Fiscal Policy and Debt Management with Incomplete Markets*

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Abstract

A Ramsey planner chooses a distorting tax on labor and manages a portfolio of securities in an environment with incomplete markets. We develop a method that uses second order approximations of the policy functions to the planner's Bellman equation to obtain expressions for the unconditional and conditional moments of debt and taxes in closed form such as the mean and variance of the invariant distribution as well as the speed of mean reversion. Using this, we establish that asymptotically the planner's portfolio minimizes an appropriately defined measure of fiscal risk. Our analytic expressions that approximate moments of the invariant distribution can be readily applied to data recording the primary government deficit, aggregate consumption, and returns on traded securities. Applying our theory to U.S. data, we find that an optimal target debt level is negative but close to zero, that the invariant distribution of debt is very dispersed, and that mean reversion is slow.

KEY WORDS: Distorting taxes. Spanning. Transfers. Optimal Portfolio. Government debt.

1 Introduction

This paper analyzes a problem of a Ramsey planner who optimally manages a portfolio of debts and other securities to smooth fluctuations in tax distortions in an incomplete markets economy hit by aggregate shocks. Within a production economy without capital, the government raises revenue by issuing securities and imposing linear taxes on labor income, then spends on exogenous government expenditures, payouts on government securities, and transfers. The government and private agents trade a specified set of risky securities whose returns depend on the aggregate state. Useful benchmarks like complete markets and one-period risk-free bonds are special cases.

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We construct a type of second-order approximations of policy functions to the planner's Bellman equation around small aggregate shocks. These approximations allow us to capture the optimal response to uncertainty at any given level of debt and under certain conditions can be expressed as linear functions of aggregate shocks. This yields a very tractable formulation of the planner's optimal policies. We confirm that these quadratic approximations are accurate by comparing them with reliable global approximations. An advantage of our quadratic approximations is that they enable analytic and interpretable expressions for the means, variances, and rates of convergence to the invariant distribution of debt, tax revenues, and tax rates.[1] Empirical counterparts to our expressions for these objects can be easily constructed using data on the primary government deficit, aggregate consumption, and returns on securities traded by the government. We show that asymptotically the government's optimal debt portfolio minimizes an appropriately defined risk criterion.

To isolate principles that give rise to key findings, we start with a simple baseline setting in which agents have quasilinear preferences and the market structure is restricted to a single security whose payout we allow to be correlated with the government consumption process. The joint distribution of returns and government expenditure is i.i.d. over time. Using properties of the government's Euler equation, we establish existence of an invariant distribution of government debt. Up to third-order terms, we show that the drift in the dynamics of debt is proportional to the covariance of total government spending (debt service plus exogenous government expenditures) and returns. A level of debt that minimizes the variance in total spending sets this covariance to zero; this level of debt becomes a point of attraction for the stochastic process for debt. The speed of mean reversion is inversely proportional to the variance of the return on the security, and the variance of the invariant distribution is proportional to the amount of risk that the government bears at its risk-minimizing debt level. Later sections of the paper show the tendency of debt to approach a level that minimizes the risk in the government's portfolio extends well beyond our baseline case.

Extending the analysis to multiple assets yields several insights. If returns satisfy a spanning condition, the planner can replicate a complete markets allocation like the one that prevails in Lucas and Stokey (1983). Allowing trade in more securities decreases the speed of convergence to the invariant distribution. The reason for this outcome is that additional securities facilitate better hedging and thereby lower the cost of being away from the long-run target level of debt. By specializing the portfolio problem to two securities, a console and short term security, we can derive prescriptions for optimal maturity management. In this two-security case, the riskiness of the return on the short-maturity asset relative to that on the long-lived console drives the average maturity of the total debt. In particular, if the return on the long maturity bond is riskier than the return on the short-maturity bill, then the optimal maturity of the planner's portfolio is inversely proportional to total public debt and most adjustment to aggregate shocks is done with the short bill. We also extend the analysis to incorporate risk aversion and more general shock processes. We show that insights from the baseline model continue to apply provided that we use concepts of "effective returns" and "effective shocks" – returns on the government debt portfolio and innovations in the present discounted value of the primary government deficit

[1] We can also use our quadratic approximation to get analytic expressions for other moments.

adjusted by marginal utilities of consumption, respectively.

In the quantitative part of the paper, we focus on two goals: (i) to compare the quality of our two numerical approximations to the Ramsey plan, one using a more accurate global method, the other using formulas derived from our quadratic approximation; and (ii) to study the predictions of the model for realistic shock and return processes. To this end, we use plausible quantities drawn from U.S. data to calibrate shock and return processes . Our analytical expressions derived under simpler environments continue to be accurate in the calibrated version of the model. We find that the optimal level of debt is close to zero and that the optimal policy displays an extremely slow rate of mean reversion (half-life of 250 years). These results are driven by the fact that a significant amount of variation in returns to the U.S. portfolio is uncorrelated with output; that means that holding large quantities of debt/assets would frustrate hedging objectives.

Our paper obviously shuts down important forces emphasized in other theories of optimal levels of government debt. For example, the literature on sovereign debt, by allowing a government each period to choose whether or not to service its debt, focuses attention on how the adverse consequences of default endogenously generates incentives to repay debt obligations. The government in our model has no such options and requires no such incentives. This eliminates the design of incentives to induce payment as determinants of government debt and its maturity composition and puts the hedging considerations on which we focus front and center. Our model describes optimal fiscal policy of a government that never contemplates dishonoring its debts. (We like to think of the U.S. and some European governments as being in this situation.) Additionally, we focus on real debt and ignore nominal issues. Extending our approach to economies with possibilities of default and monetary economies is straightforward but space-consuming as it would require us to introduce several layers of additional complications to our model. We leave that for future work.

1.1 Relationships to literatures

Our paper builds on a large literature about a Ramsey planner who chooses a competitive equilibrium with distorting taxes once-and-for-all at time zero.[2] Most of these papers use either complete or approximately complete markets as in Lucas and Stokey (1983), Buera and Nicolini (2004), Angeletos (2002), or study a singe-period risk-free bond and quasilinear preferences as in Aiyagari et al. (2002) and Barro (1979). In contrast, our analysis allows for a more general incomplete market structure and risk-aversion. In both complete market economies, as in Lucas and Stokey (1983), and quasilinear settings with a risk-free bond only, as in Barro (1979), *any* level of debt is optimal in the sense that the Ramsey planner sets expected public debt level in any future period equal to initial debt. We show that this result is non-generic; small departures from those assumptions imply that for any initial debt, driven by hedging considerations, government debt can be expected to converge to the unique risk-minimizing level.

In a related context, Barro (1999) and Barro (2003) study tax smoothing in an environment in which revenue needs are deterministic but refinancing opportunities are stochastic. In Barro's setting it is optimal for the government to issue an infinitely lived console as a way to insulate

[2]For instance Lucas and Stokey (1983), Aiyagari et al. (2002), Chari et al. (1994), Farhi (2010).

inter-temporal tax smoothing motives from concerns about rolling over short maturity debt at uncertain prices. In contrast, our analysis allows both revenue needs and returns on the debt to be stochastic. We fit empirically relevant properties of returns on debt and then find an optimal government portfolio associated with those returns.

Technically, our paper is closely related to Aiyagari et al. (2002). Those authors include an analysis of an economy in which a representative agent has quasilinear preferences. In addition to a linear labor tax, they allow a uniform positive (but not negative) lump sum transfer. There is a continuum of invariant distributions for debt, all of which feature a zero labor tax rate and debt levels that are negative and sufficiently large in absolute value to finance all government expenditures from the government's interest revenues, with transfers absorbing all aggregate fluctuations by adjusting one-to-one with the aggregate shock. A key difference in our paper is how we treat transfers. While Aiyagari et al. (2002) exogenously restrict transfers, we deduce transfers from an explicit redistribution motive by modeling agents who cannot afford to pay positive lump sum taxes. We show that so long as the utility of such agents is strictly concave and the planner puts positive weight on their well-being, the result of Aiyagari et al. (2002) about properties of the invariant distributions goes away. A benevolent planner in such settings wants to minimize fluctuations of both tax rates *and* transfers; he ultimately targets a (generally unique) level of debt that minimizes risk. The invariant distributions studied by Aiyagari et al. (2002) emerge only in the limit as the risk-aversion of all recipients of transfers goes to zero.

The equilibrium approximation tools that we apply in this paper are complementary to Faraglia et al. (2012), Lustig et al. (2008), and Siu (2004) who numerically study optimal Ramsey plans in specific incomplete markets settings. Our approximation method allows us to derive closed form expressions for the invariant distribution of debt and taxes that enlightens the analysis of the underlying forces.

Our theory of government portfolio management shares important features of the single-investor optimal portfolio theory of Markowitz (1952) and Merton (1969). Bohn (1990) and Lucas and Zeldes (2009) use some insights from the single-investor literature to study portfolio choices of a government in a partial equilibrium after having specified a loss function for the government. Common to both Merton's investor and our Ramsey planner are hedging motives that shape portfolio and savings choices. In contrast, our analysis allows both revenue needs and returns on the debt to be stochastic. However, unlike the Merton investor, our Ramsey planner is benevolent (i.e., it maximizes the utility of the agents with whom it trades) and it internalizes the general equilibrium effects of his distorting tax rate choices on equilibrium asset prices. As a consequence, the optimal portfolio does not feature the usual mean-variance trade-off but instead attempts to minimize the appropriately defined measure of risk.

The remainder of this paper is organized as follows. In section 2, we analyze a streamlined setting in which only one risky security can be traded and the representative agent has quasilinear preferences. In sections 3 and 4 we extend the analysis to include multiple assets, persistent shocks, concerns for redistribution, and risk aversion. In section 5, we study a quantitative example with parameters calibrated to U.S. data.

2 Quasilinear preferences

We begin with a streamlined setting with a representative household having quasilinear preferences. Each of a measure one of identical agents has preferences over consumption and labor supply streams $\{c_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$ that are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(c_t - \frac{1}{1+\gamma} l_t^{1+\gamma} \right), \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is a time discount factor. One unit of labor produces one unit a nonstorable single good that can be consumed by the household or the government. Feasibility requires

$$c_t + g_t = l_t, \quad t \geq 0 \tag{2}$$

where g_t denotes government consumption.

The government imposes a flat tax at rate τ_t on labor earnings and buys or sells a single one-period security having an exogenous state-contingent payoff p_t. Consumers sell or buy the same security, so it is in zero net supply each period. Let B_t be the number of securities that the the government sells in period t at price q_t. Government budget constraints are

$$g_t + p_t \mathrm{B}_{t-1} = \tau_t l_t + q_t \mathrm{B}_t, \quad t \geq 0 \tag{3}$$

A probability measure $\pi(ds)$ over a compact set $\mathcal{S}$ governs an exogenous i.i.d. shock s_t. Government purchases and payoffs on the single security are random variables g, p represented by bounded functions that map $\mathcal{S} \to \mathbb{R}_+$. We order elements in $\mathcal{S}$ using the function $p(\cdot)$, i.e., $s \geq s'$ if $p(s) \geq p(s')$ and, as such, we will use the phrase "p is increasing in s". Let $\bar{g} = \mathbb{E}g \equiv \int_{s \in \mathcal{S}} \pi(ds) g(s)$ and $\bar{p} = \mathbb{E}p$. We use $s^t = (s_0, ..., s_t)$ to denote a history of shocks. We often use z_t to denote a random variable z with a time t conditional distribution that is a function of the history s^t. It is convenient to define $B_t \equiv q_t \mathrm{B}_t$ and $R_{t+1} \equiv p_{t+1}/q_t$ and to re-write the government's budget constraint (3) as

$$g_t + R_t B_{t-1} = \tau_t l_t + B_t.$$

A representative agent's budget constraint is

$$c_t + b_t = (1 - \tau_t)\, l_t + R_t b_{t-1}, \tag{4}$$

where b_t is the representative agent's purchase of the single security. The period t market clearing condition for the security is

$$b_t = B_t. \tag{5}$$

We exogenously confine government debt to a compact set

$$B_t \in \left[\underline{B}, \overline{B}\right]. \tag{6}$$

The assumption of compactness of the feasible debt simplifies the analysis. We make the bounds sufficiently large that they do not affect the properties of the joint invariant distributions of government debt and the tax rate that we analyze below.

Definition 1. A *competitive equilibrium* given an initial government debt B_{-1} at $t = 0$ is a sequence $\{c_t, l_t, B_t, b_t, R_t, \tau_t\}_{t=0}^{\infty}$ such that (i) $\{c_t, l_t, b_t\}_{t=0}^{\infty}$ maximize (1) subject to the budget constraint (3); and (ii) constraints (2), (5), and (6) are satisfied. An *optimal competitive equilibrium* given B_{-1} is a competitive equilibrium that has the highest value of (1).

