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THE RATIONAL DISTRIBUTED LAG STRUCTURAL FORM - A GENERAL ECONOMETRIC MODEL

Kent D. Wall*

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COMPUTER RESEARCH CENTER FOR ECONOMICS AND MANAGEMENT SCIENCE
National Bureau of Economic Research, Inc.
575 Technology Square
Cambridge, Massachusetts 02139

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Contents

Abstract

The Rational Distributed Lag Structural Form of an econometric model is introduced, and its relationship to several traditional forms of representation is discussed. The traditional forms are viewed as special cases of the Rational Structural Form. Thus, the latter provides a unified framework for any treatment of the linear, time invariant modelling problem. In particular, a solution of the estimation problem for the Rational Structure Form leads to the solution of the estimation problem for all traditional forms.

1. Introduction

This paper introduces the Rational Distributed Lag Structural Form (RSF) representation of linear[1], discrete-time, constant coefficient econometric models. It is shown that this single, very flexible, representation encompasses all of the traditional linear models as special cases, thus providing a useful vehicle for the development of a unified framework for linear econometric modelling. The groundwork will then be laid for subsequent works dealing with the identification and estimation of such forms, as well as an exposition of a new model building methodology of great potential which utilizes the RSF model representation. In addition, the use of this representation allows for a generalization of the univariate techniques of Astrom and Bohlin [2], and Box and Jenkins [5] to multiple equation systems.

The second section contains a detailed description of the RSF representation and introduces appropriate notation. The third section discusses several special cases of the RSF, demonstrating by example, how each of the important traditional linear models are included within the class described by the RSF. Finally, some general observations and conclusions are made concerning the usefullness of this representation in econometrics.

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[1]Linearity is taken here to imply linearity in the variables. This is to be distinguished from linearity in the parameters, which, in general, is not found in the RSF. Thus, nonlinear estimation problems may be encountered when determining the parameters of a general RSF model.

2. The RSF Representation

The Rational Distributed Lag Structural Form representation can be described in two ways: (i) by presentation of each equation of the general multi-equation model, or (ii) by presentation of the entire system of equations using compact vector-matrix notation. Since a familiarity with both descriptions contributes to a fuller understanding of the implications of the RSF representation, both will be presented here.

Each equation of the RSF model may be written as

$$y_{it} = \sum_{\substack{j=1\\j\neq i}}^{G_i} \frac{\beta_{ij}(L)}{\alpha_{ij}(L)} L^{\delta_{ij}} y_{jt} + \sum_{j=1}^{K_i} \frac{b_{ij}(L)}{a_{ij}(L)} L^{D_{ij}} x_{jt} + \frac{c_i(L)}{d_i(L)} e_{it} + k_i \tag{1}$$

where k_i is an arbitrary constant and,

$$\alpha_{ij}(L) = 1 + \alpha^1_{ij}L + \ldots + \alpha^{\rho_{ij}}_{ij} L^{\rho_{ij}}$$

$$\beta_{ij}(L) = \beta^0_{ij} + \beta^1_{ij}L + \ldots + \beta^{\sigma_{ij}}_{ij} L^{\sigma_{ij}}$$

