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MICROECONOMIC APPROACHES TO THE THEORY OF INTERNATIONAL COMPARISONS

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ABSTRACT

The paper considers alterantive approaches to providing consistent multilateral indexes of real output, real input, real consumption or productivity across many regions, countries or industries at one point in time. The recommended approaches are based on aggregating up various bilateral indexes which in turn are based on the economic theory of index numbers, either in the producer or consumer theory context.

In order to distinguish between various competing multilateral approaches, an axiomatic or test approach to multilateral comparisons is developed. This test approach indicates that the Geary-Khamis and Van Yzeren approaches to multilateral output comparisons are dominated by the (new) own share and the Eltetö-Köves-Szulc methods.

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Microeconomic Approaches to the Theory of Intenational Comparisons

W.E. Diewert

1. Introduction

For many purposes, it is useful to compare the real outputs or real incomes of a number of countries or regions at a single point in time. This information may be required to allocate international aid or interregional transfer payments between member countries or regions. In this paper, we consider several approaches to the problem of providing international or interregional comparisons. These approaches are based on microeconomic theory. For a survey of the literature on international comparisons, see Kravis [1984]. For presentations of the axiomatic or test approach for making comparisons between two countries, see Voeller [1981] and Eichhorn and Voeller [1983].

A price deflator which converts the nominal national product ratio between two countries into a real output ratio is called a purchasing power parity function in the international comparisons literature (e.g., see Voeller [1981] and Eichhorn and Voeller [1983]). In section 2, we define a bilateral (i.e., two country) output price deflator or purchasing power parity function which is the international counterpart to the Fisher and Shell [1972] national output price deflator. This output price deflator requires the assumption of revenue maximizing behavior on the part of producers in both countries; thus this deflator is based on producer theory.[1]

Kravis [1984;10] pointed out that initially multilateral comparisons used bilateral comparisons as building blocks. Kravis notes that the simplest of these methods is the use of Laspeyres price indexes in a "star" system. One country is chosen as the base country (this is the center of the star) and a bilateral comparison is made between the base country and all other countries. Unfortunately, this relatively straightforward method is not satisfactory in practice: the resulting relative price and output levels are not even approximately invariant with respect to the choice of the base country.[2] Thus in section 3, we consider how best to use the bilateral indexes discussed in section 2 in order to make multilateral comparisons that treat all countries symmetrically.

In sections 4 and 5, we turn our attention from the bilateral producer price comparison problem to the bilateral real output comparison problem. In section 4, we pursue the between country or region counterpart to the single country intertemporal Fisher-Shell [1972] price deflation approach. In this approach, the nominal private domestic product ratio for two countries is deflated by the corresponding output price deflator (which will be discussed in section 2). In section 5, the privately produced outputs of two countries are compared using what Caves, Christensen and Diewert [1982b; 1399-1401] call the Malmquist output index. The basic idea for this theoretical index was first suggested by Malmquist [1953] in the consumer context. It was developed in the producer context by Bergson [1961], Moorsteen [1961], Samuelson and Swamy [1974; 590-591], and Hicks [1981; 256], Caves, Christensen and Diewert [1982b] and Diewert [1983b; 1064-1076]. A Malmquist index of output of country 1 relative to country 2 is the proportion that we have to deflate the output vector of country 1 so that the deflated output vector and the country 2 primary input vector are on the frontier of the technology set for country 2.

In section 6, we discuss how best to use the bilateral real output indexes in order to make multilateral real output comparisons in a symmetric manner. We find that there are at least 6 plausible methods for combining the bilateral indexes into consistent multilateral indexes. In order to distinguish between these possibilities, we consider an axiomatic or test approach to multilateral quantity indexes in sections 7 and 8. Using this test approach, we are able to eliminate 3 of our 6 methods.

In section 9, we leave the producer theory approaches behind and develop a consumer theory approach to making purchasing power parity comparisons. The basic building block in this approach is the single household Konüs [1939] cost of living index and an exact index number result for translog preferences established by Caves, Christensen and Diewert [1982b; 1410]. We then aggregate over households in the two countries whose price levels are to be compared to obtain a bilateral aggregate consumer intercountry price index. These bilateral country indexes may be made into a system of symmetric multilateral indexes using the techniques explained in section 3.

In section 10, we develop quantity counterparts to the intercountry consumer price indexes defined in section 9. The individual household Malmquist [1953] quantity index plays a crucial role in this section.

In section 11, we briefly consider two methods for making multilateral real output comparisons that are not based on bilateral building blocks. The methods are due to Geary [1958] and Khamis [1970][1972] and to Van Yzeren [1957].

Section 12 contains a few extensions to the measurement of real input and productivity, while section 13 contains proofs of new propositions.

2. A Producer Theory Approach to Bilateral Output Price Comparisons

We start off by making some basic assumptions and notational conventions which are understood to hold until we reach section 9 where we switch to consumer theory approaches to international comparisons.

We assume that there are I countries or regions that are to be compared. The private production sector of each country or region utilizes nonnegative amounts of M primary inputs. These are different types of labour, capital, land and other natural resources. There are N (net) outputs that can be produced. These are different types of consumer goods and services, exports and imports. The quantity of each imported good utilized by the private production sector of an economy is indexed with a negative sign. Classifications of net outputs and primary inputs are to be consistent across countries, and ideally, homogeneous within each classification.

The vector of primary inputs in country i is $v^i \equiv (v_1^i, v_2^i, \ldots, v_M^i) > 0_M$, where v_m^i is the amount of input m used in country i, $i = 1,\ldots,I$ and $m = 1,\ldots,M$.[3]

The positive price vector for the (net) outputs produced by country i is denoted by $p^i \equiv (p_1^i, p_2^i, \ldots, p_N^i) >> 0_N$ for $i = 1,2,\ldots,I$.[4] The corresponding net output vector for the private production sector for country i is $y^i \equiv (y_1^i, y_2^i, \ldots, y_N^i)$ with $p^i \cdot y^i \equiv \sum_{n=1}^N p_n^i y_n^i > 0$. (If $y_n^i < 0$, then good n utilized as an input in country i; this good could be an imported good or it could be an intermediate input that is produced by the government sector in country i.)

The technology set for the private production sector in country i is the set $S^i \equiv \{(y^i, v^i)\}$, a feasible set of net output vectors y^i and primary input vectors v^i. We assume that each technology set S^i is a closed subset of R^{N+M} and we shall occasionally assume additional weak regularity conditions on these sets.[5] If knowledge is freely transferrable between countries, then it is possible that $S^i = S$ for $i = 1,\ldots,I$, so that there is a common technology set across countries; but we do not require this assumption in what follows.

Define country i's <u>private national product function</u> g^i by

$$(1) \qquad g^i(p,v) \equiv \max_y\{p\cdot y : (y,v)\epsilon S^i\}, \ i = 1,2,\ldots,I$$

where $p \equiv (p_1,\ldots,p_N) >> 0_N$ is a positive vector of (net) output prices and $v \equiv (v_1,\ldots,v_M) \geqslant 0_N$ is a nonnegative vector of primary inputs. The number $g_i(p,v)$ is the maximum value of output (less the value of imports and the value of net purchases from the government sector) that country or region i can produce, given that it faces prices p and has at its disposal the primary input vector v. In what follows, we will speak only of country i but it should be understood that the analysis can be given a regional interpretation. It should also be understood that any governmental activities in country i that are organized along profit or revenue maximizing lines (e.g., national railroads or airlines) may be included in country i's "private" national product.[6]

In analogy to the Fisher and Shell [1972] <u>output price deflator</u>, $g^i(p^{t+1},v)/g^i(p^t,v)$, which is a measure of the price level in country i during period t+1 relative to the price level in period t,[7] we define the <u>output price index</u> of country i relative to j using the country i technology and primary input vector as

(2) $$P^i(p^j,p^i) \equiv g^i(p^i,v^i)/g^i(p^j,v^i);\ i,j = 1,2,\ldots,I$$

where the functions g^i are defined by (1), $p^i \gg 0_N$ is the (net) output price vector for country i and $v^i \geq 0_M$ is the corresponding primary input vector that is being utilized by country i during the period under consideration, $i = 1,2,\ldots,I$. The output price index defined by (2) is the value of country i's private national product during the reference period divided by the value of private output that country i could produce if it faced country j's price vector p^j instead of its own price vector p^i. Thus, $P^i(p^j,p^i)$ is a hypothetical "pure" measure of the level of output prices in country i relative to the level in country j. If N = 1 so that there is only one net output (and hence no index problem), then the measure (2) reduces to $P^i(p_1^j,p_1^i) = p_1^i/p_1^j$, the output price ratio between the two countries.

In definition (2), we used country i's technology set S^i and primary input vector v^i as reference quantities. An analogous <u>output price index</u> for country i relative to country j, $P^j(p^j, p^i)$, may be defined by using the country j technology set (or its dual national product function g^j) and primary input vector as reference quantities:

(3) $$P^j(p^j,p^i) \equiv g^j(p^i,v^j)/g^j(p^j,v^j);\ i,\ j = 1,2,\ldots,I.$$

$P^j(p^j, p^i)$ is also a "pure" measure of the level of output prices in country i and if N=1, it also reduces to p_1^i/p_1^j.

In the general N>1 case, the theoretical indexes defined by (2) and (3) cannot be numerically calculated unless we know the functions g^i or their dual technology sets S^i. We shall now attempt to find some bounds that can readily be calculated for the indexes defined by (2) and (3). Later in this section, we shall make some addition assumptions about the nature of the technology sets S^i or their dual national product functions g^i and this will enable us to obtain simple exact formulae for theoretical indexes of the form defined by (2) and (3).

In the remainder of this section, we assume net revenue maximizing behavior on the part of each country and we also assume that each country's private national product is positive; i.e., if $y^i \equiv (y_1^i, y_2^i, \ldots, y_N^i)$ denotes the net output vector for country i for i=1,2,...,I, then we assume that

$$(4) \qquad p^i \cdot y^i = \max_y \{p^i \cdot y = (y, v^i) \epsilon S^i\} \equiv g^i(p^i, v^i) > 0, \; i = 1, \ldots, I$$

where $p_i >> 0_N$ is the positive price vector for net outputs in country i and $v^i \geqslant 0_M$ is the nonnegative primary input vector used by the (competitive) private producers in country i.

<u>Proposition 1</u> (Hicks [1940], Samuelson [1950], Fisher and Shell [1972; 57-58], Archibald [1977; 66], Diewert [1983b; 1057-1058]): Suppose that producers are competitively optimizing and that private national product is positive in all countries (i.e., (4) holds). Then the theoretical output price indexes defined by (2) and (3) satisfy the following observable bounds:

$$(5) \qquad P^j(p^j, p^i) \geqslant p^i \cdot y^j / p^j \cdot y^j \quad \text{and}$$

$$(6) \qquad [P^i(p^j, p^i)]^{-1} \geqslant [p^i \cdot y^i / p^j \cdot y^i]^{-1} \quad \text{for } i, j = 1, 2, \ldots, I.$$

For proofs of the Propositions, see the Appendix.

The Laspeyres price index between countries i and j occurs on the right hand side of (5) while the reciprocal of the Paasche price index occurs on the right hand side of (6). If the technology set in country i is such that a zero net output vector can be produced (i.e., if $(0_N, v^i) \in S^i$), then it can be shown that $P^i(p^j, p^i) \geq 0$. Moreover, if $p^j \cdot y^i > 0$, then the inequalities (6) may be rewritten in the following more familiar form:

(7) $$p^i \cdot y^i / p^j \cdot y^i \geq P^i(p^j, p^i) \qquad \text{for } i, j = 1,2,\ldots,I.$$

Recall that if private producers in country i are importing good n or purchasing it from the government sector, then $y_n^i < 0$. If we were dealing with a collection of isolated closed economies, then we could perhaps assume that the net output vectors y^i were all nonnegative. Under this (unrealistic) condition, the following augmented versions of the inequalities (7) and (5) can be obtained:

(8) $$\min_n\{p_n^i/p_n^j : n=1, \ldots, N\} \leq P^i(p^j, p^i) \leq p^i \cdot y^i / p^j \cdot y^i; \ i,j=1,\ldots,I;$$

(9) $$p^i \cdot y^j / p^j \cdot y^j \leq P^j(p^j, p^i) \leq \max_n\{p_n^i/p_n^j : n=1,\ldots,N\}; \ i,j=1,\ldots,I.$$

For proofs of (8) and (9), see Samuelson [1947; 159] or Diewert [1983b; 1056-1058].

Since we are not dealing with isolated closed economies in empirical applications, the bounds given by (8) and (9) are not theoretically valid. Moreover, the bounds are too wide to be of much practical use in any case.

It would be of some practical use to obtain a theoretical price index between countries i and j that would lie between the Paasche index between i and j, $P_P^{ij} \equiv p^i \cdot y^i / p^j \cdot y^i$, and the Laspeyres index, $P_L^{ij} \equiv p^i \cdot y^i / p^i \cdot y^i$. This will be done in Proposition 2 below, but first we require two new definitions.

Let λ be a number between 0 and 1. Define a λ <u>weighted average of the</u>

national product functions for countries i and j for i,j = 1, 2, ..., I by:

(10) $g^{ij}(p,\lambda) \equiv \lambda g^i(p,\lambda v^i+(1-\lambda)v^j) + (1-\lambda)g^j(p,\lambda v^i+(1-\lambda)v^j)$

where g^i is the country i private product function defined by (1), $p \gg 0_N$ is a reference price vector and v^i is the country i primary input vector. If the technology sets S^i are the same across the world, then $g^i = g^j \equiv g$ and $g^{ij}(p,\lambda) = g(p,\lambda v^i+(1-\lambda)v^j)$ which is a national product function for a hypothetical country which has a weighted average, $\lambda v^i+(1-\lambda)v^j$, of the primary input vectors for countries i and j at its disposal. In general, note that

(11) $g^{ij}(p,1) = g^i(p,v^i)$ and $g^{ij}(p,0) = g^i(p,v^j)$

so that when $\lambda=1$, $g^{ij}(p,1)$ becomes the country i private product function and when $\lambda = 0$, $g^{ij}(p,1)$ becomes the country i private product function and when $\lambda=0$, $g^{ij}(p,0)$ becomes the country j private product function.

We now define the λ weighted average output price index for country i relative to j by

(12) $P_\lambda^{ij}(p^i,p^i) \equiv g^{ij}(p^i,\lambda)/g^{ij}(p^j,\lambda)$ for i, j = 1, 2, ..., I.

For each λ between 0 and 1, $P_\lambda^{ij}(p^j,p^i)$ is a "pure" measure of the level of output prices in country i relative to the level in country j. As usual, if N=1, the index reduces to p_1^i/p_1^j.

Proposition 2: Suppose: (i) producers are competitively optimizing, (ii) each country can produce a zero net output vector; i.e., $(0_N, v^i) \in S^i$ for i=1, ..., I, (iii) private national product is positive in all countries; i.e., (4) holds, (iv) $p^j \cdot y^i > 0$ for all i, j = 1, 2, ..., I and (v) the functions $g^i(p, \lambda v^i + (1-\lambda)v^j)$ are continuous functions of λ for $0 \leq \lambda \leq 1$ for i, j = 1, 2, ..., I.[8] Then for each pair of countries i and j there exists λ^* such that $0 \leq \lambda^* \leq 1$ and $P^{ij}_{\lambda^*}(p^j, p^i)$ lies between $P^{ij}_P \equiv p^i \cdot y^i / p^j \cdot y^i$ and $P^{ij}_L \equiv p^i \cdot y^j / p^j \cdot y^j$.

Proposition 2 is a counterpart to a result derived in the context of an intertemporal output price index for a single country.[9] However, the present result is not as useful as the corresponding intertemporal result. In the intertemporal case, the Paasche and Laspeyres price indexes usually differ by less than one per cent.[10] Hence if one takes a symmetric average of the two indexes, one can be reasonably certain that the appropriate theoretical index has been closely approximated. One such average index is the Fisher (1922) ideal price index P_F which is defined as the geometric average of the Paasche and Laspeyres indexes:

$$(13) \qquad P_F(p^j, p^i, y^j, y^i) \equiv (P^{ij}_P P^{ij}_L)^{1/2} = (p^i \cdot y^i \; p^i \cdot y^j / p^j \cdot y^i \; p^j \cdot y^j)^{1/2}.$$

However, in the context of international comparisons, the Paasche and Laspeyres price indexes often differ by more than 50 per cent, (e.g., see Ruggles [1967; 189-190] or Kravis, Kenessey, Heston and Summers [1975; 11]). Hence we cannot be certain that the empirically observable Fisher price index defined by (13) is close to the corresponding theoretical index $P^{ij}_{\lambda^*}(p^j, p^i)$ whose existence is asserted in Proposition 2.

