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EXISTENCE OF EQUILIBRIUM AND STRATIFICATION IN LOCAL AND HIERARCHICAL TIEBOUT ECONOMIES WITH PROPERTY TAXES AND VOTING

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ABSTRACT

We present the first fully closed general equilibrium model of hierarchical and local public goods economies with the following features: (i) multiple agent types who are endowed with both some amount of private good (income) and a house, who are mobile between houses and jurisdictions, and who vote in local and national elections; (ii) multiple communities that finance a local public good through property taxes which are set in accordance with absolute majority rule; and (iii) a national government that produces a national public good financed through an income tax whose level is determined through majority rule voting. In contrast to previous models, no overly restrictive assumptions on preferences and technologies are required to prove the existence of an equilibrium in the presence of property taxation and voting. Thus, the existence of an equilibrium is proved without any of the major restrictions used in the past, and sufficient conditions for stratification of agents into communities based on their public good preferences and their wealth levels are found. This model lays the groundwork for a positive applied analysis of local public finance and intergovernmental relations. It furthermore builds the foundation for a parameterized computable general equilibrium model of local public goods and fiscal federalism that is used elsewhere to analyze a variety of policy issues.

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I. Introduction

Since Tiebout (1956), economists have sought to formalize the effects of mobility, diverse preferences and local government behavior on the level and efficiency of public goods production in competing communities (Wooders (1978, 1980), Ellickson (1979), Greenberg (1977,1983), Bewley (1981), Henderson (1991)). With an eye toward future empirical applications, one branch of this literature has focused on developing strictly positive Tiebout models by explicitly incorporating realistic institutional and political features. Most recently, Epple, Filimon and Romer (1993) (henceforth EFR) showed in an important paper that one can find a voting equilibrium when local public spending is financed through the property tax, but they express considerable pessimism as to whether it would be possible to achieve this existence result without resorting to a number of strong assumptions.[1] Still, their equilibrium result represents significant progress in light of Rose-Ackerman's (1979) previous argument that an equilibrium in a similar environment generally does *not* exist. Models by Dunz (1985), Greenberg and Shitovitz (1988) and Konishi (1994) avoid many of the difficulties encountered by Rose-Ackerman and EFR by using proportional *income* taxes and prohibiting taxation of property. However, while local property taxes are used heavily in the US, local income taxes play no empirically important role.

The goal of this paper is to build on the work of Rose-Ackerman, EFR and Dunz in order to ascertain if it is possible, in the context of a local public goods voting model, to utilize the property tax with few restrictive assumptions. Furthermore, the paper attempts to lay the theoretic foundation for a computable general equilibrium model developed elsewhere (Nechyba (1994a,b)) which can be used in policy analysis. In the process, we seek to account for three stylized facts:

(i) Voting takes place to determine local public spending and tax levels;

(ii) The economy contains house endowments whose value is taxed by local governments;

(iii) Consumers with different preferences and endowments are freely mobile and "shop" among communities.

We find that by restricting the nature of housing in each community, a general equilibrium model with these features can be developed and the existence of an equilibrium readily proved. Furthermore, the model is sufficiently flexible to not only incorporate local public goods but also a national public good.

[1] They conclude that "in order to ensure existence, fairly severe assumptions must be placed upon individual preferences and technology" and that "standard assumptions on individual preferences are not sufficient" (p. 586).

This allows us to stipulate a fourth stylized fact which prepares the way for the integration of local public finance into the study of fiscal federalism[2]:

> (iv) There exists a hierarchical public sector in which the higher level of government employs a different revenue source, an income tax, to finance its public good.

Finally, we find sufficient conditions under which agents separate into different communities based on either their tastes for local public goods, their wealth levels, or both. This separation, or "stratification" can be found in both EFR (1993) and Westhoff (1977). While stratification is not automatic in our model and while it is not as easily defined, a weaker form of stratification does arise as assumptions very similar to some in EFR are employed. Therefore, although our model is not a strict generalization of EFR, a simplification of it looks remarkably similar to theirs without requiring their strong assumptions for the existence of an equilibrium.

Property Taxation, Voting and Local Public Finance

As indicated above, the attempts to incorporate the empirically important institutions of voting and local property taxation into a single model have met with mixed results. Non-convexities that arise in the presence of perfectly divisible land either cause the absence of equilibria (Rose-Ackerman (1979)) or necessitate the use of strong restrictions on preferences and technologies to guarantee their existence (EFR (1993)). So long as housing is modelled as perfectly divisible, these nonconvexities vanish only when the local property tax is abandoned in favor of local income taxation (Konishi (1994)). While the approach of substituting tax systems may be technically convenient, both empirical and theoretical evidence suggests that the absence of local income taxes in the US is not merely a historical accident, but rather the result of forces within a general equilibrium Tiebout world that make property taxes a *dominant* tax strategy for local governments.[3]

In light of this, rather than abandoning the property tax or resorting to restrictive assumptions, we propose to change the way land is modelled. In particular, we employ Dunz's (1985) technique of endowing agents with *heterogeneous* houses of *fixed size*. This differs from Rose-Ackerman and

[2] Thus far, the study of intergovernmental relations (see, for example, a recent review by Oates (1994)) has largely been divorced from the theoretical local public goods literature inspired by Tiebout (1956).

[3] Ninety-eight percent of all locally raised tax revenues in politically independent US school districts, for example, are raised through the property tax. Krelove (1993) and Nechyba (1994b) provide theoretical arguments that suggest it is difficult for income taxes to play any substantial role in a local Tiebout equilibrium in which tax systems are chosen endogenously. (In Nechyba (1994b) this is shown to hold for the type of Tiebout model discussed here.)

EFR in that (i) it introduces heterogeneity into the land market; (ii) it fixes the total housing stock; and (iii) it casts the problem into a fully *closed* general equilibrium framework by introducing land *endowments*.[4] The closed nature of the model makes possible an analysis of capitalization, one of the most fundamental issues in local public finance, as well as an investigation of the political implications of the effects of government policies on property owners in a general equilibrium world (Nechyba (1994a,b,1996)). Neither of these issues can be addressed when housing is supplied by outsiders according to an exogenously given supply schedule.[5]

In addition to a fixed housing stock, the existence result in this paper requires the assumption of arbitrarily fixed community boundaries.[6] While these restrictions are different from those entertained by EFR (1993), they seem at least somewhat natural and consistent with observation. Fixed house sizes imply that the only way for an agent to change his consumption of housing is to move. Given the high migration observed in the US, moving indeed seems to be the major avenue taken by individuals to changing their consumption of housing. Furthermore, whereas the only motivation for moving in the EFR and Rose-Ackerman models is a desire to switch communities, our model makes possible *intra*-jurisdictional migration (which actually accounts for much of the migration observed in the US.)[7] Finally, despite a recent trend toward school district consolidation in some states, most local boundaries are rarely altered. Therefore, since the purpose behind these positive Tiebout models is to create a framework for the empirical analysis of various policy issues in intergovernmental relations and local public finance, fixed community and house sizes can be assumed without losing too much generality. Other models with less institutional detail (for example Wooders (1978,1980) or Scotchmer (1985)) are better suited for analyzing the long run issue of how community boundaries arise.

This model of communities and housing, then, facilitates a straightforward proof of the existence of an equilibrium without resorting to the strong assumptions made in the previous literature.[8]

[4] In both Rose-Ackerman and EFR, housing is supplied by an absentee landlord.

[5] Dunz (1985) also points out that this way of modelling land avoids the internal inconsistencies inherent in many models with a continuum of consumers and a finite amount of land (see Berliant (1985)).

[6] Since boundaries can be set arbitrarily for an arbitrary number of communities, we do not require community sizes to fall within certain ranges as in Rose-Ackerman. A different branch of this literature focuses on the existence problem when the number of jurisdictions is endogenous. For two distinct approaches to endogenous club formation, see Wooders (1978, 1980) who uses approximate cores and Scotchmer (1985) who finds noncooperative equilibria.

[7] This large degree of mobility is documented in the literature on housing and tenure choice (see, for example, Hanushek and Quigley (1978) and Ioannides (1987)). Approximately 20 percent of metropolitan residents move each year, two thirds of which move within their metropolitan area.

Furthermore, we define a new fixed point correspondence (inspired by Konishi (1994)) that is applicable to other models and can be used to simplify the proof in Dunz substantially while dropping his "independence assumption" on preferences.[9]

Hierarchical Public Good Production

In addition to generalizing the Tiebout model with property taxes and voting, we add a hierarchical dimension to the public good sector. The model developed below therefore contains a national government which provides a national public good financed through proportional *income* taxation whose provision level is determined through majority voting. Note that consumers are thus taxed in two ways: first by their local governments based on their property holdings and second by the national government based on their income. This distinction in tax instruments plays an important role in many of the results obtained in later applied work on intergovernmental relations (Nechyba (1994a)). Furthermore, the existence of two different tax systems facilitates an easy comparison between their respective effects and allows us to conclude that, when local tax systems are chosen endogenously, the property tax is a *dominant* tax strategy for all communities (Nechyba (1994b)). Finally, the model provides an explanation for the endogenous formation of intergovernmental grant systems as a mechanism for local communities to escape a prisoners' dilemma created by the general equilibrium nature of the problem (Nechyba (1994b)).

In order to prove the existence of a political equilibrium with hierarchical public goods, the model makes use of the distinction made in Shepsle (1979) between structurally induced and preference induced equilibria. Preference induced equilibria in the presence of two or more issues and three or more voters have been shown to be extremely rare and to depend on precise geometric properties of individual preferences (McKelvey (1976), Plott (1969)). But when political institutions impose

8 In order to overcome the problems raised in Rose-Ackerman, EFR assume that (i) all agents have identical preferences; (ii) indifference curves of *indirect* utility functions are concave; (iii) local public good production functions are linear with positive intercept; (iv) the slopes of indifference curves change continuously with the private good endowment (Westhoff's (1977) "single crossing assumption"); (v) individual demands for the housing good have price elasticity less than or equal to 1 for any housing price level and for any level of private good endowment; (vi) the housing good is supplied by absentee landlords. None of these assumptions are made here.

9 The "independence assumption" states that an agent's most preferred level of the local public good is independent of who else resides in his community. Since Dunz uses income taxation, tax revenues and consequently local public good levels depend critically on community income. Thus, an agent's most preferred local public good level should be allowed to vary with community income (and therefore with the composition of the community). It should be noted that, by employing the method of proof in this paper, Dunz would be able to drop this independence assumption.

structure in such a way as to enable us to apply Black's median voter result (Black (1948)), structurally induced political equilibria become possible where preference induced equilibria do not exist.[10] In our case, both local and national public good levels are determined through majority rule voting. As is fairly standard in this literature, voters are myopic and vote on each issue separately holding current public goods levels (other than the one being voted on) fixed. It is shown that preferences over both local and national tax rates under this assumption are single peaked, which allows us to apply Black's median voter theorem in all local and national "elections".[11] Thus the structure imposed by the federalist institution forces agents to consider national and local public policy separately and brings forth the existence of a political equilibrium in an economy with both local and national public goods.

The paper is organized as follows: Section II introduces the model and defines an equilibrium; Section III proves the existence of an equilibrium under general conditions; Section IV discusses different definitions of stratification of an equilibrium, finds conditions sufficient to guarantee that equilibria will satisfy these definitions, and relates them to the stratified equilibria in EFR; Section V is a short conclusion, and Section IV is an appendix that contains the proofs to the lemmas in Section III.

