Rational Bubbles in Non-Linear Business Cycle Models: Closed and Open Economies

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This paper studies rational bubbles in non-linear dynamic general equilibrium models of the macroeconomy. The term 'rational bubbles' refers to multiple equilibria arising from the absence of a transversality condition (TVC) for capital. The lack of TVC can be due to an overlapping generations structure. Rational bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment. In contrast to explosive rational bubbles in linearized models (Blanchard (1979)), the rational bubbles in *non-linear* models here are stable and *bounded*. Bounded bubbles can generate persistent fluctuations of real activity, and capture key business cycle stylized facts. Both closed and open economies are analyzed. In a non-linear two-country model with integrated financial markets, bubbles must be perfectly correlated across countries.

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1. Introduction

This paper studies rational bubbles in non-linear dynamic stochastic general equilibrium (DSGE) model of the macroeconomy. The term 'rational bubbles' refers to multiple equilibria arising from the absence of a transversality condition (TVC) for aggregate capital. The lack of TVC can be due to an overlapping generations structure with finitely-lived households (see below). If a TVC is imposed, all models studied here have a unique solution. I consider models whose aggregate static equations and aggregate Euler equations are identical to those of standard Real Business Cycle (RBC) models with capital accumulation, but I assume that there is no TVC for aggregate capital. Agents have rational expectations. Rational bubbles in the models here reflect self-fulfilling fluctuations in agents' expectations about future investment.

A key finding is that rational bubbles in the non-linear macro models here are bounded. The bounded nature of bubbles, in non-linear models, makes them a novel candidate for explaining business cycles. I construct bounded bubbles that feature recurrent boom-bust cycles characterized by persistent investment and output expansions which are followed by abrupt contractions in real activity. Numerical simulations show that bounded rational bubbles in non-linear macro models can generate persistent fluctuations of real activity, and capture key business cycle stylized facts; the unconditional mean of real activity is close to the no-bubble steady state. Both closed and open economies are analyzed. A central finding for a nonlinear two-country model is that, with integrated financial markets, bounded bubbles must be perfectly correlated across countries. Global bubbles may, thus, help to explain the synchronization of international business cycles.

The boundedness of rational bubbles reflects non-linear effects. Linearized versions of the business cycle models considered here have a unique stable solution that satisfies the TVC. Rational *bubbles* (no-TVC solutions) in the linearized models are explosive, i.e. their *expected* trajectories tend to $\pm\infty$. Explosive rational bubbles were first studied by Blanchard (1979) who considered simple linear asset pricing models without TVC (see also Blanchard and Watson (1982)). Explosive rational bubbles in linear(ized) models are problematic. The accuracy of a linear model approximation can deteriorate sharply when the economy departs substantially from the point of approximation. In a macro model with decreasing returns to capital, explosive trajectories of capital and output are infeasible, as the capital stock cannot grow beyond a maximum level. A linearized model does not take this constraint into consideration, and it may also violate non-negativity constraints on consumption and output. By contrast, the present analysis of bounded rational bubbles in non-linear models takes decreasing returns and boundary conditions into account. The non-linear model solutions presented here remain accurate when the economy deviates significantly from steady state (for several models discussed below, *exact* closed form solutions are provided).

Like Blanchard (1979), I assume a bubble process with two states: the economy can either be in a 'boom' state or in a 'bust' (crash) state. In a boom, capital investment and output diverge positively from the no-bubble decision rule that holds under the TVC (saddle path). High investment during a boom is sustained by agents' belief that, with positive probability, investment will continue to grow next period, thereby depressing future consumption and raising the (expected) future marginal utility-weighted return of capital. In a bust, investment drops abruptly, and thereby reverts towards the no-bubble decision rule. Busts are triggered by self-fulfilling downward revisions of expected future investment. Transitions between booms and busts are prompted by a random sunspot, and occur with an exogenous probability.

As pointed out above, I assume economies without transversality condition (TVC). Standard DSGE models postulate an optimizing infinitely-lived representative household. The set of optimality conditions of an infinitely-lived household's decision problem includes a TVC that stipulates that the value of aggregate capital has to be zero, at infinity. The TVC (in conjunction with static equilibrium conditions and Euler equations) defines a unique equilibrium, in standard DSGE models. I present a novel overlapping generations (OLG) structure with finitely-lived households that has the same aggregate static conditions and the same aggregate Euler equations as standard DSGE models. However, there is no TVC for *aggregate* capital in that OLG structure. The key features of this OLG structure are: (i) complete risk sharing among contemporaneous generations; (ii) newborn agents receive an endowment (from older generations) such that the consumption of newborns represents a time-invariant share of aggregate consumption. This OLG structure allows to generate rational bubbles in tractable non-linear DSGE models suitable for calibration to quarterly data. (Non-linear OLG business cycle models without the two key features mentioned above are typically much more cumbersome, due to the implied heterogeneity of generations, which makes stochastic analysis very difficult.)

The results here are also relevant for research on numerical solution methods for DSGE models. Linearized DSGE models with a unique stable solution are the workhorses of modern quantitative macroeconomics (see, e.g., King and Rebelo (1999), Kollmann et al. (2011a,b) for overviews). This paper presents non-linear DSGE models (without TVC) that have multiple stable solutions, although the linearized versions of those models have a unique stable solution, as the number of eigenvalues (of the linearized state-space form) outside the unit circle *equals* the number of non-predetermined variables (Blanchard and Kahn (1980), Prop. 1). The classic Blanchard and Kahn (1980) condition for the existence and uniqueness of a stable solution for linear rational expectations models is, thus, irrelevant for non-linear models. Standard non-linear numerical solution methods for non-linear DSGE models (see overview in Judd (1998)) do not impose the TVC. Detecting TVC violations can be extremely difficult, in non-linear stochastic economies (those violations can be caused by very low-probability events in a distant future). The results presented here suggest that the set of stable non-linear model solutions, without TVC, can be much larger than hitherto understood.

A large literature has studied linearized DSGE models with multiple stationary sunspot equilibria. These multiple equilibria arise if the number of eigenvalues (of the linearized state-space form) outside the unit circle is less than the number of non-predetermined variables (Blanchard and Kahn (1980), Prop. 3).[1] Linearized models may exhibit stationary sunspot equilibria if increasing returns and/or externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999)), financial frictions (e.g., Martin and Ventura (2018)) or certain OLG structures (e.g., Woodford (1986), Galí (2018)) are assumed. The equilibria studied in these papers satisfy transversality conditions (TVC). The specific features and calibrations that deliver stationary sunspot equilibria in linearized models can be debatable.[2] By contrast, the paper here presents multiple equilibria in *non-linear* DSGE models--without the features that were just mentioned; as discussed above, the *linearized* versions of the models here have a *unique* stable solution.

The notion of a rational bubble introduced by Blanchard (1979) has been highly influential in the literature on asset prices (e.g., see Mussa (1990) and Stracca (2004) for references).[3] However, so far, this notion has had much less impact on structural macroeconomics.

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[1] See Taylor (1977) for an early example of a model with sunspots, due to the presence of 'too many' stable roots.
[2] E.g., increasing returns/externalities need to be sufficiently strong; in OLG models the steady state interest rate has to be smaller than the trend growth rate (r<g) etc. Note that r>g holds, in the novel OLG structure developed in the paper here. Linearized versions of the OLG structure here have a unique stable solution.
[3] Google Scholar records 2615 cites (01/2020) for Blanchard (1979) and its companion paper Blanchard and Watson (1982).

Ascari et al. (2019) study *temporarily* explosive bubbles, in a standard *linearized* three-equation New Keynesian macro model (without capital accumulation). The authors assume bounded rationality, and postulate that, once an explosive path reaches a threshold, the economy reverts permanently to its unique saddle path. Under fully rational expectations, the future switch to the saddle path would, from the outset, rule out the emergence of bubbles.[4] By contrast, the present analysis considers stable (bounded) bubbles in non-linear models. Limited rationality is not needed to generate stable bubble equilibria, in the present framework. The paper here considers economies with capital, but the analysis abstracts from nominal rigidities and monetary policy. In ongoing work, I am exploring rational bounded bubbles in non-linear economies with Keynesian features.

Multiple equilibria due to non-linearities are also studied by Holden (2016a,b) who shows that multiple equilibria can exist when occasionally binding constraints, OBC (such as a zero-lower-bound constraint for the interest rate) are integrated into an otherwise linear DSGE model (the linear model has a unique stable solution when the OBC is ignored). By contrast, the analysis here considers *fully* non-linear models. The multiple equilibria described here have a 'bubbly' dynamics that differs from the dynamics studied by Holden (2016a,b). [5]

The bubble equilibria discussed in this paper imply that the distribution of endogenous variables is heteroscedastic: the conditional variance of forecast errors of future endogenous variables is greater, the longer a boom driven by self-fulfilling expectations has lasted. In this sense, the present paper is related to Bacchetta et al. (2012) who study a stylized asset pricing model in which bounded stock price bubbles can arise if the sunspot shock and the asset price are heteroscedastic. The work here highlights the importance of heteroscedasticity of real activity, for generating bounded bubble equilibria, in non-linear DSGE business cycle models.

Section 2 discusses bounded rational bubbles that arise in the Long and Plosser (1983) RBC model, when the TVC is dropped. That model assumes a closed economy with log utility, a Cobb-Douglas production function and full capital depreciation. Exact closed form solutions with bubbles can be derived for that model. The subsequent Sections show how rational bubble equilibria can be constructed in richer, more realistic non-linear RBC models. Section 3

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[4] In their quantitative model, Ascari et al. (2019) set the threshold (that triggers reversion to the stable saddle path) at a very large value, so that switches to the stable saddle path occur in a distant future. The authors assume that those faraway future switches are disregarded by agents, in the model.

[5] Holden highlights indeterminacy of the length of time during which the OBC binds, and he focuses on fluctuations in the vicinity of the OBC.

considers a non-linear closed economy RBC model with incomplete capital depreciation. Sections 4 and 5 study rational bubbles in non-linear two-country RBC models.

2. Rational bubbles in a Long-Plosser RBC economy without TVC

Following Long and Plosser (1983), this Section considers a closed economy with time-separable preferences. The period utility function is $u(C_t)=\ln(C_t)$, where C_t denotes consumption in period *t*. The production function is:

$$Y_t=\theta_t K_t^{\alpha}\,,\ 0<\alpha<1\,, \tag{1}$$

where Y_t, K_t, θ_t are output, capital and exogenous total factor productivity (TFP). For simplicity, I assume that labor hours are constant and normalized to unity (the next Sections allow for variable hours). The resource constraint is

$$C_t+I_t=Y_t, \tag{2}$$

where I_t is (gross) investment. Investment equals next period's capital stock, $I_t=K_{t+1}$, as the capital depreciation rate is 100%. The Euler equation for capital is $E_t\beta\{U'(C_{t+1})/U'(C_t)\}\partial Y_{t+1}/\partial K_{t+1}=1$, where $0<\beta<1$ is the subjective discount factor. Thus,

$$E_t\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1}=1\,. \tag{3}$$

Substitution of the resource constraint into the Euler equation gives an expectational difference equation in the investment/output ratio $Z_t\equiv K_{t+1}/Y_t$:

$$E_t H(Z_{t+1},Z_t)=1, \ \text{ with } H(Z_{t+1},Z_t)\equiv\alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t\,. \tag{4}$$

Long and Plosser (1983) assume an *infinitely-lived* representative household. The necessary and sufficient optimality conditions of that household's decision problem are the household's resource constraint and Euler equation (summarized by (4)) and a transversality condition (TVC) that requires that the value of the capital stock is zero, at infinity: $\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0$. Note that $u'(C_t)K_{t+1}=K_{t+1}/C_t=Z_t/(1-Z_t)$. It can readily be seen that $Z_t=\alpha\beta\ \forall t$ satisfies (4) and the TVC. This solution corresponds to the textbook solution of the Long-Plosser model (e.g., Blanchard and Fischer (1989)). Under that solution, consumption and investment are time-invariant shares of output: $C_t=(1-\alpha\beta)Y_t,\ K_{t+1}=\alpha\beta Y_t\ \ \forall t$.

In what follows, I postulate that there is no TVC. This gives rise to multiple equilibria. I refer to a process $\{Z_t\}$ that solves (4), but that differs from the textbook solution (derived under the TVC), as a **rational bubble equilibrium**, or (rational) bubble, for short. Thus, rational bubbles feature an investment/output ratio that differs from $\alpha\beta$. Bubbles violate the TVC.[6]

Throughout this paper, the term 'rational bubbles' refers to (multiple) equilibria, due to the absence of a transversality condition (TVC) for aggregate capital. If the TVC is imposed, all models studied in this paper have a unique solution.

The lack of TVC can be justified by the assumption that the economy has an overlapping generations (OLG) population structure with finitely-lived agents. Not-for-Publication Appendix A presents a novel OLG structure with finitely-lived agents that has the *same aggregate* resource constraint and the same *aggregate* Euler equation as a Long-Plosser economy inhabited by an infinitely-lived representative agent. Thus equations (1)-(4) continue to hold in that OLG structure. At the end of her life, each individual agent holds zero assets. As agents have a finite horizon, the (infinite-horizon) TVC for *aggregate* capital is *not* an equilibrium condition, in the OLG structure. Two key features of this OLG structure are: (I) Complete risk sharing among contemporaneous generations. (II) Newborn agents receive an endowment such that the consumption of newborns represents a time-invariant share of aggregate consumption.[7] Assumptions (I) and (II) yield simple non-linear dynamic relations among **aggregate** variables that allow to easily solve for those aggregates. This OLG structure, thus, allows to generate rational bubble equilibria in tractable non-linear DSGE models suitable for calibration to quarterly data. (OLG business cycle models without the two key features mentioned above are typically much more cumbersome, due to the implied heterogeneity of generations, which makes

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[6] The decision problem of the infinitely-lived representative household assumed by Long and Plosser has a unique solution, as that problem is a well-behaved concave programming problem. Thus, $Z_t=\alpha\beta \;\; \forall t$ is the only solution that satisfies (4) and the TVC. Hence, any bubble process $Z_t \neq \alpha\beta$ satisfying (4) violates the TVC. Under the bubble process (9) below, Z_t approaches 1 if a long uninterrupted string of 'boom' sunspots is realized, which entails large positive values of $Z_t/(1-Z_t)$. This only happens with a very small probability, but it causes the TVC to be violated.

[7] The endowment of newborn is financed by transfers from older generations. See Not-for-Publication Appendix A for a discussion of transfer schemes that ensure a time-invariant consumption share of the newborn generation. In reality, all societies make significant transfers to young generations (e.g., through health and education spending as well as bequests). Those transfers are motivated by altruism and social norms (that are not explicitly modeled in the theoretical framework here). Empirically, in advanced countries, the per capital consumption of children and younger consumers (twenties) has closely tracked that of middle aged households (forties), while the consumption of older households has shown slightly faster trend growth, during recent decades. Changes in the *relative* consumption of different age groups are dwarfed by changes in the aggregate consumption *level*. For time series on consumption by age cohorts see, e.g., d'Albis et al. (2019) [data for France] and Saito (2001) [US, UK and Japan].

non-linear stochastic analysis very difficult. Without assumptions (I),(II), *approximate* aggregation across generations may still be possible, based on linearization (e.g., Galí (2018)). The focus of the present paper is on rational bubbles in non-linear models. Thus, aggregation based on linear approximations is not useful here.)

Besides assuming an OLG structure, another motivation for disregarding the TVC is that detecting TVC violations may be very difficult, in non-linear stochastic economies that are more complicated than the Long-Plosser economy, i.e. in models for which no closed form solution exists (see below). TVC violations can be caused by very low-probability events in a distant future. Agents may thus lack the cognitive/computing power to detect deviations from the TVC (see discussion in Blanchard and Watson (1982)), so that bubble equilibrium can arise.

2.1. Rational bubbles in the linearized model

Linearization of (4) around $Z=\alpha\beta$ gives:

$$E_t z_{t+1}=\lambda z_t \text{, with } z_t \equiv Z_t - Z \text{ and } \lambda \equiv 1/(\alpha\beta)>1. \tag{5}$$

λ, the eigenvalue of (5), exceeds unity. The model has one non-predetermined variable (z_t). As the number of eigenvalues greater than one equals the number of non-predetermined variables, the linearized model has a unique non-explosive solution (Blanchard and Kahn (1980), Prop. 1) given by $z_t=0$, i.e. $Z_t=\alpha\beta \; \forall t$, which corresponds to the textbook solution of the Long-Plosser model (with TVC). Blanchard (1979) pointed out that a linear model of form (5) is also solved by a bubble process $\{z_t\}$ such that

$$z_{t+1}=[\lambda/(1-\pi)]\cdot z_t \text{ with probability } 1-\pi \text{ and } z_{t+1}=0 \text{ with probability } \pi \quad (0<\pi<1). \tag{6}$$

If $z_t \neq 0$, then next period the system continues to diverge with probability $1-\pi$, while a 'bust' (return to the no-bubble solution $z=0$) occurs with probability π. Process (6) implies that after a bust, non-zero values of z never arise again, i.e. the bubble is 'self-ending'. Recurrent (never-ending) bubbles obtain if a bust implies a value $\mu \neq 0$: $z_{t+1}=(\lambda z_t - \mu\pi)/(1-\pi)$ with probability 1-π and $z_{t+1}=\mu$ with probability π.

An important feature of rational bubbles in the linearized model (5) is that the expected path of the investment/output ratio explodes: $\lim_{s\to\infty} E_t z_{t+s} = \pm\infty$ when $z_t \neq 0$. This explosiveness greatly limits the appeal of rational bubbles in the linearized model. Note that the investment/ output ratio Z_t is bounded by 0 and 1: an infinite investment ratio is not feasible. The linear approximation (on which (5) is based) neglects this constraint. A linear approximation is thus not suitable for studying rational bubbles.

2.2. Rational bubbles in the <u>non-linear</u> model

By contrast to the linearized model, the non-linear model can produce **bounded** bubbles with $0 \leq Z_t \leq 1$. Note that the non-linear model (4) holds for any process $\{Z_t\}$ such that

$$\alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1+\varepsilon_{t+1}, \tag{7}$$

where ε_{t+1} is an Euler equation forecast error with zero conditional mean: $E_t\varepsilon_{t+1}=0$. ε_{t+1} reflects unanticipated changes in Z_{t+1} that are driven by changes in households' expectations about the future path $\{Z_{t+s}\}_{s>1}$. (7) can be written as:

$$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1+\varepsilon_{t+1}). \tag{8}$$

Z_{t+1} is strictly increasing and strictly concave in both Z_t and in ε_{t+1}, for $\varepsilon_{t+1} > -1$. The strict concavity reflects decreasing returns and the convexity of the marginal utility of consumption (prudence). Figure1 plots Z_{t+1} as a function of Z_t, and that for three values of the Euler equation forecast error ε_{t+1}: $\varepsilon_{t+1}=0$ (thick black curve), $\varepsilon_{t+1}=0.5$ and $\varepsilon_{t+1}=-0.5$ (thin dashed curves). Throughout this paper, I set α=1/3 and β=0.99, so that $\alpha\beta$=0.33. These parameter values are standard in quarterly business cycle models.

