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SEQUENTIAL BARGAINING UNDER ASYMMETRIC INFORMATION

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ABSTRACT

We analyze an infinite stage, alternating offer bargaining game in which the buyer knows the gains from trade but the seller does not. Under weak assumptions the game has a unique candidate Perfect Sequential Equilibrium, and it can be solved by backward induction. Equilibrium involves the seller making an offer which is accepted by buyers with high gains from trade, while buyers with medium gains reject and make a counteroffer which the seller accepts. Buyers with low gains make an unacceptable offer, and then the whole process repeats itself. Numerical simulations demonstrate the effects of uncertainty on the length of bargaining.

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SEQUENTIAL BARGAINING UNDER ASYMMETRIC INFORMATION

By

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I. INTRODUCTION

A game of asymmetric information is analyzed where two parties are bargaining over the price at which an item is to be sold. The seller's valuation is common knowledge but the buyer's valuation is known only to the buyer. Each party, in turn, makes an offer. The other party either accepts or responds with a counteroffer. As they bargain, their payoffs are discounted over time, so that both have an incentive to come to an early agreement. If the bargaining game is one of complete information, Rubinstein [14] has shown that while almost any outcome can be supported as a Nash Equilibrium, there exists a unique Subgame Perfect Equilibrium. However, with asymmetric information, the Sequential Equilibrium concept does not lead the outcome of the game to be unique, instead it puts very little restriction on how the parties divide the surplus from their trade or how long it takes to reach an agreement. In a companion paper (Grossman-Perry [8]) we provide a new definition of equilibrium for games with asymmetric information, which we named Perfect Sequential Equilibrium (P.S.E)[1]. In this paper we show that our equilibrium concept can greatly restrict the set of outcomes to the bargaining game under asymmetric information. We also provide a method for solving infinite stage games where the uninformed player's information improves, due to his learning from the history of play, as the game unfolds.

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We analyze a bargaining game of incomplete information in which there is one type of (say) seller and the buyer's type is distributed according to some distribution. We restrict our attention to a specific sequential equilibrium that satisfies some very intuitive properties. We show that any P.S.E must satisfy these properties. Moreover, we show that under weak assumptions, there is a unique such equilibrium. In equilibrium, players communicate their private information by revealing their willingness to delay agreement. The most impatient buyers (i.e., those who expect the largest gain from trade), say, set A, accept the seller's offer immediately. The more patient buyers separate themselves into two different sets. The less patient, set B, responds with an acceptable counteroffer, while the more patient set C responds with an offer they know the seller won't accept. An acceptable offer reveals to the seller that the buyer belongs to set B. As the seller cannot revise his beliefs in an arbitrary manner after observing a deviation from the proposed equilibrium, we show that in a P.S.E he cannot credibly threaten to reject an offer above the discounted value of the game in which he moves first and buyers belong to the set B. If the seller gets an unacceptble offer he revises his belief accordingly to C, and the whole process repeats itself with the seller's new belief.

Our model can be regarded as a generalization of Rubinstein's rationalizing conjectures model [15]. He considers games which we characterize as "informationally small," which leads bargaining to last for only one round; after the seller makes an offer, any buyer who rejects surely makes an acceptable counteroffer. We consider "informationally larger" games where some remaining buyers make offers which they know the other side won't accept, and by so doing reveal that they have a relatively low willingness to pay, b. This can lead to many rounds of bargaining when the seller's information about b is sufficiently imprecise.

Section 2 sets up the structure of the bargaining game and provides a verbal definition of P.S.E. Section 3 proves some general results concerning the set of Sequential Equilibria (S.E) for the bargaining game, and then sets up a functional equation which is used to solve for the specific S.E we are interested in. Section 4 presents a constructive proof of the existence of a unique solution to the functional equation. Section 5 proves that any P.S.E. must have the structure developed in Sections 3 and 4. Section 6 contains numerical solutions for the equilibria as parameters of the problem vary. Substantive conclusions are drawn regarding the determinants of bargaining time.

The paper is organized in a way that allows the reader who is not familiar with P.S.E. to read through the paper and to skip Section 5. The assumptions we impose on the S.E are very plausible by themselves.

A number of works have appeared on noncooperative solutions to infinite-horizon sequential bargaining games with incomplete information. Cramton [3] and Fudenberg et.al. [5], provide excellent surveys. Sobel-Takahashi [17] assume that the uninformed player (the seller) makes all the offers. Bargaining stops only when the informed player (the buyer) accepts the current offer. By having the seller make all the offers, Sobel and Takahashi are able to avoid the complications involved with beliefs at out of equilibrium nodes. If the seller makes all the offers and the seller's valuation is known, the offers reveal no information. Along the equilibrium, there is always a positive probability that the buyer will respond with either "Yes" or "No." Thus, there is no out of equilibrium behaviour for the informed player. They prove that a unique sequential equilibrium exists, in which price must decrease over time. In a model similar to the Sobel-Takahashi's model, Cramton [1] analyzed the case where both bargainers have incomplete information about the other party's valuation.

The conjectures most apt to support an equilibrium are what Rubinstein [16] calls "optimistic conjectures," i.e., if an offer is made which is not an equilibrium offer, then the other bargainer updates his beliefs to assume he plays with the weakest opponent to him. Such an updating is sure to prevent any deviation. Using "optimistic conjectures," one needs to look just for a pair of strategies which are sequentially rational <u>along</u>

the equilibrium path; "optimistic conjectures" are sure to support it as a sequential equilibrium. Fudenberg et.al. [5] used this idea to show that the equilibrium derived by Sobel and Takahashi [17] for the game where only the seller is allowed to make offers, can often be supported as a S.E in a game where both sides alternate offers. Of course, there are a continuum of other equilibria which are also supported.[2]

Cramton [2], in another paper, extends his model to one, where in continuous time, each player, at every instant, can either make an offer or accept the most recent offer of his opponent. This paper is similar in structure to ours, except that the game is played in continuous time and both players have valuations which are private information. However, Cramtom chooses the "out of equilibrium conjectures" to convert this game into a concession game. With these conjectures, Cramton is able to make each player's strategy involve simply the choice of when to accept offers being made by an appropriately chosen clock which ticks away new prices in real time. If a player makes an offer not specified by the clock, then his opponent is assigned to have beliefs which makes the player's move suboptimal. As Rubinstein [16] has shown, this is at most one of a continuum of sequential equilibria.[3]

2. THE BARGAINING MODEL

There are two players -- one seller, with an indivisible object to sell, and one buyer. At the beginning of say, period t, player 1 makes an offer. At the beginning of period t+1 player 2 can respond with "Y" to accept or he can reject the offer and make a counteroffer $P \in R^+$. Acceptance of an offer terminates the game. If player 2 rejects the offer then at t+2 player 1 can accept player 2's counteroffer. This continues without a time limit.

An outcome of the game is a pair (P,t) which is interpreted as agreement on price P in period t. Perpetual disagreement is denoted by $(0,\infty)$. Both the buyer and seller have costs of delaying the bargaining process. Specifically, their payoffs in subsequent rounds are discounted according to the discount factor $0 < \delta < 1$. The players' payoff functions are $U_s(P,t) = \delta^{t-1}P$ for the seller and $U_b(P,t) = \delta^{t-1}(b-P)$ for the buyer of type b. While the seller's payoff is common knowledge, the buyer's payoff depends on b which is the buyer's private information. The seller's prior assessment on the buyer's type is a probability distribution F(b) and it is common knowledge.

An action at time $t > 0$ for a player specifies his reaction which is an element of $\{Y\} \cup R^+$, to his opponent's previous offer at t-1. At t=0, an action is just an offer $P \in R^+$.

As an acceptance of an offer terminates the bargaining, a relevant history at period t is just a sequence of unaccepted offers.

Let H be the set of all possible histories, and let H^t be the set of all possible histories up to and including t. Let $h^t \in H^t$ and $h^{t+k} \in H^{t+k}$; we write $h^t \subset h^{t+k}$ if h^{t+k} is a history evolving from h^t.

A strategy is a specification of what action a player takes at each of his information sets.

Assume that the seller starts the bargaining. The seller's set of strategies Σ_s, is then the set of all sequences of function $\sigma_s = \{\sigma_s^t\}_{t=0,2,4,\ldots,\infty}$ where:

$$\sigma_s^t: [H^{t-1}] \to \{Y\} \cup R^+$$

For the buyer of type b, whose first move is a response to the seller's offer, the set of strategies Σ_b, is the set of all sequences $\sigma_b = \{\sigma_b^t\}_{t=1,3,5,\ldots,\infty}$ where:

$$\sigma_b^t: H^{t-1} \to \{Y\} \cup R^+.$$

<u>Perfect Sequential Equilibrium.</u>

For the general setting of P.S.E, including examples and discussion, the reader is referred to our paper (Grossman-Perry [8]). Roughly speaking, the set of P.S.E. is a subset of the set of S.E., in which the belief at all nodes, especially at nodes off the equilibrium path, are constrained to be "self-fulfilling" beliefs (when such exist).

In order to make the above notion precise we require the strategies to specify an action at t, for each history h^t and for every possible belief that a player might have at h^t. This definition make the following question well posed: In whose interest is it to make a move which changes the beliefs of his opponent, when that move is not generated by the proposed equilibrium strategies?[4] More precisely, we ask: Is there a set of types K, such that if the opponent believes in the conditional distribution given that $t \in K$, then this move is in the interest of all $t \in K$ and just in their interest? If the answer to this question is yes then we require the opponent to believe in the conditional distribution on t given that $t \in K$. If the answer is no, then we require only that the belief have a support contained in the support of the prior distribution.

In the next section a S.E is constructed. It is then shown that this equilibrium is the unique one which is a candidate for P.S.E.

3. CONSTRUCTION OF THE EQUILIBRIUM

Suppose that the bargaining game begins with the seller making an offer. Let $F(b)$ be the cumulative probability distribution that the seller puts on the buyer's willingness to pay. Let the support of $F(\cdot)$ be $[\bar{b}_\ell, \bar{b}_h]$.

At the beginning of the game, $t = 0$, the seller's beliefs are given by $w^0(b) \equiv F(b)$. Throughout the paper when we say, "The seller believes the buyer is in $[b_\ell, b_h]$", we mean that the beliefs of the seller are described by the conditional distribution of b given $b \in [b_\ell, b_h]$, i.e.,

$$\frac{F(b) - F(b_\ell)}{F(b_h) - F(b_\ell)} .$$

Let $V(b_\ell, b_h)$ be the seller's expected payoff when it is his turn to make an offer and his belief is that $b \in [b_\ell, b_h]$ (note that in general $V(.)$ might be a function of the history). Thus, $V(b,b)$ is the Rubinstein [14] solution for the price charged by the seller in a game of certainty when the seller moves first. Recalling that the seller's reservation price is zero, and buyer b's reservation price is b, we get from Rubinstein [14], (3.1) $$V(b,b) = \frac{b}{1+\delta} .$$

It is useful to start by proving some general properties that hold in any equilibrium that satisfies sequential rationality, i.e. in any S.E.. [5]

3.a. General Properties of Any Sequentially Rational Equilibrium.

Lemma 3.1.

(i) If there is an equilibrium where the seller has beliefs $[b_\ell,b_h]$ after some history and the seller makes an offer $P > V(b_h,b_h)$, then no buyer will accept the offer.

(ii) If after any history the seller has beliefs $[b_\ell,b_h]$, then he will accept any offer greater than or equal to $\delta V(b_h,b_h)$.

(iii) If after any history the seller has beliefs $[b_\ell,b_h]$, he will reject any offer $P < \delta V(b_\ell,b_\ell)$.

Proof:

(i) Let $\bar{P}(b_\ell,b_h)$ be the highest price (i.e. the supremum of all such prices) that the seller can get in any equilibrium after any history when the support of his beliefs is contained in $[b_\ell,b_h]$. We now show that

(3.2) $$\bar{P}(b_\ell,b_h) \le V(b_h,b_h) = \frac{b_h}{1+\delta} .$$

Suppose (3.2) does not hold, then for some $\alpha > 0$ there is some point in the bargaining where some buyer must pay arbitrarily close to $\bar{P} \equiv \frac{b_h}{1+\delta} + \alpha$. Suppose this buyer deviates and counteroffers $P_c \equiv \delta\bar{P} + .9\frac{\alpha}{\delta}(1-\delta^2)$. It is easy to see that $b - \bar{P} < \delta(b - P_c)$ for all $b \in [b_\ell,b_h]$, since the inequality is equivalent to $\bar{P} > .9\alpha + \frac{b}{1+\delta}$, which must be true as $\bar{P} = \frac{b_h}{1+\delta} + \alpha$. Hence, all the buyers in $[b_\ell,b_h]$ are better off by the counteroffer, and the seller must accept it since it is better than $\delta\bar{P}$, the discounted value of the largest price the seller can get. Recall that by assumption any revision of the seller's beliefs must have a support contained in $[b_\ell, b_h]$.

(ii) Part (ii) follows immediately from part (i).