II. The Model

Consumer Endowments and Preferences

The model contains a measure space $(N,\mathcal{N},\mu)$ of consumers, where $N \subset R$.[12] Each consumer is endowed with one of H different types of "houses" in one of M jurisdictions or communities, where M and H are finite integers and the terms jurisdiction and community are used interchangeably. (Let M

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[10] Another way to get multiple public goods is by employing d-majority voting (Greenberg (1979)) rather than simple majority voting. Majority rule voting, however, seems to have more empirical content unless d=1, in which case the two choice rules are identical.

[11] Slutsky (1977) uses a technically similar idea of restricted majority voting to prove the existence of equilibrium in a single jurisdiction economy with more than one public good. (Slutsky's economy is not a Tiebout economy; i.e. since there is only a single jurisdiction, there is no role for "voting with feet.") In particular, a public goods vector is defined as a majority rule winner if it is not defeated by any other vector with only one coordinate changed. While this definition of voting may seem restrictive in Slutsky's setting, it is less so here because of the institutional structure of the model. In particular, it is reasonable to assume that agents view local and federal policy decisions separately since these are administered by separate institutions and determined by separate votes. A more explicit institutional structure is required to provide the same justification for the assumption in Slutsky's model.

[12] This can be generalized to more general measure spaces.

and H denote the set of communities and house types respectively.) Points $n \in N$ can therefore represent both consumers and houses in the economy. More specifically, n is defined as the consumer endowed with house n. C_{ih} is the set of type h houses in community i, and C_i is the set of houses of all types in community i. Equivalently, C_{ih} is the set of consumers initially owning a house of type h in community i, whereas C_i is the set of consumers initially owning any house in community i. Agents therefore own one house in a continuum of houses.

In addition to owning a house, each consumer n is endowed with some positive quantity $z(n) \in [\underline{z}, \overline{z}] \subseteq R_{++}$ of the private good, which will be called "income", and a utility function $u^n: M \times H \times R_+^{M+2} \rightarrow R_+$. The utility function u^n is defined over: (i) i, the community the consumer resides in; (ii) h, the type of house the consumer owns (which is not necessarily the house n he was endowed with); (iii) $x \in R_+^{M+1}$, the public goods vector where x_0 is the amount of national public good and x_i ($i \in M$) is the amount of local public good produced in community i;[13] and (iv) z, the amount of private good. For any measurable $J \in \mathcal{N}$, $z(J) \equiv \int_J z(n)\, dn$ is the total amount of private good z initially owned by members of J. The following are maintained assumptions about utility functions:

<u>Assumption 1 (A1)</u>: All utility functions are continuous, monotone (if $z>0$) and strictly quasi-concave in the public goods x and the private good z.[14]

<u>Assumption 2 (A2)</u>: For all $(i,h),(j,h') \in M \times H$, $\forall\ x,x' \in R_+^{M+1}$, $\forall\ z>0$, $\forall n \in N$,

$$u^n(i,h,x,z) > u^n(j,h',x',0) \text{ and}$$

$$u^n(i,h,x,0) = u^n(j,h',x',0).$$

<u>Assumption 3 (A3)</u>: For all $(i,h),(j,h') \in M \times H$, $\forall\ x \in R_+^{M+1}$, there exists $b_{ih,jh'}>0$ such that

$$u^n(i,h,x,\overline{z}) < u^n(j,h',x,b_{ih,jh'}).$$ [15]

A1 is standard. A2 states that no amount of the public goods and no house can compensate consumers for not owning any private good. Furthermore, consumers are indifferent between communities and house types whenever they consume no private good. A3 says that each consumer is willing to relocate if the compensation in terms of the private good is high enough.

13 Note that this definition of utility functions allows for spillover effects.

14 Strict quasi-concavity can be weakened to just quasi-concavity with some additional work.

15 An example of a utility function satisfying these assumptions is: $u(i,h,x,z)=k_{ih}x_0^{\alpha}x_i^{\beta}z^{(1-\alpha-\beta)}$ ($0<\alpha,\beta<1$ and $\alpha+\beta<1$).

Finally, there is assumed to be a finite number of types of consumers, where types are distinguished by preferences, incomes and house endowments.

Assumption 4 (A4): There exists a finite measurable partition $(N_1,\ldots,N_A)$ of N such that for all $a \in A$, $n,n' \in N_a$,

$$(u^n, z(n)) = (u^{n'}, z(n')) \text{ and } n,n' \in C_{ih}.$$

Public Goods

Local and national public goods are produced from the private good. The technology sets for public good production are given by $Y \equiv \{Y_0, Y_1,\ldots,Y_M\}$, where Y_0 is the technology set for the national public good while Y_i, $i=1,\ldots,M$ is the technology set for the local public good in community i. Each $Y_i \subset \{(x_i, -z) \in R^2 \mid z \geq 0\}$ and admits a production function $f_i: R_+ \to R_+$ specified by

$$f_i(z) \equiv \max\{x_i \in R_+ \mid (x_i, -z) \in Y_i\}.$$

Y_i and f_i are assumed to satisfy the following:

Assumption 5 (A5): For all $i \in 0 \cup M$, Y_i is convex; f_i is a continuous, strictly increasing function of z for $i \in M$. Furthermore, there exists $\tilde{z} < z(N)$ s.t. $f_0(z) = f_0(z')$ for all $z, z' > \tilde{z}$ and f_0 is a continuous, strictly increasing function of z on $[0, \tilde{z}]$.

The first part of A5 is standard, while the second part states that at some input level below the economy's endowment of private good, the marginal product of z in the production of the national public good falls to zero. (This assumption is required in order to insure that income tax rates of 1 do not occur.) Also note that since Y_i is convex for all $i \in 0 \cup M$, f_i is concave.

We are now ready to formally define the hierarchical public goods economy:

Definition: $E = \{(N, \mathcal{N}, \mu), C, U, Y, z\}$ is a *hierarchical public goods economy with fixed jurisdictions* if

(i) $(N, \mathcal{N}, \mu)$ is a measure space with $N \subset R$ and μ the Lebesgue measure;

(ii) $C \equiv \{C_{ih} \in \mathcal{N} \mid i=1,\ldots,M;\ h=1,\ldots,H\}$ is a measurable partition of N such that

$$\mu\left(\bigcup_{h \in H} C_{ih}\right) > 0 \ \ \forall\ i=1,\ldots,M;$$

(iii) $U \equiv \{u^n : M \times H \times R_+^{M+2} \to R_+ \mid n \in N\}$;

(iv) $Y \equiv \{Y_0, Y_1,\ldots,Y_M\}$ where Y_i is a closed subset of

$$\{(x_i, -z) \in R^2 \mid z \geq 0\} \ \ \forall\ i=0,1,\ldots,M;$$

(v) $z: N \to [\underline{z}, \bar{z}] \subseteq (0, \infty)$ is a measurable function.

Prices

The private good z is the numeraire good and each house is assigned a price by a measurable price function p:N$\rightarrow R_+$. Since in equilibrium houses of the same type in the same community must have the same price, p is assumed to be constant on each C_{ih}, and the price of a type h house in community i is denoted p_{ih}. Thus, for every price function p there is an equivalent price vector p = $(p_{11},\ldots,p_{MH})$ and vice versa. (We will use both the vector and function notation.) The total value of all houses in $J_i \in \mathcal{N}$ is represented by $p(J_i) \equiv \int_{J_i} p(n)\,dn$. Often we will denote $p(n)$ as p_n.

Taxes

Production of national public goods is financed through proportional income taxes, while production of local public goods is financed through proportional property taxes.[16] Taxes are denoted $t \equiv (t_0,t_1,\ldots,t_M) \in R_+^{M+1}$ where t_0 is the national income tax rate and t_i, i=1,...,M is the local property tax rate in community i. Tax rates, however, have to be consistent with absolute majority rule voting by resident consumers, and budgets have to balance. (The meaning of majority rule voting is made precise below.) The balanced budget requirement implies that each government faces a set of alternatives that is linearly ordered and equal to the boundary of its production set. It further implies that voting over tax rates is equivalent to voting over public good levels, because each different tax rate determines uniquely a public goods level associated with this tax rate and vice versa.

Consumer Decision Problem

Consumers choose houses and communities and take tax rates $t=(t_0,t_1,\ldots,t_M)$, public good levels $x=(x_0,x_1,\ldots,x_M)$ and prices $p=(p_{11},p_{12},\ldots,p_{1H},p_{21},\ldots,p_{MH})$ as given. In other words, consumer n chooses i and h to maximize u^n over his budget set

$$B^n(p,t)=\{(i,h)\in M\times H \mid p_n+(1-t_0)\,z(n)\geq(1+t_i)\,p_{ih}\}.$$

His utility of choosing a house of type h in community i,

$$u^n(i,h,x,p_n+(1-t_0)\,z(n)-(1+t_i)\,p_{ih}),$$

will be denoted $u^n(i,h)$ when it is clear what x, t and p are. Finally, let S_{ih}:$R^{MH+2M+2}\rightarrow\rightarrow$N assign to every $(p,x,t)\in R^{MH+2M+2}$ the set of agents who can afford (i,h) and *strictly* prefer (i,h) to any other

[16] We follow Dunz (1985) in defining income for tax purposes to be the private good endowment. Thus, capital gains from house sales are not part of the national tax base.

house in any other community. Similarly, let $W_{ih}: R^{MH+2M+2} \to\to N$ assign to every $(p,x,t) \in R^{MH+2M+2}$ the set of agents who can afford (i,h) and *weakly* prefer (i,h) to all other houses in any other community. More formally,

$$S_{ih}(p,x,t) = \{n \in N \mid (i,h) \in B^n(p,t) \text{ and } u^n(i,h) > u^n(j,h') \quad \forall\, (j,h') \in B^n(p,t) \setminus \{(i,h)\}\}$$

$$W_{ih}(p,x,t) = \{n \in N \mid (i,h) \in B^n(p,t) \text{ and } u^n(i,h) \geq u^n(j,h') \quad \forall\, (j,h') \in B^n(p,t)\}.$$

Absolute Majority Rule Voting

We first need to define induced preference relations over taxes. Let Q^n denote n's induced preference relation over property tax rates in community i, where $t_i\, Q^n(i,h,p,x)\, t_i'$ means consumer n weakly prefers property tax rate t_i to t_i' given (i,h,p,x).[17] Formally define Q^n as follows:

<u>Definition</u>: Let $n \in N$, $(i,h,p,x) \in M \times H \times R_+^{MH+M+1}$ and $[\bullet] = [p_n + (1 - \frac{f_0^{-1}(x_0)}{z(N)})z(n) - p_{ih}]$. Let $x_{-i}(t_i) \equiv (x_0,\ldots,x_{i-1}, f_i(t_i\, p_i(C_i)), x_{i+1},\ldots,x_M)$. Then $t_i\, Q^n(i,h,p,x)\, t_i'$ if and only if

(i) $u^n(i,h,x_{-i}(t_i), [\bullet] - t_i\, p_{ih}) \geq u^n(i,h,x_{-i}(t_i'), [\bullet] - t_i'\, p_{ih})$ and $[\bullet] - t_i\, p_{ih} > 0$ and $[\bullet] - t_i'\, p_{ih} > 0$; or

(ii) $[\bullet] - t_i\, p_{ih} > 0$ and $[\bullet] - t_i'\, p_{ih} \leq 0$; or

(iii) $t_i \leq t_i'$ and $[\bullet] - t_i\, p_{ih} \leq 0$ and $[\bullet] - t_i'\, p_{ih} \leq 0$.

