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ABSTRACT

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Shakeouts and Market Crashes*

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May 30, 2004

Abstract

Stock-market crashes tend to follow run-ups in prices. These episodes look like bubbles that gradually inflate and then suddenly burst. We show that such bubbles can form in a Zeira-Rob type of model in which demand size is uncertain. Two conditions are sufficient for this to happen: A declining hazard rate in the prior distribution over market size and a positively sloped supply of capital to the industry. For the period 1971-2001 we fit the model to the Telecom sector.

1 Introduction

Stock-market crashes tend to follow run-ups in prices. The NYSE index rose in the late 1920s, and then crashed in October 1929, and the Nasdaq rose steadily through the 80's and 90's and crashed after March 2000. These episodes therefore look like bubbles that gradually inflate and then suddenly burst.

In a learning model of the Zeira-Rob type we study the possibility of bubble-like behavior of stock prices, but driven by fundamentals. We add to Rob (1991) a rising adjustment cost for the growth of industry capacity and a declining hazard rate in the prior distribution over market size. In that form, the model generates a crash when an irreversible creation of capacity overshoots demand.

We fit the model to the Telecom sector which, in the year 2000, crashed more spectacularly than most others. Figure 1 portrays the 30-year history of the Nasdaq Telecom index. The NYSE index in the first panel shows that the Telecom crash was not an isolated event. The second panel shows two indexes of real activity along with the Nasdaq Telecom index. Evidently the real and financial indicators all fell sharply in 2000. We seek to explain this link – the slow rise in these indexes followed by their sharp decline.

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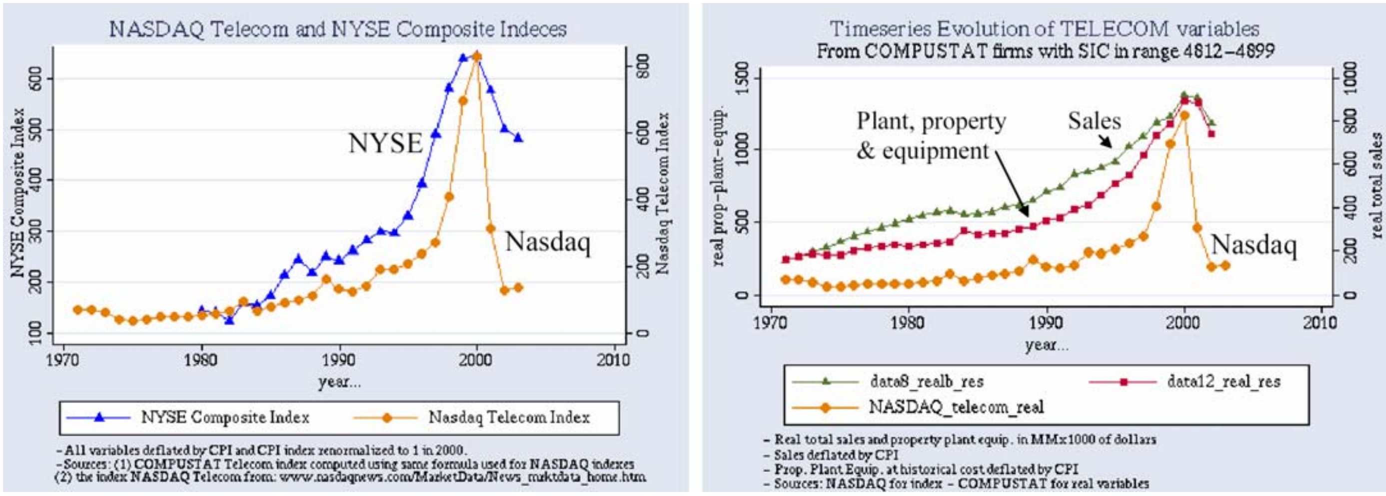

Figure 1: Financial indexes and two real series

The link between real and financial indexes exists in other sectors too, particularly around the time of the 2000 crash. Figure 2 reports the March 2000-to-March 2001 changes in prices and in fundamentals. Sectors that experienced greater drops in value also experienced larger declines in sales. Telecom underwent one of the sharpest declines.

Our explanation of the pre-crash run-up is of the "Peso Problem" type. Krasker (1980) used this term to describe rationally changing beliefs about an event that is not realized within the sample period. Until the year 2000, optimism was on the rise. Two conditions that could produce such a rise are: (i) A declining hazard rate in the prior distribution over market size and (ii) A rising cost of investment. The decreasing-hazard assumption delivers a rising optimism: As the market grows, further growth looks ever more feasible, and the likelihood of a crash ever more remote.

Two other properties of the model are noteworthy. First, optimism and the price-earnings ratio rationally rise even though earnings remain flat so that to someone who predicts earnings by extrapolating from the past, the behavior of the stock market would seem irrational. And second, along the equilibrium path the crash hazard declines, so that older markets are more likely to survive.

Contrary to popular opinion about the 90's, we find that capacity expansion was not too fast but too *slow*. Our model states that in light of what was known at the time, Telecom capacity should have expanded more rapidly and its the crash should have happened earlier. This is because expanding capacity entails a positive informational externality that a competitive firm ignores when choosing how much to invest.

Many of the ideas in our paper are also in Zeira (1987, 1999), Caplin and Leahy (1994) and in Horvath, Schivardi and Woywode (2001). Models in which a new tech-

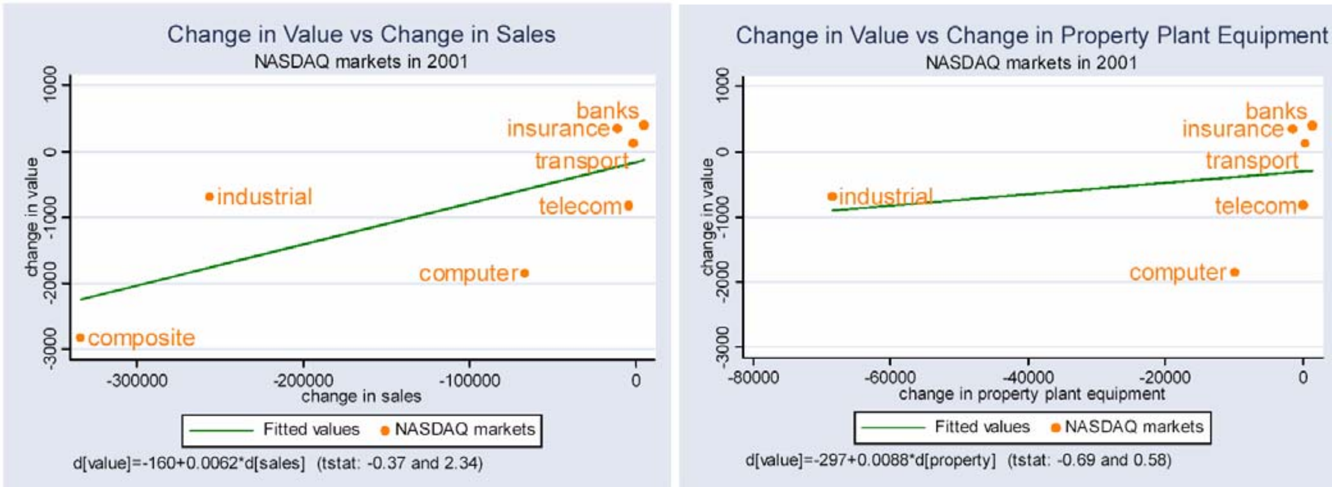

Figure 2: (March 2000-March 2001) Size of crash vs. changes in sales (panel 1) and changes in capacity (panel 2)

nology devalues an existing technology are Boldrin and Levine (2000), Greenwood and Jovanovic (1999), and Jovanovic and MacDonald (1994). Mazzucato and Semmler (1999) who find that in the automobile industry stock prices were the most volatile during the period 1918-1928 when market shares were the most unstable.

2 Model

The model is one of an industry in which market size is unknown. Investing in this market is risky because capacity may exceed market size. Firms continually update their beliefs about market size and create new capacity based on these beliefs. When capacity outstrips demand, the price falls and further growth stops. We study the dynamics leading up to the crash.

Demand.—As a function of quantity, Q, supplied to the market, price is

$$P(Q,z) = \begin{cases} p \;\; if \;\; Q \leq z \\ 0 \;\; if \;\; Q > z \end{cases}$$

Willingness to pay, p, is known, but z is unknown. We think of z as the number of new consumers, each demanding one unit of good per unit time. The parameter z does not change over time. It is a random variable drawn at time $t = 0$ from a distribution $F(z)$; Figure 3 shows a family of demand curves indexed by various values of z, and highlights the demand curve that would occur if $z = z_2$. The distribution $F(z)$ is common knowledge among the potential entrants at $t = 0$. It is the common prior distribution which is updated in light of experience.