(iii) Part (iii) is proved in a similar way as part (i), by defining $\underline{P}$ as the lowest price the seller can get in any equilibrium, and showing that all buyers will accept an offer $\underline{P}_c$ which is higher than the $\underline{P}$ and satisfies $b_\ell - \underline{P}_c \geq \delta(b_\ell - \underline{P})$, unless $\underline{P} \geq \delta V(b_\ell, b_\ell)$.

Q.E.D.

Let (p_b^e, t_b^e) be the price buyer b pays and the time of agreement with buyer b <u>along</u> the equilibrium path. We can now establish the following.

<u>Lemma 3.2.</u> Let b_1, b_2 be such that $b_1 < b_2$, then $p_1^e \leq p_2^e$ and $t_1^e \geq t_2^e$.

<u>Proof:</u>

Clearly $p_i^e > p_j^e$ and $t_i^e > t_j^e$ is not an equilibrium, as b_i should follow b_j's strategy and by doing so settle earlier on a lower price. Hence, what is left to show is that neither (a) $p_1^e \geq p_2^e$ and $t_1^e < t_2^e$ nor (b) $p_1^e > p_2^e$ and $t_1^e \leq t_2^e$, can be an equilibrium. This is done by showing that for b_2 to follow his equilibrium path it must be that:

(3.3) $\quad \delta^{t_1^e}(b_2 - p_1^e) \leq \delta^{t_2^e}(b_2 - p_2^e) \quad$ which implies

$\delta^{t_1^e}(b_1 - p_1^e) < \delta^{t_2^e}(b_1 - p_2^e)$, when either (a) or (b) hold.

Thus, b_1 should deviate and follow b_2's strategy.

Q.E.D.

Lemma 3.3.

(a) If there is an equilibrium where the seller has beliefs $[b_\ell, b_h]$ after some history, and the seller makes an offer P which is rejected by the buyer, then, along the equilibrium, at the next stage the buyer either makes a unique acceptable offer P_a, or counters with an unacceptable offer.

(b) If there is an equilibrium where the seller has beliefs $[b_\ell, b_h]$ after some history, and some buyers $b \in \bar{B}$ offer an unacceptable offer $\bar{q}$ and some $b \in \underline{B} \neq \bar{B}$ offer an unacceptable offer $\underline{q}$, which are followed by different strategies, say $\underline{s}$ and $\bar{s}$, by the seller, then there exists another equilibrium where $\bar{q}$ and $\underline{q}$ are followed by a unique strategy s, which yields the same outcome as $\underline{s}$ and $\bar{s}$.

Proof:

(a) Suppose there exists an equilibrium where some buyers counter with $\bar{P}_a$ and some with a lower offer $\underline{P}_a$. Suppose the seller's strategy is to accept both offers. Clearly every buyer is better off by offering $\underline{P}_a$ rather than $\bar{P}_a$.

(b) Let $\bar{B}_i \subset \bar{B}$ and $\underline{B}_i \subset \underline{B}$ be the groups that settle in i periods after their counteroffers $\bar{q}$, and $\underline{q}$ respectively. Let $\bar{P}_i$, $\underline{P}_i$ be the seller's offer (or buyer's accepted counteroffer) under $\bar{s}$ and $\underline{s}$ respectively. If $\bar{P}_i > \underline{P}_i$, then $\bar{B}_i$ must be empty or else all $b \in \bar{B}_i$ would have made the counteroffer $\underline{q}$ rather than $\bar{q}$. Similarly, if $\bar{P}_i < \underline{P}_i$, then $\underline{B}_i$ must be empty for the same reason. Thus, whenever $\bar{P}_i \neq \underline{P}_i$, the higher price is one which no group of buyers actually pay. Let the seller switch his strategy to $\underline{P}_i \equiv \mathrm{Min}(\bar{P}_i, \underline{P}_i)$, i.e., he chooses $\underline{P}$ irrespective of whether he receives the counteroffer $\bar{q}$ or $\underline{q}$. We claim that this leads to the same outcomes as: choosing $\bar{P}$ in response to $\bar{q}$ and $\underline{P}$ in response to $\underline{q}$. To see this, note that if, say $\bar{P}_1 > \underline{P}_1$, then buyers $b \in \bar{B}_i$ for $i > 1$ must prefer to wait until period i to receive $\bar{P}_i \leq \underline{P}_i$ than to settle at time 1 for $\underline{P}_1$ (since these buyers could always have offered $\underline{q}$ instead of $\bar{q}$ and received the offer $\underline{P}_1$). Hence the strategies of these buyers will not change if the seller responds with $\text{Min}(\bar{P}_1, \underline{P}_1)$ to the offer of either $\underline{q}$ or $\bar{q}$. An identical argument holds for any date i when $\bar{P}_i > \underline{P}_i$. . Q.E.D.

We can summarize the discussion so far with the following proposition, and by Figure 1.

Proposition 3.1.

Suppose it is the seller's turn to make an offer and the seller has beliefs $[b_\ell, b_h]$. Then along the equilibrium there are numbers b_a, b_c, P and P_a such that the seller makes an offer P which is accepted if $b \geq b_a$ and rejected otherwise. The rejecting group is then divided into two intervals (each, or both of them may be empty): one interval, $[b_c, b_a)$, such that if $b \in [b_c, b_a)$, then the buyer makes an acceptable offer $P_a \geq \delta V(b_c, b_a)$; and another interval, $[b_\ell, b_c)$, such that if $b \in [b_\ell, b_c)$, then the buyer makes an unacceptable offer, which by Lemma 3.3b, we may take to be zero. An unacceptable offer reveals to the seller that $b \in [b_\ell, b_c)$. Such an offer is rejected and the whole process repeats itself where $[b_\ell, b_c)$ becomes the new $[b_\ell, b_h)$.

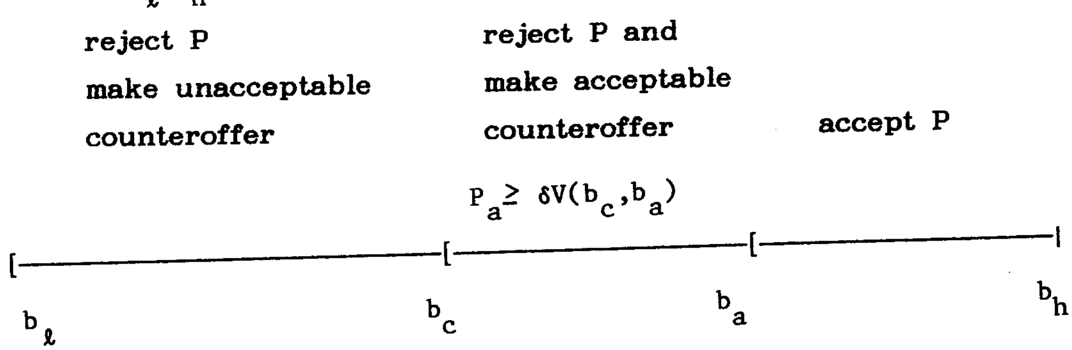

Fig.(1)

The proof of this proposition follows immediately from the above lemmae. For the rest of the paper we denote the unacceptable offer by zero.

The above discussion shows that although the notion of sequential equilibrium eliminates some Nash equilibria as possible solutions, we are still left with a huge set of eqilibria, many of which are clearly unreasonable.

It is important to note that the seller can insist on an acceptable counter offer $P_a > \delta V(b_c,b_a)$ even though he knows that the offer came from the set (b_c,b_a). For example, in the subgame where the buyer starts and the seller has beliefs $[b_\ell,b_h]$, and $b_\ell > \delta V(b_h,b_h)$, then the seller can set $P_a = \delta V(b_h,b_h)$ and all the buyers will make that offer. This is supported as a S.E. where the seller "threatens" to revise his beliefs to $[b_h,b_h]$ if he gets an "out of equilibrium counter offer" less than $\delta V(b_h,b_h)$. Rubinstein [15] called such a system of beliefs "optimistic conjectures". In such equilibria, it is as if the seller uses his lack of information to his advantage. Instead, we will be concerned with S.E. where the deviation to an out of equilibrium move q, is analyzed as follows. If the seller receives an offer q, he attempts to find a group of buyers K with the property that if the seller believed the offer was made by K and took a best response, then K is indeed the set of buyers that are better off than had they followed the equilibrium path. If such a K exists, and there is some buyer in K who is strictly better off from the deviation, then the S.E. is ruled out, and q is called a "successful deviation."

In order to find S.E. which are not subject to the above type of deviation, we will construct an S.E. with the following three important properties:

(i) After any history an individual's action depends on the history only through the effect of the history in changing the seller's beliefs.

(ii) If in equilibrium $b \in [b_c,b_a)$ is supposed to make an offer that the seller accepts, then the seller cannot credibly threaten to reject an offer above the discounted value of the continuation of the game with $[b_c,b_a)$, i.e. $P_a = \delta V(b_c,b_a)$.

(iii) If the seller receives an offer of zero, then he is not allowed to put positive probability on those buyers for whom it is a dominated move for any reasonable response.

We will show in Section 5 that every P.S.E must have these properties. In Section 4 it is shown that under some weak conditions, there exists a unique equilibrium that satisfies these three requirements.

3.b. The Stationary Equilibrium.

To help the reader follow our method of solving for the equilibrium it is helpful to present an outline of the solution first and then to explain how to solve for it.

Thus let $P(b_\ell,b_h)$ be the price the seller asks when it is his turn to make an offer and his information is that $b \in [b_\ell,b_h)$. Similarly, let $P_a(b_\ell,b_a)$ be the acceptable offer made by the buyer along the equilibrium when it is the buyer's turn to make an offer and the seller's information is that $b \in [b_\ell,b_a)$. The number $b_a(P;b_\ell,b_h)$ is defined to be the marginal buyer's type that accepts the seller's offer P when the seller's information is $b \in [b_\ell,b_h)$. Similarly we define $b_c(b_\ell,b_a)$ to be the marginal buyer that makes an acceptable counteroffer. Finally, $b \in [b_\ell,b_c)$ chooses to offer zero. When the seller receives an unacceptable offer he updates his belief to $b \in [b_\ell,b_c)$ and counters with $P(b_\ell,b_c)$ and so on.

If stationary strategies are followed the seller will be able to compute how his offer of P will affect his payoff. The following functional equation involving $V(.)$, $P(.)$, $b_c(.)$ and $b_a(.)$ describes the seller's optimization problem.

$$(3.4) \quad V(b_\ell,b_h) = \max_{P,b_a,b_c} \Big[P\frac{F(b_h)-F(b_a)}{F(b_h)-F(b_\ell)} + \delta P_a(b_\ell,b_a)\frac{F(b_a)-F(b_c)}{F(b_h)-F(b_\ell)} + \delta^2 V(b_\ell,b_c)\frac{F(b_c)-F(b_\ell)}{F(b_h)-F(b_\ell)}\Big] .$$

where the Max is subject to (3.6), (3.7), and (3.8), and where $P(b_\ell,b_h)$ is now defined as

(3.5) $P(b_\ell,b_h)$ is a P which achieves the Max in (3.4).

(3.6) $P_a(b_\ell,b_a) \equiv \delta V(b_c(b_\ell,b_a),b_a)$.

$$(3.7) \quad b_c = b_c(b_\ell,b_a) = \begin{cases} \text{Min}\{b \in [b_\ell,b_a] \mid b - \delta V(b,b_a) \geq \delta(b - P(b_\ell,b))\}. \\ b_a \text{ if } b - \delta V(b,b_a) < \delta(b - P(b_\ell,b)) \text{ for all } b\in[b_\ell,b_a] \end{cases}$$

Let

(3.8a)
$$f(b_\ell,b) = \begin{cases} b - P - \delta[b - P_a(b_\ell,b)] & \text{if } b_c(b_\ell,b) < b \\ b - P - \delta^2[b - P(b_\ell,b)] & \text{if } b_c(b_\ell,b) = b, \end{cases}$$

then

(3.8b)
$$b_a = b_a(P;b_\ell,b_h) = \begin{cases} b_h & \text{if } f(b_\ell,b) < 0 \text{ for all } b \in [b_\ell,b_h] \\ \text{Min } \{ b \mid f(b_\ell,b) \geq 0 \} \end{cases}$$

Equation (3.7) could be interpreted in the following way. Assume the players reach a subgame in which it is the buyer's turn to make an offer and the seller's belief is that $b \in [b_\ell,b_a)$. Then $b_c(b_\ell,b_a)$ is the lowest type, such that if the seller would believe that he is the lowest type among the types that prefer to settle today, then $b_c(b_\ell,b_a)$ would prefer to settle today at a price of $\delta V(b_c,b_a)$, rather than offering zero and being tomorrow in a game where the seller thinks he plays with $[b_\ell,b_c)$ and hence offers $P(b_\ell,b_c)$.

Equation (3.8) simply states that b_a is the lowest type that prefers accepting P today to waiting. How long the buyer is willing to wait depends on whether $b_c(b_\ell,b_a) < b_a$. If $b_c < b_a$ then all $b \in [b_c,b_a)$ offer P_a, and hence b_a is the marginal buyer that prefers P today to P_a tomorrow. If $b_c = b_a$, then all buyers are going to offer zero and hence b_a prefers P today to $P(b_\ell,b_a)$ two periods forward. Through (3.8) it is possible to express the Max in (3.4) with just one choice variable. It might be helpful to think of b_a as the only choice variable, where P is then determined by (3.8) as the largest price which induces b_a to be the marginal buyer.

