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## ABSTRACT

We develop tests for common values at first-price sealed-bid auctions. Our tests are nonparametric, require observation only of the bids submitted at each auction, and are based on the fact that the "winner's curse" arises only in common values auctions. The tests build on recently developed methods for using observed bids to estimate each bidder's conditional expectation of the value of winning the auction. Equilibrium behavior implies that in a private values auction these expectations are invariant to the number of opponents each bidder faces, while with common values they are decreasing in the number of opponents. This distinction forms the basis of our tests. We consider both exogenous and endogenous variation in the number of bidders. Monte Carlo experiments show that our tests can perform well in samples of moderate sizes. We apply our tests to two different types of U.S. Forest Service timber auctions. For unit-price ("scaled") sales often argued to fit a private values model, our tests consistently fail to find evidence of common values. For "lumpsum" sales, where *a priori* arguments for common values appear stronger, our tests yield mixed evidence against the private values hypothesis.

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## 1 Introduction

At least since the influential work of Hendricks and Porter (1988), studies of auction data have played an important role in demonstrating the empirical relevance of economic models of strategic interaction between agents with asymmetric information. However, a fundamental issue remains unresolved: how to choose between private and common values models of bidders' information. In a common values auction, information about the value of the object for sale is spread among bidders; hence, a bidder would update his assessment of the value of winning if he learned the private information of an opponent. In a private values auction, by contrast, opponents' private information would be of interest to a bidder only for strategic reasons—learning an opponent's assessment of the good would not affect his beliefs about his own valuation.

In this paper we propose nonparametric tests to distinguish between the common values (CV) and private values (PV) paradigms based on observed bids at first-price sealed-bid auctions. The distinction between these paradigms is fundamental in the theoretical literature on auctions, with important implications for bidding strategies and the design of markets. While intuition is often offered for when one might expect a private or common values model to be more appropriate, a more formal approach would be valuable in many applications. In fact, discriminating between common and private values was the motivation behind Paarsch's (1992) pioneering work on structural estimation of auction models. More generally, models in which strategic agents' private information leads to adverse selection (a common values auction being just one example) have played a prominent role in the theoretical economics literature, yet the prevalence and significance of this type of informational asymmetry is not well established empirically. Because a first-price auction is a market institution particularly well captured by a tractable theoretical model, data from these auctions offer a promising opportunity to test for adverse selection using structure obtained from economic theory.

Several testing approaches explored previously rely heavily on parametric assumptions about the distribution functions governing bidders' private information (e.g. Paarsch (1992), Sareen (1999)). Such tests necessarily confound evaluation of the economic hypotheses of interest with evaluation of parametric distributional assumptions. Some prior work (e.g., Gilley and Karels (1981)) has suggested examining variation in bid levels with the number of bidders as a test for common values. However, Pinkse and Tan (2002) have recently shown that this type of reduced-form test generally cannot distinguish CV from PV models in first-price auctions: in equilibrium, strategic behavior can cause bids to increase or decrease in the number of opponents under either paradigm. We overcome both of these limitations by taking a nonparametric structural approach, exploiting the relationships between observable bids and bidders' latent expectations implied by equilibrium bidding in a model that nests the private and common values frameworks. Unlike tests of particular PV or CV models (e.g., Paarsch (1992), Hendricks, Pinkse, and Porter (2003)), our approach enables testing a null hypothesis including all PV models within the standard affiliated values framework (Milgrom and Weber (1982)) against an alternative including all CV models in that framework. The price we pay for these advantages is reliance on an assumption of equilibrium bidding. This is not an innocuous assumption. However, a first-price auction is a market institution that seems particularly well suited to this structural approach.

The importance of tests for common values to empirical research on auctions is further emphasized by recent results showing that CV models are identified only under strong conditions on the underlying information structure or on the types of data available (e.g., Athey and Haile (2002)). Hence, a formal method for determining whether a CV or PV model is more appropriate could offer an important diagnostic tool for researchers hoping to use demand estimates from bid data to guide the design of markets. Laffont and Vuong (1996) have observed that any common values model is observationally equivalent to some private values model, suggesting that such testing is impossible. However, they did not consider the possibilities of binding reserve prices or variation in the numbers of bidders, either of which could aid in distinguishing between the private and common values paradigms.

Our tests exploit variation in the number of bidders and are based on detecting the effects of the *winner's curse* on equilibrium bidding. The winner's curse is an adverse selection phenomenon arising in CV but not PV auctions. Loosely, winning a CV auction reveals to the winner that he was more optimistic about the object's value than his opponents were. This "bad news" (Milgrom (1981)) becomes worse as the number of opponents increases—having the most optimistic signal among many bidders implies (on average) even greater over-optimism than does being most optimistic among a few bidders. A rational bidder anticipates this bad news and adjusts his expectation of the value of winning (and, therefore, his bid) accordingly. In a PV auction, by contrast, the value a bidder places on the object does not depend on his opponents' information, so the number of bidders does not affect his expected value of the object conditional on winning. Relying on this distinction, our testing approach is based on detecting the adjustments rational bidders make in order to avoid the winner's curse as the number of competitors changes. This is nontrivial because we can observe only bids, and variation in the level of competition affects the aggressiveness of bidding even in a PV auction. However, economic theory enables us to separate this competitive response from responses (if any) to the winner's curse.

We consider several statistical tests, all involving distributions of bidders' expected valuations (actually, particular conditional expectations of their valuations) in auctions with varying numbers of bidders. In a PV environment, these distributions should not vary with the number of bidders, whereas the CV alternative implies a first-order stochastic dominance relation. Our testing problem is complicated by the fact that we cannot compare empirical distributions of bidders' expectations directly; rather, we can only compare empirical distributions of *estimates* of these expectations, obtained using nonparametric methods recently developed by Guerre, Perrigne, and Vuong (2000) (hereafter GPV) and extended by Li, Perrigne, and Vuong (2002), Li, Perrigne, and Vuong (2000) (together, LPV hereafter) and by Hendricks, Pinkse, and Porter (2003) (hereafter HPP). This nonparametric first stage raises several issues that can significantly complicate the asymptotic distributions of test statistics. A further complication arising in many applications is the endogeneity of bidder participation. After developing our tests for the base case of exogenous participation, we consider several standard models of endogenous participation and provide conditions under which our tests can be adapted.

While our testing approach is new, we are not the first to explore implications of the winner's curse as an approach for distinguishing PV from CV models. Hendricks, Pinkse and Porter (2003, footnote 2) suggest a testing approach applicable when there is a binding reserve price, in addition to several tests of a pure common values model that are applicable when one observes, in addition to bids, the *ex post* realization of the object's value. Although our tests are applicable when there is a binding reserve price, this is not required—an important advantage in many applications, including the drilling rights auctions studied by HPP and the timber auctions we study below. In addition, our tests require observation only of the bids—the only data available from most first-price auctions. For second-price and English auctions, Paarsch (1991) and Bajari and Hortacsu (2003) have considered testing for the winner's curse using a simple regression approach. However, second-price sealed-bid auctions are uncommon in practice, and the applicability of this approach to English auctions is limited by the fact that the winner's willingness to pay is never revealed (creating a missing data problem) and further by ambiguity regarding the appropriate interpretation of losing bids (e.g., Bikhchandani, Haile, and Riley (2002), Haile and Tamer (2003)). Athey and Haile's (2002) study of identification in auction models includes sufficient conditions for discriminating between common and private values. However, they focus on cases in which only a subset of the bids is observable, consider only exogenous participation, and do not develop formal statistical tests.

The remainder of the paper is organized as follows. The next section summarizes the underlying model, the method for inferring bidders' expectations of their valuations from observed bids, and the main principle of our testing approach. In section 3 we provide the details of two types of tests and develop the necessary asymptotic theory. In section 4 we report the results of Monte Carlo experiments demonstrating the performance of our tests. In section 5 we show how the tests can be extended to environments with endogenous participation, and section 6 presents an approach for incorporating auction-specific covariates. Section 7 then presents the empirical application to U.S. Forest Service auctions of timber harvesting contracts, where we consider data from two types of sales that differ in ways that seem likely *a priori* to affect the significance of any common value elements. We conclude in section 8.

## 2 Model and Testing Principle

The underlying theoretical framework is Milgrom and Weber's (1982) general affiliated values model. Throughout we denote random variables in upper case and their realizations in lower case. We use boldface to denote vectors. An auction has $N \in \{\underline{n} \ldots \bar{n}\}$ risk-neutral bidders, with $\underline{n} \geq 2$. Each bidder $i$ has a valuation $U_i \in (\underline{u}, \overline{u})$ for the object and observes a private signal $X_i \in (\underline{x}, \overline{x})$ of this valuation. We let $\mathbf{X}_{-i}$ denote the vector of signals of $i$'s opponents. Valuations and signals have joint distribution $\tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n)$, which is assumed to have a positive joint density on $(\underline{u}, \overline{u})^n \times (\underline{x}, \overline{x})^n$. We make the following standard assumptions (see Milgrom and Weber (1982)).

**Assumption 1** *(Symmetry)* $\tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n)$ *is exchangeable with respect to the indices* $1, \ldots, n$.[1]

**Assumption 2** *(Affiliation)* $U_1, \ldots, U_n, X_1, \ldots, X_n$ *are affiliated.*

**Assumption 3** *(Nondegeneracy)* $E[U_i|X_i = x, \mathbf{X}_{-i} = \mathbf{x}_{-i}]$ *is strictly increasing in* $x$ $\forall \mathbf{x}_{-i}$.

Initially, we also assume that the number of bidders is not correlated with bidder valuations or signals:[2]

**Assumption 4** *(Exogenous Participation) For each* $n < \bar{n}$ *and all* $(u_1, \ldots, u_n, x_1, \ldots, x_n)$, $\tilde{F}_n(u_1, \ldots, u_n, x_1, \ldots, x_n) = \tilde{F}_{\bar{n}}(u_1, \ldots, u_n, \infty, \ldots, \infty, x_1, \ldots, x_n, \infty, \ldots, \infty)$.

[1] We discuss relaxation of the symmetry assumption in section 8.

[2] This assumption is not made by Milgrom and Weber (1982) because they consider fixed $n$.

Such exogenous variation in the number of bidders will arise naturally in some applications but not others (cf. Athey and Haile (2002) and section 5 below). Endogenous participation will be considered in section 5 after results for this base case are presented.

A seller conducts a first-price sealed-bid auction for a single object; i.e., sealed bids are collected from all bidders, and the object is sold to the high bidder at a price equal to his own bid.[3] Under Assumptions 1–3, in an $n$-bidder auction there exists a unique symmetric Bayesian Nash equilibrium in which each bidder employs a strictly increasing strategy $s_n(\cdot)$. Assuming equilibrium bidding by his opponents, bidder $i$ chooses his bid $b$ to maximize

$$E\left[(U_i - b)\mathbf{1}\{s_n(x_j) \leq b \quad \forall j \neq i\}|X_i = x_i\right]$$

As shown by Milgrom and Weber (1982), the first-order condition characterizing the equilibrium bid function is

$$v(x, x, n) = s_n(x) + \frac{s_n'(x)F_n(x|x)}{f_n(x|x)} \quad \forall x \tag{1}$$

where

$$v(x, x', n) \equiv E\left[U_i|X_i = x, \max_{j\neq i} X_j = x'\right], \tag{2}$$

$F_n(\cdot|x)$ is the distribution of the maximum signal among a given bidder's opponents conditional on his own signal being $x$, and $f_n(\cdot|x)$ is the corresponding conditional density.

The conditional expectation $v(x, x, n)$ in (1) gives a bidder's expectation of his valuation conditional on his signal and on his equilibrium bid being pivotal. Our testing approach is based on the fact that this expectation is decreasing in $n$ whenever valuations contain a common value element. To show this, we first formally define private and common values.[4]

**Definition 1** *Bidders have* **private values** *iff* $E[U_i|X_1, \ldots, X_n] = E[U_i|X_i]$*; bidders have* **common values** *iff* $E[U_i|X_1, \ldots, X_n]$ *strictly increases in* $X_j$ *for* $j \neq i$.

Note that the definition of common values incorporates a wide range of models with a common value component, not just the special case of *pure common values*, where the value of the object is unknown but identical for all bidders.[5]

[3] We describe the auction as one in which bidders compete to buy. The translation to the procurement setting, where bidders compete to sell, is straightforward.

[4] Affiliation implies that $E[U_i|X_1, \ldots, X_n]$ is nondecreasing in all $X_j$, and symmetry implies that when the expectation strictly increases in some $X_j, j \neq i$, it must strictly increase in all $X_j, j \neq i$. For simplicity our definition of common values excludes cases in which the winner's curse arises for some realizations of signals but not others. Without this, the results below would still hold but with weak inequalities replacing some strict inequalities. Up to this simplification, our PV and CV definitions characterize a partition of the set of models falling in Milgrom and Weber's (1982) affiliated values framework.

[5] Our terminology corresponds to that used by, e.g., Klemperer (1999) and Athey and Haile (2002), although it is not the only one used in the literature. Some authors reserve the term "common values"

The following theorem gives the key result enabling discrimination between PV and CV models.

**Theorem 1** *Under Assumptions 1–4, $v(x,x,n)$ is invariant to $n$ for all $x$ in a PV model but strictly decreasing in $n$ for all $x$ in a CV model.*

**Proof:** Given symmetry, we focus on bidder 1 without loss of generality. With private values, $E[U_1|X_1,\ldots,X_n] = E[U_1|X_1]$, which does not depend on $n$. With common values

$$\begin{aligned} v(x,x,n) &\equiv E\left[U_1|X_1 = X_2 = x, X_3 \le x, \ldots, X_{n-1} \le x, X_n \le x\right] \\ &= E_{X_n \le x} E\left[U_1|X_1 = X_2 = x, X_3 \le x, \ldots, X_{n-1} \le x, X_n\right] \\ &< E_{X_n} E\left[U_1|X_1 = X_2 = x, X_3 \le x, \ldots, X_{n-1} \le x, X_n\right] \\ &= E\left[U_1|X_1 = X_2 = x, X_3 \le x, \ldots, X_{n-1} \le x\right] \\ &\equiv v(x,x;n-1) \end{aligned}$$

with the inequality following from the definition of common values. □

Informally, in equilibrium a rational bidder adjusts his expectation of his valuation downward to reflect the fact that he wins only when his own signal is higher than those of all opponents. The size of this adjustment depends on the number of opponents: the information that the maximum signal among $n$ is equal to $x$ implies a higher expectation of $U_i$ than the information that the maximum among $m > n$ is equal to $x$. Hence, the conditional expectation $v(x,x,n)$ decreases in $n$.

### 2.1 Structural Interpretation of Observed Bids

To use Theorem 1 to test for common values, we must be able to infer or estimate the latent expectations $v(x_i,x_i,n)$ for bidders in auctions with varying numbers of participants. We assume that for each $n$ the researcher observes the bids $B_1,\ldots,B_n$ from $T_n$ $n$-bidder auctions. We let $T = \sum_n T_n$ and assume that for all $n$, $\frac{T_n}{T} \to \rho_n \in (0,1)$ as $T \to \infty$. Below we will add the auction index $t \in \{1,\ldots,T\}$ to the notation defined above as necessary. For simplicity we initially assume an identical object is sold at each auction. As shown by GPV, standard nonparametric techniques can be applied to control for auction-specific covariates. Below we will also suggest a more parsimonious alternative that may be more

for the special case we call pure common values and use the term "interdependent values" (e.g., Krishna (2002)) or the less accurate "affiliated values" for the class of models we call common values. Additional confusion sometimes arises because the partition of the Milgrom-Weber framework into CV and PV models is only one of two partitions that might be of interest, the other being defined by whether bidders' signals are independent. Note in particular that dependence of bidders' signals is neither necessary nor sufficient for common values.

useful in applications with many covariates. We assume throughout that each auction is independent of all others.[6]

As pointed out by GPV, the strict monotonicity of $s_n(\cdot)$ implies that in equilibrium the joint distribution of bidder signals is related to the joint distribution of bids through the relations

$$\begin{aligned} F_n\left(y|x\right) &= G_n\left(s_n(y)|s_n(x)\right) \\ f_n\left(y|x\right) &= g_n\left(s_n(y)|s_n(x)\right)s_n'\left(y\right) \end{aligned} \tag{3}$$

where $G_n(\cdot|s_n(x))$ is the equilibrium distribution of the highest bid among $i$'s competitors conditional on $i$'s equilibrium bid being $s_n(x)$, and $g_n\left(\cdot|s_n\left(x\right)\right)$ is the corresponding conditional density. Because $b_i = s_n(x_i)$ in equilibrium, the differential equation (1) can then be rewritten

$$v(x_i, x_i, n) = b_i + \frac{G_n(b_i|b_i)}{g_n(b_i|b_i)} \equiv \xi(b_i; n). \tag{4}$$

For simplicity we will refer to the expectation $v(x_i, x_i, n)$ on the left side of (4) as bidder $i$'s "value." Although these values are not observed directly, the joint distribution of bids is. Hence, the ratio $\frac{G_n(\cdot|\cdot)}{g_n(\cdot|\cdot)}$ is nonparametrically identified. Because $x_i = s_n^{-1}(b_i)$, equation (4) implies that each $v\left(s_n^{-1}\left(b_i\right), s_n^{-1}\left(b_i\right), n\right)$ is identified as well. This need not be sufficient to identify the model (i.e., to identify $\tilde{F}_n(\cdot)$); however, identification of the distribution of values $v(X_i, X_i, n)$ will be sufficient for our purpose.