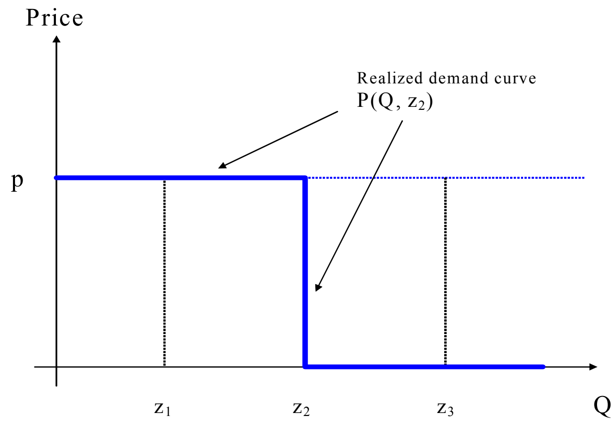

Figure 3: TYPICAL DEMAND CURVE $P(Q, z_2)$

Production.—Firms are infinitessimal and of indeterminate size. Production cost is zero, and the salvage value of production capacity is negligible. As long the price is positive, industry output is the same as the industry's capacity to produce it. Let k denote the industry's capacity and let n denote new capacity, i.e., aggregate investment. Capacity does not depreciate, and until the crash comes it evolves as follows:

$$k_{t+1} = k_t + n_t. \tag{1}$$

Initial capacity, $k_0 \geq 0$ is given.

Investment.—Adjustment costs of investment are rising at the industry level, but constant at the level of an individual firm. The unit cost, c, of adding capacity rises with *aggregate* investment:

$$c = C(n),$$

where $C'(n) \geq 0$. In other words, the industry faces a rising price of capital, but each individual firm is a price taker.

Learning.—Firms share the common prior over z; the C.D.F. $F(z)$. All firms know the history of prices and industry outputs. Based on this they revise their opinion about z. "Overshooting" happens at date T when k exceeds z for the first time. Before date T, firms know only that $z \geq k$. At date T, we assume that firms learn z exactly.

To recapitulate, there are three distinct epochs:

1. *Before date T.*—Agents know only that $k \leq z$.

2. *At date T.*—The first time that $k > z$, firms learn z, perhaps because spare capacity $k_T - z$ becomes public information. This excess capacity is at once

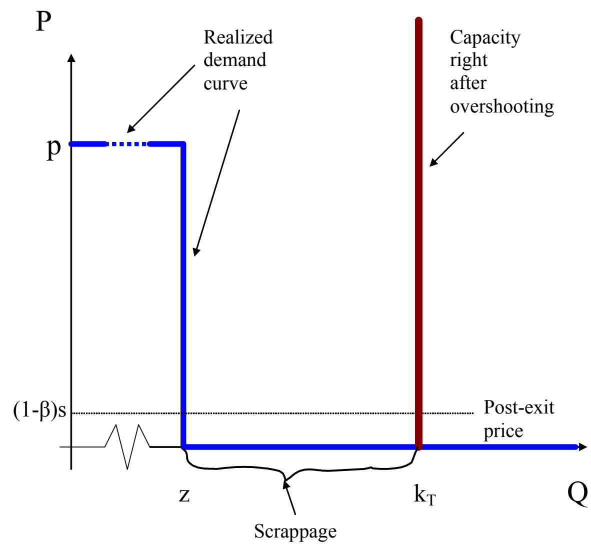

Figure 4: SCRAPPAGE AT DATE T

scrapped for a unit return of s and equilibrium product price falls to $(1-\beta)\,s$.[1] This is illustrated in Figure 4.[2]

3. After date T.—Product price remains for ever at $(1-\beta)\,s$, and the stock price remains for ever at s. There are no further dynamics.

The unit value of capacity.—Before date T, agents know only that z exceeds k. To simplify the algebra we assume that $s=0$. Then the value of a unit of capacity is the random variable

$$V_t = \begin{cases} v_t & \text{if} \quad k \le z \\ 0 & \text{if} \quad k > z, \end{cases} \tag{2}$$

where $(v_t)_0^{T-1}$ is the sequence of unit values for k that are relevant to dates before T. Now, T is a random variable that is possibly infinite. The probability that the market will crash at some point is $F(\infty) \le 1$. If the inequality is strict, the distribution is "defective" and $1-F(\infty)$ is the probability that the market will never crash, in which case we set $T=\infty$. This all means that we must solve for the infinite sequence $(v_t)_0^{\infty}$ of capital values that will obtain if z happens to be infinite. Until Section 4.1, then, we shall be dealing with worlds in which z is unbounded.

[1] An example is the post-2000 collapse in the price of on-line advertising at sites such as Yahoo and AOL – by a factor of three or more – and those of their competitors (Angwin 2002).

[2] Koeva (2000, Table 1) reports a 24-month time-to-build estimate for Telecom – roughly the economy-wide average. This is for large projects. Other types of capacity expansion such as those that entail the purchase of equipment are much shorter than that. We shall assume that time-to-build is one year.

2.1 Equilibrium

The dynamics that we now discuss is for dates $t = 1, 2, ..., T-1$, i.e., for periods during which $k_t < z$. During this period strategies depend only on time. Firms are of measure zero, so it does not matter whether capacity is created by incumbents or by new entrants, and we do not have a theory of firm size, only a model of industry size.

Stock prices.—All investors are rational.[3] The Bellman equation in the region $k < z$ is stated for the value, v_t, of a unit of capital: $v_t = p + \beta E_t V_{t+1}$ which, by (2), can be written as

$$v_t = p + \beta v_{t+1} \Pr\{z \geq k_{t+1} \mid z \geq k_t\}.$$

Using Bayes Rule,

$$v_t = p + \beta v_{t+1} \frac{1 - F(k_{t+1})}{1 - F(k_t)}. \quad (3)$$

Optimal investment.—We assume that $C(0) = 0$, and that $C(n)$ is unbounded. Then the marginal investment condition always holds with equality. It states that the purchase price of capital should equal the expected present value of a unit of capital in place:[4]

$$C(n_t) = \beta v_{t+1} \frac{1 - F(k_{t+1})}{1 - F(k_t)}. \quad (4)$$

Equation (4) holds only for $k < z$. But since z can take on any value up to $z_{\max}$, (4) must hold for all $k_{t+1} \leq z_{\max}$.

Definition 1 *Equilibrium is a pair of positive sequences $(v_t)_0^\infty$ and $(k_t)_0^\infty$ that satisfy (3) and (4).*

We now reduce the two equilibrium conditions (3) and (4) to a single (second-order) difference equation. For the three special cases, we shall prove the existence and uniqueness of the equilibrium by showing that a unique solution exists to this difference equation.[5]

Substituting from (4) into (3), the relation is

$$v_t = p + C(n_t). \quad (5)$$

If n rises over time, C will rise and, by (5), so will v. That is, stock prices will rise until date T.

[3] This distinguishes our model from many in the finance literature, e.g., Abreu and Brunnermeier (2002).

[4] The model deals only with industry capacity, k_t, and not how it is divided among firms.

[5] To obtain a unique solution to a second-order difference equation, we would need two boundary conditions. But we have only one: k_0, and in general a multiplicity of equilibria is a possibility.

With (4), (5) implies

$$C(n_t) = \beta\left(p + C\left[n_{t+1}\right]\right)\frac{1 - F\left(k_{t+1}\right)}{1 - F\left(k_t\right)}, \tag{6}$$

or, rearranged,

$$\begin{aligned} C\left(n'\right) &= -p + C\left(n\right)\frac{1}{\beta}\frac{1 - F\left(k\right)}{1 - F\left(k + n\right)} \\ &= -p + C\left(n\right)\frac{1}{\beta}\exp\left\{\int_k^{k+n} h\left(s\right) ds\right\} \\ &\equiv \Phi\left(n, k\right) \end{aligned} \tag{7}$$

where $h\left(z\right) = \frac{f(z)}{1-F(z)}$ is the hazard rate, and where a prime refers to a one-period-ahead value of a variable.

2.2 Two special cases

The two cornerstones of our model are $C' > 0$ and $h' < 0$. Relaxing either of these disables the model from generating rising stock prices. We shall now show this with the help of two examples.

1. *Constant-hazard.*—Let $F\left(z\right) = 1 - e^{-\lambda z}$. Then $h\left(z\right) = \lambda$ and (7) collapses to

$$C\left(n'\right) = -p + C\left(n\right)\beta^{-1}e^{\lambda n}$$

   The only admissible solution is a constant n_t, which solves for n the equation

$$C\left(n\right) = \beta e^{-\lambda n}\frac{p}{1 - \beta e^{-\lambda n}} \quad \text{all } t. \tag{8}$$

   The LHS of (8) is increasing in n while the RHS is decreasing therefore the solution is unique. (Moreover, any non-stationary solution (n_t) would be explosive because $\beta^{-1}e^{\lambda n} > 1$ for all $n \geq 0$). Since n is constant, (5) tells us that stock prices, v_t, will be constant until date T, and iterations of (1) imply that $k_t = k_0 + nt$ for $t \leq T$.

