TECHNICAL WORKING PAPER SERIES

INVESTMENT UNDER ALTERNATIVE RETURN ASSUMPTIONS: COMPARING RANDOM WALKS AND MEAN REVERSION

Gilbert E. Metcalf
Kevin Hassett

Technical Working Paper No. 175

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
March 1995

We are grateful for helpful advice from Avinash Dixit. We also thank the Department of Energy and the National Science Foundation for their financial support. The views expressed here do not necessarily reflect those of the Board of Governors of the Federal Reserve System. This paper is part of NBER's research program in Public Economics. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Technical Working Paper #175
March 1995

INVESTMENT UNDER ALTERNATIVE RETURN ASSUMPTIONS: COMPARING RANDOM WALKS AND MEAN REVERSION

ABSTRACT

Many recent theoretical papers have come under attack for modeling prices as Geometric Brownian Motion. This process can diverge over time, implying that firms facing this price process can earn infinite profits. We explore the significance of this attack and contrast investment under Geometric Brownian Motion with investment assuming mean reversion. While analytically more complex, mean reversion in many cases is a more plausible assumption, allowing for supply responses to increasing prices. We show that cumulative investment is generally unaffected by the use of a mean reversion process rather than Geometric Brownian Motion and provide an explanation for this result.

Gilbert E. Metcalf
Department of Economics
Tufts University
Medford, MA 02155
and NBER

Kevin Hassett
Board of Governors of the
Federal Reserve System
Mailstop 80
Washington, DC 20551

I. Introduction

Much of the recent work on investment has used continuous time stochastic processes to model returns or prices (e.g. Pindyck (1982,1988), Abel (1983), Dixit (1992), and Bertola (1990)). The bulk of this work makes the assumption that prices (or returns) follow Geometric Brownian Motion (GBM). The advantage of GBM is that it leads to tractable solutions for investment decisions. In addition, the investment decision rules are explicit and intuitive. However, it has been noted that GBM is not a plausible equilibrium price process (e.g. Lund (1993)). Geometric Brownian Motion, a continuous time random walk in logs, is unbounded above. But as prices rise, there exist incentives for new firms to enter the market (or existing firms to expand) to supply the good in question. Firms will enter until the marginal entrant earns zero profits. This supply response will tend to damp the price increase. The relevance of this point for investment models is obvious. If output prices rise, firms increase investment to expand. In equilibrium however, this supply shift will lead to a fall in output price if demand curves are downward sloping. Similarly, if prices fall, high cost firms will exit the market at some point. The supply shift will blunt the price fall. Thus, as price rises (falls), the underlying trend should begin to fall (rise). Mean reversion is a way to capture this effect.

While not as simple as GBM, mean reversion can be quite tractable. The purpose of this paper is to illustrate the use of mean reversion and to compare investment rates under Geometric Mean Reversion (GMR) with investment rates under Geometric Brownian Motion. Two related points emerge from this paper. First, there are offsetting effects when moving from GBM to GMR. Increasing mean reversion reduces the long-run variance of the price process. This suggests that investment will rise based on previous findings by researchers of an inverse relationship between uncertainty and investment (e.g. Pindyck (1982, 1988)). We will refer to this effect as the "variance" effect. On the other hand, the increasing volatility of GBM means that higher price levels may be achieved. This will induce greater amounts of investment. We call this effect the "realized price" effect. In simulations over a range of reasonable parameters that we present below, these two effects offset each other so that expected cumulative investment after a period of time is the same under GBM and GMR.

Our second point follows naturally from the first. While GBM may not follow naturally from any underlying theory and the non-stationarity of GBM processes might be troubling, cumulative investment behavior under GBM is very similar to cumulative investment behavior under GMR. Hence the additional tractability and intuitive nature of results that emerge when GBM is used can be bought at very low cost in terms of realism.

This result follows from a model in which the production function exhibits decreasing returns to scale with lumpy investment. A natural question then is how general our findings in this paper are. An alternative model of investment assumes constant returns to scale production function with convex costs of adjustment (e.g. Abel (1983)). In this model the realized price and variance effects both work to increase investment, and hence, our results would not apply. A quick look at Dixit and Pindyck (1994), however, suggests that there are a broad range of applications for which the model we analyze in this paper is appropriate.

In the next section, we motivate the use of GMR and review the theory of investment in a world with mean reversion. The following section compares cumulative investment under GMR versus GBM. The last section summarizes our findings.

II. Mean Reversion and Supply Responses

We begin by defining Geometric Brownian Motion (GBM) and Geometric Mean Reversion (GMR). We then suggest a heuristic argument for modeling prices as following GMR. A price (P_t) following Geometric Brownian Motion can be characterized by the following stochastic differential equation:

$$dP_t = \alpha P_t dt + \sigma P_t dz \qquad (1)$$

where dz is an increment to a Wiener Process with mean zero and unit variance. The parameter α measures the trend in the price process while σ measures its volatility.

Geometric Mean Reversion can be characterized by the following stochastic differential equation:

$$dP_t = (\alpha+\lambda(\bar{P}e^{\alpha t} - P_t))P_t dt + \sigma P_t dz \qquad (2)$$

where λ is a positive parameter measuring the speed of reversion Equation (2) says that prices rise exponentially at rate α from a starting price $\bar{P}$. However if shocks to prices (dz) push P_t above trend, the effective trend is pushed down, thereby driving prices back toward the trend line. In the case where α equals 0, we obtain the Geometric Ornstein-Uhlenbeck process:

$$dP_t = \lambda(\bar{P} - P_t)P_t dt + \sigma P_t dz. \qquad (3)$$

In the simulations which follow, we will assume prices follow a process as given by equation (3). If λ equals 0, prices follow GBM; if $\lambda > 0$, then prices follow GMR.