To address estimation, let $B_{it}$ denote the bid made by bidder $i$ at auction $t$, and let $B_{it}^*$ represent the highest bid among $i$'s opponents. GPV and LPV suggest nonparametric estimates of the form

$$\begin{aligned} \hat{G}_n(b;b) &= \frac{1}{T_n \times h_G \times n} \sum_{t=1}^{T} \sum_{i=1}^{n} K\left(\frac{b - b_{it}}{h_G}\right) \mathbf{1}\left(b_{it}^* < b, n_t = n\right) \\ \hat{g}_n(b;b) &= \frac{1}{T_n \times h_g^2 \times n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbf{1}(n_t = n) K\left(\frac{b - b_{it}}{h_g}\right) K\left(\frac{b - b_{it}^*}{h_g}\right). \end{aligned} \tag{5}$$

Here $h_G$ and $h_g$ are bandwidths and $K(\cdot)$ is a kernel. $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ are nonparametric estimates of

$$G_n(b;b) \equiv G_n(b|b)g_n\left(b\right) = \frac{\partial}{\partial b}\Pr(B_{it}^* \le m, B_{it} \le b)|_{m=b}$$

and

$$g_n(b;b) \equiv g_n(b|b)g_n\left(b\right) = \frac{\partial^2}{\partial m \partial b}\Pr(B_{it}^* \le m, B_{it} \le b)|_{m=b}$$

[6]This is a standard assumption, but one that serves to qualify almost all empirical studies of bidding, where data are taken from auctions in which bidders compete repeatedly over time.

respectively, where $g_n(\cdot)$ is the marginal density of bids in equilibrium. Because

$$\frac{G_n(b;b)}{g_n(b;b)} = \frac{G_n(b|b)}{g_n(b|b)} \tag{6}$$

$\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)}$ is a consistent estimator of $\frac{G_n(b|b)}{g_n(b|b)}$. Hence, by evaluating $\hat{G}_n(\cdot,\cdot)$ and $\hat{g}_n(\cdot,\cdot)$ at each observed bid, we can construct a pseudo-sample of consistent estimates of the realizations of each $V_{it} \equiv v(X_{it}, X_{it}, n)$ using (4):

$$\hat{v}_{it} \equiv \hat{\xi}(b_{it}; n_t) = b_{it} + \frac{\hat{G}_n(b_{it}; b_{it})}{\hat{g}_n(b_{it}; b_{it})}. \tag{7}$$

This possibility was first articulated for the independent private values model by Laffont and Vuong (1993) and GPV, and has been extended to affiliated values models by LPV and HPP. Following this literature, we refer to each estimate $\hat{v}_{it}$ as a "pseudo-value."

### 2.2 Main Principle of the Test

Each pseudo-value $\hat{v}_{it}$ obtained from (7) is an estimate of $v(x_{it}, x_{it}, n_t)$. If we have pseudo-values from auctions with different numbers of bidders, we can exploit Theorem 1 to develop a test. Let $F_{v,n}(\cdot)$ denote the distribution of the random variable $V_{it} = v(X_{it}, X_{it}, n)$. Because $F_{v,n}(v) = \Pr(v(X_{it}, X_{it}, n) \leq v)$, Theorem 1 and Assumption 4 immediately imply that under the PV hypothesis, $F_{v,n}(\cdot)$ must be the same for all $n$, while under the CV alternative, $F_{v,n}(v)$ must strictly increase in $n$ for all $v$.

**Corollary 1** *Under the private values hypothesis*

$$F_{v,\underline{n}}(v) = F_{v,\underline{n}+1}(v) = \ldots = F_{v,\bar{n}}(v) \quad \forall v. \tag{8}$$

*Under the common values hypothesis*

$$F_{v,\underline{n}}(v) < F_{v,\underline{n}+1}(v) < \ldots < F_{v,\bar{n}}(v) \quad \forall v. \tag{9}$$

## 3 Tests for Stochastic Dominance

Corollary 1 suggests that a test for stochastic dominance applied to estimates of each $F_{v,n}(\cdot)$ would provide a test for common values. If the values $v_{it} = v(x_{it}, x_{it}, n)$ were directly observed, a wide variety of existing tests from the statistics and econometrics literature could be used (e.g, McFadden (1989), Anderson (1996), Davidson and Duclos (2000), Barrett and Donald (2003)). The empirical distribution function

$$\hat{F}_{v,n}(y) = \frac{1}{T_n}\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}(v_{it} \leq y, n_t = n).$$

is commonly used to form test statistics.

Our testing problem has the complication that each $v_{it}$ is not directly observed but estimated. Hence, the empirical distributions we can construct are

$$\hat{F}_{\hat{v},n}(y) = \frac{1}{T_n}\frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}\left(\hat{v}_{it} \leq y, n_t = n\right).$$

Several difficulties are involved in formulating consistent tests based on these empirical distributions and the testing principle above, and in deriving the large sample properties of the test statistics. The primary difficulty is the dependence of the asymptotic distributions of test statistics on the first-step nonparametric estimation of pseudo-values. This not only adds estimation error but also introduces finite sample dependence between nearby pseudo-value estimates that must be accounted for in the asymptotic theory. A further complication is trimming, which is needed at the boundaries of the supports of the pseudo-value distributions, because nonparametric density estimates appear in (7). Trimming introduces theoretical difficulties that can depend on conditions that are hard to interpret in practice (see, for example, Lavergne and Vuong (1996)), and naive trimming rules can lead to inconsistent tests. Finally, note that the validity of inference based on the bootstrap or subsampling also relies on knowledge of convergence rates and asymptotic distributions of test statistics, so the difficulties in deriving asymptotic distributions cannot be avoided simply by employing resampling methods.

We consider two approaches that enable us to overcome these difficulties. The first involves testing the implications of stochastic dominance for finite sets of functionals of each $F_{v,n}(\cdot)$. This approach enables us to apply multivariate one-sided hypothesis tests based on tractable asymptotic approximations. The second approach uses a generalized version of familiar Kolmogorov-Smirnov statistics, with critical values approximated by subsampling.

### 3.1 Multivariate One-Sided Hypothesis Tests for Stochastic Dominance

Let $\gamma_n$ denote a finite vector of monotonic functionals of the distribution $F_{v,n}(\cdot)$. We will consider tests of hypotheses of the form

$$\begin{aligned} H_0 \text{ (PV)} : \; & \gamma_{\underline{n}} = \gamma_{\underline{n}+1} = \cdots = \gamma_{\bar{n}} \\ H_1 \text{ (CV)} : \; & \gamma_{\underline{n}} > \gamma_{\underline{n}+1} > \cdots > \gamma_{\bar{n}} \end{aligned}$$

or, letting $\delta_{m,n} \equiv \gamma_m - \gamma_n$ and $\delta \equiv \left(\delta_{\underline{n},\underline{n}+1}, , \ldots , \delta_{\bar{n}-1,\bar{n}}\right)'$,

$$\begin{aligned} H_0 \text{ (PV)}: &\ \delta = \mathbf{0} \\ H_1 \text{ (CV)}: &\ \delta > \mathbf{0}. \end{aligned} \tag{10}$$

We consider two types of functionals $\gamma_n$.[7] The first is a vector of quantiles of $F_{v,n}(\cdot)$. The second is the mean. In the next two subsections we show that for both cases we can construct consistent estimators of each $\gamma_n$ (or the difference vector $\delta$) with multivariate normal asymptotic distributions. These results rely on the following assumptions.

**Assumption 5** *1. Each $G_n(b; b)$ is $R+1$ times differentiable in its first argument and $R$ times differentiable in its second argument. Each $g_n(b; b)$ is $R$ times differentiable in both arguments. The derivatives are bounded and continuous.*

*2. $\int K(\epsilon)\, d\epsilon = 1$ and $\int \epsilon^r K(\epsilon)\, d\epsilon = 0$ for all $r < R$. $\int |\epsilon|^R K(\epsilon)\, d\epsilon < \infty$.*

*3. $h_G = h_g = h$. As $T \to \infty$, $h \longrightarrow 0$, $Th^2 \big/ \log T \longrightarrow \infty$, $Th^{2+2R} \longrightarrow 0$.*

#### *3.1.1 Tests based on Quantiles*

Let $\hat{b}_{\tau,n}$ denote the $\tau$th quantile of the empirical distribution of bids from all $n$-bidder auctions, i.e.,

$$\hat{b}_{\tau,n} = \hat{G}_n^{-1}(\tau) \equiv \inf\{b : \hat{G}_n(b) \geq \tau\}$$

where $\hat{G}_n(b) = \frac{1}{nT_n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbf{1}(b_{it} \leq b, n_t = n)$. Similarly, let $b_{\tau,n}$ denote the $\tau$the quantile of the marginal distribution of bids, $G_n(\cdot)$, while $x_\tau$ denotes the $\tau$th quantile of the marginal distribution $F_x(\cdot)$ of a bidder's signal. Equation (4) and monotonicity of the equilibrium bid function imply that $v_{\tau,n}$, the $\tau$th quantile of $F_{v,n}(\cdot)$, can be estimated by

$$\hat{v}_{\tau,n} = \hat{b}_{\tau,n} + \frac{\hat{G}_n(\hat{b}_{\tau,n}; \hat{b}_{\tau,n})}{\hat{g}_n(\hat{b}_{\tau,n}; \hat{b}_{\tau,n})}.$$

Because sample quantiles of the bid distribution converge to population quantiles at rate $\sqrt{T_n}$, the sampling variance of $\hat{v}_{\tau,n} - v(x_\tau, x_\tau, n)$ will be governed by the slow pointwise nonparametric convergence rate of $\hat{g}_n(\cdot;\cdot)$.[8] As shown in GPV, for fixed $b$, $\hat{g}_n(b; b)$ converges

[7] Because each null hypothesis we consider consists of a single point in the space of the "parameter" $\delta$, the difficulties discussed in Wolak (1991) do not arise here.

[8] Note that $G_n(b; b)$ is estimated more precisely than $g_n(b; b)$ for all bandwidth sequences $h$. For simplicity, in Assumption 5 we have chosen $h_G = h_g$, in which case the sampling variance will be dominated by that from estimation of $g_n(b; b)$. We have assumed undersmoothing rather than optimal smoothing to avoid estimating the asymptotic bias term for inference purposes. An alternative is to choose different sequences for $h_G$ and $h_g$. If we have chosen $h_G$ and $h_g$ close to their optimal range, the sampling variance will still be dominated by that of $\hat{g}_n(b; b)$ and the result of the theorem will not change. On the other hand if $h_G \approx h_g^2$ so that $\hat{G}_n(b; b)$ and $\hat{g}_n(b; b)$ share the same magnitude of variance, then the convergence rate for $G_n(b; b)$ will be far from optimal.

at rate $\sqrt{T_n h_g^2}$ to $g_n(b;b)$. Theorem 2 then describes the limiting behavior of each $\hat{v}_{\tau,n}$. The proof is given in the appendix.

**Theorem 2** *Suppose Assumption 5 holds. Then as $T_n \longrightarrow \infty$ for each $n$,*
*(i)* $\hat{b}_{\tau,n} - b_{\tau,n} = O_p\left(\frac{1}{\sqrt{T_n}}\right)$.
*(ii) For each $b$ such that $g_n(b;b) > 0$,*

$$\sqrt{T_n h^2}\left[\hat{\xi}(b;n) - v\left(s_n^{-1}(b), s_n^{-1}(b), n\right)\right] = \sqrt{T_n h^2}\left(\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b|b)}{g_n(b|b)}\right)$$

$$\xrightarrow{d} N\left(0, \frac{1}{n}\frac{G_n(b|b)^2}{g_n(b|b)^3 g_n(b)}\left[\int\int K(e)^2 K(e')^2 \, de\, de'\right]\right).$$

*(iii) For distinct values $\tau_1, \ldots, \tau_L$ in $(0,1)$, the $L$-dimensional vector with elements $\sqrt{T_n h^2}\left(\hat{\xi}\left(\hat{b}_{\tau_l,n}; n\right) - v\left(x_{\tau_l}, x_{\tau_l}, n\right)\right)$ converges in distribution to $Z \sim N(0,\Omega)$, where $\Omega$ is a diagonal matrix with $l$th diagonal element*

$$\Omega_l = \frac{1}{n}\frac{G_n(s_n(x_{\tau_l})|s_n(x_{\tau_l}))^2}{g_n(s_n(x_{\tau_l})|s_n(x_{\tau_l}))^3 g_n(s_n(x_{\tau_l}))}\left[\int\int K(e)^2 K(e')^2 \, de\, de'\right].$$

#### *3.1.2 Tests based on Means*

An alternative to comparing quantiles is to compare means of the pseudo-value distributions. We can estimate

$$E_x[v(x,x,n)] = \int v \; dF_{v,n}(v)$$

with the sample average of the pseudo-values in all $n$-bidder auctions:

$$\hat{\mu}_n = \frac{1}{n \times T_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbf{1}(n_t = n)\,\hat{v}_{it}. \qquad (11)$$

By Corollary 1, $E_x[v(x,x,n)]$ is the same for all $n$ under private values but strictly decreasing in $n$ with common values.

A complication in implementing a test is the need for a practical way of trimming boundary values of $\hat{v}_{it}$ that preserves the interpretation of the null and alternative hypotheses. We use a trimming rule that equalizes the quantiles trimmed from $F_{\hat{v},n}(\cdot)$ across all $n$. Because equilibrium bid functions are strictly monotone, the pseudo-value at the $\tau$th quantile of $F_{v,n}(\cdot)$ is that of the bidder with signal at the $\tau$th quantile of $F_x(\cdot)$. Hence, trimming bids at the same quantile for all values of $n$ also trims the same bidder types (signals) from all distributions, thereby preserving the consistency of a test based on Corollary 1.

To make this precise, let $\hat{b}_{\tau,n}$ denote the $\tau$th quantile of $\hat{G}_n(\cdot)$ and recall that $x_\tau$ is the $\tau$th quantile of the marginal distribution of a bidder's signal. The *quantile-trimmed mean* is then defined as

$$\mu_n \equiv E\left[v(x,x,n)\ \mathbf{1}\left(x_\tau \le x \le x_{1-\tau}\right)\right]$$

with sample analog

$$\hat{\mu}_{n,\tau} \equiv \frac{1}{n \times T_n} \sum_{t=1}^{T} \sum_{i=1}^{n} \hat{v}_{it}\ \mathbf{1}\left(\hat{b}_{\tau,n} \le b_{it} \le \hat{b}_{1-\tau,n}, n_t = n\right).$$

We can then test the modified hypotheses

$$H_0 : \mu_{\underline{n},\tau} = \cdots = \mu_{\bar{n},\tau} \tag{12}$$

$$H_1 : \mu_{\underline{n},\tau} > \cdots > \mu_{\bar{n},\tau} \tag{13}$$

which are implied by (8) and (9), respectively. The next theorem shows the consistency and asymptotic normality of each $\hat{\mu}_{n,\tau}$.

**Theorem 3** *Suppose Assumption 5 holds,* $\frac{(\log T)^2}{Th^3} \longrightarrow 0$ *and* $Th^{1+2R} \longrightarrow 0$. *Then*
*(i)* $\hat{\mu}_{n,\tau} \xrightarrow{p} \mu_{n,\tau}$.
*(ii)* $\sqrt{T_n h}\left(\hat{\mu}_{n,\tau} - \mu_{n,\tau}\right) \xrightarrow{d} N\left(0, \sigma_n\right)$ *where*

$$\sigma_n = \left[\int \left(\int K\left(\epsilon'\right) K\left(\epsilon' - \epsilon\right) d\epsilon'\right)^2 d\epsilon\right] \left[\frac{1}{n} \int_{b_{\tau,n}}^{b_{1-\tau,n}} \frac{G_n\left(b;b\right)^2}{g_n\left(b;b\right)^3} g_n\left(b\right)^2 db\right]. \tag{14}$$

The proof is given in the appendix. Note that the convergence rate of each $\hat{\mu}_n$ is $\sqrt{T_n h}$. While this is slower than the parametric rate $\sqrt{T_n}$, it is faster than the $\sqrt{T_n h^2}$ rate obtained for the quantile differences described above. Intuitively, the intermediate $\sqrt{T_n h}$ rate of convergence arises because although $\hat{g}_n\left(b;b\right)$ is an estimated bivariate density function, in constructing the estimate $\hat{\mu}_n$ we average along the diagonal $B^*_{it} = B_{it}$.[9]

[9] While the test based on averaged pseudo-values converges faster than that based on a vector of quantiles, the improvement of the convergence rate is not proportional because the conditions on bandwidths for the partial mean case can be different from those on the pointwise estimates. However, there are still improvements after taking this into account, and this advantage of the means-based test is evident in (unreported) Monte Carlo simulations.

#### *3.1.3 Test Statistics*

In this section we focus on a test for differences in the means of the pseudo-value distributions. An analogous test can be constructed using quantiles; however, because of the faster rate of convergence of the means test and its superior performance in Monte Carlo simulations, we focus on this approach. A likelihood ratio (LR) test (e.g., Bartholomew (1959), Wolak (1989)) or the weighted power test of Andrews (1998) provide possible approaches for formulating test statistics based on the asymptotic normality results above. Because we do not have a good a prior choice of the weighting function for Andrews' weighted power test, we have chosen to use the LR test.[10]

Let $\sigma_n$ denote the asymptotic variance given in (14) for each value of $n = \underline{n}, \ldots, \bar{n}$ and define $a_n \equiv \frac{T_n h}{\sigma_n}$. Then the asymptotic covariance matrix of the vector $\left(\hat{\mu}_{\underline{n},\tau} \ \ldots \ \hat{\mu}_{\bar{n},\tau}\right)'$ is

$$\Sigma = \begin{bmatrix} \frac{1}{a_{\underline{n}}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{\underline{n}+1}} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{a_{\bar{n}}} \end{bmatrix}.$$

The restricted maximum-likelihood estimate of the (quantile-trimmed) mean pseudo-value under the null hypothesis (12) is given by

$$\bar{\mu} = \frac{\sum_{n=\underline{n}}^{\bar{n}} a_n \, \hat{\mu}_{n,\tau}}{\sum_{n=\underline{n}}^{\bar{n}} a_n}.$$

To test against the alternative (13), let $\mu^*_{\underline{n}}, \ldots, \mu^*_{\bar{n}}$ denote the solution to

$$\min_{\mu_{\underline{n}},\ldots,\mu_{\bar{n}}} \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\hat{\mu}_{n,\tau} - \mu_n\right)^2 \quad s.t. \quad \mu_{\underline{n}} \geq \mu_{\underline{n}+1} \geq \cdots \geq \mu_{\bar{n}}. \tag{15}$$

This solution can be found using the well-known "pool adjacent violators" algorithm (Ayer, et al. (1955)), using the weights $a_n$. Define the test statistic

$$\bar{\chi}^2 = \sum_{n=\underline{n}}^{\bar{n}} a_n \left(\mu^*_{n,\tau} - \bar{\mu}\right)^2.$$

The following corollary states that, under the null hypothesis, the LR statistic $\bar{\chi}^2$ is asymptotically distributed as a mixture of Chi-square random variables. The proof is given in Bartholomew (1959, Section 3).

[10] Indeed, Monte Carlo results in Andrews (1998) comparing the LR test to his more general tests for multivariate one-sided hypotheses, which are optimal in terms of a "weighted average power," suggests that the LR tests are "close to being optimal for a wide range of [average power] weighting functions" (pg. 158).