2.2.1. Deterministic economy

Consider first a deterministic economy, in which $\varepsilon_{t+1}=0$ holds $\forall t$, so that the investment/output ratio obeys $Z_{t+1}=\Lambda(Z_t,0)$ (see thick black curve labelled '$\varepsilon_{t+1}=0$' in Fig. 1). The graph of $Z_{t+1}=\Lambda(Z_t,0)$ shows combinations of Z_t and Z_{t+1} that are consistent (in a deterministic economy) with the date t Euler equation and with the resource constraints at t and t+1. A rise in Z_t increases investment and lowers consumption at date t, which raises the marginal utility of consumption at t; output at t+1 rises too, while the marginal product of capital at t+1 falls. The household's Euler equation thus implies that the marginal utility of consumption at t+1 has to increase, which calls for a fall in consumption at t+1. Thus, a rise in Z_t has to be followed by a fall in the consumption/output ratio at t+1, and hence by an increase in Z_{t+1} (the investment/ output ratio at t+1). This explains the positive relation between Z_t and Z_{t+1}.

The function $Z_{t+1}=\Lambda(Z_t,0)$ cuts the 45-degree line at $Z=\alpha\beta$ and $Z=1$. The slope of the function is $1/(\alpha\beta)>1$, at the no-bubble solution $Z=\alpha\beta$. In a deterministic economy, a realization $Z_t<\alpha\beta$ puts the investment ratio on a monotone trajectory that reaches Z=0 in finite time; a realization $Z_t>\alpha\beta$ induces a monotone path that asymptotes to Z=1 (without ever reaching Z=1).

2.2.2. Stochastic bubbles

I now show that the Long-Plosser economy without TVC has stochastic bubble equilibria that feature recurrent, bounded fluctuations. These equilibria do not converge to Z=0 or Z=1. Trajectories that lead to Z=0 (zero capital and output: economic 'extinction'), or that converge to Z=1 (zero consumption share) seem empirically irrelevant. Standard DSGE macro analysis focuses on *recurrent* fluctuations in economic activity driven by exogenous stationary shocks to TFP (and other fundamentals). Therefore, this paper concentrates on *recurrent* bubbles, i.e. bubbles that are not self-ending and that do not lead to economic extinction.

When $Z_t<\alpha\beta$, then the law of motion (8) implies that the economy can hit a zero-capital corner solution in subsequent periods (see Fig. 1). Once the zero-capital corner is reached, output, investment and consumption remain at zero forever. Thus, a *recurrent* stochastic bubble must feature an investment/output ratio that always stays in the interval $[\alpha\beta,1)$. The bubble equilibria studied here thus exhibit capital over-accumulation (the investment/output ratio being at least as large as in the textbook no-bubble equilibrium that holds under the TVC). In contrast, (explosive) rational bubbles in the linearized model can be positive or negative.

By analogy to the Blanchard (1979) bubble, I assume that there are two possible states at t+1, with a negative and a positive realization of the Euler equation forecast error ε_{t+1}, respectively. These two states indicate a 'bust' and a 'boom' at t+1. Let ε_{t+1} take these values:

$-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\cdot\pi/(1-\pi)$ with exogenous probabilities π and $1-\pi$, respectively, where $\overline{\varepsilon_t}\in[0,1)$ and $0<\pi<1$. Z_{t+1} then takes these two values with probabilities π and $1-\pi$:

$$Z_{t+1}^L\equiv\Lambda(Z_t,-\overline{\varepsilon_t}) \text{ and } Z_{t+1}^H\equiv\Lambda(Z_t,\overline{\varepsilon_t}\pi/(1-\pi)) \text{ with } Z_{t+1}^L\leq Z_{t+1}^H\leq 1. \tag{9}$$

Recurrent rational bubbles

In the spirit of the recurrent Blanchard-style bubble process in the linearized model (see above), I assume that when an investment bust occurs in period t+1, then agents choose an investment/output ratio Z_{t+1}^L that is close to the no-bubble investment/output ratio $\alpha\beta$: $Z_{t+1}^L=\alpha\beta+\Delta$, where $\Delta>0$ is a small positive constant. $\Delta>0$ is needed to generate *recurrent* bubbles. $\Delta=0$ would imply that bubbles are self-ending, while $\Delta<0$ would entail that the economy will ultimately hit the zero-capital corner (see above).[8]

When $Z_{t+1}^L=\alpha\beta+\Delta$ is assumed, the first equation shown in (9) pins down $-\overline{\varepsilon_t}$ as a function of Z_t; this then determines Z_{t+1}^H. Let $Z_{t+1}^H=\Psi(Z_t)$ denote the (unique) value of Z_{t+1}^H that is associated with Z_t.

Under the assumed bubble process, the date *t* Euler equation (4) can be expressed as

$$\pi H(\alpha\beta+\Delta,Z_t)+(1-\pi)H(Z_{t+1}^H,Z_t)=1. \tag{10}$$

$Z_{t+1}^H=\Psi(Z_t)$ solves this Euler equation. For $Z_t\in[\alpha\beta+\Delta,1]$, the function Ψ has these properties: (i) $\Psi'>0$, $\Psi''<0$; (ii) $Z_t<\Psi(Z_t)\leq 1$; (iii) $\Lambda(Z_t,0)<\Psi(Z_t)$. Thus, Z_{t+1}^H is a strictly increasing and strictly concave function of Z_t. Property (ii) ensures that if $Z_t\in[\alpha\beta+\Delta,1)$, then $Z_\tau\in[\alpha\beta+\Delta,1)$ holds $\forall\,\tau>t$.

Consider an economy that starts in period *t*=0, with an exogenous initial capital stock K_0. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and 1-π, respectively ($0<\pi<1$). Then the following process for the investment/output ratio $\{Z_t\}_{t\geq 0}$ is a

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[8] Assume $\Delta=0$ (so that $Z_{t+1}^L=\alpha\beta$) and consider what happens when $Z_t=\alpha\beta$. The first equation shown in (9) then becomes $\alpha\beta\equiv\Lambda(\alpha\beta,-\overline{\varepsilon_t})$ which implies $\overline{\varepsilon_t}=0$, so that $Z_{t+1}^H=Z_{t+1}^L=\alpha\beta$, i.e. Z is stuck at $\alpha\beta$ in all subsequent periods. Setting $\Delta>0$ rules out that absorbing state.

recurrent rational bubble: $Z_0 \in [\alpha\beta+\Delta,1)$; $Z_{t+1}=Z_{t+1}^L \equiv \alpha\beta+\Delta$ if $u_{t+1}=0$ and $Z_{t+1}=Z_{t+1}^H$ if $u_{t+1}=1$, for $t \geq 0$, where $Z_{t+1}^H=\Psi(Z_t)$ solves the date t Euler equation (10).

The investment/output ratio in the initial period, Z_0, does not obey the recursion that governs the investment ratio in subsequent periods. Z_0 is indeterminate. However, $Z_0 \in [\alpha\beta+\Delta,1)$ has to hold to ensure that investment/output ratios in all subsequent periods are in the interval $[\alpha\beta+\Delta,1)$.[9] Given a sequence $\{Z_t\}_{t\geq 0}$, the path of capital $\{K_{t+1}\}_{t\geq 0}$ can be generated recursively (for the given initial capital stock K_0) using $K_{t+1}=Z_{t+1}\theta_t(K_t)^\alpha$ for $t \geq 0$.

In a deterministic economy, the investment-output ratio would rise steadily and converge to unity, after a value $Z_t > \alpha\beta$ has been realized. In a stochastic bubble equilibrium, an *uninterrupted* infinite sequence of investment booms (u=1) would asymptotically drive the investment/output ratio to unity. Of course, an uninterrupted boom run has zero probability. At any time, the investment output ratio can drop to $\alpha\beta+\Delta$, with probability π. This ensures that the investment/output ratio undergoes recurrent fluctuations. If the bust probability π is sufficiently big and if $\Delta>0$ is close to zero, then bubbles induce fluctuations of real activity that remain most of the time near the steady state of the no-bubble economy. This is the case in the stochastic simulations reported below.

What expectations sustain the rational bubble equilibrium? Agents expect at date t that Z_{t+1} will equal $Z_{t+1}^L=\alpha\beta+\Delta$ or $Z_{t+1}^H=\Psi(Z_t)$ with probabilities π and 1-π, respectively, where the values of Z_{t+1}^L and Z_{t+1}^H are known at t. At t+1, agents are free to select a value of Z_{t+1} that differs from Z_{t+1}^L or Z_{t+1}^H; however, in a bubble equilibrium, they chose not to do so because a choice $Z_{t+1} \in \{Z_{t+1}^L, Z_{t+1}^H\}$ is 'validated' by their date t+1 expectations about Z_{t+2}. Assume that a **bust** occurs in t+1 (u_{t+1}=0), so that agents choose $Z_{t+1}=\alpha\beta+\Delta$; in equilibrium, this choice is sustained by agents' expectation (at t+1) that Z_{t+2} will equal $\alpha\beta+\Delta$ or $\Psi(\alpha\beta+\Delta)$ with probabilities π and 1-π, respectively. By contrast, if a **boom** occurs at t+1 (u_{t+1}=1), then

[9] In the stochastic simulations discussed below, I set $Z_0=\alpha\beta+\Delta$. The effect of Z_0 on subsequent simulated values vanishes fast. Z_0 does not noticeably affect simulated moments over a long simulation run.

agents choose $Z_{t+1}=Z_{t+1}^H \equiv \Psi(Z_t)$; this choice is supported by the expectation (at t+1) that Z_{t+2} will equal $\alpha\beta+\Delta$ or $\Psi(Z_{t+1}^H)=\Psi(\Psi(Z_t))$ with probabilities π and 1-π, respectively. Note that $\Psi(\alpha\beta+\Delta)<\Psi(\Psi(Z_t))$. This shows that, in a boom (at t+1), agents are more optimistic about Z_{t+2} than in a bust (at t+1). As in Blanchard (1979), booms and busts reflect hence self-fulfilling variations in agents' expectations about the future state of the economy. An investment boom [bust] is triggered by a more [less] optimistic assessment of next period's investment/ output ratio.

2.2.3. Quantitative results: bubble equilibrium

I next discuss numerical simulations. To assess whether a rational bubble alone can generate a realistic business cycle, I assume that TFP is constant. The bust probability is set at $\pi=0.5$. I set $\Delta=10^{-6}$, as that value produces standard deviations of real activity (HP filtered) in the empirical range. As indicated above, α=1/3 and β=0.99 is assumed in all simulations.

Panel (1) of Fig. 2 plots $Z_{t+1}^L=\alpha\beta+\Delta$, $Z_{t+1}^H=\Psi(Z_t)$ and the conditional mean $E_t Z_{t+1}=\pi_t Z_{t+1}^L+(1-\pi_t)Z_{t+1}^H$, as functions of Z_t. Also shown in Panel (1) is the value of Z_{t+1} that would obtain in a deterministic economy $(\varepsilon_{t+1}=0): Z_{t+1}=\Lambda(Z_t,0)$. In the stochastic bubble equilibrium, the investment/output ratio grows between t and t+1 $(Z_{t+1}>Z_t)$ when a boom occurs at t+1 $(u_{t+1}=1)$; when there is a bust at t+1 $(u_{t+1}=0)$, the investment rate either remains unchanged at $\alpha\beta+\Delta$ (if $Z_t=\alpha\beta+\Delta$), or it drops to $Z_{t+1}=\alpha\beta+\Delta$ (if $Z_t>\alpha\beta+\Delta$).

Fig. 2 shows that $Z_{t+1}^H=\Psi(Z_t)$ is a steeply increasing function of Z_t. In a bubble equilibrium, a sequence of booms (u=1) generates, thus, a succession of rapid increases in the investment/output ratio; this is followed by an abrupt contraction once a bust (u=0) occurs. By contrast, a sequence of busts keeps the investment ratio at the lower bound $\alpha\beta+\Delta$. [10]

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[10] The variance of the forecast error ε_{t+1} is an increasing function of Z_t, i.e. ε_{t+1} is heteroscedastic. The conditional variance of Z_{t+1} is likewise increasing in Z_t.

The strict concavity of the recursion $Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1})$ with respect to the Euler equation forecast error ε_{t+1} (see (8)) implies that $E_tZ_{t+1}<\Lambda(Z_t,0)$; thus, the conditional mean of the date t+1 investment ratio E_tZ_{t+1} is strictly below the value of Z_{t+1} that would obtain in a deterministic economy $(\Lambda(Z_t,0))$. The unconditional mean of the investment ratio is $E(Z)=0.3333$ which is very close to (but greater than) $\alpha\beta+\Delta$. [11]

Business cycle statistics

Panel (2) of Fig. 2 shows representative simulated paths of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line). The Figure shows that the bubble model generates sudden, but short-lived, expansions in output and investment. During the expansion phase of a bubble, the rapid rise in investment is accompanied by a contraction in consumption.

Table 1 (Row (a)) reports model-generated standard deviations (in %) and cross-correlations of HP filtered logged time series of output (*Y*), consumption (*C*) and investment (*I*); also shown are mean values of these variables and of the investment/output ratio (*Z*). All model-generated business cycle statistics reported in Table 1 (and in subsequent Tables) are based on one simulation run of T=10000 periods. The reported theoretical business cycle statistics are median statistics computed across rolling windows of 200 periods.[12] Mean values (of *Y*,*C*,*I* and *Z*) are computed using the whole simulation run (T periods) and expressed as % deviations of the deterministic steady state (of the no-bubble economy).

To evaluate the model predictions, Table 1 also reports US historical business statistics based on HP filtered quarterly data for the period 1968q1-2017q4 (see Row (b)). The empirical standard deviations of GDP, consumption and investment are 1.47%, 1.19% and 4.96%,

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[11] E_tZ_{t+1} is an increasing and strictly concave function of Z_t: $E_tZ_{t+1}=\zeta(Z_t)$, $\zeta'>0, \zeta''<0$. The graph of E_tZ_{t+1} intersects the 45-degree line at $Z_t=0.599$. The unconditional mean $E(Z)$ is (much) smaller than that point of intersection, due to the strict concavity of ζ ($E(Z)=E(\zeta(Z))<\zeta(E(Z))$).

[12] Rolling 200-periods windows of simulated series are used to compute model-predicted moments, as the historical business cycle statistics shown in Table 1 pertain to a sample of 200 quarters (see below). For each 200-periods window of artificial data, I computed standard deviations and correlation, using logged series (HP filtered in the respective window). Table 1 reports median values, across windows, of these standard deviations and correlations.

respectively. In the data, consumption and investment are strongly procyclical; these variables and GDP are highly serially correlated.

The model-predicted standard deviations of output, consumption and investment are 1.14%, 2.35% and 3.41%, respectively (see Row (a) of Table 1). Thus, the model underpredicts slightly the empirical volatility of output and investment; however, consumption is more volatile in the model than in the data. In the model, consumption and investment are procyclical; output and investment are predicted to be positively serially correlated, while consumption is predicted to be negatively autocorrelated. In the bubble economy, average output and investment are 0.5% and 2.3% higher than in the steady state of the no-bubble economy, while consumption is 0.3% lower. Thus, the unconditional mean of these endogenous variables is close to steady state.

Capital over-accumulation (compared to the no-bubble equilibrium) implies that the bubble economy is 'dynamically inefficient', due to violation of the transversality condition (TVC). Abel et al. (1989) propose an empirical test of dynamic efficiency. Their key insight is that, in a dynamically efficient economy, income generated by capital (i.e. output minus the wage bill) exceeds investment. Abel et al. (1989) show that, in annual US data, this condition is met in all years of their sample (1929-1985). The US historical sample average of the (capital income-investment)/GDP ratio is 13.41%.

In the bubbly Long-Plosser economy, the (capital income – investment)/GDP ratio is positive in 96.4% of all quarters, but the average ratio is slightly negative, -0.12%. Note that, in the no-bubble version of the Long-Plosser economy, the (capital income – investment)/GDP ratio equals $\alpha(1-\beta)=0.33\%$, which is only slightly greater than zero, and much smaller than the empirical ratio. Thus, even modest dynamic inefficiency produces a negative mean capital income – investment gap. As shown below, RBC models with incomplete capital depreciation can generate bubble equilibria with sizable positive mean capital income – investment gaps.

3. Rational bubbles in an RBC model with incomplete capital depreciation (no TVC)

I next show how rational bubble equilibria can be constructed in a richer, more realistic non-linear RBC model with incomplete capital depreciation and variable labor.

As before, I postulate that there is no TVC for capital. The period utility function is $U(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t),\ \Psi>0,$ where $0\leq L_t\leq 1$ are hours worked. The household's total time endowment (per period) is normalized to one, so $1-L_t$ is leisure.[13] The resource constraint and the output technology are

$$C_t+K_{t+1}=Y_t+(1-\delta)K_t \text{ with } Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{11}$$

where $0<\delta<1$ is the capital depreciation rate. θ_t (TFP) is exogenous and follows the bounded AR(1) process $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon_{t+1}^{\theta}$, $0\leq\rho<1$, where $\varepsilon_{t+1}^{\theta}$ is a white noise that equals $-\sigma_{\theta}$ or σ_{θ} with probability ½ $(\sigma_{\theta}\geq 0)$. The standard deviation of the $\varepsilon_{t+1}^{\theta}$ is thus σ_{θ}.[14] The economy has these efficiency conditions

$$C_t\Psi/(1-L_t)=(1-\alpha)\theta_t(K_t)^{\alpha}(L_t)^{-\alpha} \text{ and} \tag{12}$$

$$E_t\beta\{C_t/C_{t+1}\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha}+1-\delta)=1. \tag{13}$$

(12) indicates that the household's marginal rate of substitution between leisure and consumption is equated to the marginal product of labor, while (13) is the date *t* Euler equation for capital.

(11) and (12) pin down consumption and hours worked as functions of K_{t+1},K_t,θ_t:

$$C_t=\gamma(K_{t+1},K_t,\theta_t) \text{ and } L_t=\eta(K_{t+1},K_t,\theta_t). \tag{14}$$

Substituting these expressions into the Euler equation gives:

$$E_tH(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \text{ where} \tag{15}$$

$$H(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)\equiv\beta\{\gamma(K_{t+1},K_t,\theta_t)/\gamma(K_{t+2},K_{t+1},\theta_{t+1})\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(\eta(K_{t+2},K_{t+1},\theta_{t+1}))^{1-\alpha}+1-\delta).$$

The model thus boils down to an expectational difference equation in capital. Given a process for capital that solves (15), one can use (14) to determine consumption, hours and output. The conventional no-bubble model solution (that obtains when the TVC for capital is imposed) is described by a unique decision rule $K_{t+1}=\lambda(K_t,\theta_t)$ (e.g., Schmitt-Grohé and Uribe (2004)). I assume that there is no TVC. A rational bubble equilibrium is a process $\{K_t\}$ that satisfies (15) but that deviates from the no-bubble decision rule (and violates the TVC). Throughout the following analysis, I focus on *recurrent* rational bubbles, i.e. on rational bubbles that are not self-ending and that do not lead to zero capital.

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[13]The upper bound on labor hours implies that capital and output are bounded. Some widely used preference specifications (e.g., $U(C_t,L_t)=\ln(C_t)-\Psi\cdot(L_t)^{\mu}$, $L_t\geq 0$, $\mu>1$) do not impose an upper bound on labor. Then rational bubbles may induce unbounded growth of hours, capital and output.

[14] The discrete distribution of the TFP innovation ensures that the TFP process is bounded, and it simplifies the computation of conditional expectations (Euler equations) in the numerical model solution.