2. *Constant* $C\left(n\right)$.—Let $C\left(n\right) = c$ for all n. This is Rob's (1991) case. Provided that $c < \frac{\beta p}{1-\beta}$, (4) always holds a positive n. Then (5) implies that $v_t = p + c$ for all t. I.e., stock prices are again constant. Finally, (6) reads

$$1 + \frac{p}{c} = \frac{1}{\beta}\frac{1 - F\left(k\right)}{1 - F\left(k + n\right)},$$

   which, by $c < \frac{\beta p}{1-\beta}$, implies the existence of a unique investment function $n = \psi\left(k\right) > 0$. Together with (1), this gives the sequence k_t uniquely.

Evidently, we must relax both assumptions if we are to have any chance of reconciling this model with the rising stock prices in Figure 1.

3 The case of a decreasing h.

This section derives properties that an equilibrium must satisfy if $h' < 0$ and $C' > 0$. Then we shall show that a unique equilibrium exists when F is Pareto and when $C(n) = cn$. We start with the general case.[6]

Assumptions on F.—Suppose that the support of F is $[z_{\min}, \infty)$. Suppose furthermore that $h(z) > 0$ for all $z > z_{\min}$, and that $h'(z) < 0$. That is, F has a strictly decreasing hazard. Assume, moreover, that $\lim_{z\to\infty} h(z) = 0$. We relax these assumptions in Section 4.2 where we impose a finite bound on z.

Analysis.—Let $\bar{n}$ solve the equation

$$C(\bar{n}) = \frac{\beta p}{1-\beta}. \tag{9}$$

Lemma 1

$$\lim_{t\to\infty} n_t = \bar{n}$$

Proof. Since $v_t \leq p/(1-\beta)$, the RHS of (4) is at most equal to the RHS (9). Therefore

$$n_t < \bar{n}.$$

Moreover, the conditional probability of the market surviving for another period is

$$\xi(k,n) \equiv \frac{1-F(k+n)}{1-F(k)} = \exp\left\{-\int_k^{k+n} h(z)\,dz\right\} \tag{10}$$

If h is decreasing, ξ is increasing in k and decreasing in n. Since k_t is an increasing sequence, we then must have

$$\xi(k_t, n_t) \geq \xi(k_0, \bar{n})$$

Therefore for any t, the return to a unit of incumbent capital is

$$\begin{aligned} v_t &= \sum_{j=t}^{\infty} \beta^{j-t} \prod_{\tau=t}^{j} \xi(k_\tau, n_\tau)\, p \\ &\geq p \sum_{j=t}^{\infty} \left[\beta \xi(k_0, \bar{n})\right]^{j-t} \\ &\geq \frac{p}{1-\beta\xi(k_0,\bar{n})} \end{aligned}$$

Therefore, for any t

$$n_t \geq n_{\min} > 0,$$

[6] Prat (2003) provides a survivorship-bias type of rationale for why the hazard rate may be declining, especially when the uncertainty over market types is large.

where $n_{\min}$ solves

$$C(n_{\min}) = \frac{\beta\xi(k_0,\bar{n})\,p}{1-\beta\xi(k_0,\bar{n})}$$

But then $k_t \geq t n_{\min}$, and so $\lim_{t\to\infty} k_t = \infty$. Therefore the RHS of (7) converges to $-p + C(n)/\beta$. Rearranging, we get (9). ■

The next proposition contains the result we need. The algebra simplifies if we re-state the difference equation (7) as a difference equation in $c = C(n)$. Assume that C is a one-to one increasing map from $R_+ \to R_+$ and that its range is all of R_+ so that $n(c) \equiv C^{-1}(c)$ is uniquely defined for all $c \geq 0$. Then write (7) as

$$\begin{aligned} c' &= -p + \frac{c}{\beta}\exp\left\{\int_k^{k+n(c)} h(z)\,dz\right\} \\ &\equiv \phi(c,k)\,. \end{aligned} \tag{11}$$

This difference equation is easier to work with

Proposition 1 *Before the crash (i.e., for $t < T$),*

$$n_{t+1} > n_t.$$

Proof. Since $n(c)$ is strictly increasing, $\phi(c,k)$ is strictly increasing in c. Since $h(z)$ is a decreasing function, $\phi(c,k)$ is strictly decreasing in k. Suppose, contrary to the claim, that $n_{t+1} \leq n_t$. Then $c_{t+1} \leq c_t$. Since $C(0) = 0$ and $v_t \geq p > 0$, (4) implies $n_{t+1} > 0$. But then $k_{t+1} > k_t$, and therefore

$$c_{t+2} = \phi(c_{t+1}, k_{t+1}) < \phi(c_t, k_t) = c_{t+1}.$$

Iterating this argument leads to the conclusion that

$$n_t \geq n_{t+1} \Longrightarrow n_{t+1} > n_{t+2} > .... \geq n_{\min}$$

And since the initial value $n_t < \bar{n}$, we conclude that $\lim_t n_t < \bar{n}$. But $k_t \geq t n_{\min}$ and once again $\lim_{t\to\infty} k_t = \infty$, and therefore (9) must hold, and this is a contradiction. ■

Since a bounded monotone sequence must have a limit, we conclude that (n_t) indeed does have a limit and that this unique limit solves (9).

Lemma 2 *If h is decreasing in z, ϕ is decreasing in k.*

Proof. Differentiating,

$$\frac{\partial\phi}{\partial k} = \left[h\left(k + C^{-1}[c]\right) - h(k)\right]\frac{c}{\beta}\exp\left\{\int_k^{k+C^{-1}(c)} h(z)\,dz\right\} < 0$$

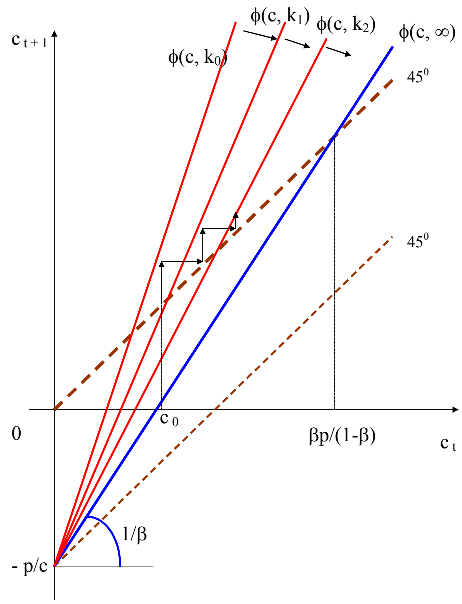

Figure 5: Behavior of the solution, c_t, to the difference equation (11).

because h is decreasing, whereas $C^{-1}(c) > 0$. ■

Since ϕ decreases with k, the mode of convergence will therefore be as shown in Figure 5.

Older markets are less likely to crash—We now show that the probability of survival for one more period rises along the equilibrium path. From (7) and (10) the conditional survival probability is

$$\xi(k_t, n_t) = \frac{1}{\beta}\left[\frac{p}{C(n_t)} + \frac{C(n_{t+1})}{C(n_t)}\right]^{-1}$$

By Proposition 1 C is increasing with t and $\frac{C(n_{t+1})}{C(n_t)} > 1$ on the transition path, converging to unity. The survival probability therefore rises from $\frac{1}{\beta}\left[\frac{p}{C(n_0)} + \frac{C(n_1)}{C(n_0)}\right]^{-1}$ to $\frac{1}{\beta}\left[\frac{p}{C(\bar{n})} + 1\right]^{-1}$, but we cannot show that the increase is monotonic.

3.1 Simulated example: Pareto F

We simulate the equilibrium of a special case of a decreasing hazard distribution, the Pareto distribution:

$$F(z) = 1 - \left(\frac{z}{z_{\min}}\right)^{-\rho}.$$

Its hazard rate is $\frac{\rho}{z}$. Although we have no evidence on the distribution of market sizes, the Pareto distribution has been found to fit well the distribution of city sizes, and the distribution of firm sizes (Axtell 2002) so that perhaps it is a natural one to analyze here.

We shall show that a unique equilibrium exists in this case, and we shall solve for it explicitly. Now (6) reads

$$C(n) = \beta(p + C[n'])\left(\frac{k}{k'}\right)^{\rho}$$

Then, since $k' = k + n$, this equation reads

$$C(n) = \beta(p + C[n'])\left(\frac{1}{1+\frac{n}{k}}\right)^{\rho}$$

which we can rearrange this into the difference equation

$$C(n') = -p + C(n)\frac{1}{\beta}\left(1 + \frac{n}{k}\right)^{\rho} \tag{12}$$

This is the specific form depicted in Figure 5.