Single-security incomplete market models such as Barro (1979) and Aiyagari et al. (2002) assume that the security's payout is risk-free, a special case of our setup in which $p(s)$ is independent of s. Two purposes induce us to allow stochastic payoffs. First, it is well known that standard real business cycle models driven by productivity (and/or expenditure shocks) fail to generate realistic holding period returns. Our more flexible choice for security markets allows us to remedy this shortcoming in parsimonious way. To offer an example for just one place and time, in section 5 we document how the real returns to U.S. government portfolios fluctuate and use those findings to discipline payoffs. Second, as we show in section 4, the optimal process for risk-free government debt when the representative consumer is risk-averse in consumption resembles the optimal behavior of state-contingent debt when the represent agent is risk-neutral. With risk-aversion and risk-free debt, a key object is an "effective" return on debt that takes into account a shadow cost of raising revenues; this influential return is stochastic even when the single security is risk-free. In this section, we identify the joint distribution of returns on government debt and other variables as a key determinant of a long-run target government debt level and also of the speed at which a government should approach it. We will use a notion of an effective return on debt to extend that insight to more general settings in section 4.

The representative consumer's first-order necessary conditions imply that

$$1 - \tau_t = l_t^{\gamma},\ \mathbb{E}_{t-1} R_t = \frac{1}{\beta}. \tag{7}$$

The security price q_t satisfies $q_t = \beta \bar{p}$ so the return on the security $R_t\left(s^t\right) = \frac{p(s_t)}{\beta \bar{p}}$. Since p is increasing in s_t, so is R.

Substitute (7) into the consumer's budget constraint to obtain

$$c_t = l_t^{1+\gamma} + R_t B_{t-1} - B_t. \tag{8}$$

Use (8) to eliminate c_t from the feasibility condition (2) to obtain

$$\left(l_t - l_t^{1+\gamma}\right) + B_t = R_t B_{t-1} + g_t. \tag{9}$$

Lemma 1. $\{c_t, l_t, B_t, b_t, R_t, \tau_t\}_{t=0}^{\infty}$ *is a competitive equilibrium given* B_{-1} *if and only if* $\{l_t, B_{t-1}\}_{t=0}^{\infty}$ *satisfies* (6) *and* (9).

The lemma allows us to obtain the optimal competitive equilibrium allocation and debt process by solving

$$\max_{\{l_t,B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(R_t B_{t-1} - B_t) + \frac{\gamma}{1+\gamma} l_t^{1+\gamma} \right], \tag{10}$$

where the maximization is subject to constraints (6) and (9). The objective function in (10) is a version of (1) in which we used (8) to eliminate c_t.

For our purposes, instead of the labor supply l_t, it is convenient to work with tax revenues $Z_t = \tau_t l_t$:

$$Z_t = Z(l_t) \equiv l_t - l_t^{1+\gamma}. \tag{11}$$

Tax revenues are bounded from above by $\overline{Z}$, the level associated with the tax rate at the peak of the Laffer curve.[3] We show in the appendix that it is not optimal to set the tax rate to a value that is to the right of the peak of the Laffer curve, which implies that the map defined in (11) is invertible, in the sense that for each $Z \leq \overline{Z}$ there are a unique labor supply $l(Z)$ and associated tax rate $\tau(Z) = 1 - l^{\gamma}(Z)$ that satisfy equation (7); $l(Z), -\tau(Z)$ are each decreasing functions of Z. Let $\Psi(Z) \equiv \frac{1}{1+\gamma} l(Z)^{1+\gamma}$ be the utility cost of supplying labor required to raise Z units of tax revenues. We show in the appendix that Ψ is strictly decreasing, strictly concave, and differentiable and that it satisfies Inada conditions $\lim_{Z\to-\infty} \Psi'(Z) = 0$ and $\lim_{Z\to\overline{Z}} \Psi'(Z) = -\infty$.

The optimal value function $V(B\_)$ for problem (10) satisfies the Bellman equation

$$V(B\_) = \max_{\{Z(s),B(s)\}_{s\in S}} \int_{s\in S} \pi(ds) \left[(R(s) B\_ - B(s)) + \gamma \Psi(Z(s)) + \beta V(B(s)) \right] \tag{12}$$

where maximization is subject to $Z(s) \leq \overline{Z}, B(s) \in [\underline{B}, \overline{B}]$ and

$$Z(s) + B(s) = R(s) B\_ + g(s) \quad \text{for all } s. \tag{13}$$

Strict concavity and differentiability of Ψ implies that V is also strictly concave and differentiable. Policy functions $\tilde{B}(s, B\_)$ and $\tilde{Z}(s, B\_)$ attain the right side of Bellman equation (12). Let $\tilde{\tau}(s, B\_)$ denote the associated optimal tax rate policy.
An important endogenous variable, gross government expenditures, $E(s, B\_)$ is defined as

$$E(s, B\_) = R(s) B\_ + g(s), \tag{14}$$

which equals government expenditures including interest and repayment of government debt. Aggregate shocks have effects on $E(s, B\_)$ that depend partly on government debt $B\_$.

We begin our analysis by stating a lemma that summarizes some key properties of the optimal policy rules.

Lemma 2. *There exists $B^{max} > B^{min}$ such that the functions $\tilde{B}$, $\tilde{Z}$, and $\tilde{\tau}$ are increasing in $B\_$ and decreasing in s if $B\_ \leq B^{min}$ and increasing in s if $B\_ \geq B^{max}$.*

Proof. The right side of expression (12) can be maximized separately for each s. The optimal

[3] The expression for the maximum revenue is $\overline{Z} = \gamma \left(\frac{1}{1+\gamma} \right)^{1+1/\gamma}$.

values of $\{Z(s), B(s)\}$ are increasing in $E(s, B\_)$.[4] As $E(s, B\_)$ is increasing in $B\_$ for all s, both $\tilde{B}$ and $\tilde{Z}$ are increasing in $B\_$ for all s. Let $B^{max} = \max_{s'>s''\in\mathcal{S}} \frac{g(s'')-g(s')}{R(s')-R(s'')}$ and $B^{min} = \min_{s'>s''\in\mathcal{S}} \frac{g(s'')-g(s')}{R(s')-R(s'')}$, then $E(s, B\_)$, and hence also $\tilde{B}$ and $\tilde{Z}$, is decreasing in s for $B\_ \leq B^{min}$, and increasing in s for $B\_ \geq B^{max}$.[5] Because $\tilde{\tau}$ is an increasing function of $\tilde{Z}$, it inherits the same properties. $\square$

The first part of Lemma 2 establishes that the optimum tax rate and debt level are both increasing in E. Total expenditures, $E(s, B\_)$ are higher when debt $B\_$ is higher, which in turn implies B and τ are increasing in $B\_$. Since R is increasing in s, E is increasing in s if $B\_$ is sufficiently large (i.e., greater than B^{max}) and decreasing in s if $B\_$ is sufficiently small (i.e., smaller than B^{min}).

Let $\left\{\tilde{Z}_t, \tilde{B}_t\right\}_{t=0}^{\infty}$ be an optimum process generated by iterations on policy functions that attain the right side of Bellman equation (12). First-order conditions associated with the problem on the right side of (12) imply that the marginal social value of assets $V'(\tilde{B}_t)$ is a risk-adjusted martingale, so that if $\tilde{B}_t$ is interior then[6]

$$V'(\tilde{B}_t) = \beta\mathbb{E}_t R_{t+1} V'(\tilde{B}_{t+1}) = \mathbb{E}_t V'(\tilde{B}_{t+1}) + \beta \mathrm{cov}_t(R_{t+1}, V'(\tilde{B}_{t+1})). \tag{15}$$

Lemma 2 also allows us to restrict the sign of the covariance appearing in (15). Since $R_{t+1}\left(s^t, s_{t+1}\right)$ is increasing in s_{t+1}, the covariance appearing on the right side of equation (15) is positive if $\tilde{B}_t \leq B^{min}$ and negative if $\tilde{B}_t \geq B^{max}$.

We shall establish the principle that the level of debt that minimizes variation in $E(s, B\_)$, so that

$$B^* = \mathrm{argmin}_B \mathrm{var}\big(R(s)B + g(s)\big), \tag{16}$$

directs the long run dynamics of the optimal plan. It is easy to verify that

$$B^* = -\frac{\mathrm{cov}\,(R, g)}{\mathrm{var}\,(R)}. \tag{17}$$

We assume debt limits are loose enough that $B^* \in [\underline{B}, \overline{B}]$.

Implications of Lemma 2 and equation (15) are particularly stark when R allows the government to hedge risk in $g(s)$ perfectly. Define

$$\mathcal{R}^* \equiv \left\{R : \text{there exists } B \in [\underline{B}, \overline{B}] \text{ that makes } R(s)B + g(s) \text{ be independent of } s\right\}. \tag{18}$$

When $R \in \mathcal{R}^*$, the debt B^* defined by equation (17) satisfies $\mathrm{var}\big(R(s)B^* + g(s)\big) = 0$ and the limits in Lemma 2 satisfy $B^{min} = B^{max}$. In this case, $V'\left(\tilde{B}_t\right)$ is a supermartingale if $\tilde{B}_t \leq B^*$ and a submartingale if $\tilde{B}_t \geq B^*$. Therefore, debt $\tilde{B}_t$ has an upward drift if it is less than B^* and a downward drift if it is greater than B^*. The tax rate $\tilde{\tau}_t$ and tax revenues $\tilde{Z}_t$ follow corresponding dynamics.

[4] This follows from the objective function being additively separable and concave in (Z, B) as well as constraint (13) taking the form $Z + B = E(s, B\_)$.

[5] Recall that we order elements in $\mathcal{S}$ using the payoff function $p(s)$.

[6] The appendix provides an analysis of the situation in which $\tilde{B}_t$ is not required to be interior.

Proposition 1. *The optimal process $\{\tilde{B}_t\}$ has a unique invariant distribution. If $R \in \mathcal{R}^*$, then $\tilde{B}_t \to B^*$ a.s.*

Proposition 1 asserts that the invariant distribution of $\tilde{B}_t$ is unique. Moreover, if the returns process R allows for perfect hedging of g, then the invariant distribution is easy to characterize: the monotonicity of policy functions established in Lemma 2 and the martingale properties of equation (15) immediately imply that $\tilde{B}_t$ converges to B^* defined in equation (17) with probability 1.

A payoff vector $R \in \mathcal{R}^*$ for which perfect hedging is possible is non-generic because it requires perfect correlation between asset returns and government expenditures. This special case nevertheless illustrates a general principle that an optimal policy drives government debt to a level that minimizes fluctuations in $E(\cdot, B)$. When perfect hedging is impossible because $R \notin \mathcal{R}^*$, shocks take government debt away from B^*. However, as we see shortly, that the martingale conditions will hold approximately and the mean of the invariant distribution of debt will still be B^*, which thus serves as a "target" debt. We shall provide good estimates of the speed of reversion to that target and of the variance of $\tilde{B}_t$ in the invariant distribution. A one-to-one mapping between the debt $\tilde{B}_t$ and government revenues $\tilde{Z}_t$ allows us to characterize probability distributions of tax revenues and the tax rate.

To analyze the more general case with $R \notin \mathcal{R}^*$, we start with the observation that any random variable g can be expressed as

$$g(s) = \bar{g} + \sigma\epsilon_g(s)$$

for some constant σ and a mean zero random variable ϵ_g with $\mathbb{E}\epsilon_g^2 = 1$. Similarly, we can represent

$$p(s) = \overline{p} + \sigma\sigma_{gp}\epsilon_p(s)$$

for some constant σ_{gp} and a mean zero random variable ϵ_p with $\mathbb{E}\epsilon_p^2 = 1$. Define $\sigma_p^2 = \sigma^2\sigma_{gp}^2$ as the variance of p.

We take a second order expansion of policy rules with respect to σ around $\sigma = 0$. Information about state s can be summarized by $(p(s), g(s))$, so we can write the policy functions as $\tilde{B}(p(s), g(s), B\_; \sigma)$, where the notation emphasizes that these are implicit functions of σ. It is easy to check that $\tilde{B}(p, g, B\_; 0) = B\_$, which motivates us to approximate around $B\_$, the *current* debt. This approach differs from one that would approximate around a mean of an ergodic distribution. We show in Appendix 7.2 that

$$\tilde{B}(g, p, B\_; \sigma) = B^* + \beta\hat{g} + \left(\frac{B\_ + \sigma_p^2 B^*}{1 + \sigma_p^2}\right)\frac{\hat{p}}{\bar{p}} + \frac{1}{1 + \sigma_p^2}(B\_ - B^*) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2), \quad (19)$$

where $\hat{p}(s) = p(s) - \overline{p}$ and $\hat{g}(s) = g(s) - \overline{g}$. The second order expansion is linear in g and p up to terms that appear in $\mathcal{O}(\cdot)$.[7] Dropping the higher order terms[8] leads us to approximate $\tilde{B}, \tilde{Z}$ with

$$\check{B}\left(g, p, B\_; \sigma\right) = B^* + \beta \hat{g} + \left(\frac{B\_ + \sigma_p^2 B^*}{1 + \sigma_p^2} \right) \frac{\hat{p}}{\bar{p}} + \frac{1}{1 + \sigma_p^2} (B\_ - B^*) \tag{20}$$

and

$$\begin{aligned} \check{Z}(g, p, B\_; \sigma) = & \left(\frac{1 - \beta}{\beta} \right) B^* + \hat{g} + \left(\frac{B\_ + \sigma_p^2 B^*}{1 + \sigma_p^2} \right) \left(\frac{\hat{p}}{\beta \bar{p}} \right) \\ & + \frac{1}{\beta} \left(B\_ - B^* \right) - \left(\check{B}(g, p, B\_) - B^* \right). \end{aligned} \tag{21}$$

These approximate policy rules can be used to compute all conditional and unconditional moments of the stochastic process of debt and tax revenues. For brevity, we focus on three key moments: the mean, variance and the speed of mean reversion to the invariant distribution of debt and taxes. Let $\left\{ \check{B}_t, \check{Z}_t \right\}_{t=0}^{\infty}$ be debt and tax revenues generated by the approximated policy functions.