Recurrent rational bubbles

By analogy to the bubble process in the Long-Plosser economy without TVC (Sect. 2), I consider bubble equilibria in which the capital stock K_{t+1} takes one of two values: $K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}$ with exogenous probabilities π and $1-\pi$, respectively ($0<\pi<1$), where $K_{t+1}^L=\lambda(K_t,\theta_t)e^{\Delta}$, for a small constant Δ. With probability π, the capital stock thus takes a value close to the no-bubble decision rule (as in the bubbly Long-Plosser model). An exogenous i.i.d. sunspot (independent of TFP) determines whether K_{t+1}^L or K_{t+1}^H is realized (see below). At date *t*, agents anticipate that K_{t+2}, too takes one of two values $K_{t+2}\in\{K_{t+2}^L,K_{t+2}^H\}$ with probabilities π and $1-\pi$, respectively, with $K_{t+2}^L=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$. The date *t* Euler equation (15) can thus be written as:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t) + (1-\pi)\cdot E_t H(K_{t+2}^H,K_{t+1},K_t,\theta_{t+1},\theta_t)=1 \text{ for } K_{t+1}\in\{K_{t+1}^L,K_{t+1}^H\}. \quad (16)$$

Throughout the following analysis, I set $\Delta>0$, as $\Delta>0$ is needed to generate *recurrent* bubbles. As in the Long-Plosser economy (without TVC), bubbles are self-ending or ultimately hit the zero capital corner, when $\Delta\leq 0$.

Consider an economy that starts in period t=0, with an exogenous initial capital stock K_0. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and $1-\pi$, respectively ($0<\pi<1$). u_t is independent of TFP. Then the following process for capital $\{K_t\}_{t\geq 0}$ is a recurrent rational bubble: $K_{t+2}=K_{t+2}^L\equiv\lambda(K_{t+1},\theta_t)e^{\Delta}$ if $u_{t+1}=0$ and $K_{t+2}=K_{t+2}^H$ if $u_{t+1}=1$, for $t\geq 0$, where K_{t+2}^H satisfies the date t Euler equation (16).[15]

As in the bubbly Long-Plosser economy (without TVC), the dynamics of capital reflects self-fulfilling variations in agents' expectations about *future* capital. Due to decreasing returns to capital and bounded TFP, the paths of capital and output are bounded. An *uninterrupted* sequence of investment booms (driven by an infinite string of u=1 sunspot realizations) would drive the capital towards its upper bound. However, an uninterrupted boom has zero probability.

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[15] Note that K_{t+2}^L depends on θ_{t+1}. The numerical simulations below consider bubbles in which, conditional on date t information, a TFP innovation at t+1 has an equiproportional effect on K_{t+2}^L and on K_{t+2}^H. Specifically, $K_{t+2}^H=s_t^H\cdot K_{t+2}^L$, where $s_t^H>0$ is in the date t information set. This greatly simplifies the computation of bubbles. Solving for s_t^H (at each date) pins down the equilibrium capital process. See Not-for-Publication Appendix B

At any time, the capital stock can revert towards the no-bubble decision rule, with probability π. For small values of Δ and a sufficiently high bust probability π (as assumed in the simulations discussed below), capital and output remain close to the range of the no-bubble equilibrium, most of the time, and the unconditional mean of endogenous variables is close to the no-bubble steady state.

Not-for-Publication Appendix B provides further discussions of the bubble model and describes the numerical method used to solve it.

3.1. Quantitative results

I again set α=1/3, β=0.99. The capital depreciation rate is set at δ=0.025. The preference parameter Ψ (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at steady state.[16] Parameters in this range are conventional in quarterly macro models (e.g., King and Rebelo (1999)). I set the autocorrelation of TFP at ρ=0.979, while the standard deviation of TFP innovations is set at σ_θ=0.72%, as suggested by King and Rebelo (1999). All numerical simulations discussed below assume $\Delta=10^{-6}$. That value generates standard deviations of real activity that are roughly in line with empirical statistics. I report results for two values of the bust probability: π=0.2 and π=0.5.

Table 2 reports simulated business cycle statistics (of HP filtered variables) for several model variants (see Cols. (1)-(10)), as well as historical US business cycle statistics (Col. (11)). Standard deviations (in %) of output (Y), consumption (C), investment (I) and hours worked (L) are shown, as well as correlations of these variables with output, autocorrelations and mean values. The Table also reports the mean of the (capital income-investment)/GDP ratio, as well as the fraction of periods in which this ratio is positive.

Cols. (1)-(4) of Table 2 pertain to bubble model variants with just bubble (sunspot) shocks (constant TFP assumed). Cols. (5)-(8) consider bubble model variants with joint bubble and TFP shocks. Cols. (9)-(10) assume a no-bubble model (TVC imposed) with TFP shocks.[17] Cols. (1),(3),(5),(7) assume a bust probability π=0.5, while Cols. (2),(4),(6),(8) assume π=0.2.

[16] (12) implies that the Frisch labor supply elasticity (LSE) with respect to the real wage (marginal product of labor) is $LSE=(1-L)/L$ at the steady state, where L are steady state hours worked. Ψ is set such that $L=0.5$, as then LSE=1.

[17] The no-bubble model is solved using a second-order Taylor approximation, as it is well-know that this approximation is very accurate for standard (no-bubble) RBC models (e.g., Kollmann et al. (2011a,b)).

Cols. labelled 'Unit Risk Aversion' ('Unit RA') assume log utility, $U(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t)$. Columns labelled 'High RA' assume greater risk aversion: $U(C_t,L_t)=\ln(C_t-\overline{C})+\Psi\cdot\ln(1-L_t)$, where $\overline{C}$ is a constant that is set at 0.8 times steady state consumption. The 'High RA' preferences imply that consumption has a strictly positive lower bound: $C_t\geq\overline{C}>0$; in the 'High RA' case, the coefficient of relative risk aversion is 5, at steady state consumption.

Fig. 3 shows simulated paths of output (*Y*, continuous black line), consumption (*C*, red dashed line), investment (*I*, dark blue dash-dotted line) and hours worked (*L*, light blue dotted line). Panel (i) (for i=1,..,10) of Fig. 3 assumes the model variant considered in Col. (i) of Table 2. The *Y*, *C* and *I* series plotted in Fig. 3 are normalized by steady state output (of the no-bubble economy); hours worked (*L*) are normalized by steady state hours. The same sequence of sunspots is fed into each of the bubble model variants with the same bust probability; also, the same sequence of TFP innovations is fed into each model variant with TFP shocks.

Col. (1) of Table 2 assumes a variant of the bubble model with unit risk aversion and a bust probability π=0.5; fluctuations are just driven by bubble shocks (constant TFP assumed). The predicted standard deviations of output, consumption, investment and hours worked are 0.49%, 1.08%, 4.29% and 0.74%, respectively. In line with the historical data, investment is predicted to be more volatile than output. However, the model (with unit risk aversion) predicts that consumption is more volatile than output, which is counterfactual. The model also predicts that consumption is negatively correlated with output (a positive bubble shock raises investment; this crowds out consumption, which raises labor supply and thereby boosts output).[18] However, the model predicts that investment and hours worked are strongly procyclical, as is consistent with the data. In the model, output, consumption, investment and hours worked are positively serially correlated, but predicted autocorrelations (about 0.35) are smaller than the empirical autocorrelations (about 0.9).

Panel (1) of Fig. 3 shows simulated paths driven just by bubble shocks, for the bubble model with unit risk aversion and π=0.5. We see that the bubble equilibrium generates booms in output, labor hours and investment that are relatively infrequent and brief. This explains the low predicted autocorrelation of real activity. In most periods, output, consumption, investment and output remain close to (slightly above) the steady state levels of the no-bubble economy.

[18] This is a familiar feature of flex-wage models driven by investment shocks; e.g., Coeurdacier et al. (2011).

A lower bust probability π=0.2 generates more persistent booms in real activity. For an economy with just bubble shocks, this is illustrated in Col. (2) of Table 2, where a unit risk aversion and π=0.2 are assumed (see also Panel (2) of Fig. 3). The autocorrelation of real activity is now about 0.6. Consumption is again predicted to be more volatile than output.

Model variants with 'High Risk Aversion (RA)' utility generate less consumption volatility—those variants capture the fact that consumption is less volatile than output; see Cols. (3) and (4) of Table 2 (and Panels (3) and (4) of Fig. 3), where π=0.5 and π=0.2 are assumed.

In summary, the bubble model with constant TFP can generate persistent fluctuations, as well as a realistic volatility of output and aggregate demand components.

The no-bubble model driven by stochastic TFP shocks underpredicts the volatility of real activity, but it captures the fact that consumption is less volatile than output, while investment is more volatile (see Table 2, Cols. (9) and (10)). In the no-bubble model, consumption and investment are pro-cyclical; furthermore, real activity is highly serially correlated

The bubble economy with joint bubble shocks and TFP shocks generates fluctuations in real activity that are more volatile than the fluctuations exhibited by the no-bubble economy (see Table 2, Cols. (5)-(8)). In this sense, the bubble equilibrium with TFP shocks is closer to the historical business cycle moments.

Panels (5)-(10) of Fig. 3 show that the effect of bubbles on simulated series is clearly noticeable (compared to the no-bubble economy with TFP shocks): the bubble economy exhibits more rapid, short-lived, increases in investment, labor hours and output.

In the bubble economies considered here, the unconditional mean of endogenous variables is again close to the no-bubble steady state (as in the Long-Plosser economy with bubbles studied in Sect. 2).[19] For all variants of the bubble economy with incomplete capital depreciation considered in Table 2, the average (capital income – investment)/GDP ratio is positive and large (unlike in the Long-Plosser model); the average ratio ranges between 8.5% and 9.2%, and it is only slightly smaller than the value of that ratio in the no-bubble steady state, 9.59%.[20] Capital income exceeds investment in close to 100% of all periods. This highlights the difficulty of detecting violations of the TVC (dynamic inefficiency), as discussed above.

[19]In Tables 2-4, mean values of Y,C,I,L are reported as % deviations from the no-bubble steady state. The mean (capital income – investment)/GDP ratio (see below) is *not* expressed as a % deviation from steady state.

[20]In the bubble economy, the steady state (capital income – investment)/GDP ratio is $\alpha r/(\delta+r)$ where $r=(1-\beta)/\beta$ is the steady state interest rate.

4. Rational bubbles in a Dellas two-country RBC economy (no TVC)

I next study bubbles in open economies. This Section considers Dellas' (1986) two-country RBC model. The Dellas model is a two-country version of the Long and Plosser (1983) model, as it also assumes log utility, Cobb-Douglas production functions and full capital depreciation. Like the Long-Plosser model, the Dellas model has an exact closed form solution. I construct rational bubbles that arise when there is no transversality condition (TVC), in the Dellas economy.

Assume a world with two symmetric countries, referred to as Home (H) and Foreign (F), respectively. The household of country $i{=}H{,}F$ has log preferences of the type assumed in the closed economy RBC model of Sect. 3. Thus, her period utility is: $U(C_{i,t},L_t)=\ln(C_{i,t})+\Psi\cdot\ln(1-L_{i,t})$, $\Psi>0$, where $C_{i,t}$ and $L_{i,t}$ are consumption and hours worked. Each country is specialized in the production of a distinct tradable intermediate good. Country i's intermediate good production function is $Y_{i,t}=\theta_{i,t}(K_{i,t})^{\alpha}(L_{i,t})^{1-\alpha}$, where $Y_{i,t}$, $\theta_{i,t}$, $K_{i,t}$ are the intermediate good output, TFP and capital in country i. Capital and labor are immobile internationally. TFP is exogenous and follows a bounded Markov process. The country i household combines local and imported intermediates into a non-tradable final good, using the Cobb-Douglas aggregator $Z_{i,t}=(y^i_{i,t}/\xi)^{\xi}\cdot(y^j_{i,t}/(1-\xi))^{1-\xi}$, $i{\neq}j$, where $y^j_{i,t}$ is the amount of intermediate j used by country i. There is local bias in final good production: $\frac{1}{2}<\xi<1$. The country i final good is used for consumption, $C_{i,t}$, and investment, $I_{i,t}$: $Z_{i,t}=C_{i,t}+I_{i,t}$. Due to full capital depreciation, the capital stock at t+1 equals investment at t: $K_{i,t+1}=I_{i,t}$. The price of country i's final good $(P_{i,t})$ equals its marginal cost: $P_{i,t}=(p_{i,t})^{\xi}\cdot(p_{j,t})^{1-\xi}$, $i{\neq}j$, where $p_{j,t}$ is the price of intermediate good j. Country i's demand functions for domestic and imported intermediates are: $y^i_{i,t}=\xi\cdot(p_{i,t}/P_{i,t})^{-1}Z_{i,t}$ and $y^j_{i,t}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-1}Z_{i,t}$, for $j{\neq}i$. Market clearing for intermediate goods requires

$$y^i_{H,t}+y^i_{F,t}=Y_{i,t},\ \text{for } i{=}H{,}F. \tag{17}$$

Country i's terms of trade and real exchange rate are $q_{i,t}\equiv p_{i,t}/p_{j,t}$ and $RER_{i,t}\equiv P_{i,t}/P_{j,t}$, with $i{\neq}j$.

The model assumes complete international financial markets, so that consumption risk is efficiently shared across countries. In equilibrium, the ratio of Home to Foreign households' marginal utilities of consumption is, thus, proportional to the Home real exchange rate

(Kollmann, 1991, 1995; Backus and Smith, 1993). With log utility, this implies that Home consumption spending is proportional to Foreign consumption spending: $P_{H,t}C_{H,t}=\Lambda\cdot P_{F,t}C_{F,t}$, where Λ is a date- and state-invariant term that reflects the (relative) initial wealth of the two countries. I assume that the two countries have the same initial wealth, i.e. $\Lambda=1$. Thus:

$$P_{H,t}C_{H,t}=P_{F,t}C_{F,t}. \tag{18}$$

Each household equates the marginal rate of substitution between leisure and consumption to the marginal product of labor, expressed in units of consumption, which implies

$$C_{i,t}\Psi/(1-L_{i,t})=(p_{i,t}/P_{i,t})(1-\alpha)(Y_{i,t}/L_{i,t}). \tag{19}$$

Country i's Euler equation for domestic physical capital is:

$$E_t\beta(C_{i,t}/C_{i,t+1})[(p_{i,t+1}/P_{i,t+1})\alpha Y_{i,t+1}/K_{i,t+1}]=1, \tag{20}$$

where the term in square brackets is country i's marginal product of capital at date t+1, expressed in units of the country i final good. Substitution of the intermediate good demand functions into the market clearing condition for intermediates (17) gives:

$$p_{i,t}Y_{i,t}=\xi\cdot(P_{i,t}C_{i,t}+P_{i,t}K_{i,t+1})+(1-\xi)\cdot(P_{j,t}C_{j,t}+P_{j,t}K_{j,t+1}) \text{ for } i,j=H,F;\ j\neq i. \tag{21}$$

Let $\kappa_{i,t}\equiv P_{i,t}K_{i,t+1}/(P_{i,t}C_{i,t})$ denote country i's investment/consumption ratio. Using (18),(21), the labor supply and Euler equations (19),(20) can be written as

$$L_{i,t}/(1-L_{i,t})=((1-\alpha)/\Psi)\cdot\{1+\xi\kappa_{i,t}+(1-\xi)\kappa_{j,t}\} \text{ for } i=H,F;\ j\neq i, \tag{22}$$

$$\alpha\beta\cdot E_t(1+\xi\kappa_{H,t+1}+(1-\xi)\kappa_{F,t+1})=\kappa_{H,t} \text{ and } \alpha\beta\cdot E_t(1+(1-\xi)\kappa_{H,t+1}+\xi\kappa_{F,t+1})=\kappa_{F,t}. \tag{23}$$

The deterministic steady state investment/consumption ratio is $\kappa\equiv\alpha\beta/(1-\alpha\beta)$. Let $\widetilde{\kappa_{i,t}}\equiv\kappa_{i,t}-\kappa$ denote the deviation of $\kappa_{i,t}$ from its steady state value. The Euler equations (23) imply:

$$\alpha\beta\cdot E_t(\xi\widetilde{\kappa_{H,t+1}}+(1-\xi)\widetilde{\kappa_{F,t+1}})=\widetilde{\kappa_{H,t}} \text{ and } \alpha\beta\cdot E_t((1-\xi)\widetilde{\kappa_{H,t+1}}+\xi\widetilde{\kappa_{F,t+1}})=\widetilde{\kappa_{F,t}}. \tag{24}$$

This gives $$\begin{bmatrix} E_t\widetilde{\kappa_{H,t+1}} \\ E_t\widetilde{\kappa_{F,t+1}} \end{bmatrix}=B\cdot\begin{bmatrix} \widetilde{\kappa_{H,t}} \\ \widetilde{\kappa_{F,t}} \end{bmatrix}, \text{ with } B\equiv\frac{1}{\alpha\beta(2\xi-1)}\cdot\begin{bmatrix} \xi & -(1-\xi) \\ -(1-\xi) & \xi \end{bmatrix}. \tag{25}$$

The eigenvalues of B are $\lambda_S\equiv 1/(\alpha\beta)$ and $\lambda_D\equiv 1/(\alpha\beta(2\xi-1))$, with $\lambda_D>\lambda_S>1$. ($1/(2\xi-1)>1$ as $\frac{1}{2}<\xi<1$.) As both eigenvalues exceed 1, the only non-explosive solution of (25) is $\widetilde{\kappa_{i,t}}=0$ i.e. $\kappa_{i,t}=\alpha\beta/(1-\alpha\beta)$,

$\forall t$, i=H,F. This solution satisfies Home and Foreign TVCs. Dellas (1986) focuses on the no-bubble solution.

4.1. Rational bubbles

I now study rational bubble equilibria with $\widetilde{\kappa_{i,t}}\neq 0$ that arise when there is no TVC. I show that the Dellas economy without TVC has bubble equilibria that feature recurrent, bounded fluctuations of capital, hours worked, output and consumption. These equilibria do not converge to zero capital or zero consumption. If the Home or Foreign capital stock ever fell to zero, then capital and output in both countries would remain stuck at zero in all subsequent periods. Such trajectories seem empirically irrelevant. The goal of the analysis here is to construct bubble equilibria with recurrent fluctuations in real activity, and thus I focus on bubbles with strictly positive capital.