Above we noted that supply responses could motivate mean reversion in prices. Leahy (1993) has modeled equilibrium behavior in a market in which agents are homogeneous. He finds that there exists a band within which prices fluctuate with barriers at top and bottom. That agents are identical is crucial to his result. With heterogeneous agents the model becomes more complicated. Some form of price reversion will occur as entry or expansion occurs (price high) or firms exit or downsize (price low). As a first cut at modeling the process, equation (3) with $\lambda > 0$ seems reasonable.

What effect will mean reversion have on prices? Figure 1 illustrates a price process for GBM and GMR where the trend in price equals zero (equation (3)). The long run price is normalized to 1 and the instantaneous volatility equals .15. The reversion adjustment parameter equals .08 in this example. These parameters mean that volatility is on the order of 15% of price and that trend will be negative 4.0% when price rises to 1.5 and positive 4.0% when price falls to 0.5. The price is followed out for 10 periods. The solid line represents the price assuming it follows GBM while the dotted line represents the price following GMR. We've constructed these two series using the same random process; the only difference is the treatment of the trend as the price deviates from $\bar{P}$ equals 1.0. As price rises above $\bar{P}$, the GMR assumption begins to pull the price back down. For example, at t slightly below 6, the trend for the GMR process is roughly -3.2% as opposed to the trend for the GBM process of zero. With the price (in both cases) above $\bar{P}$, the negative trend for the GMR price leads to a widening gap between the two price processes. By the tenth period, the GBM price is roughly 40% greater than the GMR price.

The effect of the GMR assumption is to reduce the conditional variance of price at time T computed at time $t < T$.[1] Intuition suggests that this dampening of the price response should encourage investment and increase cumulative investment. On the other hand, the increased variance for GBM means that higher trigger prices can be achieved - which in turn can lead to a greater amount of investment in a project whose return depends

---

1 The variance of the price at time 10 conditional on price equaling 1 at time 0 is .111 for the GMR process and .269 for the GBM process.

on P.[2] These two effects work in opposite directions. Hence we turn to simulations to measure their relative impact on investment. We begin by presenting the theory of irreversible investment under uncertainty when prices follow GBM and GMR and then present results of investment simulations.

III. Optimal Investment Rules Under Alternative Price Processes

In this section we model the decision at the individual level to invest in a unit of capital. We assume that the investment is irreversible and that the marginal cost of using the investment is zero. For simplicity, we assume that the size of the investment is fixed and we normalize so that maximal output per period of time equals one.[3] Firms are assumed to be heterogeneous with differing rates of productivity per unit of capital and output is reduced by a factor δ_i which ranges from 0 to 1. Hence the value of output per unit of investment will equal δP_t and the return on an investment of one unit of capital will be δP per period. The firm's problem then is to choose a

---

2 We have yet not formally stated how cumulative investment will be motivated in this model. For now assume that the return on an investment is some fraction of P, where the fraction is a variable distributed across the population of firms.

3 Making the size of the investment endogenous does not alter the results in any significant way so long as firms are price takers.

time, T, to invest in a unit of capital to maximize

$$V = E_0\left\{ \int_T^\infty \delta_i P_t e^{-\rho t} dt - K e^{-\rho T} \right\} \tag{5}$$

where ρ is the discount rate (common to all firms) and K is the cost of the investment (assumed constant)[4]. To solve this problem, we must make assumptions about the return stream. In the next sub-section, we assume that prices follow GBM with no drift and in the following sub-section, we assume GMR.

III. A. Geometric Brownian Motion

Our first assumption is that the returns on the investment follow the continuous time stochastic process

$$dP_t = \sigma P_t dz \tag{6}$$

where dz is the usual increment 5to a Wiener process (see equation 1 above). Maximizing (5) subject to (6) is a standard problem in the irreversibility literature (e.g Dixit (1992), McDonald and Siegel (1986), Pindyck (1988)) and the optimal time to invest occurs when[5]:

$$\delta_i P_{it} \geq \frac{b}{b-1} \rho K, \tag{7}$$

where

---

4 It would be easy to add depreciation to the model. Assuming exponential depreciation at rate η, the term ρ in the integral of equation (5) would be replaced by $\rho+\eta$. Nothing substantive changes.

5 See Dixit (1992) for a clear derivation of this result.

$$b = \frac{.5\,\sigma^2 + \sqrt{(.5\,\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} > 1. \tag{8}$$

In addition, the value function for the expected value in equation 5 is given by

$$V^*(P;P^*) = \begin{cases} aP^b, & P < P^* \\ \dfrac{\delta P}{\rho} - K, & P \geq P^* \end{cases} \tag{9}$$

where P^* is the price at which equation (7) is satisfied as an equality (trigger price) and a is a constant of integration. The value function has two components. Before the investment is made, the only value is the option held by the firm. The upper component of equation (9) describes the option. As price falls, the likelihood of the option ever being exercised goes down and the option becomes less valuable. The bottom component of equation (9) describes the net (of cost) present value of the investment once investment is made.

III. B. Geometric Mean Reversion

If prices follow GMR, the solution to the maximization problem in (5) subject to

$$dP_t = \lambda(\bar{P} - P_t)P_t dt + \sigma P_t dz \tag{10}$$

is relatively straightforward (see appendix for details)[6].

