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INTER-EQUATION CONSTRAINTS AND THE SPECIFICATION OF DYNAMIC STRUCTURE

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Abstract

This note considers the effect of a class of linear inter-equation constraints on the specification of the lag structure in econometric models. In particular, attention is focused on the linear summing, or "adding up", constraints which arise between equations in factor shares analysis. The consequences of such constraints on the specification of lag structures for models with dynamic adjustments and autoregressive or moving-average disturbances are presented in the form of linear restrictions which result in singular coefficient matrices. Thus, the structural (lag) specification of one equation depends on the structure of all other equations in the model.

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1.0 INTRODUCTION

Estimating simultaneous systems of equations for demand functions or cost shares has led to situations where constraints arise not only among parameters of each equation, but also among parameters of different equations. Recent examples of such efforts are Berndt and Wood [1975], and Berndt and Savi [1975]. Their applications, requiring inter-equation restrictions, come under the heading of singular systems. Singular systems do not, however, exhaust the possibilities leading to inter-equation constraints, Nadiri and Rosen [1969] being a prime example of a more general (nonlinear) type of system constraint.

When dynamic effects are introduced, such as those resulting from partial adjustment processes or adaptive expectations, the complexity of the restrictions on the parameters becomes greatly exaserbated. Yet there is a definite need for an explicit derivation of these restrictions - at least within the framework of singular systems resulting from a linear summing constraint. These concerns form the focal point for the ensuing presentation, i.e., the development of the restrictions for estimating dynamic adjustment phenomena under linear summing constraints.

First, a review of structural specification in static singular systems is given. The results are then extended to dynamic systems, where the dynamics arise from inclusion of distributed lags on the exogenous variables. Finally, these results are employed to derive the constraints for rational distributed lag systems.

2.0 THE BASIC STATIC MODEL

The origin of the ensuing analysis can be found in the static model employed by Berndt and Wood [1975]. With a slight change in notation, their model can be written as

(1) $$y_t = B_o x_t + e_t$$

where,

y_t = Gx1 vector of endogenous variables
x_t = Mx1 vector of exogenous variables
B_o = GxM matrix of coefficients
e_t = Gx1 vector of sequentially independent, jointly normal, random disturbances.

The x_t vector is further structured such that $x_{1t} \equiv 1$, i.e., the first column of B_o consists of constant, or intercept, terms. If each of the equations of (1) are interpreted as factor shares, then

(2) $$\ell' y_t = 1$$

for all t, where ℓ is a Gx1 vector of ones. This summing constraint acts across all the equations and therefore ties the coefficients of one equation to all others. The end result of (2) is a set of constraints on both B_o and the variance-covariance matrix, Ω, of e_t. Following Barten [1969], Berndt and Wood state that B_o and Ω must satisfy

(3) $$\ell' B_o = [1\ 0\ 0\ \ldots\ 0]$$

(4) $$\ell' \Omega = [0\ 0\ 0\ \ldots\ 0]$$

2.1 The Static Model and Correlated Errors

A logical extension of the basic static model includes the possibility of sequentially correlated errors. This case was considered by Berndt and Savin [1975]. Specifically, they allowed for autoregressive or moving-average error structures which could be characterized by vector difference equations. Thus the basic model (1) - (2) was modified to

$$y_t = B_o x_t + n_t \tag{5}$$

with,

$$n_t = \sum_{p=1}^{P} R_p n_{t-p} + e_t \qquad \text{(AR errors)} \tag{6}$$

or,

$$n_t = \sum_{q=1}^{Q} C_q e_{t-q} + e_t \qquad \text{(MA errors).} \tag{7}$$

In the case of models (5) and (6) or (5) and (7), Berndt and Savin show that (2) results in the additional restrictions

$$\ell' R_p = k'_p \qquad 1 \leq p \leq P \tag{8}$$

or

$$\ell' C_q = k'_q \qquad 1 \leq q \leq Q \tag{9}$$

where k_p (or k_q) is a Gx1 vector with the same unknown constant in each position.

2.2 Distributed Lag Models

In their last section, Berndt and Savin [1975] discuss extensions of the above analysis to models derived from adaptive expectations or partial adjustment behavior; but provide no formulas analogous to (3) - (4) and (8) - (9). As a first step in filling this gap, consider the effect of (2) on the specification of distributed lag models of dynamic adjustment. In this situation (5) is generalized to

$$y_t = \sum_{k=0}^{K} B_k x_{t-k} + n_t ,$$

or, in operator notation,

$$(10) \qquad y_t = B(L)\, x_t + n_t$$

with $B(L) = B_o + B_1 L + B_2 L^2 + \ldots + B_K L^K$. L is the lag operator, i.e., $L^k x_t = x_{t-k}$. Thus B(L) is a polynomial operator matrix and the B_k $(1 \leq k \leq K)$ are the GxM coefficient matrices whose elements are to be estimated. Similar use of operator matrices in expressing the autoregressive or moving-average structure of n_t permits (5) - (7) to be replaced by (1) with either,

$$(11) \qquad R(L)\, n_t = e_t \qquad (R_o = I)$$

or,

$$(12) \qquad n_t = C(L)\, e_t \qquad (C_o = I).$$

Assuming that x_t is structured as before, i.e., with its first element fixed at unity for all t, then (2) requires all the B_k to satisfy

$$(13) \qquad \ell' \sum_{k=o}^{K} B_k = \sum_{k=0}^{K} \ell' B_k = [1\ 0\ 0\ \ldots\ 0].$$

The restrictions on the structure of (11) or (12) remains unchanged in this dynamic framework since the error term can be considered separately once (13) is imposed. Note that relationships are now present both between equations and across the lags.