As shown below, any strictly positive process for Home and Foreign capital that satisfies the Euler equations (23),(24) has to be such that

$$\widetilde{\kappa_{H,t}}=\widetilde{\kappa_{F,t}}\geq 0 \ \ \forall t. \tag{26}$$

Thus, the bubbly investment/consumption ratio has to be always at least as large as the steady state ratio. Also, the bubble process has to be **identical** across the two countries. To see this, let $S_t\equiv\widetilde{\kappa_{H,t}}+\widetilde{\kappa_{F,t}}$ and $D_t\equiv\widetilde{\kappa_{H,t}}-\widetilde{\kappa_{F,t}}$ be the sum and the difference of the two countries' investment/consumption ratios, expressed as deviations from steady state. (25) implies $E_tS_{t+1}=\lambda_S\cdot S_t$ and $E_tD_{t+1}=\lambda_D\cdot D_t$, where λ_S and λ_D are the eigenvalues of B. Note that $\widetilde{\kappa_{H,t}}=\tfrac{1}{2}\cdot(D_t+S_t)$ and $\widetilde{\kappa_{F,t}}=\tfrac{1}{2}\cdot(S_t-D_t)$. Thus, $E_t\widetilde{\kappa_{H,t+s}}=\tfrac{1}{2}\cdot(\lambda_S)^s\{S_t+(1/(2\xi-1))^sD_t\}$ and $E_t\widetilde{\kappa_{F,t+s}}=\tfrac{1}{2}(\lambda_S)^s\{S_t-(1/(2\xi-1))^sD_t\}$, where I use the fact that $\lambda_D=\lambda_S/(2\xi-1)$. As $\lambda_S>1$ and $1/(2\xi-1)>1$, a necessary condition for non-negativity of $\kappa_{H,\tau},\kappa_{F,\tau}$ in all future dates and states $\tau\geq t$ is $D_t=0$ and $S_t\geq 0$. This implies (26).[21]

Intuitively, a (positive) bubble that e.g. occurs solely in the Home country ($\widetilde{\kappa_{H,t}}>0$) would trigger an improvement in the Home terms of trade, and a rise in the Home trade deficit,

[21] $D_t\neq 0$ would imply $\lim_{s\to\infty}E_t\widetilde{\kappa_{H,t+s}}=-\infty$ or $\lim_{s\to\infty}E_t\widetilde{\kappa_{F,t+s}}=-\infty$; with strictly positive probability, $\kappa_{H,\tau}$ or $\kappa_{F,\tau}$ would thus be **negative** at some date(s) $\tau\geq$t. Setting $D_t=0$ shows that $S_t<0$ would imply $\lim_{s\to\infty}E_t\widetilde{\kappa_{H,t+s}}=-\infty$ and $\lim_{s\to\infty}E_t\widetilde{\kappa_{F,t+s}}=-\infty$, so that $\kappa_{H,\tau}<0$ and/or $\kappa_{F,\tau}<0$ would hold with positive probability at some date(s) $\tau\geq$t.

due to growing intermediate imports by Home, fueled by the bubble-induced boom in Home investment. This would put Foreign investment on a downward trajectory. If the Home bubble lasted sufficiently long, the Foreign capital stock would ultimately reach zero. Thus, a recurrent bubble (with strictly positive capital) cannot occur just in one country.[22] Why do bubbles have to be *identical* in the two countries? The reason is that any difference between domestic and foreign investment/consumption ratios at date t $(D_t \neq 0)$ would trigger a larger expected difference in period t+1; thus, the expected cross-country difference would explode, and that at a faster rate than the sum of these two-country's investment/consumption ratios (as $\lambda_D > \lambda_S$). This would drive capital to zero, in one of the countries, in future periods $\tau > t$.

In what follows, I thus assume that (26) holds. Let $\kappa_t = \kappa_{H,t} = \kappa_{F,t}$ denote the common investment/consumption ratio in both countries, and let $\widetilde{\kappa_t} \equiv \kappa_t - \kappa$ be its deviation from the steady state ratio κ. The Home and Foreign Euler equations (24) imply

$$\alpha\beta E_t \widetilde{\kappa_{t+1}} = \widetilde{\kappa_t}. \qquad (27)$$

Recurrent rational bubbles

By analogy to the bubble equilibria discussed in previous Sections, I assume that $\widetilde{\kappa_{t+1}}$ takes two values: $\widetilde{\kappa_{t+1}} \in \{\Delta, \widetilde{\kappa_{t+1}^H}\}$ with exogenous probabilities π and $1-\pi$, respectively, with $0<\pi<1$ and $\Delta>0$. $\Delta>0$ ensures that the bubble is recurrent (not self-ending) and that it does not lead to zero capital. (As in the bubbly Long-Plosser model, $\Delta=0$ would imply that bubbles are self-ending; with $\Delta<0$, the capital stock would ultimately fall to zero.)

Consider a world economy that starts in period $t=0$, with exogenous initial capital stocks $K_{H,0}, K_{F,0}$. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and 1-π, respectively ($0<\pi<1$). Then the following process for the investment/ consumption ratio $\{\widetilde{\kappa_t}\}_{t\geq 0}$ is a recurrent rational bubble: $\widetilde{\kappa_{t+1}} = \Delta$ if $u_{t+1}=0$ and $\kappa_{t+1} = \widetilde{\kappa_{t+1}^H}$ if $u_{t+1}=1$, for $t\geq 0$,

[22]Note from the Foreign Euler condition shown in (24) (see second equation) that if $\widetilde{\kappa_{H,t}} > 0$ and $E_t \widetilde{\kappa_{H,t+1}} > 0$ hold, then $\widetilde{\kappa_{F,t}} = E_t \widetilde{\kappa_{F,t+1}} = 0$ is impossible. Thus a bubble cannot occur just in country H.

where $\widetilde{\kappa_{t+1}^{H}}$ solves the date t Euler equation (27). Note that (27) implies $\alpha\beta\{\pi\Delta+(1-\pi)\widetilde{\kappa_{t+1}^{H}}\}=\widetilde{\kappa_t}$, and so $\widetilde{\kappa_{t+1}^{H}}=(\widetilde{\kappa_t}-\alpha\beta\pi\Delta)/(\alpha\beta(1-\pi))$. If $\widetilde{\kappa_t}\geq\Delta$ holds, then $\widetilde{\kappa_{t+1}^{H}}>\widetilde{\kappa_t}$. [23]

Given $\{\kappa_t\}$, one can solve for hours, consumption, investment and output, using the static equilibrium conditions. $\kappa_t=\kappa_{H,t}=\kappa_{F,t}$ implies that labor hours are identical across countries (see (22)), and that investment and output, valued at market prices, are equated across countries: $P_{H,t}K_{H,t+1}=P_{F,t}K_{F,t+1}$, $p_{H,t}Y_{H,t}=p_{F,t}Y_{F,t}$. As consumption, valued at market prices, is likewise equated across countries (see (18)), net exports are zero. Country i's terms of trade equal the inverse of i's relative output: $q_{i,t}\equiv p_{i,t}/p_{j,t}=Y_{j,t}/Y_{i,t}$, $j\neq i$. Consumption and investment obey $C_{i,t}=(1/(1+\kappa_t))(p_{i,t}/P_{i,t})Y_{i,t}$ and $K_{i,t+1}=\kappa_t C_{i,t}$. As $p_{i,t}/P_{i,t}=(q_{i,t})^{1-\xi}=(Y_{j,t}/Y_{i,t})^{1-\xi}$ with $j\neq i$, we find: $K_{H,t+1}=(\kappa_t/(1+\kappa_t))(Y_{H,t})^{\xi}(Y_{F,t})^{1-\xi}$, $K_{F,t+1}=(\kappa_t/(1+\kappa_t))(Y_{H,t})^{1-\xi}(Y_{F,t})^{\xi}$. Note that the $\{\kappa_t\}$ process is unbounded. However $1/(1+\kappa_t)$ and $\kappa_t/(1+\kappa_t)$ are strictly positive and bounded; it can be seen (from preceding formulae) that this implies that capital, output and consumption are bounded.

4.2. Quantitative results

Table 3 reports simulated business statistics for the two-country Dellas model with bubbles (Cols. (1)-(3)); also shown are historical business statistics (Col. (4)). Historical standard deviations, correlations with GDP and autocorrelations are based on US data, 1968q1-2017q4; historical cross-country correlations are correlations between the US and the Euro Area, 1970q1-2017q4 Empirically, the US real exchange rate is about 2.5 times as volatile as US output; US net exports (normalized by GDP) are countercyclical. Real activity is positively correlated across the US and the Euro Area. The cross-country correlations of output and investment are close to 0.5; the cross-country correlations of consumption and employment are slightly lower (0.39).

I again set $\alpha=1/3$, $\beta=0.99$. The share of spending devoted to domestic intermediates is set at $\xi=0.9$.[24] I set the bust probability at $\pi=0.5$. Δ is set at 2.227×10^{-6}, as this parallels the

[23] The κ_0 ratio (initial period) is indeterminate. $\widetilde{\kappa_0}\geq\Delta$ has to hold to ensure that $\widetilde{\kappa_t}\geq\Delta$ $\forall t>0$.

[24] This is consistent with the fact that the mean US trade share (0.5×(imports+exports)/GDP) was 10% in 1968-2017.

calibration of the investment bust in the bubbly Long-Plosser closed economy model (Sect. 2), and generates a realistic volatility of output.[25]

Versions of the two-country model with TFP shocks assume that Home and Foreign TFP follow the autoregressive process that Backus et al. (1994) estimated using quarterly TFP series for the US and an aggregate of European economies:

$$\begin{bmatrix} \ln\theta_{H,t+1} \\ \ln\theta_{F,t+1} \end{bmatrix} = \begin{bmatrix} .906 & .088 \\ .088 & .906 \end{bmatrix} \cdot \begin{bmatrix} \ln\theta_{H,t} \\ \ln\theta_{F,t} \end{bmatrix} + \begin{bmatrix} \varepsilon^{\theta}_{H,t+1} \\ \varepsilon^{\theta}_{F,t+1} \end{bmatrix}, \tag{28}$$

where $\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}$ are white noises with $Std(\varepsilon^{\theta}_{H,t+1})=Std(\varepsilon^{\theta}_{F,t+1})=0.852\%$, $Corr(\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1})=0.258$. I assume a discrete distribution of the TFP innovations, to ensure that the TFP process is bounded.[26] (28) implies that TFP is a highly persistent process, and that there are delayed positive cross-country spillovers (positive off-diagonal elements of the autoregressive matrix).

Col. 1 of Table 3 considers a version of the bubble model with just bubble shocks (constant TFP assumed). Col. 2 assumes a bubble model with joint bubble and TFP shocks, while Col. 3 assumes a no-bubble model (TVC imposed) with TFP shocks.

The bubble model with constant TFP predicts that output, consumption, investment and hours are identical across countries, i.e. these variables are perfectly correlated across countries (see Col. 1). The dynamics of these variables corresponds, thus, to that predicted by the corresponding bubbly Long-Plosser closed economy (see Sect. 2). E.g., like its closed-economy counterpart, the Dellas economy with bubbles predicts that consumption is more volatile than output.[27] Because of the predicted perfect correlation of Home and Foreign output, the terms of trade and the real exchange rate are constant, when there are just bubble shocks.

The no-bubble Dellas model with TFP shocks generates realistic output and consumption variability (see Col. 3, Table 3); however, investment, hours worked and the real exchange rate are less volatile than in the data (hours are constant). The no-bubble model with TFP shocks generates fluctuations in output, consumption and investment that are positively correlated across

[25] $\Delta=2.227\times10^{-6}$ implies that, in a bust, the ratio of investment spending divided by nominal GDP, $Z_{i,t}\equiv P_{i,t}K_{i,t+1}/(p_{i,t}Y_{i,t})=\kappa_t/(1+\kappa_t)$ exceeds its steady state value $\alpha\beta$ by the amount 10^{-6}, as in the closed economy.

[26] $\varepsilon^{\theta}_{H,t+1}=a\cdot v_{H,t+1}+b\cdot v_{F,t+1}$, $\varepsilon^{\theta}_{F,t+1}=b\cdot v_{H,t+1}+a\cdot v_{F,t+1}$ where $v_{H,t+1}$, $v_{F,t+1}$ are independent random variables that equal 1 or -1 with probability 0.5. I set $a=0.8447\%$, $b=0.1108\%$ to match the stated standard dev. and corr. of $\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}$.

[27] The Dellas economy assumes endogenous labor. Hours worked rise in response to a positive bubble shock. This is why real activity is more volatile than in the closed economy (Long-Plosser) model with fixed labor of Sect. 2.

countries. The predicted cross-country correlation of output (0.39) is smaller than the empirical correlation (0.53), while predicted cross-country correlations of consumption and investment (0.56) are higher than the corresponding empirical correlations (about 0.4).

Note that all model variants predict a zero trade balance. The bubble economy with joint bubble shocks and TFP shocks (Col. 2) generates higher cross-country correlations of output, consumption and investment than the no-bubble economy (Col. 3). Also, the presence of TFP shocks implies that the real exchange rate shows non-negligible fluctuations (while the real exchange rate is constant in the bubble model with constant TFP, as discussed above).

5. Rational bubbles in a two-country RBC model with incomplete capital depreciation (no TVC)

This Section discusses rational bubbles in a more general two-country RBC model that resembles the classic International RBC model proposed by Backus et al. (1994). This model cannot be solved in closed form. It assumes incomplete capital depreciation, a CES final good aggregator, and it allows for non-unitary risk aversion. Other model features are identical to those of the Dellas model. Thus, each country is specialized in the production of a distinct tradable good. In each country, domestic and imported tradables are combined into a non-tradable final good used for consumption and investment. Complete global financial markets are assumed. The law of motion of Home and Foreign TFP is again given by (28).

As in the closed economy RBC model of Sect. 3, I assume the period utility function $U(C_{i,t},L_t)=\ln(C_{i,t}-\overline{C})+\Psi\cdot\ln(1-L_{i,t})$, $\overline{C}\geq 0$. The country i final good is generated from domestic and imported intermediates using a CES aggregator: $Z_{i,t}=[\xi^{1/\phi}\cdot(y^i_{i,t})^{(\phi-1)/\phi}+(1-\xi)^{1/\phi}\cdot(y^j_{i,t})^{(\phi-1)/\phi}]^{\phi/(\phi-1)}$, $j\neq i$, where ϕ is the substitution elasticity between domestic and imported intermediates. There is local bias in final good production: $\frac{1}{2}<\xi<1.$ The price of country i's final good $(P_{i,t})$ now is $P_{i,t}=[\xi\cdot(p_{i,t})^{1-\phi}+(1-\xi)\cdot(p_{j,t})^{1-\phi}]^{1/(1-\phi)}$, $j\neq i$, while country i's demand functions for domestic and imported inputs are $y^i_{i,t}=\xi\cdot(p_{i,t}/P_{i,t})^{-\phi}Z_{i,t}$ and $y^j_{i,t}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-\phi}Z_{i,t}$. The law of motion of country i's capital stock is $K_{i,t+1}=(1-\delta)K_{i,t}+I_{i,t}$, where $0<\delta<1$ is the capital depreciation rate. The final good market clearing condition in country i is $Z_{i,t}=C_{i,t}+I_{i,t}$.

The *static* equilibrium conditions allow to solve for date t consumption, labor and terms of trade $C_{i,t}, L_{i,t}, q_{i,t}$ as functions of both countries' capital stocks in t and t+1 and of date t productivity. By substituting these functions into the two countries' capital Euler equations, one can write the Euler equations as expectational difference equations in Home and Foreign capital:

$$E_t H_i(\overrightarrow{K_{t+2}}, \overrightarrow{K_{t+1}}, \overrightarrow{K_t}, \overrightarrow{\theta_{t+1}}, \overrightarrow{\theta_t}) = 1 \quad \text{for } i=H,F, \tag{29}$$

where $\overrightarrow{K_t} \equiv (K_{H,t}, K_{F,t})$ and $\overrightarrow{\theta_t} \equiv (\theta_{H,t}, \theta_{F,t})$ are vectors of Home and Foreign capital and TFP, respectively. The function H_i maps R_+^{10} into R.

The no-bubble solution of the model (that obtains when TVCs are imposed) is described by decision rules $K_{i,t+1}=\lambda_i(\overrightarrow{K_t}, \overrightarrow{\theta_t})$ that map date t capital and TFP into capital at date t+1.

Assume that there is no transversality condition (TVC) for capital, which makes rational bubbles possible. I consider a bubble process that parallels the bubbles in previous Sections. Assume that capital $K_{i,t+1}$ takes one of two values: $K_{i,t+1} \in \{K_{i,t+1}^L, K_{i,t+1}^H\}$, with probabilities π and 1-π, respectively, where $K_{i,t+1}^L=\lambda_i(\overrightarrow{K_t}, \overrightarrow{\theta_t}) \cdot e^{\Delta}$. Like in previous models, $\Delta>0$ is required to generate *recurrent* bubbles. As in the Dellas economy with complete financial markets, the bubble has to be perfectly synchronized across countries. Hence, $K_{H,t+1}^H$ and $K_{F,t+1}^H$ are realized together (and so are $K_{H,t+1}^L$ and $K_{F,t+1}^L$). (The superscripts 'H' (boom) and 'L' (bust) refer to the state of the bubble, while the subscripts 'H' (Home) and 'F' (Foreign) refer to the country.)

Consider a world economy that starts at date t=0, with exogenous initial capital stocks $K_{H,0}, K_{F,0}$. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and 1-π, respectively ($0<\pi<1$). Then the following process for Home and Foreign capital stocks $\{K_{H,t}, K_{F,t}\}_{t\geq 0}$ is a recurrent rational bubble:

(a) $K_{i,t+2}=K_{i,t+2}^L \equiv \lambda_i(\overrightarrow{K_{t+1}}, \overrightarrow{\theta_{t+1}}) \cdot e^{\Delta}$ for $i=H,F$ if $u_{t+1}=0$, for $t\geq 0$;

(b) $K_{i,t+2}=K_{i,t+2}^H$ for $i=H,F$, if $u_{t+1}=1$, for $t\geq 0$, where $K_{H,t+2}^H, K_{F,t+2}^H$ satisfy date t Euler equations (29). (Not-for-Publication Appendix C provides further discussions.)

5.1. Quantitative results

As in Sect. 3, I set α=1/3, β=0.99, δ=0.025. Ψ (utility weight on leisure) is again set so that the Frisch labor supply elasticity is unity, at the steady state. As in the calibration of the Dellas model, the local spending bias parameter is set at ξ=0.9. The substitution elasticity between domestic and imported intermediates is set at ϕ=1.5; that value is consistent with estimated price elasticities of aggregate trade flows and it has been widely used in International RBC models (e.g., Backus et al. (1994)). The parameters of the bubble process are the same as in the closed economy model (with incomplete capital depreciation) studied in Sect. 3; thus, Δ is again set at $\Delta=10^{-6}$, and two values of the bust probability are considered: π=0.2 and π=0.5.

Predicted business cycle statistics generated by the two-country RBC model with incomplete capital depreciation are shown in Table 4. Cols. labelled 'Unit Risk Aversion' (or 'Unit RA') assume log utility (minimum consumption set at $\overline{C}$=0). In Cols. labelled 'High RA', $\overline{C}$ is set at 0.8 times steady state consumption (implied risk aversion, at steady state: 5).

Cols. (9) and (10) of Table 4 show simulated business cycle statistics for versions of the no-bubble model (TVC imposed) driven by TFP shocks. The simulations confirm findings that are well known from the International RBC literature (e.g., Backus et al. (1994), Kollmann (1996)): a complete markets no-bubble model driven by TFP shocks can capture the historical volatility of output and investment, but it underpredicts the empirical volatility of the real exchange rate. The no-bubble model here reproduces the fact that net exports are countercyclical. However, the model-predicted cross-country correlations of output and investment are markedly lower than the corresponding historical correlations. By contrast, the model predicts that consumption is highly correlated across countries. The low predicted cross-country correlation of output reflects the fact that, with complete financial markets, a positive shock to Home productivity raises Foreign consumption, which reduces Foreign labor supply, and thus lowers Foreign output, on impact (while Home output increases).[28]

Simulated business cycle statistics for the bubble economy with just bubble shocks (constant TFP) are reported in Cols. (1)-(4) of Table 4. Standard deviations, correlations with domestic GDP, autocorrelations and mean values are *identical* to the corresponding statistics for

[28]The no-bubble variant of the Dellas model driven by TFP shocks generates higher cross-country output correlations (see Col. (3) of Table 3) because, in that variant, hours worked are constant.

the closed economy bubble model (with incomplete capital depreciation) studied in Sect. 3 (see Cols. (1)-(4) of Table 2). This is due to the fact that, in the two-country model with complete markets, bubbles are perfectly correlated across countries; with just bubble shocks, real activity is thus perfectly correlated across countries, the terms of trade are constant and net exports are zero. The predicted volatility of output and consumption induced by bubble shocks (Cols. (1)-(4) of Table 4) is roughly comparable to volatility in the no-bubble model with TFP shocks (Cols. (9),(10)), but the volatility of hours worked is higher in the bubble economy.