**Corollary 2** *Under the null PV hypothesis,*

$$\Pr\left(\bar{\chi}^2 \geq c\right) = \sum_{k=2}^{\bar{n}-\underline{n}+1} \Pr\left(\chi^2_{k-1} \geq c\right) w(k;\Sigma) \qquad \forall c > 0,$$

*where $\chi^2_j$ denotes a standard Chi-square distribution with $j$ degrees of freedom, and each mixing weight $w(k;\Sigma)$ is the probability that the solution to (15) has exactly $k$ distinct values when the vector $\{\hat{\mu}_{\underline{n},\tau}, \ldots, \hat{\mu}_{\bar{n},\tau}\}$ has a multivariate $N(0,\Sigma)$ distribution.*

In practice the weights $w(k;\Sigma)$ can be obtained by simulation from the $MVN(0,\hat{\Sigma})$ distribution, where $\hat{\Sigma}$ is a diagonal matrix with elements obtained from sample analogs of (14). An alternative to using equation (14), explored below, is to estimate each element of $\Sigma$ using bootstrap distributions of mean pseudo-values.

### 3.2 A Sup-Norm Test

A second testing approach is based on a Kolmogorov-Smirnov-type (KS) statistic for a $k$-sample test of equal distributions against an alternative of strict first-order stochastic dominance. Consider the sum of supremum distances between successive empirical distributions of pseudo-values:

$$\delta_T = \sum_{n=\underline{n}}^{\overline{n}-1} \sup_{v\in[\underline{v},\bar{v}]} \left\{ \hat{F}_{\hat{v},n+1}(v) - \hat{F}_{\hat{v},n}(v) \right\} \tag{16}$$

where $[\underline{v},\bar{v}]$ is a compact interval strictly bounded away from the boundaries of the support of the pseudo-value distribution for all $n$.

The KS statistic $\delta_T$ is a limiting case of a more general class of test statistics of the form

$$\bar{\delta}_T = \sum_{n=\underline{n}}^{\overline{n}-1} \sup_{v\in[\underline{v},\bar{v}]} \left\{ \frac{1}{(n+1)T_{n+1}} \sum_{t=1}^{T} \sum_{i=1}^{n+1} \mathbf{1}\left(n_t = n+1\right) \Lambda\left(\hat{v}_{it} - v\right) - \frac{1}{nT_n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbf{1}\left(n_t = n\right) \Lambda\left(\hat{v}_{it} - v\right) \right\}$$

where $\Lambda(\cdot)$ is a differentiable strictly decreasing function. If we take $\Lambda(\cdot)$ to be $1-\psi\left(\frac{\cdot}{h'}\right)$, where $\psi(\cdot)$ is a smooth distribution function with bounded support and $h'$ is a bandwidth parameter, then $\bar{\delta}_T$ provides a smooth approximation to the KS statistic in (16), with $\lim_{h'\to 0} \bar{\delta}_T = \delta_T$. We will work with this generalized statistic.

Strict monotonicity of $\Lambda(\cdot)$ and uniform consistency of each estimate of $E_n\Lambda(v_{it}-v)$ on the compact set $v \in [\underline{v},\bar{v}]$ imply that $\bar{\delta}_T \to 0$ as $T \to \infty$ under $H_0$, while $\bar{\delta}_T \to \Delta > 0$ under

$H_1$. This forms a basis for testing. In particular, define the test statistic

$$S_T = \eta_T \bar{\delta}_T$$

where $\eta_T$ is a normalizing sequence proportional to $(Th)^{1/2}$. Appendix C describes the large sample behavior of this test statistic. In particular, we show there that $S_T$ has a nondegenerate limiting distribution.

To approximate the asymptotic distribution of the test statistic, we use a subsampling approach.[11] Recall that the observables consist of the set of bids $B_t = (B_{1t}, \ldots, B_{n_t t})$ from each auction $t = 1, \ldots, T$. So we can write

$$\bar{\delta}_T = \bar{\delta}_T \left(B_1, \ldots, B_{T_{\underline{n}}}, \ldots, B_T\right).$$

Let $R_T$ denote a sequence of subsample sizes and, for each $n$, let $R_{nT}$ be a sequence proportional to $R_T$, with $R_{nT} < T_n$. Let

$$\kappa_T = \sum_{n=\underline{n}}^{\overline{n}} \binom{T_n}{R_{nT}}$$

denote the number of subsets $\left(B_1^*, \ldots B_{R_{\underline{n}T}}^*, \ldots B_{R_{\overline{n}T}}^*\right)$of $(B_1, \ldots B_T)$ consisting of all bids from $R_{nT}$ of the original $T_n$ $n$-bidder auctions, $n = \underline{n}, \ldots \overline{n}$. Let $\bar{\delta}_{T,R_T,i}$ denote the statistic $\bar{\delta}_{R_T}\left(B_1^*, \ldots B_{R_{\underline{n}T}}^*, \ldots B_{R_{\overline{n}T}}^*\right)$ obtained using the $i$th such subsample of bids. The sampling distribution $\Phi_T$ of the test statistic $S_T$ is then approximated by

$$\Phi_{T,R_T}(x) = \frac{1}{\kappa_T} \sum_{i=1}^{\kappa_T} \mathbf{1}\left(\eta_{R_T} \bar{\delta}_{T,R_T,i} \leq x\right). \tag{17}$$

The critical value for a test at level $\alpha$ is taken to be the $1-\alpha$ quantile, $\Phi_{T,R_T}^{1-\alpha}$, of $\Phi_{T,R_T}$.

**Theorem 4** *(i) Let $R_T \to \infty$ and $\frac{R_T}{T} \to 0$ as $T \to \infty$. Then under $H_0$,* $\Pr\left(S_T > \Phi_{T,R_T}^{1-\alpha}\right) \to \alpha$.
*(ii) Assume that as $T \to \infty$, $R_T \to \infty$, $\frac{R_T}{T} \to 0$, and $\liminf_T (\eta_T/\eta_{R_T}) > 1$. Then under $H_1$,* $\Pr\left(S_T > \Phi_{T,R_T}^{1-\alpha}\right) \to 1$ *as $T \to \infty$.*

The proof is omitted because the result follows from Theorem 2.6.1 of Politis, Romano, and Wolf (1999), given the discussion above and the results in Appendix C. As usual, in practice the empirical distribution in (17) is approximated using random subsampling.

[11] See Linton, Massoumi, and Whang (2002) for a recent application of subsampling to tests for stochastic dominance in a different context.

## 4 Monte Carlo Simulations

Here we summarize the results of Monte Carlo experiments performed to evaluate our testing approaches. We examine the performance of our tests on data generated by two PV models and two CV models:

**(PV1)** independent private values, $x_i \sim \mathrm{u}[0,1]$;

**(PV2)** independent private values, $\ln x_i \sim N(0,1)$;

**(CV1)** common values, i.i.d. signals $x_i \sim \mathrm{u}[0,1]$, $u_i = \frac{x_i}{2} + \frac{\sum_{j \neq i} x_j}{2(n-1)}$;[12]

**(CV2)** pure common values, $u_i = u \sim \mathrm{u}[0,1]$, conditionally independent signals $x_i$ uniform on $[0,u]$.[13]

Before reporting the results, we turn to Figure 1. Here we illustrate the empirical distributions of pseudo-values obtained by applying the first-stage nonparametric estimators using one simulated data set from each of the four models. We do this for $n = 2, \ldots, 5$, with $T_n = 200$. For the PV models, the estimated distributions are very close to each other. For the CV models these distributions clearly suggest the first-order stochastic dominance relation implied by the winner's curse. Note that in both model CV1 and model CV2, the effect of a change in $n$ on the distribution appears to be largest when $n$ is small. This is the case in many CV models and is quite intuitive: the difference between $E[U_1|X_1 = \max_{j \in \{2,\ldots,n\}} X_j = x]$ and $E[U_1|X_1 = \max_{j \in \{2,\ldots,n+1\}} X_j = x]$ typically shrinks as $n$ grows. This is important because most auction data sets contain relatively few observations for $n$ large but many observations for $n$ small—exactly where the effects of the winner's curse are most pronounced.

We first consider the LR test based on quantile-trimmed means. Tables 1 and 2 summarize the test results, using tests with nominal size 5% and 10%. The last two rows in Table 1 indicate that in the PV models there is a tendency to over-reject when sample analogs of (14) are used to construct the mixing weights in Corollary 2. For example, for tests with nominal size 10% and data generated by the PV1 model, we reject 20.5% of the time when the range of bidders is 2–4, and 39% of the time when the range of bidders is 2–5. The tests do appear to have good power properties, rejecting the CV models in 70 to 100 percent of the replications. However, the over-rejection under the null is a concern.

[12]Here $v(x,x;n) = \frac{3n-2}{4(n-1)}x$, leading to the equilibrium bid function $s_n(x) = \frac{3n-2}{4n}x$. It is easy to see that although $v(x,x;n)$ is strictly decreasing in $n$, $s_n(x)$ strictly increases in $n$.

[13]The symmetric equilibrium bid function for this model is given in Matthews (1984).

One possible reason for the over-rejections is that the asymptotic approximations of the variances of the average pseudo-values derived in Theorem 3 may be poor at the modest sample sizes we consider. We have considered an alternative of using bootstrap estimates of the elements of $\Sigma$.[14] We use a block bootstrap procedure that repeatedly selects an auction from the original sample at random and includes all bids from that auction in the bootstrap sample, thereby preserving any dependence between bids within each auction. The results, reported in Table 2, indicate that the tendency towards over-rejection is attenuated when we estimate these variances with the bootstrap. For a test with nominal size 10%, we now reject no more than 14% of the time when the range of $n$ is 2–4, and 18% of the time when the range of $n$ is 2–5. With a 5% nominal size, our rejection rates range between 4% and 12%. The power properties remain very good. These results are encouraging and suggest use of the bootstrap in practice.

Table 3 provides results for the KS tests using the smoothed step function

$$\Lambda(v_{it} - v) = \frac{exp\left((v - v_{it})/h'\right)}{1 + exp\left((v - v_{it})/h'\right)}$$

with smoothing parameter $h' = 0.01$.[15] This test appears to perform extremely well. The rejection rates for the two PV models are very close to the nominal sizes in all cases, and the rejection rates for the CV models are extremely high.

## 5 Endogenous Participation

Thus far we have assumed that variation in bidder participation across auctions is exogenous to the joint distribution of bidders' valuations and signals. Such exogenous variation could arise, for example, from shocks to bidders' costs of participation, variation in bidder populations across markets, or seller restrictions on participation—e.g., in government auctions (McAfee and McMillan (1987)) or field experiments (Engelbrecht-Wiggans, List, and Lucking-Reiley (1999)). However, in many applications participation may be endogenous. Here we explore adaptation of our testing approach to such situations, considering several different models of participation.

[14] Note that we are not bootstrapping the distribution of the test statistic given in Corollary 2, only the component $\Sigma$. Bootstrapping the distribution of the test statistic would require resampling bids under the null hypothesis on the latent values $v(x, x, n)$.

[15] We have incorporated the recentering approach suggested by Chernozhukov (2002) whereby, in each subsample, the test statistic is recentered by the original full-sample test statistic. Hence the subsample test statistic is $\mathcal{L}^s \equiv \sqrt{Rh_R}\left[\sum_{n=\underline{n}}^{\bar{n}-1} \sup_x \left(\hat{F}^s_{n+1}(x) - \hat{F}^s_n(x)\right) - \mathcal{K}\right]$, where $\mathcal{K} \equiv \sum_{\underline{n}}^{\bar{n}-1} \sup_x \left(\hat{F}_{n+1}(x) - \hat{F}_n(x)\right)$ is the full-sample statistic. The $p$-value is computed as $\frac{1}{S}\sum_{s=1}^{S} \mathbf{1}\left(\mathcal{L}^s > \sqrt{Th_T}\mathcal{K}\right)$.

### 5.1 Binding Reserve Prices

The most common model of endogenous participation is one in which the seller uses a binding reserve price $r$, so that only bidders with sufficiently favorable signals bid. We continue to let $N$ denote the number of potential bidders and will now let $A$ denote the number of *actual* bidders—those submitting bids of at least $r$. Variation in the number of potential bidders is still taken to be exogenous; i.e., Assumption 4 is maintained in this case. However, $A$ will be determined endogenously. Because we consider sealed bid auctions with private information, it is natural to assume bidders know the realization of $N$ but not that of $A$ when choosing their bids, because $A$ is determined by the realizations of the signals.[16] We still assume the researcher can observe $N$.[17] As before, we let $F_{v,n}(\cdot)$ denote the distribution of the values $v(X,X,n)$ of the $n$ potential bidders.

As shown by Milgrom and Weber (1982), given $r$ and $n$, a bidder participates if and only if his signal exceeds the "screening level"

$$x^*(r,n) = \inf\left\{x : E\left[U_i|\, X_i = x, \max_{j\neq i} X_j \leq x\right] \geq r\right\}. \tag{18}$$

That is, a bidder participates only if he would be willing to pay the reserve price for the good even when no other bidder were. In a PV auction, we may assume without loss of generality that $E[U_i|X_i = x] = x$. Because $E\left[U_i|\, X_i = x, \max_{j\neq i} X_j \leq x\right] = E[U_i|X_i = x]$ in a PV model, equation (18) gives $x^*(r,n) = r$. In a common values model, however, $E\left[U_i|\, X_i = x, \max_{j\neq i} X_j \leq x\right]$ decreases in $n$ (the proof follows that of Theorem 1), implying that $x^*(r,n)$ increases in $n$. This gives the following lemma.

**Lemma 1** *The screening level $x^*(r,n)$ is invariant to $n$ in a PV model but strictly increasing in $n$ in a CV model.*

This result implies that our baseline testing approach must be modified to account for the selection introduced by endogenous participation. For both PV and CV models, the equilibrium participation rule implies that the marginal distribution of the signals of *actual* bidders is the truncated distribution

$$F_x(x|r,n) = \frac{F_x(x) - F_x(x^*(r,n))}{1 - F_x(x^*(r,n))}.$$

[16] Schneyerov (2002) considers a different model in which bidders observe a signal of the number of actual bidders after the participation decision but before bids are made.

[17] See HPP for an example. If this is not the case, not only is testing difficult, but the more fundamental identification of bidders' values $v(x,x,n)$ generally fails. This is because bidding is based on a first-order condition for bidders who condition on the realization of $N$ when constructing their beliefs $G_n(b|b)$ regarding the most competitive opposing bid. In general, identification based on a first-order condition requires that the researcher condition on the same information available to bidders.

Hence, letting

$$v^*(r,n) = v\left(x^*\left(r,n\right), x^*\left(r,n\right), n\right)$$

the distribution of values for actual bidders is given by

$$F^A_{v,n}(v) = \frac{F_{v,n}(v) - F_{v,n}(v^*(r,n))}{1 - F_{v,n}(v^*(r,n))} \quad \forall v \geq v^*(r,n). \tag{19}$$

In a PV model, $F_x(x|r,n) = \frac{F_x(x)-F_x(r)}{1-F_x(r)}$. Because neither this distribution nor the expectation $v(x,x,n)$ varies with $n$ in a PV model, it is still the case that the distribution $F_{v,n}(\cdot)$ is invariant to $n$ in a PV model, implying that $F^A_{v,n}(\cdot)$ is too. However, the CV case does not give a clean prediction about $F^A_{v,n}(\cdot)$. Because $x^*(r,n)$ increases with $n$ under common values, changes in $n$ affect the marginal distribution of actual bidders' values in two ways: first by the fact that $v(x,x,n)$ decreases in $n$ for fixed $x$; second by the fact that as $n$ increases, only higher values of $x$ are in the sample. The first effect creates a tendency toward the FOSD relation derived in Theorem 1 for CV models, while the second effect works in the opposite direction. This leaves the effect on $F^A_{v,n}(v)$ of an exogenous change in $n$ ambiguous in a CV model. However, we can obtain unambiguous testable predictions under both the PV and CV hypotheses by exploiting the following result.

**Lemma 2** *With a binding reserve price $r$, $F_{v,n}\left(v^*\left(r,n\right)\right)$ is identified for all $n$.*

**Proof:** Let $\tilde{F}_{x,n}(\cdot)$ denote the joint distribution of signals $X_1,\ldots,X_n$ in an $n$-bidder auction. Then

$$\begin{aligned} F_{v,n}\left(v^*(r,n)\right) &= F_x(x^*(r,n)) \\ &= \tilde{F}_{x,n}\left(x^*(r,n),\infty,\ldots,\infty\right) \\ &= \sum_{k=1}^{n} \frac{k}{n} \Pr\left(A = n-k|N=n\right) \end{aligned} \tag{20}$$

where the first equality follows from nondegeneracy and the last two follow from exchangeability. □

With $F_{v,n}\left(v^*(r,n)\right)$ known for each $n$, we can then reconstruct $F_{v,n}(v)$ for all $v \geq v^*(r,n)$. In particular, from (19) we have

$$F_{v,n}(v) = \left[1 - F_{v,n}\left(v^*\left(r,n\right)\right)\right] F^A_{v,n}(v) + F_{v,n}\left(v^*\left(r,n\right)\right) \quad \forall v \geq v^*(r,n). \tag{21}$$

With this we can give the main result of this section.

**Theorem 5** *$F_{v,n}(v)$ is identified for all $v \geq v^*(r,n)$.*

**Proof:** With a binding reserve price

$$G_n(b|b) = \Pr(A = 1|B_i = b, N = n) + \sum_{j=2}^{n} \Pr(A = j, \max_{k\in\{1,2,\dots,j\}\setminus i} B_k \leq b|B_i = b, N = n).$$

Hence, the observables and the first-order condition (4) uniquely determine $v(s_n^{-1}(b), s_n^{-1}(b), n)$ for all $n$ and $b \geq s_n(x^*(r,n))$. This determines the distribution $F_{v,n}^A(\cdot)$. Lemma 2 and equation (21) then give the result. □

Testable implications of the PV and CV models for the distribution $F_{v,n}(v)$ were established in Corollary 1, and estimation is easily adapted from that for the baseline case using sample analogs of the probabilities in the identification results above.[18] However, note that we cannot compare the distributions $F_{v,n}(v)$ in their (truncated) left tails, but rather only in the regions of common support of the distributions $F_{v,n}^A(\cdot)$. In particular, because $x^*(r,n)$ is nondecreasing in $n$ we can test[19]

$$H_0 : F_{v,\underline{n}}(v) = F_{v,3}(v) = \cdots = F_{v,\bar{n}}(v) \quad \forall v \geq v^*(r,\bar{n}) \tag{22}$$

against

$$H_1 : F_{v,\underline{n}}(v) < F_{v,3}(v) < \cdots < F_{v,\bar{n}}(v) \quad \forall v \geq v^*(r,\bar{n}) \tag{23}$$

which are implied by (8) and (9), respectively.[20]

While this provides an approach for consistent testing, the fact that we must restrict the region of comparison could be a limitation in finite samples, particularly if the true model is one in which the effects of the winner's curse are most pronounced for bidders with signals in the left tail of the distribution. However, a significant difference between $v^*(r,\bar{n})$ and $v^*(r,n)$ for $n < \bar{n}$ (the reason a test of (22) vs. (23) would involve a substantially restricted support) is *itself* evidence inconsistent with the PV hypothesis but implied by the CV hypothesis. Hence, a complementary testing approach is available based on the following theorem.

