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ASSET PRICES, SUBSTITUTION EFFECTS, AND THE IMPACT OF CHANGES IN ASSET STOCKS

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Abstract

The standard result in macroeconomic models is that an increase in the stock of government debt has an ambiguous effect on aggregate demand. Models which have derived this result have assumed that all assets are gross substitutes. Some recent work within the framework of mean-variance portfolio models, however, seems to imply that the assumption that all assets are gross substitutes is sufficient to determine whether an increase in government debt is expansionary or contractionary.

This apparent inconsistency is resolved by showing that gross substitutability is sufficient to sign the impact of a change in government debt only when money is riskless.

To carry out the analysis, portfolio choice and equilibrium asset prices are characterized in a new way through the use of a distance function.

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I. Introduction

Two recent articles by Blanchard and Plantes [1977] and Roley [1979] reach conclusions which, if taken together, appear to be inconsistent with a large body of literature in macroeconomics on the impacts of changes in the stock of government debt. A standard assumption made in macroeconomic models with several financial assets is that all assets are gross substitutes. Blanchard and Plantes show that, in an intertemporal capital asset pricing model, a necessary condition for all assets to be gross substitutes is that the covariance between the future prices of any two assets be positive. Roley presents a model in which he demonstrates that the sign of the change in the price of asset i when the quantity of asset j is changed is determined by the sign of the covariance between the future prices of assets i and j. Hence, taking these two results together, the assumption that assets are gross substitutes is, in Roley's model, sufficient to determine the direction in which the price of asset i changes when the quantity of asset j is varied.

On the other hand, a standard result of macroeconomic models which assume assets are gross substitutes is that even neglecting feed-back effects from the real sector we cannot, in general, determine which way the price of asset i will move if the quantity of asset j is changed (Tobin [1963], Park [1972], Cohen and McMenamin [1978]). In particular, the impact of a change in the stock of government debt on equity prices or long-term bond prices is ambiguous. This is in apparent conflict with the results of Blanchard and Plantes and Roley.

The purpose of this paper is to reconcile these results. To do so, in Section II the investor's portfolio choice problem is formulated in terms of the distance function. This provides a convenient framework for analyzing changes in asset stocks in terms of substitution effects. It is shown that restricting only the signs of the substitution effects among assets is not sufficient to determine the sign of $\partial p_i / \partial q_j$, the change in asset i's price when the stock of asset j is changed. Section III makes use of the mean-variance approach of Roley to derive the explicit form of the distance function and to show that, in general, assuming assets are substitutes or complements, in the sense defined in this paper, does not impose a restriction on the covariances of their future prices nor does signing the covariances allow us to sign dp_i/dq_j. In Section IV, Roley's results are shown to be an implication of his assumption that money is a riskless asset. The intuitive reasoning behind the failure of Roley's conclusions to generalize to the case in which money also involves risk due to uncertainty about future inflation is briefly discussed. Section V summarizes the paper.

II. Asset Stock Changes and Portfolio Equilibrium

Suppose the problem faced by the individual investor is to choose a portfolio allocation in order to maximize a continuously differentiable concave function of the mean and variance of next period's wealth, $U(E(W),\sigma^2(W))$. If q_i denotes the number of units of asset i, $i = 1, \ldots, n$, in the individual's portfolio, we have $E(W) = \sum_i \mu_i q_i = \mu' q$ and $\sigma^2(W) = q'\Sigma q$ where

$\mu' = (\mu_1,\ldots,\mu_n)$ = vector of expected next period asset prices

$q' = (q_1,\ldots,q_n)$

$\Sigma = [\sigma_{ij}]$ = variance covariance matrix of next period asset prices.

The maximization of $U(\cdot)$ is carried out subject to the wealth constraint

$$p'q^o = W^o = p'q$$

where $p' = (1,p_2,\ldots,p_n)$ is the vector of current asset prices normalized on the price of asset 1, and q^o is the vector of initial asset holdings.