Thus, when (i) after tax private good levels are positive under both t_i and t_i', then $t_i\, Q^n(i,h,p,x)\, t_i'$ if utility (holding fixed location and other public good levels) under t_i is at least as great as under t_i'. If, on the other hand, after tax private good levels are negative under one of the tax rates, then $t_i\, Q^n(i,h,p,x)\, t_i'$ only if either (ii) after tax private good levels are positive under t_i or (iii) t_i is less than or equal to t_i'. Note that (ii) and (iii) are needed to define a complete preference ordering over tax rates.

Furthermore, let $r^n: M \times H \times R_+^{MH+M+1} \to\to R_+$ be defined as follows:

$$r^n(i,h,p,x) = \{t_i \in R_+ \mid t_i\, Q^n(i,h,p,x)\, t_i' \quad \forall\, t_i' \in R_+\}.$$

Thus, $r^n(i,h,p,x)$ is simply agent n's most preferred property tax rate given (i,h,p,x).

Similarly, let $Q_0^n(i,h,p,x)$ denote consumer n's induced preference relation over national income tax rates given (i,h,p,x):

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[17] Some readers have suggested that Q^n should not be dependent on h. In that case, however, preferences over tax rates would no longer be single peaked and it would become difficult to find conditions under which equilibria exist. We therefore make the standard though admittedly restrictive assumption that agents vote given h. (In equilibrium, of course, h is optimal and thus the two approaches are identical; they also would be if h is homogensous within C_i.)

Definition: Let $n \in N$, $(i,h,p,x) \in M \times H \times R_+^{MH+M+1}$ and let $[*] = [p_n + z(n) - (1 + \frac{f_i^{-1}(x_i)}{p(C_i)})\, p_{ih}]$. Let $x_{-0}(t_0) \equiv (f_0(t_0 z(N)), x_1, \ldots, x_M)$. Then $t_0\ Q_0^n(i,h,p,x)\ t_0'$ if and only if

(i) $u^n(i,h,x_{-0}(t_0), [*] - t_0 z(n)) \geq u^n(i,h,x_{-0}(t_0'), [*] - t_0' z(n))$ and $[*] - t_0 z(n) > 0$ and $[*] - t_0' z(n) > 0$; or

(ii) $[*] - t_0 z(n) > 0$ and $[*] - t_0' z(n) \leq 0$; or

(iii) $t_0 \leq t_0'$ and $[*] - t_0 z(n) \leq 0$ and $[*] - t_0' z(n) \leq 0$.

Also, let $t_0^n: M \times H \times R_+^{MH+M+1} \rightarrow\rightarrow R_+$ such that

$$t_0^n(i,h,p,x) = \{t_0 \in R_+ \mid t_0\ Q^n(i,h,p,x)\ t_0'\ \forall\ t_0' \in R_+\}.$$

$Q^n(i,h,p,x)$ and $Q_0^n(i,h,p,x)$ will be denoted Q^n and Q_0^n when (i,h,p,x) are understood. Now let $J \equiv \{J_{ih} \in \mathcal{N} \mid (i,h) \in M \times H\}$ partition N and let $J_i \equiv \bigcup_{h \in H} J_{ih}$ We can then formally define absolute majority rule voting in the spirit of Denzau and Parks (1975, 1983):

Definition: Fix $(p,x) \in R_+^{MH+M+1}$ and J. Then $t \in R_+^{M+1}$ is consistent with *absolute majority rule voting* if and only if

$$\text{(i)} \sum_{h \in H} \mu\{n \in J_{ih} \mid t_i\, Q^n(i,h,p,x)\, t_i'\} \geq \frac{\mu(J_i)}{2}\quad \forall\ t_i' \in R_+, \forall\ i \in M; \text{ and}$$

$$\text{(ii)} \sum_{i \in M} \sum_{h \in H} \mu\{n \in J_{ih} \mid t_0 Q_0^n(i,h,p,x)\ t_0'\} \geq \frac{\mu(N)}{2}\quad \forall\ t_0' \in R_+.$$

Also, $t_0 \in R_+$ is said to be consistent with majority rule voting if (ii) holds; and for all $i \in M$, $t_i \in R_+$ is said to be consistent with absolute majority rule voting if (i) holds.

Finally, we can formally define an equilibrium for this economy:

Definition: An *equilibrium* for the hierarchical public goods economy E is a list (J,p,x,t) where

(i) $J \equiv \{J_{ih} \in \mathcal{N} \mid i=1,\ldots,M;\ h=1,\ldots,H\}$ is a measurable partition of N;

(ii) $p: N \rightarrow R_+$ is measurable and constant on each C_{ih};

(iii) $x \in R_+^{M+1}$;

(iv) $t \in R_+^{M+1}$ such that

(E1) $\mu(J_{ih}) = \mu(C_{ih})\ \forall\ i \in M,\ h \in H;$

(E2) $x_0 = f_0(t_0 z(N));$

(E3) $x_i = f_i(t_i p\ (C_i))\ \forall\ i \in M\ ;$

(E4) $\forall\ (i,h) \in M \times H,\ \ S_{ih}(p,x,t) \subseteq J_{ih} \subseteq W_{ih}(p,x,t);$

(E5) $\sum_{i \in M} \sum_{h \in H} \mu\left\{n \in J_{ih} \mid t_0 Q_0^n(i,h,p,x)\ t_0'\right\} \geq \frac{\mu(N)}{2}\ \ \forall\ t_0' \in R_+\ ;$

(E6) $\sum_{h \in H} \mu\left\{n \in J_{ih} \mid t_i Q^n(i,h,p,x)\ t_i'\right\} \geq \frac{\mu(J_i)}{2}\ \ \forall\ t_i' \in R_+,\ \forall\ i \in M.$

E1 restricts the population in each community to the housing capacity of that community. E2 and E3 formalize the balanced budget requirement and E4 requires that consumers maximize utility. Finally, E5 and E6 specify local and national tax rates to be the result of majority rule voting.

A final assumption A6 is formally introduced later after the needed notation is developed. It states that the agent with lowest endowment $\underline{z}$ has sufficient private good left after paying the highest income tax bill to pay the highest property tax bill on a house with the lowest possible price.

III. Existence of an Equilibrium

Absolute majority rule equilibria

Single peakedness of preferences is a sufficient condition for the existence of majority rule voting equilibria. We thus show that preferences over tax rates (or equivalently preferences over public good levels) are single peaked along the dimensions that is voted on, both at the national and the local level.

Lemma 1: For all $(i,h,p,x) \in M \times H \times R_+^{MH+M+1}$, $\forall n \in N$, $t^n(i,h,p,x)$ is unique and Q^n is single peaked.

The proof to the next lemma is analogous to that of Lemma 1.

Lemma 2: For all $(i,h,p,x_i) \in M \times H \times R_+^{MH+1}$, $\forall\ n \in N$, $t_0^n(i,h,p,x)$ is unique and Q_0^n is single peaked.

Furthermore, A1 and the Maximum Theorem imply the following:

Lemma 3: The function $t^n(i,h,\cdot,\cdot)$ is continuous in p and x_0, and the function $t_0^n(i,h,\cdot,\cdot)$ is continuous in p and x_i.

Compactness

In order to apply Kakutani's Theorem, the domain and range of the fixed point mapping that will be defined shortly must be compact and convex. This requires the placing of bounds on the public goods set, the price set, and the tax rate set. Recall that by A5, there exists some maximal amount of national public good production $f_0(\tilde{z})$ where $\tilde{z}<z(N)$. Let the set of all possible public goods levels then be defined by $X = [0,f_0(\tilde{z})]\times[0,\overline{X}]^M$, where

$$\overline{X} = \max \{f_i(z(N)) \mid i \in M\}.$$

Next we need to bound house prices away from 0 in order to ensure finite t. Pick $\underline{P}\in R_{++}$ as this lower bound.[18] The upper bound $\overline{P}$ is then defined high enough so that all consumers would always prefer to own a house with price $\underline{P}$ to one with price $\overline{P}$. (This is similar to Dunz (1985).) By A3,

$$\forall\ (i,h),(j,h')\in M\times H,\ \exists\ b_{ih,jh'}\ \text{s. t.}\ \forall\ x\in\ \overline{X},\ \forall\ n\in N,$$
$$u^n(i,h,x,b_{ih,jh'}-\underline{P}) \geq u^n(j,h',x,\ \bar{z}).$$

Let $\bar{b}_{ih,jh'}=\min\{b_{ih,jh'}\}$ and define $\overline{P} \equiv 2\max\ \{\bar{z} \cup \{\bar{b}_{ih,jh'} \mid (i,h),(j,h')\in M\times H\}$. The price set is then $P=[\underline{P},\overline{P}]^{MH}$. Furthermore, note that, given $\underline{P}$, there exists $\overline{T}<\infty$ such that

$$t_i^n(i,h,p,x)\leq\overline{T}\quad \forall n\in N,\ \forall(i,h,p,x)\in M\times H\times P\times X.$$

Finally, let $\overline{T}_0 = \dfrac{\tilde{z}}{z\,(N)} <1$ be the upper bound for income taxes. The tax rate set is then defined by $T=[0,\overline{T}_0]\times[0,\overline{T}]$. By definition of these sets, for all $(p,x)\in P\times X$ and for all $n\in N$, t_i^n and t_0^n lie in $[0,\overline{T}]$ and $[0,\overline{T}_0]$ respectively. The following assumption on $\underline{z}$ is used in the proof of the next lemma:

<u>Assumption 6 (A6)</u>: $(1-\overline{T}_0)\underline{z} > \overline{T}\,\underline{P}$.

A6 simply states that the lowest income agent is able to afford to pay the highest possible property tax bill on a house of price $\underline{P}$ after paying the maximum income tax bill.

The definitions of the price and tax sets have the following useful property: [19]

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[18] Given that we rarely observe house prices less than or equal to 0, the notion of a positive lower bound on house prices does not strike us as unreasonable. Since the existence proof holds for any arbitrary lower bound on prices, this does illustrate, however, that different lower bounds on prices will lead to different equilibrium prices and possibly different equilibrium assignments of agents. When applying the model, the lower bound on prices must therefore be set carefully to reflect a realistic relation to incomes. This is done in the calibration of a computable general equilibrium version of the model in Nechyba (1994a,b,1996). Our aim for now, however, is merely to prove the general existence of an equilibrium.

[19] The proof is an adaptation of a proof in Dunz (1985).

Lemma 4: For all $(p,x,t)\in P\times X\times T$, if $p_{ih}=\underline{P}$ and $p_{jh'}=\overline{P}$, then $\forall\ n\in N$, either $u^n(i,h) > u^n(j,h')$ and/or $(j,h')\notin B^n(p,t)$.

Nonemptiness

So far, we have defined the compact and convex space of prices, public goods levels and tax rates $P\times X\times T$. In order for the fixed point correspondence defined below to be nonempty, there needs to exist a measurable partition $\{J_{11},\ldots,J_{MH}\}$ of N for every $(p,x,t)\in P\times X\times T$ such that for all $(i,h)\in M\times H$, $S_{ih}(p,x,t)\subseteq J_{ih}\subseteq W_{ih}(p,x,t)$. This does not hold since there may be combinations of prices and taxes that cause budget sets for some agents to be empty. For this reason, we define a transformed economy E' for every hierarchical public goods economy E. This transformed economy E' will be identical to E except for the addition of an empty community C_{M+1} (of measure zero), and an extension of preferences to include this new community. We will then prove that there exists an equilibrium for E' which will be easily translated to an equilibrium in E.