**Theorem 6** *Under the PV hypothesis, $F_{v,n}(v^*(r,n))$ is identical for all $n$. Under the CV hypothesis, $F_{v,n}(v^*(r,n))$ is strictly increasing in $n$.*

[18] The definitions of $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ would require the obvious modifications to account for the fact that only $a$ bids, not $n$, are observed in each auction with $n$ potential bidders.

[19] Each $v^*(r,n)$ is just the lowest value of $v(x,x,n)$ for an actual bidder and is therefore easily estimated from the pseudo-values.

[20] We have assumed here that $r$ is fixed across auctions. This is not necessary. For example, if $\overline{r}$ is the largest reserve price used, the approach here would enable identification of $F_{v,n}(v)$ for all $v > v^*(\overline{r},\overline{n})$. Furthermore, as GPV have suggested, variation in $r$ can enable one to trace out more of the distribution $F_{v,n}(\cdot)$ by extending methods from the statistics literature on random truncation. A full development of this extension is a topic unto itself and not pursued here.

**Proof:** Because $F_{v,n}\left(v^*(r,n)\right) = F\left(x^*(r,n)\right)$, the result follows from Lemma 1. □

Consistent estimation of $F_{v,n}\left(v^*(r,n)\right)$ for each $n$ is easily accomplished with sample analogs of the probabilities on the right-hand side of (20). A multivariate one-sided hypothesis test similar to those developed above could then be applied.

### 5.2 Costly Participation

Endogenous participation also arises when it is costly for bidders to participate. In some applications preparing a bid may be time consuming. In others, learning the signal $X_i$ might require estimating costs based on detailed contract specifications, soliciting bids from subcontractors, or exploratory work. Because bidders must recover participation costs on average, for $N$ large enough it is not an equilibrium for all bidders to participate, even if all are certain to place a value on the good strictly above the reserve price (if any). We consider two standard models of costly participation from the theoretical literature.

#### *5.2.1 Bid Preparation Costs*

Samuelson (1985) studied a model in which bidders first observe their signals and the reserve price $r$, then decide whether to incur a cost $c$ of preparing a bid. Samuelson considered only the independent private values model; however, for our purposes this model of costly participation is equivalent to one in which the seller charges a participation fee $c$ (a bidder's participation decision and first-order condition are the same regardless of whether the fee is paid to the seller, to an outside party, or simply "burned"). The case of a participation fee paid to the seller has been treated by Milgrom and Weber (1982) for their general affiliated values model. We will assume that their "regular case" (pp. 1112–1113) obtains.

Given $r$ and $n$, participation is again determined by the realization of signals and a screening level

$$x^*\left(r,c,n\right) = \inf\left\{x : \int_{-\infty}^{x} \left[v(x,y,n) - r\right] dF_n(y|x)\right] \geq c\right\}.$$

Unlike the model in the preceding section, this screening level varies with $n$ even with private values, because $F_n\left(x|x\right)$ varies with $n$. However, a valid testing approach can nonetheless be developed in a manner nearly identical to that in the preceding section. In particular, the argument used to prove Lemma 2 also implies the following.

**Lemma 3** *With a reserve price $r$ and bid preparation cost $c$, $F_{v,n}\left(v\left(x^*\left(r,c,n\right), x^*\left(r,c,n\right), n\right)\right)$ is identified from observation of $A$ and $N$.*

Letting $v^*(r,n)$ now denote $v(x^*(r,c,n), x^*(r,c,n), n)$, the first-order condition (4) and equation (21) can then be used to construct consistent estimates of $F_{v,n}(v)$ for all $v \geq v^*(r,\bar{n})$, enabling testing of the hypotheses (22) vs. (23) as in the preceding section.

#### *5.2.2 Signal Acquisition Costs*

A somewhat different model is considered by Levin and Smith (1994).[21] In their model, each bidder chooses whether to incur cost $c$ in order to learn (or to process) his signal $X_i$ and submit a bid. Bidders know $N$ and observe the number of actual bidders before they bid. In the symmetric equilibrium, each potential bidder's participation decision is a binomial randomization. With no reserve price, this leads to exogenous variation in $A$. Because $A$ is observed by bidders prior to bidding in their model, our analysis for the baseline case of exogenous variation in $N$ then carries through directly, substituting $A$ for $N$.[22]

### 5.3 Unobserved Heterogeneity

The last model of endogenous participation we consider is the most challenging. Here participation is determined in part by unobserved factors that also affect the distribution of bidders' valuations. There are really two problems that arise in such an environment. First, it is clear that if auctions with large numbers of bidders tend to be those in which the good is known by bidders to be of particularly high (or low) value, tests based on an assumption that variation in participation is exogenous can give misleading results. Second, unobserved heterogeneity in first-price auctions introduces serious challenges to the nonparametric identification of bidders' valuations that underlies our approach (recall footnote 17).[23] Here we describe a structure under which both problems can be overcome, using instrumental variables.

Suppose that the number of actual bidders, $A$, at each auction can be represented as a function of two factors, $Z$ and $W$. $Z$ is an index capturing the effects of factors observable both to bidders and to the researcher. $W$ summarizes the effects factors observable only to

[21] Li (2002) has considered parametric estimation of the symmetric IPV model for first-price auctions under this entry model.

[22] If, in addition, there were a binding reserve price, only bidders who paid the signal acquisition cost *and* observed sufficiently high signals would participate. The mixed strategies determining signal acquisition, however, still result in exogenous variation in the number of "informed bidders," $I$, a subset $A$ of whom would obtain sufficiently high signals to bid. In this case testing would be possible following the approach in section 5.1, but with $I$ replacing $N$.

[23] Krasnokutskaya (2003) has recently shown that methods from the literature on measurement error can be used to enable estimation of a particular private values model in which unobserved heterogeneity enters multiplicatively (or additively) and is independent of the idiosyncratic components (themselves independently distributed) of bidders' values.

bidders. We make the following assumptions.

**Assumption 6** *$Z \in \mathbb{R}$ is independent of $(U_1, \ldots, U_{\bar{n}}, X_1, \ldots, X_{\bar{n}}, W)$.*

**Assumption 7** *$A = \phi(Z, W)$, with $\phi$ increasing in $Z$ and strictly increasing in $W$.*

Assumption 6 allows the possibility that the unobserved factor $W$ is correlated with $(U_1, \ldots, U_{\bar{n}}, X_1, \ldots, X_{\bar{n}})$, but requires that $Z$ not be. We will see in the following section that observables affecting the distribution of valuations can be controlled for directly. Hence, the availability of $Z$ satisfying Assumption 6 is a standard exclusion restriction, and we will refer to $Z$ as the instrument. Changes in $Z$ will provide exogenous variation in the level of competition that is essential to our ability to detect the winner's curse. Note that in some applications $Z$ might simply be the number of potential bidders, $N$. However, it need not be. For example, we might have

$$A = \upsilon(\zeta) + W$$

where $Z = \upsilon(\zeta)$ is a function (possibly unknown) of a vector of instrumental variables $\zeta$.

In Assumption 7, monotonicity of $\phi(\cdot)$ in $Z$ is the requirement that the instrument be positively correlated with the endogenous variable $A$. Weak monotonicity of $\phi(\cdot)$ in the unobservable $W$ would be a normalization of $W$ and could be assumed without loss. The strict monotonicity assumed here is a restriction implying that $(A, Z)$ are joint sufficient statistics for $W$; in particular

$$\tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n | A, Z, W) = \tilde{F}_n(U_1, \ldots, U_n, X_1, \ldots, X_n | A, Z).$$ [24]

This property will enable us to retain identification of bidders' values despite the presence of unobservables.[25]

Define

$$v(x, x; a, z) \equiv E\left[U_i | X_i = \max_{j \neq i} X_j = x, \phi(Z, W) = a, Z = z\right]. \tag{24}$$

Let

$$G_{a,z}(b^* | b) = \Pr(\max_{j \neq i} B_j \leq b^* | B_i = b, A = a, Z = z)$$

[24] If the relationship between $A$ and $W$ were only weakly monotone, conditioning on $(A, Z)$ would be equivalent to conditioning on $(A, Z)$ and the event $W \in \mathcal{W}$ for some set $\mathcal{W}$. In some applications this may be sufficient to enable the use of the first-order condition (4) as a useful approximation.

[25] This idea is related to that taken in a very different problem by Olley and Pakes (1996).

and denote the corresponding conditional density by $g_{a,z}(b^*|b)$. When bidders observe both $Z$ and $W$ before choosing their bids, the sufficient statistic property implies that equilibrium bidding can be characterized by the first-order condition (the analog of (4))

$$v(x_i, x_i; a, z) = b_i + \frac{G_{a,z}(b_i|b_i)}{g_{a,z}(b_i|b_i)}. \tag{25}$$

Analogous to the baseline case, this first-order condition enables consistent estimation of each $v(x_i, x_i; a, z)$, using straightforward modifications of the nonparametric estimators described above.

Now observe that

$$\begin{aligned} \Pr\left(v\left(X,X;A,Z\right) \le v | Z=z\right) &= \Pr\left(v\left(X,X;\phi(z,W),z\right) \le v\right) \qquad (26) \\ &= \Pr\left(E\left[U_1 | X_1 = \max_{j\in\{2,\ldots,\phi(z,W)\}} X_j = X, W, Z=z\right] \le v\right) \\ &= \Pr\left(E\left[U_1 | X_1 = \max_{j\in\{2,\ldots,\phi(z,W)\}} X_j = X, W\right] \le v\right) \end{aligned}$$

where the final equality follows from Assumption 6. Assumption 7 and the proof of Theorem 1 imply that the final expression above is increasing in $z$ in a CV auction but invariant to $z$ in a PV auction. Hence our testing approaches are still applicable if we exploit the exogenous variation in the instrument $Z$ rather than variation in $N$ or $A$. In particular, after estimating pseudo-values using equation (25), one can pool pseudo-values over all values of $a$ while holding $z$ fixed to compare the sample analogs of the distributions in (26) across values of $z$. We emphasize that while the *comparison* of distributions of pseudo-values forming the test is done pooling over $a$, the first-stage *estimation* of pseudo-values using (25) must be done conditioning on both $a$ and $z$.

## 6 Observable Heterogeneity

For simplicity we have thus far assumed that data were available from auctions of identical goods. In practice this is rarely the case. In our application below, as in many others, we observe auction-specific characteristics that are likely to shift the distribution of bidder valuations. The results above can be extended to incorporate observables using standard nonparametric techniques. Let $\mathbf{Y}$ be a vector of observables and define $G_{n,\mathbf{y}}(b|b) = \Pr(\max_{j\neq i} B_j \le b | B_i = b, N = n, \mathbf{Y} = \mathbf{y})$ etc. One simply substitutes $\frac{G_{n,\mathbf{y}}(b|b)}{g_{n,\mathbf{y}}(b|b)}$ for $\frac{G_n(b|b)}{g_n(b|b)}$ on the right-hand side of the first-order condition (4) and

$$v(x, x, n, \mathbf{y}) \equiv E[U_i | X_i = \max_{j\neq i} X_j = x, \mathbf{Y} = \mathbf{y}]$$

on the left-hand side. Standard smoothing techniques can be used to estimate $\frac{G_{n,\mathbf{y}}(b|b)}{g_{n,\mathbf{y}}(b|b)}$, in principle enabling one to apply our testing approaches. With many covariates, however, estimation of the pseudo-values using smoothing techniques will require large data sets.

An alternative is available if we are willing to assume

$$v(x, x, n, \mathbf{y}) = v(x, x, n) + \Gamma(\mathbf{y}) \tag{27}$$

with $\mathbf{Y}$ independent of $X_1, \ldots, X_n$. This additively separable structure is particularly useful because it is preserved by equilibrium bidding.[26]

**Lemma 4** *Suppose that* $\mathbf{Y}$ *is independent of* $\mathbf{X}$ *and (27) holds. Then the equilibrium bid function, conditional on* $\mathbf{Y} = \mathbf{y}$*, has the additively separable form* $s(x; n, \mathbf{y}) = s(x; n) + \Gamma(\mathbf{y})$.

The proof follows the standard equilibrium derivation for a first-price auction (only the boundary condition for the differential equation (1) changes) and is therefore omitted. An important implication of this result is that we can account for observable heterogeneity in a two-stage procedure that avoids the need to condition on (smooth over) $\mathbf{Y}$ when estimating distributions and densities of bids. Let

$$s_0(n) = E_X[s(X; n)]$$

and

$$\Gamma_0 = E_{\mathbf{Y}}[\Gamma(\mathbf{Y})].$$

We can then write the equilibrium bidding strategy as

$$s(x; n, \mathbf{y}) = s_0(n) + \Gamma_0 + \Gamma_1(\mathbf{y}) + s_1(x; n)$$

where the stochastic term $s_1(x; n)$ has mean zero conditional on $(n, \mathbf{y})$. Now observe that

$$\beta_{it} \equiv s_0(n_t) + \Gamma_0 + s_1(x_{it}; n_t) \tag{28}$$

is the bid that bidder $i$ *would have submitted* in equilibrium in a generic (i.e., $\Gamma_1(\mathbf{y}) = 0$) $n$-bidder auction. Hence, given estimates $\hat{\Gamma}_1(\mathbf{y})$ of each $\Gamma_1(\mathbf{y})$, we can construct estimates $\hat{\beta}_{it} = b_{it} - \hat{\Gamma}_1(\mathbf{y})$ of each $\beta_{it}$. Our tests can then be applied using these "homogenized" bids.

To implement this approach, we first regress all observed bids on the covariates $\mathbf{Y}$ and a set of dummy variables for each value of $n$. The sum of each residual and the corresponding intercept estimate provides an estimate of each $\beta_{it}$. These estimates are then treated as bids

[26] If the covariates enter multiplicatively rather than additively, an analogous approach to that proposed below can be applied.

in a sample of auctions of homogeneous goods. Note that the function $\Gamma_1(\cdot)$ is estimated using all bids in the sample rather than separately for each value of $n$. This can make it possible to incorporate a large set of covariates and can make a flexible specification of $\Gamma_1(\cdot)$ feasible. Furthermore, the asymptotic distributions of the ultimate test statistics are not affected by this procedure as long as $\hat{\Gamma}_1(\mathbf{y})$ converges at a faster rate than the pseudo-value estimates. This is guaranteed, for example, if $\Gamma_1(\cdot)$ is parametrically specified.

Adapting this approach to the models of endogenous participation discussed above is straightforward. The case requiring modification is that in which instrumental variables are used. There the intercept of the equilibrium bid functions $s(x; a, \mathbf{y}, w)$ will now vary with both $a$ and $w$ (or, equivalently, with both $a$ and $z$). Under the assumptions of section 5.3, one needs only to include in the regressions separate intercepts $s_0(a, z) + \Gamma_0$ (replacing $s_0(n) + \Gamma_0$ in (28)) for each combination of $a$ and $z$. The sum of the $(a, z)$-specific intercept and the residuals from the corresponding auctions are then the homogenized bids.

## 7 Application to U.S. Forest Service Timber Auctions

### 7.1 Data and Background

We apply our tests to auctions held by the United States Forest Service (USFS). In each sale, a contract for timber harvesting on Federal land was sold by first-price sealed bid auction. Detailed descriptions of the contracts being sold and the auctions themselves can be found in, e.g., Baldwin (1995), Baldwin, Marshall, and Richard (1997), Athey and Levin (2001), Haile (2001), or Haile and Tamer (2003). Here we discuss a few key features that are particularly relevant to our analysis.

We will separately consider two types of Forest Service auctions, for which the significance of common value elements may be different. The first type is known as a *lumpsum* sale. As the term suggests, here each bidder offers a total bid for an entire tract of standing timber. The winning bidder pays his bid regardless of the volume actually realized at the time of harvest. Bidders, therefore, may face considerable common uncertainty over the volume of timber on the tract. More significant, individual bidders often conduct a “cruise” of the tract before the auction, creating a natural source of the private information essential to the CV model. Before each sale, however, the Forest Service conducts its own cruise of the tract to provide bidders with estimates of (among other things) timber volumes by species, harvesting costs, costs of manufacturing end products from the timber, and selling prices of these end products. This creates a great deal of common knowledge information about the tract. Whether scope remains for significant private information regarding determinants of tract value common to all bidders is uncertain, although our *a priori* belief was that lumpsum sales were likely possess common value elements.

The second type of auction is known as a "scaled sale." Here, bids are made on a per unit (thousand board-feet of timber) basis. The winner is selected based on these unit prices and the *ex ante* estimates of timber volumes obtained from the Forest Service cruise. However, payments to the Forest Service are based on the winning bidder's unit prices and the *actual* volumes, measured by a third party at the time of harvest. As a result, the importance of common uncertainty regarding tract values may be reduced. In fact, bidders are less likely to send their own cruisers to assess the tract value for a scaled sale (National Resources Management Corporation (1997)). This may leave less scope for private information regarding any shared determinants of bidders' valuations and, therefore, less scope for common values. Bidders may still have private information of an idiosyncratic nature, e.g., regarding their own sales and inventories of end products, contracts for future sales, or inventories of uncut timber from private timber sales. This has led several authors (e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), Haile and Tamer (2003)) to assume a private values model for scaled sales.[27] However, this is not without controversy; Baldwin (1995) and Athey and Levin (2001) argue for a common values model even for scaled sales.[28]

The auctions in our samples took place between 1982 and 1990 in Forest Service regions 1 and 5. Region 1 covers Montana, eastern Washington, Northern Idaho, North Dakota, and northwestern South Dakota. The Region 5 data consist of sales in California. The restriction to sales after 1981 is made due to policy changes in 1981 that (among other things) reduced the significance of subcontracting as a factor affecting bidder valuations, because resale opportunities can alter bidding in ways that confound the empirical implications of the winner's curse (cf. Bikhchandani and Huang (1989), Haile (1999), and Haile (2001)). For the same reason, we restrict attention to sales with no more than 12 months between the auction and the harvest deadline.[29] For consistency, we consider only sales in which the Forest Service provided *ex ante* estimates of the tract values (based on the cruise) using the predominant method of this time period, known as the "residual value method" (cf. Baldwin, Marshall, and Richard (1997)). We exclude salvage sales, sales set aside for small

[27] Other studies assuming private values at timber auctions (USFS and others) include Cummins (1994), Elyakime, Laffont, Loisel, and Vuong (1994), Hansen (1985), Hansen (1986), Johnson (1979), Paarsch (1991), and Paarsch (1997).