It is possible to provide a direct justification for the use of the Fisher price index in the international comparisons context and we now proceed to do this. Suppose that the private national product functions g^i have the following separable forms over the relevant range of output prices p and primary input vectors v:

$$(14) \qquad g^i(p, v) = (p \cdot Bp)^{1/2}\ h^i(v) \qquad \text{for } i=1,2,\ldots,I$$

where $B \equiv [b_{nk}]$ is an N by N symmetric matrix of constants,[11] $p \cdot Bp \equiv \Sigma_{n=1}^{N} \Sigma_{k=1}^{N} p_n b_{nk} p_k$ and the functions $h^i(v)$ are positive, nondecreasing functions of v for $v > 0_M$. Assumption (14) implies that outputs are separable from inputs in each country. The quantity $h^i(v^i)$ may be interpreted as an input aggregate for country i. Note that the net output price function $(p \cdot Bp)^{1/2}$ is assumed to be common across countries, but the input aggregator functions $h^i(v)$ may differ across countries. Recall our earlier definitions of the output price indexes P^i and P^j defined by (2) and (3). Under the separability assumption (14), it can be seen that

$$(15) \qquad P^i(p^j, p^i) \equiv g^i(p^i, v^i)/g^i(p^j, v^i) = (p^i \cdot Bp^i/p^j \cdot Bp^j)^{1/2} = P^j(p^j, p^i);$$

i.e., the theoretical indexes $P^i(p^j, p^i)$ and $P^j(p^j, p^i)$ coincide, and in fact, it can be seen that this common index equals $g^i(p^i, v)/g^i(p^j, v)$ for <u>any</u> reference primary input vector v. Now we can state our result which provides a justification for the use of the Fisher formula.

<u>Proposition 3</u>: Suppose that there is optimizing behavior in each country (so that (4) holds) and that the private product functions g^i have the separable form defined by (14). Then for any two countries i and j and any reference

primary input vector v,

$$P_F(p^i, p^i, y^j, y^i) = g^i(p^i, v)/g^i(p^j, v) = P^i(p^j, p^i) = P^j(p^j, p^i), \tag{16}$$

where the Fisher ideal price index P_F is defined by (13).

Caves, Christensen and Diewert [1982a] asserted the above Proposition. The proof of the Proposition requires Hotelling's [1932; 594] Lemma: if the private product function $g^i(p^i, v^i)$ is differentiable with respect to its output price arguments at the point (p^i, v^i), then the observed country i net output vector y^i is equal to the vector of first order partial derivatives of g^i with respect to its price arguments; i.e., we have

$$y^i = \nabla_p g^i(p^i, v^i) \tag{17}$$

$$\text{where } \nabla_p g^i(p^i, v^i) \equiv [\partial g^i(p^i, v^i)/\partial p_1^i, \ldots, \partial g^i(p^i, v^i)/\partial p_N^i]^T.$$

It can be shown[12] that the special functional form defined by (14) can approximate any separable function of the form f(p)h(v), where f(p) is linearly homogeneous, to the second order. Thus we have provided a reasonably strong justification for the use of the Fisher formula in applied work. However, the justification is still not as strong as one would like, since separability assumptions of this type are empirically restrictive. Hence we turn now to a translog approach that does not require any restrictive separability assumptions.

Suppose the private product function for country i has the following

<u>translog functional form</u> for i=1, 2, ..., I:

(18) $$\ell n g^i(p, v) \equiv \alpha_0^i + \Sigma_{n=1}^N \alpha_n^i \ell n p_n + (1/2) \Sigma_{n=1}^N \Sigma_{k=1}^N \alpha_{nk}^i \ell n p_n \ell n p_k$$

$$+ \Sigma_{m=1}^M \beta_m^i \ell n v_m + (1/2) \Sigma_{m=1}^M \Sigma_{j=1}^M \beta_{mj}^i \ell n v_m \ell n v_j$$

$$+ \Sigma_{n=1}^N \Sigma_{m=1}^M \gamma_{nm}^i \ell n p_m \ell n v_m$$

where $\alpha_{nk}^i = \alpha_{kn}^i$ for all i, k, n and $\beta_{mj}^i = \beta_{jm}^i$ for all i, j, m.[13]
It can be shown that the translog g^i defined by (18) can approximate an arbitrary twice continuously differentiable private product function to the second order; i.e., the translog functional form is <u>flexible</u>.

Proposition 4 below also requires the definition of the <u>translog output price index</u>[14] between countries i and j:

(19) $$P_T(p^i, p^i, y^j, y^i) \equiv \Pi_{n=1}^N (p_n^i/p_n^j)^{(1/2)(s_n^i + s_n^j)}$$

where $p^i \equiv (p_1^i, \ldots, p_N^i)$ and $s_n^i \equiv p_n^i y_n^i / p^i \cdot y^i$ for i=1,...,I and n=1,2,...,N. Thus the index is a product of the N price ratios for the same goods in countries i and j, p_n^i/p_n^j, where the nth ratio in the product is raised to a power which is equal to the average expenditure share on good n in the two countries, $(1/2)s_n^i + (1/2)s_n^j$.

<u>Proposition 4:</u> Assume: (i) country i's private product function g^i has the translog functional form defined by (18) above for i=1,2,...,I; (ii) the private production sector in country i faces the positive price vector $p^i \gg 0_N$ and has available the positive primary input vector $v^i \gg 0_M$ for i=1,2,...,I; (iii) there is optimizing behavior in each country so that

$p^i \cdot y^i = g^i(p^i, v^i) > 0$ for $i=1,2,\ldots,I$ where y^i is the revenue maximizing net output vector for economy i, and (iv) $\alpha^i_{nk} = \alpha^j_{nk}$ for all i, j, n and k; i.e., the coefficients for the quadratic terms in the logarithms of output prices are the same across all countries (but the remaining coefficients can all differ across countries). Then a geometric average of the theoretical output price indexes defined by (2) and (3) above is exactly equal to the translog output price index $P_T(p^i,p^i,y^j,y^i)$ defined by (19); i.e., we have

$$(20) \qquad [P^i(p^j, p^i)\, P^j(p^j,p^i)]^{1/2} = P_T(p^j,p^i,y^j,y^i).$$

The restriction that the primary input vectors v^i all be strictly positive can be relaxed; all we require in order that (20) hold is that each v^i be nonnegative and nonzero.[15] However, we cannot relax the positivity restrictions on the output price vectors p^i.

The result (20) is a very useful result. It tells us that we can calculate <u>exactly</u> the geometric mean of the two theoretical price indexes, $P^i(p^i,p^i)$ and $P^j(p^j,p^i)$, using only the observable price and quantity data for countries i and j, p^i, p^j, y^i and y^j. Moreover, the assumptions made in Proposition 4 are not particularly restrictive.

Proposition 3 justified the use of the Fisher formula, $P_F(p^j,p^i,y^j,y^i)$, in order to evaluate empirically an economic index of the output prices of country i relative to j, while Proposition 4 justified the use of the translog formula, $P_T(p^i,p^i,y^j,y^i)$. Which formula should we use in empirical applications? Fortunately, (limited) empirical evidence indicates that it does not matter which formula is used; the answers will be approximately the same.[16] Furthermore, Diewert [1978; 888] has shown that the indexes P_F and

P_T numerically approximate each other to the second order around any point where the two price arguments are equal and the two quantity arguments are equal; i.e., we have

$$(21)\qquad P_F(p^j,p^i,y^j,y^i) = P_T(p^j,p^i,y^j,y^i), \qquad \text{if } p^j = p^i >> 0 \text{ and } y^j = y^i \text{17}$$

$$\nabla P_F(p^j,p^i,y^j,y^i) = \nabla P_T(p^j,p^i,y^j,y^i), \text{ and}$$

$$\nabla^2 P_F(p^j,p^i,y^j,y^i) = \nabla^2 P_T(p^j,p^i,y^j,y^i)$$

where ∇P_F signifies the vector of first order partial derivatives of the function P_F with respect to all 4N arguments and $\nabla^2 P_F$ denotes the 4N by 4N matrix of second order partial derivatives of P_F with respect to all of its arguments.

In this section, we have provided solutions to the problem of comparing the level of (net) output prices in one country with the level in another country. The two formulae which we recommend for making these bilateral price comparisons, P_F and P_T, are functions of the price and quantity vectors, p^i, p^j, y^i, y^j, which occur in the two countries to be compared. Interestingly enough, these are index number formulae which occur in the test or axiomatic approach to index number theory.[18]

3. Multilateral Purchasing Power Parity Comparisons

Kravis [1984; 10] pointed out that initially, multilateral or many country price comparisons were made using bilateral index number formulae as building blocks. The simplest way to proceed is to pick a "base" or "star" country, say country I, choose an index number formula of the form $P(p^j,p^i,y^j,y^i)$, and calculate the price level of country i relative to country I as $P(p^I,p^i,y^I,y^i)$ for $i=1,2,\ldots,I-1$. Early workers in this area chose P to be the Laspeyres formula; i.e., $P(p^I,p^i,y^I,y^i)$ was chosen to be $p^i \cdot y^I/p^I \cdot y^I$. However, based on the analysis in the previous section, we would recommend that P_F or P_T be chosen as the index number formula rather than the Laspeyres formula, P_L.

The problem with using the star system to obtain a consistent set of international price levels between the I countries is that the results depend on the choice of the base or star country. Different choices for the base country will yield different relative price levels for the I countries in general.[19]

The question we wish to address in this section is: how can we achieve a consistent (or transitive or circular) set of relative price levels that make use of the binary comparisons of the form $P(p^j,p^i,y^j,y^i)$, where P is a suitable index number formula, but at the same time, we do not single out any single country to play an asymmetric role in the system of multilateral comparisons?

It seems clear that what we should do is average over the bilateral comparisons in order to achieve symmetry (and perhaps more "accuracy") but it is not clear how precisely we should go about doing this averaging.

In order to obtain some ideas on how to proceed, let us temporarily consider the case where there is only one output (i.e., N=1) and hence no index number problem. If we wish to compare the price level in country i, p_1^i say, with prices in other countries in a way that does not involve singling out any one country, it is clear that we must compare p_1^i to a symmetric function of world prices. Simple symmetric functions might be: (i) $\Sigma_{j=1}^I s_j \cdot p_1^j$

where the s_j are nonnegative weights summing to one (this corresponds to an arithmetic average of world prices) or (ii) $\Pi_{j=1}^{I}(p_1^j)^{s_j}$ (this corresponds to a geometric average of world prices). There are also several natural choices for the weights s_j: (i) $s_j=1/I$ for each j (equal weights), (ii) $s_j = p_1^j y_1^j/\Sigma_i p_1^i y_1^i$, j=1,...,I (nominal share of world output weights) and (iii) $s_j \equiv y_1^j/\Sigma_i y_1^i$, j=1,...,I (quantity weights).

With the above considerations in mind, for each country i, pick some nonnegative weights s_j^i which sum up to one; i.e., $\Sigma_{j=1}^{I}\ s_j^i = 1$. Pick a two country index number formula $P(p^j,p^i,y^j,y^i)$ of the type studied in the previous section and define δ_i as follows:

$$(22) \qquad \delta_i \equiv \Pi_{j=1}^{I}[P(p^j,p^i,y^j,y^i)]^{s_j^i}, \qquad i=1,2,\ldots,I.$$

If $P = P_F$, P_T or P_L and N=1, then $\delta_i = p_1^i/\Pi_{j=1}^{N}(p_1^j)^{s_j^i}$, so that δ_i is the price level in country i relative to the (weighted geometric mean) world average price, $(p_1^1)^{s_1^i}(p_1^2)^{s_2^i} \ldots (p_1^I)^{s_I^i}$. In the general case where N>1, we shall interpret δ_i as the level of prices in country i relative to an average world price.

In general, we would like an average of the country i relative price levels to the world price level to equal one; i.e., we would like the following equality to hold for some positive weights $s_j>0$, j=1,...,I, which sum to one:

$$(23) \qquad \Pi_{j=1}^{I}\delta_j^{s_j} = 1, \text{ where } s_j>0 \text{ and } \Sigma_{j=1}^{I}s_j=1.$$

<u>Proposition 5</u>: Equation (23) will be satisfied provided that: (i) $s_j^i = s_j$ for i=1,...,I and j=1,...,I; i.e., the country i weights s_j^i which occur in definition (22) are independent of i and equal to the weights s_j which occur in (23); (ii) $P(p^i,p^i,y^i,y^i) = 1$; i.e., the index number formula P which is

used in definition (22) satisfies the identity test, and (iii) $P(p^j,p^i,y^j,y^i) = 1/P(p^i,p^j,y^i,y^j)$; i.e., the index number formula satisfies the country reversal test.

Proposition 5 allows to solve the multilateral price comparisons problem posed at the beginning of this section. First, pick the index number formula P which occurs in (22) to be P_F or P_T (the Paasche and Laspeyres formulae do not satisfy the country reversal test). Secondly, pick some positive country weights s_j which sum up to one. Finally, calculate the I numbers δ_i defined by (22) where the s_j^i are replaced by the s_j. The resulting numbers δ_1, δ_2, ..., δ_I provide a cardinal scaling of prices in the I countries. These numbers may be multiplied by a common scalar in order to make the price level in any given country equal to unity if desired. The <u>multilateral output price index</u> of country i relative to country j may now be defined as δ_i/δ_j. Note that the resulting system of multilateral prices makes use of all of the bilateral information but yet treats all countries symmetrically.

Consider some special choices for the weights s_j which occur in (22) and (23).

<u>Case (i)</u>: $s_j \equiv 1/I$ for $j=1,\ldots,I$. These might be termed <u>democratic weights</u>: each country is given an equal weight in the formation of the hypothetical world average price level. If the Fisher formula P_F is used for P, then the resulting system of multilateral purchasing power parities $\delta_1,\ldots,\delta_I$ are scalar multiples of the Eltetö and Köves [1964] and Szulc [1964] (EKS) purchasing power parities.[20] If the translog formula P_T is used for P, then our parities $\delta_1,\ldots,\delta_I$ are scalar multiples of the Caves, Christensen and

Diewert [1982a] purchasing power parities.

Case (ii): $s_j \equiv p^j \cdot y^j / \Sigma_{k=1}^{I} p^k \cdot y^k$ for $j=1,\ldots,I$. Each country is weighted according to its share of world private output. In analogy with the cost of living index literature,[21] these weights might be termed plutocratic weights. Again, the Fisher formula P_F or the translog formula P_T[22] could be used as a P.

Case (iii): Average quantity weights. The requisite formulae are somewhat complex. Given our price index formula for prices in country i relative to country j, $P(p^j,p^i,y^j,y^i)$, we form an estimate of the real output of country i relative to country j, $Y_j^i \equiv p^i \cdot y^i / p^j \cdot y^j \; P(p^j,p^i,y^j,y^i)$. For estimates of world real output shares using country j as the numeraire country $s_j^i \equiv Y_j^i / \Sigma_{k=1}^{I} Y_j^k$ for $i,j=1,\ldots,I$. Now define the average world quantity weights by $s_j \equiv \Sigma_{i=1}^{I} s_j^i / I$ for $j=1,\ldots,I$.