Proposition 2. *The invariant distribution of $\check{B}_t, \check{Z}_t$ has:*

- **Mean**

$$\mathbb{E}\left(\check{B}_t \right) = B^*, \quad \mathbb{E}\left(\check{Z}_t \right) \equiv Z^* = \bar{g} + \frac{1 - \beta}{\beta} B^*,$$

- **Variance**

$$\text{var}(\check{B}) = \frac{\text{var}\left(g + R B^* \right)}{\text{var}(R)} \left(1 + \beta^2 \text{var}(R) \right), \quad \text{var}(\check{Z}) = \left(\frac{1 - \beta}{\beta} \right)^2 \text{var}(\check{B}),$$

- **Speed of reversion to mean**

$$\mathbb{E}_0 \left(\check{B}_t - B^* \right) = \left(\check{B}_0 - B^* \right) \left(\frac{1}{1 + \beta^2 \text{var}\left(R \right)} \right)^t,$$

$$\mathbb{E}_0 \left(\check{Z}_t - Z^* \right) = \left(\check{Z}_0 - Z^* \right) \left(\frac{1}{1 + \beta^2 \text{var}\left(R \right)} \right)^t.$$

Proposition 2 shows that the risk-minimizing level of debt B^*, defined in equation (17), is the mean of the invariant distribution of the approximating debt series $\check{B}$. It is positive if R and g are negatively correlated and negative otherwise. Thus, it is better to have debt (positive B^*) if the bond return is high in states in which expenditures are low, and to have assets (negative B^*) otherwise. The lower is the variance of returns, the slower is the speed of reversion. This makes sense because if $B_t \neq B^*$, then fluctuations in the rate of return put additional risk into $E(s)$.

[7] Here $\mathcal{O}(x^n)$ denotes a member of a class of functions

$$f(x) \in \mathcal{O}\left(x^n\right) \text{ if and only if } \lim_{x \to 0} \|f(x) x^{-n}\| < \infty.$$

We denote $\mathcal{O}(x^n) + \mathcal{O}(y^m)$ with $\mathcal{O}(x^n, y^m)$.

[8] Dropping terms of the order $(1 - \beta)\sigma^2$ from the second order expansion allows us to obtain closed form solutions for moments of the ergodic distribution. Since standard parameterizations set the discount factor beta fairly close to 1, dropping these terms is inconsequential.

This risk is increasing in the volatility of R. Therefore, the more volatile is R, the more costly it is for the government to stabilize its debt B_t and the higher is optimal speed to repay debt when $B_t > B^*$ (or to accumulate debt when $B_t < B^*$). Debt dynamics approximate a random walk when the security is nearly a risk-free bond, recovering a result of Barro (1979). Finally, the dispersion of the invariant distribution of government debt depends on the magnitude of the residual risk $\text{var}(g + RB^*)$ at the risk minimizing debt level B^*. The bigger is the variance of this component, the less hedging the government can obtain from B^*, and the probability that shocks push debt away from B^* becomes higher. If $R \in \mathcal{R}^*$, then the variance of the orthogonal component is zero and the invariant distribution puts all mass on debt B^*, consistent with an assertion of Proposition 1.

To understand the striking finding that the mean of the invariant distribution of $\check{B}_t$ is B^*, it is useful to connect the martingale (15) and the static variance minimization problem (16). By strict concavity of the value function, there is a one-to-one relationship between debt B_t and its marginal value to the planner, $V'(B_t)$. An inspection of the martingale equation (15) shows that the covariance term $\text{cov}_t(V'(B_{t+1}), R_{t+1})$ is important in determining the drift of the dynamics of debt in the long run. For a given B_t, the debt next period B_{t+1} depends only on E_{t+1} and consequently,

$$\text{cov}_t(V'(B_{t+1}), R_{t+1}) \propto \text{cov}_t(E_{t+1}, R_{t+1}) + \mathcal{O}(\sigma^3).$$

It is easy to verify that $\text{cov}_t(E_{t+1}, R_{t+1}) = \frac{1}{2}\frac{\partial \text{var}(R_{t+1}B_t + g_{t+1})}{\partial B_t}$. Thus, ignoring $\mathcal{O}(\sigma^3)$ terms, the covariance term in the martingale equation (15) is proportional to slope of the variance of E_t with respect to government debt B_t. As B^* minimizes the variation in E the slope is zero at B^*. The change in signs of the slope implies that, to second order, $V'(B_t)$ will be a sub(super)-martingale when $B_t > (<) B^*$. This explains why the approximation $\check{B}_t$ of government debt $\tilde{B}_t$ drifts towards B^*.

The preceding logic can be verified using equation (19). The drift in $\check{B}_t$ is $\mathbb{E}_t[\check{B}_{t+1} - \check{B}_t] = \frac{\sigma_p^2}{1+\sigma_p^2}(\check{B}_t - B^*)$. Substituting for B^*, this simplifies to

$$\mathbb{E}_t[\check{B}_{t+1} - \check{B}_t] = \frac{\beta^2}{1+\sigma_p^2}\text{cov}(R_{t+1}, E_{t+1}), \tag{22}$$

and thus, the drift $\mathbb{E}_t[\check{B}_{t+1} - \check{B}_t]$ is proportional to the slope of the variance curve: $\text{var}\left[E(\cdot, \check{B}_t)\right]$.

Proposition 2 also characterizes the invariant distribution of tax revenues Z. This is accomplished by first iterating equation (13) forward in time and taking expectations at $t = 0$. We deduce that

$$\frac{\mathbb{E}_0 B_t}{\beta} = \sum_j \beta^j \mathbb{E}_0(\tilde{Z}_{t+j} - g_{t+j}).$$

As $t \to \infty$, conditional means converge to means of the invariant distribution and

$$\mathbb{E}(Z) = \bar{g} + \left(\frac{1-\beta}{\beta}\right)\mathbb{E}(B).$$

The present value budget constraint implies that tax revenues *scale* with the annuity value of debt, so long run fluctuations in debt are associated with long fluctuations in tax revenues. The scaling factor $\frac{1-\beta}{\beta}$ indicates that tax revenues are much less volatile than debt.[9] In appendix 7.2, we make this intuition precise by computing the variance of Z_t in the invariant distribution using the second order approximation to the optimal policy $\tilde{Z}(g, p, B\_)$ and deriving expressions corresponding to those in Proposition 2.

As a natural extension, one can map the volatility of tax revenues Z_t into the volatility of the tax rate τ_t. This mapping helps quantify the amount of tax smoothing when perfect spanning is infeasible. Given that $l = (1-\tau)^{1/\gamma}$, tax revenue as a function of τ is $Z(\tau) = (1-\tau)^{1/\gamma}\tau$. The deviation of τ from the ergodic mean τ^* is then approximately

$$\tau - \tau^* = \frac{1}{Z'(\tau^*)}(Z - Z^*).$$

We can approximate the variance of τ as

$$\text{var}(\tau) \approx \left(\frac{1}{Z'(\tau^*)}\right)^2 \text{var}(Z).$$

The value of $Z'(\tau^*)$ cannot be obtained in closed form but is easy to compute.[10] To illustrate: for $\gamma = 2$ and $\tau^* \in [0., 0.4]$, the coefficient $\left(\frac{1}{Z'(\tau^*)}\right)^2$ varies between 1 and 3.7. The standard deviation of τ is then up to twice that of tax revenues, but still much less than that of government debt.

Figure I illustrates the accuracy of the quadratic approximation. Here we assume that $\gamma = 2$, $\beta = 0.98$. Government expenditure g is a approximated using a 5 state Gaussian quadrature with mean 20% and standard deviation 1.8% . The payoff vector p was defined by

$$p(s) = 1 + \chi(g(s) - 0.2) + \sigma_e \epsilon(s).$$

We solved where $\epsilon(s)$ is a 2 state i.i.d. process independent of $g(s)$. The parameter χ was set to 0.7 to make the asset payoff process be slightly correlated with g.[11] Figure I plots the ergodic distribution of the global solution, solved using projection methods, and our quadratic approximation for 3 levels of σ_e: 0.001,0.02, and 0.04. σ_e is smallest in the topmost panel and largest in the bottommost panel. The figure indicates that invariant distributions generated by approximated policy rules (solid red lines) provide good approximations to the invariant distribution generated by the global solution (dashed black lines).

3 Extensions

Forces that we isolated within the section 2 economy prevail under alternative assumptions about numbers of securities, motives for redistribution, preferences that express aversion to consumption risk, and persistence of shocks to government expenditures. We discuss them in greater detail now.

[9] For a β=0.98, $\frac{1-\beta}{\beta} \approx \frac{1}{50}$

[10] See the proof of Proposition 2 for more details.

[11] This matches the correlation that would occur if the representative agent were risk averse. We discuss this case more in section 4

3.1 More assets

We begin by allowing K one-period securities. Let $\vec{p} = \left\{p^k\right\}_{k=1}^K$ be payoffs of the K securities and $\vec{R} = \beta^{-1}\vec{p}$ be the corresponding matrix of returns. The return of asset k in state s is $R^k(s)$. The government budget constraint at time t is

$$g_t + \sum_{k=1}^{K} R_t^k B_{t-1}^k = \tau_t l_t + \sum_{k=1}^{K} B_t^k.$$

We restrict the value of the government portfolio to be in a compact set

$$B_t \equiv \sum_{k=1}^{K} B_t^k \in \left[\underline{B}, \overline{B}\right].$$

Temporarily suppose that the government portfolio weights are fixed, meaning that there are constants $\alpha_1, ..., \alpha_K$ such that $\frac{B_t^k}{\sum_k B_t^k} = \alpha_k$ for all t. Define $R(s) \equiv \sum_{k=1}^K \alpha_k R^k(s)$. Then the optimal policy problem is equivalent to the one analyzed in section 2. Thus, if the government fixes its portfolio weights, then all insights about optimal debt management and fiscal policy from section 2 apply.

Now suppose that the government optimally chooses portfolio weights each period. The planners' value function $V(B\_)$ satisfies a Bellman equation

$$V(B\_) = \max_{\left\{Z(s), B(s), B^k\_\right\}_{s,k}} \int_{s\in S} \pi(ds) \left[\left(\sum_k R^k(s) B^k\_ - B(s)\right) + \gamma\Psi(Z(s)) + \beta V(B(s))\right] \tag{23}$$

where maximization is subject to $Z \leq \overline{Z}, B(s) \in [\underline{B}, \overline{B}]$, and

$$B(s) + Z(s) = \sum_k R^k(s) B^k\_ + g(s) \quad \forall s, \tag{24}$$

$$\sum_k B^k\_ = B\_.$$

Here $B\_$ is the value of the government portfolio before portfolio choice in the current period. Counterparts of the section 2 martingale equation (15) now hold for derivatives of the value function with respect to every payoff vector R^k. Paralleling Proposition 1, the invariant distribution of the total value of government portfolio $\tilde{B}_t$ is unique.

Let $\mathbb{C}[\vec{R}, \vec{R}]$ be the covariance of returns and $\mathbb{C}[\vec{R}, g]$ be the vector of covariances between returns and exogenous expenditures g. The vector $\vec{B}^*$ that minimizes the variance of $E(s, B\_)$ is

$$\vec{B}^* = \min_{\vec{B}} \text{var}\left[\left(\sum_k R^k(\cdot) B^k + g(\cdot)\right)\right]. \tag{25}$$

When $\mathbb{C}[\vec{R}, \vec{R}]$ has full rank, the optimal portfolio problem (25) has a unique solution with $\vec{B}^* = -\mathbb{C}[\vec{R}, \vec{R}]^{-1}\mathbb{C}[\vec{R}, g]$.[12] With multiple securities, we use the notation B^* introduced before

[12] The matrix $\mathbb{C}[\vec{R}, \vec{R}]$ fails to have full rank if there exists a linear combination of $\{R^k\}$ that yields a risk free bond. The minimum variance in equation (25) is then unchanged by the amount invested in a risk-free bond

to denote value of the total portfolio, $\sum_k B^{*,k}$.