Definition: Let $E = \{(N,\mathcal{N},\mu),C,U,Y,z\}$ be a hierarchical public goods economy with fixed jurisdictions. Then $E' = \{(N,\mathcal{N},\mu),C',U',Y',z\}$ is a *transformed hierarchical public goods economy with fixed jurisdictions* if

(i) $C'\equiv\{C'_{ih}\in\mathcal{N}\mid C'_{ih}=C_{ih}\ \forall\ (i,h)\in M\times H$ and $C_{(M+1)h}=\varnothing\ \forall h\in H\}$;

(ii) $U'\equiv\{u'^n:\{M\cup\{M+1\}\}\times H\times R_+^{M+3}\to R_+ \mid n\in N\}$ where

$$\forall\ n\in N,\ \forall\ i\in M,\ \forall\ x\in R_+^{M+1}\times\{0\},\ \forall\ h,h''\in H,\ \forall\ z,z''\in R_+,$$
$$u'^n(i,h,x,z) = u^n(i,h,x,z) \text{ and}$$
$$u^n(i,h,x,0) = u'^n(M+1,h'',x_0,\ldots,x_M,0,z'') = \underline{u}^n.$$

(iii) $Y'\equiv Y\cup\{Y_{M+1}\}$ where $Y_{M+1}=\{(0,-z)\in R^2 \mid z\geq 0\}$.

Note that agents strictly prefer all other communities over M+1 so long as private good consumption in the other communities is strictly positive.[20] Furthermore, there is no production technology for public goods in the new community.

House prices, local public good levels and property taxes will be restricted to zero in community M+1. Thus, for all p,t_0, and h, $u^n(M+1,h) = \underline{u}^n$. Also note that this implies

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[20] The definitions of S_{ih} and W_{ih} can be extended straightforwardly to include the additional community.

$$(M+1,h) \in B^n(p,t) \;\forall\, h \in H,\; \forall\, n \in N,\; \forall\, (p,t) \in P \times T.$$

We can then extend the definition of an equilibrium:

Definition: An *equilibrium* for the transformed hierarchical public goods economy E' is a list (J',p',x',t') where

(i) $J' \equiv \{J'_{ih} \in \mathcal{N} \mid (i,h) \in (M \cup \{M+1\}) \times H\}$ is a partition of N;

(ii) $p': N \to R_+$ is measurable and constant on each C_{ih} and $p'(\varnothing)=0$;

(iii) $x' \in R_+^{M+2}$;

(iv) $t' \in R_+^{M+2}$ such that

(E1') $\mu(J'_{ih}) = \mu(C_{ih}) \;\forall\, i \in M \cup \{M+1\},\, h \in H$;

(E2') $x'_0 = f_0(t'_0 z(N))$;

(E3') $x'_i = f_i(t'_i p'(C_i)) \;\forall\, i \in M \cup \{M+1\}$;

(E4') $\forall\, (i,h) \in (M \cup \{M+1\}) \times H,\; S_{ih}(p',x',t') \subseteq J'_{ih} \subseteq W_{ih}(p',x',t')$;

(E5') $\displaystyle\sum_{i \in M \cup \{M+1\}} \sum_{h \in H} \mu\{n \in J'_{ih} \mid t'_0 Q_0^n(i,h,p',x')\, \hat{t}_0\} \geq \frac{\mu(N)}{2} \;\forall\, \hat{t}_0 \in R_+$;

(E6') $\displaystyle\sum_{h \in H} \mu\{n \in J'_{ih} \mid t'_i Q^n(i,h,p',x')\, \hat{t}_i\} \geq \frac{\mu(J'_i)}{2} \;\forall\, \hat{t}_i \in R_+,\; \forall\, i \in M \cup \{M+1\}$.

Note that since $p'(C_{M+1}) = 0$, $x'_{M+1} = 0 = t'_{M+1}$. It is easily seen that if (J',p',x',t') is an equilibrium for E', then $(J,p,x,t) = (\{J'_{11},\dots,J'_{MH}\}, p', (x'_0,x'_1,\dots,x'_M), (t'_0,t'_1,\dots,t'_M))$ is an equilibrium for the economy E. We will proceed by defining a correspondence whose fixed point is an equilibrium for the economy E', which will then translate into an equilibrium for the economy E.

Fixed Point Correspondence

First we extend the price, public good and tax sets restricting all coordinates for community M+1 to 0; i.e. we let $\tilde{P} \equiv P \times [0]^H$, $\tilde{X} \equiv X \times [0]$ and $\tilde{T} \equiv T \times [0]$. Furthermore, let v_{ih}^a denote the measure of agents of type a assigned to a house of type h in community i. Then the set of all possible assignments of agents is

$$V \equiv \{v \in R_+^{(M+1)HA} \mid \sum_{i \in M \cup \{M+1\}} \sum_{h \in H} v_{ih}^a = \mu(N_a) \;\forall\, a \in A\}.[21]$$

[21] By including v as an argument in the utility function, the model can easily be expanded to include population

Thus, define

$$\mathcal{J}(v) \equiv \{(J_{11},\ldots,J_{(M+1)H}) \mid J_{ih} \in \mathcal{N} \text{ and } \mu\{J_{ih} \cap N_a\} = v_{ih}^a \;\; \forall\ (i,h,a) \in (M \cup \{M+1\}) \times H \times A\}.$$

The following lemma shows that this set is always nonempty.

Lemma 5: For all $v \in V$, $\mathcal{J}(v) \neq \varnothing$.

Thus, for each $v \in V$ we can form at least one partition $J(v)$ of N such that and $\mu\{J_{ih}(v) \cap N_a\} = v_{ih}^a$. Since v is the same for all members of $\mathcal{J}(v)$, all $J(v)$, $J'(v) \in \mathcal{J}(v)$ are equivalent in the sense that J_{ih}, J'_{ih} contain the same measure of each consumer type. Note that $\forall\ (i,h,p,x) \in M \times H \times P \times X$, $\forall\ a \in A$, $\forall\ n,n' \in N_a$, $t^n(i,h,p,x) = t^{n'}(i,h,p,x)$ and $Q^n(i,h,p,x) = Q^{n'}(i,h,p,x)$. Therefore define $Q^a(i,h,p,x) = Q^n(i,h,p,x)$ and $t^a(i,h,p,x) = t^n(i,h,p,x)$ for $n \in N_a$.

We now define a mapping $\xi: V \times \widetilde{P} \times \widetilde{X} \times \widetilde{T} \rightarrow V \times \widetilde{P} \times \widetilde{X} \times \widetilde{T}$ and note that by construction, $V \times \widetilde{P} \times \widetilde{X} \times \widetilde{T}$ is convex, compact and non-empty, and by Lemma 5, $\mathcal{J}(\hat{v}) \neq \varnothing$:

$\xi(v,p,x,t) = \{ (\hat{v},\hat{p},\hat{x},\hat{t}) \in V \times \widetilde{P} \times \widetilde{X} \times \widetilde{T} \mid$

(i) $\forall\ (i,h) \in (M \cup \{M+1\}) \times H$, $S_{ih}(p,x,t) \subseteq J_{ih}(\hat{v}) \subseteq W_{ih}(p,x,t)$ for $J(\hat{v}) \in \mathcal{J}(\hat{v})$;

(ii) $\forall\ (i,h) \in M \times H$, $\forall\ p'_{ih} \in [\underline{P},\overline{P}]$, $\hat{p}_{ih}((\sum_{a \in A} v_{ih}^a) - \mu(C_{ih})) \geq p'_{ih}((\sum_{a \in A} v_{ih}^a) - \mu(C_{ih}))$ and $\forall\ h \in H$, $\hat{p}_{(M+1)h} = 0$;

(iii) $\hat{x}_{M+1} = 0$ and $\forall\ i \in M$, $\hat{x}_i = f_i(t_i p(C_i))$;

(iv) $\hat{x}_0 = f_0(t_0 z(N))$;

(v) $\hat{t}_{M+1} = 0$ and $\forall\ i \in M$, $\forall\ t'_i \in [0,\overline{T}]$, $\left\{ \sum_{a \in A} \sum_{h \in H} v_{ih}^a \mid \hat{t}_i\ Q^a(i,h,p,x)\ t'_i \right\} \geq \dfrac{\sum_{a \in A} \sum_{h \in H} v_{ih}^a}{2}$;

(vi) $\forall\ t'_0 \in [0,\overline{T}_0]$, $\left\{ \sum_{a \in A} \sum_{i \in M} \sum_{h \in H} v_{ih}^a \mid \hat{t}_0\ Q_0^a(i,h,p,x)\ t'_0 \right\} \geq \dfrac{\mu(N)}{2} \}$.

The next two lemmas state that any fixed point is an equilibrium in the transformed economy and that the fixed point correspondence is nonempty.

Lemma 6: If $(v,p,x,t) \in \xi(v,p,x,t)$, then $(\mathcal{J}(v),p,x,t)$ is an equilibrium of E'.

Lemma 7: $\xi(v,p,x,t) \neq \varnothing\ \ \forall\ (v,p,x,t) \in V \times \widetilde{P} \times \widetilde{X} \times \widetilde{T}$.

externalities. All proofs will go through with some additional notation.

Upper hemi-continuity and Convexity

As in Dunz (1985), the following lemma is used to prove upper hemi-continuity of ξ:

Lemma 8: For all sequences $\{(p^k,x^k,t^k)\}_{k=1}^{\infty}$ with $(p^k,x^k,t^k)\in \widetilde{P}\times\widetilde{X}\times\widetilde{T}$ and $\lim_{k\to\infty}(p^k,x^k,t^k) = (p,x,t)\in \widetilde{P}\times\widetilde{X}\times\widetilde{T}$, $\exists\ \bar{k}<\infty$ such that $\forall\ k\geq\bar{k}$

$$\forall\ (i,h)\in(M\cup\{M+1\})\times H,\ S_{ih}(p,x,t)\subseteq S_{ih}(p^k,x^k,t^k) \text{ and } W_{ih}(p^k,x^k,t^k)\subseteq W_{ih}(p,x,t).$$

Lemma 9: ξ is upper hemi-continuous.

Lemma 10: $\xi(\bar{v},\bar{p},\bar{x},\bar{t})$ is convex for all $(\bar{v},\bar{p},\bar{x},\bar{t})\in V\times\widetilde{P}\times\widetilde{X}\times\widetilde{T}$.

Existence

Theorem: If an economy E satisfies assumptions A1–A6, there exists an equilibrium.

Proof: As noted earlier, $V\times\widetilde{P}\times\widetilde{X}\times\widetilde{T}$ is constructed to be a compact, convex subset of $R_+^{2((M+1)H+M+2)}$. By Lemmas 7, 9 and 10, $\xi(v,p,x,t)$ is non-empty, upper hemi-continuous and convex valued. Thus, Kakutani's Theorem implies there exists $(v',p',x',t')\in V\times\widetilde{P}\times\widetilde{X}\times\widetilde{T}$ such that $(v',p',x',t')\in\xi(v',p',x',t')$. Lemma 6 states that $(J(v')p',x',t')$ is an equilibrium for the transformed economy E' which implies $(J,p,x,t) = (\{J'_{11},\ldots,J'_{MH}\},p',(x'_0,x'_1,\ldots,x'_M),(t'_0,t'_1,\ldots,t'_M))$ is an equilibrium for the original economy E.

Q.E.D.