The definition of P_a in equation (3.6) reflects an important feature of any P.S.E. which is proved in Section 5. It implies that the seller cannot use his lack of information to his advantage. The seller cannot credibly threaten to reject an offer which is above the discounted value of the game against those who are supposed to make this offer, i.e., $P_a=\delta V(b_c,b_a)$.

Equation (3.4) is a complicated functional equation in $P(\cdot)$ and $V(\cdot)$. Note that the functions $P(\cdot)$ and $V(\cdot)$ enter the right-hand-side of (3.4) through $b_c(\cdot)$ and $P_a(\cdot)$ as well as explicitly appearing there. In Section 4, existence of a solution for this functional equation is proved, and we present a method for solving it. The rest of this section is devoted to showing how this functional equation can be used to describe a vector of strategies.

Players' Strategies

We first define the seller's strategy which gives his action as a function of the history and his beliefs

$$(3.9)\qquad \sigma_s(b_\ell,b_h) = \begin{cases} \text{offer } P(b_\ell,b_h) \text{ when it is the seller's turn} \\ \text{to move, accept any offer above} \\ P_a(b_\ell,b_a(P;b_\ell,b_h)) \text{ when the buyer makes an} \\ \text{offer which follows the seller's offer of } P. \end{cases}$$

The definition of equilibrium requires us to give the buyer's action at h^t as a function only of the history h^t. Here, however, it is more convenient to define the buyer's action as a function of the seller's beliefs rather than on the history which generated that belief.

Thus, given the buyer's strategy expressed as a function of the history of offers, we can write the buyer's strategy as a function of the seller's beliefs.

$$(3.10)\qquad \sigma_b(P;b_\ell,b_h) = \begin{cases} \text{if } b \in [b_a(P;b_\ell,b_h),b_h) \text{ then accept } P; \\ \text{if } b \in [b_c(b_\ell,b_a(P;b_\ell,b_h)),b_a(P;b_\ell,b_h)) \text{ then offer} \\ \qquad P_a(P;b_\ell,b_h); \\ \text{if } b \in [b_\ell,b_c(b_\ell,b_a(P;b_\ell,b_h))) \text{ then offer zero.} \end{cases}$$

Thus, for any belief by the seller $[b_\ell,b_h)$ and any offer P he has made, the buyer can find out the seller's belief, and hence his reaction to any counteroffer he makes. The buyer then chooses whether to accept or reject P, and how to counter in case of rejection.

Having established the players' strategies (3.9) and (3.10), let us now show they form an equilibrium. Fix the buyer's strategy to (3.10), and it is immediate from the construction of the seller's strategy that (3.9) is an optimal response at any node. Fix the seller's strategy and consider the response of a buyer to the seller's offer of P. Again by construction (3.10) is an optimal response. It remains for us to show that there exists a system of beliefs for the seller such that his strategy (3.9) is a best response. There are many systems of beliefs that support this S.E. One of them is the "Passive" one. That is, if a node is reached where the seller's belief is that $b \in [b_\ell, b_h]$ and the seller's offer is P then for any response q by the buyer, the seller believes that $b \in [b_c(b_\ell, b_a), b_a(P; b_\ell, b_h)]$. It is clear that given these beliefs the seller will reject any offer below $P_a = \delta V(b_c, b_a)$, and accept otherwise.

4. SOLVING THE FUNCTIONAL EQUATION

The existence of a stationary equilibrium depends on the existence of a solution to (3.4)-(3.8). In this section we describe a method for constructing a solution to that system.

Our method involves in "Step 1" solving for the equilibrium of games where the seller has very good information about the buyer's type, then at "Step 2," solving for the equilibrium of games where the seller has slightly worse information. Once the equilibrium at "Step 1" can be found, the equilibrium at "Step 2" can be found since over time the seller's information must improve and thus force them into a node of the type solved for in Step 1. That is, first we shall solve for the equilibrium of an informationally "small" game, and then use that solution to solve an informationally "larger" game.[6]

The Two period Game:

The informationally smallest game is one of perfect certainty. From Rubinstein [14] we know that such a game will have a unique equilibrium where one party makes an offer, which the other party immediately accepts. The next larger game is where, say, the seller has information that

$b \in [b_\ell, b_h]$ and makes an offer P, which would be accepted by a buyer if, say, his $b \in [b_a, b_h]$, while if $b \in [b_\ell, b_a)$ the buyer makes a counteroffer P_a, which the seller accepts. That is, bargaining lasts two periods, and $b_c(b_\ell, b_h) = b_\ell$. If all buyers choose to make an acceptable counteroffer we can rewrite (3.4) as:

$$(4.1) \qquad V(b_\ell, b_h) = \max_{P, b_a, P_a} \left[P \frac{F(b_h)-F(b_a)}{F(b_h)-F(b_\ell)} + \delta P_a \frac{F(b_a)-F(b_\ell)}{F(b_h)-F(b_\ell)}\right].$$

subject to (4.2), (4.3) and $b_\ell \leq b_a \leq b_h$.

$$(4.2) \qquad b_a - P = \delta(b_a - P_a), \text{ where}$$

$$(4.3) \qquad P_a = \delta V(b_\ell, b_a).$$

Note that if a buyer of type b_a finds it optimal to accept P, (i.e., (4.2) holds) then all buyers of type $b \in [b_a, b_h]$ will also find it optimal to accept (i.e. $b - P \geq \delta(b - P_a)$ for all $b \in [b_a, b_h]$). We can use (4.2) and (4.3) to write (4.1) as

$$(4.4) \qquad V^1(b_\ell, b_h) = \max_{b_\ell \leq b_a \leq b_h} \left[(1-\delta)b_a \frac{F(b_h)-F(b_a)}{F(b_h)-F(b_\ell)} + \delta^2 V^1(b_\ell, b_a)\right].$$

Remark 4.1

(i) It is easily verified that $V^1(b_\ell, b_h)$ is non decreasing in b_h and in b_ℓ.

(ii) If $b_\ell = b_h$, then (4.4) is to be interpreted as $V^1(b_\ell, b_\ell) = (1-\delta)b_\ell + \delta^2 V^1(b_\ell, b_\ell)$, or solving for V^1, we get $V^1(b_\ell, b_\ell) = \frac{b_\ell}{1+\delta}$.

Note that $V^1(b_\ell, b_\ell)$ is the **Rubinstein [14]** solution for the price charged by the seller in a game of certainty when the seller moves first.

We will now show that (4.4) has a solution $V^1(.)$.

Recalling that $[\bar{b}_\ell, \bar{b}_h]$ is the support of $F(\cdot)$ let:

$$B = \{(b_\ell, b_h) \in R^2 \mid [b_\ell, b_h] \subset [\bar{b}_\ell, \bar{b}_h]\}$$

so $B \subset R^2$ is just the set of points which can serve as a lower and upper bound for the interval on which the seller's beliefs are concentrated. Let $C(B)$ be the space of continuous functions from B to R^+, normed by the "sup" norm $\left(\text{i.e., } \|g\| = \sup_{x \in B} |g(x)|\right)$ and satisfying $V(x,x) = \frac{x}{1+\delta}$.

Lemma 4.1. If $F(\cdot)$ is continuous and $0 \leq \delta < 1$, then there exists a unique $V \in C(B)$ which solves (4.4).

Proof. Let $V \in C(B)$, then the right-hand-side of (4.4) is an operator T which takes V into a new function on B, say $T \bullet V$, i.e. $T \bullet V$ maps B into R^+. $T \bullet V$ is obviously continuous at any point in the interior of B since $F(\cdot)$ and V are continuous. To see that it is continuous at the boundary, suppose (b_ℓ^n, b_h^n) is a sequence which converges to (b_ℓ, b_ℓ) but $a_1 \equiv \lim_{n\to\infty} T \bullet V(b_\ell^n, b_h^n) \neq T \bullet V(b_\ell, b_\ell) \equiv a_2$. (Note that $T \bullet V$ is bounded.) Let b_a^n be a maximizer of the R.H.S. of (4.4) with respect to b_a, when $(b_\ell, b_h) = (b_\ell^n, b_h^n)$. Clearly we cannot have $a_1 < a_2$, because it is always feasible to set $b_a^n = b_\ell^n$. On the other hand, if $a_1 > a_2$, it must be because $\frac{F(b_h^n)-F(b_a^n)}{F(b_h^n)-F(b_\ell^n)} \equiv q_n$ does not converge to 1. This is because $b_a^n \to b_\ell$, so if $q_n \to 1$, we would have $a_1 = a_2$. But then q_n converges to something less than 1 and we must have $a_1 < a_2$. Hence $T{:}C(B) \to C(B)$. It is easily shown that T is a contraction:

Note that $T\cdot V_1(b_\ell,b_h)-T\cdot V_2(b_\ell,b_h) \leq \delta^2(V_1(b_\ell,b_a)-V_2(b_\ell,b_a))$

$$\leq \delta^2\|V_1-V_2\|,$$

where the first inequality follows from the fact that if b_a is a maximizer in the R.H.S. of (4.4) when $V = V_1$, then the R.H.S. of (4.4) has a lower value when b_a is used as a maximizer when $V = V_2$. Q.E.D.

Now if P_a is "too large" then a buyer of type b_ℓ may prefer to offer zero and by so doing identifying himself as a buyer with a low willingness to pay, which leads the seller to make a low counteroffer. We will show that if b_a is sufficiently small relative to b_ℓ then we are in a two period game and all buyers in $[b_\ell,b_a)$ will indeed find it optimal to terminate bargaining when it is their move. To define "sufficiently small" we let

$$(4.5) \qquad \tilde{b}_h(b_\ell) \equiv \max\{b_h \mid b_\ell - \delta V^1(b_\ell,b_h) \geq \delta[b_\ell - V(b_\ell,b_\ell)]\}.$$

The continuity of V^1 assures existence of $\tilde{b}_h(b_\ell)$.

Remark 4.2

It is easily verified that:

(i) if $V \in C(B)$, $\delta < 1$, and $b_\ell > 0$, then $\tilde{b}_h(b_\ell) > b_\ell$.

(ii) if $[b_\ell,b_h] \subset [b_\ell,\tilde{b}_h(b_\ell)]$ in a subgame where the seller has beliefs $[b_\ell,b_h]$ the buyer will always make an acceptable offer and the game will end in one round of bargaining in equilibrium.

Thus (4.1) was constructed from (3.4) by the assumption that we are in a subgame $[b_\ell,b_h]$ where all buyers make an acceptable counteroffer immediately following the seller's first offer, i.e. $b_c(b_\ell,b_h) = b_\ell$. If we use the definition of $b_c(b_\ell,b_h)$ given in (3.7) we see that if $V = V^1$ and $\tilde{b}_h(b_\ell) > b_h$ then it will indeed be the case that $b_c(b_\ell,b_h) = b_\ell$.

Thus if $\tilde{b}_h(b_\ell) > b_h$ our equilibrium is similar to that of Rubinstein [15]. In his game a low type buyer can distinguish himself from a high type buyer by delaying the agreement one period through rejecting the seller's offer and countering with an offer which is accepted. We allow the buyer to make counteroffers which he knows will be rejected and by doing so signal that his b is very low. For the kind of uncertainty to which we have restricted ourselves up until now (and for the type of uncertainty to which Rubinstein restricted himself) the buyer does not find it optimal to use this option. However, in what follows we consider informationally larger subgames where $\tilde{b}_h(b_\ell) < b_h$ and there buyers will find it optimal to screen themselves by making unacceptable offers in equilibrium.

Larger Game:

When $\tilde{b}_h(b_\ell) < b_h$, then offering zero becomes an equilibrium move, and we are in a game which is larger than the two period game. We have already solved the system for small games, i.e., games where the seller believes the buyers are in $[b_\ell, \tilde{b}_h(b_\ell)]$. Let $V^1 \varepsilon C(B)$ and $P^1(b_\ell, b_h)$ be a solution (4.4), and hence a solution to (3.4)-(3.8) for $[b_\ell, b_h] \subset [b_\ell, \tilde{b}_h(b_\ell)]$.