The next proposition is a counterpart of Proposition 2. $\check{B}_t$ now denotes an approximation to the total value of government portfolio analogous to the one in section 2 and $\mathbf{1}$ denotes a K dimensional (column) vector of ones.

Proposition 3. *The invariant distribution of $\check{B}_t$ has*

- **Means**
$$\mathbb{E}\left(\check{B}_t\right) = B^* = -\mathbf{1}^\intercal \mathbb{C}[\vec{R},\vec{R}]^{-1}\mathbb{C}[\vec{R},g],$$
$$\mathbb{E}(\check{Z}) = Z^* = \overline{g} + \frac{1-\beta}{\beta}B^*,$$

- **Variances**
$$\operatorname{var}\left(\check{B}_t\right) = (1+\beta^{-2}\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1})\beta^2 \operatorname{var}\left(-\mathbf{1}^\intercal \mathbb{C}(\vec{R},g)^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\left[\vec{R}-\frac{\mathbf{1}}{\beta}\right] + [g-\overline{g}]\right),$$
$$\operatorname{var}\left(\check{Z}_t\right) = \left(\frac{1-\beta}{\beta}\right)^2 \operatorname{var}(\check{B}_t),$$

- **Speeds of mean reversion**
$$\mathbb{E}_0(\check{B}_t - B^*) = (\check{B}_0 - B^*)\left(\frac{\beta^{-2}\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}{1+\beta^{-2}\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}\right)^t,$$
$$\mathbb{E}_0(\check{Z}_t - Z^*) = (\check{Z}_0 - Z^*)\left(\frac{\beta^{-2}\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}{1+\beta^{-2}\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}\right)^t.$$

For a government portfolio with total value $B\_$, holdings of individual securities $\vec{B}(B\_)$ are approximately

$$\vec{B}(B\_) \approx -\mathbb{C}(\vec{R},\vec{R})^{-1}\mathbb{C}(\vec{R},g) + \frac{\mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}{\mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbf{1}}\left(B_- + \mathbf{1}^\intercal \mathbb{C}(\vec{R},\vec{R})^{-1}\mathbb{C}(\vec{R},g)\right). \tag{26}$$

When $K > 1$, the planner's problem includes designing an optimal portfolio. However, more is involved in the planner's problem than the standard mean-variance trade-offs featured in single-agent portfolio choice problems like those studied by Merton (1969) because the benevolent planner internalizes competitive equilibrium restrictions when he designs portfolios. With quasilinear preferences, the planner knows that in equilibrium all assets bear the same expected return $\frac{1}{\beta}$. That makes both the total savings decisions and the portfolio choice be driven by the goal of minimizing risk rather than by the risk-return trade-offs emphasized by Merton. In section 4, we show that this insight extends to settings in which consumers are averse to consumption risk.

Next we illustrate Proposition 3 and formula (26) with some examples.

Example 1. *Suppose that there are two securities and that security 1 allows perfect hedging in the sense that $R^1 \in \mathcal{R}^*$, where $\mathcal{R}^*$ is defined in* (18). *Assume that the return on security 2 is*

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and, consequently, the solution to problem (25) is undetermined up to $\sum_k B^{*,k}$.

orthogonal to return on security 1. Proposition 3 implies that the invariant distribution has all of its mass on $B^* = -\frac{\text{cov}(R,g)}{\text{var}(R)}$ *and that the speed to convergence to* B^* *is*

$$\mathbb{E}_{t-1}[\check{B}_t - B^*] = \frac{1}{1 + \beta^2 \frac{var(R^2)}{var(R^1)+var(R^2)} var\left(R^1\right)} (\check{B}_{t-1} - B^*).$$

Formula (26) *implies that the portfolio composition satisfies*

$$\check{B}^1(B_t) = \frac{var\left(R^2\right)}{var\left(R^1\right) + var\left(R^2\right)} B_t + \frac{var\left(R^1\right)}{var\left(R^1\right) + var\left(R^2\right)} B^*,$$

$$\check{B}^2(B_t) = \frac{var\left(R^1\right)}{var\left(R^1\right) + var\left(R^2\right)} B_t - \frac{var\left(R^1\right)}{var\left(R^1\right) + var\left(R^2\right)} B^*.$$

In example 1, the government chooses a debt portfolio that in the long-run fully hedges its shocks, just as in the section 2 one-security case. Because complete hedging can be achieved at the steady state with security 1 only, using security 2 there would introduce risk in $E(s)$. So the government's long run portfolio has only security 1. If $B^1_{-1} \neq B^*$, the government uses security 2 to get some additional hedging while it gradually approaches B^* along a transition path. Convergence to the long-run portfolio is slower than in section 2 case in which agents can only trade security 1, since

$$\frac{1}{1 + \beta^2 \frac{\text{var}(R^2)}{\text{var}(R^1)+\text{var}(R^2)} \text{var}\left(R^1\right)} > \frac{1}{1 + \beta^2 \text{var}\left(R^1\right)}.$$

The availability of more securities slows convergence to the government's long run target portfolio.

Example 2. *Consider a setting in which it is possible perfectly to hedge government expenditures shocks by using both of two securities, so that* $R^1, R^2 \in \mathcal{R}^*$ *and* $R^1 \neq R^2$*. In this case, markets are complete. For any initial debt, the government can choose a portfolio to hedge aggregate shocks perfectly. Let* $B^{*,k} = -\frac{\text{cov}(R^k,g)}{\text{var}(R^k)}$*. For an initial debt* $B_{-1} \in [\underline{B}, \overline{B}]$*, find* B^1 *and* B^2 *that satisfy* $\sum_k \frac{B^k}{B^{*,k}} = 1$ *and* $\sum_k B^k = B_{-1}$.[13] *At this portfolio, government expenditures gross of debt service for any state* s *are*

$$E(s|B^1, B^2) = \sum_k B^k R^k(s) + g(s) = \sum_k \frac{B^k}{B^{*,k}} \left[B^{*,k} R^k(s) + g(s)\right].$$

$R^k \in \mathcal{R}^*$ *and equation* (18) *imply that* $\left[B^{*,k} R^k(s) + g(s)\right]$ *is constant and independent of* s.[14]

Farhi (2010) studies a Ramsey problem for a government that invests claims to physical capital and one-period risk-free debt. The main difference between his setting and ours is that Farhi's Ramsey planner affects returns on the government's portfolio by varying a distorting tax rate on capital income. In Farhi's model it is optimal for the government to hold large positions of risk-free debt and equity of opposite signs. That lets the government implement a complete

[13] A unique solution exists so long as $B^{*,1} \neq B^{*,2}$ which is guaranteed by $R^1 \neq R^2$

[14] It can be shown that the optimal policy satisfies $\tilde{B}(s, B\_) = B\_$ and a $\tilde{Z}(s, B\_) = Z^*$, where Z^* is a constant.

market allocation with arbitrarily small fluctuations in the capital tax rate. When we make one of the two securities approach a risk-free bond in the example 2 setting, the optimal portfolio that supports a complete markets allocation attains arbitrarily large gross positions.

Example 3. *Suppose that every asset is an infinitely lived console with a per-period coupon* $p^k(s)$ *so that the government budget constraint is*

$$g_t + \sum_{k=1}^{K} \left(p_t^k + q_t^k\right) \mathrm{B}_{t-1} = \tau_t l_t + \sum_{k=1}^{K} q_t^k \mathrm{B}_t^k.$$

The price q_t^k *of the long-lived asset satisfies*

$$q_t^k = \beta \mathbb{E}_t \left[q_{t+1}^k + p_{t+1}^k\right],$$

which because s *is i.i.d. implies that* $q_t^k = \bar{q}^k \equiv \left(\frac{\beta}{1-\beta}\right) \mathbb{E} p^k$. *Redefine* $B_t^k = \bar{q}^k \mathrm{B}_t^k$ *and* $R_t^k = \frac{\bar{q}^k + p^k(s)}{\bar{q}^k}$, *so that the optimal policy problem with long-lived assets reduces to one with short lived assets. The logic outlined in this example extends to assets that are in positive net supply, such as Lucas trees.*

Example 4. *Example 3 can be interpreted as a theory of an optimal maturity of government debt. Assume that there are two securities, a risk-free bond with one period maturity and an infinitely lived console having a per-period stochastic coupon* $p(s)$. *The amount of government debt allocated to one security relative to another indicates the "maturity" of government debt. Example 3 shows that in this setting the government issues or holds two assets with returns* $R^1(s) = \frac{1}{\beta}$ *and* $R^2(s) = \frac{\bar{q} + p(s)}{\bar{q}}$ *where* $\bar{q} \equiv \left(\frac{\beta}{1-\beta}\right) \mathbb{E} p$. *The variance minimization problem* (25) *yields minimizers*

$$B^1(B_t) = B_t - B^{*,2} \text{ and } B^2(B_t) = B^{*,2},$$

with $B^{*,2} = -\frac{\mathrm{cov}(R^2, g)}{\mathrm{var}(R^2)}$. *The effective maturity of government debt is* $\frac{B^{*,2}}{B_t - B^{*,2}}$. *For any government debt* B_t *with which it enters a period, the government issues a time-invariant amount* $B^{*,2}$ *of the console and adjusts its issues of the risk-free bond each period in response to shocks that period. The effective maturity of government debt is an inverse function of total government debt, a pattern consistent with empirical findings of Missale and Blanchard (1994), who studied European countries during 1960-1990.*

3.2 Persistent shocks

In this section we return to the settings of section 2 but assume that shocks are first order Markov rather than i.i.d. The planner's optimal value function satisfies the Bellman equation

$$V(s\_, B\_) = \max_{\{Z(s), B(s)\}_{s \in S}} \int_{s \in S} \pi(ds | s\_) \left[(R(s, s\_) B\_ - B(s)) + \gamma \Psi(Z(s)) + \beta V(B(s), s) \right] \tag{27}$$

where maximization is subject to $Z \leq \overline{Z}, B(s) \in [\underline{B}, \overline{B}]$ and

$$Z(s) + B(s) = R(s, s\_) B\_ + g(s) \text{ for all } s\_, s, \tag{28}$$

where $\pi(ds|s\_)$ denotes a transition probability kernel and

$$R(s, s\_) = \frac{p(s)}{\beta \int_{s'} \pi(ds'|s\_)p(s')}.$$

In the rest of the section we will generalize the static variance minimization problem in equation (16) to obtain counterparts of B^* when shocks are persistent. A useful construct is the expected present discounted value of net of interest government expenditures, $PV_g(s) = \mathbb{E}\sum_t \left(\beta^t g(s_t)|s_0 = s\right)$.[15] For a given level of Z, define a function $B^{**}(Z, s)$ as

$$B^{**}(Z, s\_) = \frac{\beta Z}{1-\beta} - \beta\mathbb{E}\left(PV_g|s\_\right). \tag{29}$$

For initial conditions $B_{-1} = B^{**}(Z, s\_)$, and $s_{-1} = s\_$, the policy of setting the tax rate such that the total revenues are constant through time at Z is optimal under complete markets. With returns $R(s, s\_)$, we can compute the residual in the budget constraint (28) when tax revenues are $Z(s) = Z$ and debt choices are given by $B(s) = B^{**}(Z, s)$,

$$D(Z, s\_, s) \equiv B^{**}(Z, s\_)R(s\_, s) - (Z - g(s)) - B^{**}(Z, s) = B^{**}(Z, s\_)R(s\_, s) - \frac{Z}{1-\beta} + PV_g(s), \tag{30}$$

where the last equality follows from equation (29).

We now discuss the appropriate variance minimization problem that generalizes (16). Let Z^* be the level of tax revenues that minimizes the ergodic variance of D. To calculate Z^*, define λ as invariant distribution associated with the transition kernel π.[16] We use $\mathbb{E}^\lambda(\cdot)$ and $\text{var}^\lambda(\cdot)$ to denote the mean and variance of a random variable under the stationary measure λ. The level Z^* solves

$$Z^* \in \text{argmin}_z \mathbb{E}^\lambda \left(\text{var}(D(Z)|s\_)\right). \tag{31}$$

Using equation (30), and $\hat{R}(s\_, s) = R(s\_, s) - \mathbb{E}\left(R|s\_\right)$ and $\hat{PV}_g(s\_, s) = PV_g(s) - \mathbb{E}\left(PV_g|s\_\right)$ we can simplify the objective function in equation (31) to

$$\mathbb{E}^\lambda \left(\text{var}(D(Z)|s\_)\right) = \mathbb{E}^\lambda \left(\mathbb{E}\left(B^{**}(Z, s\_)\hat{R}(s\_, s) + \hat{PV}_g(s\_, s)\right)^2 |s\_\right).$$

Taking the first order condition with respect to Z and reorganizing terms we obtain,

$$\mathbb{E}^\lambda \left(B^{**}(Z^*, s\_)\right) = -\frac{\text{cov}^\lambda\left(R, PV_g\right)}{\text{var}^\lambda\left(R\right)} - \frac{\text{cov}^\lambda\left(\text{var}\left(R|s\_\right), \beta\mathbb{E}\left(PV_g|s\_\right)\right)}{\text{var}^\lambda\left(R\right)}. \tag{32}$$

When shocks are i.i.d, $\mathbb{E}^\lambda\left(B^{**}(Z^*, s\_)\right)$ in equation (32) equals B^* defined in equation (17). To show that $B^{**}(Z^*, s)$ captures the same long-run hedging motives when shocks are persistent, we begin with a simple case in which perfect hedging is possible. We extend the

[15]When $\mathcal{S}$ is finite, π is an $S \times S$ stochastic matrix $PV_g = (I - \beta\pi)^{-1}g$.