Assume that

$$C(n) = cn$$

Then (12) reads

$$n' = -\frac{p}{c} + \frac{1}{\beta}n\left(1 + \frac{n}{k}\right)^{\rho} \tag{13}$$

The second difference equation is

$$k' = k + n \tag{14}$$

The Appendix proves the existence and uniqueness of the solution to (13), (14) for the sequence (k_t). It also describes in detail the algorithm used to find the simulated solution to the second-order difference equation implied by (13) and (14):

$$k_{t+2} = k_{t+1} + \frac{1}{\beta}(k_{t+1} - k_t)\left(\frac{k_{t+1}}{k_t}\right)^{\rho} - \frac{p}{c}.$$

The simulation in Figure 5A is based on the following parameter values:

parameter	p	c	ρ	β
chosen value	15.49	0.007	.57	0.9

The parameter β was chosen to reflect the high earnings risk of the Telecom industry. The other three parameters were chosen so that the model would fit (i) the mean Telecom price-earnings ratio of almost 18, (ii) The average growth rate of the Nasdaq Telecom index of about six percent, and (iii) The average growth of real

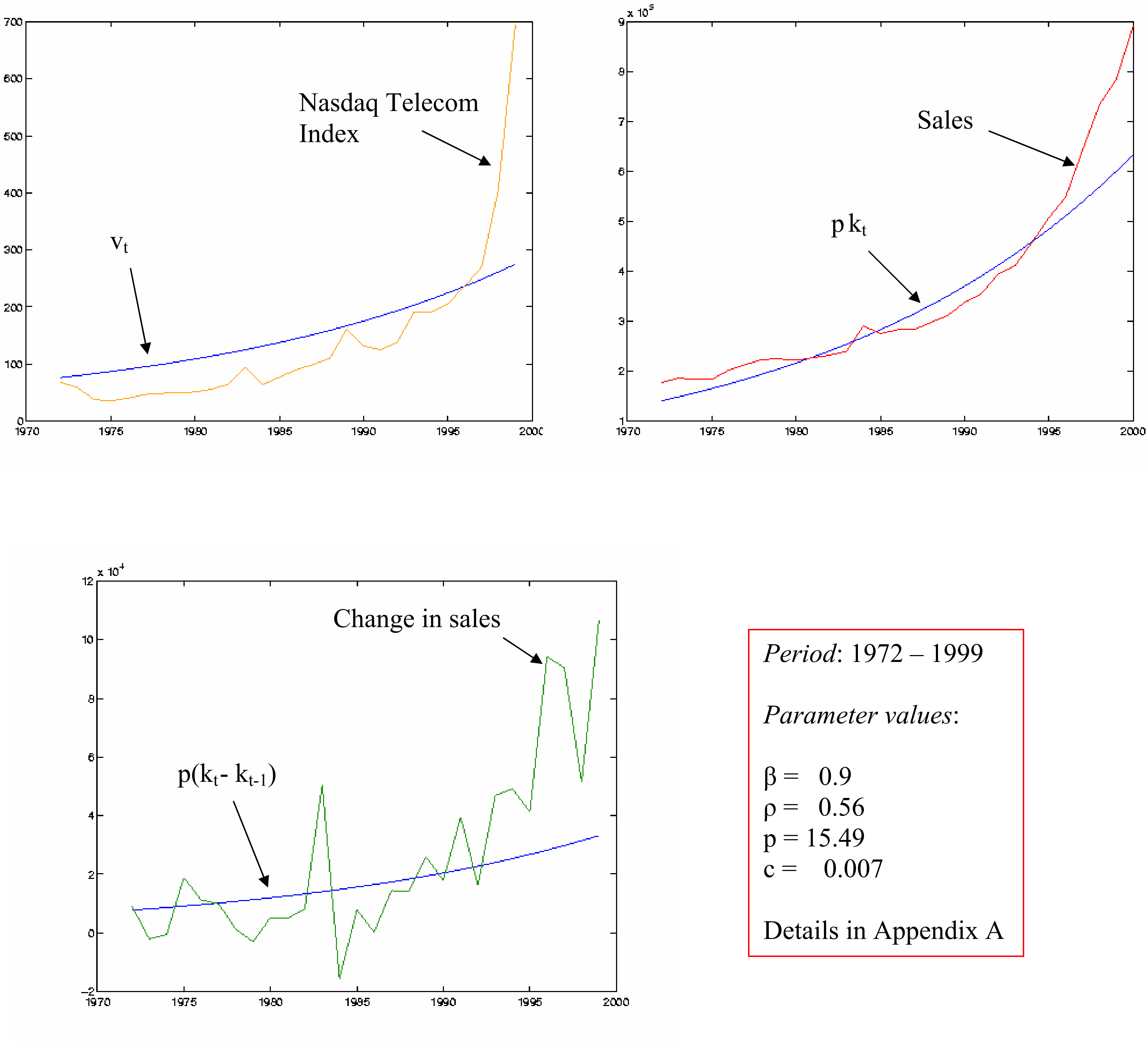

Figure 5A: Simulation of the model

telecom sales. We describe the procedure in Appendix A. The Figure portrays the simulated time paths of n_t and k_t against their observed proxies: the actual Telecom sales series used as the proxy for k and the fist difference of sales as the proxy for n_t.

The model overpredicts the initial growth of two of the three the series, and underpredicts each series's growth in the late 90's, but otherwise fits well. To fine tune the model to fit these series better, we would need to deviate from the Pareto assumption. The parameter T is the year 2000. We omit the years 2001 and beyond because we have set the scrap value of capital, s, to zero, so as to allow us to solve the model analytically. But s is also the post-crash stock-market value of capital, and so the model predicts that stock prices would remain at zero for ever after the year 2000.

4 Three extensions

The first extension shows that the results we obtained for the Pareto F extend to a situation in which z is bounded. The second extension models $C(n)$ as the supply price in the capital-goods industry and traces out the implications there of a crash in the downstream industry. The third extension reports results for increasing hazard rates.

4.1 An upper bound on z

This subsection shows that if we start with the Pareto-distributed z of Section 3.1, and impose a finite limit, Z, on market size z, (thereby making the hazard rate non-monotonic, rising sharply at $z = Z$) the equilibrium does not change much as long as Z is large. Let

$$\theta \equiv F(Z),$$

and define the CDF of the new truncated distribution of z as

$$G(z,\theta) = \begin{cases} \frac{1}{\theta}F(z) & \text{for } z \le F^{-1}(\theta) \\ 1 & \text{for } z > F^{-1}(\theta) \end{cases}$$

Let $\left(k_t^\theta\right)_{t=0}^\infty$ be an equilibrium sequence for the distribution $G(z,\theta)$, so that $\left(k_t^1\right)_0^\infty$ is the equilibrium sequence of Section 2. In this subsection we shall show that for each t,

$$\lim_{\theta \to 1} k_t^\theta = k_t^1.$$

In this sense, a large finite world in which $z \le Z$ approximates the infinite world of the Pareto prior over z. The general idea is portrayed in Figure 6.

Substituting G for F in (7), it reads

$$C(n') = -p + C(n)\frac{1}{\beta}\frac{1 - G(k,\theta)}{1 - G(k+n,\theta)}$$

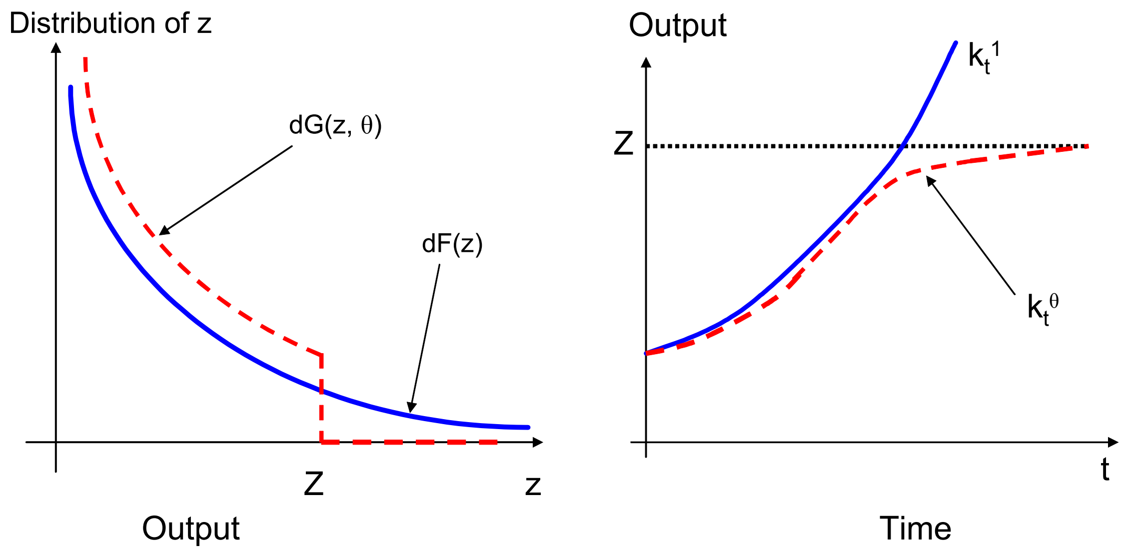

Figure 6: Finite (dashed) vs. infinite (solid) equilibrium

When F is Pareto with $\rho = 1$,

$$G(z, \theta) = \frac{1}{\theta}\left[1 - \left(\frac{z}{z_{\min}}\right)^{-1}\right]$$

for $z < Z$, where

$$Z = \frac{z_{\min}}{1-\theta}.$$

Then (7) reads

$$k'' = k' + \frac{(k'-k)}{\beta}\left(\frac{\theta - 1 + \left(\frac{k}{z_{\min}}\right)^{-\rho}}{\theta - 1 + \left(\frac{k'}{z_{\min}}\right)^{-\rho}}\right) - \frac{p}{c} \equiv \psi(k', k, \theta) \tag{15}$$