Consider the right-hand-side of (3.4). That side depends on $V(\cdot)$ and $P(\cdot)$, not only through the explicit appearance of the term $\delta^2 V(b_\ell, b_c)$, but also because $b_c(\cdot)$ and $P_a(\cdot)$ depend on $V(\cdot)$ and $P(\cdot)$. Suppose that we are given a $V^i \varepsilon C(B)$ and a $P^i \varepsilon C(B)$, and we use V^i and P^i in (3.6) and (3.7) to define $b_c^i(b_\ell, b_a)$, $P_a^i(b_\ell, b_a)$, and to constrain b_a and P in (3.8). Fix V^i and P^i and consider the (RHS) right-hand-side of (3.4). When the RHS is maximized, a new function $V^{i+1}(b_\ell, b_h)$ is generated, i.e., if $X^i \equiv (V^i, P^i)$, write

$$(4.6) \qquad V^{i+1}(b_\ell, b_h) = \max_{b_\ell \leq b_a \leq b_h} Q(b_a, b_\ell, b_h; X^i)$$

where (3.8) is used to eliminate P from the maximand by taking the largest P such that $b_a = b_a(P; b_\ell, b_h)$, and $P_a(.)$ and $b_c(.)$ are computed using X^i. We will let $b_a^{i+1}(b_\ell, b_h)$ denote a maximizer of the Q on the RHS of (4.6). Further, if the maximizer P, is unique, a new function $P^{i+1}(b_\ell, b_h)$ is also generated. If V^{i+1}, P^{i+1} are used in (3.6) and (3.7) then new functions b_c^{i+1}, P_a^{i+1}, are also generated. This procedure can be iterated to generate a sequence $(V^i, P^i, P_a^i, b_c^i)_i$. We will show that if we start the procedure with the solution to a "small game" in (4.4), then the sequence converges in a <u>finite</u> number of iterations to functions (V, P, P_a, b_c) which solve (3.4) to (3.8). Further we will show that for each n there is an interval $[b_\ell, b_h^n]$ so that if $[b_1, b_2]$ lies in that interval, then $V(b_1, b_2) = V^n(b_1, b_2)$, and the game will last for at most n rounds of bargaining.

There is a technical difficulty with the iteration. Note that $V^1(\cdot)$ exists and is continuous. However it need not be the case that $P^1(b_\ell,b_h)$ is continuous in b_ℓ and b_h, where P^1 is a maximizer of the R.H.S. of (4.1). This creates a difficulty in the definition of $b_c^1(b_\ell,b_h)$, and $b_a^1(P;b_\ell,b_h)$ in (3.7) and (3.8), since $P^1(\cdot)$ appears on the R.H.S. of (3.7) and (3.8), inside a "Min" operator. A discontinuity in $P^1(\cdot)$ implies that a marginal buyer need not exist. Following the insight in Gul, Sonnenschein and Wilson [9] a marginal buyer could probably be found if the seller followed a mixed strategy, but the analysis of such strategies is beyond the scope of this paper. We will instead simply assume that $P^1(\cdot)$ is continuous. This problem repeats itself with every iteration of (4.6). Thus we assume

<u>Assumption 4.1</u>: On each iteration of (4.6) a maximizer $P^{i+1}(b_\ell,b_h)$ is generated which is continuous in b_h.

We now show how to compute the interval in which bargaining will last i rounds. Let $\bar{b}_c^i(b_\ell,b_h) \equiv b_c^{i-1}(b_\ell,b_a^i(b_\ell,b_h))$. Thus, if the seller's belief if that $b \in [b_\ell,b_h]$, then <u>along</u> the equilibrium $\bar{b}_c^i(b_\ell,b_h)$ is the lowest type to counter the seller's offer with an acceptable offer. We now define

$$(4.7a) \quad \bar{b}_h^1(x) = \sup\{y | b_a^2(b_\ell,b_h) < \tilde{b}_h(b_\ell) \text{ for all } (b_\ell,b_h) \subset [x,y]\}$$

$$(4.7b) \quad \underline{b}_h^1(x) = \sup\{y | b_a^1(b_\ell,b_h) < \tilde{b}_h(b_\ell) \text{ for all } (b_\ell,b_h) \subset [x,y]\} .$$

Thus $\bar{b}_h^1(x)$ is the largest b_h such that in the program defining $V^2(x,b_h)$, an optimizer b_a will be chosen which leads to one round of bargaining. Similarly $\underline{b}_h^1(x)$ is the largest b_h such that in the program defining $V^1(x,b_h)$, an optimizer b_a is chosen which leads to one round of bargaining. Define

(4.8a) $b_h^1(x) = \mathrm{Min}(\underline{b}_h^1(x), \bar{b}_h^1(x))$.

Next, iteratively define

(4.8b) $b_h^i(x) = \sup\{y | \mathrm{Max}(b_a^i(b_\ell,b_h), b_a^{i+1}(b_\ell,b_h)) < b_h^{i-1}(b_\ell)$

for all $(b_\ell,b_h) \subset [x,y]\}$.

The next proposition shows that if $b_h^i(b_\ell) > b_h$, then $V^i(b_\ell,b_h)$ is the solution to the bargaining problem.

Theorem 4.1. If $b_h^i(b_\ell) > b_h$ then $V^{i+1}(b_\ell,b_h) = V^i(b_\ell,b_h)$ and $P^{i+1}(b_\ell,b_h) = P^i(b_\ell,b_h)$.

Proof. We first prove the statement for i = 1.

By (4.7b), the program in (4.4) has the same value as would occur if the maximization was done subject to the constraint $b_a < \tilde{b}_h(b_\ell)$. By (4.7a), the program in (4.6) when i = 1, has a value which would be unchanged if it is done subject to the constraint that $b_a < \tilde{b}_h(b_\ell)$. But the two constrained programs are the same, and thus $V^1(b_\ell,b_h) = V^2(b_\ell,b_h)$ and $P^1(b_\ell,b_h) = P^2(b_\ell,b_h)$.

Now assume that the theorem is true for $i \leq n-1$. Artificially constrain the maximization in (4.6) for i = n-1 to satisfy $b_a \leq b_h^{n-1}(b_\ell)$. Similarly, constrain the maximization in (4.6) for i=n to satisfy $b_a \leq b_h^{n-1}(b_\ell)$. By the hypothesis that $b_h < b_h^n(b_\ell)$, it follows that these artificial constraints do not affect the value of their respective programs. Further, by the induction hypothesis, the two constrained programs are identical. This proves that $V^{n+1}(b_\ell,b_h) = V^n(b_\ell,b_h)$ and $P^{n+1}(b_\ell,b_h) = P^n(b_\ell,b_h)$, since X^n can be replaced by X^{n-1} in (4.6) when i=n. Q.E.D.

The following corollary proves the very important result that if $[b_c,b_a]$ pool to make an acceptable counteroffer, then the interval $[b_c,b_a]$ is sufficiently small that had the seller rejected and played against $[b_c,b_a]$ bargaining would end after one round.

<u>Corollary 4.1</u>. (i) $V^i(b_\ell,b_h)$ is increasing b_h.

(ii) If $b_h < b_h^i(b_\ell)$ and $b_a^i \equiv b_a^i(b_\ell,b_h)$ and $b_c^i \equiv b_c^i(b_\ell,b_a^i)$,

then (*) $V^i(b_c^i,b_a^i) = V^1(b_c^i,b_a^i)$

(**) $b_a^i \leq \tilde{b}_h(b_c^i)$

<u>Proof</u> (i) Suppose $z < y$, then $b_a^i(b_\ell,z)$, $P^i(b_\ell,z)$ are feasible choices for b_a and P in the problem where $b_h = y$. Next, note that $Q(b_a^i(b_\ell,z),b_\ell,y,X^i) \geq Q(b_a^i(b_\ell,z), b_\ell,z,X^i)$ since if Q is evaluated at $b_h = y$, it is a weighted average of a declining sequence of prices with more weight put on the earlier prices, than occurs when Q is evaluated at $b_h = z$.

(ii) Note that if $b_h < b_h^i(b_\ell)$, then $V^i(b_\ell,b_h) = V(b_\ell,b_h)$ and $P^i(b_\ell,b_h) = P(b_\ell,b_h)$ which solves the functional equations (2.4) and (4.6).

Let $b_c \equiv b_c^i(b_\ell,b_h)$ and $b_a \equiv b_a^i(b_\ell,b_h)$, then by the definition of $b_c^i(\cdot)$: $b_c - \delta V(b_c,b_a) \geq \delta(b_c - P(b_\ell,b_c))$.

But $P(b_\ell,b_c) \leq P(b_c,b_c)$, so $b_c - \delta V(b_c,b_a) \geq \delta(b_c - P(b_c,b_c))$.

Thus by (i)

(***) $b_c - \delta V(b_c,x) \geq \delta(b_c - P(b_c,b_c))$ for $b_c \leq x \leq b_a$.

Next note that $V(b_c,x)$ solves the functional equation (4.4) when b_c is thought of as a fixed parameter and x varies in $[b_c,b_a]$, since (***) assures that $b_c^i(b_c,x) = b_c$. Thus $V(b_c,x) = V^1(b_c,x)$ and $b_a \leq \tilde{b}_h(b_c)$. Q.E.D.

By Theorem 4.1, we know that a subgame $[b_\ell,b_h] \subset [b_\ell,b_h^i)]$ can be solved using our iterative procedure. If we show that for some finite n, $b_h^n(\bar{b}_\ell) \geq \bar{b}_h$, where $[\bar{b}_\ell,\bar{b}_h]$ is the support of $F(.)$, then all subgames which will arise from the initial node $[\bar{b}_\ell,\bar{b}_h]$ can be solved. This is proved next. We will require the following assumption:

Assumption 4.2

If there exists a convergent sequence $\{x_n\}$ such that $\lim_{n\to\infty}(b_a^n(b_\ell,x_n) - x_n) = 0$, then $\lim_{n\to\infty}(\bar{b}_c^n\ (b_\ell,x_n)-x_n) = 0$.

We believe this assumption unnecessary because we conjecture that $b_a^i(b_\ell,b_h)$ is bounded away from b_h; however, we have not proved this conjecture. The conjecture states that the seller never "throws away his move" by making an offer which a negligible group of buyers accepts.

Theorem 4.2. Assume that $\bar{b}_\ell > 0$ and $\delta < 1$. Then there exists a finite N such that $b_h^N(\bar{b}_\ell) \geq \bar{b}_h$, and $V^N(b_\ell,b_h)$, $P^N(b_\ell,b_h)$ satisfy (3.4)-(3.8) for all $[b_\ell,b_h] \subset [\bar{b}_\ell,\bar{b}_h]$.

Proof:

Note that $b_h^n(b_\ell) \geq b_h^{n-1}(b_\ell)$ by (4.8b). Fix $b_\ell \in [\bar{b}_\ell,\bar{b}_h]$ and use (4.8b) to generate the sequence $\{b_h^n(b_\ell)\}_n$. Since $(b_h^n)_n$ is a monotone sequence, if we show that it doesn't converge, then there exists an N such that $b_h^N \geq \bar{b}_h$. Suppose by contradiction that $\lim_{n\to\infty} b_h^n = b^*$. This means that $b_a^n(b_\ell,b_h^{n-1}+\varepsilon) \geq b^*$ for $\varepsilon > 0$ and n large. Therefore, there is a subsequence b_h^n so that $\lim b_a^n(b_\ell,b_h^{n-1}) = b^*$. From Assumption 4.2, this means that $\lim \bar{b}_c^n(b_\ell, b_h^{n-1}) = b^*$. So the intervals $[b_a^n, b_h^{n-1}]$ and $[\bar{b}_c^n, b_a^n)$ used in the functional equation (3.4) are becoming very small, while $[b_\ell, \bar{b}_c^n(b_\ell, b_h^{n-1})] \to [b_\ell, b_h^{n-1}]$. Therefore, for every small $\varepsilon > 0$ there exists an n such that (3.4) can be written:

$$(4.9) \qquad V^n(b_\ell, b_h^{n-1}) = \varepsilon + \delta^2 V^{n-1}(b_\ell, \bar{b}_c^n) \frac{F(\bar{b}_c^n) - F(b_\ell)}{F(b_h^{n-1}) - F(b_\ell)} .$$

See Remark 4.2, for the fact that $b_\ell > \bar{b}_\ell > 0$ implies $\tilde{b}_h(b_\ell) > b_\ell$, so $b^* > b_\ell$. Hence as $n \to \infty$, the fact that $\delta^2 < 1$, and $\bar{b}_c^n \to b^*$ is used in (4.9) to conclude that $\lim V^n(b_\ell, b_h^{n-1}) = \frac{\varepsilon}{1-\delta^2}$. This is impossible since:

$$V^n(b_\ell, b_h^{n-1}) \geq V^n(b_\ell, b_\ell) = V^1(b_\ell, b_\ell) = \frac{b_\ell}{1+\delta} .$$

The proof of the rest of the Theorem follows immediately from Theorem 4.1.

Q.E.D.

The previous Theorems also imply that there exists a unique V, and P which satisfy the functional equation, as long as at each stage i there is a unique maximizer in (4.6). This is because V^1 is unique (by Lemma (4.1)) and Theorems 4.1 and 4.2 show that there is a unique extension of V^1 for intervals larger than $[b_\ell, b_h^1(b_\ell)]$, given by (4.1). That is, any other function $\underline{V}(b_\ell, b_h)$ which solved the system (3.4)-(3.8) say in $[b_\ell, b_h^1(b_\ell) + \varepsilon]$ must satisfy (4.6) where V^{i+1} is replaced by $\underline{V}$ and X^i is replaced by X^1. A similar argument holds at all larger intervals.

These remarks are summarized in the following theorem.

<u>Theorem 4.3</u> If for each $[b_\ell, b_h] \subset [b_\ell, b_h^i(b_\ell)]$, the right-hand-side of (4.6) has a unique maximizer b_a, and a unique b_c, then there is a unique solution for V(.) and P(.) to (3.4)-(3.8).