Predicted business cycle statistics for the bubble economy, with simultaneous bubble shocks and TFP shocks, are shown in Cols. (5)-(8) of Table 4. With joint bubble shocks and TFP shocks, the predicted volatility of real activity is higher, and thus generally closer to the data, than the volatility generated by the no-bubble model with TFP shocks. The model with joint bubble and TFP shocks is especially successful at matching the positive empirical cross-country correlations of output and investment, and the counter-cyclicality of the trade balance; however the predicted cross-country consumption correlation is too high, when compared to the data.

Fig. 4 shows simulated sample paths for the model version with 'High Risk Aversion' and a bust probability π=0.2. Panels (1) and (2) of the Figure show results for the bubble economy with just bubble shocks, and for the bubble economy with joint bubble and TFP shocks, respectively. Panel (3) of Fig. 4 pertains to a no-bubble economy with TFP shocks; in that variant, the negative cross-country correlation of high-frequency output and investment fluctuations is clearly discernible. Bubble shocks induce relatively widely spaced output and investment booms that are perfectly correlated across countries (see Panel (1)). In the bubble economy with joint bubble and TFP shocks, output and investment are markedly more synchronized across countries than in the no-bubble economy with TFP shocks (see Panel (2)).

6. Conclusion

This paper constructs bounded rational bubbles in non-linear DSGE models of the macroeconomy. The term 'rational bubbles' refers to multiple equilibria due to the absence of a transversality condition (TVC) for capital. The lack of TVC can be justified by assuming an OLG structure with finitely-lived agents. Bounded rational bubbles provide a novel perspective on the drivers and mechanisms of business cycles. This paper studies bubble equilibria in which the economy undergoes boom-bust cycles characterized by persistent investment and output expansions which are followed by abrupt contractions in real activity. Importantly, the existence of multiple stable bubble equilibria is due to *non-linear* effects. *Linearized* versions of the models considered here have a unique stable solution. In contrast to explosive rational bubbles in linear models (Blanchard (1979)), the rational bubbles in non-linear models considered here are bounded. Both closed and open economies are analyzed. It is shown that rational bubbles in non-linear models can generate persistent fluctuations of real activity and capture key business cycle stylized facts. In a two-country model with integrated financial markets, rational bubbles must be perfectly correlated across countries. Global bubbles may, thus, help to explain the international synchronization of international business cycles.

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Table 1. Long-Plosser model (closed economy) with bubbles: business cycle statistics

	Standard dev. %			Corr. with Y		Autocorrelations			Mean [% deviation from SS]			
	Y	*C*	*I*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*	*Z*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Predicted business cycle statistics												
	1.14	2.35	3.41	0.38	0.43	0.42	-0.19	0.42	0.54	-0.34	2.35	1.01
(b) Historical business cycle statistics												
	1.47	1.19	4.96	0.87	0.92	0.87	0.89	0.92				

Notes: Row (a) reports simulated business cycle statistics for a Long-Plosser economy with bubbles (no transversality condition); see Sect. 2 of paper. *Y*: output; *C*: consumption; *I*: investment; *Z*: investment/output ratio.

In the simulated model, fluctuations are just driven by bubble shocks (constant TFP assumed). Bust probability π=0.5.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods. The reported simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Simulated series were logged and HP filtered (the HP filter was applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods; means are expressed as % deviations from the deterministic steady state of the no-bubble economy.

Row (b) reports US historical business cycle statistics (quarterly data), 1968q1-2017q4. The empirical data are taken from BEA NIPA (Table 1.1.3). *Y*: GDP; *C*: 'Personal consumption expenditures'; *I*: 'Fixed investment'.

Table 2. Closed economy RBC model (incomplete capital depreciation): business cycle statistics

	Bubble model (no TVC)								*No-bubble model*		
	Bubble shocks; no TFP shocks				Bubble & TFP shocks				TFP shocks		
	Unit Risk aversion		High RA		Unit RA		High RA				
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.27	1.60	0.98	1.57	1.14	0.72	1.47
C	1.08	2.63	0.29	0.61	1.16	2.71	0.38	0.72	0.49	0.26	1.19
I	4.29	9.38	3.22	6.51	5.38	9.85	3.86	6.72	3.33	2.20	4.96
L	0.74	1.73	1.04	2.18	0.82	1.70	1.05	2.22	0.34	0.30	1.06
Correlations with GDP											
C	-0.97	-0.95	-0.99	-0.98	0.04	-0.54	0.01	-0.62	0.95	0.99	0.87
I	0.98	0.96	0.99	0.99	0.89	0.86	0.97	0.98	0.99	0.99	0.92
L	0.99	0.97	0.99	0.99	0.79	0.81	0.45	0.82	0.98	-0.96	0.82
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.65	0.68	0.57	0.66	0.71	0.70	0.87
C	0.33	0.60	0.35	0.62	0.43	0.62	0.53	0.65	0.76	0.72	0.89
I	0.36	0.63	0.37	0.64	0.53	0.65	0.51	0.65	0.70	0.70	0.92
L	0.34	0.61	0.35	0.62	0.45	0.62	0.41	0.63	0.70	0.74	0.92
Means [% deviation from no-bubble steady state]											
Y	1.41	2.80	1.25	2.12	1.37	2.75	1.31	2.17	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.68	1.34	0.33	0.55	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.61	7.28	4.44	7.40	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.34	0.73	0.01	-0.03	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.16	8.78	8.92	8.53	9.58	9.58	13.42
Fraction of periods with (capital income > investment) [in %]											
	99.20	96.31	99.55	97.72	99.20	96.43	99.37	97.74	100	100	100

Notes: This Table reports simulated business cycle statistics for a closed economy RBC model with full capital depreciation (see Sect. 3 of paper). *Y*: output (GDP); *C*: consumption ; *I*: investment; *L*: hours worked.

Cols. (1)-(4) pertain to versions of the bubble model (no transversality condition, TVC) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Cols. (5)-(8) pertain to versions of the bubble model, driven by simultaneous bubble and TFP shocks. Cols. (9)-(10) pertain to versions of the no-bubble model, driven by TFP shocks. 'Unit Risk Aversion': log utility; 'High Risk Aversion (RA)': consumption utility given by $\ln(C_t-\overline{C})$, with $\overline{C}>0$. π: bust probability of bubble process.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods with (capital income > investment)' likewise pertains to the whole simulation run of T periods.

Col. (11) reports US historical statistics (quarterly data). Statistics for *Y*,*C*,*I*: see Table 1. The empirical measure for '*L*' is: 'Total Employment' (Source: CPS, as reported by FRED database, series CE160V). Historical statistics about 'capital income – investment': based on US annual data 1929-1985 reported by Abel et al. (1989)).

Table 3. Two-country Dellas model: business cycle statistics

	Bubble model (no TVC)		*No-bubble Model*	
	Bubble shocks; no TFP shocks	Bubble & TFP shocks	TFP shocks	Data
	(1)	*(2)*	*(3)*	*(4)*
Standard deviations [in %]				
Y	1.52	1.96	1.36	1.47
C	1.86	2.22	1.28	1.19
I	3.95	4.01	1.28	4.96
L	0.97	0.97	0.00	1.06
NX	0.00	0.00	0.00	0.43
RER	0.00	1.23	1.23	3.66
Correlations with domestic GDP				
C	0.25	0.57	0.99	0.87
I	0.76	0.88	0.99	0.92
L	0.50	0.31	--	0.82
NX	--	--	---	-0.51
RER	--	-0.41	-0.54	-0.27
Autocorrelations				
Y	0.63	0.77	0.80	0.87
C	-0.17	0.48	0.81	0.89
I	0.41	0.66	0.81	0.92
L	0.10	0.10	--	0.92
NX	--	--	--	0.78
RER	--	0.75	0.75	0.81
Cross-country correlations				
Y	1.00	0.68	0.39	0.53
C	1.00	0.84	0.56	0.39
I	1.00	0.95	0.56	0.45
L	1.00	1.00	--	0.39
Means [% deviation from no-bubble steady state]				
Y	0.95	1.18	0.22	--
C	-0.01	0.12	0.22	--
I	3.07	3.33	0.22	--
L	0.42	0.42	0.00	--
Mean (capital income – investment)/GDP [in %]				
	-0.02	-0.02	0.33	13.42
Fraction of periods with (capital income > investment) [in %]				
	97.01	97.01	100.00	100.00

Notes: This Table reports simulated business cycle statistics for a two-country RBC world (Dellas) with full capital depreciation (see Sect. 4 of paper). *Y*: GDP; *C*: consumption ; *I*: investment; *L*: labor input. *NX*: net exports/GDP; *RER*: real exchange rate. A rise in R*ER* represents an appreciation.

Table 3. (continued)

Col. (1) pertains to a version of the bubble model (no transversality condition, TVC) in which fluctuations are just driven by bubbles shocks (constant TFP assumed). Col. (2) pertains to a version of the bubble model, driven by simultaneous bubble and TFP shocks. The bubble process (Cols. 1 and 2) assumes a bust probability π=0.5. Col. (3) pertains to a no-bubble model, driven by TFP shocks.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations with domestic GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods with (capital income > investment)' likewise pertains to the whole simulation run of T periods.

Col. (4) reports historical statistics. Historical standard deviations, correlations with domestic GDP and autocorrelations of GDP, consumption, investment, employment, net exports and the real exchange rate are based on quarterly US data, 1968q1-2017q4 (see Tables 2 and 3). The empirical measure of *NX* is: US nominal exports-imports (goods and services) divided by nominal GDP (from BEA NIPA Table 1.1.5). Empirical measure of the US real exchange rate: real effective exchange rate, REER (from BIS; 1968:q1-1993q4: 'narrow index'; 1994q1-2017q4: 'broad index'; a quarterly average of the monthly BIS REER series is used). Historical statistics about 'capital income – investment': based on US annual data 1929-1985 reported by Abel et al. (1989)).

Historical cross-country correlations (of Y,C,I,L) are correlations between US series and series for an aggregate of the Euro Area for 1970q1-2017q4 (logged and HP filtered quarterly series). (Euro Area data are only available from 1970q1.) Source for EA data: ECB Area-wide Model (AWM) database (version Aug. 2018). (EWM series for Y,C,I,L: YER, PCR, ITR, LNN.)

Table 4. Two-country RBC model (incomplete capital depreciation): business cycle statistics

	Bubble model (no TVC)								*No-bubble model*		
	Bubbles shocks; no TFP shocks				Bubble & TFP shocks				TFP shocks		
	Unit Risk aversion		High RA		Unit RA		High RA				
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.46	1.78	1.18	1.65	1.32	0.97	1.47
C	1.08	2.63	0.29	0.61	1.18	2.79	0.41	0.70	0.56	0.31	1.19
I	4.29	9.38	3.22	6.51	6.36	10.54	4.95	7.34	4.60	3.90	4.96
L	0.74	1.73	1.04	2.18	0.88	1.79	1.13	2.24	0.44	0.62	1.06
NX	0.00	0.00	0.00	0.00	0.16	0.16	0.13	0.13	0.16	0.13	0.43
RER	0.00	0.00	0.00	0.00	0.32	0.32	0.44	0.44	0.32	0.44	3.66
Correlations with domestic GDP											
C	-0.97	-0.95	-0.99	-0.98	0.09	-0.46	0.03	-0.55	0.85	0.61	0.87
I	0.98	0.96	0.99	0.99	0.90	0.88	0.97	0.98	0.95	0.96	0.92
L	0.99	0.97	0.99	0.99	0.81	0.81	0.46	0.78	0.94	-0.01	0.82
NX	--	--	--	--	-0.53	-0.46	-0.58	-0.46	-0.58	-0.68	-0.51
RER	--	--	--	--	-0.44	-0.35	-0.58	-0.39	-0.48	-0.68	-0.27
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.63	0.67	0.57	0.65	0.67	0.64	0.87
C	0.33	0.60	0.35	0.62	0.46	0.62	0.57	0.65	0.75	0.71	0.89
I	0.38	0.63	0.37	0.64	0.54	0.64	0.55	0.64	0.63	0.61	0.92
L	0.34	0.61	0.35	0.62	0.46	0.62	0.48	0.64	0.63	0.69	0.92
NX	--	--	--	--	0.61	0.61	0.66	0.66	0.61	0.66	0.78
RER	--	--	--	--	0.84	0.84	0.81	0.81	0.84	0.81	0.82
Cross-country correlations											
Y	1.00	1.00	1.00	1.00	0.29	0.54	-0.00	0.52	0.17	-0.46	0.53
C	1.00	1.00	1.00	1.00	0.96	0.99	0.98	0.99	0.84	0.96	0.39
I	1.00	1.00	1.00	1.00	0.27	0.74	-0.07	0.53	-0.35	-0.83	0.45
L	1.00	1.00	1.00	1.00	0.63	0.92	0.85	0.96	-0.35	0.46	0.39
Means [% deviation from no-bubble steady state]											
Y	1.41	2.80	1.25	2.12	1.65	3.02	1.45	2.29	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.95	1.60	0.44	0.65	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.93	7.61	4.72	7.61	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.35	0.73	0.09	0.05	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.15	8.78	8.89	8.51	9.55	9.58	13.42
Fraction of periods with (capital income > investment) [in %]											
	99.20	96.31	99.55	97.72	99.20	96.45	99.44	97.75	100	100	100

Notes: This Table reports simulated business cycle statistics for a two-country RBC model with incomplete capital depreciation (see Sect. 5 of paper). *Y*: GDP; *C*: consumption ; *I*: investment; *L*: labor input; *NX*: net exports/GDP; *RER*: real exchange rate. A rise in R*ER* represents an appreciation.

Table 4. (continued)

Cols. (1)-(4) pertain to versions of the bubble model (no transversality condition, TVC) in which fluctuations are just driven by bubbles (constant TFP assumed). Cols. (5)-(8) pertain to versions of the bubble model, driven by simultaneous bubble and TFP shocks. Cols. (9)-(10) pertain to versions of the no-bubble model, driven by TFP shocks.

'Unit Risk Aversion': log utility; 'High Risk Aversion (RA)': consumption utility given by $\ln(C_t - \overline{C})$, with $\overline{C}>0$.

π: bust probability of bubble process.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods with (capital income > investment)' likewise pertains to the whole simulation run of T periods.

Col. (11) reports historical statistics (see Table 3).

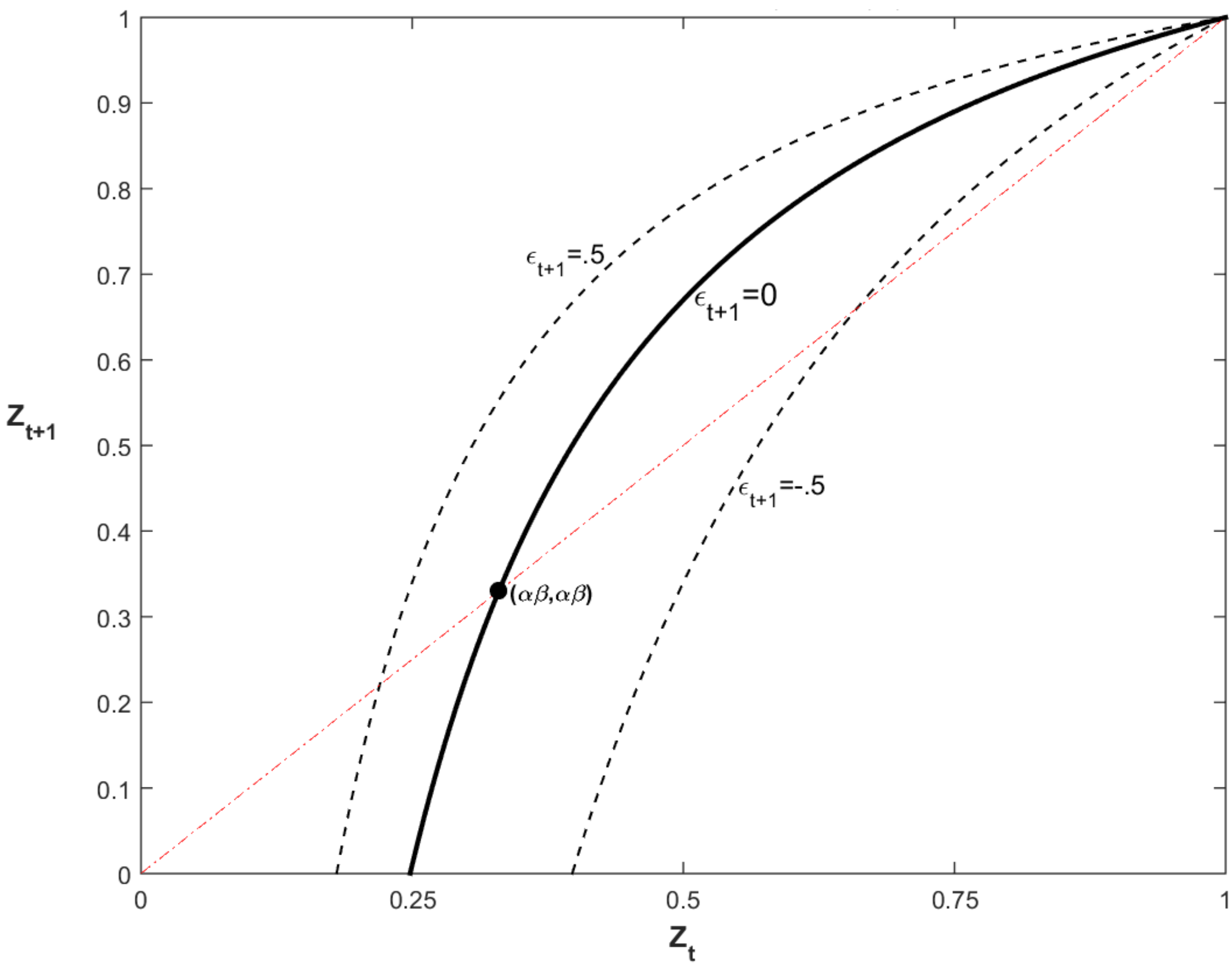
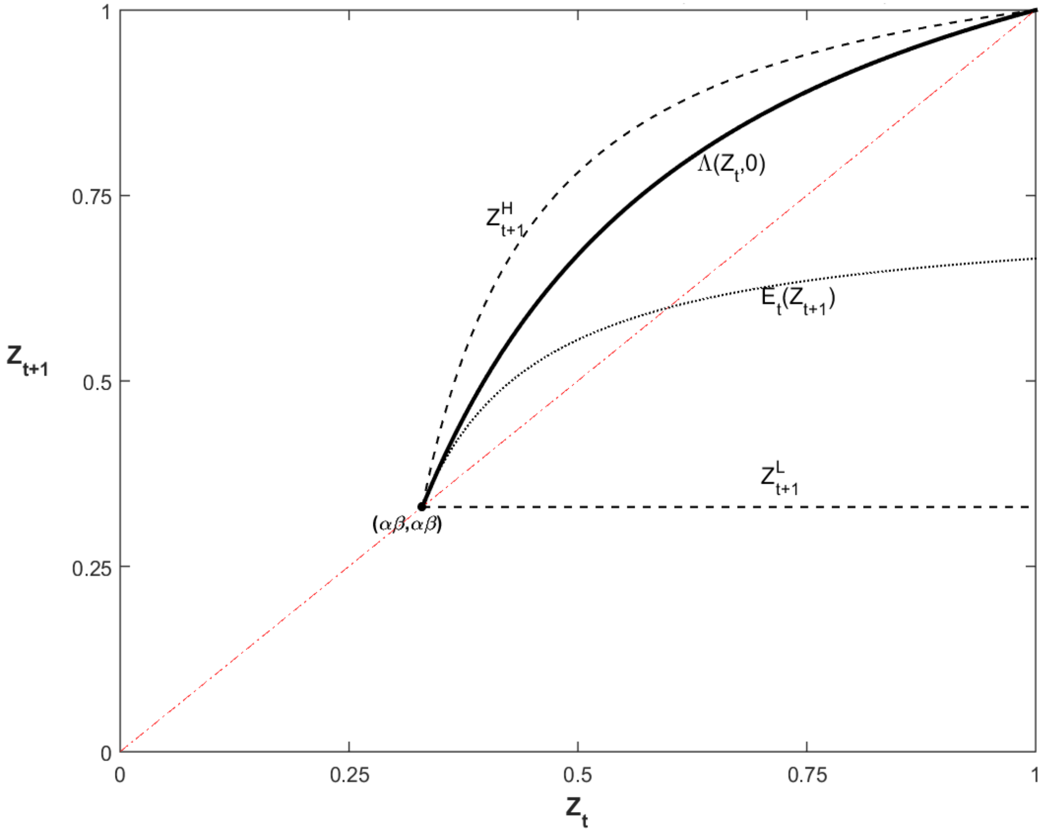
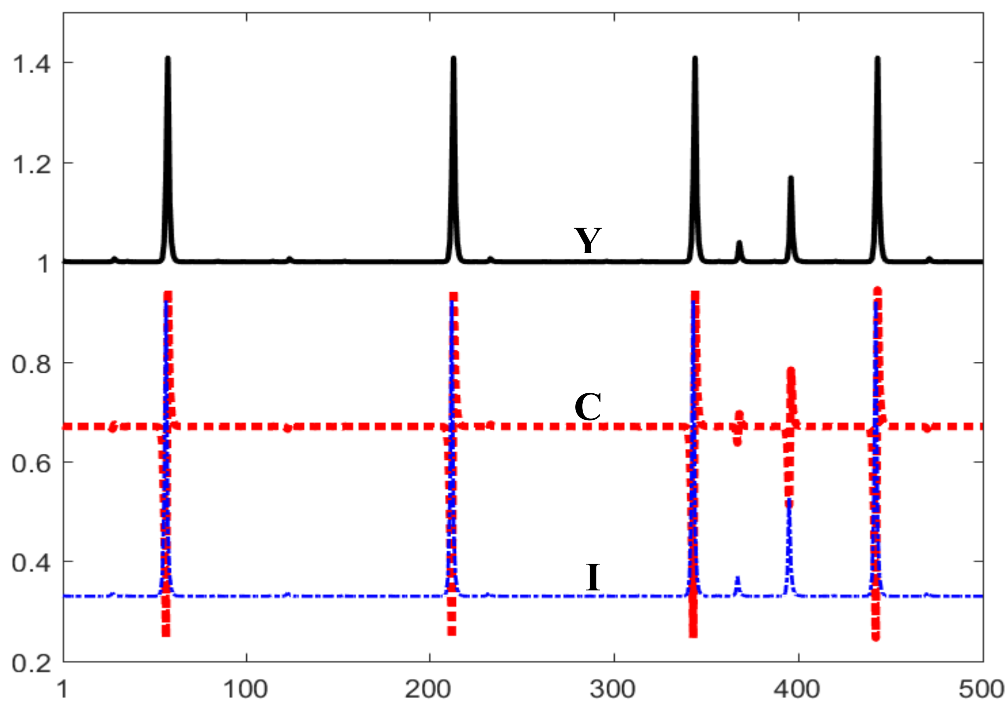