---

[6] Dixit and Pindyck (1994) provide an excellent discussion of investment models in which the value function follows GMR. As they note in their book (p. 162) they provide no rationale for why the value function is mean reverting. Our model differs in

However, there is no intuitive way to write the expression for the trigger price as in equation (7) above. The value function is given by

$$V^*(P;P^*) = \begin{cases} AP^{\nu} H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)), & P < P^* \\ \delta\left\{BP^{\nu}H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)) + \sum_{i=1}^{\infty} c_i P^i\right\} - K, & P \geq P^* \end{cases} \tag{11}$$

where H(x,a,b) is the confluent hypergeometric function (see appendix), $Z(\nu) = 2\nu + \frac{2\bar{P}\lambda}{\sigma^2}$, ν is the positive root to the quadratic $Q(x) = .5\sigma^2 x(x-1) + \lambda\bar{P}x - \rho$, and the c_i's are defined in equation A13 in the appendix. Smooth pasting and value matching conditions can be invoked to eliminate the constants A and B yielding an equation which implicitly defines P^*, the trigger price at which point it is optimal to invest. Letting $G^P = \sum_{i=1}^{\infty} c_i P^i$, we get:

$$(\delta G^P - K)(\nu H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)) \tag{12}$$

---

modeling mean reversion in the output price and in providing a rationale for why the price is mean reverting. In addition, modeling mean reversion in prices rather than the value function leads to a somewhat more complicated solution methodology than the methodology described in Dixit and Pindyck. We note the differences in our derivation in the appendix.

$$+ PH'(\frac{2\lambda}{\sigma^2} P, \nu, Z(\nu))) = \delta G^{P\prime} PH(\frac{2\lambda}{\sigma^2} P, \nu, Z(\nu)).$$

where H' refers to $\frac{\partial H(x,y,z)}{\partial x}$ and G^P is a particular solution to the differential equation A11 in the appendix (see appendix for details). While seemingly different, the value functions in equations (9) and (11) collapse to the same form as λ approaches 0. The confluent hypergeometric function evaluated at 0 (H(0,a,b)) equals 1 and B approaches 0. Furthermore, G^P equals P/ρ for λ equals 0. Hence the value functions converge to the same value as we'd expect. Using the fact that $\frac{\partial H(ax,b,c)}{\partial x} = \frac{ab}{c}$ H(ax,b+1,c+1), it is easy to verify that equation (12) reduces to equation (7) as λ approaches 0.

Equations (7) and (12) govern investment at the individual level for price processes following GBM and GMR respectively. In the next section, we embed the individual investment decision in a model of aggregate investment and provide results from simulations to compare and contrast cumulative investment under the two price process assumptions.

IV. Cumulative Investment

The model of section III describes how an individual firm would compute the optimal rule for determining the optimal time to install new capital. Moving to the aggregate level, we explain cumulative investment by assuming heterogeneity in the return that the investment will earn (δ).[7] Firms are modeled to

---

7 This gives us a "Probit" type model of diffusion (viz.

have the same discount rate and complete information about the price process they face (though no information about the specific price realization in the future) as well as complete information on the productivity of the investment (δ). The parameter δ is normally distributed but truncated at zero and one. At low levels of P, only those firms with high values of δ - those for whom the potential return from an investment are the greatest - will invest. If prices rise, more firms will invest. Investment will only occur when the current price exceeds the maximum price that has occurred up to that date. Hence, if prices are increasing over some interval, cumulative investment may increase, while if prices are decreasing over some interval, there will be no increase in cumulative investment. In the simulations that follow, we assume that δ is (truncated) normally distributed with mean .50 and standard deviation .15. Assuming that each firm can make one investment, cumulative investment can range from 0 to 100% - meaning that from 0 to 100% of the firms purchase capital.

Intuitively, we would expect that as mean reversion increases (λ increases), firms should be more eager to invest in an irreversible investment. While the upside gain diminishes with increased mean reversion, so does the downside loss. Mean reversion has the effect of reducing the variance of the future return. As noted above, increased uncertainty reduces investment. Changes in the trigger price for investment provide

---

Stoneman (1983)).

some support for this idea. Table 1 shows the trigger price for an investment as λ increases from 0 to .090 for different values of σ. The investment is assumed to have a cost of 5 and will provide a return equal to 50% of the output price. Prices follow GMR with $\bar{P}$ equal to 1.0. The discount rate is .10. At $\sigma = 0.05$, the trigger price ranges from 1.118 ($\lambda = 0$) to 1.103 ($\lambda = .09$). The trigger price also falls with increasing λ for $\sigma = 0.15$ and 0.25. Table 1 also shows that increasing uncertainty discourages investment by driving up the trigger price for a given value of λ.

To illustrate how changing λ affects investment, consider figure 2. This set of price realizations hovers around the mean reversion price so that the GMR price and the GBM price are nearly coincident. Cumulative investment at the end of 10 periods is slightly less than 37% if price follows GBM while cumulative investment is roughly 41% if price follows GMR (figure 3).

That the trigger price is falling does not prove that investment will occur sooner. The expected time to any price above the reversion price will also increase as λ increases. Put differently, the maximum price that will be achieved over a given time period will decrease as λ increases. Given our model assumptions, cumulative investment at time t will be determined by $\hat{P}_t = \max(P_s)$, $s \in [0,t]$. Substituting $\hat{P}_t$ into equation (12) gives a critical value of $\hat{\delta}$ which is a lower bound on the return parameter that must be achieved before investment will occur.

All firms with a value of $\delta \geq \hat{\delta}$ will have invested by time t. For the model parameterized in table 1, Monte Carlo simulations (1000 replications) show that $\frac{\partial \hat{P}}{\partial \lambda} < 0$. For $\sigma = 0.15$, the average maximum price falls from 1.440 at λ equals 0 to 1.367 at λ equals .09.[8]

We can illustrate the realization effect with the price process shown in figure 1. These particular realizations of GBM and GMR show an upward trend and the maximum price under GBM is roughly 40% higher than the maximum price under GMR[9]. The capital diffuses more rapidly in this case: cumulative investment is about 83% under GBM by the end of period 10 while only 67% under GMR (figure 4).

