i\to\infty}\left|c_{T-n(i)}(m^*) - c^*\right| = 0$ and

$$\left|m_{T-n(i)}(m^*, c^*_{T-n(i)}) - m_{T-n(i)}(m^*, c^*)\right| \leq \frac{R}{G\underline{\Psi}}\left|c^*_{T-n(i)} - c^*\right| \tag{180}$$

This implies

$$\lim_{i\to\infty}\left|w(m^*, c^*_{T-n(i)+1}) - w(m^*, c^*)\right| = 0 \tag{181}$$

From (178) and (181), we obtain $\lim_{i\to\infty} v_{T-n(i)}(m^*) = w(m^*, c^*)$ and this implies $w(m^*, c^*) = v(m^*)$. This implies that $c(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

D $E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}] < 1$

If Ψ_{t+1} were distributed lognormally with no bounds, we would have that

$$\begin{aligned}\log \Psi^{-\rho}_{t+1} &= -\rho \log \Psi_{t+1} &\quad (182)\\ &\sim N(\rho\sigma^2_\psi/2, \rho^2\sigma^2_\psi) &\quad (183)\end{aligned}$$

[17][**Dini's theorem**] For a monotone sequence of continuous functions $\{v_n(m)\}_{n=1}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty} v_n(m) = v(m)$ where $v(m)$ is continuous, convergence is uniform.

implying

$$E_t[\tilde{\Psi}_{t+1}^{-\rho}] = e^{\rho\sigma_\psi^2/2+\rho^2\sigma_\psi^2/2}$$

so (32) can be rewritten

$$\begin{aligned}
R\beta G^{-\rho}e^{\rho\sigma_\psi^2/2+\rho^2\sigma_\psi^2/2} &< 1 &(184)\\
(R\beta)^{1/\rho}G^{-1}e^{\sigma_\psi^2/2+\rho\sigma_\psi^2/2} &< 1 &(185)\\
(R\beta)^{1/\rho}G^{-1}e^{\sigma_\psi^2/2+\sigma_\psi^2/2+(\rho-1)\sigma_\psi^2/2} &< 1 &(186)\\
(R\beta)^{1/\rho}G^{-1}E_t[\tilde{\Psi}_{t+1}^{-1}]e^{(\rho-1)\sigma_\psi^2/2} &< 1 &(187)\\
E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}]e^{(\rho-1)\sigma_\psi^2/2} &< 1 &(188)\\
E_t[(R\beta)^{1/\rho}/\tilde{\Gamma}_{t+1}] &< 1 &(189)
\end{aligned}$$

where the last line follows under the maintained assumption $\rho > 1$.

We have assumed Ψ is distributed as a truncated lognormal; clearly if we specify the truncation points $[\underline{\Psi}, \bar{\Psi}]$ sufficiently widely (189) will hold. We therefore assume that the truncation points are sufficiently wide.