$$a_{ij}(L) = 1 + a^1_{ij}L + \ldots + a^{R_{ij}}_{ij} L^{R_{ij}}$$

$$b_{ij}(L) = b^0_{ij} + b^1_{ij}L + \ldots + b^{S_{ij}}_{ij} L^{S_{ij}}$$

$$c_i(L) = 1 + c^1_iL + \ldots + c^{Q_i}_i L^{Q_i}$$

$$d_i(L) = 1 + d^1_iL + \ldots + d^{P_i}_i L^{P_i}.$$

Thus the i^{th} endogenous variable of the model at time t, y_{it} $(1\leq i\leq G)$, can be related to G_i other endogenous variables, y_{jt} $(j\neq i)$, K_i exogenous variables, x_{jt} $(1\leq j\leq K)$, and a random disturbance term, e_{it}. The relationships between all of these variables take the form of rational distributed lags in the "lag operator" L (i.e., $L^{\ell}Z_t = Z_{t-\ell}$). The exact form of each rational operator is quite arbitrary, being solely determined by the orders of each polynomial operator involved, i.e., by $\{\rho_{ij}, \sigma_{ij}, \delta_{ij}\}$, $\{R_{ij}, S_{ij}, D_{ij}\}$, or $\{P_i, Q_i\}$ for each i and each j. Each of the leading coefficients of the $\alpha_{ij}(L)$, $a_{ij}(L)$, and $d_i(L)$ polynomials is fixed at unity in order that the conventional normalization rule be satisfied.[2] The model is seen to be recursive in that explicit dependence upon time is preserved in (1). This is somewhat different from the traditional models where usually y_{it} is replaced by y_i, a 1XT ($1 \leq t \leq T$) vector. The principle reasons for keeping the representation recursive are: (i) the recursive representation reduces computational requirements in large models used for estimation work, and (ii) the recursive representation emphasizes the dynamic nature of the model - a characteristic so important in policy design work where an understanding of the evolutionary behavior is critical.

By "stacking" each equation of the model, as given in (1), on top of one another and resorting to vector-matrix notation, the RSF may alternately be written as

$$T(L)y_t = U(L)x_t + V(L)e_t, \tag{2}$$

where y_t is now a GX1 vector of endogenous variables observed at time t, x_t is a KX1 vector of exogenous variables observed at time t, and e_t is a GX1 vector of random disturbances at time t. The rational matrix operators T(L), U(L), and V(L) are dimensioned respectively as GXG, GXK, and GXG. In view of (1)

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[2] The leading coefficient in the $c_i(L)$ polynomials, c_i^0, is not fixed at unity to satisfy the usual normalization rule. This constraint is imposed only for use of the RSF in estimation problems where it is then required for unique determination of the estimated residual variances.

it is clear that the ij^{th} elements of these matrices are given by

$$[T(L)]_{ij} = \begin{cases} 1 & ; i=j \\ \dfrac{-\beta_{ij}(L)}{\alpha_{ij}(L)} L^{\delta_{ij}} & ; i \neq j \end{cases}$$

$$[U(L)]_{ij} = \frac{b_{ij}(L)}{a_{ij}(L)} L^{D_{ij}} \quad ; \text{all } i, j$$

$$[V(L)]_{ij} = \begin{cases} \dfrac{c_i(L)}{d_i(L)} & ; i=j \\ 0 & ; i \neq j \end{cases}$$

The representation (2) is now more clearly related to the traditional structural form

$$B \;\; y_t = \Gamma x_t + u_t.$$

It is seen that the constant structure matrix, B, has been replaced by a rational operator structure matrix, T(L); the exogenous multiplier matrix, Γ, has been replaced by a rational operation matrix, U(L); and the uncorrelated random disturbance vector, u_t, has been replaced by $V(L)e_t$ (thus allowing for a very general corrleation structure in the disturbance term). If the exogenous vector of the traditional structural form has been expanded to include lagged exogenous and endogenous variables then it may be argued that the traditional model is really of the form

$$B(L) \; y_t = \Gamma(L) x_t + u_t$$

This is more general than the seemingly static form, and it may more easily be identified with the RSF in (2), however, it is still less general than (2) because B(L) and Γ(L) are polynomial matrix operators which are special cases of rational matrix operators. In general terms, then, the RSF is obtained from the traditional structural form by exchanging constant or polynomial operator matrices for rational operator matrices. Once again, as with (1), the recursive time dependent nature of the phenomena being modelled is emphasized by (2) through the explicit use of the index t and operator L.

Before investigating several interesting special cases of the RSF representation, it is appropriate here to define two important concepts resulting from the description of the RSF given above. Each deals with a part of the overall specification of the RSF representation:

(a) Economic Specification. An economic specification defines the overall size of the model (number of inputs, K, and outputs, G) and the input quantities for each equation of the model (the integers K_i and G_i). Thus a priori information and economic theory are employed to define all the variables of the model and define which variables effect which other variables.

(b) Structure Specification. The structure specification, or structure, of the RSF is that quantity of information which assigns particular values to the set of integers $I \stackrel{\Delta}{=} \{\rho_{ij}, \sigma_{ij}, \delta_{ij}$, for $1 \le j \le G_i$; R_{ij}, S_{ij}, D_{ij}, for $1 \le j \le K_i$; P_i, Q_i; for each i, $1 \le i \le G\}$. Thus a structure defines the orders of all polynomial operators and pure delays in each rational transfer function of each equation of the complete model.

The economic specification fixes the dimensions and the locations of zero elements in the matrix operators, $T(L)$ and $U(L)$, of (2). The structure specification then fixes the exact form of the remaining nonzero rational elements.