[16]The measure λ satisfies $\lambda(A) = \int_S \lambda(ds)\pi(A|s)$ for any measurable set A. For finite S, one can compute λ using the equation $\lambda' = \lambda'\Pi$ and normalized to sum to 1.

notion of perfectly spanning returns and define

$$\mathcal{R}^* = \left\{ R(s, s\_) \equiv \frac{p(s)}{\beta \int_{s'} \pi(ds'|s\_)p(s')} \middle| \exists B \in [\underline{B}, \overline{B}] \text{ s.t } Bp(s) + PV_g(s) \text{ is independent of } s \right\}. \tag{33}$$

To explain why (33) is an appropriate extension of definition (18), we introduce,

Definition 2. A stochastic steady state is a function $\mathcal{B} : \mathcal{S} \to \mathbb{R}$ that satisfies

$$\tilde{B}(s\_, \mathcal{B}(s\_), s) = \mathcal{B}(s) \quad \forall s\_, s.$$

With persistent shocks, perfect tax smoothing occurs when there exists a stochastic steady state. Unlike the outcome with i.i.d. shocks, government debt is not be constant when tax revenues are held constant. A stochastic steady state $\mathcal{B}$ is connected to the solution $B^{**}(Z^*, s)$:

Proposition 4. *For $R \in \mathcal{R}^*$, $B^{**}(Z^*, s)$ is the stochastic steady state. Furthermore,*

$$B^{**}(Z^*, s\_) = -\frac{\text{cov}(R, PV_g|s\_)}{\text{var}(R|s\_)} \quad \forall s\_.$$

We now extend Proposition 2 to allow for both persistence and $R \notin \mathcal{R}^*$. For the remainder of this section, we assume a continuous state space with $s = (g, p)$ and that $\{g_t, p_t\}$ obey

$$g_t = (1 - \rho)\bar{g} + \rho g_{t-1} + \varepsilon_{g,t}, \tag{34a}$$

$$p_t = \bar{p} + \varepsilon_{p,t}, \tag{34b}$$

where $\varepsilon_{g,t}, \varepsilon_{p,t}$ are i.i.d over time with zero means and an arbitrary covariance structure. We use the technique described in section 2 to construct an approximation $\left\{\check{B}_t\right\}_{t=0}^{\infty}$ of an optimal process $\left\{\tilde{B}_t\right\}_{t=0}^{\infty}$.

Proposition 5. *Let $\mathbb{E}$ and* var *represent the expectation and variance in the invariant distribution of $\check{B}_t$. The invariant distribution of $\check{B}_t$ has*

- **Mean**

$$\mathbb{E}\left(\check{B}_t | s_t\right) = B^{**}(Z^*, s),$$

- **Variance**

$$\text{var}(\check{B}_t - B_t^{**}) = \left(\frac{1 + \beta^2 \text{var}^\lambda(R)}{\text{var}^\lambda(R)}\right) \text{var}^\lambda\left(D_t^*\right),$$

- **Speed of mean reversion**

$$\mathbb{E}_0(\check{B}_t - \mathbb{E}^\lambda B^{**}) = (\check{B}_0 - \mathbb{E}^\lambda B^{**}) \left(\frac{1}{1 + \beta^2 \text{var}^\lambda(R)}\right)^t.$$

When net-of-interest government expenditures are persistent, inter-temporal revenue needs are linked. In the long run, the planner uses asset income to hedge not only contemporaneous but also future government expenditures. When shocks are AR(1) as in (34), equation (32) for the mean of the ergodic distribution simplifies and becomes

$$\mathbb{E}^{\lambda}\left(B^{**}(Z^*, s\_)\right) = -\frac{\text{cov}^{\lambda}(R, PV_g)}{\text{var}^{\lambda}(R)} = -\left(\frac{1}{1-\rho\beta}\right)\frac{\text{cov}^{\lambda}(R, g)}{\text{var}^{\lambda}(R)}.$$

Comparing this with equation (17) shows that planner accumulates more debt (or assets) relative to the no persistence $\rho = 0$ case.

3.3 Transfers and redistribution

Optimal debt management in our model in section 2 differs significantly from that in other incomplete markets models (e.g., Aiyagari et al. (2002) and Farhi (2010)). In our model, the invariant joint distribution of debt and taxes is unique. In the long run, debt and tax rates minimize fluctuations in gross government expenditures including debt service requirements, $E(s)$. By way of contrast, optimal plans in those other models have a continuum of invariant distributions of debt levels. In all of them, tax rates are zero and debt levels are negative and big enough in absolute value that to finance all net of interest government expenditures from earnings on the government's portfolio. This results in large fluctuations in transfers in the ergodic distribution. A key difference between those models and ours is that we prohibit lump-sum taxes or transfers, while Aiyagari et al. (2002) and Farhi (2010) allow positive but not negative lump-sum transfers. In this section, we extend our analysis to an economy with lump sum transfers. We show that our section 2 results carry over essentially unchanged. We then discuss the mechanism that drives the long-run tax rate in Aiyagari et al. (2002) to zero and explain how to reconcile their results with ours.

A standard justification for ruling out lump-sum taxes in representative agent models is to refer to unmodelled "poor" agents who cannot afford to pay a lump-sum tax. In this section, we study optimal anonymous transfers in an economy with such poor agents. We extend the section 2 economy to have just enough heterogeneity across agents to make the analysis interesting. In particular, we assume that in addition to a measure 1 of agents of type 1 with quasilinear preferences $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$, there is a measure n of agents (type 2 agents) who cannot work or trade securities and who enjoy utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{2,t}\right),$$

where $c_{2,t}$ is consumption of a type 2 agent in period t; U is strictly concave and differentiable on $\mathbb{R}_+$ and satisfies the Inada condition $\lim_{c\to 0} U'(c) = \infty$.

The government and type 1 agent trade the same security described in section 2. The government imposes a linear tax rate τ_t on labor income and lump-sum transfers T_t that cannot depend on the type of agent. Negative transfers are not feasible because a type 2 agent has no income other than transfers. T_t denotes aggregate transfers, so each agent receives a per-capita transfer $\frac{T_t}{1+n}$. Since agent 2 lives hand-to-mouth, his budget constraint is

$$c_{2,t} = \frac{T_t}{1+n}.$$

The planner ranks allocation according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\left(c_t - \frac{1}{1+\gamma} l_t^{1+\gamma} \right) + \omega U \left(c_{2,t} \right) \right],$$

for some $\omega > 0$.

The time t government budget constraint is now

$$g_t + T_t + R_t B_{t-1} = \tau_t l_t + B_t.$$

With only minimal modifications, the budget constraint of a type 1 agent, Definition 1 of the competitive equilibrium, and the section 2 recursive formulation of the optimal policy problem extend to this environment. The planner's optimal value function satisfies the Bellman equation

$$V\left(B\_\right) = \max_{\{Z(s),B(s),T(s)\}_{s\in S}} \int_{s\in S} \pi\left(ds\right) \left[\left(R\left(s\right) B\_ - B\left(s\right) + \frac{T\left(s\right)}{1+n} \right) + \gamma \Psi\left(Z\left(s\right)\right) + \omega U\left(\frac{T\left(s\right)}{1+n}\right) + \beta V\left(B\left(s\right)\right) \right] \tag{35}$$

subject to,

$$Z \leq \overline{Z}, B(s) \in [\underline{B}, \overline{B}], \tag{36}$$

$$Z\left(s\right) - T\left(s\right) + B\left(s\right) = R\left(s\right) B\_ + g\left(s\right) \text{ for all } s. \tag{37}$$

Most section 2 results apply to this economy. We define B^* and $\mathcal{R}^*$ exactly as in section 2.

Proposition 6. *There exists a unique invariant distribution for $\tilde{B}_t$. $V'\left(\tilde{B}_t\right)$ satisfies the martingale equation* (15). *If $R \in \mathcal{R}^*$, then $\tilde{B}_t \to B^*$ almost surely and $\tilde{\tau}_t \to \tau^*$, $\tilde{T}_t \to T^*$ almost surely for some constant τ^*, T^*, and $\tau^* \neq 0$ generically. Both τ^*, T^* are increasing in ω.*

Proposition 6 extends Proposition 1 to an economy with endogenous transfers. The invariant distribution of government debt is unique. If perfect hedging is feasible at some level of government debt, government debt eventually reaches that level. Because B^* is independent of transfers and the Pareto weights, so is the long-run debt level. Long-run levels of transfers and tax revenues do depend on the Pareto weight and are generally not zero.

Following the same steps used to prove Proposition 2, we use a second order approximations of policy functions to obtain

Proposition 7. *The invariant distribution of $\check{B}_t$ satisfies properties stated in Proposition 2. The invariant distribution of $\check{Z}_t - \check{T}_t$ has the same properties as the invariant distribution of $\check{Z}_t$ in Proposition 2. Let $F(\check{T};\omega)$ be the cumulative distribution function of the ergodic distribution of $\check{T}_t$. If $\omega > \omega'$ then $F(\check{T};\omega)$ first order stochastically dominates $F(\check{T};\omega')$.*

The insights in section 2 about optimal debt management carry over to this heterogeneous economy. In this economy, fluctuations in both the tax rate and (non-agent-specific) lump-sum transfers are costly, so an optimal policy smooths both. Adjusting the tax rate in response to aggregate shocks is costly because the dead-weight loss of taxation is convex in tax rates, as stressed by Barro (1979). Adjusting transfers is also costly because that induces fluctuations in inequality. For any fixed weight ω, there is an optimal level of transfers $T^*(\omega)$ that is increasing in ω and that achieves the socially optimal amount of inequality. Deviations from that level of transfers reduce social welfare. That is why an optimal policy smooths transfers.

In Aiyagari et al. (2002) and Farhi (2010) the government eventually sets tax rates to zero and thereafter adjusts transfers one-to-one with government expenditures. A main difference between our setup and theirs is that they do not model heterogeneity explicitly but only appeal to it informally as a justification for imposing a restriction $T_t \geq 0$. This restriction puts a kink in the cost of using transfers – the marginal cost of an increase in transfers is zero, while a marginal cost of a decrease in transfers is infinite at $T = 0$. A high marginal cost of negative transfers creates an incentive for the government in Aiyagari et al. (2002) and Farhi (2010) to accumulate enough assets to ensure that the constraint $T_t \geq 0$ is eventually slack. Since fluctuations in positive transfers are costless, in the long-run the government uses those transfers to offset all fluctuations in expenditures g_t.

By way of contrast, in our economy, the cost of using transfers is endogenous. It is smooth (so that the marginal cost from both increasing and decreasing transfers around the optimal level $T^*(\omega)$ is the same, $\frac{\omega}{1+n}U'(T^*(\omega)))$ and welfare costs of departing from the optimal inequality level are strictly convex. This implies very different long run dynamics than those emphasized by Aiyagari et al. (2002). As we show in a companion paper, Bhandari et al. (2015b), this insight carries over to richer economies with much more heterogeneity and in which no agent is excluded from the financial markets.

4 Risk aversion and persistent government expenditures

We extend our analysis to a setting in which a representative agent has preferences that display risk-aversion and in which fluctuations in net-of-interest government deficits are persistent. We assume that a representative agent's one-period utility function , $U : \mathbb{R}^2_+ \to \mathbb{R}$, is given by

$$U(c,l) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}. \tag{38}$$

The constant elasticity of substitution assumption simplifies exposition. Our results extend to U's which are strictly concave in $(c, -l)$ and twice continuously differentiable. We let $U_{x,t}$ or $U_{xy,t}$ denote first and second derivatives of U with respect to $x, y \in \{c, l\}$. We assume that natural debt limits apply to the consumer, which ensures that first-order conditions are satisfied off corners.

The state $s \in S$, where S is finite, is Markov with transitions described by a stochastic matrix whose entries are $\pi(s_t|s_{t-1})$. Both government expenditures g and a representative agent's labor productivity θ can depend on s. Feasibility at t requires

$$c_t + g_t = \theta_t l_t. \tag{39}$$

The government's budget constraint at t is

$$g_t + \frac{p_t}{q_{t-1}}\mathrm{B}_{t-1} = \tau_t \theta_t l_t + \mathrm{B}_t, \tag{40}$$

where $q_{t-1} = \frac{\beta \mathbb{E}_{t-1} U_{c,t} p_t}{U_{t,t-1}}$ is the price of the security issued in period $t-1$.