To sum up this section, we recommend that the empirical investigator interested in making multilateral price comparisons use formula (22) to form country prices with the use of the Fisher or translog bilateral price index and the country weights given by cases (ii) or (iii) above. We do not recommend the use of democratic weights since the resulting purchasing power parities are not invariant to superficial repackaging of the countries. By this, we mean the following: suppose country I is split into two countries where each new country has the old price vector p^I and (1/2) of the old quantity vector y^I. We would not expect such a reorganization of countries to affect the average level of prices in countries $1,2,\ldots,I-1$, but this will generally happen if we use democratic weights.[23]

Clearly, there are many variations that are possible; e.g., arithmetic averages could be used in place of geometric averages in (22), different weights s_j could be chosen, different bilateral index number formulae $P(p^j,p^i,y^j,y^i)$ could be chosen, etc. It is also clear that a more systematic axiomatic approach to our suggested method for building up a system of multilateral indexes form a set of bilateral comparisons should be undertaken, but we leave this for future research.

We turn now to the problem of making real output comparisons.

4. Bilateral Real Output Comparisons: The Price Deflation Approach.

Our first approach to the problem of constructing bilateral indexes of real output based on production theory is the Fisher-Shell [1972; 53] approach. In this approach, an index of relative real output between the private production sectors of two countries is obtained by dividing the nominal value of output ratio for the two countries by the corresponding true price index. Thus if we recall our earlier definitions of the two theoretical price indexes between countries i and j, (2) and (3), we may define the corresponding Fisher-Shell quantity indexes by:

$$(24)\qquad Q^i(p^j,p^i,y^j,y^i) \equiv p^i\cdot y^i/p^j\cdot y^j\ P^i(p^j,p^i)$$

$$= p^i\cdot y^i/p^j\cdot y^j[g^i(p^i,v^i)/g^i(p^j,v^i)] \qquad \text{using (2)}$$

$$= g^i(p^i,v^i)/g^i(p^i,v^j)[g^i(p^i,v^i)/g^i(p^j,v^i)] \quad \text{using (4)}$$

$$= g^i(p^j,v^i)/g^j(p^j,v^j)$$

$$(25)\qquad = g(p^j,v^i)/g(p^j,v^j) \qquad , i,j=1,2,\ldots,I$$

where (25) follows if the technology sets S^i are identical across countries, and hence $g^i(p,v) = g(p,v)$ for $i=1,\ldots,I$. Similarly:

$$(26) \qquad Q^j(p^j,p^i,y^j,y^i) \equiv p^i \cdot y^i/p^j \cdot y^j \; P^j(p^j,p^i)$$

$$= g^i(p^i,v^i)/g^j(p^j,v^j)[g^j(p^i,v^j)/g^j(p^j,v^j)] \quad \text{using (3) and (4)}$$

$$= g^i(p^i,v^i)/g^j(p^i,v^j)$$

$$(27) \qquad = g(p^i,v^i)/g(p^i,v^j)$$

where (27) follows if the technology sets are identical across countries.

From (25) and (27), we see that in the case of identical technologies across countries, the Fisher-Shell output indexes reduce to Samuelson-Swamy [1974; 588] and Sato [1976; 438] output indexes, which are of the form $g(p,v^i)/g(p,v^j)$ for some reference output price vector p.

It is clear that under the hypotheses of Proposition 3, the theoretical output indexes defined by (24) and (26) are exactly equal to the Fisher quantity index $Q_F(p^j,p^i,y^j,y^i) \equiv [Q_L Q_P]^{1/2}$ where $Q_L(p^j,p^i,y^j,y^i) \equiv p^j \cdot y^i/p^j \cdot y^j$ and $Q_P(p^j,p^i,y^j,y^i) \equiv p^i \cdot y^i/p^i \cdot y^j$ are the Laspeyres and Paasche quantity indexes; i.e., we have under the hypotheses of Proposition 3,

$$(28) \qquad Q_F(p^j,p^i,y^j,y^i) = Q^i(p^j,p^i,y^j,y^i) = Q^j(p^j,p^i,y^j,y^i), \; i,j=1,\ldots,I.$$

Is is also clear that under the hypotheses of Proposition 4, we have the following equalities:

$$(29) \qquad [Q^i(p^j,p^i,y^j,y^i)Q^j(p^j,p^i,y^j,y^i)]^{1/2} = p^i \cdot y^i/p^j \cdot y^j \; P_T(p^j,p^i,y^j,y^i)$$

$$\equiv \tilde{Q}_T(p^i,p^i,y^j,y^i), \; i,j=1,\ldots,I$$

where Q^i and Q^j are the theoretical output indexes defined by (24) and (26), P_T is the translog price index defined by (19), and the implicit translog output index $\tilde{Q}_T$ is defined by the right hand side of (29).

We have obtained two empirically implementable methods for calculating bilateral theoretical output indexes that are based on production theory. The first method utilizes the Fisher quantity index Q_F while the second method uses the implicit translog quantity index $\tilde{Q}_T$.

As in section 2, we may ask which formula should we use in empirical applications? The answer is the same as in section 2: in most applications, it will not make a great deal of difference.[24]

5. Bilateral Real Output Comparisons: The Distance Function Approach.

The approach used in the previous section to define real output indexes was rather indirect. In this section, we shall outline a more direct approach that is also based on production theory.[25]

It is first necessary to define the country i distance or output deflation function d^i: for $v > 0_M$ and $y \epsilon R^N$, define[26]

(30) $$d^i(y, v) \equiv \min_\delta\{\delta: (y/\delta, v) \epsilon S^i, \delta > 0\}, i=1,\ldots,I$$

where S^i is the technology set for country i. Thus $d^i(y, v)$ tells us by what proportion we have to deflate the net output vector y so that the deflated output vector and the reference input vector v are just on the frontier of the country i production possibilities set S^i.

The Malmquist[27] output index of country i relative to country j using the country k technology set and primary input vector v^k is

(31) $$Q_M^k(y^j, y^i) \equiv d^k(y^i, v^k)/d^k(y^j, v^k), \quad i,j,k=1,2,\ldots,I.$$

Let us assume technical efficiency in each country so that the observed country i output vector y^i is on the country i production possibilities frontier for each i. This implies:

(32) $$d^i(y^i, v^i) = 1, \quad i=1,2,\ldots,I.$$

Some special cases of the Malmquist indexes defined by (31) will be of interest to us; namely, the cases where k equals i or j: for $i,j=1,\ldots,I$,

(33) $$\begin{aligned} Q_M^i(y^j, y^i) &\equiv d^i(y^i, v^i)/d^i(y^j, v^i) \\ &= 1/d^i(y^j, v^i) \qquad \text{if (32) holds;} \end{aligned}$$

(34) $$\begin{aligned} Q_M^j(y^j, y^i) &\equiv d^j(y^i, v^j)/d^j(y^j, v^j) \\ &= d^j(y^i, v^j) \qquad \text{if (32) holds.} \end{aligned}$$

The Malmquist indexes defined by (33) and (34) appear in the Proposition which follows.

<u>Proposition 6</u>: Assume: (i) $p \gg 0_N$ and $p \cdot y > 0$ for $i=1,\ldots,I$, (ii) y solves $\max_y \{p^i \cdot y : (y, v^i) \in S^i\}$ for $i=1,\ldots,I$ and (iii) $\delta_{ij} > 0$ solves $\min_\delta \{\delta : (y^j/\delta, v^i) \in S^i, \delta > 0\} = d^i(y^j, v^i)$ for $i,j=1,\ldots,I$. Then

(35) $$Q_M^j(y^j,y^i) \geq p^j \cdot y^i / p^j \cdot y^j \equiv Q_L(p^j,p^i,y^j,y^i) \text{ (the Laspeyres quantity index);}$$

(36) $$Q_M^i(y^j,y^i) \leq p^i \cdot y^i / p^i \cdot y^j \equiv Q_P(p^j,p^i,y^j,y^i) \text{ (the Paasche quantity index).}$$

Inequality (35) tells us that the theoretical Malmquist index of the output of country i relative to country j, using the inputs and technology of country j as reference quantities, is bounded from below by the (observable) Laspeyres quantity index for country i's output relative to country j's. Inequality (36) tells us that another theoretical Malmquist index of the output of country i relative to country j, using the inputs and technology of country i as reference quantities, is bounded from above by the (observable) Paasche quantity index.

What is new in Proposition 6 is the absence of positivity restrictions on the output vectors y^i and y^j. Under the positivity hypothesis, inequality (35) may be extended to:

$$(37) \qquad Q_L(p^j,p^i,y^j,y^i) \leq Q_M^j(y^j,y^i) \leq \max_n\{y_n^i/y_n^j : n=1,\ldots,N\}$$

and the inequality (36) may be extended to[28]

$$(38) \qquad \min_n\{y_n^i/y_n^j : n=1,\ldots,N\} \leq Q_M^i(y^j,y^i) \leq Q_P(p^j,p^i,y^j,y^i).$$

However, without the positivity restrictions on the y^i, we cannot deduce (37) or (38).

It would be desirable to find a theoretical output index that would be between the Laspeyres and Paasche quantity indexes that occur in (35) and (36) respectively. This can be done, but first we must define a country ij deflation function, d^{ij}, that uses as reference quantities, a convex combination of the input vectors for countries i and j, $(1-\lambda)v^i + \lambda v^j$, and a convex combination of the technology sets for the two countries, $(1-\lambda)S^i + \lambda S^j$, where λ is a scalar between 0 and 1; i.e., define

(39) $$d^{ij}(y,\lambda) \equiv \min_{\delta>0}\{\delta:(y/\delta,(1-\lambda)v^i+\lambda v^j)\epsilon[(1-\lambda)S^i+\lambda S^j]\},\ i,j=1,\ldots,I.$$

Note that

(40) $$d^{ij}(y,0) = d^i(y,v^i) \text{ and } d^{ij}(y,1) = d^j(y,v^j)$$

where d^i and d^j were the country i and j deflation functions that occurred in definitions (33) and (34) respectively. Thus as λ goes from 0 to 1, d^{ij} maps the country i deflation function into the country j deflation function in a continuous manner.

For any $0\leq\lambda\leq 1$, we may now define the <u>Malmquist λ weighted average output index</u> for country i relative to j as

(41) $$Q_\lambda^{ij}(y^j,y^i) \equiv d^{ij}(y^i,\lambda)/d^{ij}(y^j,\lambda),\ i,j=1,\ldots,I.$$

The meaning of the index (41) compared to our earlier theoretical indexes defined by (33) and (34) is that we now use a λ weighted average of the input vectors and a λ weighted average of the country i and j technology sets as our reference quantities. We then determine the proportional deflation factor $d^{ij}(y^i,\lambda)$ that it takes to put the deflated country i net output vector, $y^i/d^{ij}(y^i,\lambda)$, on the frontier of the average technology set, $(1-\lambda)S^i+\lambda S^j$, using the average primary input vector $(1-\lambda)v^i + \lambda v^j$. Similarly we determine the proportional deflation factor $d^{ij}(y^j,\lambda)$ that it takes to put the deflated country j net output vector, $y^j/d^{ij}(y^j,\lambda)$, on the frontier of the same average technology set, $(1-\lambda)S^i + \lambda S^j$, again using the average input vector $(1-\lambda)v^i + \lambda v^j$. The theoretical output index is the ratio of the two deflation factors, $d^{ij}(y^i,\lambda)/d^{ij}(y^j,\lambda)$.

Using definitions (33), (34) and (41) and the equalities (40), we have the following relationships between our new family of index numbers $Q_{\lambda}^{ij}(y^j,y^i)$ and our old indexes Q_M^i and Q_M^j:

(42) $$Q_0^{ij}(y^j,y^i) = Q_M^i(y^j,y^i) \text{ and } Q_1^{ij}(y^j,y^i) = Q_M^j(y^j,y^i).$$

<u>Proposition 7</u>: Suppose the hypotheses of Proposition 6 hold and in addition, the Malmquist λ weighted average output index for country i relative to j, $Q_{\lambda}^{ij}(y^j,y^i)$, defined by (41) is a continuous function of λ for $0 \leq \lambda \leq 1$. Then there exists a λ^* such that $0 \leq \lambda^* \leq 1$ and $Q_{\lambda^*}^{ij}(y^j,y^i)$ lies between the Laspeyres and Paasche quantity indexes, $p^j \cdot y^i/p^j \cdot y^j$ and $p^i \cdot y^i/p^i \cdot y^j$ respectively.

As was the case at the end of Proposition 2, again it seems reasonable to approximate the theretical index $Q_{\lambda^*}^{ij}(y^j,y^i)$, whose existence is given above in Proposition 7, by a symmetric average of the Paasche and Laspeyres quantity indexes, such as the Fisher ideal quantity index, $Q_F(p^j,p^i,y^j,y^i) \equiv [p^j \cdot y^i p^i \cdot y^i/p^j \cdot y^j p^i \cdot y^j]^{1/2}$.

The production theory approaches we have taken in this section and the previous section to the problem of making bilateral real output comparisons between two countries or regions have led us to two concrete index number formulae: the Fisher quantity index, $Q_F(p^j,p^i,y^j,y^i)$ defined above, and the implicit translog quantity index, $\tilde{Q}_T(p^j,p^i,y^j,y^i)$ defined by (29) and (19). As we mentioned before, the two index number formulae approximate each other to the second order, so it should not make too much difference which of these two formulae the empirical investigator uses.[29]

We now turn to the problem of making multilateral quantity comparisons in a consistent manner using the bilateral Fisher or implicit translog comparisons as building blocks.

6. Multilateral Comparisons of Real Output: Preliminary Approaches.

Just as was the case in section 3, the star system could be used to form a consistent, transitive ordering of relative real output levels over all I countries. More explicitly, the star system works as follows. First, pick an appropriate bilateral quantity index number formula, which gives the real output of country i relative to country j, $Q(p^j,p^i,y^j,y^i)$ say. Secondly, pick a numeraire country, say j=1. Then country i's share of real world private product is defined by

$$(43)\qquad \sigma_i^j = Q(p^j,p^i,y^j,y^i)/\Sigma_{k=1}^I Q(p^j,p^k,y^j,y^k),\ i=1,2,\ldots,I$$

where j=1. Note that $\Sigma_{i=1}^I \sigma_i^j = 1$.

Of course, the problem with the star system is that the results depend on the choice of the numeraire country j; i.e., all countries are not treated in a symmetric manner in the star system. Thus we need to consider alternative methods for making multilateral comparisons.

Our first alternative method starts off with the following definition:

$$(44)\qquad \alpha_i \equiv [\Sigma_{j=1}^I [Q(p^j,p^i,y^j,y^i)]^{-1}]^{-1},\ i=1,2,\ldots,I.$$

Country i's share of real world private product is now defined as:

$$(45)\qquad s_i \equiv \alpha_i/\Sigma_{k=1}^I \alpha_k.$$

The intuitive interpretation of the above system for achieving multilateral comparisons of real output in a transitive and symmetric way can be grasped if we consider the case of one output good; i.e., N=1. In this case, if Q is Q_F or $\tilde{Q}_T$, (44) becomes

$$(46)\qquad \alpha_i = y_1^i/\Sigma_{j=1}^I y_1^j \quad , \qquad i=1,2,\ldots,I,$$

i.e., α_i becomes country i's share of total world output. In this N=1 case, we will have $\Sigma_{k=1}^I \alpha_k = 1$, so that in this case, $s_i = \alpha_i$ = country i's share of world private product.

In the general N > 1 case, α_i defined by (44) defines country i's share of world product in the "metric" of country i. Since these metrics are not quite compatible in general, we make a further adjustment via definitions (45) to ensure that the shares sum to one. Let us call the resulting method for making multilateral comparisons of real output, the <u>own share system</u>.

It turns out that the own share system, using the Fisher formula, is related to the Eltetö, Köves [1964] and Szulc [1964] (EKS) method for making multilateral comparisons. This can be seen as follows: using the Fisher formula, we have

$$\alpha_i = I^{-1}[\Sigma_{j=1}^I I^{-1}[Q_F(p^j,p^i,y^j,y^i)]^{-1}]^{-1}$$

$$\cong I^{-1}\ \Pi_{j=1}^I Q_F(p^j,p^i,y^j,y^i)]^{1/I}$$

where we have approximated the harmonic mean by a geometric mean. Thus

$$\alpha_i/\alpha_k = s_i/s_k$$

$$(47)\qquad \cong \Pi_{j=1}^I Q_F(p^j,p^i,y^j,y^i)^{1/I}/\Pi_{j=1}^I Q_F(p^j,p^k,y^j,y^k)^{1/I}$$

= the EKS output of country i relative to j.