Corresponding to this portfolio allocation problem we can define the distance function (Deaton [1979a], [1979b], Walsh [1980])

$$d(u,q) = \max_\lambda \{\lambda: U(\mu'q/\lambda,\ (q'/\lambda)\Sigma(q/\lambda)) \geq u\}$$

Given a portfolio q, $d(u,q)$ tells us by how much we must proportionately blow up or shrink q in order to leave the individual with utility equal to u. The function $d(u,q)$ is continuous, first and second differentiable, decreasing in u, and increasing in q in the neighborhood of the optimal portfolio. Also, d is homogeneous of degree one in q. Since we must have $d(u,q) = 1$ when q is the portfolio actually held by the individual, we can use the distance function to implicitly characterize the dependence of utility on asset holdings. This allows us to express the optimal portfolio allocation as the solution to the following maximization problem:

(1) $$\max_{u,q} \quad u + \lambda_1[1-d(u,q)] + \lambda_2[p'(q^o - q)]$$

First order conditions for (1) are given by

(2) $$1 - \lambda_1 d_u = 0$$

(3) $$-\lambda_1 d_i - \lambda_2 p_i = 0 \qquad i=1,\ldots,n$$

(4) $$p'(q^o - q) = 0$$

where $d_u = \partial d/\partial u$ and $d_i = \partial d/\partial q_i$. Using (2) and (3) we have

(5) $$d_i = -\frac{\lambda_2}{\lambda_1} p_i = -\lambda_2 p_i d_u \ .$$

Multiplying both sides by q_i and summing over i yields

(6) $$1 = \Sigma q_i d_i = -\lambda_2 d_u \Sigma p_i q_i = -\lambda_2 d_u p'q^o$$

where we have used (4) and the homogeneity of the distance function. Substituting (6) into (5):

(7) $$d_i = p_i/p'q^o \ .$$

Equation (7) implies that d_i , evaluated at $d(u,q) = 1$, is equal to the price of asset i relative to initial wealth. This means that given a vector of asset holdings $\bar{q}$, we can use (7) to determine the vector of asset prices relative to initial wealth which would induce theindividual to hold the portfolio $\bar{q}$.

Since we have chosen the normalization $p_1 = 1$, the n equations in (7), of which $n-1$ are independent,[1] determine

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[1]Multiplying both sides of (7) by q_i, summing over i , and using the budget constraint and the homogeneity of $d(u,q)$ yields $\Sigma d_i q_i = \Sigma p_i q_i/p'q^o = 1$, or

$$d_n = (1 - \sum_{i=1}^{n-1} d_i q_i)/q_n \ .$$

the n-1 asset prices $p_2,\ldots,p_n$. We can write these n-1 equations as

(8) $d_i/d_1 = p_i \quad i=2,\ldots,n$

by using the fact that $d_1 = 1/p'q^{\circ}$.

We will now assume that we can aggregate equation (8) over all individuals to obtain a similar expression that applies at the market level. Given the stock of assets in the economy then, we can use (8) to determine a vector of asset prices which would induce individuals to hold that stock of assets.

Suppose now that there is a small change in the quantity of the kth asset. This will throw portfolios out of equilibrium and asset prices will have to adjust to re-establish portfolio equilibrium. The required change in p_i can be found by differentiating equation (8):

$$(9) \quad dp_i/dq_k = \frac{d_i}{d_1}\left(\frac{d_{ik}}{d_i} - \frac{d_{1k}}{d_1}\right) + \frac{d_i}{d_1}\left(\frac{d_{iu}}{d_i} - \frac{d_{1u}}{d_1}\right)u_k$$

where $d_{ik} = \partial^2 d/\partial q_i \partial q_k$ and $u_k = \partial u/\partial q_k$.

In order to evaluate equation (9), note that d_{ik} is equal to the change in the marginal valuation of asset i in response to a change in the quantity of asset k , holding utility constant. It thus corresponds to a substitution effect, defined however in terms of quantity changes rather than price changes. If an extra unit of asset k reduces the value of an extra unit of asset i, $d_{ik} < 0$ and i and k are called q-substitutes. If $d_{ik} > 0$, i and k are q-complements (Hicks [1956], Deaton [1979a]).