We employ a primal approach to formulate the optimal policy problem. We use the feasibility constraint and the household's first-order necessary conditions to eliminate taxes and prices from either the government's or the household's budget constraint to deduce the following *implementability restrictions* on competitive equilibrium allocations:

$$U_{c,t}\mathrm{B}_t + U_{c,t}\left[\theta_t l_t + \frac{U_{l,t}}{U_{c,t}} l_t - g_t\right] = \frac{p_t U_{c,t}}{\beta \mathbb{E}_{t-1} p_t U_{c,t}} U_{c,t-1}\mathrm{B}_{t-1} \quad t \geq 1, \tag{41}$$

$$c_0 + b_0 = -\frac{U_{l,0}}{U_{c,0}} l_0 + p_0 \beta^{-1} \mathrm{B}_{-1}.$$

The optimal policy problem is to maximize $\mathbb{E}_0 \sum_t \beta^t U(c_t, l_t)$ by choosing $\{c_t, l_t, \mathrm{B}_t\}_{t=0}^{\infty}$ subject to implementability constraints (41) and the feasibility constraints (39).

The shadow cost to the planner of raising revenues varies with the marginal utility of consumption. We define *effective government debt* $X_t \equiv U_{c,t}\mathrm{B}_t$, an *effective security return* $\mathrm{R}_t \equiv \frac{U_{c,t} p_t}{\beta \mathbb{E}_{t-1} U_{c,t} p_t}$, and an *effective primary deficit* $\Phi_t \equiv U_{c,t}\left[g_t - Z_t\right] = -U_{c,t}\left[c_t + \frac{U_{l,t}}{U_{c,t}}\left(\frac{c_t + g_t}{\theta_t}\right)\right]$, where Z_t denotes tax revenues. The variables X_t, R_t and Φ_t serve as counterparts to B_t, R_t and $g_t - Z_t$ in the section 2 economy with quasilinear preferences.

It is possible to construct an optimal policy recursively by taking (X, s) as state variables, meaning that an optimal allocation for $\tau \geq t > 0$ depends only on the values of (X_{t-1}, s_{t-1}). We can define

$$\mathrm{R}(s, s\_, c) = \frac{U_c(s) p(s)}{\beta \sum_{s' \in S} \pi\left(s' | s\_\right) U_c(s) p(s')}.$$

For $t > 0$, the planner's optimal value function $V\left(X\_, s\_\right)$ satisfies a Bellman equation

$$V\left(X\_, s\_\right) = \max_{\{c(s), X(s)\}_{s \in S}} \sum_{s \in S} \pi\left(s | s\_\right)\left[U\left(c(s), \frac{c(s) + g(s)}{\theta(s)}\right) + \beta V\left(X\left(s\right), s\right)\right] \tag{42}$$

subject to

$$X(s) = \mathrm{R}\left(s, s\_, c\right) X\_ + \Phi(s, c). \tag{43}$$

Paralleling the analysis in section 3.2, let $\tilde{c}(s, X\_, s\_), \tilde{X}\left(s, X\_, s\_\right)$ denote policy functions, and $\tilde{\mathrm{R}}(s, X\_, s\_)$, $\tilde{\Phi}\left(s, X\_, s\_\right)$, $\tilde{\mathrm{B}}\left(s, X\_, s\_\right)$, $\tilde{Z}\left(s, X\_, s\_\right)$ and $\tilde{\tau}(s, X\_, s\_)$ denote effective returns, an effective net-of-interest deficit, and government debt, tax revenues and tax rates implied by the policy functions that attain the right side of (42). We can use the first-order conditions for the problem on the right side of (42) to show that the marginal cost of government debt $V_t' = V'\left(\tilde{X}_t, s_t\right)$ satisfies the martingale equation

$$V'(\tilde{X}_t, s_t) = \beta \mathbb{E}_t \mathrm{R}_{t+1} V'(\tilde{X}_{t+1}, s_{t+1}) = \mathbb{E}_t V'(\tilde{X}_{t+1}, s_{t+1}) + \beta \mathrm{cov}_t(\mathrm{R}_{t+1}, V'(\tilde{X}_{t+1}, s_{t+1})). \tag{44}$$

That effective returns depend on the marginal utility of consumption, an endogenous object, complicates the present economy relative to the section 2 economy with quasilinear preferences. Nevertheless, many insights can be gleaned by proceeding along the lines of sections 2 and 3.2. We extend the analysis of returns in set $\mathcal{R}^*$ from the quasilinear environment and study properties of payoffs that allow perfect hedging. Since returns are endogenous, we will characterize perfect spanning returns in terms of a set of exogenous payoffs $\mathcal{P}^*$. Extending Definition 2 of stochastic steady state to marginal utility adjusted debt, let $\{\mathcal{X}(s)\}_{s\in\mathcal{S}}$ be a vector satisfying

$$\tilde{X}(s, \mathcal{X}(s\_), s\_) = \mathcal{X}(s) \quad \forall s, s\_.$$

Proposition 8. *There exists a non-empty set of exogenous payoffs $\mathcal{P}^*$ such that for each $p \in \mathcal{P}^*$ there is a stochastic steady state $\mathcal{X}$ associated with a constant tax rate. The mean X^* of the invariant distribution of $\mathcal{X}$ satisfies*

$$X^* = -\mathbb{E}^{\lambda}\left(\frac{\operatorname{cov}\left(\tilde{\mathrm{R}}\left(s, \mathcal{X}(s\_), s\_\right), PV_{\tilde{\Phi}}\left(s, \mathcal{X}(s\_), s\_\right)|s\_\right)}{\operatorname{var}\left(\tilde{\mathrm{R}}\left(s, \mathcal{X}(s\_), s\_\right)|s\_\right)}\right),$$

Furthermore, there exists a class of economies for which the risk-free payoffs, $p^{rf}(s) = 1$ are in $\mathcal{P}^*$.

Payoffs in $\mathcal{P}^*$ satisfy the spanning property similar to equation (33),

$$p(s)U_c\left(s, \mathcal{X}(s\_), s\_,\right)\mathcal{X}(s\_) + PV_{\tilde{\Phi}}\left(s, \mathcal{X}(s\_), s\_,\right) \text{ is constant for all } s, s\_.$$

As in the section 2 economy with quasilinear preferences, the tax rate is constant when government debt equals $\mathcal{X}(s\_)$. Unlike the section 2 economy, here income effects cause tax revenues $\tilde{Z}\left(s, \mathcal{X}(s\_), s\_\right)$ and consumption $\tilde{c}\left(s, \mathcal{X}(s\_), s\_\right)$ to fluctuate. Therefore, debt choices $\{\tilde{\mathrm{B}}\left(s, \mathcal{X}(s\_), s\_\right)\}_{s\in\mathcal{S}}$ vary with $s\_$ but in ways that make the vector of marginal utility adjusted debt choices $\left[\tilde{c}\left(s, \mathcal{X}(s\_), s\_\right)\right]^{-\sigma}\tilde{\mathrm{B}}\left(s, \mathcal{X}(s\_), s\_\right)$ be constant for all $s\_, s$. The tax rate, consumption, and labor supply behave as in the complete market economy of Lucas and Stokey (1983).

The expression for X^* implies that $sign\left(\tilde{\mathrm{B}}\left(\cdot, \mathcal{X}\cdot,\right)\right) = -sign\left(\tilde{\mathrm{R}}\left(\cdot, \mathcal{X}, \cdot\right), PV_{\tilde{\Phi}}\left(\cdot, \mathcal{X}, \cdot\right)\right)$. This sign can be easily computed. For example, consider a simple economy with a risk-free bond in which shocks s affect only government expenditures g and but not the payoff p or productivity θ and are i.i.d over time. It is easy to confirm that if perfect hedging is attained, then $\operatorname{cov}\left(\tilde{\mathrm{R}}, PV_{\tilde{\Phi}}\right) > 0$ and that therefore perfect hedging can be achieved only with negative government debt. The intuition for this result is as follows. Even if the payoff p does not depend on s, the effective return R is positively correlated both with g and with the effective primary deficit Φ. Therefore, the government can hedge fluctuations in Φ by holding a positive quantity of this security.

Computing optimal policies in more general cases (for example when $p \notin \mathcal{P}$) requires approximating the value function in Bellman equation (42) numerically. However, it is possible to find the target level of debt, i.e., the (effective) level of debt that provides the maximum amount of hedging without having to solve Bellman equation (42) and then computing an invariant distribution implied by the associated optimal policies. We describe the procedure next.

For a given τ solve for a consumption and labor, $c(\tau, s), l(\tau, s)$ allocation using

$$(1-\tau)\theta(s)U_c(\tau, s) + U_l(\tau, s) = 0 \tag{45a}$$

$$\theta(s)l(\tau, s) - c(\tau, s) - g(s) = 0. \tag{45b}$$

From this consumption and labor allocation, construct effective primary deficits and effective returns as

$$\Phi(\tau, s) = \Phi(c(\tau, s), l(c, s)) \text{ and } \mathrm{R}(\tau, s\_, s) = \frac{U_c(\tau, s)p(s)}{\beta\mathbb{E}\left(U_c(\tau)p|s\_\right)}. \tag{46}$$

Under a policy that sets tax rate equal to τ every period, we can iterate on the budget constraint to compute the marginal utility weighted debt as the expected present value of the marginal utility weighted primary surplus

$$X(\tau, s\_) = -\beta\mathbb{E}\left(PV_{\Phi}(\tau)|s\_\right). \tag{47}$$

As in equation (30) we compute the residual in budget constraint

$$D(\tau, s\_, s) = X(\tau, s\_)\mathrm{R}(\tau, s\_, s) + \Phi(\tau)(s) - X(\tau, s) = X(\tau, s\_)\mathrm{R}(\tau, s\_, s) + PV_{\Phi}(\tau)(s). \tag{48}$$

The counterpart of the problem (31) is

$$\tau^* = \arg\min_{\tau} \mathbb{E}^{\lambda}\left(\mathrm{var}(D(\tau)|s\_)\right), \tag{49}$$

where λ is the invariant distribution over s. As before, for $p \in \mathcal{P}^*$, the stochastic steady state coincides with $X(\tau^*, s)$. For more general payoffs we provide counterparts of the expressions in Proposition 2 and Proposition 5. Let $\mathbb{E}$ and var be the expectation and variance taken over the ergodic distribution generated by $\tilde{X}\left(s, X\_, s\_\right)$. The estimates for mean, speed of convergence and variance of the ergodic distribution are

$$\mathbb{E}[\tilde{X}_t|s_t = s] \approx X(\tau^*, s), \tag{50a}$$

$$\mathbb{E}_0[\tilde{X}_t - X(\tau^*, s_t)] \approx (X_0 - X(\tau^*, s_0))\left(\frac{1}{1 + \beta^2\mathrm{var}^{\lambda}(\mathrm{R}(\tau^*))}\right)^t, \tag{50b}$$

$$\mathrm{var}(\tilde{X}_t - X(\tau^*, s_t)) \approx \left(\frac{1 + \beta^2\mathrm{var}^{\lambda}(\mathrm{R}^*)}{\mathrm{var}^{\lambda}(\mathrm{R}(\tau^*))}\right)\mathrm{var}(D(\tau^*)). \tag{50c}$$

In section 5, we show that this procedure does a good job of approximating the mean, variance and speed of convergence to the invariant distribution for $\tilde{X}_t$ in a variety of circumstances.

4.1 Optimal portfolio choice with risk aversion

In this section, we return to the connection with Merton (1969) to emphasize that, even with risk aversion, the relevant objective for the government is minimizing a notion of risk and not a mean-variance trade-off. We then apply this insight in a simple example with two securities, a risk free bond and a risky asset that mimics the "stock market", to show that it is may not be optimal for the government to go long in the stock market even if it yields a higher expected return.[17]

To extend problem (42) to $K>1$ securities with $\{p^k(s_t)\}_k$ as the vector of payoffs we only need to modify the implementability constraint (43). Let $\mathrm{R}^k(s,s\_,c)=\frac{U_c(s)p^k(s)}{\beta\sum_{s'\in S}\pi(s'|s\_)U_c(s)p^k(s')}$ be the effective returns on each of the K securities, the implementability constraint with $K>1$ securities is given by

$$X(s)=\sum_k \mathrm{R}^k\left(s,s\_,c\right)X^k\_+\Phi(s,c), \tag{51}$$

with $\sum_k X^k\_=X\_$ is the marginal utility adjusted total debt. It is easy to check that the Euler equation for the planner holds with respect to each security

$$V'(\tilde{X}_t,s_t)=\mathbb{E}_t V'(\tilde{X}_{t+1},s_{t+1})+\beta\mathrm{cov}_t(\mathrm{R}^k_{t+1},V'(\tilde{X}_{t+1},s_{t+1})). \tag{52}$$

We briefly describe the appropriate static risk-minimization problem in the case with $K>1$ assets that captures dynamic hedging concerns. As in equation (46), construct effective returns for the security with payoff $p^k(s)$ using the allocation defined by equation (45) and $\vec{\mathrm{R}}^k(\tau,s\_,s)=\frac{U_c(\tau,s)p^k(s)}{\beta\mathbb{E}\left(\bar{U}_c(\tau)p^k|s\_\right)}$. We use a two step approach to compute the residual deficit in the government budget constraint. Using $X(\tau,s)$ as in equation (47), we pose the following risk-minimization problem that generalizes problem (25) in the quasilinear setting and extends problem (49). Let $\vec{X}(\tau,s\_)$ be a vector of size K that denotes how a given level of effective assets $X(\tau,s\_)$ is split into effective holdings of the K possible securities.

$$\vec{X}(\tau,s\_)=\arg\min_X\sum_s\pi(s|s\_)\left[\sum_k\vec{\mathrm{R}}^k(\tau,s\_)(s)X^k+PV_{\bar{\Phi}}(\tau)(s)\right]^2 \tag{53}$$

subject to

$$\sum_k X^k=X(\tau,s\_).$$

When $\mathbb{C}[\vec{\mathrm{R}},\vec{\mathrm{R}}]$ has full rank, the optimal portfolio is given by

$$\vec{X}(\tau,s\_)\propto\mathbb{C}[\mathrm{R}(\vec{\tau,s}\_),\mathrm{R}(\vec{\tau,s}\_)|s\_]^{-1}\mathbb{C}[\mathrm{R}(\vec{\tau,s}\_),PV_{\Phi}(\tau)|s\_].$$

To pin down τ^* we compute the residual deficit in the government budget constraint as

$$D(\tau,s\_,s)=\sum_k\vec{\bar{R}}^k(\tau,s\_)(s)\vec{X}^k(\tau,s\_)+PV_{\bar{\Phi}}(\tau)(s)$$

and τ^* minimizes the ergodic variance of $D(\tau)$ exactly as in equation (49).