Thus we have provided an economic justification for the use of the EKS method for making multilateral comparisons of real output.

Instead of using the Fisher formula Q_F in (47), Caves, Christensen and Diewert [1982a] advocated the use of the direct translog formula Q_T in place of Q_F. In the present context where some outputs can be negative, we would recommend the use of the implicit translog quantity index $\tilde{Q}_T$ in place of Q_T.

For any choice of a binary quantity index number formula Q, let us define the <u>Generalized EKSCCD</u> method for making multilateral quantity comparisons as follows. Define

$$(48) \qquad \beta_i \equiv \Pi_{j=1}^{I} Q(p^j,p^i,y^j,y^i)^{i/I} \qquad , \ i=1,\ldots,I.$$

Country i's EKSCCD share of world private product is now defined as:

$$(49) \qquad S_i \equiv \beta_i/\Sigma_{k=1}^{I}\beta_k \qquad , \ i=1,\ldots,I.$$

We note that the own share method (45) satisfies the following <u>consistency in aggregation</u> property whereas the EKSCCD method does not: suppose any one country is split into two coutnries, each of which has the same price vector and one half of the original quantity vector. Then the relative outputs of the countries which are not split up should remain constant.[30]

We now consider another class of methods for making multilateral comparisons of real output.

Recall that the shares σ_i^j, i=1,...,I, defined by (43) allowed us to partition up world product using the bilateral metric of country j. What we now propose to do is to average over these country j metrics:

$$(50) \qquad \sigma_i(s_1,\ldots,s_I) \equiv \Sigma_{j=1}^{I} s_j\sigma_i^j \qquad , \ i=1,\ldots,I.$$

We still must choose the country weights s_j which occur in (50). A natural first choice would be to give each country's metric the same weight and choose <u>democratic weights</u> where $s_j \equiv 1/I$ for $j=1,\ldots,I$. However, this method seems to give tiny countries a comparitively large influence in the formation of the averages defined by (44). Moreover, the resulting method does not satisfy the consistency in aggregation (or invariance under country partitioning) property mentioned above. Thus it may be preferable to use <u>plutocratic weights</u>, where $s_j \equiv p^j \cdot y^j / \Sigma_{k=1}^I p^k \cdot y^k$ is country j's share of world private product. This plutocratic averaging method satisfies the consistency in aggregation property but it has a drawback as well: the resulting shares defined by (44) are not invariant to scale changes in the prices of any one country. This is a very serious drawback, since we would not want country i's share of world real output to depend on the inflation rate in country j. A final variant for (50) would be to use <u>quantity weights</u>, where the s_j which appear in (50) are defined by (45), the own shares. The resulting method satisfies the consistency in aggregation property and is also invariant to scale changes in country prices.

In this section, we have considered six different methods for making multilateral comparisons of real output, using bilateral index number formulae based on producer theory as building blocks. The six methods were: (i) the star system, (ii) the own share system, (iii) the EKSCCD method, (iv) the democratic weights method, (v) the plutocratic weights method and (vi) the (own share) quantity weights method. However, we have not been able to recommend any of these methods as being clearly the best. Thus in the next two sections, we shall turn to the test or axiomatic approach to index number theory and see whether this approach leads to any clear cut recommendations. In the following section, we momentarily digress and consider the axiomatic properties of bilateral index number formulae. Then in section 8, we shall look for multilateral counterparts to the bilateral properties studied in section 7.

7. Bilateral Comparisons of Real Output: The Test Approach.

The test or axiomatic approach to bilateral index number theory has its origins in the work of Walsh [1901] and Fisher [1922]. More recently, the test approach has been more definitively studied by Eichhorn and Voeller [1976][1983] and Vartia [1985]. The reason why we are devoting a section of the present paper to this well studied topic is that the existing literature does not deal with the problems that arise from the fact that in our framework, some quantities can be negative. The existing literature on the test approach assumes that all quantities are at least nonnegative. Thus it is necessary to review existing index number tests (or axioms or properties) and see if they are sensible when some quantities can be negative.

We regard a bilateral index number formula as a function, $Q(p^1,p^2,y^1,y^2)$, of 4N variables. The N dimensional vectors p^1 and p^2 represent prices in countries 1 and 2 and the vectors y^1 and y^2 represent the net output vectors in countries 1 and 2 (if $y_n^i < 0$, then the nth good is being utilized as an input into the private production sector of country i). We shall denote the tests or properties that we would like our bilateral quantity index Q to satisfy by BT1, BT2, etc. Our first test is:

<u>BT1</u>: <u>Positivity</u>: $Q(p^1,p^2,y^1,y^2) > 0$ for $p^i \gg 0_N$, y^i such that $p^i \cdot y^i > 0$, $i=1,2$. Moreover, Q is a continuous function.

Although the positivity test seems very reasonable, it can be verified that the Fisher ideal index Q_F does <u>not</u> satisfy it. (The implicit translog quantity index $\tilde{Q}_T$ does satisfy it however). In order to have the Fisher formula satisfy a positivity test, we must strengthen BT1 to the following property:

<u>BT1'</u>: <u>Strong Positivity</u>: Q is continuous and $Q(p^1,p^2,y^1,y^2) > 0$ for $p^1 \gg 0_N$, $p^2 \gg 0_N$ and y^i such that $p^j \cdot y^i > 0$ for $i,j=1,2$.

In words, we restrict the domain of definition of Q to strictly positive price vectors and quantity vectors which have a positive inner product with both price vectors.[31]

The following test is stronger than the usual identity test (e.g., see Eichhorn and Voeller [1976][1983]). It is a slight modification of a test due to Vartia [1985]:

<u>BT2</u>: <u>Identity</u>: $Q(\alpha p, \beta p, y, y) = 1$ for $\alpha > 0$, $\beta > 0$.

The above test says that if the price vectors in the two countries are proportional and the quantity vectors are identical, then the output of country 2 relative to 1 should equal unity. A stronger identity test arises if we drop the hypothesis of price proportionality; i.e., $Q(p^1,p^2,y,y)=1$ for

$p^i \gg 0_N$, $p^i \cdot y > 0$ for $i=1,2$. Q_F satisfies this stronger identity property but $\tilde{Q}_T$ does not. However both $\tilde{Q}_T$ and Q_F satisfy the weaker property BT2.

The following property is a fundamental one for binary index number formulae.

<u>BT3</u>: <u>Proportionality</u>: $Q(p^1,p^2,y^1,\lambda y^2) = \lambda Q(p^1,p^2,y^1,y^2)$ for $\lambda > 0$.

Thus if we double all outputs in country 2, the quantity index doubles.

If there is only one good, then it is easy to show that tests BT2 and BT3 determine the functional form for Q; i.e., if N=1, then $Q(p^1,p^2,y^1,y^2) = y^2/y^1$ which is the output of country 2 divided by the output of country 1.

The next test is also a variant of a test due to Vartia [1985].

<u>BT4</u>: <u>Strong Monetary Unit Test</u>: $Q(\alpha p^1,\beta p^2,\gamma y^1,\gamma y^2) = Q(p^1,p^2,y^1,y^2)$ for $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.

The above test says that if we simultaneously change the level in prices in each country by a (possibly different) proportional factor and we change all quantities in both countries by the same proportional factor γ, then the quantity index remains unchanged. This property implies that the quantity index does not depend on inflation rates in the two countries. Furthermore, if both countries experience output growth at the same rate, then the quantity index remains unchanged since it is supposed to represent the output of country 2 relative to that of country 1.

The following 3 tests are invariance or symmetry tests.

<u>BT5</u>: <u>Commensurability</u> (<u>or</u> <u>Invariance</u> <u>to</u> <u>Scale</u> <u>Changes</u> <u>in</u> <u>Units</u>): Let D be a diagonal N by N matrix with positive elements on its main diagonal. Then $Q(Dp^1,Dp^2,D^{-1}y^1,D^{-1}y^2) = Q(p^1,p^2,y^1,y^2)$ where D^{-1} denotes the inverse matrix of D.

<u>BT6</u>: <u>Country</u> <u>Reversal</u> (<u>or</u> <u>Symmetric</u> <u>Treatment</u> <u>of</u> <u>Countries</u>): $Q(p^2,p^1,y^2,y^1) = 1/Q(p^1,p^2,y^1,y^2)$.

The above test says that if we interchange the role of the countries in the index number formula Q, then the resulting value of the index equals the reciprocal of the original value of the index. It is the country counterpart to Irving Fisher's [1922] time reversal test.

<u>BT7</u>: <u>Commodity</u> <u>Reversal</u> <u>Test</u> (<u>or</u> <u>Symmetric</u> <u>Treatment</u> <u>of</u> <u>Commodities</u>): Let P denote an N by N permutation matrix; i.e., each row and column of P contains one unit element and the remaining components are zeros. Then $Q(Pp^1,Pp^2,Py^1,Py^2) = Q(p^1,p^2,y^1,y^2)$.

This test was considered by Irving Fisher [1922;81] and more recently by Vartia [1985].

It is straightforward to show that our two desirable bilateral index number formulae, Q_F and $\tilde{Q}_T$, that arose in our producer theory approaches to making bilateral output comparisons satisfy all of the above tests, except that Q_F does not satisfy BT1.

Some additional tests that have been considered in the bilateral context are:

BT8: Strong Proportionality: For $\lambda > 0$, $Q(p^1,p^2,y,\lambda y) = \lambda$;

BT9: Monotonicity: For $y^2 \geq y^3$, $Q(p^1,p^2,y^1,y^2) \geq Q(p^1,p^2,y^1,y^3)$;

BT10: Mean Value Test: $\min_n\{y_n^2/y_n^1\} \leq Q(p^1,p^2,y^1,y^2) \leq \max_n\{y_n^2/y_n^1\}$

where $p^i >> 0_N$, and $y^i \equiv [y_1^i,y_2^i,\ldots,y_N^i]$, $i=1,2$.

The Fisher quantity index Q_F satisfies BT8 and BT9 but the implicit translog quantity index $\tilde{Q}_T$ satisfies neither. Neither Q_F nor $\tilde{Q}_T$ satisfy BT10. (If we could restrict quantities to be nonnegative, then Q_F would also satisfy BT10).

All of the above tests seem to be intuitively desirable with the exception of BT10: when the quantity vectors y^i can have negative components, the mean value test is no longer reasonable.

To sum up this section, we have considered 9 tests or properties that we would like our bilateral quantity index Q to have. In comparing our two desirable formulae Q_F and $\tilde{Q}_T$ (derived from the viewpoint of economic theory), we found that both satisfied tests BT2 through BT7. In addition, Q_F satisfied BT8 and BT9 while $\tilde{Q}_T$ satisfied BT1. We leave it up to the reader to decide which formulae is preferable.

8. Multilateral Comparisons of Real Output: The Test Approach.

We shall denote desirable properties for a multilateral system for making real output comparisons by MT1, MT2, etc. As in earlier sections, we denote the N dimensional price and net output vectors for country i by p^i and y^i for $i=1,\ldots,I$. In order to economize on space, we now define the N by I matrices of all prices and all quantities by $p \equiv [p^1,p^2,\ldots,p^I]$ and $y \equiv [y^1,y^2,\ldots,y^I]$ respectively. A multilateral system of output indexes is a set of I functions, $S_i(p,y)$, $i=1,\ldots,I$, where S_i is to be interpreted as country i's share of world private product.

In general, there will be domain restrictions on the $S_i(p,y)$ functions. As a minimal set of restrictions, we assume that $p^i \gg 0_N$ and $p^i \cdot y^i > 0$ for $i=1,\ldots,I$. At times, we will also want to add additional restrictions such as $p^i \cdot y^j > 0$ for all i and j, particularly when the share functions S_i are functions of bilateral Fisher quantity indexes.

The first property we want the S_i to satisfy is a fundamental one, since it allows us to interpret the functions S_i as shares.

<u>MT1</u>: <u>Share Test</u>: $S_i(p,y) > 0$, $i=1,\ldots,I$ and $\Sigma_{i=1}^{I} S_i(p,y) = 1$.

Furthermore, the functions $S_i(p,y)$ are continuous in p,y.

The following test is an approximate counterpart to the bilateral identity test, BT2:

<u>MT2</u>: <u>Weak Proportionality</u>: Let $\alpha_i > 0$, $\beta_i > 0$ for $i=1,\ldots,I$ with

$\Sigma_{i=1}^{I}\beta_i = 1$. Let $p^1=p^2=\ldots=p^I$ and $y^1=y^2=\ldots=y^I$. Then

$$S_i(\alpha_1 p^1,\alpha_2 p^2,\ldots,\alpha_I p^I,\beta_1 y^1,\beta_2 y^2,\ldots,\beta_I y^I) = \beta_i \text{ for } i=1,\ldots,I.$$

The above property is a very fundamental one and we would expect every reasonable multilateral system to satisfy it. The test says the following:

suppose that the price and quantity vectors in each country are (separately) proportional to the price and quantity vectors of all other countries. The country i's share of world output is equal to its (common) share of world output on all N output markets. This test contains an identity test as a special case: if all output vectors are identical and all price vectors are proportional, we have

$$S_i(\alpha_1 p^1,\ldots,\alpha_I p^1,y^1,\ldots,y^I) = S_i(\alpha_1 p^1,\ldots,\alpha_I p^1,I^{-1}\Sigma_{j=1}^I y^1,\ldots,I^{-1}\Sigma_{j=1}^I y^1)$$

$$= 1/I \text{ for } i=1,\ldots,I,$$

so that in this case, each country has an equal share of world output. Test MT2 also enables us to deduce what the functional form for $S_i(p,y)$ must be if N=1 so that there is only one net output. In this case, we have

$$S_i(p^1,\ldots,p^I,y^1,\ldots,y^I) = S_i(1p^1,\ldots,(p^I/p^1)p^1,\beta_1\Sigma_{j=1}^I y^j,\ldots,\beta_I\Sigma_{j=1}^I y^j)$$

$$= \beta_i$$

$$\equiv y^i/\Sigma_{j=1}^I y^j \quad \text{for } i=1,\ldots,I.$$

The next multilateral test is a counterpart to the bilateral proportionality test, BT3.

MT3: Proportionality: Let $\lambda_i > 0$. Then for $i=1,2,\ldots,I$:

$$(52) \quad S_j(p,y^i,\ldots,y^{i-1},\lambda_i y^i,y^{i+1},\ldots,y^I) = \lambda_i S_i(p,y)/[1+S_i(p,y)(\lambda_i-1)] \text{ for } j=i$$

$$= S_j(p,y)/[1+S_i(p,y)(\lambda_i-1)] \text{ for } j\neq i.$$

On the left hand side of (52), we have the country j share function after the original country i quantity vector y^i has been multiplied by the positive scalar λ_i; on the right hand side of (52), we have some algebraic expressions involving the original (before multiplication by λ_i) share functions $S_i(p,y)$. Equations (52) are supposed to hold for all I choices for the quantity vector y^i which is multiplied by the scalar λ_i.

We derived test MT3 in analogy to the N=1 case when the share functions are defined by (51); in this case, it can be verified that equations (52) will hold.

In order to obtain a better intuitive feeling for the test MT3, one should consider the test in ratio form. Denote the left hand side of (52) by $S_j(\lambda_i)$. Then (52) is equivalent to $S_i(\lambda_i)/S_k(\lambda_i) = \lambda_i S_i(p,y)/S_k(p,y)$ for $k \neq i$ and $S_j(\lambda_i)/S_k(\lambda_i) = S_j(p,y)/S_k(p,y)$ for $j \neq i$ and $k \neq i$. These multilateral proportionality properties seem to be just as reasonable as the bilateral proportionality property BT3. Unfortunately, as we shall soon see, it proves to be very difficult to simultaneously satisfy MT3 along with other reasonable multilateral tests.

The following four tests are counterparts to the corresponding bilateral tests, BT4, BT5, BT6 and BT7.

<u>MT4</u>: <u>Monetary Unit Test</u>: Let $\alpha_i > 0$, i=1,...,I and $\beta > 0$. Then

$$S_i(\alpha_1 p^1,\ldots,\alpha_I p^I,\beta y^1,\ldots,\beta y^I) = S_i(p,y),\ i=1,\ldots,I.$$

This property says that inflation levels in each country and a uniform growth in outputs over all countries does not affect each country's share of world output.