Figure 1. Long & Plosser model: The Figure plots the date t+1 investment/output ratio, Z_{t+1}, as a function of Z_t, for $\varepsilon_{t+1} \in \{-0.5;0;0.5\}$. ε_{t+1} : Euler equation forecast error. The law of motion of Z is: $Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1})$.



(1) 'Boom' value, Z^H_{t+1}, and 'Bust' value, Z^L_{t+1}, of investment/output ratio at t+1, and expected value $(E_t Z_{t+1})$, shown as functions of $Z_t \in [\alpha\beta + \Delta, 1)$. $\Lambda(Z_t, 0)$ is value of Z_{t+1} in a deterministic economy (zero Euler equation forecast error).



(2) Simulated paths

Figure 2. Long & Plosser economy with bubbles (no transversality condition)
Simulated series of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line) are normalized by steady state output (see Panel (2)). Y C I

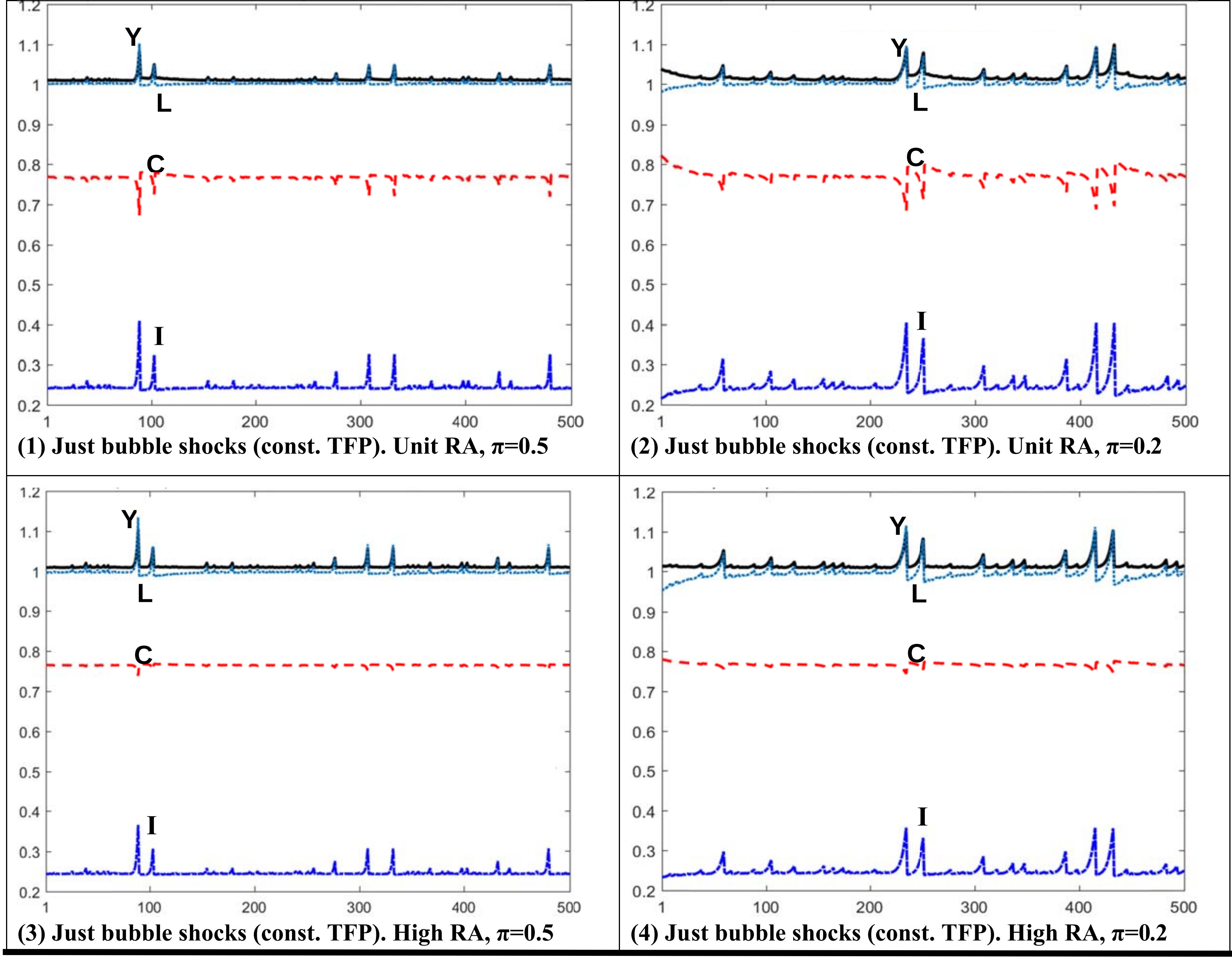
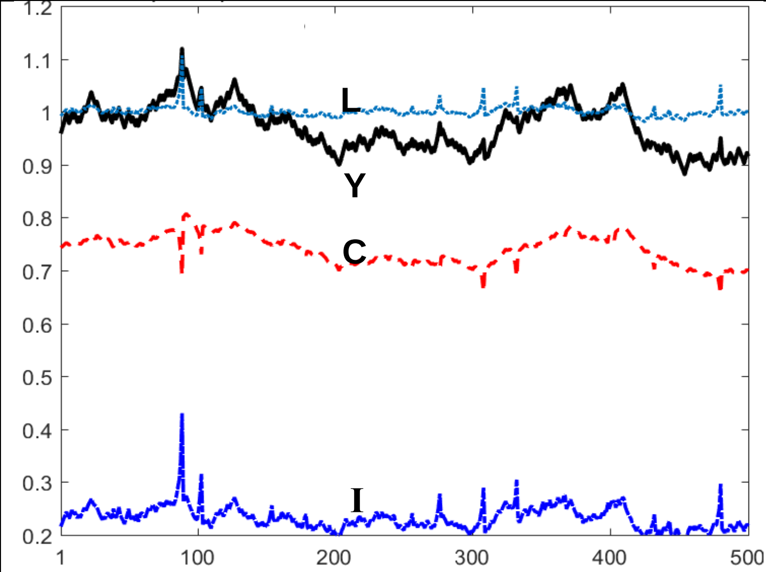


Figure 3. Closed economy RBC model (incomplete capital depreciation): simulated paths

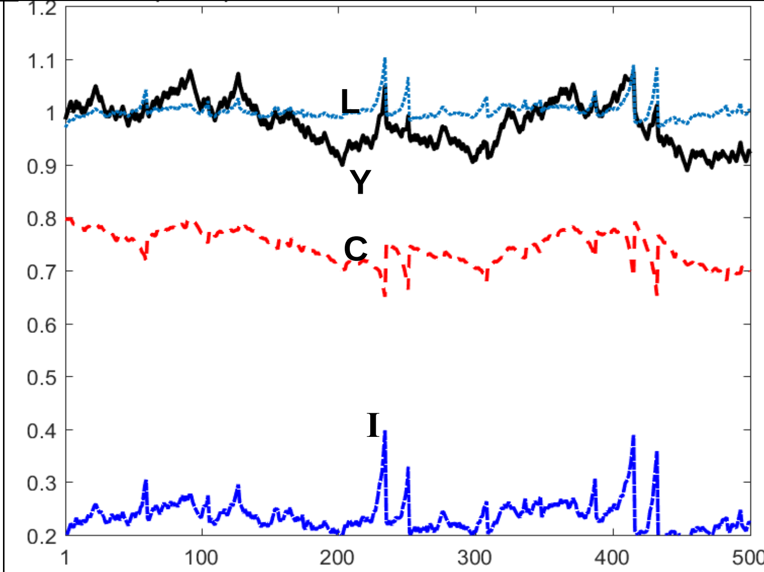
For 10 versions of the closed economy RBC model with incomplete capital depreciation described in Sect. 3, simulated paths of GDP (Y, continuous black line), consumption (C, red dashed line), investment (I, dark blue dash-dotted line) and hours worked (L, light blue dotted line) are shown. The plotted Y, C and I series are normalized by steady state GDP. The plotted hours worked (L) series is normalized by steady state hours. Y C I L

Panel (i) of this Figure assumes the model version considered in Col. (i) of Table 2. RA: risk aversion. π: bust probability of bubble process.

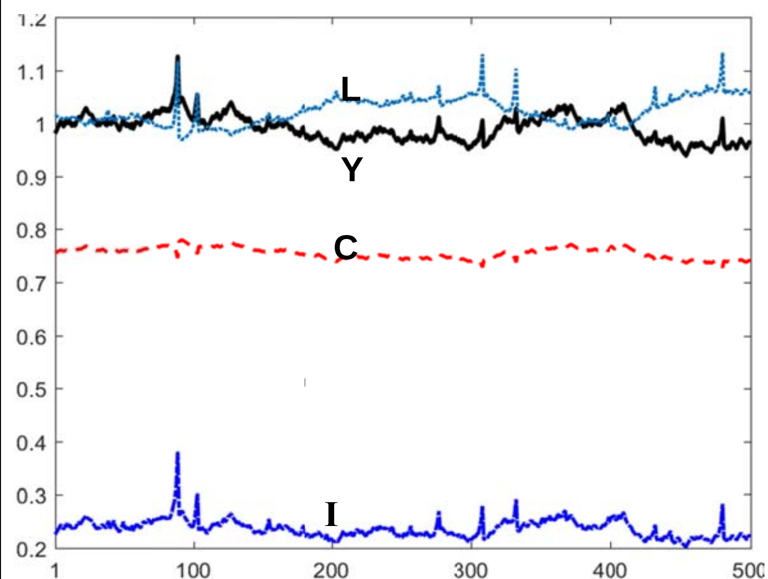
Panels (1)-(4) pertain to versions of the bubble model (no transversality condition) in which fluctuations are just driven by bubbles (constant TFP assumed). Panels (5)-(8) pertain to versions of the bubble model, driven by simultaneous bubble and TFP shocks. Panels (9)-(10) pertain to versions of the no-bubble model, driven by TFP shocks.