IV. Stratified Equilibria

As mentioned in the introduction, both Westhoff (1977) and EFR (1993) use single crossing assumptions to find "stratified equilibria". EFR, for example, assume that the marginal rate of substitution at any point in the public good/price space is a continuous function of wealth which implies that, since all agents have identical preferences, indifference curves of individuals with differing incomes cross only once. This yields an equilibrium in which agents stratify into communities based on both their wealth and their marginal willingness to pay for the local public good. A sufficient condition for this single crossing assumption is that wealth is entirely exogenous, a condition which is satisfied in EFR because land is owned by absentee landlords. In our closed general equilibrium model, however, each agent owns a house whose equilibrium value is determined endogenously. This implies that, despite the exogenous endowments of private good, private wealth is endogenous to the model. Additional complexity is added by the possibility of a large number of different types of agents endowed with different preferences, as well as a large number of heterogeneous house types. Defining notions of stratification in our model therefore becomes more challenging than in the previous framework. We will define several such notions and then proceed to find sufficient conditions for these to hold in equilibrium.

First, an equilibrium is *preference stratified by house type across communities* if agents who live in the same house type sort themselves across communities by their equilibrium marginal willingness to pay for discrete amounts of the local public good. Similarly, we say that an equilibrium satisfies *preference stratification across communities* if agents sort themselves into communities by their equilibrium marginal willingness to pay for discrete amounts of the local public good.[22]

Next, an equilibrium satisfies *wealth stratification by house type* if the ordering of agents is the same whether we order them by the tax inclusive price of their equilibrium home or by their after tax endowments. Thus, in an equilibrium that is wealth stratified by house type, agents sort themselves into house types by their after tax endowments. Similarly, we say that an equilibrium is *wealth stratified by community* if agents sort themselves into communities by their after tax endowments. Finally, we say an equilibrium satisfies *complete stratification* if agents sort themselves into communities by both their equilibrium marginal willingness to pay for discrete amounts of the local public good *and* their after tax endowments in such a way that wealthy communities produce more local public goods than poor communities.

To define these notions more formally, we need the following additional notation. If $x_i<x_j$, let $MWTP^n_{ij}(i,h,p,x,t)$ be agent n's marginal willingness to pay for (x_j-x_i) *additional* units of the local public good when residing in (i,h) and facing (p,x,t). Similarly, if $x_i>x_j$, let $MWTP^n_{ij}(i,h,p,x,t)$ be agent n's marginal willingness to pay for the *last* (x_i-x_j) units of the local public good when residing in (i,h) and facing (p,x,t). More formally, let $\bar{z}(n,i,h,p,t) = (1+t_0)z(n)+p_n-(1+t_i)p_{ih}$ and $x(i,j) = (x_0,x_1,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_M)$. Then

$$MWTP^n_{ij}(i,h,p,x,t) \equiv \{|c| \in R_+ \mid u^n(i,h,x,\bar{z}(n,i,h,p,t)) = u^n(i,h,x(i,j),\bar{z}(n,i,h,p,t)-c)\}.$$[23]

[22] Note that stratification here means that agents sort themselves according to their marginal willingness to pay *evaluated at the equilibrium (J,p,x,t)*, which is a weaker notion of stratification than that employed in Westhoff and EFR who assume an initial ordering of agents by their MRS. As argued before, it is impossible to make such an assumption when wealth is endogenous and preferences are not identical.

Definition: An equilibrium (J,p,x,t) for an economy E is

(i) *preference stratified by house type across communities* if

$\forall i,j$ s.t. $x_i<x_j$, $\forall\ h\in H$, $\forall\ n\in J_{ih}$, $n'\in J_{jh}$,

$$MWTP^{n}_{i,j}\ (i,h,p,x,t) \le MWTP^{n'}_{j,i}\ (j,h,p,x,t).$$

(ii) *preference stratified across communities* if

$\forall i,j$ s.t. $x_i<x_j$, $\forall\ h,h'\in H$, $\forall\ n\in J_{ih}$, $\forall\ n'\in J_{jh'}$,

$$MWTP^{n}_{i,j}\ (i,h,p,x,t) \le MWTP^{n'}_{j,i}\ (j,h',p,x,t).$$

(iii) *wealth stratified by house type* if $\forall i,j\in M$, $\forall\ h,h'\in H$,

$(1+t_i)p_{ih} < (1+t_j)p_{jh'} \Rightarrow \forall\ n\in J_{ih}$, $\forall\ n'\in J_{jh'}$, $(1+t_0)z(n) + p_n \le (1+t_0)z(n') + p_{n'}$.

(iv) *wealth stratified by community* if (iii) holds and if $\forall i,j\in M$,

$(1+t_i)p_{ih} < (1+t_j)p_{jh'}$ for some $h,h'\in H \Rightarrow$

$$\max \{(1+t_i)p_{ih''} \mid h''\in H\} \le \min \{(1+t_j)p_{jh''} \mid h''\in H\}.$$

(v) *completely stratified* if $\forall i,j$ s.t. $x_i<x_j$, $\forall\ h,h'\in H$, $\forall\ n\in J_{ih}$, $\forall\ n'\in J_{jh'}$,

$MWTP^{n}_{i,j}\ (i,h,p,x,t) \le MWTP^{n'}_{j,i}\ (j,h,p,x,t)$ and $(1+t_0)z(n)+p_n \le (1+t_0)z(n')+ p_{n'}$.

Our notion of stratification is, therefore, different from that of EFR in that it is based on equilibrium marginal willingness to pay rather than an initial ordering of agents by marginal rates of substitutions. We note in passing that if all agents had identical preferences (as in EFR) and the quality of an agent's house endowment was an increasing function of his income, we could impose the same kind of ordering on agents and thus achieve a result analogous to the EFR notion of stratification. This is because the ordering would then be the same regardless of whether we ordered agents by their income endowments or their equilibrium wealth levels.

Preference Stratification

Next we state two further assumptions. A7 assumes that all communities are endowed with some positive measure of each house type. A8 states that agents are indifferent between living in the same type of house in different communities whenever the local public good levels are the same and they consume the same amount of private good in both communities. In other words, A8 disallows spillover effects as well as intrinsic differences (like beauty and air-quality) between communities.

Assumption 7 (A7): For all $(i,h)\in M\times H$, $\mu(C_{ih}) \neq \varnothing$.

[23] Note that by the earlier assumptions on utility functions, this exists and is unique.

Assumption 8 (A8): For all $n \in N$, $\forall$ $i,j \in M$, $\forall$ $h \in H$, $\forall$ $z \in R_+$, $\forall$ x,x' s.t. $x_i = x'_j$,

$$u^n(i,h,x,z) = u^n(j,h,x',z).$$

Theorem 2: Let (J,p,x,t) be an equilibrium for an economy E satisfying A1–A8. Then (J,p,x,t) is *preference stratified by house type across communities.*

Proof: Let (J,p,x,t) be an equilibrium for an economy satisfying A1–A8.

(i) Claim: For all $h \in H$, $\forall$ $i,j \in M$ such that $x_i < x_j$, $(1+t_i)p_{ih} < (1+t_j)p_{jh}$.

Suppose not; i.e. suppose $(1+t_i)p_{ih} \geq (1+t_j)p_{jh}$ for some $h \in H$. Then by A8 and the monotonicity of u^n (A1),

$$u^n(i,h,x,(1+t_0)\,z(n) + p_n - (1+t_i)p_{ih}) < u^n(j,h',x,(1+t_0)\,z(n) + p_n - (1+t_j)p_{jh}) \quad \forall\, n \in N,$$

which contradicts (J,p,x,t) being an equilibrium.

(ii) The rest of the theorem then follows straightforwardly from this claim. Note that since the only aspect of a community he cares about is its local public good level (by A8), agent n could purchase $(x_j - x_i)$ additional units at a price of $(1+t_j)p_{jh}-(1+t_i)p_{ih}$ (which is greater than 0 by the previous claim) by migrating to (j,h'). Similarly, agent n' could "sell" $(x_j - x_i)$ of his local public good at the same price. Since (J,p,x,t) is an equilibrium,

$$MWTP^n_{i,j}\ (i,h,p,x,t) \leq (1+t_j)p_{jh}-(1+t_i)p_{ih} \leq MWTP^{n'}_{j,i}\ (j,h,p,x,t).$$

Q.E.D.

The following is immediate from the theorem above:

Corollary 2: Let (J,p,x,t) be an equilibrium for an economy E satisfying A1–A8. Then if $|H|=1$, (J,p,x,t) is preference stratified by community.

Theorem 2 states that if each house type is available in every community, if there are no spillover effects and if agents place no intrinsic value on a particular community, then agents living in the same house type will sort themselves across communities according to their equilibrium marginal willingness to pay for discrete amounts of the local public good. Note that we made no ordering assumption of the Westhoff or EFR kind to get this stratification result which is a direct consequence of the equilibrium price structure (see the claim in the proof to Theorem 2). Furthermore, note that under the relatively weak conditions of Theorem 2, we get a partially stratified equilibrium where the heterogeneity of house types accounts for differences among agents *within* communities in their marginal willingness to pay. Only by removing the heterogeneity of houses, i.e. only by assuming homogeneity of housing, do we get complete stratification closer to EFR (Corollary 2), the kind of stratification Epple and Platt (1992) try to remove from the EFR framework by introducing a taste parameter. Here, the heterogeneity of the land market produces heterogeneity in equilibrium marginal willingness to pay for the local public good within communities. Equilibrium prices for houses, however, place strict limits on the degree to which agents in communities differ in terms of their marginal willingness to pay since agents who live in the same house type segregate themselves into communities according to their preferences for the local public good.[24]

Wealth Stratification

We proceed by defining the "normality" of a house: Suppose an agent prefers house (j,h') to house (i,h) given public goods vector x despite the fact that he has less private good in (j,h') than in (i,h). Then we say that (j,h') is *normal* if, when the agent gets an additional amount c in both locations, he still prefers (j,h') to (i,h). More formally:

Definition: A house h' in community j is *normal* if $\forall$ $n\in N$, $\forall$ $x\in X$, $\forall$ $z>z'$, $\forall$ $c>0$, $\forall$ $(i,h)\neq(j,h')$,

$$u^n(i,h,x,z) \leq u^n(j,h',x,z') \Rightarrow u^n(i,h,x,z+c) < u^n(j,h',x,z'+c).$$ [25]

Finally we state A9 which says that all agents are endowed with identical preferences.

Assumption 9 (A9): For all $n,n'\in N$, $\forall$ $(i,h)\in M\times H$, $\forall$ x,z,

$$u^n(i,h,x,z) = u^{n'}(i,h,x,z).$$

Theorem 3: Let (J,p,x,t) be an equilibrium for an economy E satisfying A1–A6 and A9. If houses are normal, then (J,p,x,t) is wealth stratified by house type.

Proof: Let (J,p,x,t) be an equilibrium. Pick $(i,h),(j,h')\in M\times H$ such that $(1+t_i)p_{ih} < (1+t_j)p_{jh'}$ and let $n\in J_{ih}$, $n'\in J_{jh'}$. Define

$$c \equiv (1-t_0)\,(z(n)-z(n')) + p_n - p_{n'}$$

$$z' \equiv (1-t_0)\,z(n') + p_{n'} - (1+t_j)p_{jh'}$$

[24] Note that, at the present time, this intra-community heterogeneity of houses is exogenously given and can be thought of as the result of some history. An interesting question raised by this is to what extent, when histories are treated endogenously, intra-community heterogeneity of housing would arise. A dynamic version of this model that endogenizes the housing stock is required to answer this question.