3. Important Special Cases

In order to substantiate the claims made concerning the generality of the RSF representation in econometrics, several important special cases of (1) and (2) will now be discussed. First only univariate or single equation models will be considered using the description provided in (1).

A. Single Equation Models: G=1. In the univariate model the first subscript, i, on the left-hand variable becomes immaterial so it is dispensed with. Moreover, all of the right-hand side variables become exogenous so that the first term on this side in (1) may be dropped. Thus (1) reduces to

$$y_t = \sum_{j=1}^{K} \frac{b_j(L)}{a_j(L)} L^{D_j} x_{jt} + \frac{c(L)}{d(L)} e_t + k. \tag{3}$$

(i) *The Univariate Time Series (ARMA) Model*. By specifying K=0 all purely exogenous variables are excluded leaving

$$y_t = \frac{c(L)}{d(L)} e_t + k$$

or,

$$d(L)\, \tilde{y}_t = c(L)\, e_t$$

where $\tilde{y}_t \triangleq y_t - k$. This is just the autoregressive-moving average univariate time series model treated so extensively by Box & Jenkins [5], Phillips [20], and Aigner [1].

(ii) *The Univariate Autoregressive-Moving Average Model with Exogenous Inputs (ARMAX)*. By specifying K=1 and c(L) = d(L) = 1 there results

$$\tilde{y}_t = \frac{b(L)}{a(L)} x_{t-D} + e_t.$$

This is the ARMAX model treated by Jorgenson [16], Steiglitz & McBride [24], and Dhrymes, Klein, & Steiglitz [11] (where reference to many other works are cited).

(iii) *The Univariate Distributed Lag Model*. Coupling the specification given in (ii) above with the additional constraint that a(L)=1 produces the distributed lag model.

$$\tilde{y}_t = b(L)\, x_{t-D} + e_t .$$

or,

$$\tilde{y}_t = \sum_{k=0}^{S} b_k x_{t-D-k}.$$

previously treated by Maddalo & Rao [19], and Solow [23].

(iv) *The Autoregressive Multiple Input Model*. By specifying $K \geq 1$ and $a_j(L) = d(L)$ for $1 \leq j \leq K$ there results the autoregressive, or polynomial, form model which allows for lagged endogenous variables,

(4) $$a(L)\, \tilde{y}_t = \sum_{j=1}^{K} b_j(L) x_{t-D_j} + c(L) e_t.$$

This model has experienced wide use in the engineering field notably by Astrom & Bohlin [2]. Later work by Bohlin [3] and then Bohlin & Wensmark [4] dealt directly with the general single equation model (3), but their efforts have only been reported in the engineering literature.

B. Simultaneous Models: $G>1$. For multiple equation models the more compact description (2) will be used, although recourse to (1) will be made where appropriate. It will now be important to retain both the subscript, i, and the first right-hand side summation in (1) because of the obvious need to allow for endogenous interaction.

(i) *The Polynomial Structural Form (PSF).* Perhaps the most widely investigated simultaneous model is that which may be classed as a polynomial structural form. This is obtained as a special case of (2) by constraining each equation of the model to have a polynomial form similar to that of (4) above. In this situation each equation in (1) is constrained such that $\alpha_{ij}(L) = a_{ij}(L) = d_i(L) \triangleq a_i(L)$, i.e., the denominator polynomials across any one equation are identical, but not between equations. Thus each equation of (2) may be written as

$$a_i(L)y_{it} = \sum_{\substack{j \neq i \\ j=1}}^{G_i} \beta_{ij}(L)L^{\delta_{ij}} y_{jt} + \sum_{j=1}^{K_i} b_{ij}(L)L^{D_{ij}}x_{jt} + c_i(L)e_{it} + k_i.$$

where $a_i(L) \triangleq 1 + a_i^1 L + \ldots + a_i^{R_i} L^{R_i}$. If all the coefficients of like powers of L are collected together in coefficient matrices then (2) takes on the special PSF form:

$$(5) \qquad A(L)y_t = B(L)x_t + C(L)\, e_t + \underline{k}$$

where $\underline{k}$ is GX1 vector of constants and,

$$A(L) = \sum_{k=0}^{R} A_k L^k \quad , \quad \det A_0 \neq 0, \quad R \triangleq \max_{i,j} \{R_i, \delta_{ij}+\sigma_{ij}\}$$

$$B(L) = \sum_{k=0}^{S} B_k L^k \quad , \quad S \triangleq \max_{i,j} \{S_i, D_{ij}+S_{ij}\}$$

$$C(L) = \sum_{k=0}^{Q} C_k L^k \quad , \quad C_0 = I \quad , \quad Q \triangleq \max_{i} \{Q_i\}.$$