Which of these two effects (lower variance vs. lower realized price) dominates? Table 2 presents results from a Monte Carlo analysis where cumulative investment is simulated 1,000 times for different values of λ and σ. As in table 1, we assume $P_0 = \bar{P} = 1.0$ and $\rho = .10$. We vary λ between 0 and 0.090.[10]

---

8 The variation is quite substantial. The standard deviation of $\hat{P}$ for GBM is .44 and .27 for GMR. The maximum value of $\hat{P}$ over the 1000 replications is 4.436 for GBM and 2.601 for GMR.

9 While there is an upward trend in the realized price series, the process itself has no trend.

10 The condition $\rho > \lambda\bar{P}$ ensures that the value function is bounded. This puts an upper limit on λ. Note though that λ is scaled by $\bar{P}$. Higher values of λ are possible if the reversion

Consider the case where $\sigma = 0.15$. The average value of cumulative investment by period 10 varies between 47 and 48%. We cannot reject the joint hypothesis that all the expected cumulative investment levels are the same across the different values of λ for a given σ.

We test the robustness of these results in Table 3. In this table, we report results from different simulations for extreme values of λ and σ for different values of the discount rate (ρ) and the cost of the capital investment (P_k). In the first set of results we cut the cost of the investment in half from 5.0 to 2.5. The first column reports mean cumulative investment for the case in which price follows GBM (λ=0.0) while the second column reports mean cumulative investment for the case in which price follows GMR with a reversion parameter of .09. For values of σ between .05 and .25 cumulative investment is roughly the same for the GBM and the GMR cases.

In the third and fourth rows, we double the purchase price. For reasonable values of σ, our results hold. However, for high values of σ, there is a difference between the GBM and the GMR cases. This result holds for the following simulations in which we first increase and then reduce the discount rate: for very high levels of σ mean cumulative investment can differ markedly depending on the price process[11]. Such extreme variation in the

---

price is reduced.

[11] In the last two sets of results, we report GMR results for the

price process is unusual and the similarity between the GBM and GMR cumulative investment holds at lower levels of σ. For cases where annual variation in returns is on the order of 25% or more, these results suggest that the approximating GMR with a GBM process can affect mean investment results to some degree.

Considering extreme ranges for the volatility parameter leads to an additional interesting finding. In general, increasing uncertainty should lead to lower investment. However, in cases where there is a low level of investment when σ is low, increasing σ substantially can lead to higher mean levels of investment. For example, in the GBM case where P_k equals 10, mean investment rises from .003 when σ equals .05 to .069 when σ equals .25. While the trigger for investment is rising with σ, the likelihood that price will hit that trigger by a given time also rises with σ. In this case the second effect dominates the first.

The results in tables 2 and 3 suggest that in general we do not sacrifice significant realism by assuming prices follow Geometric Brownian Motion. Over a reasonable range of parameters of the processes, the use of Geometric Brownian Motion will not lead to results substantially different than when Geometric Mean Reversion is assumed. Crucial to this supposition is the offsetting nature of the variance and price realization effects that we identify above. In cases where one of these effects

---

case where λ equals .07 to avoid violating the condition that ρ must exceed $\lambda\bar{P}$.

dominates, then the greater computational work associated with GMR may be important as results may diverge from results obtained assuming GBM.

V. Conclusion

In this paper we compare aggregate investment when the return follows Geometric Brownian Motion (GBM) or Geometric Mean Reversion (GMR). While GMR is analytically more burdensome, it is possible to compute investment trigger prices and simulate cumulative investment when returns are mean reverting. Theory suggests that the effect on investment of moving from GBM to GMR is ambiguous.

There are two offsetting effects on investment when moving from a model where prices follow GBM to a model assuming mean reversion in prices. First, the variance of the price process will decline. This will have the effect of reducing the trigger price for investment which in turn will increase cumulative investment over time (holding a price path constant). However, the realized price path is not held constant. There is a lower probability of achieving some particular price above the mean reversion level by a given time t. Holding a trigger price constant, the chance of hitting that price falls as one moves from GBM to GMR. We find that for reasonable parameter values these two effects offset each other in simulations that we present. The average level of cumulative investment at the end of a finite period is essentially the same under either GBM or GMR. This suggests that Geometric Brownian Motion may be a reasonable simplifying assumption to make in models of irreversible investment under uncertainty.

Appendix

Investment Assuming Geometric Mean Reversion

We sketch out the solution to the investment problem of choosing an optimal time to make a one-time investment which pays δP_t per period forever assuming P follows GMR. We begin by noting that once the investment is made the net expected return on the investment is given by

$$\delta E_T\left\{ \int_T^{\infty} P_s e^{-\rho(s-T)} ds \right\} - K \tag{A1}$$

Define $G(P_T)$ as the expected value at time T of the integral in A1. Hence the value of the project at the time of the investment is given by $\delta G(P_T) - K$. The investment will be made only if P exceeds a trigger price P^*. At prices below P^* the only value to the investment is in its option value, V(P). Thus the value of the project $V^*(P,P^*)$ is given by

$$V^*(P;P^*) = \begin{cases} V(P) & \text{if } P < P^* \\ \delta G(P) - K & \text{if } P \geq P^* \end{cases} \tag{A2}$$

By the usual arbitrage argument[12]

$$\rho V(P) = \frac{E(dV)}{dt}. \tag{A3}$$

Applying Ito's Lemma to dV, substituting the result into A3, taking expectations and letting dt go to zero yields the differential equation