[25] A similar definition appears as assumption (E) in Kaneko (1983).

$$z \equiv (1-t_0)\, z(n') + p_{n'} - (1+t_i)p_{ih}$$

Suppose $(1-t_0)z(n) + p_n > (1-t_0)z(n') + p_{n'}$. Note that this implies c>0 and $(i,h),(j,h') \in B^{n'}(p,t) \subseteq B^n(p,t)$. This further implies z>0 and z'>0. Finally note z>z'. By A9 we can let $u \equiv u^n = u^{n'}$. Since (J,p,x,t) is an equilibrium, $n \in W_{ih}(p,x,t)$ and $n' \in W_{jh'}(p,x,t)$ and therefore

$$u(i,h,x,z) \leq u(j,h',x,z')$$
$$u(i,h,x,z+c) \geq u(j,h',x,z'+c)$$

which contradicts the normality of (i,h).

Q.E.D.

The following is trivial given Theorem 3:

Corollary 3: Let (J,p,x,t) be an equilibrium for an economy E satisfying A1–A6 and A9. If houses are normal and $|H|=1$, then (J,p,x,t) is wealth stratified by community

Theorem 3 states that if preferences are identical and houses are normal, agents will sort themselves into house types by their post-tax endowments. In other words, if preferences are identical as in EFR, poor residents will segregate themselves from wealthy residents within communities. If we further assume that there is only one house type in the economy (Corollary 3), then agents are stratified across communities by their post tax endowments.

Complete Stratification

Theorem 4: Let (J,p,x,t) be an equilibrium for an economy E satisfying A1–A6, A8-A9 and $|H| = 1$. Then if houses are normal, (J,p,x,t) is completely stratified.

Proof: Let $H=\{h\}$ and note that $|H| = 1$ implies A7 is satisfied. Then Corollary 2 implies (J,p,x,t) is preference stratified across communities; i.e.

$$x_i < x_j \Rightarrow \forall\, n \in J_i,\ \forall\, n' \in J_j,\ MWTP^{n}_{i,j}\ (i,h,p,x,t) \leq MWTP^{n'}_{j,i}\ (j,h,p,x,t)$$

Corollary 3 then implies (J,p,x,t) is wealth stratified by community. Furthermore, by the claim in the proof to Theorem 2, if $x_i < x_j$, then $(1+t_i)p_{ih} < (1+t_j)p_{jh}$. Thus,

$$x_i < x_j \Rightarrow \forall\, n \in J_i,\ \forall\, n' \in J_j,\ (1+t_0)z(n) + p_n \leq (1+t_0)z(n') + p_{n'}.$$

Q.E.D.

Theorem 4 combines the previous two corollaries to state that if preferences are identical, if there are no spillover effects and no intrinsically valuable characteristics of communities, if houses are normal and if there is only one house type in the economy, then agents sort themselves into communities by both their equilibrium marginal willingness to pay and by their post-tax endowments, and wealthy communities produce a higher local public good level than poor communities. Note that these restrictive conditions required to ensure stratification are analogous to some of those used by EFR, although none of the their more severe restrictions on preferences are necessary.

V. Conclusion

In this paper, we have proved the existence of an equilibrium for an economy with local and national governments, majority rule voting over local property and national income taxes, and mobile agents who are endowed with houses. In doing so, we have not been required to make overly restrictive assumptions about preferences or utility functions, and have made use of the structure of the model to find a political equilibrium on both the local and the national level. Furthermore, we have demonstrated conditions under which equilibria will be stratified in several senses: under different conditions, agents are shown to segregate themselves by wealth endowment, marginal willingness to pay for the local public good, or both, and market prices for houses are shown to play an important role in producing heterogeneity among agents within communities.

This model is used elsewhere to take a fresh look at issues in fiscal federalism and local public finance. It offers empirically verifiable predictions on the political effects of intergovernmental grants and provides an explanation for the unresolved puzzle of the "flypaper effect" (Nechyba (1994a). Furthermore, it explains why local communities tend to use property rather than income in their tax bases and how national grant programs may arise as a coordination mechanisms for incorporating income into local bases (Nechyba (1994b)), and it facilitates an evaluation of different types of national policies in terms of equity and efficiency goals (Nechyba (1996)). Finally, the model presented here lays the foundation for the first parameterized computable general equilibrium model of local public finance and fiscal federalism found in the literature (Nechyba (1994a,b,1996)).

Finally, future work may generalize the framework in a variety of directions and thus provide further theoretical foundations to applied analysis. The addition of a dynamic dimension would facilitate an investigation of the endogenous development of jurisdictions and their housing stocks.

This could shed light on the emergence of jurisdiction boundaries and zoning regulations as well as facilitate explanations for the recent trend of school district consolidations in some states. Expanding the model to include an intermediate "state" level or additional local governments seems technically feasible and may shed light on the evolution of the complex web of local and state governments in the US. The addition of commercial and industrial players could result in a new investigation of strategic community behavior aimed at attracting these players to the local tax base. Finally, the often used local revenue enhancing system of user fees could be added and analyzed in the Tiebout context.

VI. Appendix

Proof of Lemma 1: Fix $n \in N$, $(i,h,p,x) \in M \times H \times R_+^{MH+M+1}$ and let $\tilde{p} = p_i(C_i)$. Let $t_i, t_i' \in R_+$ such that $t_i Q^n(i,h,p,x)\, t_i'$ and $t_i'\, Q^n(i,h,p,x)\, t_i$. Let $[\bullet] = [p_n + (1 - \frac{f_0^{-1}(x_0)}{z(N)})z(n) - p_{ih}]$. By the definition of Q^n, $t_i Q^n t_i'$ and $t_i' Q^n t_i$ implies that either

(i) $u^n(i,h,x_{-i}(t_i), [\bullet] - t_i p_{ih}) = u^n(i,h,x_{-i}(t_i'), [\bullet] - t_i' p_{ih})$ and
$[\bullet] - t_i p_{ih} > 0$ and $[\bullet] - t_i' p_{ih} > 0$; or

(ii) $t_i' = t_i$ and $[\bullet] - t_i p_{ih} \le 0$.

Let $t^\alpha \equiv \alpha t_i + (1-\alpha)t_i'$ for $\alpha \in [0,1]$. First assume (i). To show that t^n is unique, we show that t^α is strictly preferred to both t_i and t_i'. By strict quasi-concavity of $u^n(i,h,\cdot,\cdot,\cdot)$ (A1),

$$(1.1) \qquad u^n(i,h,x_0,x_1,\ldots,x_{i-1}, \alpha f_i(t_i\tilde{p}) + (1-\alpha)f_i(t_i'\tilde{p}), x_{i+1},\ldots x_M, [\bullet] - t^\alpha p_{ih}) >$$
$$u^n(i,h,x_{-i}(t_i), [\bullet] - t_i p_{ih}) = u^n(i,h,x_{-i}(t_i'), [\bullet] - t_i' p_{ih}).$$

A5 implies f_i is concave which implies

$$(1.2) \qquad f_i(t^\alpha \tilde{p}) = f_i((\alpha t_i + (1-\alpha)\, t_i')\tilde{p}) \ge \alpha f_i(t_i\tilde{p}) + (1-\alpha)f_i(t_i'\tilde{p}).$$

Next (1.1), (1.2) and monotonicity of u^n (A1) imply

$$u^n(i,h,x_{-i}(t^\alpha), [\bullet] - t^\alpha p_{ih}) > u^n(i,h,x_{-i}(t_i), [\bullet] - t_i p_{ih}),$$
$$u^n(i,h,x_{-i}(t^\alpha), [\bullet] - t^\alpha p_{ih}) > u^n(i,h,x_{-i}(t_i'), [\bullet] - t_i' p_{ih}).$$

$\Rightarrow t^n(i,h,p,x)$ is unique. Now assume (ii). Then $t_i = t_i'$. Thus, $t^n(i,h,p,x_0)$ is again unique. It follows straightforwardly from this proof that Q^n is single peaked.

Q.E.D.

Proof of Lemma 4: If $p_n < (1+t_j)\overline{P} - (1-t_0)z(n)$, then $(j,h') \notin B^n(p,t)$. $\Rightarrow$ If $p_n < \overline{P} - \overline{z}$, then $(j,h') \notin B^n(p,t)$. Thus, we only need to check $p_n \in [\overline{P} - \overline{z}, \overline{P}]$. Fix $(x,t) \in X \times T$. Note that A6 $\Rightarrow$ $(1-t_0)z(n) \ge t_j\underline{P}$. Then by monotonicity, (4.1) $$u^n(i,h,x,p_n-\underline{P}) \le u^n(i,h,x,(1-t_0)z(n)+p_n-(1+t_i)\underline{P}) = u^n(i,h) \text{ and}$$

(4.2) $$u^n(j,h',x,\bar{z}+p_n-\overline{P}) \ge u^n(j,h',x,(1-t_0)z(n)+p_n-(1+t_j)\overline{P}) = u^n(j,h').$$

Suppose $\overline{P} \ge p_n > \frac{\overline{P}}{2}$ and $(j,h')\in B^n(p,t)$. Then, by the definition of $\overline{P}$, $p_n > \bar{b}_{ih,jh'}$ and by monotonicity,

$$u^n(i,h,x,p_n-\underline{P}) > u^n(i,h,x,\bar{b}_{ih,jh'}-\underline{P}) \ge u^n(j,h',x,\bar{z}) \ge u^n(j,h',x,\bar{z}+p_n-\overline{P}).$$

Thus, by (4.1) and (4.2), $u^n(i,h) > u^n(j,h')$.

Now suppose $\frac{\overline{P}}{2} \ge p_n \ge \overline{P}-\bar{z}$. Then $\bar{z} \ge \frac{\overline{P}}{2} \Rightarrow \bar{z} \ge b_{ih,jh'}$. Then by definition, $\overline{P} = 2\bar{z} \Rightarrow p_n = \bar{z}$ $\Rightarrow \bar{z}+p_n-\overline{P} = 0$. If $t_0>0$ and/or $t_j>0$,

$$(1-t_0)z(n) + p_n - (1+t_j)\overline{P} < 0 \Rightarrow (j,h')\notin B^n(p,t).$$

If $t_0=t_j=0$, then by A2,

$$u^n(i,h,x,p_n-(1+t_i)\underline{P})) > u^n(j,h,x,\bar{z}+p_n-\overline{P}) \Rightarrow u^n(i,h) > u^n(j,h').$$

Q.E.D.

Proof of Lemma 5: For all $a\in A$, $D\in \mathcal{N}$, let $\gamma_a(D) \equiv \mu(D\cap N_a)$. Then $\gamma_1,\dots,\gamma_A$ are A nonatomic measures on $(N,\mathcal{N})$ which implies by Theorem 1 in Dubins and Spanier (1961)[26] that the set of matrices

$$\left\{\begin{bmatrix} \gamma_1(J_{11}) & \cdots & \gamma_1(J_{(M+1)H}) \\ \vdots & \ddots & \vdots \\ \gamma_A(J_{11}) & \cdots & \gamma_A(J_{(M+1)H}) \end{bmatrix} \mid (J_{11},\dots,J_{(M+1)H}) \text{ is a measurable partition of N} \right\}$$

is convex and compact. Matrices where each row is composes of all 0's except one $\mu(N_a)$ are in this set. (Just let $N_a\subseteq J_{ih}$ for some $a\in A$, $(i,h)\in (M\cup\{M+1\})\times H$ and let $N_a\cap J_{jh'} = \emptyset$ for all $(j,h')\ne(i,h)$.) This implies that $\mathcal{J}(v) \ne \emptyset$ for all $v\in V$ s.t. for each $a\in A$, $v_{ih}^a = \mu(N_a)$ for some (i,h). But then the convexity of the above matrix implies that $\mathcal{J}(v) \ne \emptyset$ for all $v\in V$.