Then if $\left(k_t^\theta\right)_{t=0}^\infty$ is an equilibrium sequence, it must be obtainable by iterating ψ from the pair $\left(k_0, k_1^\theta\right)$. Note that

$$\lim_{\theta \to 1} \psi(k', k, \theta) = k' + \frac{(k'-k)}{\beta}\left(\frac{k'}{k}\right)^{\rho} - \frac{p}{c} \tag{16}$$

Lemma 3 *For θ sufficiently close to unity,*

$$\frac{\partial}{\partial k'}\left(\frac{k''}{k'}\right) > 0.$$

Proof. At $\theta = 1$,

$$\frac{k''}{k'} = 1 + \frac{\left(1 - \frac{k}{k'}\right)}{\beta} \left(\frac{k'}{k}\right)^{\rho} - \frac{p}{k'c}$$

so that $\frac{\partial}{\partial k'}\left(\frac{k''}{k'}\right) > 0$ at $\theta = 1$. But $\partial^2 \psi / \partial k' \partial \theta$ exists in the neighborhood of $\theta = 1$ and the claim follows. ■

Then if k' rises (thereby raising k''/k' as well), it turns out that k'''/k'' will rise too:

Lemma 4 *For θ sufficiently close to unity, $\frac{\partial}{\partial k'}\left(\frac{k''}{k'}\right) > 0$ implies that*

$$\frac{\partial}{\partial k'}\left(\frac{k'''}{k''}\right) > 0.$$

Proof. At $\theta = 1$,

$$\frac{k'''}{k''} = 1 + \frac{\left(1 - \frac{k'}{k''}\right)}{\beta} \left(\frac{k''}{k'}\right)^{\rho} - \frac{p}{k''c}$$

so that $\frac{\partial}{\partial k'}\left(\frac{k'''}{k''}\right) > 0$ at $\theta = 1$. Since the cross-partial derivatives exist in the neighborhood of $\theta = 1$, the claim follows. ■

Corollary 1 *For θ sufficiently close to unity,*

$$\frac{\partial k_t^{\theta}}{\partial k_1} > 0$$

Proof. For any t, Lemmas 3 and 4 imply that a simultaneous rise in k_1 will raise k_2/k_1 which, in turn, implies a rise in k_3/k_2 and so on. This proves the claim. ■

Proposition 2 *For each t,*

$$\lim_{\theta \to 1} k_t^{\theta} = k_t^1.$$

Proof. The proof is in four steps.

(i) ψ is continuous on the set $A^{\theta} = \{(k, k') \mid k \leq k' \text{ and } F(k') < \theta.\}$.

(ii) If (k, k') are continuous in θ, so is k''. This follows from *(i)*

(iii) If n_0 is continuous in θ, so is k_t for any t, as long $(k_{t-1}, k_t) \in A^{\theta}$. This is because k_0 is fixed, and because we can iterate the result in *(ii)* using ψ.

(iv) n_0^{θ} is continuous in θ at $\theta = 1$. Suppose not. Then n_0^{θ} would jump at $\theta = 1$. Suppose the jump was positive. Then k_1/k_0 would jump up. Then by Lemmas 3 and 4 and Corollary 1, k_t would jump up for each t. But then (4) (which must hold at $\theta = 1$) would fail to hold at some $\theta < 1$. To see why re-write it as

$$cn_0^{\theta} = p \sum_{t=0}^{\infty} \beta^t \left(\frac{1 - G\left(k_t^{\theta}, \theta\right)}{1 - G\left(k_0^{\theta}, \theta\right)}\right) \tag{17}$$

The LHS of (17) would jump up. But the RHS is continuous in θ and, as the entire (k_t) sequence jumps up, the RHS would exhibit a downward jump, a contradiction. Similar logic works if n_0^θ has a negative jump.

Putting $(i) - (iv)$ together proves the claim. ■

Via (5), this result implies that stock prices, v_t, and industry output, k_t, also converge to the equilibrium we described in Section 3.1. Thus the main thrust of the results of Section 3 holds up in finite worlds in which the demand hazard eventually starts to rise.

4.2 An upstream industry

This section serves a dual purpose. First, it shows how the external effect implied by the presence of n in the unit-cost function $C(n)$ can be a purely pecuniary effect. In this case the welfare statement made in the introduction – that investment is too slow and the crash comes too late – is correct. And, second, it shows how the crash can be contagious across sectors.

We have so far modelled a single industry and applied it to the Telecom sector. The Telecom sector is a part of the Nasdaq, yet the entire Nasdaq crashed. This section shows how a crash in one sector can spread to an upstream sector. Assume that k is produced competitively by a fixed number μ of firms in the upstream industry. For a given capital-goods firm the cost of producing x units of capital is $g(x)$, where $g' > 0$ and $g'' > 0$. Consider the symmetric situation in which every capital-goods firm produces the same amount, x. Suppose that k is purchased only by one industry. That is, the only downstream buyers of k are in the industry we have modelled in the previous sections. Equilibrium then requires that

$$n = \mu x.$$

The price per unit of capital is $C(n)$, where n is investment in the downstream industry. The first-order condition for optimal production of k is

$$C(n) = g'\left(\frac{n}{\mu}\right)$$

Therefore the market value of each capital-good producer is

$$V_{k,t} = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[C(n_\tau) - g\left(\frac{n_\tau}{\mu}\right) \right].$$

and the value of the industry is $\mu V_{k,t}$. Now $C(n) - g\left(\frac{n}{\mu}\right) = g'\left(\frac{n}{\mu}\right) - g\left(\frac{n}{\mu}\right)$ is increasing in n.

Simultaneous crashes upstream and downstream.—As n rises, so does $V_{k,t}$. When the downstream industry crashes, so does the upstream industry. When n_t permanently falls to zero, $V_{k,t}$ does too. If the industry supplies capital to more than one final-good industry, then the equilibrium condition changes; $V_{k,t}$ would still fall, but not to zero.

4.3 Increasing hazard rates

Although increasing hazards do not generate rising stock prices, it is easier to prove general results for that case. This subsection displays those results.

Existence and uniqueness of the equilibrium. Let $n_t(n_0, k_0)$ be the t'th iterate of Φ from (n_0, k_0). That is,

$$n_1(n_0, k_0) = \Phi(n_0, k_0), \quad n_2(n_0, k_0) = \Phi(n_1[n_0, k_0], k_0 + n_1[n_0, k_0]),$$

etc. Let $k_t(n_0, k_0) = \sum_{\tau<t} n_\tau(n_0, k_0)$.

Lemma 5 *If h is increasing, $n_t(n_0, k_0)$ and $k_t(n_0, k_0)$ are strictly increasing in n_0.*

Proof. We first show that $\Phi(\cdot)$ is increasing in both arguments. By assumption, $C' > 0$. Since $h \geq 0$, $\int_k^{k+n} h(s)\,ds$ is increasing in n. This shows that Φ is increasing in n. Next, $\frac{d}{dk}\int_k^{k+n} h(s)\,ds = h(k+n) - h(k) \geq 0$ because h is increasing and because $n \geq 0$. Therefore Φ is also increasing in k.

Since C is increasing, so is C^{-1}, and so it suffices to show that $C(n_t)$ rises with n_0 for all t. But this is equivalent to showing that the t'th iterate of Φ is increasing in n_0. Since $k_{t+1} = k_t + C^{-1}[\Phi(n_t, k_t)]$, the feedback effect on Φ that a rise in n_{t-1} induces on k_t is positive. Since Φ is increasing in both arguments we are done. ■

We pause in order to observe that an increasing-hazard distribution cannot be defective – $F(\infty) = 1$.