The reason why the optimal portfolio is different from that of a typical Merton (1969) investor is that our government is benevolent and operates in general equilibrium. Benevolence implies that both the representative consumer and the government have the same attitudes towards risk and end up discounting risky payoffs in exactly the same way. With general equilibrium,

[17] This result contrasts with partial equilibrium arguments of Lucas and Zeldes (2009) that the government should always have a positive position in the security (stock market in their application) that has a positive risk premium.

expected risk-adjusted returns adjusted by marginal utility are equalized across securities. Thus any gains to the government by holding securities with higher expected revenues are offset by losses to the consumer. This is reflected by effective holding period returns R_t and not the returns R_t showing up in the government's Euler equation.

We now construct a simple example to demonstrate that the government may not want to *chase an equity premium*. The representative agent is risk averse with period utility $U(c,l) = \log(c) - \frac{l^2}{2}$ with Frisch elasticity set to 0.5. For simplicity, we only have i.i.d. productivity and payoff shocks. Productivity θ_t distributed as $\log(\theta) \sim \mathcal{N}(0, 0.03)$. For the market structure, the government can trade two assets. The first is a risk free bond with payoff $p^f = 1$. The second is a "stock market" asset which has a payoff with a loading, $\chi > 0$ on $\log(\theta_t)$ plus an orthogonal component

$$p^S = 1 + \chi \log(\theta_t) + 0.01\epsilon_t,$$

where $\epsilon_t \sim \mathcal{N}(0, 1)$ is independent of θ and i.i.d. over time.[18]

Let $R_t^f = \frac{U_{c,t}}{\beta \mathbb{E}_t U_{c,t+1}}$ be the risk-free rate and $R_{t,t+1}^S = \frac{U_{c,t} p_{t+1}^S}{\beta \mathbb{E}_t p_{t+1}^S U_{c,t+1}}$ be the holding period return on the risky asset. This environment features an equity premium, $\mathbb{E}_t R_{t+1}^S > R_t^f$, but the government does not choose invest in the stock market asset. Instead, when the government is initially in debt, the government begins by shorting the stock market and in the long run holding all of it's assets in the risk free bond. Although the initial short position in the stock market exposes the government to the orthogonal variation ϵ_t, it provides a good (but temporary) hedge by delivering higher returns in times of low TFP. Eventually the government only uses the risk free bond to hedge. We see the dynamic portfolio re-balancing in Figure II. The two lines show the marginal utility adjusted positions in the risk free security (blue line) and the risky security (green line).

5 A quantitative example

We assume that a representative household has separable, isoelastic preferences $U(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. Labor productivity follows an AR(1) process

$$\log \theta_t = \rho_\theta \log \theta_{t-1} + \sigma_\theta \epsilon_{\theta,t}, \tag{54}$$

where $\epsilon_{\theta,t}$ is an i.i.d. standard normal variable. A stochastic payoff p_t from a single security is

$$p_t = 1 + \chi_p \epsilon_{\theta,t} + \sigma_p \epsilon_{p,t}, \tag{55}$$

where $\epsilon_{p,t}$ is an i.i.d. standard normal variable; χ_p and σ_p parameterize variance and correlation of asset returns with output. Analogous to equation (55), we parametrize government expenditures as

$$\log g_t = \log \bar{g} + \chi_g \epsilon_{\theta,t} + \sigma_g \epsilon_{g,t}, \tag{56}$$

where $\epsilon_{g,t}$ is another i.i.d. standard normal variable.

[18] We approximate $\log(\theta)$ and ϵ_t with a finite state Markov process using Gaussian quadrature.

Baseline calibration

We set $\sigma = 1$ and $\gamma = 2$ to attain a Frisch elasticity of labor supply of 0.5. We set $\beta = 0.98$, which implies a 2% annual interest rate in an economy without shocks. We calibrate parameters $\rho_\theta, \sigma_\theta, \chi_p, \chi_g, \sigma_g, \sigma_p$ in equations (54)-(56) using moments from a competitive equilibrium with an exogenous government policy that fits some stylized features of actual U.S. debt in the post war period.

We estimate policy rules that the U.S. government uses for (B_t, τ_t) as follows. We assume that B_t follows a rule

$$\log B_t = (1 - \rho_{B,B}) \log \bar{B} + \rho_{B,B} \log B_{t-1} + \rho_{B,Y} \log Y_t + \rho_{B,Y\_} \log Y_{t-1}, \tag{57}$$

where we estimate coefficients $\rho_{B,B}, \rho_{B,Y}, \rho_{B,Y\_}$ using market value of gross federal debt (annual) series for the period 1947-2010 published by the Federal Reserve Bank of Dallas for B_t.[19] For Y_t, we use the annual aggregate labor earnings from the Bureau of Economic Analysis.[20] Both B_t and Y_t are Hodrick-Prescott pre-filtered (after taking logs) using a smoothing parameter of 6.25. Then we set τ_t as a residual to make sure that the government budget constraint is satisfied.[21] Table I reports the estimates and figure III shows the fit.[22]

Using the government policy described in (57), we set the parameters $\rho_\theta, \sigma_\theta, \chi_p, \chi_g, \sigma_g, \sigma_p$ to jointly match moments for output, returns, and government expenditures for the period 1947-2010.

We impute returns R_t using data on the federal primary surplus PS_t [23] and market value of debt as follows. In the data the duration of government debt is approximately constant,[24] therefore the budget constraint of the government can be written as

$$(p_t + q_t)\mathrm{B}_{t-1} = q_t\mathrm{B}_t + PS_t. \tag{58}$$

Multiply and divide the first term by q_{t-1} and use the fact that the holding period return for long term debt is given by $R_t = \frac{q_t + p_t}{q_{t-1}}$ (see example 3 in section 3.1) to rewrite equation (58) as

$$R_t = \frac{B_t + PS_t}{B_{t-1}}, \tag{59}$$

where $B_t = q_t\mathrm{B}_t$ is the market value of government debt observed in the data. We use this formula to compute the holding period return.

The average return annual over the sample 1947-2010 is 5% with an annual standard devia-

[19] The calculation of this series takes into account outstanding marketable and non-marketable debt of different maturities issued by the Treasury and uses current market prices to convert par value to market value.

[20] Since in our model we abstract from capital, our measure of output Y is aggregate labor earnings. The results remain essentially unchanged if we use GDP per capita instead.

[21] We obtain very similar result if we estimate policy rules for τ_t along the lines of equation (57) using Barro and Redlick (2011) data on average marginal income tax rates

[22] Our results are essentially unchanged by adding more lags debt (or taxes) or output. We report the results with the specification that has one lag of debt, contemporaneous output, and one lag of output because it fits the data well (see figure III) and also parsimoniously captures short-run dynamics of the optimal policy. This allows us to compare an optimal policy and our approximation to the U.S. policy.

[23] This is measured as total federal receipts net of federal consumption and transfer payments, both obtained from the Bureau of Economic Analysis.

[24] From 1985 to 2010 the average maturity of U.S. federal debt has been around 60 months.

tion of 5%. A key empirical fact that drives a lot of our results is that a substantial component of fluctuations in returns is uncorrelated with fundamentals.[25] For government expenditures, we use federal government consumption expenditures plus transfer payments obtained from the Bureau of Economic Analysis. Table II summarizes parameters values and the fit of the competitive equilibrium to U.S. data.

Using this calibration, we compute a global approximation to the Ramsey allocation.[26]

Results

Table III reports the long run properties of debt and taxes under the optimal plan. Convergence is slow (half life of 250 years). In the long run under an optimal plan, government debt has large fluctuations (standard deviation of 20%) while there are small movements in tax rates and tax revenues (standard deviations of 0.5% and 0.6% respectively). Figures IV and V plot the ergodic distribution and mean path across 30,000 simulations.

Sections 2 and 4 showed that an appropriately posed static variance minimization problem anchors the long run debt level. Further, we used insights from quadratic approximations of the policy rules to obtain simple expressions that map moments of the joint distribution of returns and deficits to the speed of convergence and variance around the long run levels. We now verify that these descriptions apply to our calibrated economy.

Expressions in equations (50) are now compared against the mean, speed of convergence and variance computed from long simulations using our global approximation to the optimal policy rules. Table IV summarizes the results for effective debt, X_t. The simple formulas do well in predicting the ergodic outcomes. Moreover, we see in Figure VI that formula (50b) gives a very accurate approximation for the evolution of $\mathbb{E}_0 X_t$.

Formula (50) is evaluated at the optimal policy τ^*, and an intermediate step is to compute τ^* using (49). We show that a simple back-of-the-envelope calculation that constructs effective returns and deficits directly from the data and that evaluates optimal debt level using (50) directly (thus implicitly ignoring the fact that tax rates in the data may not be set optimally) gives a good approximation to the target debt level, the speed of convergence, and the variance around the target debt level.

To make this operational, we exploit the relationship between the one period holding returns R_t and the effective return R_t,

$$R_t = \mathrm{R}_t \left(\frac{U_{c,t-1}}{U_{c,t}} \right) \tag{60}$$

as well the relationship between the primary surplus PS_t and the marginal utility weighted surplus Φ_t

$$\Phi_t = U_{c,t} PS_t. \tag{61}$$

Imputing R_t from equation (59) and time series on real personal consumption expenditure, we

[25] Alternative ways of measuring returns yield similar conclusions. In Bhandari et al. (2015a), we study holding period returns across debts of different maturities and document the presence of a component that is orthogonal to fundamentals like productivity across several maturities.

[26] We follow Kopecky and Suen (2010) to obtain a 5 state discrete approximation for the AR(1) process in equation (54). For the shocks ϵ_p, ϵ_g we use a 5 state Gaussian quadrature. The Bellman equation is solved using a value function iteration method where we approximate the value function using a 30 point grid with linear splines as basis functions.

can use (60) to infer R_t. For our sample, the returns R_t and R_t have a correlation coefficient of 0.91. We compute PV_Φ by estimating a first order VAR using $[\log Y_t, \Phi_t]$. The estimates allow us to express $PV_{\Phi,t} = \alpha_y \log Y_t + \alpha_\Phi \Phi_t$. We use these estimates in the formula $X^* = -\frac{\mathrm{cov}(\mathrm{R}_t, PV_{\Phi,t})}{\mathrm{var}(\mathrm{R}_t)}$ to obtain,

$$X^* = -\frac{\alpha_y \mathrm{cov}(\mathrm{R_t}, Y_t) + \alpha_\Phi \mathrm{cov}(\mathrm{R_t}, \Phi_t)}{\mathrm{var}(\mathrm{R}_t)} = -0.08,$$

where $\alpha_y = 1.138$ and $\alpha_\Phi = -0.80$ from our VAR estimation. Similarly, using the sample variance of effective returns in equation (50b) yields a half life of 244 years. These alternative computations compares well with our estimates previous reported in Table IV

As our final exercise, we compare the optimal plan to actual U.S. policies. We use short sample simulations of length 63 periods and compare the volatility and persistence of debt and the tax rate for the optimal plan and U.S. data. Table V reports the comparison with U.S. data. From this exercise we conclude that tax rates responded little to shocks under both the optimal plan and in the U.S. data, but that they are more persistent under an optimal plan. As a consequence, government debt is more volatile and is repaid more slowly under the optimal plan.