Q.E.D.

Proof of Lemma 6: Note (i), (iii) and (iv) $\Rightarrow$ (E2'), (E3') and (E4') are satisfied. Furthermore, $\forall$ $(i,h)\in (M\cup\{M+1\})\times H$, $\forall$ $a\in A$, $\forall$ $J(v)\in \tilde{J}(v)$,

$$\mu\{J_{ih}(v)\cap N_a\} = v_{ih}^a \Rightarrow \mu\{J_{ih}(v)\} = \sum_{a\in A} v_{ih}^a .$$

Thus, (v) and (vi) $\Rightarrow$ (E5') and (E6'). We then only need to show that (E1') holds. Note that

[26] Theorem 1 in Dubins and Spanier (1961) is used extensively in Dunz (1985). It is an extension of Lyapunov's Theorem and states:

Theorem: Let $\mu = \mu_1,\dots,\mu_n$ be an n-tuple of countably additive, finite, real-valued functions defined on a σ-algebra $\mathcal{U}$ of subsets of U. For every ordered partition P of U into k measurable sets $A_1,\dots,A_k$ (with $A_j\in\mathcal{U}$ for $1\le j\le k$), define the n×k matrix of real numbers $M(P) = (\mu_i(A_j))$. Then if μ is nonatomic, the range R of the matrix-valued function M is a compact convex set of matrices.

(E1') is satisfied if $m_{ih} \equiv (\sum_{a\in A} v_{ih}^a) - \mu(C_{ih}) = 0 \;\; \forall\; (i,h)\in M\cup\{M+1\}\times H$. Since m_{ih} is the net migration into (i,h), $\sum_{i\in M\cup\{M+1\}} \sum_{h\in H} m_{ih} = 0$. Thus, if $m_{jh'} < 0$, $\exists\; (i,h)\in (M\cup\{M+1\})\times H$ s.t. $m_{ih} > 0$. Then assume without loss of generality that $m_{ih}>0$. Since $\mu(C_{(M+1)h}) = 0 \;\forall\; h\in H$, $\exists\; (j,h')\in M\times H$ s. t. $m_{jh'} < 0$. First suppose $i\neq M+1$. Then by the definition of ξ, $p_{ih} = \overline{P}$ and $p_{jh'} = \underline{P}$, which implies by Lemma 4 that $W_{ih}(p,x,t) = \emptyset \Rightarrow \sum_{a\in A} v_{ih}^a = 0 \;\Rightarrow\; m_{ih} = -\mu(C_{ih})<0$ which is a contradiction. Finally assume $i=M+1$. Since $p_{jh'} = \underline{P}$, A6 $\Rightarrow \forall\; h\in H$, $\forall\; n\in N$, $(j,h')\in B^n(p,t)$ and $u^n(M+1,h) < u^n(j,h')$. But then $W_{ih}(p,x,t) = \emptyset \Rightarrow \sum_{a\in A} v_{ih}^a = 0 \;\Rightarrow$ $m_{ih} = 0$ which is another contradiction.

Q.E.D.

<u>Proof of Lemma 7</u>: Note that $\xi(v,p,x,t) \neq \emptyset$ if $B^n(p,t) \neq \emptyset$ for all $n\in N$. This holds because

$$(M+1,h)\in B^n(p,t) \;\forall\; h\in H,\; \forall\; n\in N,\; \forall\; (p,t)\in \widetilde{P}\times\widetilde{T}.$$

Q.E.D.

<u>Proof of Lemma 8</u>: Let $\{(p^k,x^k,t^k)\}_{k=1}^{\infty}$ be such a sequence and let S_{ih} and S_{ih}^k denote $S_{ih}(p,x,t)$ and $S_{ih}(p^k,x^k,t^k)$ respectively. Note that by A2, $n\in S_{ih} \Rightarrow (1-t_0)z(n)+p_n-(1+t_i)p_{ih}>0$. Then by the continuity of u^n, $\exists\; \overline{k} <\infty$ s. t. $\forall\; k\geq\overline{k}$, $\forall\; (i,h)\in M\times H$, $\forall\; n\in S_{ih}$, $\forall\; (j,h')\in B^n(p^k,t^k)\backslash(i,h)$,

$$u^n(i, h, x^k,\; p_n^k + (1-t_0^k)z(n) - p_{ih}^k - t_i^k p_{ih}^k) > u^n(j, h', x^k,\; p_n^k + (1-t_0^k)z(n) - p_{jh'}^k - t_j^k p_{jh'}^k)$$

with $(i,h)\in B^n(p^k,t^k)$. By definition of E', $u'^n(M+1,h)=u'^n(M+1,h') \;\forall\; h,h'\in H$. This implies $S_{(M+1)h}=\emptyset \;\forall\; h\in H$, $(p,x,t)\in \widetilde{P}\times\widetilde{X}\times\widetilde{T}$. Thus, $\forall\; (i,h)\in (M\cup\{M+1\})\times H$, if $n\in S_{ih}$, $k\geq\overline{k} \Rightarrow n\in S_{ih}^k$, which implies $S_{ih}\backslash S_{ih}^k = \emptyset \Rightarrow S_{ih}\subseteq S_{ih}^k$.

Similarly, let W_{ih} and W_{ih}^k denote $W_{ih}(p,x,t)$ and $W_{ih}(p^k,x^k,t^k)$ respectively. By the continuity of u^n and A2, $\exists\; \overline{k} <\infty$ s.t. $\forall\; k\geq\overline{k}$, $\forall\; (i,h)\in M\times H$, $\forall\; n\notin W_{ih}$, either $(i,h)\notin B^n(p^k,t^k)$ or $\exists$ $(j,h')\in B^n(p^k,t^k)$ with

$$u^n(j, h', x^k,\; p_n^k + (1-t_0^k)z(n) - p_{jh'}^k - t_j^k p_{jh'}^k) > u^n(i, h, x^k,\; p_n^k + (1-t_0^k)z(n) - p_{ih}^k - t_i^k p_{ih}^k).$$

Furthermore, if $n\notin W_{(M+1)h}$, then $\exists\; (j,h')\in M\times H$ s.t. $(1+t_0)z(n) + p_n - (1+t_j)p_{jh'} > 0$. Then $\exists\; \overline{k} <\infty$ s.t. $\forall\; k\geq\overline{k}$, $(1-t_0^k)z(n) + p_n^k - (1+t_j^k)p_{jh'}^k > 0$ which implies $u^n(j, h') > u^n(M+1, h)$ $\Rightarrow n\notin W_{(M+1)h}^k$. Thus $\forall\; (i,h)\in (M\cup\{M+1\})\times H$, if $n\notin W_{ih}$, $k\geq\overline{k} \Rightarrow n\notin W_{ih}^k$. Therefore, if $n\in W_{ih}^k$, $k\geq\overline{k} \Rightarrow n\in W_{ih}$, which implies $W_{ih}^k\backslash W_{ih} = \emptyset \Rightarrow W_{ih}^k\subseteq W_{ih}$.

Q.E.D.

Proof of Lemma 9: Suppose $\{(\bar{v}^k,\bar{p}^k,\bar{x}^k,\bar{t}^k)\}_{k=1}^{\infty}$ and $\{(v^k,p^k,x^k,t^k)\}_{k=1}^{\infty}$ are sequences s.t. $\lim_{k\to\infty}(\bar{v}^k,\bar{p}^k,\bar{x}^k,\bar{t}^k) = (\bar{v},\bar{p},\bar{x},\bar{t})$, $\lim_{k\to\infty}(v^k,p^k,x^k,t^k) = (v,p,x,t)$ and $(\bar{v}^k,\bar{p}^k,\bar{x}^k,\bar{t}^k)\in\xi(v^k,p^k,x^k,t^k)$ for all $k\geq 1$. We must show $(\bar{v},\bar{p},\bar{x},\bar{t})\in\xi(v,p,x,t)$.

(i) Let $S_{ih} = S_{ih}(p,x,t)$ and $S_{ih}^k = S_{ih}^k(p^k,x^k,t^k)$. Note that (i) is satisfied if for $n\in N_a$, $n\in S_{ih} \Rightarrow \bar{v}_{ih}^a = \mu(N_a)$ and $n\notin W_{ih} \Rightarrow \bar{v}_{ih}^a = 0$. Lemma 8 $\Rightarrow \exists\, \bar{k}$ s.t. $\forall\, k\geq\bar{k}$, if $n\in S_{ih}$, then $n\in S_{ih}^k$. This implies that if $n\in N_a$, $n\in S_{ih}^k \Rightarrow (\bar{v}_{ih}^a)^k = \mu(N_a)\ \forall\, k\geq\bar{k} \Rightarrow \lim_{k\to\infty}(\bar{v}_{ih}^a)^k = \bar{v}_{ih}^a = \mu(N_a)$. Similarly, Lemma 8 $\Rightarrow \exists\, \bar{k}$ s.t. $\forall\, k\geq\bar{k}$, if $n\notin W_{ih}$, then $n\notin W_{ih}^k$. This further implies that if $n\in N_a$, $n\notin W_{ih}^k \Rightarrow (\bar{v}_{ih}^a)^k = 0\ \forall\, k\geq\bar{k} \Rightarrow \lim_{k\to\infty}(\bar{v}_{ih}^a)^k = \bar{v}_{ih}^a = 0$.

(ii) Note that (ii) is satisfied if $\forall\,(i,h)\in M\times H$, (1) $\sum_{a\in A} v_{ih}^a > \mu(C_{ih}) \Rightarrow \bar{p}_{ih} = \overline{P}$; (2) $\sum_{a\in A} v_{ih}^a < \mu(C_{ih}) \Rightarrow \bar{p}_{ih} = \underline{P}$; and (3) $\sum_{a\in A} v_{ih}^a = \mu(C_{ih}) \Rightarrow \bar{p}_{ih}\in[\underline{P},\overline{P}]$. If (1) $\sum_{a\in A} v_{ih}^a > \mu(C_{ih})$, $\exists\, \bar{k}$ s.t. $\forall\, k\geq\bar{k}$, $\sum_{a\in A}(v_{ih}^a)^k > \mu(C_{ih}) \Rightarrow \bar{p}_{ih}^k = \overline{P} \Rightarrow \lim_{k\to\infty}\bar{p}_{ih}^k = \bar{p}_{ih} = \overline{P}$. Similarly, if (2) $\sum_{a\in A} v_{ih}^a < \mu(C_{ih})$, $\exists\, \bar{k}$ s.t. $\forall\, k\geq\bar{k}$, $\sum_{a\in A}(v_{ih}^a)^k < \mu(C_{ih}) \Rightarrow \bar{p}_{ih}^k = \underline{P} \Rightarrow \lim_{k\to\infty}\bar{p}_{ih}^k = \bar{p}_{ih} = \underline{P}$. Finally, if (3) $\sum_{a\in A} v_{ih}^a = \mu(C_{ih})$, $\exists\, \bar{k}$ s.t. $\forall\, k\geq\bar{k}$, (a) $\sum_{a\in A}(v_{ih}^a)^k > \mu(C_{ih})$ or (b) $\sum_{a\in A}(v_{ih}^a)^k < \mu(C_{ih})$ or (c) $\sum_{a\in A}(v_{ih}^a)^k = \mu(C_{ih})$. If (a), $\bar{p}_{ih}^k = \overline{P} \Rightarrow \lim_{k\to\infty}\bar{p}_{ih}^k = \bar{p}_{ih} = \overline{P}$; if (b), $\bar{p}_{ih}^k = \underline{P} \Rightarrow \lim_{k\to\infty}\bar{p}_{ih}^k = \bar{p}_{ih} = \underline{P}$; and if (c), $\lim_{k\to\infty}\bar{p}_{ih}^k = \bar{p}_{ih}\in[\underline{P},\overline{P}]$.