[28] Other studies assuming common values models for Forest Service timber auctions include Chatterjee and Harrison (1988), Lederer (1994), and Leffler, Rucker, and Munn (1994).

[29] This is the same rule used by Haile and Tamer (2003) and the opposite of that used by Haile (2001) to focus on sales with significant resale opportunities.

firms, and sales of contracts requiring the winner to construct roads.

Table 4 describes the resulting sample sizes for auctions with each number of bidders $n = 2, 3, \ldots, 12$. There are fairly few auctions with more than four bidders, particularly in the sample of lumpsum sales. However, the unit of observation, both for estimation of the pseudo-values and for estimation of the distribution of pseudo-values, is a bid. Our data set contains 75 or more bids for auctions of up to seven bidders in both samples.

Our data include all bids[30] for each auction, as well as a large number of auction-specific observables. These include the year of the sale, the appraised value of the tract, the acreage of the tract, the length (in months) of the contract term, the volume of timber sold by the USFS in the same region over the previous six months, and USFS estimates of the volume of timber on the tract, harvesting costs, costs of manufacturing end products, selling value of the end products, and an index of the concentration of the timber volume across species (cf. Haile (2001)). All dollar values are in constant 1983 dollars per thousand board-feet of timber. Table 5 provides summary statistics.

### 7.2 Results

We first perform our tests on each sample under the assumption of exogenous participation. We consider comparisons of auctions with up to 7 bidders, although we look at ranges of 2–3, 2–4, 2–5, and 2–6 bidders as well. We use the method described in section 6 to eliminate the effects of observable heterogeneity with an initial linear regression of bids on the covariates listed above. Figures 2 and 3 show the estimated distributions of pseudo-values for each of these comparisons. The distributions compared appear to be roughly similar, although there is certainly some variation. Table 6 reports the formal test results, where we set the number of resampling draws at 5000 for both types of tests. For each specification we report the $R^2$ from the regressions of bids on auction covariates, the means of each estimated distribution of pseudo-values, and the p-value associated with each test of the private values null hypothesis.

The fit of the bid regressions are generally very good (recall that bids are already normalized by the size of the tract), leaving plausible residual variation to be attributed to bidders' private information. The formal tests provide little evidence of common values. For the scaled sales, only one of the ten test statistics suggests rejection of the PV null at the 10 percent level. For the lumpsum sales, the p-values are generally smaller; however,

[30] In practice separate prices are bid for each identified species on the tract. Following, e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003), we consider only the total bid of each bidder, which is also the statistic used to determine the auction winner. See Athey and Levin (2001) for an analysis of the distribution of bids across species.

only two of ten tests indicate rejection at a 10 percent level. Only in the sample of auctions with 2–4 bidders do we obtain results suggesting rejection of the PV null from both types of test.

One possible reason for a failure to reject the null is the presence of unobserved heterogeneity correlated with the number of bidders.[31] If tracts of higher value in unobserved dimensions also attracted more bidders, for example, there would be a tendency for the distributions compared to shift in the direction opposite that predicted by the winner's curse. There is some suggestion of this in the graphs. Hence, we also perform the tests using the model of endogenous participation with unobserved heterogeneity discussed in section 5.3.[32] As instruments, $\zeta$, we use the numbers of sawmills and logging firms in the county of each sale and its neighboring counties (cf. Haile (2001)). This approach adds a second least-squares projection used to estimate $Z = v(\zeta) = E[A|\zeta]$. We then construct a discrete instrument $W$ by splitting the sample into thirds (halves when we compare only 2- and 3-bidder auctions) based on the number of predicted bidders.

Figures 4 and 5 show the resulting empirical distributions of pseudo-values compared in each test. For the scaled sales, the distributions are generally close and exhibit no clear ordering. For the lumpsum sales the distributions also appear to be fairly similar, although most comparisons suggest the stochastic ordering predicted by a CV model. The formal test results are given in Table 7. For the scaled sales we again find only one of ten tests suggesting rejection of the PV null at a 10 percent level. Furthermore, the two samples for which p-values below 0.15 are obtained are also the two samples for which the two types of test yield substantially different results. An examination of the corresponding graphs in Figure 4 reveals that these are cases in which the empirical distributions are shifting with $n$, but in a nonmonotonic fashion. For the lumpsum sales, three of ten tests yield p-values below 0.05, and six of ten tests give p-values below 0.20. Among the KS tests, which appeared to be the best performers in the Monte Carlo experiments, the p-values are below 0.05 in two of five cases, and below 0.20 in four of five cases. However, while this provides a much stronger suggestion of common values than the tests on the scaled sale data, the results are clearly mixed.

As a specification check, we have examined the relationship between the estimated pseudo-values and the associated bids. Under the maintained assumption of equilibrium

[31] Haile (2001) provides some evidence using a different sample of USFS auctions.

[32] We continue to assume the absence of a binding reserve price. See, e.g., Mead, Schniepp, and Watson (1981), Baldwin, Marshall, and Richard (1997), Haile (2001), Haile and Tamer (2003) for arguments that Forest Service reserve prices are nonbinding, explanations for why this might be the case, and supporting evidence.

bidding in the Milgrom-Weber model, $v(s_n^{-1}(b), s_n^{-1}(b), n)$ must be strictly monotone in $b$. While testing this restriction has been suggested by GPV and LPV, we are not aware of any formal testing approach that is directly applicable. However, this does not appear to be essential in our case. The importance of a formal test is in giving the appropriate allowance for deviations from strict monotonicity that would arise from sampling error. In most cases we find no deviations from strict monotonicity whatsoever, so that no formal test could reject. In particular, we have examined the relation between bids and estimated pseudo-values in each subset of the data examined above. For the case in which no instrumental variables are used (so that the samples are divided based on the value of $n$) we find violations only in the case of lumpsum sales with $n = 6$, and here only in the right tail. When instrumental variables are used, the samples are split based on the value of both $n$ and the instrument, leading to smaller samples and greater sampling error. Nonetheless, even here there are only a few violations. For scaled sales, violations occur at no more than 2 points (i.e., 2 bids) per subsample, and the magnitudes of the violations are extremely small—on the order of 0.03 to 0.3 percent of the pseudo-values themselves. The handful of larger violations for lumpsum sales again occur only when auctions with $n = 6$ are examined. These subsamples also account for two of three cases in which the means test and KS test give qualitatively different results.

## 8 Conclusions and Extensions

We have developed nonparametric tests for common values in first-price sealed-bid auctions. The tests are nonparametric, require observation only of bids, and are consistent against all common values alternatives within Milgrom and Weber's (1982) general framework. The tests perform well in Monte Carlo simulations and can be adapted to incorporate auction-specific covariates as well as several models of endogenous participation. In addition to providing an approach for formal testing, our approach of comparing distributions of pseudo-values obtained from auctions with different numbers of bidders provides one natural way for quantifying the *magnitude* of any deviation from a private values model. For example, our estimates can be used to describe how much bidders adjust their expectations of the value of winning in response to an exogenous increase in competition (on average, or at various quantiles, etc.). This provides a natural measure of the severity of the winner's curse. Of course, in some applications one would like to address questions like how far wrong a particular policy prescription would go if a private values model were incorrectly assumed. Unfortunately, answering such a question will generally require identification of the model, and such identification generally fails without strong functional form assumptions once the

PV hypothesis is dropped (see, e.g., Li, Perrigne and Vuong (2000) and Athey and Haile (2002)). Indeed, the lack of nonparametric identification of CV models is one motivation for developing formal tests for common values.

In our application to USFS timber sales, we consistently fail to find evidence of common values at scaled sales. This is consistent with *a priori* arguments for private values at these auctions offered in the literature. We obtain mixed evidence against the PV hypothesis for lumpsum sales, where the *a priori* case for common values seemed stronger. The estimates published following the Forest Service cruise may be sufficiently precise that they leave little role for private information of a common values nature.[33] In fact, the cruises performed by the Forest Service for lumpsum sales are more thorough than those for scaled sales, a fact reflected in the name "tree measurement sale" given to such sales by the Forest Service. Hence, the intuitive argument for common values at the lumpsum sales might simply be misleading. It is, of course, a desire to avoid relying on intuition alone that led us to pursue a formal testing approach in the first place.

However, our tests are not without limitations that should be kept in mind when interpreting our empirical results and applying our tests elsewhere. While we have allowed a rich class of models in our underlying framework, we have maintained the assumption of equilibrium competitive bidding in a static game, ruling out collusion and dynamic factors that might influence bidding decisions. While a verification of the monotonicity restriction our assumptions imply provides some comfort, this specification test cannot detect all violations of these assumptions. Even if these assumptions are satisfied, our techniques for dealing with endogenous participation and auction heterogeneity have required additional assumptions and finite sample approximations. Finally, while our tests are consistent, the effects of the winner's curse in the USFS auctions may be sufficiently small that they are difficult to detect in the moderate sample sizes available. In this case a failure to reject the PV null (here or elsewhere) should be viewed as evidence that any CV elements are fairly small relative to other sources of variation in the data.

A further limitation of the approach as we have described it above is an assumption of symmetry. However, this is not essential. One can extend our methods to detect common value elements with asymmetric bidders (i.e., dropping the exchangeability assumption) as long as at least one bidder participates in auctions with different numbers of competitors. A full treatment of this topic is left for future work. However, two basic modifications of our approach are required. The first is that we must examine one bidder at a time. For example,

[33]The fact that bidders conduct their own tract cruises does not contradict this, because the information obtained from private cruises could relate primarily to firm-specific (private value) factors.

a test for the presence of common values for bidder 1 can be based on the distributions of his values $v_1(x, x, n)$, given in equilibrium by the first-order condition

$$v_1(x_{1t}, x_{1t}, n_t) = b_{1t} + \frac{\frac{\partial}{\partial b} \Pr(\max_{j \neq 1} B_{jt} \leq b^*, B_{1t} \leq b | N_t = n_t)|_{b=b^*=b_{1t}}}{\frac{\partial^2}{\partial b \partial b^*} \Pr(\max_{j \neq 1} B_{jt} \leq b^*, B_{1t} \leq b | N_t = n_t)|_{b=b^*=b_{1t}}}. \tag{29}$$

The nonparametric estimators described previously are easily adapted to this case, using the joint distribution of bids from the auctions bidder 1 participates in to estimate the right-hand side of (29). Under the PV hypothesis, $v_1(x, x, n)$ is constant across $n$. In order to obtain a stochastic ordering under the CV alternative, however, we require the second modification: in considering auctions with $n = 2, 3, \ldots$, we must construct a sequence of sets of opponents faced by bidder 1 in which the winner's curse is becoming unambiguously more severe, e.g., {bidder 2}, {bidder 2, bidder 3}, {bidder 2, bidder 3, bidder 4}, etc. This structure ensures that the severity of the winner's curse faced by bidder 1 is greater in auctions with larger numbers of participants, even though opponents are not perfect substitutes for each other. While estimation using a long sequence would require a great deal of data, doing so for a shorter sequence (where the change in the severity of the winner's curse is typically largest anyway) may be feasible in some applications.

# Appendix

## A Proof of Theorem 2

1. This is a standard result on the $\sqrt{T_n}$-convergence of sample to population quantiles (cf. van der Vaart (1999), Corollary 21.5).

2. For simplicity we introduce the notation $\mathcal{I}_t^n = \mathbf{1}(n_t = n)$, $G_n \equiv G_n(b;b)$, $g_n \equiv g_n(b;b)$, $\hat{G}_n \equiv \hat{G}_n(b;b) = \frac{1}{nT_nh}\sum_{t=1}^T \mathcal{I}_t^n \sum_{i=1}^n \mathbf{1}(b_{it}^* < b) K\left(\frac{b-b_{it}}{h}\right)$ and

$$\hat{g}_n \equiv \hat{g}_n(b;b) = \frac{1}{nT_nh^2}\sum_{t=1}^T \mathcal{I}_t^n \sum_{i=1}^n K\left(\frac{b-b_{it}}{h}\right) K\left(\frac{b-b_{it}^*}{h}\right).$$

Then we can use a first-order Taylor expansion to write

$$\hat{v}\left(s^{-1}(b), s^{-1}(b), n\right) - v\left(s^{-1}(b), s^{-1}(b), n\right) = \frac{\hat{G}_n}{\hat{g}_n} - \frac{G_n}{g_n}$$

$$= \frac{\hat{G}_n - G_n}{g_n} - \frac{G_n}{g_n^2}(\hat{g}_n - g_n) + o\left(\hat{G}_n - G_n\right) + o(\hat{g}_n - g_n)$$

$$= \frac{\hat{G}_n - E\hat{G}_n}{g_n} + \frac{E\hat{G}_n - G_n}{g_n} - \frac{G_n}{g_n^2}(\hat{g}_n - E\hat{g}_n) - \frac{G_n}{g_n^2}(E\hat{g}_n - g_n) + o\left(\hat{G}_n - G_n\right) + o(\hat{g}_n - g_n).$$

Standard bias calculations for kernel estimators yield, by Assumption 5,

$$|E\hat{G}_n - G_n| \le |\int (G_n(b; b+h\epsilon) - G_n(b;b)) K(\epsilon)\, d\epsilon| \le Ch^R \int |\epsilon|^R K(\epsilon)\, d\epsilon = o\left(\frac{1}{\sqrt{Th^2}}\right)$$

and

$$|E\hat{g}_n - g_n| \le \left|\int\int (g_n(b+h\epsilon; b+h\epsilon') - g_n(b;b)) K(\epsilon) K(\epsilon')\, d\epsilon\, d\epsilon'\right| \le C'h^R = o\left(\frac{1}{\sqrt{Th^2}}\right)$$

where $C$ and $C'$ are constants. Next it will be shown that

$$\sqrt{T_nh^2}(\hat{g}_n - E\hat{g}_n) \xrightarrow{d} N\left(0, \frac{1}{n}\left(\int\int K(e)^2 K(e')^2\, de\, de'\right) g_n(b;b)\right).$$

For this purpose it suffices to show that

$$\lim_{T_n\to\infty} Var\left(\sqrt{T_nh^2}(\hat{g}_n(b;b) - E\hat{g}_n(b;b))\right) = \frac{1}{n}\left(\int\int K(e)^2 K(e')^2\, de\, de'\right) g_n(b;b).$$

This is verified by the following calculation:

$$Var\left(\frac{1}{\sqrt{T_nh^2}\cdot n}\sum_{t=1}^T\sum_{i=1}^n \left[K\left(\frac{b_{it}-b}{h}\right) K\left(\frac{b_{it}^*-b}{h}\right)\right]\mathcal{I}_t^n\right)$$

$$= T_n\left(\frac{1}{T_nn^2h^2} Var\left(\sum_{i=1}^n \left[K\left(\frac{b_{it}-b}{h}\right) K\left(\frac{b_{it}^*-b}{h}\right)\right]\right)\right)$$

$$= \frac{1}{nh^2}\left\{Var\left[K\left(\frac{b_{it}-b}{h}\right) K\left(\frac{b_{it}^*-b}{h}\right)\right]\right.$$

$$\left. + (n-1)\, Cov\left[K\left(\frac{b_{it}-b}{h}\right) K\left(\frac{b_{it}^*-b}{h}\right), K\left(\frac{b_{jt}-b}{h}\right) K\left(\frac{b_{jt}^*-b}{h}\right)\right]_{j\ne i}.\right\}$$

It is a standard result that

$$E\left(K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)\right)=O\left(h^{2}\right)$$

and it can be verified that for $j\neq i$

$$E\left[K\left(\frac{b_{it}-b}{h}\right)K\left(\frac{b_{it}^{*}-b}{h}\right)K\left(\frac{b_{jt}-b}{h}\right)K\left(\frac{b_{jt}^{*}-b}{h}\right)\right]=O\left(h^{4}\right).$$

Therefore we can write

$$Var\left(\sqrt{T_nh^2}\left(\hat{g}_n\left(b;b\right)-E\hat{g}_n\left(b;b\right)\right)\right)=\frac{1}{n\,h^2}E\left[K\left(\frac{b_{it}-b}{h}\right)^2K\left(\frac{b_{it}^{*}-b}{h}\right)^2\right]+O\left(h^{4}\right)$$

$$=\frac{1}{n}\int\int\frac{1}{h^2}K\left(\frac{\epsilon-b}{h}\right)^2K\left(\frac{\epsilon'-b}{h}\right)^2g_n\left(\epsilon,\epsilon'\right)d\epsilon d\epsilon'+O\left(h^{4}\right)$$

$$=\frac{1}{n}\left(\int\int K\left(e\right)^2K\left(e'\right)^2de\,de'\right)g_n\left(b;b\right)+o\left(1\right)$$

where the last equality uses the substitutions $e=\left(\epsilon-b\right)/h$ and $e'=\left(\epsilon'-b\right)/h$. Finally the same type of variance calculation shows that

$$Var\left(\sqrt{T_nh^2}\left(\hat{G}_n-E\hat{G}_n\right)\right)\longrightarrow 0.$$

Hence the proof for part 2 is complete.