<u>MT5</u>: <u>Commensurability (or Invariance to Scale Changes in Units)</u>: Let D be a diagonal N by N matrix with positive elements on its main

diagonal. Then $S_i(Dp,D^{-1}y) = S_i(p,y)$, $i=1,\ldots,I$.

<u>MT6</u>: <u>Symmetric Treatment of Countries (Multilateral Country Reversal Test)</u>: Let Π be an I by I permutation matrix and let $S(p,y)$ be the column vector of share functions; i.e., $S(p,y) \equiv [S_1(p,y),\ldots,S_I(p,y)]^T$. Then $S(p,y)\Pi = S(p\Pi,y\Pi)$.

The above property says that if we evaluate our system of share functions using a permutation of the data, then the resulting share functions are the same permutation of the original share functions. This property means that no country can be singled out to play a special role in generating the country share functions. This property is termed base country invariance in Kravis, Kenessey, Heston and Summers [1975].

<u>MT7</u>: <u>Symmetric Treatment of Commodities (Commodity Reversal Test)</u>: Let P denote an N by N permutation matrix. Then $S_i(Pp,Py) = S_i(p,y)$ for $i=1,\ldots,I$.

The next three tests have no counterparts in the bilateral context. This is due to the fact that they are consistency in aggregation tests; i.e., we ask that the system of share functions changes in a consistent manner as the number of countries I changes. In the bilateral context, it is not natural to think of changing the number of countries (which is two of course).

The following test formalizes the consistency in aggregation property that was mentioned in earlier sections.

<u>MT8</u>: <u>Country Partioning Test</u>: Let $S_j^I \equiv S_j(p^1,\ldots,p^I,y^1,\ldots,y^I)$ for $j=1,\ldots,I$ and let $0<\lambda<1$. Define $S_j^{I+1} \equiv S_j(p^1,\ldots,p^{i-1},p^i,p^{i+1},\ldots,p^I,p^i,y^1,\ldots,y^{i-1},\lambda y^i,y^{i+1},\ldots,y^I,(1-\lambda)y^i)$ for $j=1,\ldots,I,I+1$. Then $S_j^{I+1} = S_j^I$ for $j=1,\ldots,i-1,i+1,\ldots,I$, $S_i^{I+1} = \lambda S_i^I$ and $S_{I+1}^{I+1} = (1-\lambda)S_i^I$. This property is to hold no matter which country i is partitioned.

The functions S_j^I are the share functions for the initial world economy that consists of I countries. The functions S_j^{I+1} correspond to a new world economy, where the original country i, which had price vector p^i and quantity vector y^i, is partioned into two countries. The new country i has price vector p^i and quantity vector λy^i where λ is a positive fraction and the separatist offshoot of the old country i is now country I+1 with price vector p^i and quantity vector $(1-\lambda)y^i$. Test MT8 says that under these conditions the old country i share S_i^I splits into λS_i^I and $(1-\lambda)S_i^I$ and the remaining shares are unaffected by this partitioning of countries.

<u>MT9</u>: <u>Irrelevance of Tiny Countries Test</u>: Let $\lambda_i > 0$ and define $S_j^I(\lambda_i) \equiv S_j(p^1,\ldots,p^I,y^1,\ldots,y^{i-1},\lambda_i y^i,y^{i+1},\ldots,y^I)$ for $j=1,\ldots,I$. Define $S_j^{I-1} \equiv S_j(p^1,\ldots,p^{i-1},p^{i+1},\ldots,p^I,y^1,\ldots,y^{i-1},y^{i+1},\ldots,y^I)$ for $j=1,\ldots,I-1$. Then $\lim_{\lambda_i\to 0} S_j^I(\lambda_i) = S_j^{I-1}$ for $j=1,\ldots,i-1$ and $\lim_{\lambda_i\to 0} S_{j+1}^I(\lambda_i) = S_j^{I-1}$ for $j=i,i+1,\ldots,I-1$. This property is to hold for all choices of the disappearing country i.

The above property may be explained in words as follows. Consider an initial world economy with I countries in it. Deflate the quantity vector for country i down to zero in a proportional manner and consider the resulting system of limiting share functions. For all countries except country i, the limiting share is equal to the corresponding share we would get if we simply deleted the data for country i and defined the resulting system of shares for the world economy consisting of only I-1 countries. Roughly speaking, MT9 says that the existence of tiny countries should not materially change the output shares of the remaining countries.

The following test is also a consistency in aggregation test, but it is not as compelling as the prevous two tests.

MT10: Strong Dependence on a Bilateral Formula: Let $Q(p^i,p^j,y^i,y^j)$ be a bilateral index number formula and define the bilateral raitos $Y_{ij} \equiv Q(p^i,p^j,y^i,y^j)$, $i,j=1,2,2,\ldots,I$. The share functions $S_i(p,y)$ dewpend only on the bilateral ratios; i.e., we have

$$(53) \qquad S_i(p,y) = F_i0y_{11},Y_{12},\ldots,Y_{I-1,I},Y_{II}), \quad i=1,\ldots,I$$

for some functions of I^2 variables, F_i, $i=1,\ldots,I$. Moreover, as the quantity vectors for all countries except i and j are deflated proportionally to zero, we have for $1 \leq i < j \leq I$,

$$(54) \quad \lim_{\lambda\to 0} \frac{S_i(p,\lambda y^1,\ldots,\lambda y^{i-1},y^i,\lambda y^{i+1},\ldots,\lambda y^{j-1},y^j,\lambda y^{j+1},\ldots,\lambda y^I)}{S_j(p,\lambda y^1,\ldots,\lambda y^{i-1},y^i,\lambda y^{i+1},\ldots,\lambda y^{j-1},y^j,\lambda y^{j+1},\ldots,\lambda y^I)}$$

$$= Q(p^j,p^i,y^j,y^i).$$

A weaker version of MT10 would drop (54). The justification for this test depends on how strongly one feels about the accuracy of the bilateral formula Q. If one feels strongly that Q is best for making bilateral comparisons, then property (54) is a very natural one. It says that the relative outputs for countries i and j should approach the value given by the bilateral index number formula as the outputs of all other countries shrink to zero.

Many additional multilateral tests could be devised. However, the above 10 tests are sufficient to enable us to discriminate between the six multilateral systems developed in section 6 above. We shall now verify which of the above tests each system satisfies.

Proposition 8: Suppose the bilateral quantity index $Q(p^i,p^j,y^i,y^j)$ satisfies the bilateral tests BT1 through BT7. Then: (i) the multilateral star system defined by (43) using country j as the numeraire country satisfies all of the above ten multilateral tests except MT6, MT9 (actually the tiny country test fails only when the tiny country is chosen to be the numeraire country j) and MT10; (ii) the own share multilateral system defined by (44) and (45) satisfies all tests except MT3; (iii) the EKSCCD system defined by (48) and (49) satisfies all tests except MT8 and MT10; (iv) the democratic weights system defined by (50) with $s_j \equiv 1/I$ satisfies all tests except MT3, MT8, MT9 and MT10; (v) the plutocratic weights system defined by (50) with $s_j \equiv p^j \cdot y^j / \Sigma_{k=1}^I p^k \cdot y^k$ satisfies all tests except MT3, MT4 and MT10, and (vi) the quantity weights system defined by (50) with the shares s_j defined by the own shares (45) satisfies all tests except MT3 and MT10.

The above Proposition is very useful since it enables us to eliminate 3 of the 6 multilateral systems from contention: the last 3 systems are dominated by the own share system (and system (iv) is dominated by the EDSCCD system). Furthermore, the failure of the plutocratic system (v) to satisfy the invariance under inflation test MT4 seems to be a fatal defect.

Turning now to the relative merits of the first 3 systems, we see that the star system is quite a reasonable one to use if there is a natural numeraire country. Thus if a single country is interested in comparing its real output with a possibly varying number of other countries, it is quite natural to give up on the symmetry test MT6, and use the star system, choosing the numeraire country to be the home country.

However, if a group of countries wish to jointly determine their relative real outputs, then it is natural to demand that the symmetric treatment of countries test MT6 hold, and thus the choice is between systems (ii) and (iii). The tradeoff between the two systems is that the own share system fails to satisfy the proportionality test MT3, while the EDSCCD system fails to satisfy the country partitioning test MT8 and the strong dependence on the bilateral formula test MT10. At present, I would advocate the use of the own share system since its weighting or consistency in aggregation properties seem to be better than the EDSCCD properties. However, reasonable people could certainly differ on the relative importance of satisfying the tests MT3, MT8 and MT10, so my advocacy is not a strong one. Fortunately, from a numerical point of view, the two systems should yield very similar results, depending on how closely various harmonic means approximate the corresponding geometric means (recall (47)).

We conclude this section by mentioning an obvious point. Once we have decided what the "correct" quantity shares $S_i(p,y)$ for each country are we can obtain consistent country i price indexes $P_i(p,y)$ by dividing the country i nominal private product, $p^i \cdot y^i$, by the share function, $S_i(p,y)$; i.e.,

(55) $$P_i(p,y) \equiv p^i \cdot y^i / S_i(p,y), \quad i=1,\ldots,I.$$

The resulting system of price index and share functions will have the following property:

$$\Sigma_{i=1}^{I} P_i(p,y)S_i(p,y) = \Sigma_{i=1}^{I} p^i \cdot y^i$$

$$= \text{nominal world private product.}$$

We now leave the realm of production theory and turn our attention to consumer theory approaches to multilateral comparisons.

9. Consumer Theory Approaches to Purchasing Power Parity Comparisons.

Consider household h in country i. Let $F_h^i(z,x)$ denote the preference or utility function of this household over nonnegative combinations of market goods $x \equiv (x_1,\ldots,x_N) \geqslant 0_N$ and other variables $z \equiv (z_1,\ldots,z_L) >> 0_L$. It should be understood that the N used in this section is in general not equal to the N of previous sections. The z variables could be demographic variables or consumptions of public goods. We assume that $x^{ih} > 0_N$ is a solution to the following expenditure minimization problem: for i=1,...,I and $h=1,\ldots,H_i$,

(56) $$\min_x \{w^i \cdot x : F_h^i(z^{ih},x) \geqslant u_h^i,\ x \geqslant 0_N\} \equiv C_h^i(u_h^i,z^{ih},w^i) > 0$$

where $w^i >> 0_N$ is the positive vector of consumer prices that each household in country i faces and C_h is the <u>expenditure</u> <u>function</u> of household h in country i. Note that we are assuming that there are I countries and H_i households in country i.

The household h Konüs [1924] <u>cost</u> <u>of</u> <u>living</u> or <u>price</u> <u>index</u> of country i relative to country j can be defined as

$$(57) \qquad K_h^{ij} \equiv C_h^i(u_h^i, z^{ih}, w^i)/C_h^i(u_h^i, z^{ih}, w^j), \quad i,j=1,\ldots,I, h=1,\ldots,H_i$$

$$= w^i \cdot x^{ih}/C_h^i(u_h^i, z^{ih}, w^j) \qquad \text{using (56).}$$

In definition (57), we used the preferences and demographic variables of household h in country i as reference quantities. In the following definition of the household k <u>cost</u> <u>of</u> <u>living</u> or <u>price</u> <u>index</u> of country i relative to j, we use the preferences and demographic variables of household k in country j as reference quantities:

$$(58) \qquad K_k^{*ij} \equiv C_k^j(u_k^j, z^{jk}, w^i)/C_k^j(u_k^j, z^{jk}, w^j), \quad i,j=1,\ldots,I, k=1,\ldots,H_j$$

$$= C_k^j(u_k^j, z^{jk}, w^i)/w^j \cdot x^{jk} \qquad \text{using (56).}$$

The following Proposition shows that under certain restrictions on preferences (which do not seem to be too restrictive), we can compute the geometric mean of the theoretical price indexes K_h^{ij} and K_k^{*ij} given only the observable price vectors in the two countries, w^i and w^j, the consumption vector of household k in country i, x^{ih}, and the consumption vector of household k in country j, x^{jk}.

<u>Proposition 9</u> (Caves, Christensen and Diewert [1982b; 1410]): If the

expenditure functions for household h in country i, C_h, and household k in country j, C_k^j, are translog[32] with identical coefficients on the second order terms in commodity prices, then

$$(59)\qquad [K_h^{ij}K_k^{*ij}]^{1/2} = \Pi_{n=1}^{N}(w_n^i/w_n^j)^{(1/2)[(w_n^i x_n^{ih}/w^i \cdot x^{ih}) + (w_n^j x_n^{jk}/w^j \cdot x^{jk})]}$$

$$\equiv P_T(w^j,w^i,x^{jk},x^{ih})$$

where P_T is the translog or Törnqvist price index of consumer prices in country i relative to j, using the consumption vectors of household h in country i and household k in country j as quantity weighting vectors.

Proposition 9 gives us a reasonable approach for comparing prices in country i to those of country j through the prefences of single households in each country. At this stage, it seems reasonable to average over households to obtain an average index of country i consumer prices relative to those of country j. Thus fix i and j and choose H_iH_j weights α_{hk}^{ij} which satisfy

$$(60)\qquad \alpha_{hk}^{ij} \geqslant 0,\ \Sigma_{h=1}^{H_i}\Sigma_{k=1}^{H_j}\alpha_{hk}^{ij} = 1$$

and form the following <u>average</u> <u>index</u> of country i prices relative to those of country j:

$$(61)\qquad \begin{aligned} K^{ij} &\equiv \Pi_{h=1}^{H_i}\Pi_{k=1}^{H_j}[K_h^{ij}K_k^{*ij}]^{(1/2)\alpha_{hk}^{ij}} \\ &= \Pi_{h=1}^{H_i}\Pi_{k=1}^{H_j}[P_T(w^j,w^i,x^{jk},x^{ih})]^{\alpha_{hk}^{ij}} \end{aligned}$$

where the equality (61) follows under the hypotheses of Proposition 9, hypotheses which we assume for the remainder of this section.

Two special cases of the general weighting scheme defined by (60) are of some interest.

Case (i): Democratic Weighting. In this case, each household in each country gets an equal weight, i.e., we have

(62) $$\alpha_{hk}^{ij} \equiv 1/H_i H_j.$$

In this case, the equality (61) may be rewritten as

(63) $$\begin{aligned} K^{ij} &= [\Pi_{h=1}^{H_i}(K_h^{ij})^{1/H_i}]^{1/2}[\Pi_{k=1}^{H_j}(K_k^{*ij})^{1/H_j}]^{1/2} \\ &= \Pi_{h=1}^{H_i}\Pi_{k=1}^{H_j}[P_T(w^j,w^i,x^{jk},x^{ih})]^{1/H_i H_j}. \end{aligned}$$

Note that the right hand side of (63) may be evaluated empirically provided that we have information on the individual household consumption vectors, x^{ih} and x^{jk}, in each country.

Case (ii): Plutocratic Weighting. In this case, each household gets a weight that is proportional to its share of country consumption; i.e., we have

(64) $$\alpha_{hk}^{ij} \equiv s_h^i s_k^j \quad ; \; i,j=1,\ldots,I; \; h=1,\ldots,H_i; \; k=1,\ldots,H_j;$$

where $s_h^i \equiv w^i \cdot x^{ih}/\Sigma_{m=1}^{H_i} w^i \cdot x^{im}$ and $s_k^j \equiv w^j \cdot x^{jk}/\Sigma_{m=1}^{H_j} w^j \cdot x^{jm}$. In this case, the equality (61) becomes

(65) $$\begin{aligned} K^{ij} &= [\Pi_{h=1}^{H_i}(K_h^{ij})^{s_h^i}]^{1/2} \; [\Pi_{k=1}^{H_j}(K_k^{*ij})^{s_k^j}]^{1/2} \\ &= P_T(p^j,p^i,x^j,x^i) \end{aligned}$$

where the country i and j aggregate consumption vectors, x^i and x^j, appear in the translog price index formula on the right hand side of (65); i.e., we have defined

$$(66) \qquad x^i \equiv \Sigma_{h=1}^{H_i} x^{ih} \qquad \text{for } i=1,\ldots,I.$$

Thus the advantage of the plutocratic weighting system is that we can evaluate the theoretical average Konüs price index defined by the left hand side of (65) using only aggregate data. Note that the theoretical index is a geometric mean of two terms. The first term represents a plutocratically weighted geometric mean of individual consumer h Konüs price indexes for country i relative to j prices where the average is taken over all households h in country i while the second term represents a similar plutocratically weighted geometric mean of conumser k price indexes, where the average is taken over all households k in country j.