5. UNIQUENESS OF PERFECT SEQUENTIAL EQUILIBRIUM

Theorem 4.3 establishes existence and gives a condition for the uniqueness of a solution to the functional equation. Here we will show that every vector of P.S.E strategies must be derived from the solution to the functional equation.[7] See the end of Section 3.a for discussion of P.S.E. Thus we will show in this section that in every P.S.E:

(i) $P_a = \delta V(b_c, b_a)$,

(ii) if $\tilde{b}_h(b_\ell) < b_h$ then offering zero is an equilibrium move, and,

(iii) nonexistence of "nonstationary" equilibria, i.e., strategies which are not derived from the functional equation.

We will thus have shown that the stationary Sequential Equilibrium constructed in Section 3 and 4 is the unique candidate for a P.S.E.. However, there are distribution functions F(.) and values for δ where a P.S.E. does not exist in pure strategies. In an earlier version of this paper we analyzed the uniform distribution in detail and showed parameter values for which a P.S.E. did and did not exist. This work is available from the authors.[8]

In what follows we will show that a Sequential Equilibrium is not a P.S.E. by finding a deviation for the buyer's counter offer, say q, and a set of buyers K such that if the seller observes q, revises his beliefs to K, and chooses a best response given this belief, then K is the set of buyers who are better off than in the proposed equilibrium. We will call such a q a "succesful deviation." Note that throughout this Section we use the idea of a metastrategy. That is we compute the seller's move after a deviation by finding a best response when his belief at that node is altered from what the Sequential Equilibrium assigns to the node.

We begin with the following Lemma which, in effect, rules out S.E. supported by "optimistic conjectures."

Lemma 5.1. Let the seller have beliefs $[b_\ell, b_h]$. If there is a P.S.E. where a buyer $b \in [b_\ell, b_h]$ accepts an offer P, then $V(b,b) \geq P$.

Proof:

Suppose to the contrary that there exists a $\tilde{b} \in [b_\ell, b_h]$ who accepts an offer P with $P > V(\tilde{b},\tilde{b})$. By Proposition 3.1, there is an interval $[b_a, b_h]$ which accepts P. Let b' satisfy $P = V(b',b')$; clearly $b' > \tilde{b}$. By Lemma 3.1(i), $b' \leq b_h$ or else no buyer would have accepted the offer P. By Lemma 3.2, $b' > b_a$, since $\tilde{b} \geq b_a$. Note that in order for $\tilde{b}$ to have paid P it must be the case that the lowest acceptable offer to the seller at the next node P_a, satisfies $\tilde{b} - P \geq \delta(\tilde{b} - P_a)$, so that $b' - P > \delta(b' - P_a)$, and hence

$$P_a > \frac{V(b',b') - (1-\delta)b'}{\delta} \equiv \delta V(b',b').$$

Let a buyer $b \in [b_a, b']$ deviate and reject P and counteroffer $P_c = \delta V(b',b')$; clearly $P_c < P_a$. Let $K = \{b \mid b$ is better off when the seller accepts P_c than b is in the proposed equilibrium$\}$. Next, note that if $b > b'$ then $b \notin K$. This is because $b - P > \delta(b - \delta V(b',b'))$ when $b > b'$. Hence by Lemma 3.1 (ii), the seller will accept $P_c = \delta V(b',b')$ if he believes that he plays against K. This shows that if the seller updates to K, the deviation is successful. Q.E.D.

The next Lemma shows that if the seller has sufficiently precise information then there is a unique P.S.E. where the seller makes an offer of $V(b_\ell, b_\ell)$ and all buyers accept. Note that there are S.E where this is false. For example, if b_ℓ and b_h are sufficiently close, the seller can make an offer $P > V(b_\ell, b_\ell)$ which satisfies $b_\ell - P = \delta(b_\ell - \delta V(b_h, b_h))$ and all buyers will accept when the seller revises his belief to $[b_h, b_h]$ after observing rejection. Note that Kreps' [10] restriction of S.E. will not eliminate this equilibrium.

Lemma 5.2. If $b_\ell > 0$ and $\frac{\partial F(b_\ell)}{\partial b} > 0$, then there exists a $b_h > b_\ell$ such that the unique P.S.E. for the game where the seller moves first and has beliefs $[b_\ell, b_h]$ is the following stationary equilibrium: the seller offers $V(b_\ell, b_\ell)$ and all buyers accept the offer.

Proof: By Lemmas 3.1(i) and 5.1, the seller can do no better than

$$(5.2) \quad \tilde{V}(b_\ell,b_h) = \max_{P,b_a} \left[P \frac{F(b_h)-F(b_a)}{F(b_h)-F(b_\ell)} + \delta(\delta V(b_a,b_a)) \frac{F(b_a)-F(b_\ell)}{F(b_h)-F(b_\ell)}\right]$$

$$\text{subject to} \quad b_a - P = \delta(b_a - \delta V(b_a,b_a)),$$

since the best that can happen is that all buyers in $[b_\ell, b_a)$ counteroffer with $\delta V(b_a,b_a)$. P can be eliminated and (5.2) can be written as

$$(5.3) \quad \tilde{V}(b_\ell,b_h) = \max_{b_a} H(b_a;b_\ell,b_h),$$

where $H(b_a;b_\ell,b_h) = (1-\delta)b_a \frac{F(b_h)-F(b_a)}{F(b_h)-F(b_\ell)} + \delta^2 V(b_a,b_a).$

We now show that if b_h is sufficiently close to b_ℓ, then the maximizer in (5.3) is $b_a = b_\ell$. To do so, we show that $\frac{\partial H}{\partial b_a}(b_a;b_\ell,b_H) < 0$ for all $b_a \in [b_\ell,b_h]$ when b_h is close to b_ℓ. By direct calculation $[F(b_h)-F(b_\ell)] \cdot \frac{\partial H}{\partial b_a} = (1-\delta)[F(b_h)-F(b_a)] -$

$$(1-\delta)b_a \frac{\partial F(b_a)}{\partial b} + \frac{\delta^2}{1+\delta}[F(b_h)-F(b_\ell)].$$

When b_h is close to b_ℓ, so also is b_a near b_h. So the two positive terms; $F(b_h)-F(b_a)$ and $F(b_h)-F(b_\ell)$ can be made arbitrarily small, so that the negative term: $-(1-\delta)\frac{\partial F(b_a)}{\partial b} b_a$ dominates. This shows that the optimal choice for b_a is b_ℓ. Hence

$$\tilde{V}(b_\ell,b_h) = (1-\delta)b_\ell + \delta^2 \frac{b_\ell}{1+\delta} = \frac{b_\ell}{1+\delta} \equiv V(b_\ell,b_\ell).$$

Note that if the seller followed the stationary equilibrium in (3.4), then by the same argument used above he will get $b_a = b_\ell$, for b_h sufficiently close to b_ℓ.

Thus the seller can do no better than to follow the strategy which he would follow in the stationary equilibrium. Further, if the seller charges a price $P > V(b_\ell,b_\ell)$, then he must do strictly worse than $V(b_\ell,b_\ell)$ because there is a probability that his first offer will be rejected. Next note that by Lemma 3.1. no buyer in $[b_\ell,b_h]$ can do better than to get an offer $V(b_\ell,b_\ell)$ on the first move. Therefore, in any equilibrium when the seller moves first and b_h is sufficiently close to b_ℓ, the seller will offer $V(b_\ell,b_\ell)$ and the buyer will accept. Q.E.D.

The following lemma is central in our theory. Essentially, it will help us to show that the seller cannot use his lack of information to his advantage. That is, he cannot credible threaten to reject an offer which is above the discounted value of the game with the group that, in equilibrium, makes this offer. It shows that the value function of small games $V^1(.)$ can be used to compute P_a. We also show that when b_h becomes larger and larger, relative to b_ℓ (i.e., $\tilde{b}_h(b_\ell) < b_h$), then there is a point where the bargaining must go on for more than two periods. This is because in a P.S.E buyers that are more patient can signal thier low willingness to pay by making an unacceptable offer. Unlike in some S.E., here the seller cannot ignore such a signal. In what follows we use $\tilde{V}(b_\ell,b_h)$ to denote the history dependent value of the game to the seller in an equilibrium other than the one of Sections 3 and 4. Similarly a "tilde" over P or P_a will indicate alternate equilibrium prices, however $\tilde{b}_h(b_\ell)$ refers to the function in (4.5).

<u>Lemma 5.3</u>: Let the seller have beliefs $[b_\ell,b_h]$ before making an offer $\tilde{P}$. Let the equilibrium specify that if $\tilde{P}$ is offered then buyers of type $[b_c,b_a]$ reject the offer and counteroffer $\tilde{P}_a$ which the seller accepts. Assume there exists a neighborhood $A \subset R$ of $[b_c,b_a]$ such that for every $[b_1,b_2] \subset A$, $\tilde{V}(b_1,b_2) = V(b_1,b_2)$. Then in any P.S.E. $\tilde{P}_a = \delta V^1(b_c,b_a)$, and if $\tilde{b}_h(b_\ell) < b_a$ then $b_c > b_\ell$, i.e. some buyers will offer zero.

<u>Proof:</u> In order for it to be an equilibrium for the seller to accept $\tilde{P}_a$ it must be that

(i) $\tilde{P}_a \geq \delta\tilde{V}(b_c,b_a) = \delta V(b_c,b_a) \geq \delta V^1(b_c,b_a)$

where $\tilde{V}(b_c,b_a)$ is the history dependent value of the continuation game where the seller moves first, and the second inequality follows from Corollary (4.1).

<u>Case (1)</u> $b_c > b_\ell$.

This implies that along the equilirium path a buyer of type b_c will be indifferent between making the offer $\tilde{P}_a$ or waiting until the next time at which an acceptable offer is made by either the buyer or seller. Let n periods elapse until an acceptable offer denoted by $\tilde{P}$ is made. Then $b_c - \tilde{P}_a = \delta^n(b_c-\tilde{P})$. From Lemma 3.1, $\tilde{P} \leq V(b_c,b_c)$. Thus, if $\tilde{P}_a > \delta V^1(b_c,b_a)$ then $b_c - \delta V^1(b_c,b_a) > \delta^n(b_c-V(b_c,b_c))$. From the fact that $V^1(.)$ is a continuous function (by Lemma 4.1), it follows that $b_1 - \delta V^1(b_1,b_2) > \delta^n(b_1-V(b_1,b_1))$ in a neighborhood of $[b_c,b_a]$. We may take this neighborhood to be A. Hence, by the definition of $\tilde{b}_h(.)$ in (4.5) and corollary 4.1 (ii) it follows that $V(b_1,b_2) = V^1(b_1,b_2)$ for all $(b_1,b_2) \varepsilon A$, and we use this to evaluate the following deviation. In order for it to be an equilibrium for any $b \varepsilon [b_c,b_a)$ to offer $\tilde{P}_a$, the seller's strategy must involve rejecting any offer below a number $\tilde{P}_a$, and accepting any offer above $\tilde{P}_a$. Suppose $\tilde{P}_a > \delta V^1(b_c,b_a)$. Consider a deviation from $\tilde{P}_a$ to $q = \tilde{P}_a - \varepsilon > \delta V^1(b_c,b_a)$. The set of buyers who are better off if q is accepted is $K = [b_c-\varepsilon_1(\varepsilon),\ b_a+\varepsilon_2(\varepsilon)]$, where ε_1, ε_2 are non-decreasing continuous functions of ε, with $\varepsilon_1(0) = \varepsilon_2(0) = 0$, given by

(ii) $b_c - \varepsilon_1 - q = \delta^n(b_c-\varepsilon_1-\tilde{P})$ and

$$b_a + \varepsilon_2 - \tilde{P}(b_\ell,b_h) = \delta(b_a+\varepsilon_2-q).$$

By Lemma 4.1, there exists $\varepsilon > 0$ and $\delta V^1[b_c-\varepsilon_1(\varepsilon),\ b_a + \varepsilon_2(\varepsilon)] \leq q$. Thus, if the seller revises his belief to K it is a best response for him to accept q. Further, when he accepts, all $b \varepsilon K$ are better off than they would have been had they followed the proposed equilibrium. Clearly q is a successful deviation.

Case (2) $\quad b_c = b_\ell$

(A) Suppose $\tilde{b}_h(b_\ell) < b_a$. We prove the last part of the Lemma by showing that there is a successful deviation to a counteroffer of zero. Let $\tilde{V}_b(b_\ell,b)$ be the value to buyer b of a continuation game where he offers zero and the seller believes he plays against $[b_\ell,b]$. Let

(iii) $b' = \text{Sup } \{b \ \varepsilon\ [b_\ell,b_h] \mid b - \tilde{P}_a \leq \tilde{V}_b(b_\ell,b)\}$.

Recall that $\tilde{b}_h(b_\ell) < b_a$ means that $b_\ell - \delta V^1(b_\ell,b_a) < \delta(b_\ell-P(b_\ell,b_\ell))$. Hence, since $\tilde{P}_a \geq \delta V^1(b_\ell,b_a)$ it must be the case that the "Sup" is taken over a set which includes b_ℓ. Next note that from Lemma 5.2, there exists a $b > b_\ell$ such that $\tilde{V}_b(b_\ell,b) = \delta(b_\ell-P(b_\ell,b_\ell))$. Hence $b' > b_\ell$. This shows that if $K = [b_\ell,b']$ and there is a deviation to a counteroffer of zero by the buyer, then the best response of the seller will make all buyers in K better off than had they followed their equilibrium strategy. Thus $q = 0$ is a successful deviation.