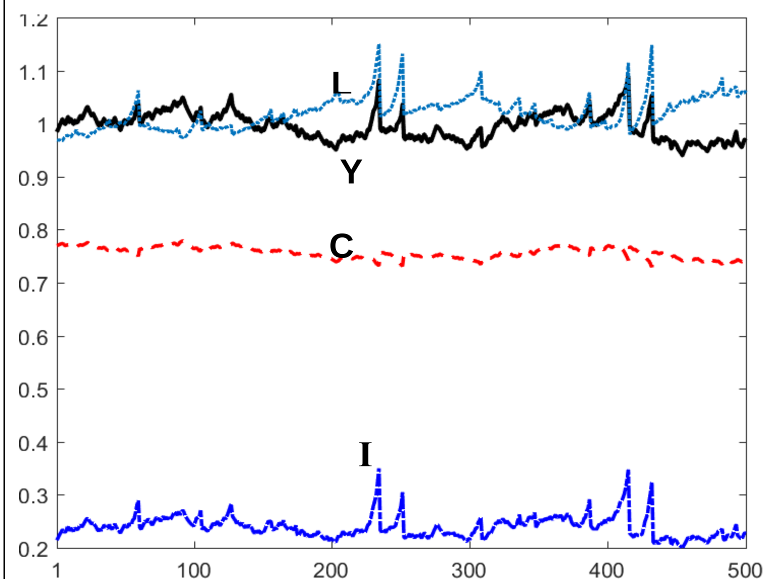
(5) Bubble & TFP shocks. Unit RA, π=0.5



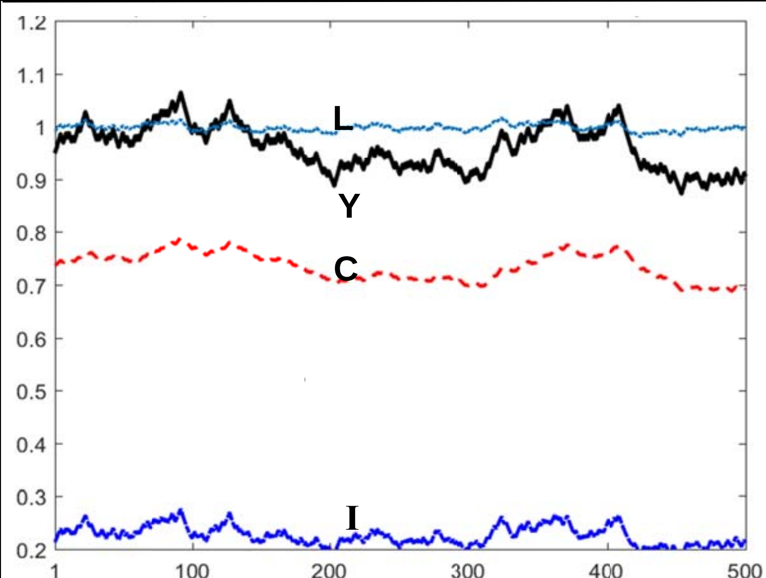
(6) Bubble & TFP shocks. Unit RA, π=0.2



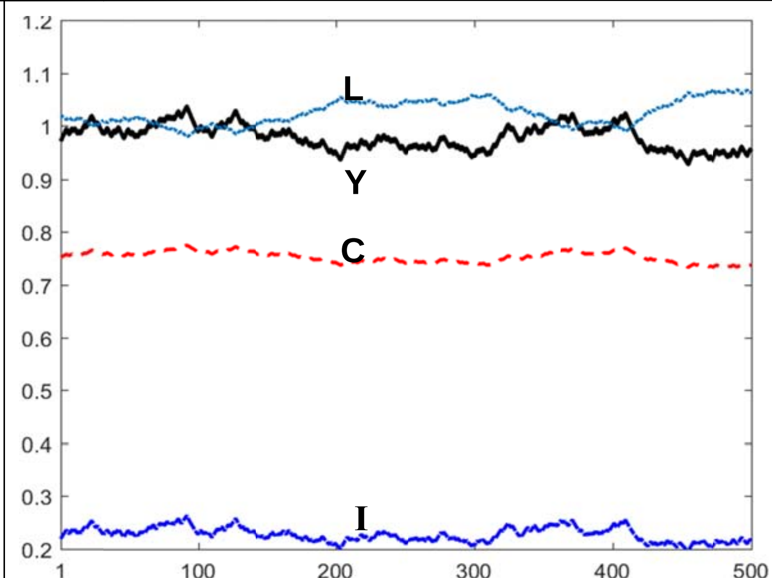
(7) Bubble & TFP shocks. High RA, π=0.5



(8) Bubble & TFP shocks. High RA, π=0.2



(9) NO-BUBBLE MODEL, TFP shocks. Unit RA



(10) NO-BUBBLE MODEL, TFP shocks. High RA

Figure 3. (continued) Y C I L

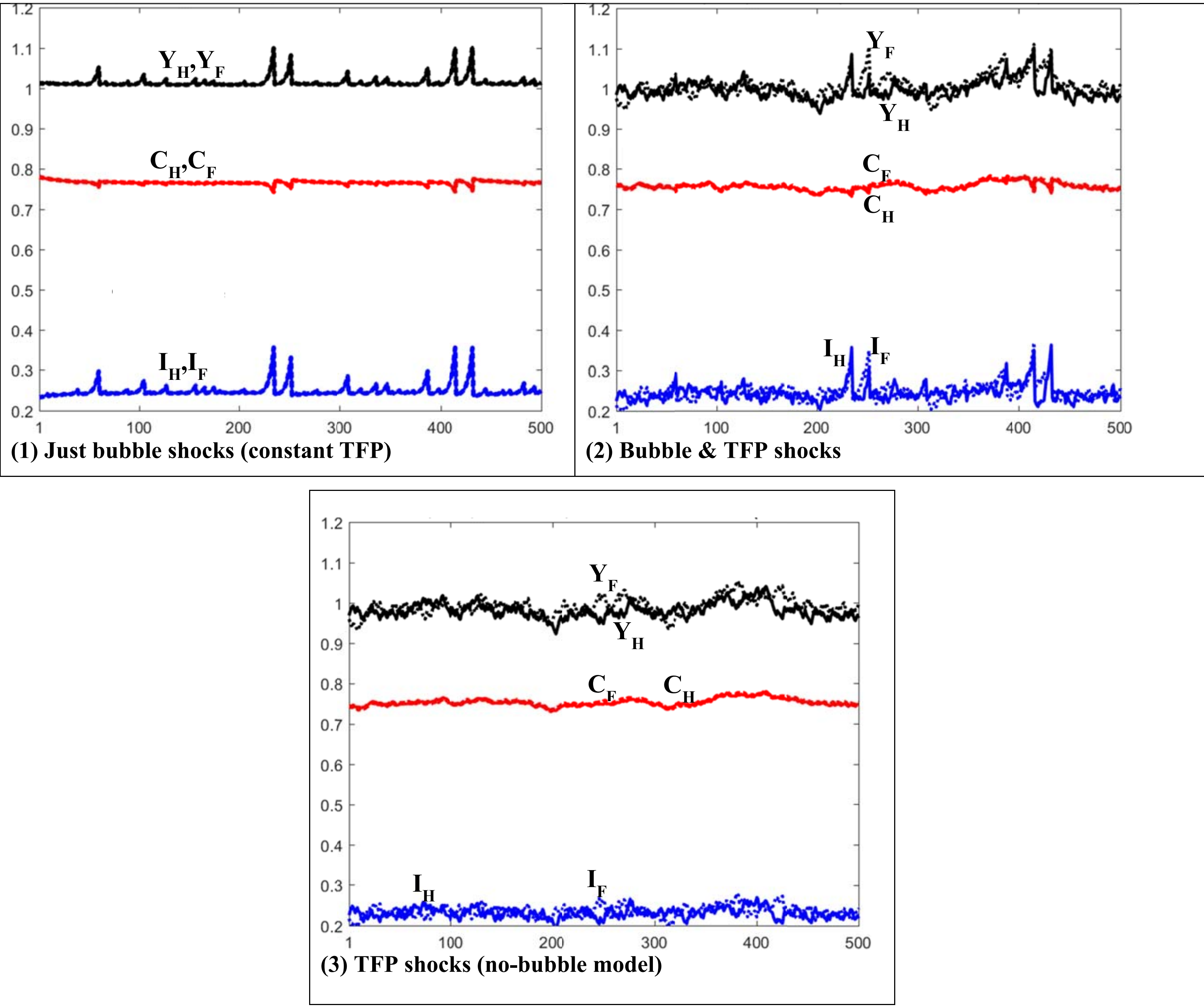


Figure 4. Two-country RBC model (incomplete capital depreciation): simulated paths

This Figure assumes the two-country RBC model with incomplete capital depreciation, 'High risk aversion' and a bust probability π=0.20 (see Sect. 5). Simulated paths of Home and Foreign GDP (Y_H, Y_F: continuous and dotted black lines), Home and Foreign consumption (C_H, C_F: continuous and dotted red lines) and investment (I_H, I_F: continuous and dotted blue lines) are shown. The plotted series are normalized by steady state GDP. Y_H Y_F C_H C_F I_H I_F

Panel (1) pertains to a bubble model (no transversality condition) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Panel (2) pertains to a bubble model, driven by joint bubble and TFP shocks. Panel (3) pertains to a no-bubble model, driven by TFP shocks.

NOT FOR PUBLICATION APPENDICES

● Appendix A (Not for Publication): OLG model

The key contribution of this paper is to construct bounded bubbles in non-linear DSGE models without transversality condition (TVC) for aggregate capital. The lack of TVC can be justified by an overlapping generations (OLG) structure with finitely-lived households. This Appendix presents a novel OLG structure with finitely-lived agents that has the *same* aggregate static equations and the same aggregate Euler equation as a Long-Plosser economy inhabited by an infinitely-lived representative agent. At the end of her life, each agent holds zero assets, in the OLG structure. As agents have a finite horizon, the (infinite-horizon) TVC for *aggregate* capital is *not* an equilibrium condition in the OLG structure.

The OLG structure provides thus a motivation for exploring rational bubbles (i.e. multiple equilibria due to the lack of TVC) in the Long-Plosser economy, and in other DSGE models. Two key features of the OLG structure considered here are: (I) complete risk sharing between contemporaneous generations. (II) Newborn agents receive an endowment such that consumption by newborns represents a time-invariant share of aggregate consumption. (Assumption (I) is also used by Galí (2018). Assumption (II) is novel, to the best of my knowledge.)

OLG version of a Long-Plosser economy

Following the Long-Plosser model discussed in Sect. 2, this Appendix assumes a closed economy with log utility, a Cobb-Douglas production function and full capital depreciation. Labor hours are constant; aggregate labor hours are normalized at unity. Generalization to CRRA utility, to the 'High Risk Aversion' utility function of Sect. 3.1, to incomplete capital depreciation, variable labor, and to a multi-country model is straightforward.

Assume that a measure 1 of agents is born each period. All agents live $N<\infty$ periods. I refer to agents who are in the i-th period of their life at date t as 'generation i' at date t. Generation i at date t thus becomes generation i+1 at t+1. At each date, generation i accounts for a fraction 1/N of the population (for $1\leq i\leq N$). All members of the same generation are identical. All agents have time-additive (log) utility and the same subjective discount factor, β. Let $c_{i,t}$

denote the consumption of generation i at date t. The expected life-time utility of the generation born at date t is $E_t \sum_{s=1}^{N} \beta^{s-1} \ln(c_{s,t+s-1})$. Aggregate consumption at t is $C_t = \sum_{i=1}^{N} c_{i,t}$.

Assume that technology and the aggregate resource constraint are the same as in the Long-Plosser model described in Sect. 2. Thus, $Y_t = \theta_t (K_t)^{\alpha} (L_t)^{1-\alpha}$, where $K_t = \sum_{i=1}^{N} K_{i,t}$ is the aggregate capital stock; $K_{i,t}$ is the capital stock owned by generation i, at the beginning of period t. L_t is aggregate hours worked. Without loss of generality, assume that each generation supplies the same amount of labor, 1/N, so that $L_t = 1$, in equilibrium. The aggregate resource constraint is $Y_t = C_t + I_t$, where I_t denotes aggregate (gross) investment. Because of full capital depreciation, the aggregate capital stock is $K_{t+1} = I_t$. Competitive firms rent capital from households and hire household labor. Thus the rental rate of capital, denoted $r_{K,t}$, equals the marginal product of capital: $r_{K,t} = \alpha Y_t / K_t$.

Assume that, at each date t, a complete set of state-contingent one-period bonds is traded. This implies that, in equilibrium, the individual consumption growth rate between t and t+1 is equated, for all states of the world, across all agents who are alive in both periods (efficient risk sharing):

$$c_{i+1,t+1}/c_{i,t} = c_{2,t+1}/c_{1,t} \text{ for } i=2,..,\text{N-1}. \quad \text{(A.1)}$$

Let $\lambda_{i,t} \equiv c_{i,t}/C_t$ denote the ratio of generation i's consumption divided by aggregate consumption at t. I refer to $\lambda_{i,t}$ as the 'consumption share' of generation i, in period t. (A.1) implies

$$\lambda_{i+1,t+1}/\lambda_{i,t} = \lambda_{2,t+1}/\lambda_{1,t} \text{ for i=2,..,N-1}. \quad \text{(A.2)}$$

(A.2) and the adding up constraint

$$\sum_{i=1}^{N} \lambda_{i,t+1} = 1 \quad \text{(A.3)}$$

provide a system of N-1 equations in the N consumption shares at date t+1: $\lambda_{1,t+1}, \lambda_{2,t+1}, ..., \lambda_{N,t+1}$. Assume, henceforth, that the consumption share of newborn agents, during the first period of their life, is time-invariant: $\lambda_{1,t} = \lambda_1 \ \forall t$. (A constant newborn consumption share can be sustained by allocating to newborns a suitable endowment financed by transfers from older generations;

see below.) Then we can use (A.2) and (A.3) to solve for the date t+1 consumption shares $\{\lambda_{i,t+1}\}_{i=1,..,N}$ for given values of the date t shares $\{\lambda_{i,t}\}_{i=1,..,N}$:

$$\lambda_{i+1,t+1} = (1-\lambda_1)\lambda_{i,t}/(1-\lambda_{N,t}) \text{ for i=1,..,N-1.} \tag{A.4}$$

(A.4) defines a system of difference equations in consumption shares. Given a time-invariant generation 1 consumption share λ_1, the consumption shares of generation i=2,..,N converges asymptotically to a constant consumption share λ_i (numerical experiments show that convergence to steady state shares is fast). The N steady state consumption shares obey

$$\lambda_{i+1} = (1-\lambda_1)\lambda_i/(1-\lambda_N) \text{ for i=1,..,N-1.}$$

Given λ_1, these equations pin down unique consumption shares of generations i=2,..,N that are consistent with the adding up constraint $\sum_{i=1}^{N}\lambda_i=1$. The following discussion assumes that all generations' consumption shares equal their steady state values.

Euler equation

The date t capital Euler equation of generation i=1,..,N-1 is $E_t\rho_{t,t+1}r_{K,t+1}=1$, where $r_{K,t+1}$ is the gross rate of return (between t and t+1) on capital, while $\rho_{t,t+1}=\beta c_{i,t}/c_{i+1,t+1}$ is the common intertemporal marginal rate of substitution (IMRS) between t and t+1 of these generations. (Full risk sharing implies that the IMRS is equated across contemporaneous generations; see (A.1)). Note that

$$\rho_{t,t+1}=\beta\sum_{i=1}^{N-1}c_{i,t}/\sum_{i=2}^{N}c_{i,t+1} = \beta(C_t-c_{N,t})/(C_{t+1}-c_{1,t+1}) = \beta[(1-\lambda_N)/(1-\lambda_1)]\cdot C_t/C_{t+1}.$$

The Euler equation can thus be expressed as

$$E_t\tilde{\beta}(C_t/C_{t+1})\times r_{K,t+1} = 1, \text{ with } \tilde{\beta}\equiv\beta\times(1-\lambda_N)/(1-\lambda_1).$$

We hence see that, up to a rescaling of the subjective discount factor when $\lambda_1\neq\lambda_N$, this OLG model implies an 'aggregate' Euler equation (in terms of aggregate consumption) that has the same form as the Euler equation in an economy with an infinitely-lived representative household. If $\lambda_1=1/N$, then $\lambda_i=1/N$ holds for i=1,..,N, which implies $\tilde{\beta}\equiv\beta$. In the special case where all generations have equal consumption shares, $\lambda_i=1/N$, the aggregate Euler equation of the OLG economy is thus *identical* to the Euler of an infinitely-lived representative household.

In the OLG structure, each agent holds zero assets, at the end of her life (see below). However, the (infinite-horizon) TVC for *aggregate* capital, $\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0,$ is not an equilibrium condition in the OLG structure.

Transfers to newborn agents: a simple example

A resource transfer from older generations to the newborn generation is needed to support time-invariant consumption shares. A variety of transfer schemes can be envisaged. I now present a simple transfer scheme that sustains *equal* consumption shares, $\lambda_i=1/N$ for 1≤i≤N. I assume lump sum transfers, and denote the transfer made by generation *i*>1 to generation 1 by $T_{i,t}$.

Assume that aggregate consumption and capital satisfy the resource constraint and Euler equation (1)-(3) of the Long-Plosser model (Sect. 2). Then, the capital Euler equations of generations *i*=1,..,N-1 are satisfied under equal consumption shares ($E_t\beta(C_t/C_{t+1})r_{K,t+1}=1$ implies that $E_t\beta(c_{i,t}/c_{i+1,t+1})\cdot r_{K,t+1}=1$ holds, when $c_{i,t}=C_t/N$ and $c_{i+1,t+1}=C_{t+1}/N$).

As postulated above, each generation provides 1/N units of labor. All generations thus have the same wage income, denoted by $W_t\equiv(1-\alpha)Y_t/N.$ However, the newborn generation *i*=1 has zero initial capital, $K_{1,t}=0,$ and thus zero capital income, while older generations have capital income. As households can invest their wealth in capital or in a complete set of state-contingent bonds, the composition of individual portfolios is indeterminate. I here discuss a symmetric equilibrium in which all generations hold zero state-contingent bonds, and in which generations *i*=1,..,N-1 make identical capital investments: $K_{i+1,t+1}=K_{t+1}/(N-1)$. The oldest generation, *i*=N, does not invest: $K_{N+1,t+1}=0.$

In the symmetric equilibrium, the generations' budget constraints at date *t* are as follows:

For *i*=1: $K_{t+1}/(N-1)+C_t/N = W_t+\sum_{i=2}^{N}T_{i,t}.$

For *i*=2,..,N-1: $K_{t+1}/(N-1)+C_t/N = W_t - T_{i,t} + r_{K,t}K_t/(N-1).$

For *i*=N: $C_t/N = W_t - T_{N,t} + r_{K,t}K_t/(N-1).$

Recall that $r_{K,t}=\alpha Y_t/K_t$. Solving these budget constraints for transfers gives: [29]

$T_{i,t}=(\alpha Y_t-K_{t+1})/(N\times(N-1))$ for i=2,..,N-1 and $T_{N,t}=\alpha Y_t/(N\times(N-1))+K_{t+1}/N.$

Equivalently:

$T_{i,t}=\{r_{K,t}K_t/(N-1)-K_{t+1}/(N-1)\}/N$ for i=2,..,N-1 and $T_{N,t}=\{r_{K,t}K_t/(N-1)+K_{t+1}\}/N.$

Thus, generations *i*=2,..,N-1 transfer a fraction 1/N of their capital income net of capital investment to the newborn generation. The transfer by generation *i*=N corresponds to a fraction 1/N of her capital income, plus a fraction 1/N of the aggregate capital stock. The total transfer received by the newborn generation is $\sum_{i=2}^{N}T_i=\{\alpha Y_t+K_{t+1}/(N-1)\}/N.$ Hence, the total transfer is strictly positive. When N is large, the total transfer is close to capital income per generation, $\alpha Y_t/N.$ That transfer compensates the newborn generation for her lack of capital income and allows her to consume and invest the same amount as older generations.

More general analysis of wealth and transfers

I now provide a more general analysis of budget constraints, transfers and wealth. Taking account of state-contingent bonds, the date *t* budget constraint of generation *i*=1,..,N is:

$$\int p_t(S_{t+1})F_{i+1,t+1}(S_{t+1})dS_{t+1}+K_{i+1,t+1}+c_{i,t}=F_{i,t}+K_{i,t}r_{K,t}+W_t-T_{i,t}, \qquad \text{(A.5)}$$

where $p_t(S_{t+1})$ is the date *t* price of a bond that pays one units of output in period *t*+1 if and only if state of the world S_{t+1} is realized at date *t*+1. $F_{i+1,t+1}(S_{t+1})$ is the number of claims of this type held by generation *i*, at the end of period *t*. [30] For *i*>1, $T_{i,t}$ is the lump sum transfer made by generation *i* to the newborn generation (*i*=1). $-T_{1,t}=\sum_{i=2}^{N}T_{i,t}$ is the total transfer received by generation 1.

Newborn agents have zero income from capital and state-contingent bonds, as $K_{0,t}=F_{0,t}=0.$ Agents in the last period of their life cannot issue claims that oblige them to make

[29] Note that there are N generational budget constraints at date *t*. However, only N-1 of the budget constraints are linearly independent, when the aggregate resource constraint $K_{t+1}+C_t=Y_t$ holds.

[30] $c_{i,t},K_{i+1,t+1},T_{i,t},r_{K,t},W_t$ and $F_{i,t}$ depend on the state of the world S_t. To simplify notation, I suppress the dependence of choice variables on the state of the world, unless confusion arises.

future payments, and it is not in their interest to acquire claims to future payments or to invest in physical capital. Thus, $K_{N+1,t+1}=0$ and $F_{N+1,t+1}(S_{t+1})=0\ \ \forall S_{t+1}$.

Generations i=1,..,N-1 equate their probability-weighted intertemporal marginal rate of substitution to the prices of state-contingent claims: $p_t(S_{t+1})=\pi_t(S_{t+1})\beta u'(c_{i+1,t+1}(S_{t+1}))/u'(c_{i,t})$ $\forall S_{t+1}$ where $\pi_t(S_{t+1})$ is the conditional probability (density) of state S_{t+1}, given date t information. Under the assumed log utility, $u(c)=\ln(c)$, this first-order condition can be written as $p_t(S_{t+1})=\pi_t(S_{t+1})\beta c_{i,t}/c_{i+1,t+1}(S_{t+1})$ for i=1,..,N-1. Financial market completeness implies thus the risk-sharing condition (A.1) among contemporaneous generations.

Note that the price of state-contingent assets can be expressed as: $p_t(S_{t+1})=\pi_t(S_{t+1})\rho_{t,t+1}$, which allows to write the budget constraint (A.5) as $E_t\rho_{t,t+1}F_{i+1,t+1}+K_{i+1,t+1}+c_{i,t}=F_{i,t}+K_{i,t}r_{K,t}+W_t-T_{i,t}$. Let $H_{i,t}=W_t+E_t\rho_{t,t+1}H_{i+1,t+1}$ and $Q_{i,t}=-T_{i,t}+E_t\rho_{t,t+1}Q_{i+1,t+1}$ denote the present value of generation i's wage income and of her transfer income, respectively, with $H_{i+1,t+1}=Q_{i+1,t+1}=0$ for i≥N. $H_{i,t}$ and $Q_{i,t}$ are generation i's human capital and her 'transfer capital', respectively. Using these definitions, we can write generation i's budget constraint as

$$E_t\rho_{t,t+1}A_{i+1,t+1}+c_{i,t}=A_{i,t} \text{ with } A_{i,t}\equiv F_{i,t}+H_{i,t}+Q_{i,t}+K_{i,t}r_{K,t}\,. \tag{A.6}$$

$A_{i,t}$ is generation i's total wealth at the beginning of period t. Agents hold zero assets at the end of their life, and thus generation N consumes her beginning-of-period assets: $c_{N,t}=A_{N,t}$.