Given that the K^{ij} defined by (63) or (65) forms a satisfactory approximation to the level of consumer prices in country i relative to those in country j, we still have to address the issue of achieving a consistent multilateral ranking of consumer prices among all I countries. Fortunately, it is not necessary to engage in a lengthy discussion of this problem. All we have to do is reinterpret the previous analysis presented in section 3 in the following way: replace the old $P(p^j,p^i,y^j,y^i)$ which occurs in (22) by K^{ij} defined by (63) or (65). The rest of the analysis in section 3 then carries through. We leave the details to the reader.

We turn now to the related problems involved in making interhousehold and intercountry comparisons of real consumption.

10. Bilateral and Multilateral Real Consumption Comparisons.

Suppose we wished to compare the real consumption of household h in country i with household k in country j. Then a first approach to measuring this real consumption ratio would be to deflate the actual expenditure ratio, $w^i \cdot x^{ih}/w^j \cdot x^{jk}$, by one of the Konüs price deflators defined by (57) or (58). In order for this procedure to give meaningful theoretical results, it is necessary to assume that <u>each household in each country has the same preferences</u> over market goods and the vectors of demographic variables can be ignored. Under these conditions, the cost functions $C_h^i(u_h^i, z^{ih}, w^i)$ defined by (56) become $C(u_h^i, w^i)$. Now we may define the following Pollak [1971; 64] <u>implicit real consumption indexes</u> for household h in country i relative to household k in country j as

$$(67) \qquad Q_{hk}^{ij} \equiv w^i \cdot x^{ih}/w^j \cdot x^{jk} \; K_h^{ij} \quad ; \quad i,j=1,\ldots,I; \; h=1,\ldots,H_i; \; k=1,\ldots,H_j;$$

$$= C(u_h^i, w^i)/C(u_k^j, w^j)[C(u_h^i, w^i)/C(u_h^i, w^j)] \quad \text{using (56) and (57)}$$

$$= C(u_h^i, w^j)/C(u_k^j, w^j) \quad ;$$

$$(68) \qquad Q_{hk}^{*ij} \equiv w^i \cdot x^{ih}/w^j \cdot x^{jk} \; K_k^{*ij}$$

$$= C(u_h^i, w^i)/C(u_k^j, w^i) \qquad \text{using (56) and (58).}$$

The extreme right hand sides of (67) and (68) are <u>Allen</u> [1949; 199] <u>quantity indexes</u>, and the reader is referred to his article for their properties.

The following Proposition is a straightforward consequence of Proposition 9.

<u>Proposition 10</u>: Under the hypotheses of Proposition 9 plus the additional hypothesis that each household in each country has the same preferences, then a geometric mean of the Allen quantity indexes defined by (67) and (68) is

equal to an implicit translog price index; i.e., for $i,j=1,\ldots,I$; $h=1,\ldots,H_i$; $k=1,\ldots,H_j$:

$$(69) \qquad [Q_{hk}^{ij} Q_{hk}^{*ij}]^{1/2} = w^i \cdot x^{ih} / w^j \cdot x^{jk} \; P_T(w^j,w^i,x^{jk},x^{ih})$$
$$= \tilde{Q}_T(w^j,w^i,x^{jk},x^{ih})$$

where P_T is the translog price index defined in (59).

We defined the right hand side of (69) to be $\tilde{Q}_T(w^j,w^i,x^{jk},x^{ih})$, the implicit translog index of consumption of household h in country i to the consumption of household k in country j. Note that if the number of consumption goods equals one (N=1), then the right hand side of (69) reduces to the "right" answer, x_1^{ih}/x_1^{jk}.

Proposition 10 allows us to make bilateral real consumption comparisons between any pair of households in any two countries. If individual household consumption data x^{ih} are available, then Proposition 10 provides a solution to the bilateral comparison problem. Solutions to the problem of making consistent multilateral comparisons in a symmetric way can now be obtained by adapting the techniques outlined in section 6.

The <u>own share method</u> for making multilateral consumption comparisons may be defined as follows. First define, for $i=1,\ldots,I$ and $h=1,\ldots,H_i$:

$$(70) \qquad \alpha_{ih} \equiv [\Sigma_{j=1}^{I} \Sigma_{k=1}^{H_j} [\tilde{Q}_T(w^j,w^i,x^{jk},x^{ih})]^{-1}]^{-1}.$$

The share of household h in country i of real world consumption is defined as

$$(71) \qquad s_{ih} \equiv \alpha_{ih}/\Sigma_{j=1}^{I} \Sigma_{k=1}^{H_j} \alpha_{jk}, \quad i=1,\ldots,I;\ h=1,\ldots,H_i.$$

The <u>democratic share method</u>[33] for making multilateral consumption comparisons requires the following definitions for $i=1,\ldots,I$; $h=1,\ldots,H_j$:

$$(72)\qquad \sigma_{ih}^{jk} \equiv \tilde{Q}_T(w^j,w^i,x^{jk},x^{ih})/\Sigma_{m=1}^{I}\ \Sigma_{n=1}^{H_m}\ \tilde{Q}_T(w^j,w^m,x^{jk},x^{mn}).$$

σ_{ih}^{jk} is the share of household h in country i of world consumption, using the preferences of household k in country j as a metric. The democratic share of world consumption of household h in country i is obtained by averaging the σ_{ih}^{jk} over all households jk:

$$(73)\qquad \sigma_{ih} \equiv \Sigma_{j=1}^{I}\ \Sigma_{k=1}^{H_j}\ \sigma_{ih}^{jk}/(H_1 + H_2 + \ldots + H_I);\ i=1,\ldots,I;\ h=1,\ldots,H_i.$$

We now prefer a democratic share method to the corresponding plutocratic share method because it seems fair to let each household count equally when forming the averages in (73); i.e., we now have a natural indivisible measure of size (the household) which was missing when we were making output comparisons.

In both the own share and democratic share method for making multilateral consumption comparisons, we essentially treated each household in each country as a separate comparison unit. To obtain country shares of world consumption, simply sum over households in the country; e.g., $s_i \equiv \Sigma_{h=1}^{H_i} s_{ih}$ and $\sigma_i \equiv \Sigma_{h=1}^{H_i} \sigma_{ih}$. However, in practise, it will be very difficult if not impossible to implement these methods empirically due to the unavailability of the individual household consumption data.[34] Hence we will develop a method in the remainder of this section that requires less empirical information to implement, (but at the same time, it is not quite as satisfactory from a theoretical point of view).

Let the preferences of household h in country i be represented by the nondecreasing, continuous from above utility function $F_{ih}(x)$, defined for $x \geqslant$

0_N. (We have absorbed any demographic or public good variables into F_{ih}, which can differ arbitrarily across households). For i=1,...,I and h=1,...,H_i, define the household h in country i <u>deflation</u> <u>function</u> $D_{ih}(u,x)$ for $x \geqslant 0_N$ and u belonging to the range of F_{ih} by

$$(74) \qquad D_{ih}(u,x) \equiv \max_\delta \{\delta : F_{ih}\,(x/\delta) \geqslant u,\ \delta>0\}.$$

Then for any reference utility level u belonging to the range of F_{ih}, the <u>Malmquist</u> [1953] <u>quantity</u> <u>index</u> of $x^* \geqslant 0_N$ relative to $x \geqslant 0_N$ is defined by

$$(75) \qquad Q_{ih}(u,x,x^*) \equiv D_{ih}(u,x^*)/D_{ih}(u,x).$$

Let the observed household h in country i in consumption vector be $x^{ih} > 0_N$ and define the corresponding utility level by

$$(76) \qquad u_{ih} \equiv F(x^{ih}),\ i=1,\ldots,I;\ h=1,\ldots,H_i.$$

Now define the following index of average household consumption in country i relative to country j:

$$(77) \qquad Q^{ij} \equiv \Sigma_{h=1}^{H_i}\ H_i^{-1}\ Q_{ih}(u_{ih},\ \Sigma_{k=1}^{H_j}\ H_j^{-1}x^{jk},x^{ih}),\quad i,j=1,\ldots,I.$$

To explain the meaning of (77), note that $Q_{ih}(u_{ih},\ \Sigma_{h=1}^{H_j}\ H_j^{-1}x^{jk},x^{ih})$ is the household h in country i Malmquist quantity index which compares the household ih consumption vector x^{ih} to the average per household consumption vector in country j, $\Sigma_{k=1}^{H_j}\ H_j^{-1}x^{jk}$, where the household ih indifference surface through the household ih consumption vector is used as the reference indifference surface. Q^{ij} is the average (over all households in country i) of these individual ih Malmquist quantity indexes Q_{ih} just described.

Let us assume expenditure minimizing behavior on the part of each household; i.e., assume for $i=1,\ldots,I$ and $h=1,\ldots,H_i$:

$$(78) \qquad w^i \cdot x^{ih} = \min_x \{w^i \cdot x : F_{ih}(x) \geq u_{ih},\ x \geq 0_N\} \equiv C_{ih}(u_{ih}, w^i).$$

<u>Proposition 11</u>: Assume: (i) $w^i >> 0_N$, $i=1,\ldots,I$; (ii) $x^{ih} > 0_N$ for $i=1,\ldots,I$ and $h=1,\ldots,H_i$, (iii) (76) and (78). Then Q^{ij} defined by (77) has the following lower bound:

$$(79) \qquad Q^{ij} \geq w^i \cdot \Sigma_{h=1}^{H_i} H_i^{-1}\, x^{ih} / w^i \cdot \Sigma_{k=1}^{H_j} H_j^{-1} x^{jk} \equiv w^i \cdot \bar{x}^i / w^i \cdot \bar{x}^j$$

where we have defined the per household average consumption vectors by $\bar{x}^i \equiv \Sigma_{h=1}^{H_i}\ H_i^{-1} x^{ih}$.

It can be seen that $(Q^{ji})^{-1}$ is also an index of the average consumption of households in country i relative to the average consumption of households in country j. In fact, if N=1, we have

$$(80) \qquad Q^{ij} = [Q^{ji}]^{-1} = \Sigma_{h=1}^{H_i}\ H_i^{-1} x_1^{ih} / \Sigma_{k=1}^{H_j}\ H_j^{-1} x_1^{jk} \equiv \bar{x}_1^i / \bar{x}_1^j$$

where the right hand side of (80) is the average country i consumption of the good divided by the average country j consumption.

Using (79) for i=j and j=i, we derive

$$(81) \qquad [Q^{ji}]^{-1} \leq w^j \cdot \Sigma_{h=1}^{H_i}\ H_i^{-1} x^{ih} / w^j \cdot \Sigma_{k=1}^{H_j}\ H_j^{-1} x^{jk} = w^j \cdot \bar{x}^i / w^j \cdot \bar{x}^j.$$

Having established the bounds (79) and (81), we can prove the following counterpart to Proposition 2, using the same technique of proof.

<u>Proposition 12</u>: Assume the hypotheses of Proposition 11. Then for each pair of countries i and j there exists a λ^*_{ij} such that $0 \leq \lambda^*_{ij} \leq 1$ and $\lambda^*_{ij}Q^{ij} + (1-\lambda^*_{ij})[Q^{ji}]^{-1}$ lies between the average household Paasche and Laspeyres quantity indexes for country i relative to j, $w^i \cdot \bar{x}^i / w^i \cdot \bar{x}^j$ and $w^j \cdot \bar{x}^i / w^j \cdot \bar{x}^j$, respectively.

Proposition 12 suggests that we approximate the theoretical index $\lambda^*_{ij}Q^{ij} + (1-\lambda^*_{ij})[Q^{ji}]^{-1}$ by a symmetric average of the Paasche and Laspeyres quantity indexes such as the Fisher index Q_F defined by

$$(82) \qquad Q_F(w^j,w^i,\bar{x}^j,\bar{x}^i) \equiv [w^i \cdot \bar{x}^i / w^i \cdot \bar{x}^j]^{1/2}[w^j \cdot \bar{x}^i / w^j \cdot \bar{x}^j].^{1/2}$$

It is important to note that the consumption index defined by (82) should <u>not</u> be interpreted as a welfare index for country i relative to j; rather it approximates an average of the theoretical per household or per capita real consumption of country i relative to country j indexes defined by (78) and (80). We have not taken into account any inequality in the distribution of consumption in each country.

At this point, we could largely duplicate the material presented in section 6: simply replace our old $Q(p^j,p^k,y^j,y^k)$ by the new relative total quantity index, $Q_F(w^j,w^i,H_j\bar{x}^j, H_i\bar{x}^i)$, which converts the per capita or per household relative consumption index defined by (82) into an index of total consumption in country i relative to total consumption in country j. With these changes, the 6 methods for making multilateral quantity comparisons described in sections 6 and 8 can be repeated.

We can summarize our analysis up to this point as follows. Bilateral international or interregional quantity comparisons can be made either on the basis of producer theory or consumer theory. Each approach leads to one or two "ideal" bilateral index number formulae. The problem of aggregating up the bilateral comparison information to yield a consistent multilateral ranking of real outputs or real consumptions can then be approached in a common axiomatic manner. We have considered six different multilateral systems and discussed their relative advantages and disadvantages.

Before concluding the paper, we devote the next section to an exposition of two multilateral systems that are not based on averaging over bilateral indexes.

11. Multilateral Systems that are Not Based on Bilateral Formulae.

Let us revert to the multilateral output comparison problem discussed above in sections 6 to 8.[35] We use the definitions and notation explained there.

The first new multilateral system we wish to consider is the Geary [1958] Khamis [1970][1972] or GK system. Consider the following two sets of equations:

$$(83)\qquad \Pi_n = \Sigma_{i=1}^{I} p_n^i y_n^i P_i^{-1} / \Sigma_{i=1}^{I} y_n^i \qquad ;\ n=1,\ldots,N \text{ and}$$

$$(84)\qquad P_i^{-1} = \Sigma_{n=1}^{N} \Pi_n y_n^i / \Sigma_{n=1}^{N} p_n^i y_n^i \qquad ;\ i=1,\ldots,I$$

Equations (83) and (84) are to be regarded as N+I simultaneous equations in the N unknown "international prices" Π_n and the I purchasing power parity (or country i price index) functions $P_i = P_i(p,y)$. It is easy to show that the system of equations defined by (83) is homogeneous and linearly dependent in the unknowns $\Pi_1,\ldots,\Pi_N,P_i,\ldots,P_I$. Hence we may drop any one of the equations and we may impose any normalization on the P_i. Khamis [1970][1972] shows that the resulting system of equations has a unique strictly positive solution (up to a positive scalar multiple) provided that the price and quantity data, $p_n^i,y_n^i,i=1,\ldots,I,n=1,\ldots,N$, are all positive.[36] Given a set of solution functions, $P_1(p,y),\ldots,P_I(p,y)$, we shall choose to normalize these functions so that the share functions $S_i(p,y)$ defined by

$$(85) \qquad S_i(p,y) \equiv p^i \cdot y^i/P_i(p,y) \qquad , i=1,\ldots,I,$$

sum up to unity. Remember that $p \equiv [p^1,\ldots,p^I]$ and $y \equiv [y^1,\ldots,y^I]$ are N by I price and quantity matrices.

<u>Proposition 13</u>: The GK system for making multilateral real output comparisons defined by (83)-(85) satisfies all multilateral tests except MT1, MT3 and MT10.