(B) Suppose $\tilde{b}_h(b_\ell) \geq b_a$ and $\tilde{P}_a > \delta V^1(b_\ell, b_a)$. Consider a deviation to $q = \tilde{P}_a - \varepsilon > \delta V^1(b_\ell, b_a+\varepsilon_2)$, where ε_2 is as given in (ii) of Case (1) above. By an argument similar to that given in Case (1), there is a successful deviation by buyers when the seller revises his belief to $K = [b_\ell, b_a+\varepsilon_2(\varepsilon)]$. Q.E.D.

Corollary 5.3 Assume a node is reached where the seller believes the buyer's type is in $[b_c, b_a)$ and in equilibrium any buyer of type $[b_c, b_a) \equiv A$ makes an offer $\tilde{P}_a$ which the seller accepts. If $\tilde{P}_a > \delta V(b_c, b_a)$, then in any neighborhood of (b_c, b_a) there exists a subgame $S \subset A$ such that $\tilde{V}(S) > V(S)$ and $\tilde{P}_a \leq \delta\tilde{V}(S)$.

Proof: Consider a deviation to $q = \tilde{P}_a - \varepsilon$. Let $S(\varepsilon) = [b_c - \varepsilon_1(\varepsilon), b_a + \varepsilon_2(\varepsilon)]$ as defined in (ii) of the previous proof. In order for the deviation not to be successful, it must be the case that

(iv) $\tilde{P}_a - \varepsilon \leq \delta\tilde{V}(S(\varepsilon))$ for all $\varepsilon > 0$,

since otherwise the seller would accept q. But since V(.) is continuous, and $\tilde{P}_a > \delta V(b_c, b_a)$, there is an $\alpha > 0$ such that $\tilde{P}_a > \delta V(S(\varepsilon)) + \alpha$, for some range of $\varepsilon > 0$. Hence, from (iv) $V(S(\varepsilon)) < \tilde{V}(S(\varepsilon))$ and $\tilde{P}_a \leq \delta\tilde{V}(S(\varepsilon))$ for ε sufficiently small. Q.E.D.

Having proved these Lemmae, we can proceed now to prove that there is no other P.S.E. but the one derived from the functional equation. The idea of the proof is that, suppose there is some subgame $[b_\ell, b_h)$ where the seller can follow a strategy which does better for him than $V(b_\ell, b_h)$. Then this will be possible only if there is some subinterval $[L_1,H_1] \subset [b_\ell,b_h]$ where the seller will do better than $V(L_1,H_1)$. This argument is iterated to generate intervals $[L_n,H_n] \subset [L_{n-1},H_{n-1}]$ which collapse to a single point, i.e., a game of certainty. Thus we show that the seller can do better than $V(b_\ell, b_h)$ only because there is some subgame in which he has perfect information where he does better than the unique equilibrium to the perfect information bargaining game.

Theorem 5.1. If $b_\ell > 0$, $\frac{\partial F}{\partial b} > 0$ for $b \in [\bar{b}_\ell, \bar{b}_h]$, and there is a unique maximizer b_a and a unique b_c on the right-hand-side of (4.6) for each $[b_\ell, b_h] \subset [\bar{b}_\ell, \bar{b}_h]$, then there is at most one P.S.E. to the bargaining game.

Proof:

Recall that the support of $F(\cdot)$ is $[\bar{b}_\ell, \bar{b}_h]$. Let

$$b_h^o = \sup\{b_h \in [\bar{b}_\ell, \bar{b}_h] \mid \text{every subgame of } [\bar{b}_\ell, b_h] \text{ has a unique equilibrium}\}.$$

From Lemma 5.2, $b_h^o > \bar{b}_\ell$. We begin by showing that every subgame of $[\bar{b}_\ell, b_h^o]$ has a unique equilibrium. Suppose not, then by definition of b_h^o, either $[b_\ell, b_h^o]$ or $(b_\ell, b_h^o]$ does not have a unique equilibrium for some $b_\ell \geq \bar{b}_\ell$. Let $\tilde{V}(b_\ell, b_h^o)$ be the value of the subgame where the seller moves first and believes either $[b_\ell, b_h^o]$ or $(b_\ell, b_h^o]$. Note that

$$(5.3) \quad \tilde{V}(b_\ell, b_h^o) = \max_P [P \text{ Prob}(P \text{ is accepted}) + \delta \tilde{P}_a(P) \text{Prob(acceptable counteroffer)} + \delta^2 \tilde{V}(b_\ell, \tilde{b}_c) \text{Prob (seller gets to make another offer)}].$$

where $\tilde{P}_a(P)$ is the lowest acceptable price given the history, and $\tilde{V}(b_\ell, \tilde{b}_c)$ is the value of the game to the seller when he does not accept a counteroffer and revises his beliefs to be that he plays against $[b_\ell, \tilde{b}_c)$. Recall from Proposition 3.1, that in any equilibrium there is an offer made by the seller which a group $[\tilde{b}_a, b_h^o]$ accepts, and another group $[\tilde{b}_c, \tilde{b}_a)$ makes a counteroffer $\tilde{P}_a$ which the seller accepts -- where either group can be empty, i.e. $\tilde{b}_a = b_h^o$ and/or $\tilde{b}_c = \tilde{b}_a$. We will consider separately the following three cases: (i) $\tilde{b}_a < b_h^o$; (ii) $\tilde{b}_a = b_h^o$ and $\tilde{b}_c = b_h^o$; (iii) $\tilde{b}_a = b_h^o$ and $\tilde{b}_c < \tilde{b}_a$.

Case (i) If in the proposed equilibrium $\tilde{b}_a < b_h^o$, then it is impossible to have $\tilde{V}(b_\ell, b_h^o) > V(b_\ell, b_h^o)$. This is because $\tilde{V}(\tilde{b}_c, \tilde{b}_a) = V(\tilde{b}_c, \tilde{b}_a)$ since $[\tilde{b}_c, \tilde{b}_a) \subset [\bar{b}_\ell, b_h^o)$, and $\tilde{V}(b_\ell, \tilde{b}_c) = V(b_\ell, \tilde{b}_c)$ since $[b_\ell, \tilde{b}_c) \subset [\bar{b}_\ell, b_h^o)$. Note that if $\tilde{V}(\tilde{b}_c, \tilde{b}_a) = V(\tilde{b}_c, \tilde{b}_a)$, and $\tilde{b}_a < b_h^o$, then by Lemma 5.3 $\tilde{P}_a = \delta V(\tilde{b}_c, \tilde{b}_a)$. That is, the only way that the seller can do better than $V(b_\ell, b_h^o)$ after a history h^t which leaves him with beliefs $[b_\ell, b_h^o]$ is if there is some subgame where he does better than the $V(\cdot)$ for that subgame. Similarly if $\tilde{V}(b_\ell, b_h^o) < V(b_\ell, b_h^o)$ it must be that in some subgame of h^t where beliefs are $[b_\ell, b_h] \subset [b_\ell, b_h^o]$, the seller does worse than the $V(\cdot)$ for that subgame, but this is impossible as $\tilde{b}_a < b_h^o$.

Case (ii) Consider now the case where no buyer accepts the seller's offer of P, and all buyers counter with an unacceptable counteroffer. That is, in the proposed equilibrium $b_h^o = \tilde{b}_a = \tilde{b}_c$. Thus we can write (5.3) as

$$(5.4) \quad \tilde{V}_t(b_\ell, b_h^o) = \delta^2 \tilde{V}_{t+2}(b_\ell, b_h^o).$$

Case (iii): In this case $b_h^o = b_a > \tilde{b}_c$. When the first term on the RHS of (5.3) is absent we can write (5.3) as

$$(5.5) \quad \tilde{V}(b_\ell, b_h^o) = \delta \tilde{P}_a \cdot \text{Prob}(b \in [\tilde{b}_c, b_h^o]) + \delta^2 V(b_\ell, \tilde{b}_c) \cdot \text{Prob}(b \in [b_\ell, \tilde{b}_c)).$$

Note that in (5.5) $\tilde{V}(b_\ell, b_h^o) > V(b_\ell, b_h^o)$ only if $\tilde{P}_a > \delta V(\tilde{b}_c, b_h^o)$. By Lemma 5.3 and its corollary this can be true only because in some subgame S which is close to $[\tilde{b}_c, b_h^o]$, $\tilde{V}(S) > V(S)$, and $\tilde{P}_a \leq \delta\tilde{V}(S)$. This can happen only if $b_h^o \in S$. Clearly

(5.6) $\tilde{V}(b_\ell, b_h^o) \leq \delta^2 \tilde{V}(S)$

Thus in both case (ii) and case (iii) a given game $[b_\ell, b_h^o]$ generates a subgame $[b_1, b_h^o]$ such that $\tilde{V}(b_\ell, b_h^o) \leq \delta^2 \tilde{V}(b_1, b_h^o)$ and $\tilde{V} \neq V$. The same argument can now be applied to the subgame $[b_1, b_h^o]$, to generate a new subgame $[b_2, b_h^o]$ such that $\tilde{V}(b_2, b_h^o) \neq V(b_2, b_h^o)$ and $\tilde{V}(b_1, b_h^o) \leq \delta^2 \tilde{V}(b_2, b_h^o)$. This argument must be repeatable an arbitrary number of times, and since $\delta^2 < 1$ it implies that $\tilde{V}(b_\ell, b_h^o) = 0$, which contradicts $\bar{b}_\ell > 0$ and Lemma 3.1.

We have now shown that every subgame of $[\bar{b}_\ell, b_h^o]$ has a unique equilibrium. We next show that if $b_h^o < \bar{b}_h$ then there exists $\varepsilon > 0$ such that every subgame of $[\bar{b}_\ell, b_h^o + \varepsilon]$ has a unique equilibrium for $\varepsilon > 0$ and sufficiently small. The argument is similar to that of the last paragraph. Suppose that $\tilde{V}(\bar{b}_\ell, b_h^o + \varepsilon) \neq V(\bar{b}_\ell, b_h^o + \varepsilon)$. This is clearly impossible if in this subgame the seller makes an offer P such that $(\tilde{b}_a, b_h^o + \varepsilon]$ accept and $\tilde{b}_a < b_h^o$, since all such subgames of the subgame starting from $[\bar{b}_\ell, b_h^o + \varepsilon]$ will have unique equilibria. Hence it must be that $\tilde{b}_a > b_h^o$. Therefore, by the continuity of F, the first term on the RHS of (5.3) can be made **arbitrarily** small, so (5.3) can be written as

$$(5.7)\quad \tilde{V}(b_\ell, b_h^o+\varepsilon) = \alpha + \delta\tilde{P}_a \text{Prob}(b \in [\tilde{b}_c, \tilde{b}_a)]) + \delta^2\tilde{V}(b_\ell, \tilde{b}_c)\text{Prob}(b \in [b_\ell, \tilde{b}_c]),$$

where $\varepsilon > 0$ can be chosen to make α arbitrarily small. As in the last paragraph, it must be the case that there exists a subgame S in which $\tilde{V}(S) \neq V(S)$, and $\tilde{V}(b_\ell, b_h^o+\varepsilon) \leq \alpha + \delta^2\tilde{V}(S)$, and a similar statement must hold for an infinite sequence of subgames of this game. Using the fact that $\delta^2 < 1$ we get the same contradiction as before Q.E.D.

6. CONCLUSIONS

We have developed a model of the losses associated from bargaining over the division of a surplus. Buyers who get a small surplus out of the relationship delay settlement to reveal their willingness to pay the seller. If the seller begins bargaining believing that the buyer's willingness to pay is $b \in [b_\ell, b_h]$, then after rejection of his offer, he will believe either: (i) that $b \in [b_c, b_a)$ if the rejection is followed by an "acceptable" counteroffer, or (ii) that $b \in [b_\ell, b_c)$ if rejection is not followed by an acceptable counteroffer, where $b_\ell \le b_c \le b_a \le b_h$. If we let b_a^1 be such that $b \in [b_a^1, b_h]$ accepts the seller's first equilibrium offer and b_c^2 is such that $b \in [b_c^2, b_a^1)$ makes an acceptable counteroffer, then we can iteratively generate $b_a^{i+1} = b_a(b_\ell, b_c^i)$ and $b_c^{i+1} = b_c(b_\ell, b_a^i)$, using the functions $b_a(\cdot)$ and $b_c(\cdot)$ which solve the functional equation. If $b \in [b_c^{i+1}, b_a^i]$, then there will be i periods of bargaining.

Thus, the equilibrium determines a function $t(b;F)$ which specifies how many periods a buyer of type b will wait before agreeing to a trade, as a function of the seller's prior distribution F. If $F(\cdot)$ is concentrated on single b, (i.e., there is certainty) then $t = 1$; bargaining ends immediately. As the "uncertainty" about b increases $t(b;F)$ will increase. For example, if we start with an F with support $[\bar{b}_\ell, \bar{b}_h]$ and a positive density therein, and generate a new F, say F_1 as $F_1(b) = \text{Max}\ [1, \frac{F(b)}{F(b_c^2)}]$, then $t(b;F_1) = t(b;F) - 2$. That is, if the seller improves his information by learning that $b \le b_c^2$ then the play of the game with beliefs F_1 is as if it is the third move of the original game where he had information F, rather than F_1.