The second term in (9) involves the change in the marginal valuation of an asset as the level of utility changes. This can be interpreted as a wealth effect arising from the change in q_k. If preferences are homothetic, we can write $d_i(u,q) = g(u)f_i(q)$ in which case $(d_{iu}/d_i) - (d_{1u}/d_1) = 0$ and the wealth effect disappears. Consequently, we can in this case rewrite (9) as

$$(9') \quad dp_i/dq_k = p_i \left(\frac{d_{ik}}{d_i} - \frac{d_{1k}}{d_1}\right) .$$

From equation (9') it immediately follows that even when wealth effects are ignored, assuming all assets are q-substitutes (q-complements), that is, assuming that all d_{ij} are negative (positive), is not sufficient to determine the sign of the change in the ith asset price in response to a change in the quantity of the kth asset. We need to know the substitutability of asset i for asset k relative to the substitutability of asset 1 for asset k in order to determine the sign of dp_i/dq_k.

For example, suppose p_i is the price of equities, q_k the quantity of government debt, and asset 1 is money. Then an increase in government debt is expansionary in the sense of increasing the price of equity if and only if government debt and money are closer q-substitutes than are government debt and equities. This result is usually expressed in terms of substitution effects defined with reference to price, not quantity, changes (for example, see Tobin [1963], Cohen and McMenamin [1978], Friedman [1978]).

In the next section we will make use of our assumption that utility depends only upon the mean and variance of next period's wealth in order to derive an explicit expression for d_i . Blanchard and Plantes' [1977] result that positive covariances between asset returns are a necessary condition for all assets to be gross p-substitutes is shown not to carry over to the q-substitutes case. Assuming assets are q-substitutes or q-complements places no restrictions on the covariances of future asset prices. Equation (9) for the mean-variance model will be derived to show that, contrary to Roley's results, the sign of dp_i/dq_j cannot be determined solely by the sign of σ_{ij}

III. Asset Stock Changes and Asset Price Covariances

The value of the distance function for given u and q is implicitly defined by the solution to

$$(10) \quad U(\ \mu'(q/d)\ ,\ (q/d)'\ \Sigma(q/d)) = u.$$

By differentiating this expression with respect to d and q_i for a fixed value of u we find that

$$(11) \quad d_i = \frac{\mu_i d^2 - (1/\rho)\ \beta_i d}{\mu' qd - (1/\rho) q'\Sigma q} = \frac{\mu_i - (1/\rho)\beta_i}{\mu' q - (1/\rho) q'\Sigma q} = \frac{p_i}{w^o}$$

evaluated at $d = 1$ where $\rho = -2u_2/u_1 > 0$ and $\beta_i = \sum_j \sigma_{ij} q_i$.

Differentiating d_i in (11) with respect to q_k we find that[2]

$$(12)\quad d_{ik} + d_{iu}u_k = \frac{-(1/\rho)\sigma_{ik}d + d_k\mu_i d + (1/\rho)\beta_k d_i - \mu' q d_i d_k}{\mu' q d - (1/\rho)q'\Sigma q} + \frac{2\mu_i d - (1/\rho)\beta_i - d_i\mu' q}{\mu' q d - (1/\rho)q'\Sigma q}\, d_u u_k .$$

At d = 1 then

$$(13)\quad d_{ik} = \frac{-(1/\rho)\sigma_{ik} + d_k\mu_i + (1/\rho)\beta_k d_i - \mu' q d_i d_k}{\mu' q - (1/\rho)q'\Sigma q}$$

and

$$(14)\quad d_{iu} = \frac{2\mu_i - (1/\rho)\beta_i - d_i\mu' q}{\mu' q - (1/\rho)q'\Sigma q}\, d_u = \frac{\mu_i - (1/\rho)d_i q'\Sigma q}{\mu' q - (1/\rho)q'\Sigma q}\, d_u .$$

From equation (13) it is apparent that an assumption about the sign of d_{ik} imposes no restriction on the sign of σ_{ik}. In Walsh [1980] it was argued that our standard stories about the transmission process of monetary policy are implicitly based upon the assumption that assets are q-complements. For example, an increase in the supply of money leaves individuals with "too much" money in their portfolios. This sets off a substitution process of portfolio allocation as individuals attempt to increase their

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[2] We are maintaining Roley's assumption that ρ is constant. If superscript j denotes the j^{th} individual, (11) can be viewed as an aggregate relationship with $\rho = \Sigma\rho^j$.

holdings of non-money assets. In this view, assets are q-complements and d_{ik} is positive. From (13) we can see that this has no implications for the sign of σ_{ik} .