The interpretation of the $\{A_k, B_k, \text{ and } C_k\}$ in terms of the coefficients of the polynomials $a_i(L)$, $\beta_{ij}(L)$, $b_{ij}(L)$, and $c_i(L)$ is somewhat complicated because of the pure delays, $L^{\delta_{ij}}$ and $L^{D_{ij}}$. However in general it may be seen that the A_k are constructed from the a_i^k and $-\beta_{ij}^k$ coefficients, the B_k are constructed from the b_{ij}^k, and the C_k are constructed using the c_i^k's. The PSF (5) has been examined by Hannan [13], [14] and Preston & Wall [20] with regards to a general identification problem when moving-average errors are present.

(ii) *The Polynomial Reduced Form (PRF)*. By inverting A_0 and pre-multiplying (5) with the inverse yeilds the polynomial reduced form. This may also be obtained from (5) by constraining $A_0=I$. Such models have been used by Chow [6], [7], [8] in econometric developments of optimal control work. The PRF model has also been used extensively in engineering for the estimation of multiple equation systems. See for example Kayshap [17] and Rowe [22].

(iii) *The Simultaneous Distributed Lag Model*. By further constraining (5) to exclude all lagged endogenous variables, i.e., $A(L)\equiv I$ (or by constraining $A_k=0$ for $k\geq 1$ in (5)), there results the simultaneous distributed lag model

$$y_t = B(L)\, x_t + C(L)e_t$$

or

$$y_t = \sum_{k=0}^{S} B_k x_{t-k} + \sum_{k=0}^{Q} C_i e_{t-k}.$$

Finally, by specializing to the case of uncorrelated distribances, $C(L)=I$, i.e., $C_k=0$ for $k\geq 1$, the usual distributed lag model, treated so widely in econometrics, appears. See Drhymes [10] for example.

(iv) *The Final Form (FF)*. Returning to (2) in its original rational description, it is seen that by specifying T(L)=I the traditional final form (FF) model is obtained[3],

$$y_t = U(L)\ x_t + V(L)e_t$$

A model such as the above could be estimated directly, however the extremely complicated and dense nature of U(L) might make estimation impracticable. Thus both T(L) and U(L) are estimated (each usually rather sparse), and the equivalent FF obtained by inversion (when the inverse exists) of T(F) and premultiplication of (2) by this inverse. Then the FF maybe determined by

$$y_t = T^{-1}(L)U(L)x_t + T^{-1}(L)V(L)e_t.$$

In terms of polynomial form models, the FF is computed by inversion of A(L) (when it exists) and premultiplication of (5) by $A^{-1}(L)$:

$$y_t = A^{-1}(L)B(L)x_t + A^{-1}(L)C(L)e_t.$$

Note that now a rational form has been obtained.

(v) *The Multivariate Time Series Model*. With the model (2) or (5) it becomes possible to talk of multivariate time series analysis - by excluding all exogenous explanators. Thus, with U(L)=0 there results

$$T(L)y_t = V(L)e_t.$$

Since V(L) has been constrained in (1) and (2) to be a diagonal operator, T(L) must be included if it is desired to remove any between equation correlation in

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[3]The terminology FF is used here in the same spirit as in Klein [18], where a definite distinction is made between the final and reduced form. Note that a true final form expresses the current endogenous vector in terms of current and lagged exogenous variables only. In the traditional reduced form the current endogenous variables are expressed in terms of current and lagged exogenous variables as well as lagged endogenous variables.

the residuals. A similar model may be constructed using the polynomial forms above. Namely,

$$A(L)y_t = C(L)e_t$$

or, $y_t = A^{-1}(L)C(L)e_t$. Such models constitute a straightforward generalization of the early work of Box & Jenkins [5], and have interesting applications in questions dealing with causality as defined by Granger [12]. An example of such an application can be found in Wall [25].