3. Because the sample quantiles of the bid distribution converge at rate $\sqrt{T_n}$ to the population quantile, which is faster than the convergence rate for the pseudo-values, for large $T_n$ the sampling error in the $\tau$th quantile of the bid distribution does not affect the large sample properties of the estimated quantiles of the pseudo-value distribution. Hence, for each $\tau\in\{\tau_1,\ldots,\tau_l\}$

$$\left(\hat{v}\left(s_n^{-1}\left(\hat{b}_{\tau_l,n}\right),s_n^{-1}\left(\hat{b}_{\tau_l,n}\right),n\right)-\hat{v}\left(x_\tau,x_\tau,n\right)\right)=O_p\left(\frac{1}{\sqrt{T_n}}\right)=o_p\left(\frac{1}{T_nh^2}\right).\tag{30}$$

This implies that the limiting distribution of the vector with elements

$$\sqrt{T_nh^2}\left(\hat{\xi}\left(\hat{b}_{\tau_l,n};n\right)-v\left(F_x^{-1}\left(\tau\right),F_x^{-1}\left(\tau\right),n\right)\right)\qquad\tau=\{\tau_1,\ldots,\tau_L\}$$

is the same as that of the vector with elements

$$\sqrt{Th^2}\left(\hat{\xi}\left(s_n\left(x_\tau\right);n\right)-v\left(x_\tau,x_\tau,n\right)\right)\qquad\tau=\{\tau_1,\ldots,\tau_L\}.$$

In part 2 we showed that each element of this vector is asymptotically normal with limit variance given by the corresponding diagonal element of $\Omega$. It remains to show that the off-diagonal elements are 0. For this purpose it suffices to show, using the standard result that kernel estimates at two distinct points (here, two quantiles $b_\tau\equiv s(x_\tau)$ and $b_{\tau'}\equiv s(x_{\tau'})$) are asymptotically independent, i.e., that

$$\lim_{T_n\to\infty}Cov\left(\sqrt{T_nh^2}\left(\hat{\xi}\left(b_\tau;n\right)-v\left(x_\tau,x_\tau,n\right)\right),\sqrt{T_nh^2}\left(\hat{\xi}\left(b_{\tau'};n\right)-v\left(x_{\tau'},x_{\tau'},n\right)\right)\right)=0.$$

Using the bias calculation and convergence rates derived in part 2, it suffices for this purpose to show that

$$\lim_{T_n\to\infty} Cov\left(\sqrt{T_nh^2}\left(\hat{g}_n\left(b_\tau;b_\tau\right)-Eg_n\left(b_\tau;b_\tau\right)\right),\sqrt{T_nh^2}\left(\hat{g}_n\left(b_{\tau'};b_{\tau'}\right)-Eg_n\left(b_{\tau'};b_{\tau'}\right)\right)\right)=0$$

To show this, first observe that the left-hand side can be written

$$Cov\left[\frac{1}{\sqrt{T_nh^2}n}\sum_{t=1}^{T}\sum_{i=1}^{n}K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right)\mathcal{I}^n_t,\frac{1}{\sqrt{T_nh^2}n}\sum_{t=1}^{T}\sum_{i=1}^{n}K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b^*_{it}-b_{\tau'}}{h}\right)\mathcal{I}^n_t\right]$$

$$=\frac{1}{n^2h^2}Cov\left[\sum_{i=1}^{n}K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right),\sum_{i=1}^{n}K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b^*_{it}-b_{\tau'}}{h}\right)\right].$$

Using the fact that for each $i$

$$E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right)\right]=O\left(h^2\right)$$

and for each $i\neq j$

$$E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right)K\left(\frac{b_{jt}-b_{\tau'}}{h}\right)K\left(\frac{b^*_{jt}-b_{\tau'}}{h}\right)\right]=O\left(h^4\right)$$

we can further rewrite the covariance function as

$$\frac{1}{n^2h^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right)K\left(\frac{b_{jt}-b_{\tau'}}{h}\right)K\left(\frac{b^*_{jt}-b_{\tau'}}{h}\right)\right]+O\left(h^2\right)$$

$$=\frac{1}{n^2h^2}\sum_{i=1}^{n}E\left[K\left(\frac{b_{it}-b_\tau}{h}\right)K\left(\frac{b^*_{it}-b_\tau}{h}\right)K\left(\frac{b_{it}-b_{\tau'}}{h}\right)K\left(\frac{b^*_{it}-b_{\tau'}}{h}\right)\right]+O\left(h^2\right)$$

$$=\frac{1}{n}\int\int K\left(e\right)K\left(e'\right)K\left(e+\frac{b_\tau-b_{\tau'}}{h}\right)K\left(e'+\frac{b_\tau-b_{\tau'}}{h}\right)g_n\left(b_\tau+he,b_\tau+he'\right)de\,de'+O\left(h^2\right)\to 0.$$

## B Proof of Theorem 3

First note that Assumption 5 directly implies the following uniform rates of convergence for $\hat{G}_n\left(b;b\right)$ and $\hat{g}_n\left(b;b\right)$ (see Horowitz (1998) and Guerre, Perrigne, and Vuong (2000)).

$$\sup_{b\in\mathbb{R}}\left|\tilde{G}_n\left(b;b\right)\right|\equiv\sup_{b\in\mathbb{R}}\left|\hat{G}_n\left(b;b\right)-G_n\left(b;b\right)\right|=O_p\left(\sqrt{\frac{\log T}{Th}}\right)+O\left(h^R\right)$$

$$\sup_{b\in R}\left|\tilde{g}_n\left(b;b\right)\right|\equiv\sup_{b\in R}\left|\hat{g}_n\left(b;b\right)-g_n\left(b;b\right)\right|=O_p\left(\sqrt{\frac{\log T}{Th^2}}\right)+O\left(h^R\right).$$

Since part (i) is an immediate consequence of part (ii), we proceed to prove part (ii) directly. Letting $\xi\left(b;n\right)=v\left(s^{-1}\left(b\right),s^{-1}\left(b\right),n\right)$, we can decompose the left side of part (ii) as

$$\sqrt{T_nh}\left(\hat{\mu}_{n,\tau}-E\left[\xi\left(b;n\right)\mathbf{1}\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right)\right]\right)$$

$$=\sqrt{T_nh}\left(\frac{1}{T_nn}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathcal{I}^n_t\left(b_{it}+\frac{\hat{G}_n\left(b_{it};b_{it}\right)}{\hat{g}_n\left(b_{it};b_{it}\right)}\right)\mathbf{1}\left(\hat{b}_{\tau,n}\le b_{it}\le\hat{b}_{1-\tau,n}\right)-E\left[\xi\left(b;n\right)\mathbf{1}\left(b_{\tau,n}\le b\le b_{1-\tau,n}\right]\right)\right)$$

$$=\hat{\mu}^1_{n,\tau}+\hat{\mu}^2_{n,\tau}+\hat{\mu}^3_{n,\tau}+\hat{\mu}^4_{n,\tau}$$

where we have again let $\mathcal{I}_t^n = \mathbf{1}(n_t = n)$, and

$$\hat{\mu}^1_{n,\tau} = \sqrt{T_n h}\frac{1}{T_n n}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})} - \frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\left(\mathbf{1}\left(\hat{b}_{\tau,n} \le b_{it} \le \hat{b}_{1-\tau,n}\right) - \mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n})\right)\mathcal{I}_t^n$$

$$\hat{\mu}^2_{n,\tau} = \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\left[\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})} - \frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right]\mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n})\mathcal{I}_t^n$$

$$\begin{aligned}\hat{\mu}^3_{n,\tau} &= \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(b_{it} + \frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\left(\mathbf{1}\left(\hat{b}_{\tau,n} \le b_{it} \le \hat{b}_{1-\tau,n}\right) - \mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n})\right)\mathcal{I}_t^n \\ &= \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\xi(b_{it};n)\left(\mathbf{1}\left(\hat{b}_{\tau,n} \le b_{it} \le \hat{b}_{1-\tau,n}\right) - \mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n})\right)\mathcal{I}_t^n\end{aligned}$$

$$\begin{aligned}\hat{\mu}^4_{n,\tau} &= \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\left(b_{it} + \frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n}) - E\left[\xi(b;n)\mathbf{1}(b_{\tau,n} \le b \le b_{1-\tau,n})\right]\right)\mathcal{I}_t^n \\ &= \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\xi(b_{it};n)\mathbf{1}(b_{\tau,n} \le b_{it} \le b_{1-\tau,n}) - E\left[\xi(b;n)\mathbf{1}(b_{\tau,n} \le b \le b_{1-\tau,n})\right]\right)\mathcal{I}_t^n.\end{aligned}$$

We consider the properties of each of these terms in turn. For $\hat{\mu}^4_{n,\tau}$, the law of large numbers gives

$$\hat{\mu}^4_{n,\tau} = \sqrt{h}\ O_p(1) = o_p(1).$$

The function in the summand of $\hat{\mu}^3_{n,\tau}$ satisfies stochastic equicontinuity conditions (a type I function of Andrews (1994)). Hence using the parametric convergence rates of $\hat{b}_\tau$ and $\hat{b}_{1-\tau}$,

$$\begin{aligned}\hat{\mu}^3_{n,\tau} &= \sqrt{T_n h}\left(E_b\xi(b;n)\mathbf{1}\left(\hat{b}_{\tau,n} \le b \le \hat{b}_{1-\tau,n}\right) - E_b\left[\xi(b;n)\mathbf{1}(b_{\tau,n} \le b \le b_{1-\tau,n})\right]\right) + o_p(1) \\ &= C\sqrt{T_n h}\left(O\left(\hat{b}_{\tau,n} - b_{\tau,n}\right) + O\left(\hat{b}_{1-\tau,n} - b_{1-\tau,n}\right)\right) + o_p(1) = \sqrt{T_n h}O_p\left(\frac{1}{\sqrt{T_n}}\right) + o_p(1) = o_p(1).\end{aligned}$$

Similarly, the function in the summand of $\hat{\mu}^1_{n,\tau}$ also satisfies stochastic equicontinuity conditions (product of type I and type III functions in Andrews (1994)), and hence

$$\begin{aligned}\hat{\mu}^1_{n,\tau} &= \sqrt{T_n h}E_b\left(\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b;b)}{g_n(b;b)}\right)\left(\mathbf{1}\left(\hat{b}_\tau \le b \le \hat{b}_{1-\tau}\right) - \mathbf{1}(b_{\tau,n} \le b \le b_{1-\tau,n})\right) + o_p(1) \\ &= O_p\left(\sup_{b\in[b_\tau-\tilde{\epsilon},b_{1-\tau}+\tilde{\epsilon}]}\left|\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)} - \frac{G_n(b;b)}{g_n(b;b)}\right|\right)\sqrt{T_n h}\ \left(O\left(\hat{b}_{\tau,n} - b_{\tau,n}\right) + O\left(\hat{b}_{1-\tau,n} - b_{1-\tau,n}\right)\right) + o_p(1) \\ &= o_p(1)\sqrt{T_n h}O_p\left(\frac{1}{\sqrt{T_n}}\right) + o_p(1) = o_p(1).\end{aligned}$$

Combining the above results, we have thus far shown that

$$\sqrt{T_n h}\left(\hat{\mu}_{n,\tau} - E\left[\xi(b;n)\mathbf{1}(b_{\tau,n} \le b \le b_{1-\tau,n})\right]\right) = \hat{\mu}^2_{n,\tau} + o_p(1).$$

The term $\hat{\mu}^2_{n,\tau}$ can be further decomposed using a second order Taylor expansion:

$$\hat{\mu}^2_{n,\tau} = \hat{\mu}^5_{n,\tau} + \hat{\mu}^6_{n,\tau} + \hat{\mu}^7_{n,\tau}$$

where

$$\hat{\mu}^5_{n,\tau} = \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{1}{g_n(b_{it};b_{it})}\left(\hat{G}_n(b_{it};b_{it}) - G_n(b_{it};b_{it})\right)\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\mathcal{I}^n_t$$

$$\hat{\mu}^6_{n,\tau} = -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(\hat{g}_n(b_{it};b_{it}) - g_n(b_{it};b_{it})\right)\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\mathcal{I}^n_t$$

$$\hat{\mu}^7_{n,\tau} = \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\underline{\mathrm{h}}^1_n(b_{it})\left(\hat{G}_n(b_{it};b_{it}) - G_n(b_{it};b_{it})\right)^2\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\mathcal{I}^n_t$$
$$+\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\underline{\mathrm{h}}^2_n(b_{it})\left(\hat{g}_n(b_{it};b_{it}) - g_n(b_{it};b_{it})\right)^2\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\mathcal{I}^n_t.$$

Here the functions $\underline{\mathrm{h}}^1_n(\cdot)$ and $\underline{\mathrm{h}}^2_n(\cdot)$ denote the second derivatives with respect to $G_n(\cdot)$ and $g_n(\cdot)$ evaluated at some mean values between $\hat{G}_n(\cdot)$ and $G_n(\cdot)$ and between $\hat{g}_n(\cdot)$ and $g_n(\cdot)$. We first bound $\hat{\mu}^7_{n,\tau}$ using the uniform convergence rates of $\hat{G}_n(\cdot)$ and $\hat{g}_n(\cdot)$:

$$\left|\mu^7_{n,\tau}\right| \le C\sqrt{T_n h}\left(O_p\left(\frac{\log T}{T_n h} + h^{2R}\right) + O_p\left(\frac{\log T}{T_n h^2} + h^{2R}\right)\right)$$
$$= O_p\left(\frac{\log T}{\sqrt{T_n h}} + \frac{\log T}{\sqrt{T_n h^3}} + \sqrt{T_n h^{1+4R}}\right) = o_p(1).$$

Now consider

$$\hat{\mu}^6_{n,\tau} = -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(\hat{g}_n(b_{it};b_{it}) - E\left[\hat{g}_n(b_{it};b_{it})\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\right]\right)\mathcal{I}^n_t$$
$$-\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(E\left[\hat{g}_n(b_{it};b_{it})\right] - g_n(b_{it};b_{it})\right)\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\mathcal{I}^n_t$$
$$= -\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(\hat{g}_n(b_{it};b_{it}) - E\left[\hat{g}_n(b_{it};b_{it})\mathbf{1}(b_{\tau,n}\le b_{it}\le b_{1-\tau,n})\right]\right)\mathcal{I}^n_t + o_p(1)$$
$$\equiv \hat{\mu}^8_{n,\tau} + o_p(1)$$

because by assumption the bias in the second term on the right-hand side of the first line is of order

$$\sqrt{T_n h}O\left(h^R\right) = O\left(\sqrt{T_n h^{1+2R}}\right) = o(1).$$

Next we show that

$$\hat{\mu}^8_{n,\tau} \xrightarrow{d} N\left(0, \Omega = \left[\int\left(\int K(\epsilon')K(\epsilon'-\epsilon)\,d\epsilon'\right)^2 d\epsilon\right]\left[\frac{1}{n}\int_{F_b^{-1}(\tau)}^{F_b^{-1}(1-\tau)}\frac{G_n(b;b)^2}{g_n(b;b)^3}g_n(b)^2\,db\right]\right).$$

This follows from a limit variance calculation for $U$-statistics. Letting $\mathbf{b}_t$ represent the vector of all bids at auction $t$, we can write

$$\hat{\mu}^8_{n,\tau} = \sqrt{T_n h}\frac{1}{n^2 T_n^2}\sum_{t=1}^{T}\sum_{s=1}^{T}m(\mathbf{b}_t,\mathbf{b}_s)\mathcal{I}^n_t\mathcal{I}^n_s$$

where

$$m(\mathbf{b}_t, \mathbf{b}_s) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{G_n(b_{it}; b_{it})}{g_n^2(b_{it}; b_{it})}\left[\frac{1}{h^2}K\left(\frac{b_{sj}-b_{ti}}{h}\right)K\left(\frac{b_{sj}^*-b_{ti}}{h}\right)\right.$$
$$\left. - E\frac{1}{h^2}K\left(\frac{b_{sj}-b_{ti}}{h}\right)K\left(\frac{b_{sj}^*-b_{ti}}{h}\right)\right]\mathbf{1}(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}).$$

Using lemma 8.4 of Newey and McFadden (1994), we can verify that

$$\sqrt{T_n h}\frac{E|m(\mathbf{b}_t, \mathbf{b}_t)|}{T_n} = O_p\left(\sqrt{T_n h}\frac{1}{T_n h}\right) = O_p\left(\frac{1}{\sqrt{T_n h}}\right) = o_p(1), \quad \text{and}$$
$$\sqrt{T_n h}\frac{\sqrt{Em(\mathbf{b}_t, \mathbf{b}_s)^2}}{T_n} = O_p\left(\sqrt{T_n h}\frac{1}{T_n\sqrt{h^3}}\right) = O_p\left(\frac{1}{\sqrt{T_n h^2}}\right) = o_p(1).$$

It then follows from Lemma 8.4 of Newey and McFadden (1994) that

$$\hat{\mu}_{n,\tau}^8 = \sqrt{T_n h}\frac{1}{n^2 T_n}\left[\sum_{t=1}^{T} E(m(\mathbf{b}_t, \mathbf{b}_s)|\mathbf{b}_t)\mathcal{I}_t^n + \sum_{s=1}^{T} E(m(\mathbf{b}_t, \mathbf{b}_s)|\mathbf{b}_s)\mathcal{I}_s^n\right] + o_p(1).$$

The first term is asymptotically negligible, because

$$\sqrt{T_n h}\frac{1}{n^2 T_n}\sum_{t=1}^{T} E(m(\mathbf{b}_t, \mathbf{b}_s)|\mathbf{b}_t)\mathcal{I}_t^n$$
$$= \sqrt{T_n h}\frac{1}{n T_n}\sum_{t=1}^{T}\left[g_n(b_{it}; b_{it})\mathbf{1}(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n}) - E[g_n(b_{it}; b_{it})\mathbf{1}(b_{\tau,n} \leq b_{it} \leq b_{1-\tau,n})]\mathcal{I}_t^n + O(h^R)\right]$$
$$= \sqrt{T_n h}O_p\left(\frac{1}{\sqrt{T_n}}\right) + O\left(\sqrt{T_n h^{1+2R}}\right) = o_p(1).$$

It remains only to verify by straightforward though somewhat tedious calculation that

$$Var\left(\sqrt{T_n h}\frac{1}{n^2 T_n}\sum_{s=1}^{T} E(m(\mathbf{b}_t, \mathbf{b}_s)|\mathbf{b}_s)I_s^n\right)$$
$$= hVar\left(\frac{1}{n}\sum_{j=1}^{n}\int_{b_\tau}^{b_{1-\tau}} \frac{G_n(b; b)}{g_n^2(b; b)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n(b)\,db\right)$$
$$= h\frac{1}{n}Var\left(\int_{b_\tau}^{b_{1-\tau}} \frac{G_n(b; b)}{g_n^2(b; b)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n(b)\,db\right) + o(1)$$
$$= h\frac{1}{n}E\left(\int_{b_\tau}^{b_{1-\tau}} \frac{G_n(b; b)}{g_n^2(b; b)}\frac{1}{h^2}K\left(\frac{b_{sj}-b}{h}\right)K\left(\frac{b_{sj}^*-b}{h}\right)g_n(b)\,db\right)^2 + o(1)$$
$$\longrightarrow \Omega \equiv \left[\int\left(\int K(\epsilon')K(\epsilon'-\epsilon)\,d\epsilon'\right)^2 d\epsilon\right]\left[\frac{1}{n}\int_{G_n^{-1}(\tau)}^{G_n^{-1}(1-\tau)} \frac{G_n^2(b; b)}{g_n^3(b; b)}g_n^2(b)\,db\right].$$

Finally, we note that if we apply the calculations performed for $\hat{\mu}^6_{n,\tau}$ to $\hat{\mu}^5_{n,\tau}$, we see that

$$E\left(\hat{\mu}^5_{n,\tau}\right) = o\left(1\right) \qquad \text{and } Var\left(\hat{\mu}^5_{n,\tau}\right) = o\left(1\right)$$

which then implies that $\hat{\mu}^5_{n,\tau} \xrightarrow{p} 0$. The proof is now completed by putting these terms together.□

## C Large Sample Behavior of the Generalized KS Statistic

Here we describe the asymptotic behavior of the generalized KS statistic $S_T = \eta_T \bar{\delta}_T$ under $H_0$. As we show, $S_T$ converges to a maximum functional of a zero-mean Gaussian process with variance-covariance function described below.