---

12 Alternatively, the Bellman equation for the dynamic optimization problem will give us this result.

$$.5\sigma^2P^2V'' + \lambda(\bar{P}-P)PV' - \rho V = 0. \tag{A4}$$

A power series solution of the form $V = \sum_{i=0}^{\infty} a_i P^{i+\nu}$ provides a solution to the differential equation. Substituting into A4 yields

$$\begin{aligned} &P^{\nu}a_0(.5\sigma^2\nu(\nu-1) + \bar{P}\lambda\nu - \rho) + \\ &P^{\nu+1}\left\{(.5\sigma^2\nu(\nu+1) + \bar{P}\lambda(\nu+1) - \rho)a_1 - \lambda\nu a_0\right\} + \\ &P^{\nu+2}\left\{(.5\sigma^2(\nu+1)(\nu+2) + \bar{P}\lambda(\nu+2) - \rho)a_2 - \lambda\nu a_1\right\} + \ldots = 0 \end{aligned} \tag{A5}$$

We choose ν as the roots of the quadratic $Q(x) = .5\sigma^2x(x-1) + \bar{P}\lambda x - \rho$. Note that one root ($\nu_1$) is negative and the other (ν_2) positive. Noting that $-.5\sigma^2\nu(\nu-1) = \bar{P}\lambda\nu - \rho$, we obtain the recurrence relation

$$a_n = \frac{\frac{2\lambda}{\sigma^2}(\nu+n-1)}{n(2\nu + \frac{2\bar{P}\lambda}{\sigma^2} - 1 + n)} a_{n-1}, \quad n \geq 1 \tag{A6}$$

Defining $Z(\nu) = 2\nu + \frac{2\bar{P}\lambda}{\sigma^2}$, we get

$$a_n = \frac{\left(\frac{2\lambda}{\sigma^2}\right)^n \nu(\nu+1)\ldots(\nu+n-1)}{n!Z(Z+1)\ldots(Z+n-1)} a_0 \tag{A7}$$

where a_0 is determined as a constant of integration. Let $H(x,a,b)$ be the confluent hypergeometric function

$$H(x,a,b) = 1 +5 \frac{a}{b} x + \frac{a(a+1)}{2!b(b+1)} x^2 + \ldots \tag{A8}$$

Then a general solution for V is given by

$$V = A_1 P^{\nu_1} H(\frac{2\lambda}{\sigma^2} P, \nu_1, Z(\nu_1)) + A_2 P^{\nu_2} H(\frac{2\lambda}{\sigma^2} P, \nu_2, Z(\nu_2)) \tag{A9}$$

As with GBM, P equals 0 is an absorbing state and V(0) = 0. Hence, A_1 must equal zero since $\nu_1 < 0$. Thus

$$V(P) = AP^{\nu} H(\frac{2\lambda}{\sigma^2} P, \nu, Z(\nu)), \quad \nu > 0. \tag{A10}$$

Now we must solve $G(P) = E_T\left\{ \int_T^{\infty} P_s e^{-\rho(s-T)} ds \right\}$.[13]

By the usual dynamic programming argument

$$.5\sigma^2 P^2 G'' + \lambda(\bar{P}-P)PG' - \rho G + P = 0. \tag{A11}$$

Equation A11 is the same as A4 except for the extra P on the left hand side of the equation. A solution to this equation is given by the sum of the solution to the homogeneous differential equation (A4) plus a particular solution. The solution to the homogeneous differential equation is given by A10 (note that G(0) equals 0 which allows us to eliminate one constant of

---

13 Dixit and Pindyck's formulation allows them to fix V upon investment (at the trigger value V^*). When prices (rather than V) follow GMR, one must explicitly solve for G(P) and apply value matching and smooth pasting conditions at P^* in equation A2. The following derivation of G(P) is not found in their book. The derivation is of interest in its own right as equation A14 below gives the expected present discounted value of an infinite stream of returns from a return process following GMR.

integration). For the particular solution, we try a power series of the form $G^P = \sum_{i=0}^{\infty} c_i P^i$. We take the derivative of the power series, substitute into A11 and group powers of P to yield

$$-\rho c_0 + \left\{(\lambda\bar{P}-\rho)c_1 + 1\right\}P + \sum_{i=2}^{\infty}\left\{(.5\sigma^2 i(i-1)+\lambda\bar{P}i-\rho)c_i - \lambda(i-1)c_{i-1}\right\} = 0 \quad \text{(A12)}$$

Therefore,

$$c_0 = 0$$

$$c_1 = \frac{1}{\rho-\lambda\bar{P}} \quad \text{(A13)}$$

$$c_i = \frac{2\lambda(i-1)}{\sigma^2(i-\nu_1)(i-\nu_2)} c_{i-1} \quad , i = 2,3,\ldots$$

where the ν_i's are the roots to the quadratic $Q(x) = .5\sigma^2 x(x-1) + \bar{P}\lambda x - \rho$. Combining the solution to the homogeneous problem and the particular solution gives us an expression for G(P)

$$G(P) = BP^{\nu}H(\frac{2\lambda}{\sigma^2} P, \nu, Z(\nu)) + \sum_{i=1}^{\infty} c_i P^i, \quad \text{(A14)}$$

where the c_i's are defined in A13[14].