Consider now the effect on p_i of a small change in q_k . Using equations (9), (13), and (14), we find, after some manipulation,

$$(15) \quad \frac{dp_i}{dq_k} = \frac{-(1/\rho)(\sigma_{ik} - p_i\sigma_{1k})}{\mu_1 - (1/\rho)\beta_1} + \frac{p_i d_k}{\mu' q - (1/\rho)q'\Sigma q}\left(\frac{\mu_i}{d_i} - \frac{\mu_1}{d_1}\right)$$

$$+ \frac{p_i}{\mu' q - (1/\rho)q'\Sigma q}\left(\frac{\mu_i}{d_i} - \frac{\mu_1}{d_1}\right) d_u u_k$$

$$= \frac{-(1/\rho)(\sigma_{ik} - p_i\sigma_{1k})}{\mu_1 - (1/\rho)\beta_1}$$

since $d_k = -d_u u_k$.[3] The sign of dp_i/dq_k depends upon the sign of $\sigma_{ik} - p_i\sigma_{1k}$. Knowledge of the sign of σ_{ik} is not sufficient to determine the sign of dp_i/dq_k .

We can apply equation (15) to the question of whether or not an increase in government debt is expansionary ($\sigma_{ik} - p_i\sigma_{1k} < 0$). Roley presents some empirical evidence that $\sigma_{ik} > 0$ for bonds and equities; the expected future price of money is approximately equal to one minus the expected rate of inflation so we might expect $\sigma_{1k} > 0$ where asset k is government debt. A priori, the sign of $\sigma_{ik} - p_i\sigma_{1k}$ cannot be determined. Loosely speaking, government debt will be expansionary if bond prices are more closely correlated with the inflation rate than they are with equity prices.

In the next section we will show how Roley's conclusion that sign dp_i/dq_k = sign $- \sigma_{ik}$ is a result of his assumption that money is a riskless asset.

[3] $d(u,q) = 1$ implies $d_k + d_u u_k = 0$.

IV. The Role of Money as a Riskless Asset

As is well known (Cass and Stiglitz [1970]), mean variance portfolio models with a riskless asset have the property that we can separate the portfolio problem into two parts. In the first, investors determine the optimal proportions in which to hold the risky assets. In the second, investors decide how much of their wealth to invest in risky assets. Market equilibrium requires that the proportions in which the risky assets are held as determined in the first part of the portfolio problem equal the proportions in which the risky assets appear in the market. Current asset prices adjust in order to insure that this equilibrium requirement is satisfied. Relative prices of the risky assets are thus determined independently of the riskless asset. When money is the riskless asset then, a change in the supply of money has no effect on the relative prices of the remaining assets:

$$(16) \quad \frac{\partial(p_i/p_j)}{\partial q_1} = \frac{p_i}{p_j}\left(\frac{\sigma_{11}-\sigma_{i1}}{\mu_i-(1/\rho)\beta_i} - \frac{\sigma_{11}-\sigma_{i1}}{\mu_k-(1/\rho)\beta_k}\right)(1/\rho) = 0$$

since $\sigma_{1k} = 0$, $k = 1, \ldots, n$ if money is riskless.

Equation (16) means that the marginal rate of substitution of asset i for asset j $(i, j = 2,\ldots,n)$ is independent of the quantity of money along an indifference curve. This implies that we can write the distance function in the form $d(u,q) = b(u,q_1, \phi(\bar{q},u))$ where $\bar{q} = (q_2,\ldots,q_n)$ and the function b is homogeneous of degree 1 in q_1 and ϕ while ϕ is homogeneous of degree one in $\bar{q}$. Differentiating the distance function with respect to the asset stocks we have

(17 a) $d_1 = b_1(u, q_1, \phi)$

(17 b) $d_i = b_2 \phi_i(\bar{q}, u)$, $i = 2,\ldots,n,$

where $b_1 = \partial b/\partial q_1$ and $b_2 = \partial b/\partial \phi$. For risky assets, the marginal rate of substitution between assets i and j is given by $\phi_i(\bar{q},u)/\phi_j(\bar{q},u)$ independent of q_1 . The marginal rate of substitution between money and any risky asset is given by $b_1/b_2\phi_i$.

To find the equilibrium prices of the risky assets, we can divide equation (17b) by (17a) to obtain

(18) $p_i = d_i/d_1 = b_2\phi_i/b_1, \quad i=2,\ldots,n$.