Proposition 3 *If h is increasing, a unique equilibrium exists.*

Proof. Choose $n_0 > 0$. Since k_0 is given, this conjecture gives rise to the pair of sequences $n_t(n_0, k_0)$ and $k_t(n_0, k_0)$. By construction, the sequences $n_t(n_0, k_0)$ and $k_t(n_0, k_0)$ satisfy (5) and (6). The only remaining question is finding the right n_0. If we can find that n_0 at which the cost, $C(n_0)$, equals the the expected present value of a unit of capital at date 1, we will be done. Now, the expected value of a unit of capital is the discounted sum of expected future profits stemming from that unit:

$$v_0(n_0) = p\sum_{t=1}^{\infty} \beta^t \left(\frac{1 - F(k_t[n_0, k_0])}{1 - F(k_0)}\right)$$

Now by Lemma 5, v is decreasing in n_0. Since $k_1(0, k_0) = k_0$, $v_0(0) \geq \beta p$, and $v_0(\infty) = 0$. On the other hand, $C(n_0)$ is strictly increasing with $C(0) = 0$. Therefore a unique n_0 exists at which $v_0(n_0) = C(n_0)$. Call this value n_0^*. Then the equilibrium capital sequence is $k_t(n_0^*, k_0)$. ■

Intuitively, equilibrium is unique because when h is increasing, the likelihood of a crash rises with industry capacity. The more is invested, the less attractive the market looks to future investors conditional on its survival. When h is *de*creasing that may no longer be so. Equilibrium may still be unique, as in the Pareto example, but Φ may not be increasing in k so that the argument, as stated in Lemma 5, fails.

5 Conclusion

This paper has linked the phenomenon of market crashes and excess capacity. As Figure 2 shows, Telecom is not the only industry where stock prices and output both fell suddenly. Other markets, including the market for PC, unexpectedly became "saturated" making further investment unprofitable (Wall Street Journal 2000, and Angwin 2002). Yet these were the very markets where, prior to the crash, there was a stock-price run-up.

The driving force behind the price run-up was, we argue, a rational rise in optimism in the face of flat earnings. The argument holds if the supply of capital to the industry in question was upward sloping, and if prior beliefs over market size had a decreasing hazard. The latter is reasonable if the distribution of market sizes for new products is skewed to the right. Axtell and others have found that such distributions arise with city sizes and firm sizes, and we conjecture that they also fit the actual distribution of market sizes. Our work on this question will be the subject of a separate paper.

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6 Appendix A: Data and simulation details

6.1 The data

Definition of TELECOM industry - We download from COMPUSTAT `data8` and `data12` for all COMPUSTAT firms with SIC code in the range 4812-4899. Such range is defined by the Census Bureau as the "Communications Sector" and the table below describes what it refers to.

1987 SIC	1987 U.S. SIC Description	1997 NAICS	1997 NAICS U.S. Description
48	Communications		
4812@	Radiotelephone Communications		
	. Paging Carriers	513321	Paging
	. Cellular Carriers	513322	Cellular and Other Wireless Telecommunications (pt)
	. Paging and Cellular Resellers	51333	Telecommunications Resellers (pt)
4813@	Telephone Communications, Except Radiotelephone		
	. Except Resellers	51331	Wired Telecommunications Carriers (pt)
	. Wired Resellers	51333	Telecommunications Resellers (pt)
	. Satellite Resellers	51334	Satellite Telecommunications (pt)
4822@	Telegraph and Other Message Communications	51331	Wired Telecommunications Carriers (pt)
4832	Radio Broadcasting Stations		
	. Networks	513111	Radio Networks
	. Stations	513112	Radio Stations
4833	Television Broadcasting Stations	51312	Television Broadcasting
4841	Cable and Other Pay Television Services		
	. Cable Networks	51321	Cable Networks
	. Except Cable Networks	51322	Cable and Other Program Distribution
4899	Communications Services, NEC		
	. Taxi Cab Dispatch	48531	Taxi Service (pt)
	. Ship-to-Shore Broadcasting Communications	513322	Cellular and Other Wireless Telecommunications (pt)
	. Satellite Communications	51334	Satellite Telecommunications (pt)
	. Except Taxi Cab Dispatch Ship-to-Shore Communications and Satellite Communications	51339	Other Telecommunications

Such micro-data set should be representative of the TELECOM sector at large. It includes all companies listed on any stock exchange in the US from 1971 on, and in the TELECOM sector[7].

Consumer Price Index - Found on http://data.bls.gov/cgi-bin/surveymost?cu by clicking on the box: "U.S. All Items, 1967=100 - CUUR0000AA0" it has the id `CUUR0000AA0`. It has been re-normalized to 2000=1 by us generating our variable `CPI_annual_norm`. Documentation on how the index is computed can be found on the web-site above.

Definition of NASDAQ Telecommunications Index - The NASDAQ Telecommunications Index contains all types of telecommunications companies, including point-to-point communication services and radio and television broadcast, and companies that manufacture communication equipment and accessories (SIC Nos. 4812-4899). On November 1, 1993, the NASDAQ Utility Index was renamed the NASDAQ Telecommunications Index. The former NASDAQ Utility Index was reset to a base of 200.00, using a factor of 5.74805. We take care of this in the preparation of the data.

Sources for the Definition of other NASDAQ Indexes - The definitions of the other NASDAQ Indexes used in Figure 1 can be found on the web-site:

http://www.nasdaq.com/reference/IndexDescriptions.stm

or

http://www.marketdata.nasdaq.com/mr4a.html

[7] As usual there is some selection bias since COMPUSTAT contains only listed companies. Still we know that listed companies account for the majority of the value of the economy.

Computation of NASDAQ Indexes - Some definitions first:

- The **market capitalization** is obtained by multiplying the number of shares in issue by the mid price.
- The **mid price** of a security is obtained by taking the average between the best bid price and the best offer price available on the market during normal business hours.
- The **number of shares outstanding** are used to calculate the market capitalization for each component of the index. These shares represent capital invested by the firm's shareholders and owners, and may be all or only a portion of the number of shares authorized[8].
- **Constituent** is any firm listed on NASDAQ

The Nasdaq Composite Index is weighted arithmetically where the weights are the market capitalizations of its constituents. The index is the summation of the market values (or capitalizations) of all companies within the index and each constituent company is weighted by its market value (shares in issue multiplied by the mid price). The formula used for calculating the index is straightforward. However, determining the capitalization of each constituent company and calculating the capitalization adjustments to the index are more complex. The index value itself is simply a number which represents the total market value of all companies within the index at a particular point in time compared to a comparable calculation at the starting point. The daily index value is calculated by dividing the total market value of all constituent companies by a number called the divisor. The divisor is then adjusted when changes in capitalization occur to the constituents of the index (see Revision of the Divisor) allowing the index value to remain comparable over time.

$$I_t = I_0 \frac{total\, market\, value_t}{divisor_t} = I_0 \frac{\sum_{i=1}^{N_t} P_{it} S_{it}}{D_t}$$

where t is the date at which we want to calculate the index I, $t = 0$ is a reference date or base date we start with (like February 1971 for the composite index which is set to 100) P_{it} is the price of a share of company i at date t, S_{it} is the total number of shares outstanding for company i at date t and D_t is a divisor, introduced to make

[8] Shares that have been issued and subsequently repurchased by the company for cancellation are called treasury shares, because they are held in the corporate treasury pending reissue or retirement. Treasury shares are legally issued but are not considered outstanding for purposes of voting, dividends, or earnings per share calculation. Shares authorised but not yet issued are called un-issued shares. Most companies show the amount of authorised, issued and outstanding, and treasury shares in the capital section of their annual reports. It is possible to back out the total number of outstanding shares of each company from the balance sheet. In COMPUSTAT it is possible to obtain market capitalization by using the following DATA items: (DATA6+DATA199*DATA25-DATA60)

the index comparable over time (basically keeps track of changing in the pool of firms or their share policies and allows the composite index only to track growth rates over periods) and defined below:

$$D_t = \sum_{i=1}^{N_0} P_{i0} S_{i0} + \sum_{j=1}^{t} G_{j-1} \frac{I_0}{I_{j-1}}$$

where G_{j-1} is net new money raised at time $j-1$ through the issue of new companies, new shares, rights issues, capital reorganizations or even capital repayments. This figure may be negative. If G is zero between periods the index boils down to:

$$I_t = \frac{total\,market\,value_t}{total\,market\,value_0} I_0 = I_0 \sum_{i=1}^{N_t} \frac{P_{it} S_{it}}{\sum_{i=1}^{N_0} P_{i0} S_{i0}} \qquad (18)$$

Again, all these adjustment are made by NASDAQ analysts, including all the necessary adjustments if a company is discarded from the index. The resulting variables, deflated by `CPI_annual_norm` are `NASDAQ_composite_real` and `NYSE_telecom_real`.