For dynamic adjustment as represented by distributed lags, (2) results in restrictions (13), (4), and either (8) or (9).

3.0 RATIONAL DISTRIBUTED LAG MODELS

A final generalization in terms of dynamical models can be achieved by introducing lagged endogenous variables resulting in an implied rational distributed lag model. This type of model can be described using operator notation as

$$(14) \qquad A(L)\, y_t = B(L)\, x_t + n_t$$

where

$$A(L) = I + A_1 L + A_2 L^2 + \dots + A_R L^R.$$

The disturbance term, n_t, may be described by either (11) or (12).

Using the definition of A(L), the model (14) can be written in terms of current and lagged variables as follows:

$$y_t = -A_1\, y_{t-1} - \dots - A_R y_{t-R} + B_o\, x_t + \dots + B_K\, y_{t-k} + n_t.$$

Imposition of (2), together with the previous results on the B_k and n_t terms ((13) with either (8) or (9)), gives

$$1 = -\ell' A_1\, y_{t-1} - \dots - \ell'\, A_R\, y_{t-R} + 1 + 0.$$

Therefore, the A_r must be such that $\ell' \sum_1^R A_r\, y_{t-r} = 0$. If it was known that the y_{t-r} are completely arbitrary then the only way to ensure (2) would be to select A_r such that

$$(15) \qquad \ell'\, A_r = 0 \qquad 1 \leq r \leq R.$$

At first glance (2) appears to invalidate the conclusion given in (15) since it implies dependencies between the elements of y_{t-r}. This problem is, however, quite transparent because (2) can itself be employed to bring about the conclusion captured in (15): Since $\ell'\, y_{t-r} = 1$, one element of y_{t-r}, say $y_{1,\, t-r}$, can be eliminated and written in terms of all the other elements. Call the result of this transformation the Gx1 vector $\tilde{y}_{t-r}$:

$$\tilde{y}'_{t-r} = [1 - \sum_{2}^{G} y_{t,t-1}, y_{2,t-1}, y_{3,t-1}, \ldots, y_{G,t-1}].$$

Then each term $\ell' A_r y_{t-r} = \ell' A_r \tilde{y}_{t-r} = 0$ can be expressed as

$$(16) \qquad 0 = (\sum a^r_{i1}) + [(\sum^r_{i2}) - (\sum^r_{i1})] y_{2,t-r} + \cdots$$

$$\cdots \qquad + [(\sum a^r_{iG}) - (\sum a^r_{i1})] y_{G,t-r}$$

where a^r_{ij} denotes the $(i,j)^{th}$ element of A_r and the individual sums are taken from $1 \leq i \leq G$. Each $y_{2,t-r}, \ldots, y_{G,t-r}$ can now be considered completely arbitrary so that satisfaction of (2) can be ensured if and only if each of the sums in (16) are zero. But this is just another way of saying that each column of A_r must sum to zero; this is exactly what (16) states.

The consequences of the restrictions given in (15) appear to be rather strong in certain special cases. Consider the situation where there is only first order lags in the endogenous variables, $R = 1$. If it is desired to explain each share with only its own lagged variable (plus whatever exogenous variables appear relevant), then A_1 is diagonal. The only way (15) can then be met is with $A_1 = 0$! Thus, shares cannot be explained by their own lagged values unless other endogenous lagged variables are included in each equation.

A more illustrative example of the restrictions to be encountered as a result of (15) is as follows. Suppose it is desired to estimate a three share factor demand relationship using only one lag on y_t; $R = 1$. Further assume that for theoretical reasons, the structure on A_1 is given by

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

Only one element in the first and third columns, and only two elements of the second column, can be freely estimated. Thus, if a_{11}, a_{22}, a_{33}, and a_{12} are chosen as free, A_1 must be restricted to

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ -a_{11} & a_{22} & -a_{33} \\ 0 & -(a_{12}+a_{22}) & a_{33} \end{bmatrix}$$

in order to satisfy (2). For reasons pointed out in Berndt and Wood [1975], it is possible for the estimation results to depend on the restrictions employed. Therefore it appears that iterative techniques, such as I3SLS, will always have to be employed.

In summation, the complete set of restrictions on the general dynamic model if (14) is given by (15), (13), and (14), together with either (8) or (9).

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