Summing up, the solution for the value function $V^*(P;P^*)$ is given

---

14 This solution is valid so long as the positive root to the quadratic $Q(x) = .5\sigma^2 x(x-1) + \lambda\bar{P}x - \rho$ is not an integer. Unless otherwise specified, our parameter values do not lead to a violation of this condition.

by

$$V^*(P;P^*) = \begin{cases} AP^{\nu} H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)), & P < P^* \\ \delta\left\{BP^{\nu} H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)) + \sum_{i=1}^{\infty} c_i P^i\right\} - K, & P \geq P^* \end{cases} \quad \text{(A15)}$$

We invoke value matching and smooth pasting conditions at P^*. Combining the two resulting equations allows us to eliminate terms involving A and B from the equation. Letting $G^P = \sum_{i=1}^{\infty} c_i P^i$, we get:

$$(\delta G^P - K)(\nu H(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)) + PH'(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu))) = \delta G^{P\prime} PH(\frac{2\lambda}{\sigma^2} P,\ \nu,\ Z(\nu)). \quad \text{(A16)}$$

This equation can be solved for P^*.[15]

---

[15] The reader may have noted that the value matching and smooth pasting conditions only give us two equations to solve for three unknowns: A, B, and P^*. Usually a limiting condition as P approaches infinity would allow us to solve for A or B. There is no obvious condition to invoke here. One can easily solve for B by picking a value of P, and computing G(P) by Monte Carlo methods. Since the other elements of equation A14 are either known or convergent power series, we can solve the resulting equation for B.

Bibliography

Abel, A., 1983, Optimal Investment Under Uncertainty, American Economic Review 73, 228-33.

Bertola, G., 1990, Irreversible Investment, mimeo, Department of Economics, Princeton University.

Dixit, A., 1992, Investment and Hysteresis, Journal of Economic Perspectives 6, 107-132.

Dixit, A. and R. Pindyck, 1994, Investment Under Uncertainty, (Princeton University Press, Princeton NJ).

Leahy, J., 1993, Investment in Competitive Equilibrium: The Optimality of Myopic Behavior, mimeo, Department of Economics, Harvard University.

Lund, D., With Timing Options and Heterogeneous Costs, the Lognormal Diffusion is Hardly an Equilibrium Price Process for Exhaustible Resources, Journal of Environmental Economics and Management, forthcoming.

McDonald, R. and D. Siegel, 1986, The Value of Waiting to Invest, Quarterly Journal of Economics 101, 707-727.

Pindyck, R., 1982, Adjustment Costs, Uncertainty, and the Behavior of the Firm, American Economic Review 72, 415-27.

Pindyck, R., 1988, Irreversible Investment, Capacity Choice, and the Value of the Firm, American Economic Review 78, 969-985.

Stoneman, P.,1983, The Economic Analysis of Technological Change, (Oxford University Press, Oxford).

Table 1. Trigger Prices Assuming Geometric Mean Reversion

λ	Trigger Price		
	σ = 0.05	σ = 0.15	σ = 0.25
0.000	1.118	1.396	1.737
0.050	1.109	1.376	1.720
0.060	1.107	1.372	1.716
0.070	1.105	1.368	1.711
0.080	1.104	1.364	1.707
0.090	1.103	1.360	1.702

This table reports the optimal trigger price for investment in an asset costing 5 with a productivity level (δ) of 0.50. Prices follow GMR with the reversion level equal to 1.0. The discount rate is .10.

Table 2. Cumulative Investment Over
A Ten Year Period Assuming
Geometric Mean Reversion

λ	Cumulative Investment		
	σ = 0.05	σ = 0.15	σ = 0.25
0.000	.505 (.004)	.470 (.006)	.372 (.011)
0.050	.506 (.002)	.473 (.004)	.375 (.009)
0.060	.506 (.002)	.477 (.004)	.376 (.009)
0.070	.506 (.002)	.473 (.004)	.378 (.008)
0.080	.506 (.002)	.478 (.004)	.380 (.008)
0.090	.506 (.002)	.479 (.003)	.382 (.008)

This table reports the results of Monte Carlo simulations of a price process following GMR. Each simulation used 1,000 replications. The mean reversion level was 1.0, the discount rate equals .10 and the cost of the investment is 5.0. The starting price in each replication was 1.0. δ is normally distributed with mean .5 and standard deviation .15. Standard errors are reported in parentheses.

Table 3. Cumulative Investment Over
A Ten Year Period: Sensitivity Analysis

σ	ρ	P_k	λ = 0.00	λ = 0.09
.05	.10	2.5	.951 (.0004)	.953 (.0003)
.25	.10	2.5	.885 (.003)	.916 (.002)
.05	.10	10.0	.0027 (.0002)	.0010 (.00004)
.25	.10	10.0	.069 (.006)	.014 (.001)
.05	.15	5.0	.080 (.002)	.066 (.001)
.25	.15	5.0	.190 (.009)	.130 (.006)
			$\lambda = .07$	
.05	.075	5.0	.784 (.002)	.792 (.001)
.25	.075	5.0	.545 (.010)	.618 (.007)

This table reports the results of Monte Carlo simulations of a price process following GMR. Each simulation used 1,000 replications. The mean reversion level and starting price in each replication was 1.0. δ is normally distributed with mean .5 and standard deviation .15. Standard errors are reported in parentheses.