Generation i=1,..,N thus faces the present value budget constraint $A_{i,t}=E_t\sum_{s=0}^{N-i}\rho_{t,t+s}c_{i+s,t+s}$ where $\rho_{t,t}\equiv 1$ while the stochastic discount factor $\rho_{t,t+s}$ is a product of the one-period-ahead stochastic discount factors defined above: $\rho_{t,t+s}\equiv\Pi_{\tau=0}^{\tau=s-1}\rho_{t+\tau,t+\tau+1}$ for s≥1. Given efficient risk sharing across contemporaneous generations (see (A.1)), $\rho_{t,t+s}=\beta^s c_{i,t}/c_{i+s,t+s}$ holds for $0<s\le N\text{-}i$. Thus $A_{i,t}=c_{i,t}\sum_{s=0}^{N-i}\beta^s$ and hence

$$c_{i,t}=\phi_i\cdot A_{i,t},\ \text{with}\quad \phi_i\equiv(1-\beta)/(1-\beta^{N-i+1})\quad \text{for i=1,..,N.}$$

Each period, generation i consumes thus a fraction ϕ_i of her wealth. ϕ_i is generation-specific, but time invariant. In an equilibrium with time-invariant generational consumption shares, the period t wealth of generation i equals thus $A_{i,t}=(\lambda_i/\phi_i)C_t$. The wealth share of generation i is:

$$A_{i,t}/\sum_{s=1}^{N} A_{s,t} = (\lambda_i/\phi_i)/\sum_{s=1}^{N}(\lambda_s/\phi_s) \equiv \kappa_i \,. \tag{A.7}$$

Note that this wealth share is time-invariant. Thus, an equilibrium with time-invariant generational consumption shares exhibits time-invariant generational wealth shares. As pointed out above, the consumption share of newborn generations, λ_1, pins down the (steady state) consumption shares of older generations, i.e. λ_i is a function of λ_1: $\lambda_i=\Lambda_i(\lambda_1)$ for i=2,..,N. There is, hence, a unique mapping from λ_1 to κ_i, the wealth share of generation i:

$$\kappa_i = (\Lambda_i(\lambda_1)/\phi_i)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s) \,.$$

If the new-born generation is allocated a wealth share $\kappa_1 = (\lambda_1/\phi_1)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)$, then this sustains an equilibrium in which the consumption share of the new-born generation is λ_1. A consumption allocation in which all generations have an identical consumption share $\lambda_i=1/N$ is sustained by allocating to the newborn generation a wealth share $\kappa_1=(1/\phi_1)/\sum_{s=1}^{N}1/\phi_s$. As an example, assume that life lasts 80 years, i.e. N=320 quarters, and that the quarterly subjective discount factor is β=0.99; then the consumption allocation with equal consumption shares $\lambda_i=1/N=0.3125\%$ requires a newborn wealth share of $\kappa_1=0.4267\%$.

The newborn generation holds zero initial bonds and zero initial capital. Thus, the initial wealth of the newborn generation is the sum of her human capital and of her 'transfer capital' (see (A.6)): $A_{1,t}\equiv H_{1,t}+Q_{1,t}$. Because $c_{1,t} = \phi_1 A_{1,t}$, a time-invariant newborn consumption share λ_1 obtains when newborn wealth equals $A_{1,t}=(\lambda_1/\phi_1)C_t$. Thus, a time-invariant newborn consumption share λ_1 obtains if the present value of the newborn generation's transfer income is $Q_{1,t}=(\lambda_1/\phi_1)C_t-H_{1,t}$.

● Appendix B (Not for Publication)

Rational bubble equilibria in the closed economy RBC model with incomplete capital depreciation (Sect. 3)

This Appendix provided further discussions of the closed economy RBC model with incomplete capital depreciation (Sect. 3), and it also explains the numerical solution method.

Bubble equilibrium

A rational bubble equilibrium is a process for capital $\{K_t\}$ that satisfies Euler equation (15) *and* that deviates from the no-bubble decision rule $K_{t+1}=\lambda(K_t,\theta_t)$. A rational bubble violates the TVC. By analogy to the bubble process in the Long-Plosser economy without TVC (see Sect. 2.2), I consider bubble equilibria in which the capital stock K_{t+1} takes one of two values: $K_{t+1}\in\{K^L_{t+1},K^H_{t+1}\}$ with exogenous probabilities π and 1-π, respectively ($0<\pi<1$), where $K^L_{t+1}=\lambda(K_t,\theta_t)e^{\Delta}$, for a small constant Δ. With probability π, the capital stock thus takes a value close to the no-bubble decision rule (as in the bubbly Long-Plosser model). An exogenous i.i.d. sunspot (that is assumed independent of TFP) determines whether K^L_{t+1} or K^H_{t+1} is realized at *t*.

At date *t*, agents anticipate that the capital stock set in *t*+1, K_{t+2}, likewise takes one of two values: $K_{t+2}\in\{K^L_{t+2},K^H_{t+2}\}$ with probabilities π and 1-π, respectively, where $K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$. The date *t* Euler equation (15) can thus be written as:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)+(1-\pi)\cdot E_t H(K^H_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)=1 \text{ for } K_{t+1}\in\{K^L_{t+1},K^H_{t+1}\}. \quad (16)$$

$K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$ depends on θ_{t+1}. The numerical simulations consider bubble equilibria in which, conditional on date *t* information, a TFP innovation at *t*+1 has an equiproportional effect on K^L_{t+2} and K^H_{t+2}. Specifically, I postulate that $K^H_{t+2}=s^H_t\cdot K^L_{t+2}$, where

$s_t^H>0$ is in the date *t* information set. Thus, $K_{t+2}^H = s_t^H \cdot \lambda(K_{t+1}, \theta_{t+1})e^{\Delta}$.[31] This greatly simplifies the computation of bubbles. Substituting the formula for K_{t+2}^H into the Euler equation (16) gives:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) + (1-\pi)\cdot E_t H(s_t^H \cdot \lambda(K_{t+1},\theta_{t+1})e^{\Delta}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1. \quad \text{(B.1)}$$

Solving for a bubble equilibrium requires solving the Euler equation for the scalar s_t^H. The Euler equation (B.1) implies that s_t^H is a function of K_{t+1}, K_t, θ_t: $s_t^H \equiv s^H(K_{t+1}, K_t, \theta_t)$. Solving for s_t^H pins down the equilibrium capital process. Given the equilibrium capital process, consumption, hours and output can be determined using (14).

I set $\Delta>0$, because a strictly positive Δ is needed to generate *recurrent* bubbles. As in the Long-Plosser economy (without TVC), bubble are self-ending when $\Delta=0$; by contrast, $\Delta<0$ implies that the capital stock ultimately reaches zero.[32]

Consider an economy that starts in period *t*=0, with an exogenous initial capital stock K_0. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and $1-\pi$, respectively ($0<\pi<1$). Assume that the sunspot is independent of TFP. Then the following process for capital $\{K_t\}_{t\geq 0}$ is a recurrent rational bubble: $K_{t+2} = K_{t+2}^L \equiv \lambda(K_{t+1}, \theta_t)e^{\Delta}$ if $u_{t+1}=0$ and $K_{t+2} = K_{t+2}^H$ if $u_{t+1}=1$, for $t\geq 0$, where K_{t+2}^H satisfies the date *t* Euler equation.

K_1 (the capital stock set at *t*=0) does not obey the recursion that governs the capital stock in subsequent periods. K_1 is indeterminate. In the numerical simulations below, I assume that agents choose $K_1 = \lambda(K_0, \theta_0)e^{\Delta}$. (The effect of K_0 and K_1 on endogenous variables in later periods vanishes as time progresses.)

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[31] The AR(1) specification of TFP implies $\theta_{t+1} = (\theta_t)^{\rho} \cdot \exp(\varepsilon_{t+1}^{\theta})$, where $\varepsilon_{t+1}^{\theta}$ is the TFP innovation at t+1. The chosen specification of K_{t+2}^L, K_{t+2}^H implies that $\partial \ln(K_{t+2}^H)/\partial \varepsilon_{t+1}^{\theta} = \partial \ln(K_{t+2}^L)/\partial \varepsilon_{t+1}^{\theta}$; thus, an unexpected change in date t+1 productivity affects K_{t+2}^H and K_{t+2}^L by the same (relative) amount.

[32] Consider the dynamics that obtains when $\Delta=0$. Assume a sunspot realization $u_t=0$, so that (with $\Delta=0$) $K_{t+1} = K_{t+1}^L \equiv \lambda(K_t, \theta_t)$. Then Euler equation (B.1) is solved by $s_t^H=1$, so that $K_{t+2}^H = \lambda(K_{t+1}, \theta_{t+1})$. This follows from the fact that $E_t H(\lambda(\lambda(K_t,\theta_t),\theta_{t+1}), \lambda(K_t,\theta_t), K_t) = 1$ (Schmitt-Grohé and Uribe (2004), eqn. (4)). Thus $K_{t+2} = K_{t+2}^H = K_{t+2}^L = \lambda(K_{t+1}, \theta_{t+1})$ and $K_{t+s+1} = \lambda(K_{t+s}, \theta_{t+s})$ also holds $\forall\, s>1$. In all periods after a sunspot realization $u_t=0$, the dynamics of the capital stocks is hence governed by the no-bubble decision rule, i.e. the bubble has ended.

What expectations sustain the rational bubble equilibrium?

As in the bubbly Long-Plosser economy (see Sect. 2), the dynamics of capital reflects self-fulfilling variations in agents' expectations about *future* capital. In a bubble equilibrium, the capital stock evolves in the following sequence:

At date t=0, agents select the capital stock $K_1=\lambda(K_0,\theta_0)e^{\Delta}$ (by assumption; see above). They expect (at t=0) that the capital stock K_2 (chosen at date t=1) will equal $K_2^L=\lambda(K_1,\theta_1)e^{\Delta}$ or $K_2^H=s^H(K_1,K_0,\theta_0)\cdot\lambda(K_1,\theta_1)e^{\Delta}$, with probabilities π and 1-π, respectively. The indicated value of K_2^H solves the date t=0 Euler equation (by construction). Thus, the stated date t=0 expectations (about K_2) sustain the chosen capital stock K_1.

At t=1, agents select the values of the capital stock K_2^L (if u_1=0) or K_2^H (if u_1=1) that were just stated. That choice is driven by agents' expectations (at t=1) about K_3, the capital stock selected *next* period (t=2). When the sunspot is $u_1=0$, then agents expect that K_3 will equal $K_3^L=\lambda(K_2^L,\theta_2)e^{\Delta}$ or $K_3^H=s^H(K_2^L,K_1,\theta_1)\cdot\lambda(K_2^L,\theta_2)e^{\Delta}$, with probabilities π and 1-π, respectively; given these expectations, a choice K_2^L is consistent with the date t=1 Euler equation for u_1=0. When the u_1=1 is realized, a choice K_2^H is sustained by agents' expectation that K_3 will equal $K_3^L=\lambda(K_2^H,\theta_2)e^{\Delta}$ or $K_3^H=s^H(K_2^H,K_1,\theta_1)\cdot\lambda(K_2^H,\theta_2)e^{\Delta}$, with probabilities π and 1-π; given these expectations, a choice K_2^H is consistent with the date t=1 Euler equation for u_1=1.

The same process is repeated in all subsequent periods.

Computational aspects

I) Solving for consumption and labor hours using the static equations

The static equations can be used to solve for consumption and labor hours as functions of capital and TFP (see (14) in Main text). Note that the labor supply equation (12) can be written as

$$C_t=[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t). \tag{B.3}$$

The date t resource constraint of the economy is $C_t+K_{t+1}=Y_t+(1-\delta)K_t$, where $Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}$.

Substituting (B.3) into the resource constraint gives:

$$[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t)=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}+(1-\delta)K_t-K_{t+1}.$$

Equivalently: $1=A_{1,t}\cdot(L_t)^{\alpha}+A_2\cdot L_t$, with $A_{1,t}\equiv-[K_{t+1}-(1-\delta)K_t]/\{[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}\}$, $A_2\equiv[1+\Psi/(1-\alpha)]$.

For the assumed capital elasticity of output α=1/3, this (cubic) equation has a unique closed form solution for date t hours worked L_t as a function of K_{t+1},K_t,θ_t. Substitution of the formula for hours into (B.3) gives a closed form formula for consumption C_t (see (14)).

II) Euler equation

TFP is assumed to follow the AR(1) process $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon^{\theta}_{t+1}$, $0\le\rho<1$, where $\varepsilon^{\theta}_{t+1}$ is a discrete innovation that equals $\varepsilon^{\theta}_{t+1}=-\sigma_\theta$ or $\varepsilon^{\theta}_{t+1}=\sigma_\theta$ with probability 1/2, respectively, where $\sigma_\theta\ge0$. θ_{t+1} thus equals $\theta_{t+1}=(\theta_t)^{\rho}e^{\sigma_\theta}$ or $\theta_{t+1}=(\theta_t)^{\rho}e^{-\sigma_\theta}$ with probability 1/2. The Euler equation (B.2) can, thus, be written as:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\&(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned}\tag{B.4}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}=\lambda(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}K^L_{t+1}$.

In the numerical simulations, I approximate the no-bubble decision rule λ using a second-order (log-quadratic) Taylor expansion. Let $\hat{\lambda}(K_t,\theta_t)$ be the second-order Taylor expansion of the no-bubble decision rule λ. In the numerical simulations, I thus define K^L_{t+1} as $K^L_{t+1}\equiv\hat{\lambda}(K_t,\theta_t)$ $\forall t$. The simulations are hence based on a version of Euler equation (B.4) in which λ is replaced by $\hat{\lambda}$:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\&(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}=\hat{\lambda}(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}K^L_{t+1}$.

Conditional on K_t,K_{t+1},θ_t, this equation can be used to determine s^H_t. I employ a bisection method for that purpose.

Like the value of the bust probability π, the specification of the bust capital stock K^L is not tied down by economic theory. The only restriction is that the resulting law of motion for capital has to be bounded and strictly positive. I verified that the bubble equilibrium constructed using $\hat{\lambda}$ meets this criterion. For model variants with constant TFP, I also computed the no-bubble decision rule $K_{t+1}=\lambda(K_t,\theta)$ using a shooting algorithm (Judd (1998), ch.10). The second-order approximation and the shooting algorithm give no-bubble decision rules that are very close, even when capital K_t is far from the steady state. The resulting bubble equilibria too are very similar. Computing $\hat{\lambda}$ is much faster.

● Appendix C (Not for Publication)

Rational bubble equilibria in the two-country economy RBC model with incomplete capital depreciation (Sect. 5)

This Appendix provides further discussion of the two-country RBC model with incomplete capital depreciation (Sect. 5). The construction of rational bubbles parallels that in the closed economy RBC model with incomplete capital depreciation (Sect. 3).

The static model equations allow to solve for date *t* consumption, hours worked and terms of trade $C_{i,t}, L_{i,t}, q_{i,t}$ as functions of both countries' capital stocks in *t* and *t*+1 and of date *t* productivity. By substituting these functions into the two countries' capital Euler equations, one can write these Euler equations as expectational difference equations in Home and Foreign capital:

$$E_t H_i(\overrightarrow{K_{t+2}}, \overrightarrow{K_{t+1}}, \overrightarrow{K_t}, \overrightarrow{\theta_{t+1}}, \overrightarrow{\theta_t}) = 1 \quad \text{for } i=H,F, \tag{29}$$

where $\overrightarrow{K_t} \equiv (K_{H,t}, K_{F,t})$ and $\overrightarrow{\theta_t} \equiv (\theta_{H,t}, \theta_{F,t})$ are vectors of Home and Foreign capital and TFP, respectively. The function H_i maps R_+^{10} into R.

The no-bubble solution of the model (that obtains when transversality conditions are imposed) is described by decision rules $K_{i,t+1}=\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})$ for *i*=H,F. Let $\overrightarrow{K_{t+1}}=\vec{\lambda}(\overrightarrow{K_t},\overrightarrow{\theta_t})$ be the no-bubble decision rule for the *vector* of Home and Foreign capital at t+1 ($\vec{\lambda}$ maps R_+^4 into R_+^2).

Assume that there is no transversality condition (TVC) for capital, which makes rational bubbles possible. I consider a bubble process that parallels the bubbles in previous Sections. Assume that capital $K_{i,t+1}$ takes one of two values: $K_{i,t+1} \in \{K^L_{i,t+1}, K^H_{i,t+1}\}$, with probabilities π and 1-π, respectively, where $K^L_{i,t+1} = \lambda_i(\overrightarrow{K_t}, \overrightarrow{\theta_t}) \cdot e^{\Delta}$, with $\Delta > 0$. Like in previous models, $\Delta > 0$ is required to generate *recurrent* bubbles. An exogenous i.i.d. sunspot (that is assumed independent of TFP) determines whether $K^L_{i,t+1}$ or $K^H_{i,t+1}$ is realized (see below). Numerical experiments show that, as in the bubbly two-country Dellas model (Sect. 4), the bubble has to be perfectly synchronized across countries (bubbles that are not synchronized ultimately hit the zero capital corner). Thus, $K^L_{H,t+1}$ and $K^L_{F,t+1}$ must be realized together (and the same must be true of $K^H_{H,t+1}$ and $K^H_{F,t+1}$). (Note that the superscripts 'L' and 'H' refer to the state of the bubble, while the subscripts 'H' (Home) and 'F' (Foreign) refer to the country.)

Consider a world economy that starts at date t=0, with exogenous initial Home and Foreign capital stocks $\overrightarrow{K_0} = (K_{H,0}, K_{F,0})$. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and 1-π, respectively ($0 < \pi < 1$). Then the following process for Home and Foreign capital $\{K_{H,t}, K_{F,t}\}_{t \geq 0}$ is a recurrent rational bubble:

(a) $K_{i,t+2} = K^L_{i,t+2} \equiv \lambda_i(\overrightarrow{K_{t+1}}, \overrightarrow{\theta_{t+1}}) \cdot e^{\Delta}$ for $i=H,F$ if $u_{t+1} = 0$, for $t \geq 0$;

(b) $K_{i,t+2} = K^H_{i,t+2}$ for $i=H,F$, if $u_{t+1} = 1$, for $t \geq 0$, where $K^H_{H,t+2}, K^H_{F,t+2}$ satisfy date t Euler equations (29).

The capital stocks set in period 0, $K_{H,1}, K_{F,1}$, do not obey the recursion that governs the capital stocks in subsequent periods. Thus, $K_{i,1}$ (i=H,F) is indeterminate. In the numerical simulations, I set $K_{i,1} = \lambda_i(\overrightarrow{K_0}, \overrightarrow{\theta_0}) \cdot e^{\Delta}$ for i=H,F.

Following the specification of the closed economy RBC model in Sect. 3, I focus on equilibria in which, conditional on date t information, productivity innovations at t+1 have equiproportional effects on $K^H_{i,t+2}$ and $K^L_{i,t+2}$. Thus: $K^H_{i,t+2} = s^H_{i,t} \cdot K^L_{i,t+2}$ for $i=H,F$, where $s^H_{i,t} > 0$ is in the date t information set. This assumption greatly simplifies the computation of bubble equilibria. Using the formulae for $K^L_{i,t+2}$ and $K^H_{i,t+2}$, the date t Euler equation (29) can be expressed as:

$$\pi E_t H_i((\lambda_H(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},\lambda_F(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta}),\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})+$$

$$(1-\pi)E_t H_i((s^H_{H,t}\cdot\lambda_H(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},s^H_{F,t}\cdot\lambda_F(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta}),\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})=1 \quad \text{for } i=H,F. \qquad \text{(C.1)}$$

Given $K_{H,t+1}, K_{F,t+1}$, the date t Euler equations of both countries only feature two unknown endogenous variables in period t: $s^H_{H,t}$ and $s^H_{F,t}$. Computing a bubble equilibrium requires solving the Home and Foreign Euler equations (C.1) for $s^H_{H,t}, s^H_{F,t}$, at each date t.

The TFP innovations are assumed to have a discrete distribution (see (28)). This makes it easy to compute the conditional expectations appearing in the Euler equation. In the numerical simulations, I approximate the no-bubble decision rule λ using a second-order (log-quadratic) Taylor expansion (the same approach was used to solve the model of Sect. 3).