The GK system fails MT1 because in general one cannot guarantee a positive solution to (83) and (84) when some quantities y_n^i are negative. Khamis [1984; 204] recognized this problem but did not exhibit a solution to it. However, it turns out to be relatively straightforward to solve this problem by adapting a technique used by Van Ijzeren [1983; 43]. First, substitute equations (83) into (84). It is then easy to show that the resulting system of equations in $[P_1^{-1},\ldots,P_I^{-1}]^T \equiv P^{-1}$ is equivalent to the following system of equations,

$$(86) \qquad (M-I_I)P^{-1} = 0_I$$

where I_I is an I by I identity matrix and the i,jth element of the I by I matrix M is defined by

$$(87) \qquad m_{ij} \equiv (p^i \cdot y^i)^{-1} \Sigma_{n=1}^{N} (y_n^i [\Sigma_{k=1}^{I} y_n^k]^{-1} p_n^j y_n^j) \quad , \; i,j=1,\ldots,I.$$

If the matrix M has all elements positive (or even weaker, if M is nonnegative but irreducible), then by the Theorem of Frobenius in matrix algebra, there is a unique (up to a positive scale factor) positive solution $P^{-1} \gg 0_I$ to (86). This is the solution we are looking for. Hence even if some quantities y_n^i are negative, as long as the m_{ij} defined by (87) are positive, the GK system will satisfy MT1.

Necessary and sufficient conditions for the GK shares to satisfy MT3 are that the GK price functions $P_j(p,y)$ be homogeneous of degree zero in their y^i arguments; i.e., for $\lambda_i > 0$, $P_j(p,y^1,\ldots,y^{i-1},\lambda_i y^i,y^{i+1},\ldots,y^I) = P_j(p,y)$ for $j=1,\ldots,I$ and for $i=1,\ldots,I$. In words, these homogeneity properties imply that the country price indexes are invariant to scale changes in country quantity vectors. Drechsler [1973; 26] noted that the GK price indexes do not have these desirable invariance properties. Moreover, Drechsler also observed that the GK system fails MT3 in a relatively spectacular manner:

> "If an infinitely great country is compared with an infinitely small country, the GK quantity index will be the same as the index using the big country's prices."
>
> (Drechsler [1973; 26]).

Another way of expressing the idea behind the above quotation is to replace the country i quantity vector y^i by λy^i for $\lambda > 0$ in (83) and (84) and then let λ tend to infinity. The limiting system of GK relative price parities turns out to be

$$(88) \qquad P_j/P_1 = (p^j \cdot y^j/p^i \cdot y^j)/(p^1 \cdot y^1/p^i \cdot y^1) \qquad , \; j=2,\ldots,I$$

which is equal to the Paasche price index for country j relative to country i divided by the Paasche price index for country 1 relative to country i. This point was orginally made by Geary [1958].

We mentioned earlier that the own share system defined by (44) and (45) also failed to pass the multilateral test MT3. However, its failure is not nearly as "bad" as the failure of the GK system exhibited in (88). Again replacing y^i by λy^i and taking limits as λ tends to infinity yields the following limiting system of own share price parities, provided the underlying bilateral formula Q satisfies BT1, BT3 and BT6:

$$(89) \qquad P_j/P_1 = [p^j \cdot y^j/Q(p^i,p^j,y^i,y^j)]/[p^1 \cdot y^1/Q(p^i,p^1,y^i,y^1)] \qquad , \; j=2,\ldots,I.$$

The results given by (89) are quite reasonable from the viewpoint of economic theory if Q is chosen to be Q_F or $\tilde{Q}_T$.

Of course, it is not surprising that the GK system fails the strong dependence on a bilateral formula test, MT10, since the GK system has no bilateral foundation. How important is it for a multilateral system to have a bilateral foundation? While there can be no definitive answer to this question, it seems worthy of some discussion. Our test MT10 is one way of formalizing a property that Drechsler [1973] termed "characteristicity." It is perhaps best to let Drechsler explain this concept in his own words:

"In general, this requirement means that the weights used for any index computations should be characteristic of the given two countries. . . . To use Indian weights in a Netherlands-Belgium comparison would be considered wrong by everybody just as if in an Indian-Pakistan comparison Dutch weights were used. In the latter cases, the weights would be very uncharacteristic; their use would amount to the same as if in the case of the computation of a 1971-1970 inter-temporal index 1920 (or 2020) prices were used."

(Drechsler [1973; 19]).

Gerardi [1985] criticizes the GK method for its lack of characteristicity; he feels that the method is equivalent to the use of the price structure of a recent year in order to evaluate the quantities of all years of the century.

The first part of our test MT10 imposes a certain amount of characteristicity on the multilateral system: the country shares are to depend only on the "best" bilateral quantity indexes, each of which has a maximal degree of characteristicity. The second part of MT10 says that as the world economy shrinks to a two country economy, the relative share of the two nonshrinking economies tends to the "best" bilateral quantity index between the two countries. Thus MT10 can be interpreted as a characteristicity test.

Turning now to other criticisms of the GK system, Marris [1984; 52] notes that in the consumer theory context, the GK quantity index will be biased upwards for countries with price structures far from the Π_n international prices (the bias will go in the opposite direction in the producer context). This is perhaps an obvious point, since the derivation of the GK system does not draw on either consumer or producer theory. Thus from the viewpoint of being consistent with economic theory, methods (i)-(vi) described earlier in section 6 have a clear advantage over the GK method.[39]

A final criticism of the GK method is the following one due to Gerardi [1985]: it is not stable with respect to the entry and withdrawal of countries; i.e., suppose we implement the method for I countries and then drop country I and redo the method. Then the relative output shares of the first I-1 countries can change markedly with the deletion of the last country. Empirical work cited by Gerardi has shown that the GK method can be quite unstable. The reason for this instability can be seen by considering the system of I equations defined by (86). Small changes in the country quantity vectors y^i can generate relatively large changes in the M matrix and these changes in turn can generate even larger changes in the P^{-1} vector.

It is obvious that the star system will not suffer from this kind of instability; in fact the method will be completely stable with respect to the entry and withdrawal of nonnumeraire countries. While methods (ii)-(vi) explained in section 6 are not completely stable, they are all much more stable than the GK method, because they all involve taking averages of bilateral quantity indexes of the form $Q(p^i,p^j,y^i,y^j)$ in a symmetric manner. As an additional country is added, these averages will not change by very much.

From a narrow point of view, the GK system is dominated by the own share system, since the latter satisfies MT1 and MT10 whereas the former satisfies neither. However, proponents of the GK system would probably be correct in asserting that prices P_i and shares S_i will be positive in real life situations, so that in paractice, MT1 will be satisfied. Proponents could also dismiss the failure of MT10 since it is a test that is biased in terms of weighting up bilateral formulae.

From a broader point of view, it also seems that the use of the GK system is not warranted for three reasons discussed above: (i) it fails MT3 in a somewhat disastrous manner, (ii) it has no reasonable producer or consumer theory interpretation and (iii) it is not stable with respect to the addition or deletion of countries.

The final multilateral method, which is not based on aggregating bilateral indexes, that we wish to consider is Van Yzeren's [1956; 25] balanced method. It is similar to the GK method in that we first define unique (up to a proportional factor) purchasing power parities, $P_1,\ldots,P_I$, and then define the corresponding share functions by (85). The parities P_i may be found by iterative substitution into the following system of equations:

(90) $$P_i = [\Sigma_{j=1}^{I}(p^i \cdot y^j/p^j \cdot y^j)P_j]^{1/2}[\Sigma_{j=1}^{I}(p^j \cdot y^i/p^i \cdot y^i)P_j^{-1}]^{-1/2},\ i=1,\ldots,I.$$

If the I^2 inner products $p^i \cdot y^j$ are all positive, Van Yzeren [1956; 25-26] shows that there is a unique positive solution ray to (90); his proof is based on showing that the solution to (90), subject to a positive normalization, minimizes the function $f(P_1,\ldots,P_I)$ over the positive orthant, subject to the same positive normalization, where f is defined by

(91) $$f(P_1,\ldots,P_I) \equiv \Sigma_{i=1}^{I}\Sigma_{j=1}^{I}P_i^{-1}(p^i \cdot y^j/p^j \cdot y^j)P_j.$$

Thus the balanced method is what Diewert [1981; 179] calls a neostatistical approach to the construction of indexes: one tries to obtain price and quantity indexes, P_i and Y_i say for $i=1,\ldots,I$, which satisfy the equations

$p^i \cdot y^j = P_i Y_j + e_{ij}$, $i,j = 1$, for some errors e_{ij} which are minimal in some norm.

The following Proposition sums up the mathematical properties of the balanced method due to Van Yzeren.

Proposition 14: The balanced method for making multilateral output comparisons defined by (90) and (85) satisfies all of the multilateral tests considered in section 8 except MT8, MT9, and MT10, provided that the inner products $p^i \cdot y^j$ are all positive.

It is of course, not surprising that the balanced method fails MT10, since that test is biased in favor of aggregative bilateral methods over genuinely multilateral methods such as the GK or balanced methods. The failure of the two country weighting tests, MT8 and MT9, is more troublesome.

From a narrow point of view, the balanced method is dominated by the EKSCCD system since it satisfies more multilateral tests. From the viewpoint of consistency with microeconomic theory, the EKSCCD system also seems preferable.

12. Extensions and Conclusion

It is straightforward to rework our material on output indexes into corresponding indexes for inputs. Thus if w^i denotes a positive vector of input prices for country or region i and x^i the corresponding nonnegative, nonzero input vector, we may rework our analysis and obtain a system of input share functions, $s_i(w^1,\ldots,w^I,x^1,\ldots,x^I) \equiv s_i(w,x) > 0$ for $i=1,\ldots,I$ such that $\Sigma_{i=1}^I s_i(w,x) = 1$. The six multilateral methods discussed in section 6 may now be reinterpreted as multilateral real input indexes.

Given a system of multilateral real output share functions $S_i(p,y)$ and a corresponding system of multilateral real input share functions $s_i(w,x)$, it is natural to define the following system of multilateral productivity functions, Π_i, by

$$(92) \qquad \Pi_i(p,y,w,x) \equiv S_i(p,y)/s_i(w,x), \; i=1,\ldots,I.$$

Such a system of productivity functions should be very useful in the context of cross sectional industry or plant data which are to be treated in a symmetric manner.

Another extension of our analysis can be made to situations where it is desirable to achieve consistency over time and space. If there are output data for I countries and T time periods, then consistent relative outputs over time and space can be achieved by treating the country data for each time period as separate data for an artificial country. Thus there will be IT separate multilateral share functions. However, in practice, I would not recommend this method. From the viewpoint of economic theory, the most accurate way of determining the output of country i in period t relative to its output in period 1 is to use the best bilateral quantity index available and the chain principle; i.e., the desired quantity index would be $Q(p^{i1},p^{i2},y^{i1},y^{i2})Q(p^{i2},p^{i3},y^{i2},y^{i3})\ldots Q(p^{it-1},p^{it},y^{it-1},y^{it})$ where p^{it} and y^{it} denote the price and quantity vectors for country i in period t and Q is the "best" bilateral quantity index. Thus in practice, I would recommend constructing a system of multilateral indexes for the first time period and then using individual country bilateral indexes for a number of time periods until a new multilateral system of relative country outputs could be constructed using the data for a single period.

The multilateral methods we have developed above in this paper may be used to construct subaggregates of aggregate output. We should not expect these multilateral subaggregates to add up to the corresponding multilateral aggregate. As a limiting case, consider a system of subaggregates that was so disaggregated, that each subaggregate consisted of a single homogeneous good. No one would expect these subaggregates to simply add up to aggregate utility or output. The point is that subaggregates cannot simply be added up to equal an aggregate -- an index number formula must be used in order to combine the subaggregates into an aggregate.

We would like to stress that we have developed our best multilateral methods (the star, the EKSCCD, and the own share systems) by aggregating up bilateral indexes. The bilateral indexes (Q_F and $\tilde{Q}_T$) were chosen for their consistency with microeconomic theory. It is this emphasis on microeconomic theory that distinguishes our work from the pioneering work of others.

Microeconomic theory enables us to provide at least tentative answers to a number of vexing problems. For example, should government expenditures, investment expenditures, stocks of consumer durables, exports and or imports be included in the international or interregional comparison? If we are doing an international comparison of real consumption, then obviously the nonconsumption items should not be included. Moreover, existing stocks of consumer durables should be added to new purchases of consumer durables and the total stock of durable should be priced out at its <u>rental price</u>, not its (stock) purchase price. If we are doing an international comparison of real outputs, then new sales of consumer goods (including consumer durables), exports and production of investment goods by private (i.e., nongovernmental) producers should be included as outputs. Import purchases by nongovernmental agencies should be included as negative outputs. All activities by non profit maximizing governmental agencies should be excluded from the comparisons.[40] Note that in the output context, consumer stock prices are used rather than rental prices.

Another vexing problem is: should prices be before or after tax prices? If we are making real consumption comparisons, the prices should be the after tax prices that consumers face. If we are making real output comparisons, the prices should be before tax prices in the case of outputs and after tax (and tariff) prices in the case of intermediate inputs and imports; i.e., the prices should be the prices that producers face. In the case of subsidized goods (e.g., health care and education in many countries), the prices consumers face are the lower after subsidy prices, while producers face the higher before subsidy prices.

We conclude by noting some additional novel features of our analysis: (i) aggregation over consumer issues are not ignored in our analysis, (ii) our best bilateral quantity indexes, Q_F and $\tilde{Q}_T$, do not satisfy some traditional tests such as the mean value test due to the existence of negative quantities and (iii) we have devised some new multilateral tests which should prove to be useful to researchers whether they share our microeconomic approach or not.

13. Appendix: Proofs of Propositions:

Proof of Proposition 1:

$$P^j(p^j,p^i) \equiv g^j(p^i,v^j)/g^j(p^j,v^j) \qquad \text{using definition (3)}$$

$$= \max_y\{p^i\cdot y : (y,v^j)\epsilon S^j\}/p^j\cdot y^j \qquad \text{using (1) and (4)}$$

$$\geq p^i\cdot y^j/p^j\cdot y^j$$

since y^j is feasible for the maximization problem. The proof of (6) is similar.

Proof of Proposition 2: Let i and j be given and define the function $h(\lambda) \equiv P_\lambda^{ij}(p^j,p^i)$. Then definitions (10) and (12) and assumption (v) imply that $h(\lambda)$ is a continuous function for $0 \leq \lambda \leq 1$. Note that $h(0) \equiv g^{ij}(p^i,0)/g^{ij}(p^j,0) = g^j(p^i,v^j)/g^j(p^j,v^j) \equiv P^j(p^j,p^i)$ and $h(1) \equiv g^{ij}(p^i,1)/g^{ij}(p^j,1) = g^i(p^i,v^i)/g^i(p^j,v^i) \equiv P^i(p^j,p^i)$. There are 24 possible a priori inequalities between the 4 numbers $h(0)$, $h(1)$, P_P^{ij} and P_L^{ij}. However, assumptions (i)-(iv) imply that the inequalities (5) and (7) hold, which may be rewritten as follows: $P_P^{ij} \equiv p^i\cdot y^i/p^j\cdot y^i \geq P^i(p^j,p^i) = h(1)$ and $P_L^{ij} \equiv p^i\cdot y^j/p^j\cdot y^j \leq P^j(p^j,p^i) = h(0)$. This means that only the following 6 inequalities are possible between the 4 numbers: (1) $h(0) \geq P_L^{ij} \geq P_P^{ij} \geq h(1)$, (2) $h(0) \geq P_P^{ij} \geq P_L^{ij} \geq h(1)$, (3) $h(0) \geq P_P^{ij} \geq h(1) \geq P_L^{ij}$, (4) $P_P^{ij} \geq h(0) \geq P_L^{ij} \geq h(1)$, (5) $P_P^{ij} \geq h(1) \geq h(0) \geq P_L^{ij}$ and (6) $P_P^{ij} \geq h(0) \geq h(1) \geq P_L^{ij}$. Since $h(\lambda)$ is continuous over $0 \leq \lambda \leq 1$, it assumes all intermediate values and hence there exists $0 \leqslant \lambda^* \leqslant 1$ such that

$$\text{(A1)} \qquad P_P^{ij} \equiv p^i \cdot y^i / p^j \cdot y^i \leqslant h(\lambda^*) \equiv P_{\lambda^*}^{ij}(p^j, p^i) \leqslant p^i \cdot y^j / p^j \cdot y^j \equiv P_L^{ij}$$

is true if case (1) occurs or such that

$$\text{(A2)} \qquad P_L^{ij} \leqslant P_{\lambda^*}^{ij}(p^j, p^i) \leqslant P_P^{ij}$$

is true if cases (2)-(6) occur. Our method of proof is due to Konüs [1939].