(iii) For all $i\in M$, the continuity of f_i implies that since $\lim_{k\to\infty} t_i^k = t_i$ and $\lim_{k\to\infty} p^k(C_i) = p(C_i)$,

$$\bar{x}_i = \lim_{k\to\infty}\bar{x}_i^k = \lim_{k\to\infty} f_i(t_i^k\, p^k(C_i)) = f_i(t_i\, p(C_i)).$$

Furthermore, $\bar{x}_{M+1}^k = 0\ \forall\, k\geq 1 \Rightarrow \bar{x}_{M+1} = 0$.

(iv) The continuity of f_0 implies that since $\lim_{k\to\infty} t_0^k = t_0$,

$$\bar{x}_0 = \lim_{k\to\infty}\bar{x}_0^k = \lim_{k\to\infty} f_0(t_0^k\, z(N)) = f_0(t_0\, z(N)).$$

(v)[27] Let $(i,h)\in M\times H$, $t^a(i,h,p,x) \equiv t_a$ and $t^a(i,h,p^k,x^k) \equiv t_a^k$. The uniqueness of $t^a(i,h,\cdot,\cdot)$ (Lemma 1) $\Rightarrow$ we can order A such that $t_{a_1}\leq t_{a_2}\leq\ldots\leq t_{a_A}$. Suppose first

[27] Denzau and Parks (1975) have previously shown the continuity of majority rule equilibria. Their result is not directly applicable here because the set of voters in communities is not fixed.

$$\text{(9.1)} \qquad \forall\, \dot{t}_i \in [0,\bar{T}],\ \left\{ \sum_{a \in A} \sum_{h \in H} v_{ih}^a \mid \dot{t}_i\, Q^a(i,h,p,x)\ \dot{t}_i \right\} > \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2}.$$

Since preferences are single peaked and the inequality is strict, Black's Theorem implies $\emptyset \neq \{a \in A \mid t_a = \hat{t}_i\} \equiv B$. By the continuity of $t^a(i,h,\bullet,\bullet)$ (Lemma 3), $\lim_{k \to \infty} t_a^k = t_a$ for all $a \in A$ which implies $\exists\, \bar{k}$ s.t. $\forall\, k \geq \bar{k}$, $\forall\, a_j \in B$, if $a < \min B$, $t_a^k < t_{a_j}^k$ and if $a > \max B$, $t_a^k > t_{a_j}^k$. Furthermore, since $\lim_{k \to \infty} (v_{ih}^a)^k = v_{ih}^a$, $\exists\, \bar{\bar{k}}$ s.t. $\forall\, k \geq \bar{\bar{k}}$,

$$\sum_{h \in H} \sum_{a < \min B} (v_{ih}^a)^k < \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2} \text{ and } \sum_{h \in H} \sum_{a > \max B} (v_{ih}^a)^k < \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2}.$$

Let $\hat{k} = \max\{\bar{k}, \bar{\bar{k}}\}$. Then $\forall\, k \geq \hat{k}$, $\bar{t}_i^k = t_{a_j}^k$ for some $a_j \in B$. Thus, since $\lim_{k \to \infty} t_{a_j}^k = \hat{t}_i\ \forall\, a_j \in B$, $\bar{t}_i = \lim_{k \to \infty} \bar{t}_i^k = \hat{t}_i$.

Next suppose (9.1) holds with equality instead. Then since preferences are single peaked, $\hat{t}_i \in [t_{a_j}, t_{a_{j+1}}]$ for some $a_j, a_{j+1} \in A$ where $t_{a_j} < t_{a_{j+1}}$. As before, $\exists\, \bar{k}$ s.t. $\forall\, k \geq \bar{k}$, if $a < a_j$, $t_a^k < t_{a_j}^k$ and if $a > a_{j+1}$, $t_a^k > t_{a_{j+1}}^k$. Similarly, $\exists\, \bar{\bar{k}}$ s.t. $\forall\, k \geq \bar{\bar{k}}$, $t_{a_j}^k < t_{a_{j+1}}^k$. Finally $\exists\, \bar{\bar{\bar{k}}}$ s.t. $\forall\, k \geq \bar{\bar{\bar{k}}}$,

$$\sum_{h \in H} \sum_{a < a_j} (v_{ih}^a)^k < \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2} \text{ and } \sum_{h \in H} \sum_{a > a_{j+1}} (v_{ih}^a)^k < \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2}.$$

Let $\hat{k} = \max\{\bar{k}, \bar{\bar{k}}, \bar{\bar{\bar{k}}}\}$. Then $\forall\, k \geq \hat{k}$, $\bar{t}_i^k \in [t_{a_j}^k, t_{a_{j+1}}^k]$. Since $\lim_{k \to \infty} [t_{a_j}^k, t_{a_{j+1}}^k] = [t_{a_j}, t_{a_{j+1}}]$, $\lim_{k \to \infty} \bar{t}_i^k = \bar{t}_i \in [t_{a_j}, t_{a_{j+1}}]$.

Finally note that $\bar{t}_{M+1}^k = 0$ for all k, which implies $\bar{t}_{M+1} = 0$.

(vi) Similar reasoning to that in (v) applies to $\bar{t}_0$. Thus, $(\bar{v}, \bar{p}, \bar{x}, \bar{t}) \in \xi(v,p,x,t)$.

Q.E.D.

<u>Proof of Lemma 10</u>: Fix $(\bar{v}, \bar{p}, \bar{x}, \bar{t}) \in V \times \tilde{P} \times \tilde{X} \times \tilde{T}$. Let $\xi_V(\bar{v}, \bar{p}, \bar{x}, \bar{t})$, $\xi_P(\bar{v}, \bar{p}, \bar{x}, \bar{t})$, $\xi_X(\bar{v}, \bar{p}, \bar{x}, \bar{t})$, and $\xi_T(\bar{v}, \bar{p}, \bar{x}, \bar{t})$ denote the projections of $\xi(\bar{v}, \bar{p}, \bar{x}, \bar{t})$ onto the sets V, $\tilde{P}$, $\tilde{X}$, and $\tilde{T}$ respectively. Note that the definition of $\xi(\bar{v}, \bar{p}, \bar{x}, \bar{t})$ implies that if these projections are convex, then $\xi(\bar{v}, \bar{p}, \bar{x}, \bar{t})$ itself is also convex.

(i) Let $v \in \xi_V(\bar{v}, \bar{p}, \bar{x}, \bar{t})$. Suppose $n \in N_a$ and $n \in S_{ih}(\bar{p}, \bar{x}, \bar{t})$. Then $v \in \xi_V(\bar{v}, \bar{p}, \bar{x}, \bar{t}) \Rightarrow N_a \subseteq J_{ih}(v) \Rightarrow v_{ih}^a = \mu(N_a)$. Next suppose $n \in N_a$ and $n \notin W_{ih}(\bar{p}, \bar{x}, \bar{t})$. Then $v \in \xi_V(\bar{v}, \bar{p}, \bar{x}, \bar{t}) \Rightarrow N_a \cap J_{ih}(v) = \emptyset \Rightarrow v_{ih}^a = 0$. Finally, if $n \in N_a$, $n \notin S_{ih}(\bar{p}, \bar{x}, \bar{t})$ and $n \in W_{ih}(\bar{p}, \bar{x}, \bar{t})$, $v \in \xi_V(\bar{v}, \bar{p}, \bar{x}, \bar{t}) \Rightarrow v_{ih}^a \in [0, \mu(N_a)]$. Thus,

$$\xi_V(\bar{v},\bar{p},\bar{x},\bar{t}) = \{v \in V \mid \text{(i) if } n \in N_a,\ n \in S_{ih}(\bar{p},\bar{x},\bar{t}),\ v_{ih}^a = \mu(N_a), \text{ and}$$
$$\text{(ii) if } n \in N_a,\ n \notin W_{ih}(\bar{p},\bar{x},\bar{t}),\ v_{ih}^a = 0\}\ .$$

Therefore, $\xi_V(\bar{v},\bar{p},\bar{x},\bar{t})$ is a subset of V with some coordinates held fixed. Since V is convex, so is $\xi_V(\bar{v},\bar{p},\bar{x},\bar{t})$.

(ii) Note that $\xi_P(\bar{v},\bar{p},\bar{x},\bar{t}) = \{p \in \tilde{P} \mid p_{(M+1)h} = 0\ \forall\ h \in H;$ and $\forall\ (i,h) \in M \times H,$

$$\text{(i) if } \sum_{a \in A} \bar{v}_{ih}^a > \mu(C_{ih}),\ p_{ih} = \bar{P},$$

$$\text{(ii) if } \sum_{a \in A} \bar{v}_{ih}^a < \mu(C_{ih}),\ p_{ih} = \underline{P}; \text{ and}$$

$$\text{(iii) if } \sum_{a \in A} \bar{v}_{ih}^a = \mu(C_{ih}),\ p_{ih} \in [\underline{P},\bar{P}]\},$$

which is clearly a convex subset of $\tilde{P}$.

(iii) Note that $\xi_X(\bar{v},\bar{p},\bar{x},\bar{t}) = (f_0(\bar{t}_0 z(N)), f_1(\bar{t}_1 \bar{p}(C_1)), f_2(\bar{t}_2 \bar{p}(C_2)), \ldots f_M(\bar{t}_M \bar{p}(C_M)), 0)$. Thus, if $x, x' \in \xi_X(\bar{v},\bar{p},\bar{x},\bar{t})$, then $x = x'$ which implies $\xi_X(\bar{v},\bar{p},\bar{x},\bar{t})$ is convex.

(iv) Suppose first that for $i \in M\ \exists\ \hat{t}_i \in [0,\bar{T}]$ s.t.

$$\left\{\sum_{a \in A} \sum_{h \in H} \bar{v}_{ih}^a \mid \hat{t}_i\ Q^a(i,h,\bar{p},\bar{x})\ t_i\right\} > \frac{\sum_{h \in H} \sum_{a \in A} v_{ih}^a}{2}\quad \forall\ t_i \in [0,\bar{T}]. \tag{10.1}$$

Then single peakedness of $Q^a(i,h,\bar{p},\bar{x})$ and uniqueness of $t^a(i,h,\bar{p},\bar{x})$ (Lemmas 1,2) imply that if $t \in \xi_T(\bar{v},\bar{p},\bar{x},\bar{t})$, $t_i = \hat{t}_i$. Now suppose (10.1) holds with equality instead.Then $\exists\ t_a = t^a(i,h,\bar{p},\bar{x})$, $t_{a'} = t^{a'}(i,h,\bar{p},\bar{x})$, s.t. if $t \in \xi_T(\bar{v},\bar{p},\bar{x},\bar{t})$, $t_i \in [t_a, t_{a'}]$. Similar reasoning applies to t_0. Finally, t_{M+1} is always fixed at zero. Thus, $\xi_T(\bar{v},\bar{p},\bar{x},\bar{t})$ is the cartesian product of convex sets and thus convex.

Q.E.D.

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