For notational simplicity we consider the case in which only one value of $n$ is used in calculating $\bar{\delta}_T$ (i.e., when only two distributions are compared). The general case is a straightforward extension. We have

$$\begin{aligned}\bar{\delta}_T &= \sup_{v\in[\underline{v},\bar{v}]}\left\{\frac{1}{(n+1)T_{n+1}}\sum_{t=1}^{T}\sum_{i=1}^{n+1}\mathcal{I}_t^{n+1}\Lambda\left(\hat{v}_{it}-v\right)-\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\mathcal{I}_t^{n}\Lambda\left(\hat{v}_{it}-v\right)\right\}\\ &\equiv \sup_{v\in[\underline{v},\bar{v}]} L\left(v\right).\end{aligned}$$

We can then write

$$\begin{aligned}L\left(v\right) = &\underbrace{\left\{\frac{1}{(n+1)T_{n+1}}\sum_{t=1}^{T}\sum_{i=1}^{n+1}\Lambda\left(v_{it}-v\right)I_t^{n+1}-\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\Lambda\left(v_{it}-v\right)\mathcal{I}_t^{n}\right\}}_{L_1(v)}\\ &+\underbrace{\left\{\frac{1}{(n+1)T_{n+1}}\sum_{t=1}^{T}\sum_{i=1}^{n+1}\Lambda\left(\hat{v}_{it}-v\right)I_t^{n+1}-\frac{1}{(n+1)T_{n+1}}\sum_{t=1}^{T}\sum_{i=1}^{n+1}\Lambda\left(v_{it}-v\right)I_t^{n+1}\right\}}_{L_2^{n+1}(v)}\\ &-\underbrace{\left\{\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\Lambda\left(\hat{v}_{it}-v\right)\mathcal{I}_t^{n}-\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\Lambda\left(v_{it}-v\right)\mathcal{I}_t^{n}\right\}}_{L_2^{n}(v)}\end{aligned}$$

Because $L_1\left(v\right) = O_p\left(\frac{1}{\sqrt{T}}\right)$ uniformly in $v$, we can write

$$\eta_T\bar{\delta}_T = \eta_T \sup_{v\in[\underline{v},\bar{v}]}\left(-L_2^n\left(v\right)+L_2^{n+1}\left(v\right)\right)+o_p\left(1\right).$$

We begin by studying the behavior of $L_2^n\left(v\right)$ for a fixed $v$. Using a first-order Taylor approximation, we have

$$\eta_T L_2^n\left(v\right) = \nu_n\sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\lambda\left(v_{it}-v\right)\left(\hat{v}_{it}-v_{it}\right)\mathcal{I}_t^n + o_p\left(1\right) \equiv \nu_n\tilde{L}_2^n\left(v\right)+o_p\left(1\right)$$

where $\lambda\left(\cdot\right) = \Lambda'\left(\cdot\right)$ and $\nu_n = \lim_{T\to\infty}\sqrt{T/T_n}$. $\tilde{L}_2^n\left(v\right)$ is very similar in structure to partial mean statistic examined in appendix B, and we therefore only reproduce the key steps of the analysis of this term. Observe that

$$\begin{aligned}
\tilde{L}_2^n(v) =& \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T}\sum_{i=1}^{n}\lambda(v_{it}-v)\left(\frac{\hat{G}_n(b_{it};b_{it})}{\hat{g}_n(b_{it};b_{it})}-\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})}\right)\mathcal{I}_t^n \\
=& \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\lambda(v_{it}-v)\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(\hat{g}_n(b_{it};b_{it})-g_n(b_{it};b_{it})\right)\mathcal{I}_t^n+o_p(1) \\
=& \sqrt{T_n h}\frac{1}{nT_n}\sum_{t=1}^{T_n}\sum_{i=1}^{n}\lambda(v_{it}-v)\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\left(\hat{g}_n(b_{it};b_{it})-E\hat{g}_n(b_{it};b_{it})\right)\mathcal{I}_t^n+o_p(1) \\
\equiv& Q^n(v)-EQ^n(v)+o_p(1),
\end{aligned}$$

where, after rewriting $v_{it}$ as $v(b_{it})$, $Q^n(v)$ is defined as

$$Q^n(v)=\sqrt{T_n h}\frac{1}{n^2T_n^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(v(b_{it})-v)\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\frac{1}{h^2}K\left(\frac{b_{it}-b_{js}}{h}\right)K\left(\frac{b_{it}-b_{js}^*}{h}\right)\mathcal{I}_t^n\mathcal{I}_s^n.$$

Using U-statistic projection results, we can approximate $Q^n(v)-EQ^n(v)$ by $P^n(v)-EP^n(v)+o_p(1)$, where

$$P^n(v)\equiv\sqrt{T_n h}\frac{1}{nT_n}\sum_{s=1}^{T}\mathcal{I}_s^n\sum_{j=1}^{n}\int\lambda(v(b_{it})-v)\frac{G_n(b_{it};b_{it})}{g_n(b_{it};b_{it})^2}\frac{1}{h^2}K\left(\frac{b_{it}-b_{js}}{h}\right)K\left(\frac{b_{it}-b_{js}^*}{h}\right)g_n(b_{it})\,db_{it}$$

$$\begin{aligned}
=&\frac{1}{n\sqrt{T_n}}\sum_{s=1}^{T}\mathcal{I}_s^n\sum_{j=1}^{n}\int\lambda(v(b_{js}+h\epsilon)-v)\frac{G_n(b_{js}+h\epsilon;b_{js}+h\epsilon)}{g_n(b_{js}+h\epsilon;b_{js}+h\epsilon)^2}\frac{1}{\sqrt{h}}K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)g_n(b_{js}+h\epsilon)\,d\epsilon \\
=&\frac{1}{n\sqrt{T_n}}\sum_{s=1}^{T}\mathcal{I}_s^n\sum_{j=1}^{n}\lambda(v(b_{js})-v)\frac{G_n(b_{js};b_{js})}{g_n(b_{js};b_{js})^2}g_n(b_{js})\frac{1}{\sqrt{h}}\int K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)d\epsilon+o_p(1) \\
\equiv&\bar{P}^n(v)+o_p(1).
\end{aligned}$$

It remains to calculate the limit of the covariance function of the process $\bar{P}^n(v):v\in[\underline{v},\bar{v}]$. First observe that

$$\begin{aligned}
Var\left(\bar{P}^n(v)\right)=&\frac{1}{n}Var\left(\lambda(v(b_{js})-v)\frac{G_n(b_{js};b_{js})}{g_n(b_{js};b_{js})^2}g_n(b_{js})\frac{1}{\sqrt{h}}\int K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)d\epsilon\right)(1+o(1)) \\
=&\frac{1}{n}E\left[\left(\lambda(v(b_{js})-v)\frac{G_n(b_{js};b_{js})}{g_n(b_{js};b_{js})^2}g_n(b_{js})\frac{1}{\sqrt{h}}\int K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)d\epsilon\right)^2\right](1+o(1)).
\end{aligned}$$

The expectations term above is given by

$$\begin{aligned}
&\int\left\{\left(\lambda(v(b_s)-v)\frac{G_n(b_s;b_s)}{g_n(b_s;b_s)^2}g_n(b_s)\right)^2\int\frac{1}{h}\left(\int K(\epsilon)K\left(\frac{b_s-b_s^*}{h}+\epsilon\right)d\epsilon\right)^2g_n(b_s^*|b_s)\,db_s^*\;g_n(b_s)\right\}db_s \\
=&\int\left\{\left(\lambda(v(b_s)-v)\frac{G_n(b_s;b_s)}{g_n(b_s;b_s)^2}g_n(b_s)\right)^2\int\left(\int K(\epsilon)K(\epsilon-z)\,d\epsilon\right)^2g_n(b_s+zh|b_s)\,dz\;g_n(b_s)\right\}db_s \\
=&\left[\int\left(\int K(\epsilon)K(\epsilon-z)\,d\epsilon\right)^2dz\right]\left[\int\lambda(v(b_s)-v)^2\frac{G_n(b_s;b_s)^2g_n(b_s)^2}{g_n(b_s;b_s)^3}db_s\right]+o(1).
\end{aligned}$$

Next, we calculate $Cov\left(\bar{P}^n(v), \bar{P}^n(v')\right)$ for $v \neq v'$:

$$Cov\left(\bar{P}^n(v), \bar{P}^n(v')\right) = Cov\Bigg\{\frac{1}{n}\sum_{i=1}^{n}\lambda\left(v\left(b_{is}\right)-v\right)\frac{G_n\left(b_{is};b_{is}\right)}{g_n\left(b_{is};b_{is}\right)^2}g_n\left(b_{is}\right)\frac{1}{\sqrt{h}}\int K(\epsilon)K\left(\frac{b_{is}-b_{is}^*}{h}+\epsilon\right)d\epsilon,$$

$$\frac{1}{n}\sum_{j=1}^{n}\lambda\left(v\left(b_{js}\right)-v'\right)\frac{G_n\left(b;b_{js}\right)}{g_n\left(b_{js};b_{js}\right)^2}g_n\left(b_{js}\right)\frac{1}{\sqrt{h}}\int K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)d\epsilon\Bigg\}.$$

The cross covariance terms for $i \neq j$ will be of order $h$. Therefore we can write

$$Cov\left(\bar{P}^n(v), \bar{P}^n(v')\right) =$$

$$\frac{1}{n}E\left\{\lambda\left(v\left(b_{js}\right)-v\right)\lambda\left(v\left(b_{js}\right)-v'\right)\left[\frac{G_n\left(b_{js};b_{js}\right)}{g_n\left(b_{js};b_{js}\right)^2}g_n\left(b_{js}\right)\right]^2\frac{1}{h}\left[\int K(\epsilon)K\left(\frac{b_{js}-b_{js}^*}{h}+\epsilon\right)d\epsilon\right]^2\right\}+o(1).$$

Using the same change of variables used above (and omitting subscripts for simplicity), we obtain

$$\begin{aligned}&Cov\left(\bar{P}^n(v), \bar{P}^n(v')\right)\\ =&\frac{1}{n}\int\left\{\lambda\left(v(b)-v\right)\lambda\left(v(b)-v'\right)\left[\frac{G_n(b;b)}{g_n(b;b)^2}g_n(b)\right]^2\int\left[\int K(\epsilon)K(\epsilon-z)\,d\epsilon\right]^2 g_n(b+zh|b)\,dz\right\}g_n(b)\,db+o(1)\\ =&\frac{1}{n}\int\lambda\left(v(b)-v\right)\lambda\left(v(b)-v'\right)\frac{G_n^2(b;b)\,g_n(b)^2}{g_n(b;b)^3}db\int\left[\int K(\epsilon)K(\epsilon-z)\,d\epsilon\right]^2 dz+o(1).\end{aligned}$$

Hence, $\lim_{T\to\infty} Var\left(\bar{P}^n(v)\right)$ is a special case of $\lim_{T\to\infty} Cov\left(\bar{P}^n(v), \bar{P}^n(v')\right)$ when $v = v'$.

Combining the results above, we derive that, under the PV null hypothesis,

$$\eta_T\bar{\delta}_T \xrightarrow{d} \sup_{v\in[\underline{v},\bar{v}]}\mathcal{G}(v)$$

where $\mathcal{G}(v)$ is a zero-mean Gaussian process defined on $v \in [\underline{v}, \bar{v}]$ with

$$Var\left(\mathcal{G}(v)\right) = \nu_n^2\lim_{T\to\infty} Var\left(\bar{P}_n(v)\right) + \nu_{n+1}^2\lim_{T\to\infty} Var\left(\bar{P}_{n+1}(v)\right)$$

and

$$Cov\left(\mathcal{G}(v), \mathcal{G}(v')\right) = \nu_n^2\lim_{T\to\infty} Cov\left(\bar{P}_n(v), \bar{P}_n(v')\right) + \nu_{n+1}^2\lim_{T\to\infty} Cov\left(\bar{P}_{n+1}(v), \bar{P}_{n+1}(v')\right).$$

The existence of this nondegenerate limit distribution is the essential result validating the use of resampling methods to construct the critical values of the test. In practice, resampling methods are typically the only feasible methods for approximating the asymptotic distributions of functionals of general Gaussian processes which are difficult to simulate directly.

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Table 1: Monte Carlo Results
200 replications of each experiment.

| | PV1 | | CV1 | | PV2 | | CV2 | |
|---|---|---|---|---|---|---|---|---|
| Range of $n$: | 2–4 | 2–5 | 2–4 | 2–5 | 2–4 | 3–5 | 3–5 | 3–6 |
| $T_n$ | 200 | 200 | 200 | 200 | 200 | 200 | 200 | 200 |
| share of p-values $< 10\%$ | 0.21 | 0.39 | 1.00 | 1.00 | 0.12 | 0.27 | 0.94 | 0.99 |
| share of p-values $< 5\%$ | 0.11 | 0.29 | 1.00 | 1.00 | 0.05 | 0.18 | 0.91 | 0.99 |

Table 2: Monte Carlo Results
Bootstrap Estimation of $\Sigma$
200 replications of each experiment.

| | PV1 | | CV1 | | PV2 | | CV2 | |
|---|---|---|---|---|---|---|---|---|
| Range of $n$: | 2–4 | 2–5 | 2–4 | 3–6 | 2–4 | 3–5 | 3–5 | 3–6 |
| $T_n$ | 200 | 200 | 200 | 200 | 200 | 200 | 200 | 200 |
| share of p-values $< 10\%$ | 0.14 | 0.18 | 1.00 | 1.00 | 0.13 | 0.21 | 0.80 | 0.91 |
| share of p-values $< 5\%$ | 0.10 | 0.12 | 1.00 | 1.00 | 0.04 | 0.11 | 0.70 | 0.83 |

Table 3: Monte Carlo Results
K-S Test using subsampled critical values.[a]
200 replications of each experiment.

| | PV1 | | | CV1 | | |
|---|---|---|---|---|---|---|
| Range of $n$: | 2–3 | 2–4 | 2–5 | 2–3 | 2–4 | 2–5 |
| $T^b$ | 200 | 200 | 200 | 200 | 200 | 200 |
| $R^c$ | 50 | 50 | 50 | 50 | 50 | 50 |
| $S^d$ | 151 | 151 | 151 | 151 | 151 | 151 |
| %(reject at 5%) | 0.01 | 0.01 | 0.01 | 0.59 | 0.92 | 0.80 |
| %(reject at 10%) | 0.06 | 0.07 | 0.11 | 0.88 | 0.99 | 0.99 |

| | PV2 | | | CV2 | | |
|---|---|---|---|---|---|---|
| Range of $n$: | 2–3 | 2–4 | 2–5 | 2–3 | 2–4 | 2–5 |
| $T$ | 200 | 200 | 200 | 200 | 200 | 200 |
| $R$ | 50 | 50 | 50 | 50 | 50 | 50 |
| $S$ | 151 | 151 | 151 | 151 | 151 | 151 |
| %(reject at 5%) | 0.01 | 0.01 | 0.02 | 0.44 | 0.82 | 0.91 |
| %(reject at 10%) | 0.02 | 0.04 | 0.15 | 0.86 | 1.00 | 1.00 |

[a]The bandwidth sequence is $h_T = O(T^{-\frac{1}{4}})$.
[b]Number of auctions.
[c]Number of auctions in each subsampled dataset.
[d]Number of subsamples taken.