Real Variables - In the model there is variable k_t which represents productive capital. We decide to proxy it with two distinct variables: sales and property plant and equipment.

The Variables have been extracted from the COMPUSTAT data set for all firms with SIC code in the range 4812-4899 for each year. Such data derived from COMPUSTAT are accessible with user login on the web-site

http://wrds.wharton.upenn.edu/

The two variables extracted are

- `data12` - Sales (Net): This item represents gross sales (the amount of actual billings to customers for regular sales completed during the period) reduced by cash discounts, trade discounts, and returned sales and allowances for which credit is given to customers. The result is the amount of money received from the normal operations of the business (i.e., those expected to generate revenue for the life of the company)[9].

- `data8` - Property, Plant, and Equipment – Total (Net): represents the cost, of tangible fixed property used in the production of revenue, less accumulated depreciation[10].

[9]See the DATA Manual in Compustat for a list of items included and excluded in this data variable.

[10]In COMPUSTAT there is the availability of data7 which is the same as data8 without amortization and depreciation. Even if the model does not include depreciation for simplicity we prefer to include it in the empirical part. Amortization expresses (in theory) the real usage per period of capital and (again in theory) we are willing to include it in our proxy.

Then these variables are aggregated by simple sum to industry level and then deflated. We deflate the sales series by the Telecom price index and the property plant and equipment series by CPI to produce our `real_property_plant_equip` and `real_sales_total`.

Construction of Figure 1 - Panel 1: `NASDAQ_telecom_index deflated by CPI` and `NYSE_composite index deflated by CPI`. **Panel 2**: `NASDAQ_telecom_index deflated by CPI, real_property_plant and_equipment and real_sales`.

6.2 Simulation and Calibration Details

The simulation exercise is an extra check on the validity of our theoretical results, a better understanding of the dynamics of the model and, since we use the analysis of the Pareto case, a grasp on how well the Pareto case captures the dynamics of the real data[11].

Calibration of the Prior - From (3) we have

$$c_t = v_{t+1}\beta\frac{1-F_{t+1}}{1-F_t}$$

and from (5) we have

$$v_t = p + c\left(n_t\right).$$

We interpret $\frac{v_t}{p}$ as the price-earning ratio that is $\left(\frac{P}{E}\right)_t$ and (assuming $c_t = c\left(n_t\right)$)

$$c_t = v_t\left(1-\left(\frac{E}{P}\right)_t\right).$$

Equating the two and dividing by v_t we obtain

$$\left(1-\left(\frac{E}{P}\right)_t\right) = \frac{v_{t+1}}{v_t}\beta\frac{1-F_{t+1}}{1-F_t} \tag{19}$$

Let $(1+g) \equiv \frac{1}{T}\sum_{t=1}^{T}\frac{v_{t+1}}{v_t}$ be the average growth factor of v_t, let $\overline{\left(\frac{E}{P}\right)}$ be the sample average of $\left(\frac{E}{P}\right)_t$. Replacing these variables in (19) by their sample averages and rearranging we have

$$\frac{1-F_{t+1}}{1-F_t} = \frac{\left(1-\overline{\left(\frac{E}{P}\right)}\right)}{(1+g)\,\beta} \equiv R. \tag{20}$$

Let $S_t \equiv 1-F_t$ be the survivor function. Then (20) implies

$$S_{t+1} = RS_t \Longrightarrow S_t = S_0R^t \tag{21}$$

We compute R by:

[11] This last point is presented in the main body of the paper.

- setting $\beta = 0.90$,
- computing $(1+g) = \frac{1}{T}\sum_{t=1}^{T}\frac{v_{t+1}}{v_t} = 1.06$ using the Nasdaq TELECOM index for v_t,
- computing a $\overline{\left(\frac{E}{P}\right)} = 0.056$ constructed by first taking the average earning-price ratio across firms in each year and then taking its time-series average.

Plugging these into (20) the result is $R = 0.96$. Then picking the natural[12] initial condition $S_0 = 1$ (21) gives us the predicted value for S_t of

$$\widehat{S}_t = (0.96)^t$$

We call such sequence the *calibrated* $\widehat{S}_t$ sequence.

Estimation of Pareto Prior - We have calibrated the implied non-parametric sequence of survivor probabilities given our time-series averages. Now, from these calibrated survivor probabilities we need to generate a predicted k_t sequence. To do this, we shall need a specific functional form for S. As in the paper, we shall assume that S_t comes from a Pareto distribution and estimate its parameters. Since the Pareto survivor function is $S(z) = \left(\frac{z_{\min}}{z}\right)^{\rho}$,

$$S_t = \left(\frac{k_{\min}}{k_t}\right)^{\rho} \tag{22}$$

We now seek that value of ρ such that the calibrated $\widehat{S}_t$ sequence delivers a capital series that resembles most to the observed sequence k_t. By taking logs in (22) we get

$$\log \widehat{S}_t = \rho \log k_{\min} - \rho \log k_t$$

We estimate ρ by fitting $\widehat{S}_t$ to the observed capital series k_t via OLS in the equation

$$\log \widehat{S}_t = \rho \log k_{\min} - \rho \log k_t + \varepsilon_t,$$

where ε is approximation/fitting error. This yields the estimates

data set used	$\widehat{\rho}$	$\rho\widehat{\log k_{\min}}$
COMPUSTAT	.57	6.78

That implies a $\ln(k_{\min}) = 11.8$ or $k_{\min} = 133291$. We could have set the parameter $k_{\min} = k_0$ the first observation of `data12_real` (or `data8_real`) and then proceed to find the ρ that matches the implied path for S_t. Since $\texttt{data12\_real}_0 = 239821.08$ (and $\texttt{data8\_real}_0 = 162517$) we were not too far off the mark. Now it is possible to back out the corresponding hazard since the Pareto hazard is $\frac{\widehat{\rho}}{k_t}$.

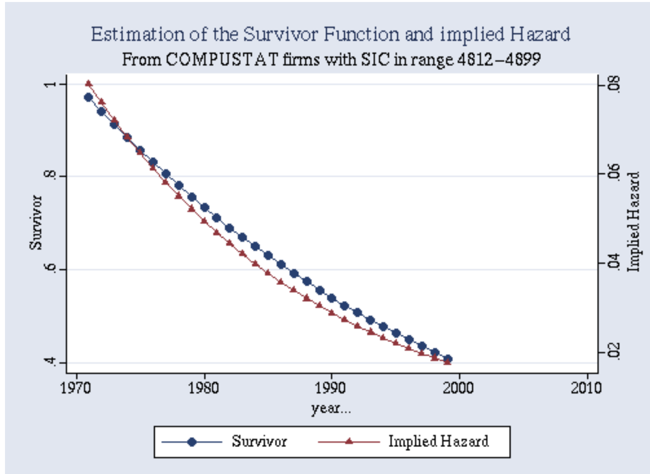


Figure 7: ACTUAL AND FITTED S_t

Simulated Capital and Value Series - It is also possible to find out the implied k_t sequence from the calibrated $\widehat{S}_t$ sequence and our estimation of ρ. In fact

$$k_t = \frac{\widehat{k_{\min}}}{\left(\widehat{S}_t\right)^{\frac{1}{\rho}}}$$

Figure 7 plots such sequence.

Then it is possible to back out implied v_t and n_t sequences. The n_t sequence simply as

$$n_t = \widehat{k}_{t+1} - \widehat{k}_t$$

For the v_t sequence we assume linearity of the cost function like in the Pareto example in the text and calibrate the values of c and p with the following calibration strategy. We use again the conditions

$$v_t = p + cn_t$$
$$\frac{v}{p} = 1 + \frac{c}{p}n_t$$

to form

$$\frac{1}{T}\sum v_t = p + c\frac{1}{T}\sum n_t$$

and

$$\frac{1}{T}\sum \frac{v_t}{p_t} = 1 + c\frac{1}{T}\sum n_t$$

[12] The condition $S_0 = 1$ implies $S(k_0) = 1$ that is the market before beginning its life has probability of surviving the crash =1 since has not crashed at date 0.