4. Summary and Conclusions

A very general representation for linear, constant, discrete-time econometric models has been presented which includes almost all of the traditional econometric models as special cases. It has been shown how, by proper structure specification (a priori restrictions on the parameters), the RSF can be employed to represent five different single equation models:

- (i) The univariate time series ARMA model,
- (ii) The univariate ARMAX model,
- (iii) The univariate distributed lag model,
- (iv) The autoregressive multiple input model,
- (v) The rational multiple input model,

and five different multiple equation models:

- (vi) The polynomial structural form,
- (vii) The polynomial reduced form model,
- (viii) The simultaneous distributed lag model,
- (ix) The final form model,
- (x) The multivariate (simultaneous) time series model.

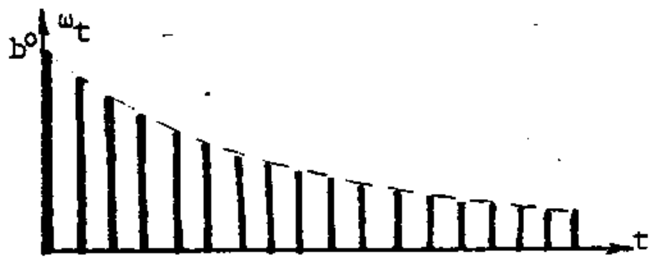
Other models with special autocorrelated error structures such as those employed by Chow & Fair [9] or Hendry [15] have not been included here but, nonetheless, can also be viewed as special cases of the RSF.

The unification of linear modelling provided by the RSF is of major importance. By allowing each of the traditional econometric models to be viewed as a specialization of the general descriptions (1) and (2), the capabilities and limitations of each model are more easily understood. For example, the traditional static simultaneous econometric model

$$B\, y_t = \Gamma\, x_t + u_t$$

can be seen as the most limited, least flexible description of how x_t effects y_t. This is because the model can only be obtained from (2) by: (a) restricting the rational form to the polynomial form; (b) restricting the lag structure to exclude lagged endogenous variables; (c) excluding lagged exogenous variables; and finally,(d) restricting the residual corrleation structure to the simplest possible case - no correlation. Tabulating the number of restrictions needed to reduce (1) or (2) to the desired form gives a very precise way of judging that model's capabilities, i.e., the more restrictions required, the more the model is limited. Not only are the limitations of a model revealed, but also where in the model the limitations occur. Furthermore, an examination of the tabulated restrictions can be used to point to possible weakness of the model when confronted with real data. Such an analysis of linear models can also elucidate the interrelationships between the various models: Instead of viewing one model with respect to the parent RSF representation, specific models may be viewed with respect to one another. For example, the distributed lag model may be depicted as a more restricted form of the polynomial reduced form by constraining $A(L)=I$ (which implies $A_k=0$ for $k\geq 1$).

The conceptual advantages of the RSF discussed in the previous paragraphs appear to be accompanied by several practical advantages in terms of parameter estimation. First, observe that the combination of T(L) and V(L) allows the RSF to admit very general error structures. The autoregressive-moving average structure of each element of V(L) permits the modelling of almost all types of error correlations encountered in real data. Second, the rational distributed lag elements of U(L) provide for a parsimonious [5] parameterization of almost all realistic weighting functions experienced in econometrics. A classic example of this is the ability of the rational lag to realize the infinite geometric lag, w_t, given in the figure below



with just two parameters: $b^0/(1+a^1L)$ and $|a^1|<1$. Finally, by allowing for different denominator polynomials in each element on the righthand side of (1), different dynamic effects for each explanator can be estimated. For example consider,

$$y_{1t} = \frac{b_{11}^0}{1+a_{11}^1 L} x_{1t} + \frac{b_{12}^0}{1+a_{12}^1 L} x_{2t} + \frac{1}{1+d_1^1 L} e_t.$$

Because $a_{11}^1 \neq a_{12}^1$ it is possible to determine whether y_{1t} responds more rapidly to x_{1t} than to x_{2t}. Such an investigation is impossible with the more simple polynomial form for this equation,

$$y_{1t} = \frac{b_{11}^0}{1+a_1^1 L} x_{1t} + \frac{b_{12}^0}{1+a_1^1 L} x_{2t} + \frac{1}{1+a_1^1 L} e_{1t}.$$

Questions concerned with a rigorous treatment of the identification, estimation, and modelling methodology using the RSF are deferred to later works. However, it should be clear from the preceding discussions that if an efficient estimation algorithm can be obtained for the general RSF representation, then the estimation problem will have been completely solved for all useful linear econometric models.

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