Figure 1. Price Realization with Upward Trend

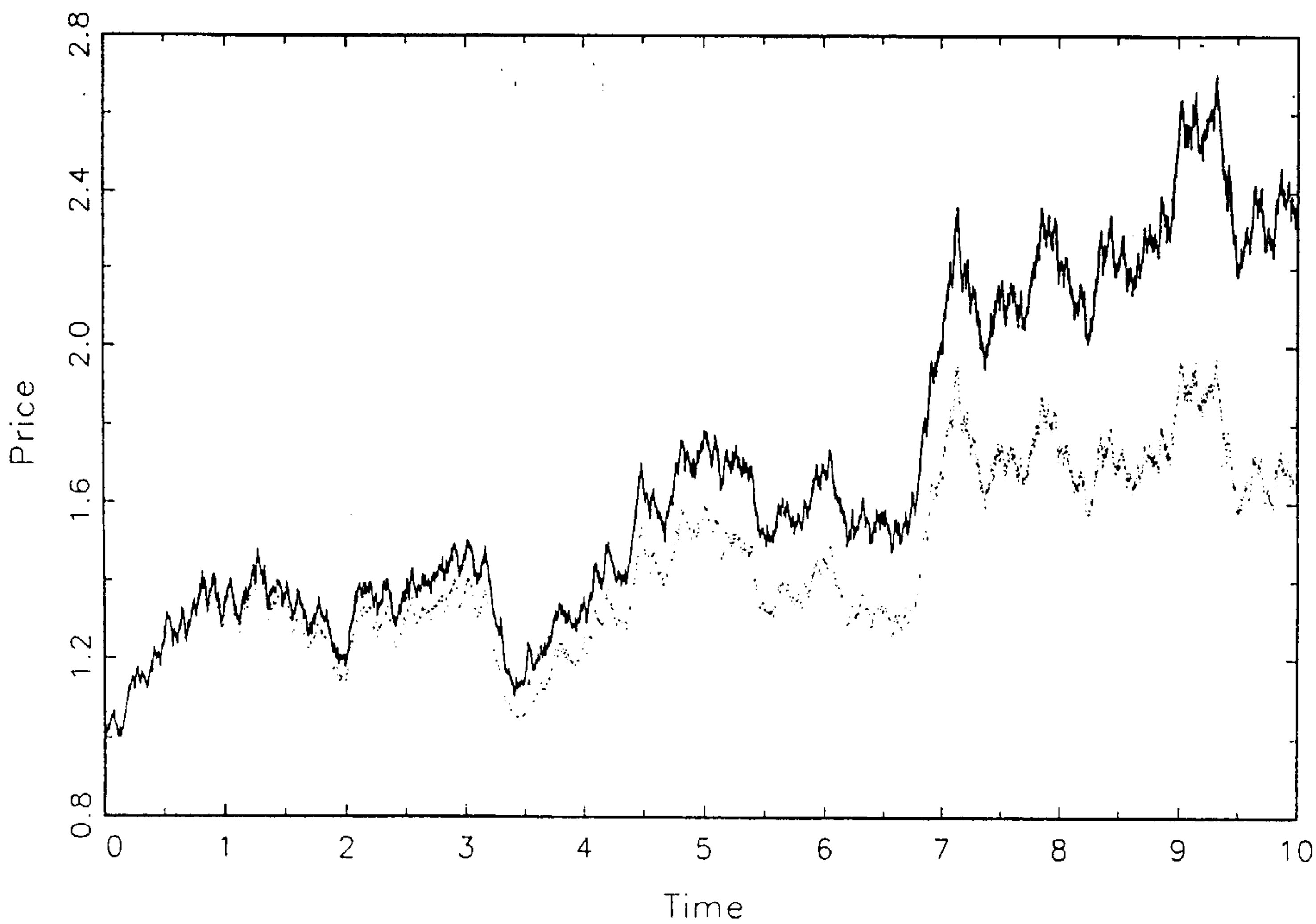

Figure 2. Price Realization with No Trend

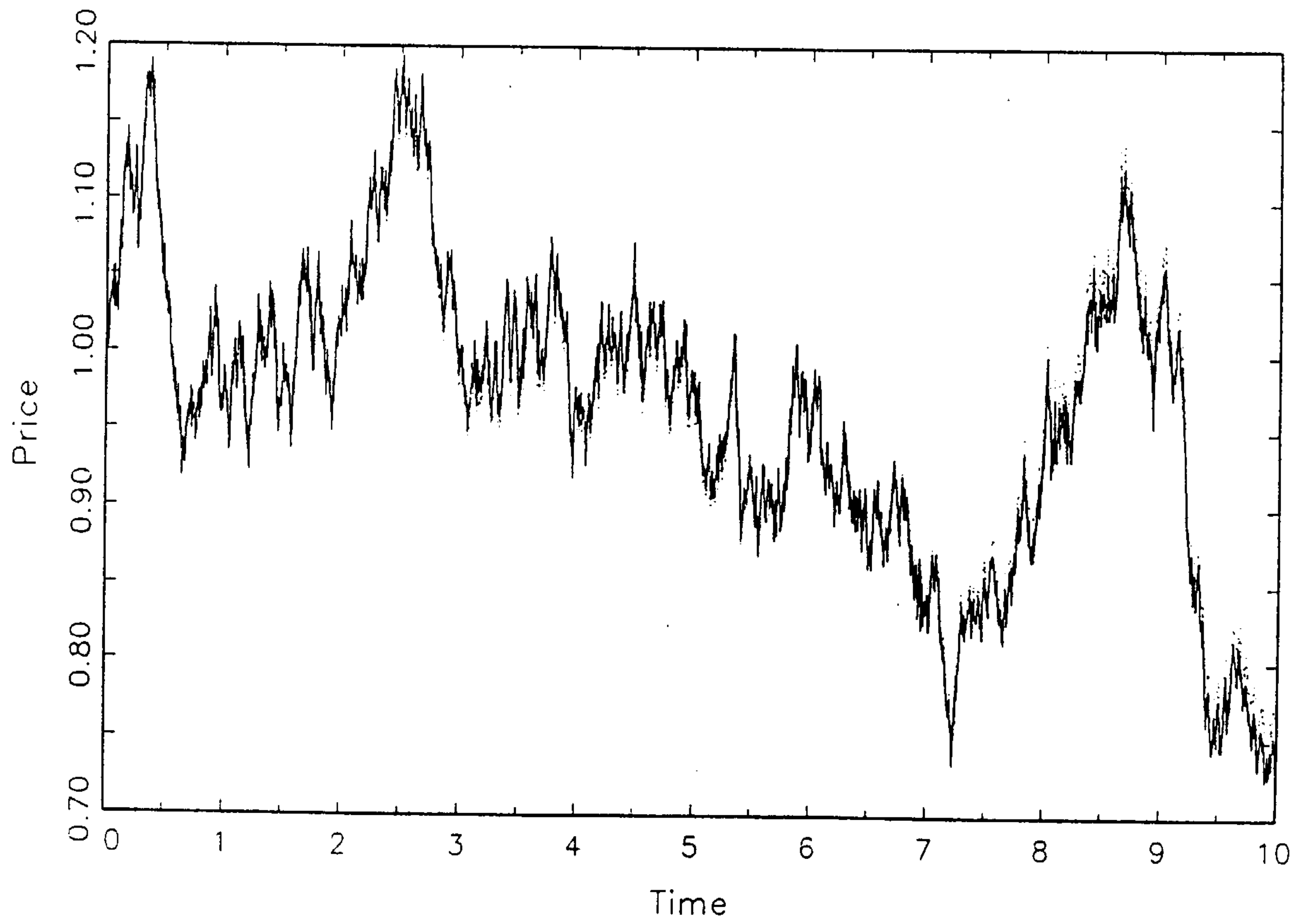