The change in p_i that results from a change in the quantity of asset k is then equal to

$$\frac{\partial p_i}{\partial q_k} = (b_2/b_1)\phi_{ik} + (b_2/b_1)[(b_{22}/b_2) - (b_{12}/b_1)]\phi_i\phi_k$$

(19) $= (b_2/b_1)\phi_{ik}$

where the bracketed expression is zero by the homogeneity properties of b .

In the mean-variance model with money treated as a riskless asset, equation (19) is equivalent to equation (15) with $\sigma_{1k} = \beta_1 = 0$:

(20) $\partial p_i/\partial q_k = (b_2/b_1)\phi_{ik} = -(1/\rho)\sigma_{ik}/\mu_1$.

Equation (20) is identical to the expression Roley [1979] derives with the exception that he assumes $\mu_1 = 1$. It follows immediately that the sign of $\partial p_i / \partial q_k$ is uniquely determined by the sign of σ_{ik}.

When there is risk associated with the holding of money, we can no longer treat money asymmetrically in analyzing the portfolio allocation problem. Money does retain one unique feature though; we have normalized asset prices so that $p_1 = 1$. It is this normalization rule, along with the riskiness of money, which destroys the simple form of equation (20). Suppose for example, that a rise in q_k requires, for equilibrium to be re-established, that all asset prices rise relative to p_k. Since p_1 is fixed, p_k must fall if p_1/p_k is to rise. Hence, knowing that p_i/p_k must rise does not imply that p_i itself will rise. What happens to $p_i = p_i/p_1$ will depend, in general, upon the correlation of asset k's future price with asset i's future price relative to the correlation of asset k's future price with the future price of money. This is exactly what equation (15) shows. In the special case considered by Roley, money is riskless and the response of asset prices to a change in q_k takes a very special form. For $i = 2,\ldots,n$, $\partial \log(p_i/p_k)/\partial q_k = \alpha_{ik} - \gamma_k$ where α_{ik} depends upon σ_{ik} and γ_k is the same for all i. On the other hand, $\partial \log(p_k/p_1)/\partial q_k = \gamma_k$. Hence,

$$\partial \log p_i / \partial q_k = \partial \log(p_i/p_k)/\partial q_k + \partial \log(p_k/p_1)/\partial q_k$$

$$(23) \quad = \alpha_{ik} - \gamma_k + \gamma_k = \alpha_{ik}$$

and the sign of α_{ik} is determined once σ_{ik} is known.

Finally, Blanchard and Plantes show that a necessary condition for all assets to be gross substitutes is that $\sigma_{ik} > 0$ for all i, k = 2,...,n. (They also assume money is riskless.) They prove this assertion by considering the relationship between partial correlation coefficients, but their result follows directly from the fact that in their model the matrix of substitution effects among the risky assets is proportional to $-V^{-1}$ where V is the n-1 x n-1 submatrix of Σ (the variance covariance matrix of q) obtained by deleting the first row and column since money is assumed riskless. If all assets are gross substitutes, the diagonal elements of $-V^{-1}$ are negative while the off-diagonal elements are all positive. In addition, the sum of each row is negative. By an application of Theorem 4 in McKenzie [1959], all the elements of $(V^{-1})^{-1} = V$ are positive, and, hence, a necessary condition for all assets to be gross substitutes is that $\sigma_{ik} \geqq 0$ for all assets i and k.

V. Summary

The analysis in this paper has been carried out largely in terms of substitution effects defined with reference to quantity changes rather than price changes. For studying the impacts of a change in the quantity of an asset this seems a natural way to define substitutes and complements. It has been shown that Blanchard and Plantes' result does not carry over to the case in which assets are q-complements, the assumption implicitly made in most discussions of the transmission process of monetary policy.

The assumption of q-complementarity does not impose restrictions on the sign of σ_{ik}.

It was shown that Roley's conclusion that the sign of σ_{ik} determined the sign of dp_i/dq_k holds only when money is treated as a riskless asset. In general, the sign of dp_i/dq_k depends upon the relative complementarity of assets i, j and money. This result, unlike Roley's is consistent with standard macroeconomic analyses of the impact of changing the size of the government's debt.

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