Proof of Proposition 3:

$$[P_F(p^j, p^i, y^j, y^i)]^2 = p^i \cdot y^i p^i \cdot y^j / p^j \cdot y^i p^j \cdot y^j \qquad \text{using (13)}$$

$$= g^i(p^i, v^i) p^i \cdot y^j / p^j \cdot y^i g^j(p^j, v^j) \qquad \text{using (4)}$$

$$= (p^i \cdot Bp^i)^{1/2} h^i(v^i) p^i \cdot y^j / p^j \cdot y^i (p^j \cdot Bp^j)^{1/2} h^j(v^j) \qquad \text{using (14)}$$

$$= \frac{(p^i \cdot Bp^i)^{1/2} h^i(v^i) p^i \cdot Bp^j (p^j \cdot Bp^j)^{-1/2} h^j(v^j)}{p^j \cdot Bp^i (p^i \cdot Bp^i)^{-1/2} h^i(v^i) (p^j \cdot Bp^j)^{1/2} h^j(v^j)} \qquad \text{using (17) and (14)}$$

$$= p^i \cdot Bp^i / p^j \cdot Bp^j \qquad \text{using } p^i \cdot Bp^j = p^j \cdot Bp^i$$

$$= [(p^i \cdot Bp^i)^{1/2} h^i(v) / (p^j \cdot Bp^j)^{1/2} h^i(v)]^2 \qquad \text{for any reference } v$$

$$= [P^i(p^j, p^i)]^2 \qquad \text{for } v = v^i.$$

Proof of Proposition 4:

$(1/2)\ \ln P^i(p^j,p^i) + (1/2)\ln P^j(p^j,p^i)$

$= (1/2)\ln[g^i(p^i,v^i)/g^i(p^j,v^i)] + (1/2)\ln[g^j(p^i,v^j)/g^j(p^j,v^j)]$

using (2) and (3)

$= (1/2)\Sigma_{n=1}^{N}[p_n^i \partial \ln g^i(p^i,v^i)/\partial p_n + p_n^j \partial \ln g^j(p^j,v^j)/\partial p_n][\ln p_n^i - \ln p_n^j]$

using the translog identity in Caves, Christensen and Diewert [1982b, 1412]

$= (1/2)\Sigma_{n=1}^{N}[(p_n^i y_n^i/p^i \cdot y^i) + (p_n^j y_n^j/p^j \cdot y^j)][\ln p_n^i - \ln p_n^j]$ using (17) and (4)

$\equiv \ln P_T(p^j,p^i,y^j,y^i)$ using definition (19).

Proof of Proposition 5:

$\Pi_{i=1}^{I}\delta_i^{s_i} = \Pi_{i=1}^{I}[\Pi_{j=1}^{I}P(p^j,p^i,y^j,y^i)^{s_j}]^{s_i}$ using (22) and (i)

$= \Pi_{i=1}^{I}\Pi_{j=1,i\neq j}^{I}[P(p^j,p^i,y^j,y^i)]^{s_i s_j}$

$= \Pi_{1 \leq i < j \leq I}\ [P(p^j,p^i,y^j,y^i)P(p^i,p^j,y^i,y^j)]^{s_i s_j}$

$= \Pi_{1 \leq i < j \leq I}\ [P(p^j,p^i,y^j,y^i)/P(p^j,p^i,y^j,y^i)]^{s_i s_j}$ using (iii)

$= 1.$

Proof of Proposition 6: Assumption (ii) implies that (y^i,v^i) is on the frontier of S^i and hence (32) will hold. Thus $Q_M^j(y^j,y^i) = d^j(y^i,v^j) \equiv \min_{\delta_i}\{\delta : (y^i/\delta,v^j)\epsilon S^j,\ \delta>0\} = \delta_{ji}>0$ by (iii). Thus $(y^i/\delta_{ji},v^j)\ \epsilon S^j$ and so y^i/δ_{ji} is feasible for the maximization problem which follows:

$$p^j \cdot y^j = \max_y \{p^j \cdot y : (y, v^j) \in S^j\} \geqslant p^j \cdot y^i/\delta_{ji}.$$

Therefore, $\delta_{ji} \geqslant p^j \cdot y^i/p^j \cdot y^j$ using (i) which is (35). (36) follows in an analogous manner.

Proof of Proposition 7: Define $h(\lambda) \equiv Q_\lambda^{ij}(y^j, y^i)$. Using (42), (35) and (36), we have $h(0) = Q_M^i(y^j, y^i) \leqslant Q_P(p^j, p^i, y^j, y^i)$ and $h(1) = Q_M^j(y^j \cdot y^i) \geqslant Q_L(p^j, p^i, y^j, y^i)$. This reduces the 24 a priori possible inequalities between the 4 numbers h(0), h(1), Q_P and Q_L to six inequalities. The remainder of the proof follows the proof of Proposition 2.

Proof of Proposition 8: Omitted due to its length and straightforward nature.

Proof of Proposition 11: Adapting a method of proof due to Malmquist [1953; 231] shows that $Q_{ih}(u_{ih}, \Sigma_{k=1}^{H_j} x^{jk}, x^{ih}) \geqslant w^i \cdot x^{ih}/w^i \cdot \Sigma_{k=1}^{H_j} H_j^{-1} x^{jk}$ for $h=1,\ldots,H_i$. Now sum these inequalities, divide both sides by H_i^{-1} and obtain (79).

Proof of Proposition 13: Proof of MT2: Suppose $p \gg 0_N$, $p \cdot y > 0$, $\alpha_i > 0$,

$\beta_i > 0$ for $i=1,\ldots,I$, $\Sigma_{i=1}\beta_i = 1$, $p \equiv \alpha_i p$, $y \equiv \beta_i y$ for $i=1,\ldots,I$. We show that $S_i(p,y) = \beta_i$ for $i=1,\ldots,I$ by showing that $\Pi_n = p_n$, $n=1,\ldots,N$, $P_i = \alpha_i$, $i=1,\ldots,I$ and $S_i = \beta_i$, $i=1,\ldots,I$ satisfy (83), (84), and (85) subject to the normalization $\Sigma_{i=1}^{I} S_i = 1$. If in addition, $y_n \neq 0$ for $n=1,\ldots,N$, then the above solution is the unique solution.

The proofs of the other tests are made in a similar manner. In each case, conditions sufficient to imply the uniqueness and positivity of the P_i, S_i part of the solution to (83)-(85) must be made.

Proof of Proposition 14: Straightforward computations. In this case, the conditions $p^i \cdot y^j > 0$ are sufficient to guarantee the uniqueness and positivity of the solution to (90) and (85), subject to the normalization $\Sigma_{i=1}^{I} S_i = 1$.

Footnotes

1. Note that since this theoretical purchasing power parity function requires the assumption of competitive (i.e., price taking) revenue maximizing behavior on the part of producers, the government sector in each country should be excluded in the international comparison. Moreover, appropriate marginal prices must be estimated for any noncompetitive sectors in the two countries. Thus the more traditional national income accounting approaches to international comparisons as outlined in Kravis [1984] perhaps have some advantages over the producer theory approaches explained in this paper, since government sectors are included and noncompetitive behavior is ignored.

2. Ruggles [1967; 189-190] constructed bilateral consumer price indexes for 19 Latin American countries for the year 1961. He found that Paasche and Laspeyres indexes differed by about 50 percent per observation (in constructing a Paasche and a Laspeyres index between two countries, the role of the base and comparison country are interchanged).

3. Notation: 0_M denotes an M dimensional vector of zeros, $v \geqslant 0_M$ means the vector v is nonnegative, $v > 0_M$ means v is nonnegative with at least one component positive and $v >> 0_M$ means each component of v is positive.

4. If country i is not producing the nth net output (or utilizing it as an input) in its private production sector, then we must estimate a positive shadow price for this good, $p_n^i > 0$, which would just induce the private production sector in country i to produce a zero amount of this nth good. Proposition 4 requires that all prices be positive.

5. We shall assume that the private national product functions g^i defined by (1) are well defined as maximums (rather than supremums) for the relevant net

output price vectors $p >> 0_N$.

6. On the other hand, sectors of country i that behave monopolistically on (net) output markets should be excluded from g^i (or appropriate marginal prices should be used).

7. See also Samuelson and Swamy [1974; 588-592], Archibald [1977; 60-61] and Diewert [1980; 461] [1983b; 1055].

8. If the sets S^i are convex, then the functions $g^i(p,v)$ are concave and (normally) continuous functions in v. For the formal mathematical properties that the national product functions g^i will have under various assumptions about the sets S^i, see Diewert [1973] or McFadden [1978].

9. See Diewert [1983b; 1060]. It should also be noted that our present Proposition 2 seems to be somewhat controversial in the literature on international comparisons. Consider the following quotation from Khamis [1984; 195]: "Also, in spite of the many reminders in the literature on index number methodology calling attention to the misconception that the Laspeyres and Paasche indices provide respectively upper and lower bounds to the 'true' index (if amenable to measurement), many proponents of the ideal Fisher formula and/or its extensions still commit this error."

10. For example, see Fisher [1922; 489].

11. We also assume that the matrix B is such that $(p \cdot Bp)^{1/2}$ is a positive, convex function of p over the relevant range of net output price vectors.

12. See Diewert [1976; 130].

13. The α_n^i, α_{nk}^i and γ_{nm}^i must also satisfy certain restrictions which will ensure that $g^i(p,v)$ is linearly homogeneous in p; see Diewert [1974; 139].

14. This corresponds to the terminology used in Christensen, Cummings and

Jorgenson [1980]. Other terms for it include the Törnqvist index and the Divisia index; see Diewert [1976; 120]. Note that the s_n^i will be negative if the nth good is an imported good which is used by the private production sector in country i, since in this case $y_n^i < 0$.

15. Under these conditions, definitions (18) must be changed so that terms involving the zero v_m^i are dropped from the summations.

16. Ruggles [1967; 189-190] showed that P_F and P_T differed by about 1 to 2 per cent per observation using his 1961 data for 19 Latin American countries. The Paasche and Laspeyres indexes differed by about 50 per cent per observation.

17. Diewert [1978; 888] proved (21) under the assumptions that $p^i=p^j\equiv p>>0_N$ and $y^i=y^j\equiv y>0_N$. However, it can be verified that (21) is still valid provided that $p^i=p^j>>0_N$, $p^i\cdot y^i>0$, $p^j\cdot y^j>0$, $p^i\cdot y^j>0$ and $p^j\cdot y^i>0$; i.e., we now allow for possible negative components in the vectors y^i and y^j.

18. See Eichhorn and Voeller [1976] [1983] and Fisher [1922] on the test approach.

19. However, it seems to be true empirically that the choice of the base country is relatively unimportant if the Fisher or translog functional form, P_F or P_T, is used. In the Ruggles [1967; 189-190] paper, changing the base country when using the Fisher (translog) formula changed the numerical results by about 2 (3) per cent per observation. In the Kravis, Kenessey, Heston and Summers [1975; 52] book, changing the base country when using the Fisher formula changed the numerical results by less than 1 per cent.

20. See Dreschler [1973; 28-29] and Mukherjee and Rao [1973; 37] for English language derivations of their method.

21. The terms democratic and plutocratic are due to Prais [1959]. See also Diewert [1983a; 188-190].

22. In this case, a straightforward computation shows that the country i parity δ_i defined by (22) is numerically equal to a (world) translog price index which compares the world price and quantity vectors $p^1,p^2,\ldots,p^I$ and $y^1,y^2,\ldots,y^I$ respectively to an artificial country i economy which has the country i price vectors replicated I times, i.e., $p^i,p^i,\ldots,p^i$, and the quantity vector $s_1y^i,s_2y^i,\ldots s_Iy^i$ where $s_j \equiv p^j \cdot y^j/\Sigma_{k=1}^I p^k \cdot y^k$ for $j=1,2,\ldots,I$. Thus we have $\delta_i = P_T(p^1,\ldots,p^I;\ p^i,\ldots,p^i;\ y^1,\ldots,y^I;\ s_1y^i,\ldots,s_Iy^i)$.

23. Hill [1984; 131] raises this consistency in aggregation issue.

24. Diewert [1978; 896] also shows that $Q_F(p^j,p^i,y^j,y^i)$ and $\tilde{Q}_T(p^j,p^i,y^j,y^i)$ differentially approximate each other to the second order around any point where the two price vectors are positive and equal and where the two quantity vectors y^i and y^j are positive and equal. The positivity restrictions on the y^i and y^j may be replaced by $y^i=y^j\equiv y$ and $p\cdot y>0$ where $p\equiv p^i=p^j \gg 0_N$.

25. Much of the analysis in this section is adapted from Diewert [1983b; 1064-1075], who assumed that output quantities were always positive. In our present context, we must allow for the possibility that certain outputs can be positive in country i and negative in country j. This seemingly minor change complicates the analysis considerably.

26. The minimum may have to be replaced by an infimum in certain cases. Also, if there is no feasible solution for the minimization problem in (30), define $d^i(y,v)=+\infty$.

27. Malmquist [1953] developed the basic idea in the consumer theory context. It has been used in the producer theory context by Bergson [1961], Moorsteen

[1961], Samuelson and Swamy [1974; 590-591], Hicks [1981; 256], Caves, Christensen and Diewert [1982b; 1399-1401] and Diewert [1983b; 1064-1077].

28. See Diewert [1983b; 1072].

29. The direct translog quantity index, $Q_T(p^j,p^i,y^j,y^i) \equiv P_T(y^j,y^i,p^j,p^i)$ (the role of prices and quantities has been interchanged) cannot be used in the present circumstances because the y^i and y^j vectors will generally have negative components, and so the usual justification (see Caves, Christensen and Diewert [1982b; 14011]) for this index is not valid.

30. Van Ijzeren [1983] and Hill [1984; 131] informally discuss this consistency in aggregation property when they ask whether averages of bilateral indexes should be weighted according to the "importance" of the country.

31. In what follows, when we speak of $Q(p^1,p^2,y^1,y^2)$, we shall always assume that the p^i and y^i satisfy the domain restrictions listed in BT1. When we specify that Q is Q_F, we assume that the p^i and y^i satisfy the domain restrictions listed in BT1'.

32. C_h^i is defined in a manner analogous to the definition of g^i in (18) except that w replaces p and z replaces v.

33. It is possible to use the direct translog quantity index Q_T in place of $\tilde{Q}_T$ in definitions (70) and (72), provided that the individual household consumption vectors x^{ih} are positive in each component (i.e., $x^{ih} \gg 0_N$ for all i and h). Under these circumstances, it can be shown that (see Diewert [1976; 123-124]) Q_T equals a certain Malmquist quantity index, provided that preferences can be represented by a translog distance function. We have not developed this result in detail, since for disaggregated consumption data, it

is highly unlikely that the positivity restrictions $x^{ih} >> 0_N$ would be satisfied.

34. We were able to suggest a practical procedure for forming multilateral consumer price comparisons; recall formula (65) which just involves aggrgate data. However, that technique worked because we were able to asssume that prices were constant over individuals within a country. We cannot assume that consumption vectors are identical over individuals within a country or region without distorting empirical facts considerably.

35. Problems involving the definitions of multilateral indexes are more complicated in the producer context due to the existence of negative quantities.

36. These positivity restrictions can be relaxed somewhat to nonnegativity restrictions.

37. Moreover, since the own share system closely approximates the corresponding EKSCCD system which satisfies MT3, the own share system will be approximately consistent with MT3.

38. Since these indexes satisfy the circularity property to a high degree of approximation, (89) will approximately equal $p^j \cdot y^j / p^1 \cdot y^1 \; Q(p^1, p^j, y^1, y^j)$, which is the "correct" bilateral answer.

39. Incidentally, Khamis [1984; 195] makes some unwarranted attacks on the use of the Fisher index; Propositions 3, 7 and 12 in the present paper present strong economic justifications for its use in the bilateral context.

40. This means that our methods are not strictly applicable to output and input comparisons between socialist countries, since profit maximizing or cost minimizing behavoir cannot be assumed. However, if correct shadow or marginal product prices could be estimated for each quantity, then our methods could be

applied. Our methods could be applied to real consumption comparisons between socialist countries, provided that rationing and queuing problems are not too severe.

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