so that

$$\overline{v} = p + c\overline{n}$$

and

$$\overline{\left(\frac{P}{E}\right)} = 1 + \frac{c}{p}\overline{n}$$

delivering

$$\widehat{p} = \frac{\overline{v}}{\overline{\left(\frac{P}{E}\right)}} \quad \text{and} \quad \widehat{c} = \frac{\overline{v} - \frac{\overline{v}}{\overline{\left(\frac{P}{E}\right)}}}{\overline{n}}$$

then we can compute the implied series

$$\widehat{v}_t = p + c\left(\widehat{k}_{t+1} - \widehat{k}_t\right)$$

7 Appendix B: Existence, uniqueness and simulation of solutions to (13), (14)

The pair of difference equations [(13), (1)] in (n, k) has no finite steady state for k. In order to be able to linearize around a steady state, we change variables from k, the level of capacity, to its rate of growth

$$x = \frac{n}{k}.$$

We shall now analyze the evolution of the pair (n, x). The change of variables transforms the pair (13) and (1) into the following pair of difference equations

$$n' = -\frac{p}{c} + n\frac{1}{\beta}(1+x)^{\rho}, \tag{23}$$

$$x' = \frac{x}{(1+x)}\left[-\frac{p}{cn} + \frac{1}{\beta}(1+x)^{\rho}\right]. \tag{24}$$

Lemma 6 *(13) and (1) are equivalent to (23) and (24).*

Proof. In the law of motion for k,

$$k' = k + n,$$

divide by n' to obtain

$$\begin{aligned}\frac{k'}{n'} &= \frac{k}{n'} + \frac{n}{n'} \\ &= \frac{k}{n}\frac{n}{n'} + \frac{n}{n'} \\ &= \left(1 + \frac{k}{n}\right)\frac{n}{n'}.\end{aligned}$$

Inverting both sides,

$$\begin{aligned}\frac{n'}{k'} &= \frac{n'}{n}\frac{1}{\left(1+\frac{k}{n}\right)} \\ &= \frac{1}{\left(1+\frac{k}{n}\right)n}\left[-\frac{p}{c}+n\frac{1}{\beta}\left(1+\frac{n}{k}\right)^{\rho}\right] \\ &= \frac{1}{\left(1+\frac{k}{n}\right)}\left[-\frac{p}{cn}+\frac{1}{\beta}\left(1+\frac{n}{k}\right)^{\rho}\right] \\ &= \frac{x}{(1+x)}\left[-\frac{p}{cn}+\frac{1}{\beta}(1+x)^{\rho}\right]\end{aligned}$$

Therefore (13) and (1) are equivalent to (23) and (24). ■

These equations have the unique steady state. Now the steady state of the system is

$$\begin{pmatrix} n \\ x \end{pmatrix} = \begin{pmatrix} \frac{\beta p}{(1-\beta)c} \\ 0 \end{pmatrix}$$

So let us linearize around it. The Jacobian evaluated at the steady state is

$$\begin{bmatrix} \frac{1}{\beta} & \frac{p\rho}{(1-\beta)c} \\ 0 & 1 \end{bmatrix}$$

The characteristic roots are $\left(\frac{1}{\beta},1\right)$. As is standard we set $n'=n$ and $x'=x$ to find two curves crossing in the steady state

$$n'=n \Longrightarrow n=\frac{p}{c}\left(\frac{\beta}{[1+x]^{\rho}-\beta}\right)\equiv\Phi(x)$$

$$x'=x \Longrightarrow n=\frac{p\beta}{c}\left(\frac{1}{[1+x]^{\rho}-\beta(1+x)}\right)\equiv\Psi(x)$$

$\Psi'(x)=-\frac{1}{((x+1)^{\rho}-x\beta-\beta)^2}\left(\rho(x+1)^{\rho-1}-\beta\right)$. These are the two demarcation curves in the phase diagram. The schedule $\Phi(x)$ is downward sloping, whereas $\Psi(x)$ may or may not be, depending on x and ρ. What matters, however, is that

$$\frac{\Psi(x)}{\Phi(x)}=\frac{[1+x]^{\rho}-\beta}{[1+x]^{\rho}-\beta-\beta x}=\begin{cases}=1 & \text{if } x=0 \\ >1 & \text{if } x>0\end{cases}.$$

So, $\Psi(x)$ is more positively sloped than $\Phi(x)$, and the two curves cross at $x=0$, as shown in Figure 8.[13]

The area where either $n<0$ or $x<0$ is not relevant for the pre-overshooting stage of the game, hence it is shaded.

[13] Mathematica was used to draw the stable manifold.

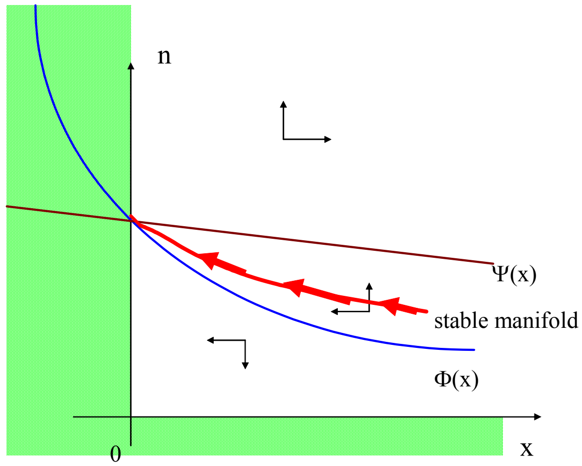


Figure 8: PHASE DIAGRAM FOR n AND $x = \frac{n}{k}$.

To be able to draw the typical arrows on the phase diagram rewrite the system as

$$\begin{aligned} n' &= -\frac{p}{c} + n\frac{1}{\beta}(1+x)^{\rho} \equiv A(x,n), \\ x' &= \frac{x}{(1+x)}\left[-\frac{p}{cn} + \frac{1}{\beta}(1+x)^{\rho}\right] \equiv B(x,n). \end{aligned}$$

Then,

$$A(x, \Phi[x]) = n \tag{25}$$

and

$$B(x, \Psi[x]) = x \tag{26}$$

The vertical arrows.—First we show that if $n > \Phi(x)$, we move even higher. And the opposite if $n < \Phi(x)$. That is,

Claim 1

$$n \gtrless \Phi(x) \Longrightarrow A(x,n) \gtrless n.$$

Proof. The relevant portion of the phase diagram is that for $x \geq 0$. For all such x,

$$\frac{\partial A(x,n)}{\partial n} \geq \frac{1}{\beta} > 1$$

Together with (25) the claim follows. ■

The horizontal arrows.—Next we show that if $n > \Psi(x)$, we move to the right. And if $n < \Psi(x)$, we move to the left. That is,

Claim 2

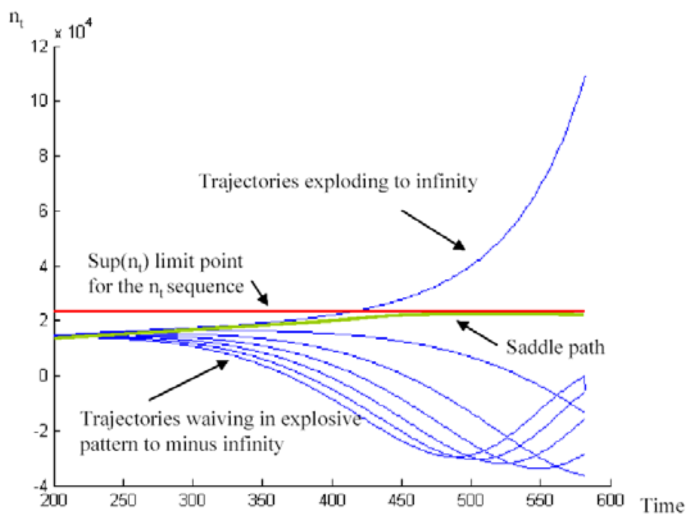
$$n \gtreqless \Psi(x) \Longrightarrow B(x,n) \gtreqless x.$$

Proof. We have
$$\frac{\partial B(x,n)}{\partial n} = \frac{x}{(1+x)}\frac{p}{cn^2} > 0$$
if $x > 0$. Together with (26) the claim follows. ■

These two claims pin down the arrows that are displayed in Figure 8 along with the saddle path. The evolution of n and x is valid only before the overshooting date. The shaded area is not admissible because x cannot be negative.

7.0.1 Simulating the Model

In the figure below we plot the time path for n. This figure shows in greater detail what would happen to the trajectory if it did not start on the saddle path.



EQUILIBRIUM TIME PATH FOR n_t AS IT APPROACHES n_∞

We now describe the algorithm we used to simulate the model:

1. Choose an initial condition for k_0
2. Posit a rather coarse initial grid for initial values n_0
3. For each n_0 from the grid simulate the system equations to generate values for the sequences $\{n,k\}_t$ for many periods
4. Discard any n_0 for which the n_t sequence either explodes, or is not not monotone increasing

5. Once all the n_0 in the initial grid are tried we choose the n_0 with the longest number of observations (before explosion or violation of monotonicity cut off the simulation) and make up a *finer* grid around it

6. The process starts over at point 3 up to the point we can find an initial condition that can generate a monotone sequence for "enough" periods.

The simulations portrayed in Section 3.1 plot the time-paths simulated n_t and k_t against their observed proxies: the actual Telecom sales series used as a proxy for capital (`sales_total`) and the fist difference of `sales_total` (a proxy for n_t).