Table 4: Data Configuration
USFS Timber Auctions

| | Scaled Sales | | Lumpsum Sales | |
|---|---|---|---|---|
| | number of auctions | number of bids | number of auctions | number of bids |
| $n = 2$ | 63 | 126 | 54 | 108 |
| $n = 3$ | 39 | 117 | 40 | 120 |
| $n = 4$ | 42 | 168 | 33 | 132 |
| $n = 5$ | 33 | 165 | 16 | 80 |
| $n = 6$ | 23 | 138 | 18 | 108 |
| $n = 7$ | 14 | 98 | 11 | 77 |
| $n = 8$ | 4 | 32 | 6 | 48 |
| $n = 9$ | 9 | 81 | 7 | 63 |
| $n = 10$ | 11 | 110 | 3 | 30 |
| $n = 11$ | 1 | 11 | 0 | 0 |
| $n = 12$ | 4 | 48 | 3 | 36 |
| TOTAL | 243 | 1094 | 191 | 802 |

Table 5: Summary Statistics
USFS Timber Auctions

| | Scaled Sales | | Lumpsum Sales | |
|---|---|---|---|---|
| | mean | std dev | mean | std dev |
| number of bidders | 4.50 | 2.47 | 4.20 | 2.30 |
| winning bid | 80.50 | 51.49 | 77.53 | 46.57 |
| appraised value | 36.12 | 32.56 | 36.10 | 29.08 |
| estimated volume | 609.89 | 640.50 | 390.04 | 555.86 |
| est. manuf cost | 141.51 | 45.79 | 153.46 | 43.08 |
| est. harvest cost | 120.57 | 29.55 | 118.36 | 24.92 |
| est. selling value | 312.04 | 75.85 | 335.74 | 96.88 |
| species concentration | 0.5267 | 0.5003 | 0.5497 | 0.4988 |
| 6-month inventory | 334161 | 120445 | 389821 | 139625 |
| contract term | 7.31 | 3.27 | 6.39 | 3.63 |
| acres | 697.78 | 2925.45 | 266.82 | 615.28 |
| region 5 dummy | 0.8519 | | 0.6806 | |

Table 6: Test Results
Without Instrumental Variables

Scaled Sales

| | | | | | |
|---|---|---|---|---|---|
| range of $n$ | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 |
| bid regression $R^2$ | .730 | .668 | .753 | .712 | .702 |
| means | 27.06 | 31.25 | 31.07 | 29.45 | 26.14 |
| | 28.25 | 33.46 | 33.51 | 32.78 | 30.14 |
| | | 41.35 | 40.32 | 37.79 | 35.34 |
| | | | 37.23 | 34.87 | 30.31 |
| | | | | 39.29 | 35.79 |
| | | | | | 49.76 |
| p-values: | | | | | |
| means test | .505 | .670 | .737 | .795 | .845 |
| K-S test | .223 | .708 | .065 | .597 | .630 |

Lumpsum Sales

| | | | | | |
|---|---|---|---|---|---|
| range of $n$ | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 |
| bid regression $R^2$ | .752 | .736 | .627 | .574 | .566 |
| means | 23.73 | 22.02 | 3.52 | 8.09 | 9.67 |
| | 23.85 | 24.60 | 5.85 | 10.62 | 12.09 |
| | | 17.97 | 0.90 | 6.48 | 8.15 |
| | | | 15.12 | 17.32 | 18.68 |
| | | | | 17.35 | 18.67 |
| | | | | | 10.71 |
| p-values: | | | | | |
| means test | .499 | .090 | .621 | .807 | .788 |
| K-S test | .314 | .114 | .197 | .705 | .039 |

Table 7: Test Results
With Instrumental Variables

Scaled Sales

| range of $n$ | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 |
|---|---|---|---|---|---|
| IV regression $R^2$ | .172 | .134 | .178 | .190 | .215 |
| bid regression $R^2$ | .740 | .671 | .758 | .722 | .724 |
| means | 24.34 | 37.13 | 34.53 | 32.92 | 34.27 |
| | 23.74 | 46.55 | 37.45 | 29.35 | 29.66 |
| | | 36.77 | 41.09 | 32.75 | 37.74 |
| p-values: | | | | | |
| means test | .442 | .343 | .668 | .488 | .627 |
| K-S test | .301 | .034 | .675 | .451 | .138 |

Lumpsum Sales

| range of $n$ | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 |
|---|---|---|---|---|---|
| IV regression $R^2$ | .387 | .258 | .291 | .292 | .321 |
| bid regression $R^2$ | .754 | .745 | .642 | .639 | .633 |
| means | 28.98 | 35.43 | 7.15 | 33.15 | 28.33 |
| | 24.11 | 24.84 | 3.48 | 29.84 | 24.94 |
| | | 24.18 | 10.29 | 28.39 | 22.82 |
| p-values: | | | | | |
| means test | .135 | .028 | .586 | .376 | .287 |
| K-S test | .172 | .021 | .452 | .103 | .043 |

Figure 1. Empirical Distributions of Pseudo-values
From One Monte Carlo Sample

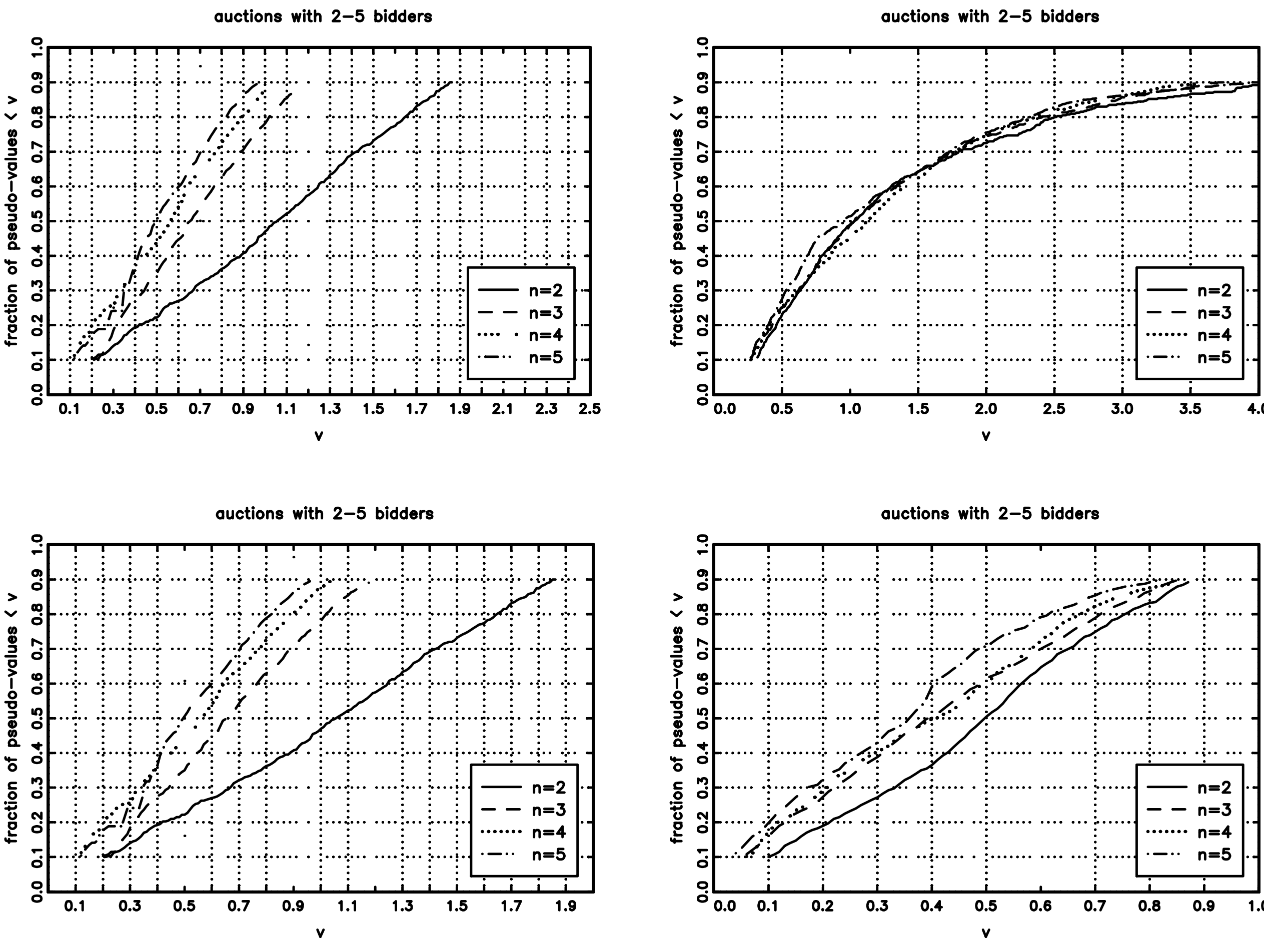

Figure 2. Empirical Distributions of Pseudo-values

Scaled Sales

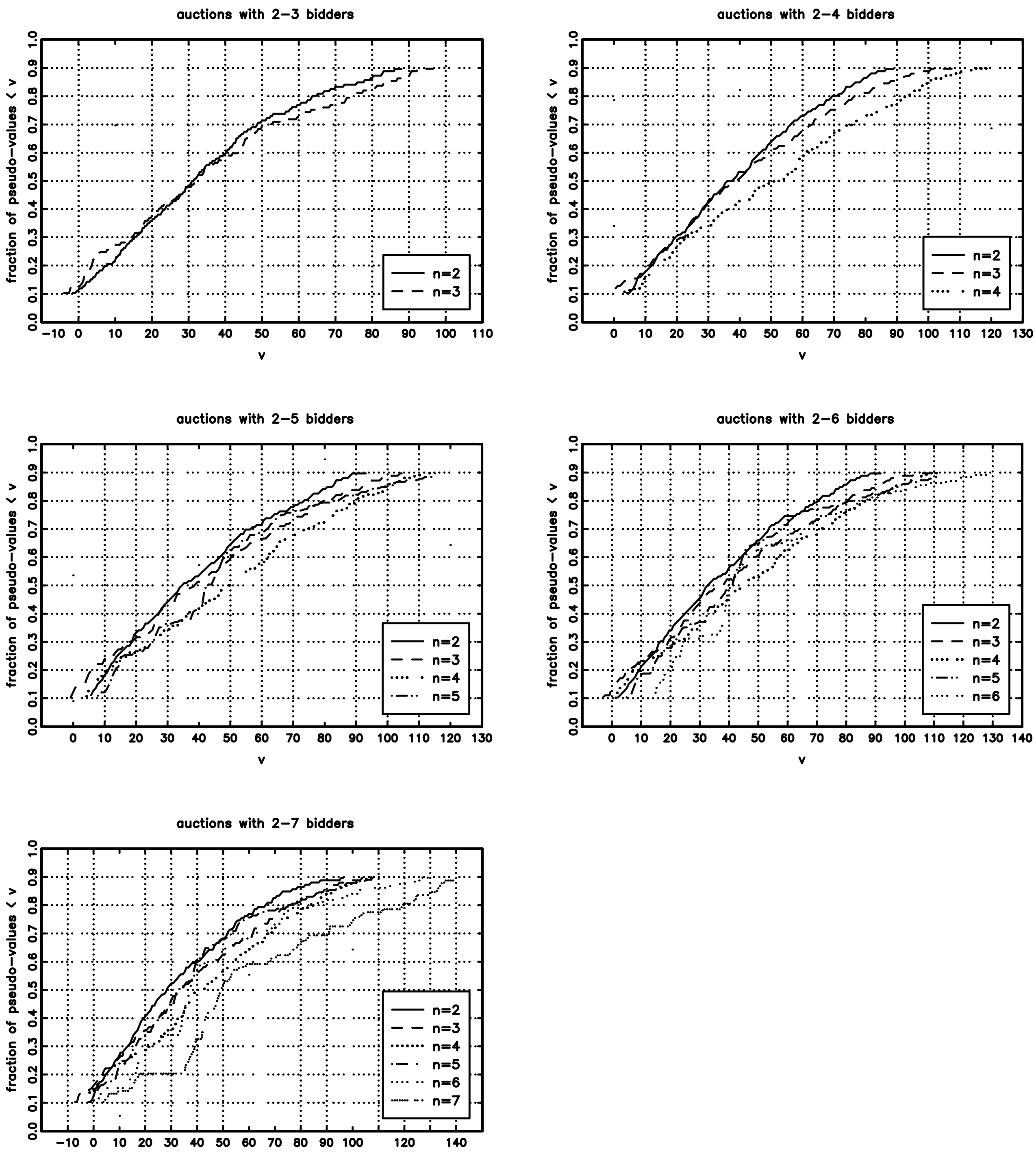

Figure 3. Empirical Distributions of Pseudo-values

Lumpsum Sales

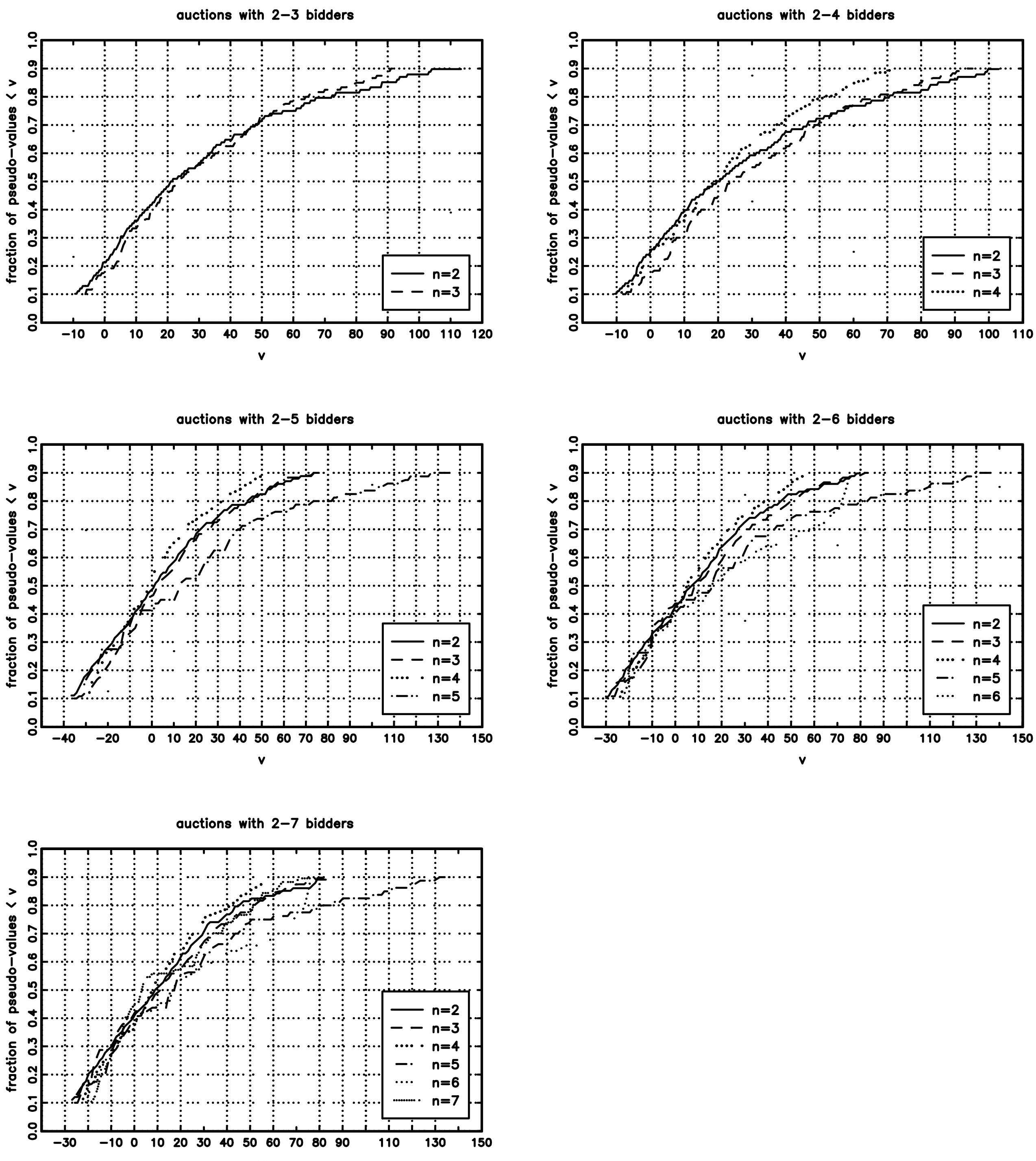

Figure 4. Empirical Distributions of Pseudo-values

Scaled Sales, Using Instrumental Variables

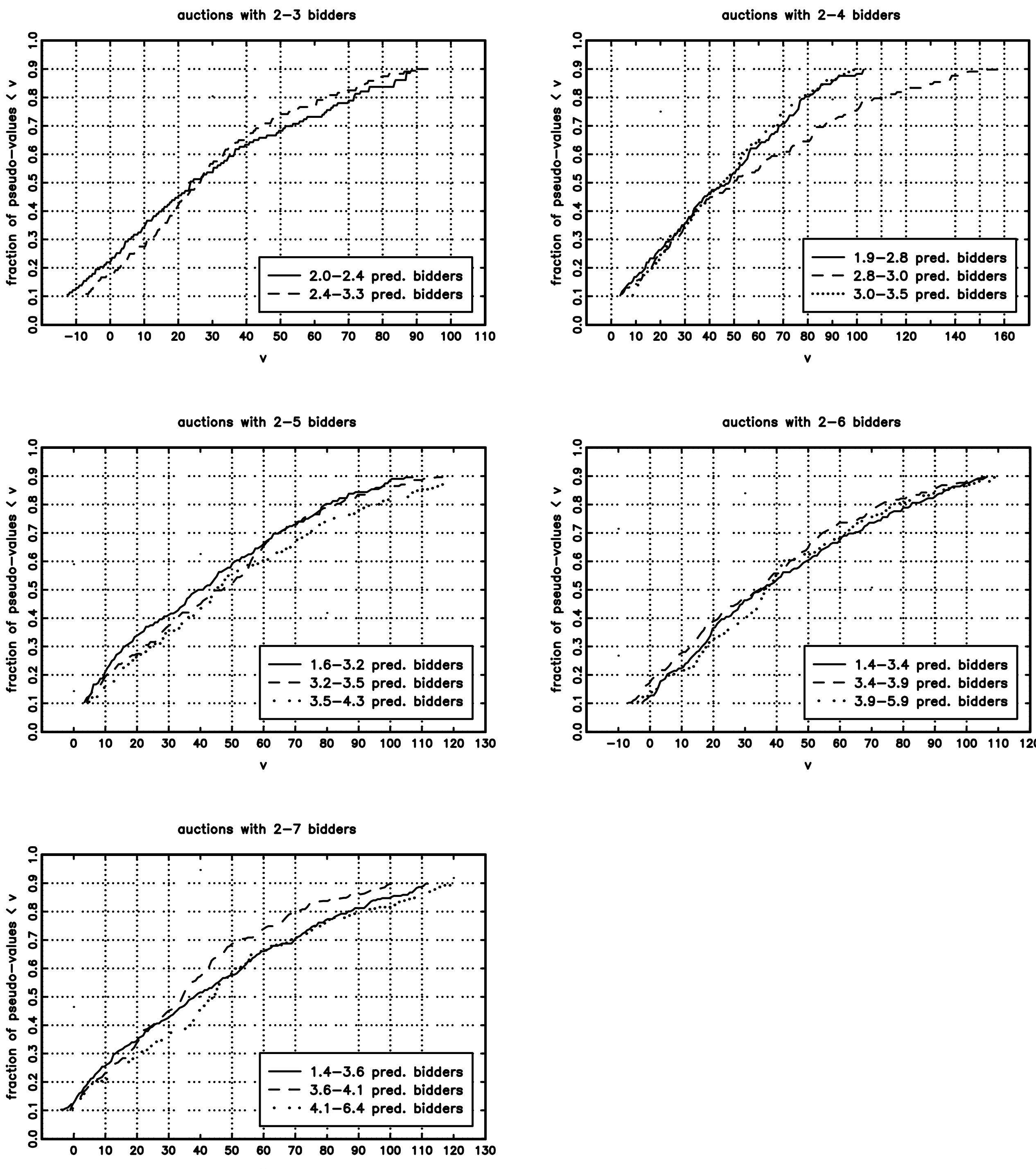

Figure 5. Empirical Distributions of Pseudo-values

Lumpsum Sales, Using Instrumental Variables