Figure 3. Cumulative Investment: Variance Effect

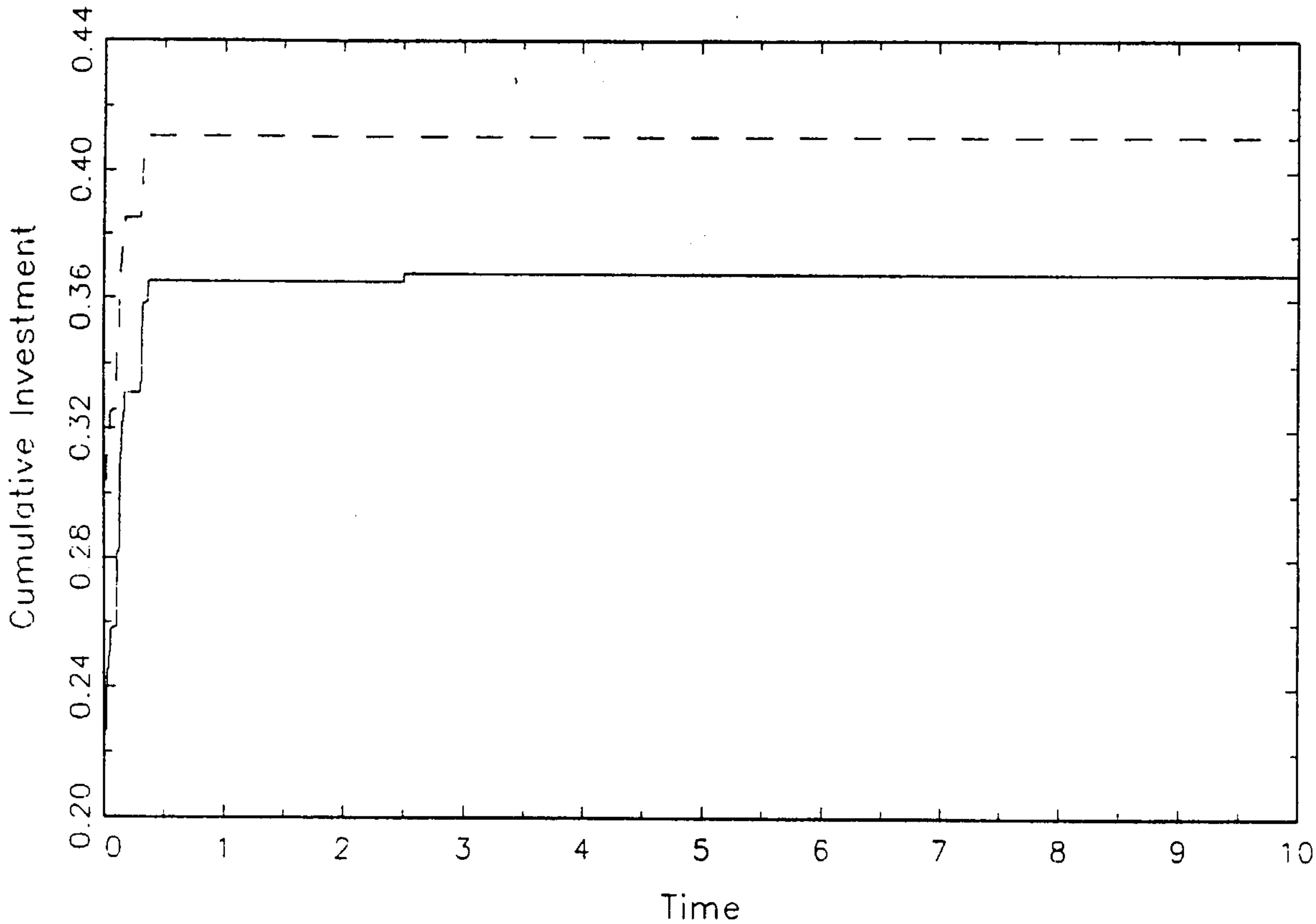

Figure 4. Cumulative Investment: Realization Effect