In general, it is difficult to prove a statement about how a change in F, changes $t(b;F)$. However, the algorithm described in Section 4 can be used to compute the solution to the functional equation for various distributions F. Figure 2 presents solutions for various truncated Normal distributions. Values for the support of F, b_ℓ and b_h are given near the upper left and right hand corners respectively. The x-axis is divided into 201 equally spaced grid points between b_ℓ and b_h inclusive. The game is discritized so that the i^{th} grid point x_i has mass $F(x_i)-F(x_{i-1})$, with $F(b_\ell) = 0$. In solving the functional equation $V(b_\ell,b_h)$ and $P(b_\ell,b_h)$ are represented as 201x201 matrices of real numbers, and in the maximization in equation (4.6) the choice of b_a is restricted to the 201 grid points. Similarly, the choice of b_c is restricted to those grid points.

In Figure 2 we consider the case where the discount factor is 0.75 and buyer's type is distributed normally with a mean of 50. We let the standard deviation vary from 100 in the first graph to 5 in the last and we were interested in how the length of the bargaining is affected. The numbers inside each graph give the respective probabilities of each segment $[b_c^{i+1},b_a^i]$, and thus the probability of i period bargaining. The scale of the χ-axis is given on the upper left and upper right corners of each graph. The scale on the first graph is of equally spaced points between 0 and 100, while the scale on the last graph is of equally spaced points between 30 and 50. Thus, if the standard deviation is 100, the probability of 1 period bargaining is 0.2777, 2 periods is 0.6088, and so on. While if the standard deviation is 5, the probability of less than 1 period bargaining is 0.91. In general, one can see in the Figure that the level of uncertainty has a positive effect on the length of the bargaining.

Figures 3 and 4 present the results of numerical simulations for δ=0.8 and δ=0.85 respectively. As δ increases there are usually more periods of bargaining, since less surplus is lost from delaying agreement.

------ put Figure 2 here -----

------ put Figure 3 next -----

------ put Figure 4 next -----

Table 1 presents results on the outcome of the bargaining as a function of the length of time between offers. If r is the annual interest rate, and L is the number of days between successive offers, then we set $\delta = 1 - rL/365$. We use r = .10, and L varies from two days to .1 day. The Table reports the expected number of days of bargaining assuming that each bargaining period last L days. It also reports the maximum number of periods over which bargaining occurs, i.e., how many periods it takes the lowest b in the support to settle. In addition, the Table reports the average price at which trade takes place, relative to the price which would have been obtained had the seller made the offer which all buyers would accept, namely $V(b_\ell,b_\ell)$. This information is repeated for the one-sided bargaining problem studied in [5,9]. In that problem the buyer is prevented from making a counteroffer in the even periods, and the seller makes offers in odd-numbered periods which the buyer either accepts or rejects. In order to help with the interpretation of the Table, consider the first three numbers of the column labelled "2". In the two-sided bargaining game, with a uniform distribution on [1,25] with each period lasting two days, (i) bargaining is expected to last 73 days, (ii) the seller expects to receive a price 32% higher than he would get if he made the offer $V(b_\ell,b_\ell)$ for which bargaining would end in one round, and (iii) it takes 62 periods for bargaining to end with probability one. Moving down to the next three numbers in the same column of the Table, we see that if the buyer is constrained not to make a counteroffer, then bargaining can be expected to last 39 days, and the expected price is 8.6% higher than the price which would end bargaining immediately, namely b_ℓ (i.e. 1).

----- put Table 1 here -----

We have examined in detail the equilibrium strategies for the two-sided bargaining game reported in the Table, and they have the property that the buyer chooses not to make an acceptable counteroffer until the seller's beliefs are such that the lowest b in the support of F(·) makes an acceptable counteroffer, i.e., $b_c(b_\ell,b_h)$ equals either b_ℓ or b_h. We have found this property in all simulations where δ is close to one. That is, even though buyers are permitted to make counteroffers, the seller chooses his offers to discourage the buyers from making the offers. The seller's desire to prevent counteroffers makes it credible for him to delay settlement by longer than occurs in the one-sided bargaining game (in which the buyer is exogenously restricted from making offers). Note that the value of the informationally smallest game (which is the basis of the backward induction) to the seller is about twice as large in the one-sided as in the two-sided bargaining game, and this may be the reason for the substantially longer period of delay in the latter game. The Table shows that the two-sided bargaining model predicts that settlement takes about twice as long (and that the expected percent price increment from bargaining is about four times as large) as in the one-sided bargaining model.

The result that settlement occurs sooner in the model where the buyer is constrained not to make a counteroffer implies that it is a more efficient mechanism than the one where the buyer is permitted to make a counteroffer. This is because, from an efficiency point of view, the division of the surplus is irrelevant; only the length of time required to reach a settlement affects the efficiency of the outcome.

By comparing the expected duration of bargaining in the problem with support [1,25] to the problem with support [1,50], it is clear that increasing the uncertainty has substantial effects. The "largest" uncertainty is in the game $[1,\infty]$. Unfortunately, as b_h gets large, the algorithm we are using for the two-sided bargaining problem is unable to solve $[1,b_h]$ on the computer in a reasonable amount of time with reasonable accuracy. However, "large" one-sided bargaining games can be accurately solved, and results for the game [1,100] are reported. It is also possible to analytically solve for a solution to the game [0,1] in the one-sided bargaining model. The solution in which the marginal buyer b_a is a linear function of b_h in the subgame $[0,b_h]$ can be computed using the results in [5], to be $b_a(b_h) = xb_h$, where $x = [1 + \sqrt{1-\delta_2}]^{-1}$. If we maintain compatibility with the previous analysis by requiring two periods to elapse between each of the seller's offers, then the expected number of periods of bargaining is

$$1\cdot(1-x) + 3\cdot(x-x^2) + 5(x^2-x^3) + \dots = (1+x)/(1-x).$$

Further, since two periods elapse between the seller's offers, we must set $\delta_2 = \delta^2$, if δ is the discount rate per period in the two-sided bargaining game. The expected price at which trade will take place can be computed to be $y/(1+x)$ where $y = x(1-\delta_2)/(1-\delta_2 x)$, and yb_h is the price charged by the seller when he believes that he plays against $[0,b_h]$.

The numerical simulations for the one-sided bargaining model illustrate the Theorem in [9] which states that the duration of bargaining goes to zero as the time between offers goes to zero. This result has recently been extended by F. Gul and H. Sonnenschein to two-sided bargaining games. The rate of convergence is "slow"; from the Table, even if there are 10 bargaining periods per day, bargaining can have substantial duration. In some situations, such as on the trading floor of a stock or commodity exchange, bargaining duration is very short (say, less than five minutes) and traders talk sufficiently rapidly so that many offers can be exchanged in a period of five minutes. In other situations, labor negotiations, for example, bargaining can last many weeks, and the ability of any side to make more than one binding offer per day is severely limited.

It is important to recognize that the seller must be bound to his offer for our model to be relevant. If the seller could take back his offer after a buyer accepts, then he would usually do so. This is because acceptance reveals that the buyer has a relatively high willingness to pay. In labor negotiations, or takeover bid negotiations, the negotiators often need approval from a "higher" authority, and it can be costly for the "higher" authority to meet and deliberate very frequently, especially when there is a fixed cost to holding a meeting.

The sequential bargaining model is capable of explaining the duration of disagreement in bargaining situations as a function of the asymmetry of information. This model is to be contrasted with the mechanism design approach of Myerson [13] and in particular to the model of unemployment in, e.g., Grossman and Hart [7]. In the latter it is the ex-ante desire to share risks that lead parties to choose ex-post inefficient mechanisms when there is asymmetric information ex-post, but symmetric information ex-ante. In such models parties are able to commit themselves ex-ante to engage in ex-post inefficient outcomes. In the bargaining model studied here such commitments do not exist. Our results thus seem applicable to situations where, for example, workers and firms sign 2-year contracts which do not bind their behaviour after the second year. Once the initial contract ends a new bargaining problem begins, and there can be strikes at the expiration of a contract. An interesting open question is to explain why it is that the parties choose to put themselves in a position where after 2 years they will face an unconstrained bargaining problem rather than writing a very long term contract which constrains their future renegotiations.

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FOOTNOTES

1. The equilibrium concept is a game theoretic version of the one in Grossman [6]. Joe Farrell [4] has independently proposed an equilibrium concept similar to ours.

2. We do not analyze a concession game, i.e., a game where only one party can make offers and the other only accepts or rejects, because the party that is unable to make counteroffers is artifically restricted. It is easy to show that in a game where both types have valuations which are common knowledge, the party that makes the offers gets all the surplus in the unique Subgame Perfect Equilibrium.

3. In Cramton's formulation, after the seller reveals his type (i.e., where Cramton's subgame is identical to our game) the seller chooses a price to offer at each time $P(t)$ and the only choice permitted to the buyer is to accept $P(t)$ or wait until, say t', and accept $P(t')$; Cramton [2]. Cramton does not appear to consider the possibility that the buyer could make a counteroffer q which would induce the seller to accept q rather than wait and play his continuation strategy. Of course, this behavior can be made a sequential equilibrium, if the seller revises his beliefs "optimistically" about the buyer when he receives an offer $q < P(t)$ at time t. Cramton chooses $P(t)$ to be the certainty Rubinstein solution in the game between the seller and the particular buyer at t who in equilibrium offers $P(t)$. Thus he constructs the equilibrium to be the solution to a particular concession game.

4. This argument has also been used by Grossman [6], Kreps [10], Kreps and Wilson [11], Milgrom and Roberts [12] and Myerson [13].

5. In what follows we only consider S.E. where a revision in beliefs does not increase the support of the distribution representing the seller's beliefs. Further we only consider pure strategy equilibria.

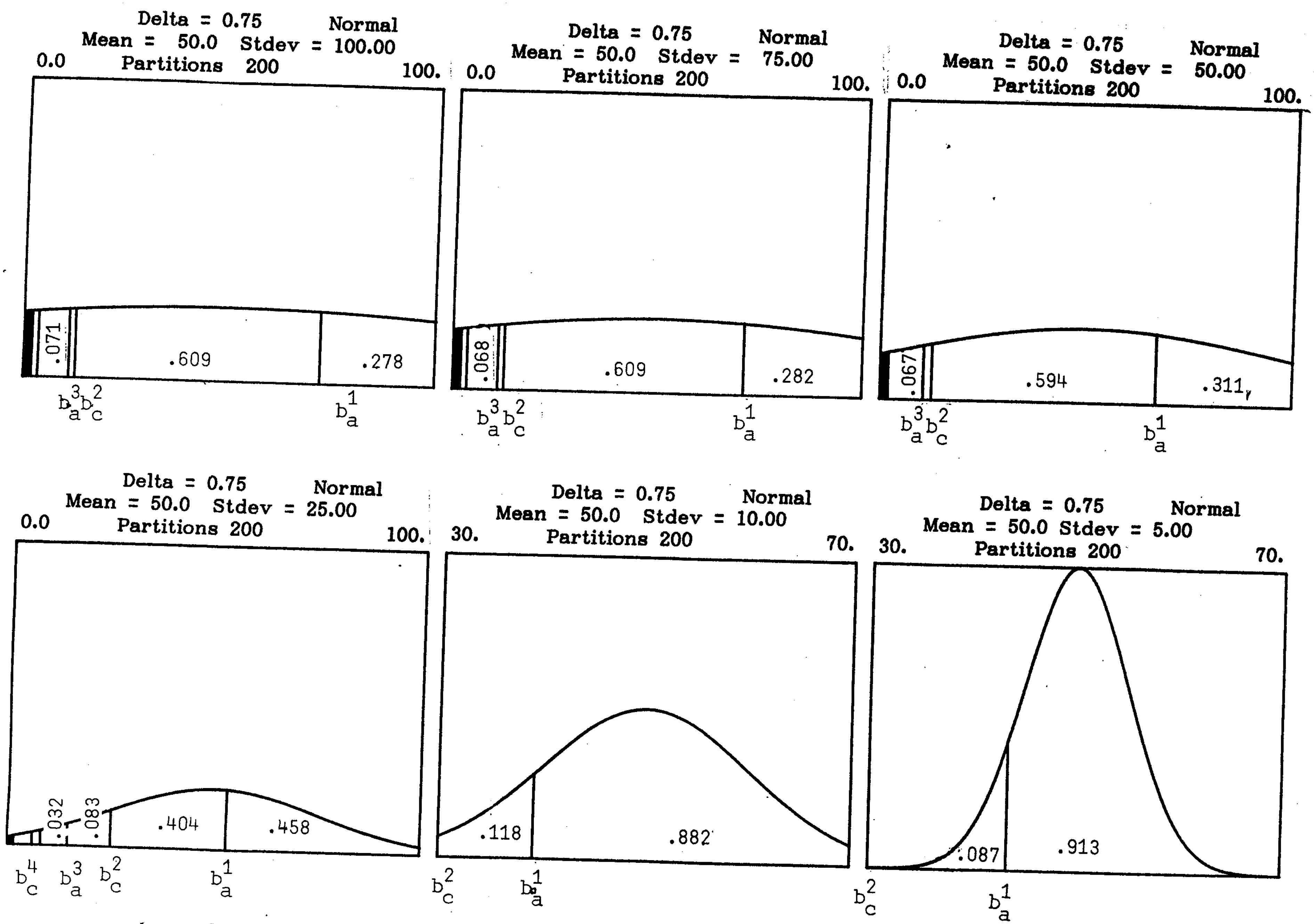
6. Our method is thus different from that followed by Sobel and Takahashi [17], who first solve a game where the players live for 1 period and then inductively solve the game where players live for N periods. Our method is similar to that used by Fudenberg, Levine and Tirole to analyze bargaining games where only the uninformed player makes offers.

7. When we say uniqueness, we mean uniqueness of the equilibrium outcome. It might be that there is another equilibrium, where the unacceptable offers are different. Yet, the price and the time of agreement with each buyer are the same as in the proposed equilibrium.

8. Our notion of equilibrium is much stronger than Sequential Equilibrium, and hence there are parameter values (of e.g., δ) such that a buyer can deviate from an equilibrium where $P_a = \delta V(b_c, b_a)$. In particular a buyer can offer $q < \delta V(b_c, b_a)$ such that if the seller believed that he played against a particular $K \subset [b_c, b_a]$, then $q \geq \delta V(K)$, so that he accepts, and all $b \in K$ are better off than along the equilibrium path. One weakening of our equilibrium concept for which the functional equation gives the unique P.S.E. is as follows. Instead of requiring that the seller, upon observing a deviation by a set K of buyers, revises his beliefs as if every one in K is equally likely to have deviated, suppose instead that he is allowed to choose any posterior distribution w_k with support K as his belief. Further, suppose we impose the continuity requirement that if K is close to the set $[b_c, b_a]$, members of which in equilibrium are supposed to make an acceptable counteroffer P_a, then w_k should be close to the equilibrium posterior. This preserves uniqueness of equilibrium because the P.S.E. updating arguments were always used to rule out situations where $P_a > \delta V(b_c, b_a)$. If the seller gets a counteroffer $q = P_a - \varepsilon$, and ε is small then a group K of buyers close to $[b_c, b_a]$ will be made better off, and by the continuity assumption the seller's posterior will be close to the equilibrium one, and hence the value of the continuation game $\delta V(K)$ will be close to $\delta V(b_c, b_a)$, so he will accept the offer. On the other hand, the non-existence problem arises when $P_a = \delta V(b_c, b_a)$ and buyers deviate to $q = P_a - \varepsilon$. Here the seller could choose a posterior which is optimistic about the top group of buyers in $K = [b_c - \varepsilon_1, b_a + \varepsilon_2]$, where K is the set of buyers who are made better off if q were accepted. He can then reject q as less than the optimistic value of the game with K, so the deviation is not successful.

9. Where the support is [0,1], the seller would have to offer price 0 to ensure that all buyers would accept without further bargaining. Therefore, the seller's percentage price-gain from bargaining is infinite.

Figure 2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b_a^1
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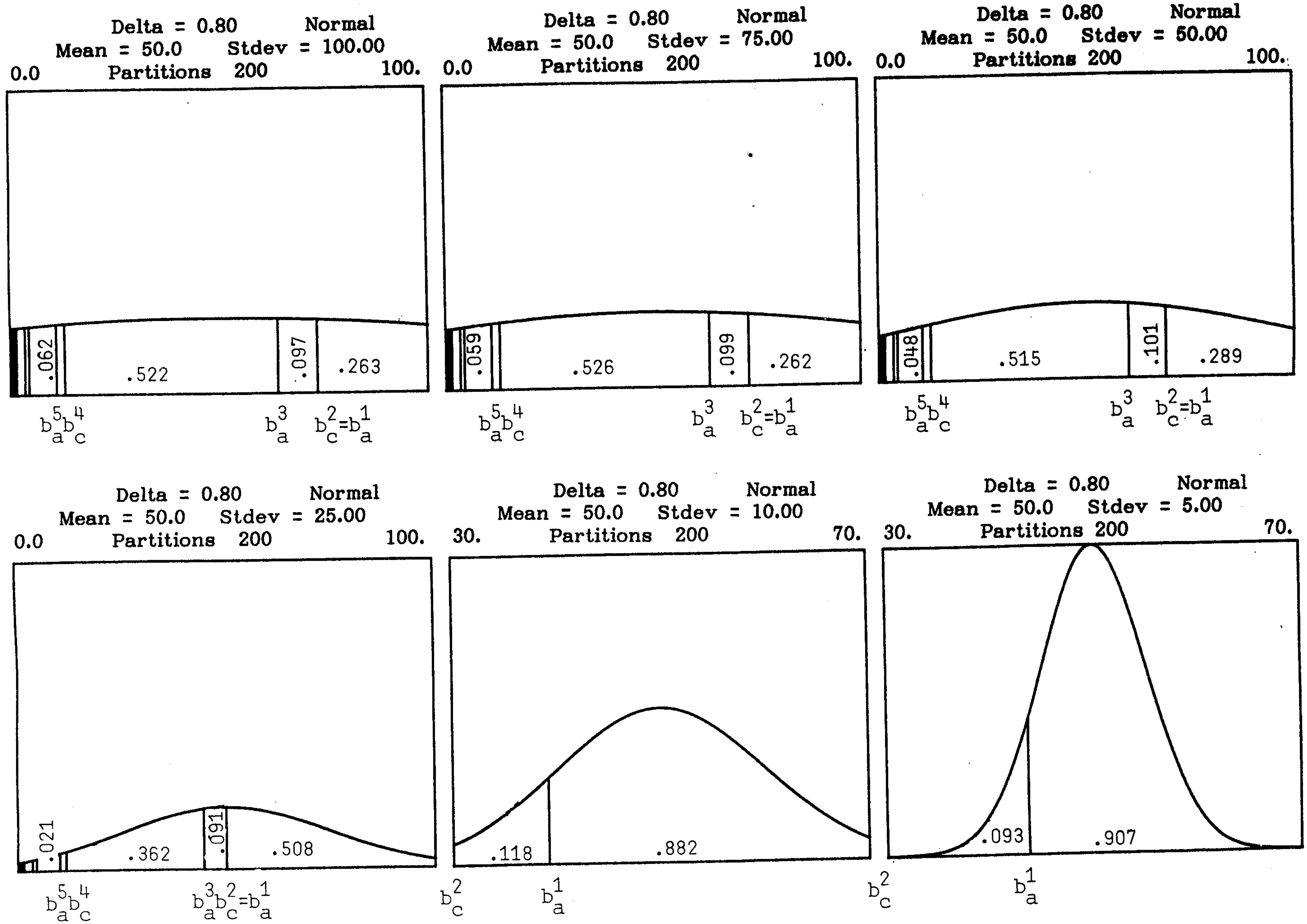
b_c^2
b_a^1

Figure 3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Partitions 200
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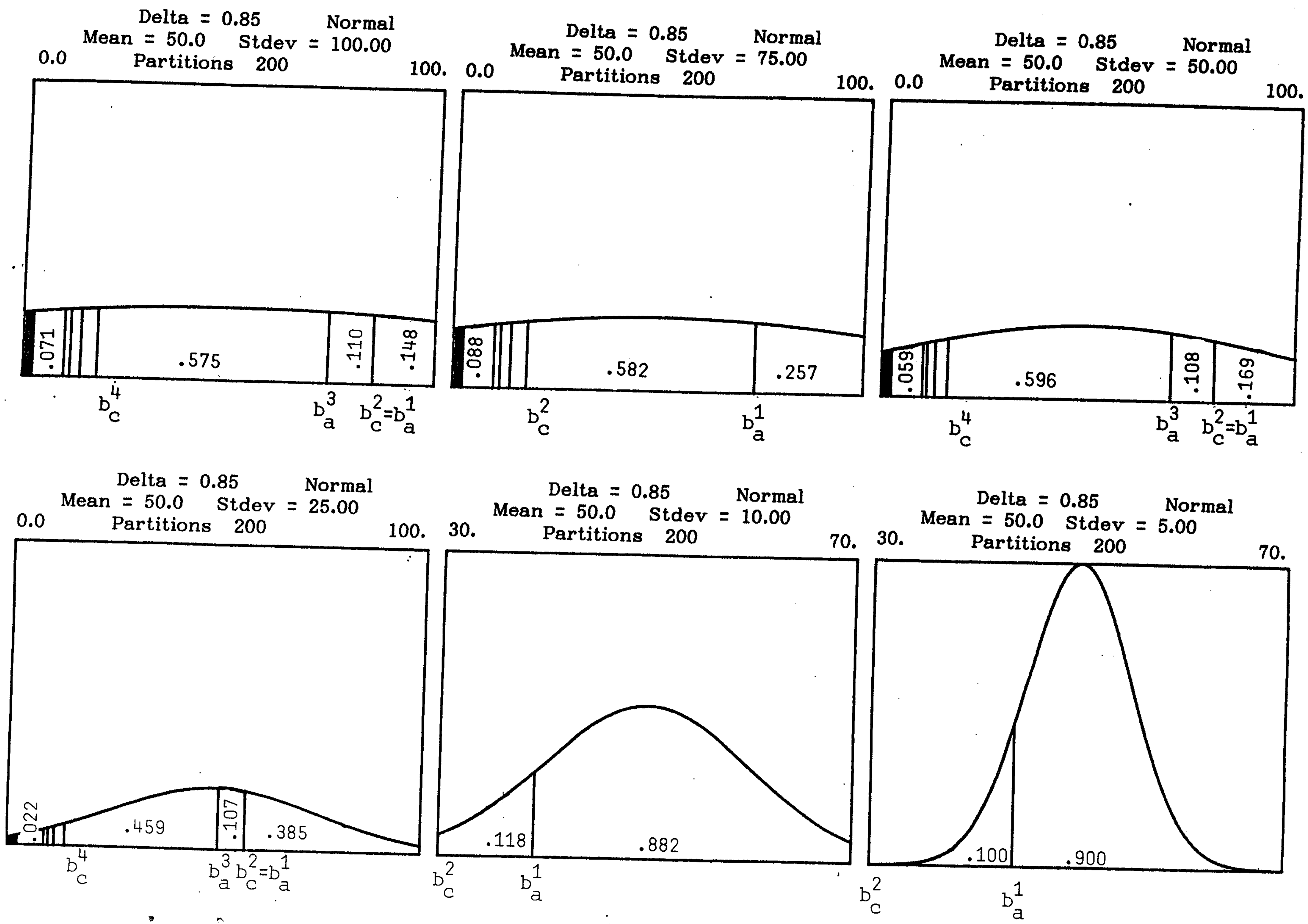
70.
.118
.882
b_c^2
b_a^1
b_c^2
Delta = 0.80 Normal
Mean = 50.0 Stdev = 5.00
Partitions 200
30.
70.
.093
.907
b_a^1

Figure 4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b_c^4
b_a^3 $b_c^2=b_a^1$
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Partitions 200
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b_c^2
b_a^1
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Mean = 50.0 Stdev = 5.00
Partitions 200
30.
70.
.100
.900
b_c^2
b_a^1

TABLE 1

DAYS BETWEEN OFFERS		2	1	.5	.1
Uniform Distribution* Support is [1,25] Buyers May Make Counteroffers	Expected Days of Bargaining	73	38	20	4.1
	Expected Price Increase from Bargaining	32%	16%	8.4%	1.7%
	Maximum Periods of Bargaining	62	64	66	68
Uniform Distribution Support is [1,25] No Buyer Counteroffer Permitted	Expected Days of Bargaining	39	21	11	2.1
	Expected Price Increase from Bargaining	8.6%	4.4%	2.2%	.45%
	Maximum Periods of Bargaining	41	43	43	43
Uniform Distribution Support is [1,50] Buyers May Make Counteroffers	Expected Days of Bargaining	110	64	35	7.7
	Expected Price Increase from Bargaining	100%	56%	30%	6.3%
	Maximum Periods of Bargaining	110	120	130	140
Uniform Distribution Support is [1,50] No Buyer Counteroffer Permitted	Expected Days of Bargaining	72	37	20	3.9
	Expected Price Increase from Bargaining	30%	16%	7.9%	1.6%
	Maximum Periods of Bargaining	75	77	79	79
Uniform Distribution Support is [1,100] No Buyer Counteroffer Permitted	Expected Days of Bargaining	120	72	41	9.2
	Expected Price Increase from Bargaining	130%	71%	39%	8.5%
	Maximum Periods of Bargaining	140	160	170	190
Uniform Distribution Support is [0,1] No Buyer Counteroffer Permitted	Expected Days of Bargaining	120	86	61	27
	Expected Price[9]	.016	.012	.0082	.0037
	Maximum Periods of Bargaining	∞	∞	∞	∞

*All games were solved on the computer using a 400 point discretization, except that the game with support of [1,100] used 10,000 points, and the game [0,1] was solved analytically using the formula given in the text.