6 Concluding remarks

This paper develops a theory of optimal government debt management. We show a generic insight that the dynamic hedging considerations are captured by a simple problem that minimizes a particular measure of fiscal risk. Our analysis delivers simple formulas for the mean, variance and speed of convergence to the ergodic distribution of debt. Here we also analyze some extensions of our basic environment, an endeavor we pursue more in Bhandari et al. (2015b) that studies economies whose substantial *ex ante* heterogeneity coming from permanent or persistent differences in skills that unleashes motives for redistribution and social insurance. Our analysis in this paper sets the stage for such work – not only by providing appropriate tools for approximating equilibria well and for formulating Ramsey problems in mathematically convenient ways, but also in terms of highlighting forces that ultimately determine transient and long-run dynamics of debt and taxes. For example, appropriately adjusted fiscal risk minimization problems continue to shape the mean of an ergodic distribution of government debt, while the hedging costs of being away from that fiscal risk-minimizing debt level shape the speed of convergence. In another extension Bhandari et al. (2015a), we shall use the empirical properties of returns across maturities to compute an optimal term structure of government debt.

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7 Appendix

7.1 Proof of Proposition 1

We first show preliminary results that we discussed in Section 2. Let $\left\{\tilde{B}_t, \tilde{l}_t\right\}_{t=0}^{\infty}$ be a solution to (10) and $\tilde{Z}_t = \tilde{l}_t - \tilde{l}_t^{1+\gamma}$.

Lemma 3. *$\tilde{l}_t^{\gamma} \geq \frac{1}{1+\gamma}$ for all t and there is one to one map between $\tilde{l}_t$ and $\tilde{Z}_t$ with $\tilde{Z}_t \leq \overline{Z}$ for all t. The function $\Psi : (-\infty, \overline{Z}] \to \mathbb{R}$ is strictly decreasing, strictly concave, differentiable and satisfies $\lim_{Z \to -\infty} \Psi'(Z) = 0$ and $\lim_{Z \to \overline{Z}} \Psi'(Z) = -\infty$.*

Proof. First we show that $\tilde{l}_t^{\gamma} \geq \frac{1}{1+\gamma}$ for all t. Suppose there exists a k such that $\tilde{l}_k^{\gamma} < \frac{1}{1+\gamma}$. Let total revenues $Z(l) = l - l^{1+\gamma}$. The maximum value of $Z(\cdot)$ is achieved at $l^* = \frac{1}{(1+\gamma)^{\frac{1}{\gamma}}}$. Since $Z(\infty) = -\infty < Z(\tilde{l}_k) < Z(l^*)$, applying the Intermediate Value Theorem, we can find a $l_k^{'} > \frac{1}{(1+\gamma)^{\frac{1}{\gamma}}} > \tilde{l}_k$ such that $Z(l_k^{'}) = Z(\tilde{l}_k)$. Construct an alternative sequence of $\left\{\tilde{B}_t, \hat{l}_t\right\}_{t=0}^{\infty}$ where $\hat{l}_t = l_k^{'}$ for $t = k$ and $\hat{l}_t = \tilde{l}_t$ for $t \neq k$. This sequence $\left\{\tilde{B}_t, \hat{l}_t\right\}_{t=0}^{\infty}$ also satisfies constraints (9) for all t but has a strictly higher welfare as the objective function (10) is strictly increasing in l_t. Thus, we obtain a contradiction that $\{\tilde{B}_t, \tilde{l}_t\}$ is a optimal solution.

Since $\tilde{l}_t^{\gamma} \geq \frac{1}{1+\gamma}$, it implies that $\tilde{Z}_t$ is bounded above by

$$\bar{Z} \equiv \gamma \left(\frac{1}{1+\gamma}\right)^{1+1/\gamma} \tag{62}$$

and the relationship between $\tilde{Z}_t$ and $\tilde{l}_t$ is one to one in the relevant range of $\tilde{l}_t$. In the text we used the function, $\Psi : (-\infty, \bar{Z}] \to \mathbb{R}$ that satisfies

$$\Psi(l - l^{1+\gamma}) = \frac{1}{1+\gamma} l^{1+\gamma}$$

to denote the utility cost of raising revenues Z. Differentiating with respect to l we get

$$\Psi'(l - l^{1+\gamma}) = \frac{l^{\gamma}}{[1 - (1+\gamma)l^{\gamma}]}.$$

On the domain where $\Psi(\cdot)$ is defined $l^{\gamma} \geq \frac{1}{1+\gamma}$ and thus $\Psi'(\cdot) < 0$. As $l^{\gamma} \to \frac{1}{1+\gamma}$ from above, $Z \to \bar{Z}$ and $\Psi'(\cdot) \to \infty$. Similarly $l^{\gamma} \to \infty$, $Z \to -\infty$ and $\Psi'(\cdot) \to 0$. Differentiating twice with respect of l we get

$$\Psi''(\cdot)\left[1 - (1+\gamma)l^{\gamma}\right] - \Psi'(\cdot)\left[\gamma(1+\gamma)l^{\gamma-1}\right] = \gamma l^{\gamma-1}.$$

Using both $\Psi'(\cdot) \leq 0$ and $l^{\gamma} \geq \frac{1}{1+\gamma}$ we get that $\Psi''(\cdot) < 0$.

We can now prove Proposition 1. The optimality conditions to (12) are

$$V'(B\_) = \beta \int R(s)\, V'\left(\tilde{B}(s, B\_)\right) \pi(ds) - \bar{\kappa} + \underline{\kappa} \tag{63}$$

and

$$\gamma \Psi'\left(\tilde{Z}(s, B\_)\right) = -1 + \beta V'\left(\tilde{B}(s, B\_)\right) - \bar{\kappa}(s) + \underline{\kappa}(s), \tag{64}$$

where $\bar{\kappa}(s)$ and $\underline{\kappa}(s)$ are the Lagrange multipliers on $B \leq \overline{B}$ and $B \geq \underline{B}$, and $\bar{\kappa} = \int \bar{\kappa}(s) R(s) \pi(ds)$, $\underline{\kappa} = \int \underline{\kappa}(s) R(s) \pi(ds)$. We begin with the following lemma that shows that $\tilde{B}(s, B\_)$ is strictly increasing in E. $\square$

Lemma 4. *If $\tilde{B}(s, B\_) \in (\underline{B}, \overline{B})$ and $E(s, B\_) > (<) E(s', B\_)$ then $\tilde{B}(s, B\_) > (<) \tilde{B}(s', B\_)$*

Proof. We will prove the case where $E(s, B\_) > E(s', B\_)$, the other case is symmetric. From Lemma 2, $\tilde{B}(s, B\_)$ is weakly increasing in $E(s, B\_)$, and thus, $\tilde{B}(s, B\_) \geq B(s', B\_)$. Suppose $\tilde{B}(s', B\_) = \tilde{B}(s, B\_)$ then $\tilde{Z}(s', B\_) < \tilde{Z}(s, B\_)$ from the budget constraint (13). Strict concavity of $\Psi(\cdot)$ and an interior solution to (64) then implies

$$
\begin{aligned}
-1 + \beta V'(\tilde{B}(s', B\_)) &= -1 + \beta V'(\tilde{B}(s, B\_)) \\
&= \gamma \Psi'(\tilde{Z}(s, B\_)) \\
&< \gamma \Psi'(\tilde{Z}(s', B\_)).
\end{aligned}
$$

As $\tilde{B}(s', B\_) \in (\underline{B}, \overline{B})$, this violates (64) for $(s', B\_)$ yielding a contradiction. Thus $\tilde{B}(s, B\_) > \tilde{B}(s', B\_)$. $\square$

First, suppose $R \in \mathcal{R}^*$. Since in this case $B^* = B^{min} = B^{max}$, Lemma 2 implies that $\tilde{B}(s, B^*) = B^*$ for all s. Since $\tilde{B}(s, \cdot)$ is increasing by Lemma 2, it satisfies $\tilde{B}\left(s, B\_\right) \geq B^*$ for all $B\_ \geq B^* > \underline{B}$ and $\tilde{B}\left(s, B\_\right) \leq B^* < \overline{B}$ for all $B\_ \leq B^*$, where strict inequalities follow from the assumption that $B^* \in \left(\underline{B}, \overline{B}\right)$. If $B_0 \geq B^*$ then $\tilde{B}_t \geq B^* > \underline{B}$ for all t implies that $\underline{\kappa}_t = 0$ and from (63)

$$
\begin{aligned}
V'\left(\tilde{B}_t\right) &\leq \beta \mathbb{E} R(\cdot) V'\left(\tilde{B}\left(\cdot, \tilde{B}_t\right)\right) \\
&= \mathbb{E} V'\left(\tilde{B}\left(\cdot, \tilde{B}_t\right)\right) + \beta \operatorname{cov}\left(R(\cdot), V'\left(\tilde{B}\left(\cdot, \tilde{B}_t\right)\right)\right) \\
&\leq \mathbb{E} V'\left(\tilde{B}\left(\cdot, \tilde{B}_t\right)\right) \\
&= \mathbb{E} V'(\tilde{B}_{t+1})
\end{aligned}
$$

where the last inequality follows from the fact that $R(s)$ is increasing by construction, while $V'\left(\tilde{B}\left(\cdot, \tilde{B}_t\right)\right)$ is decreasing in s by concavity of V and the fact that $\tilde{B}\left(s, \tilde{B}_t\right)$ is increasing in s by Lemma 2. $\tilde{B}_t \geq B^*$ implies that $V'(\tilde{B}_t) \leq V'(B^*)$ for all t. Therefore $V'\left(\tilde{B}_t\right)$ is a bounded submartingale, which means it must converge by the Martingale Convergence Theorem. Continuity of V' and strict concavity of V implies that there exists a $\mathcal{B}$ such that $\lim_{t\to\infty} \tilde{B}_t = \mathcal{B}$ a.s. We claim that $\tilde{B}(s, \mathcal{B}) = \mathcal{B}$ for all s in the support of π. Suppose $\tilde{B}(s', \mathcal{B}) > \mathcal{B}$ (the argument for the other case is symmetric) for some s' then $\tilde{B}(s'', \mathcal{B}) > \mathcal{B}$ for all $s'' > s'$ as $\mathcal{B} > B^*$. As $\tilde{B}_t \to \mathcal{B}$ almost surely there exists a subsequence t_n with $s_{t_n} > s'$ for all n. But as

$$
\lim_{n\to\infty} \tilde{B}_{t_n} = \lim_{n\to\infty} \tilde{B}(s_{t_n}, \tilde{B}_{t_n - 1}) \geq \lim_{n\to\infty} \tilde{B}(s', \tilde{B}_{t_n-1}) = \tilde{B}(s', \lim_{n\to\infty} \tilde{B}_{t_n-1}) = \tilde{B}(s', \mathcal{B}) > \mathcal{B},
$$

we immediately obtain a contradiction. As $\tilde{B}(s, \mathcal{B}) = \mathcal{B}$ for all s, we can conclude, from Lemma 4, that either $\mathcal{B} = \overline{B}$ or $E(s, \mathcal{B})$ is independent of s. From (63), $\tilde{B}(s, \overline{B}) = \overline{B}$ for all s if and only if $\overline{\kappa} = 0$. The same arguments as in Lemma 4 then shows that $\mathcal{B} = \overline{B}$ if and only if $E(s, \overline{B})$ is independent of s. We have therefore shown the limit $\mathcal{B}$ must satisfy the property that $E(s, \mathcal{B})$ is independent of s for all s. As $R \in \mathcal{R}^*$, there is a unique point B^* that satisfies this property. Thus, if $B_0 > B^*$ then $\tilde{B}_t \to B^*$ a.s. The case when $B_0 < B^*$ is symmetric.

Suppose $R \notin \mathcal{R}^*$, which implies that $E(\cdot, B\_)$ is not a constant for all $B\_$. We want to show that there are sets $\mathcal{S}', \mathcal{S}'' \subset \mathcal{S}$ of positive measure such that $\tilde{B}(s', B\_) \geq B\_$ for all $s' \in \mathcal{S}'$ and $B\_ \geq \tilde{B}(s'', B\_)$ for all $s'' \in \mathcal{S}''$ with at least one inequality strict and both inequalities strict if $B\_ \in (\underline{B}, \overline{B})$. Suppose $\overline{B} > B\_ \geq \tilde{B}(s, B\_)$ for almost all s, which by strict concavity of V and strict monotonicity of $\tilde{B}$ in $E(s, B\_)$ implies $V'(B\_) \leq \beta \mathbb{E} R(\cdot) V'\left(\tilde{B}(\cdot, B\_)\right)$ with equality only if $B\_ = \tilde{B}(s, B\_) = \underline{B}$ for almost all s. In the previous paragraph we showed that the latter can occur if and only if $E(s, B)$ is independent of s. As $R \notin \mathcal{R}^*$, this cannot be the case, and thus $V'(B\_) < \beta \mathbb{E} R(\cdot) V'\left(\tilde{B}(\cdot, B\_)\right)$. On the other hand equation (63) implies that $V'(B\_) \geq \beta \mathbb{E} R(\cdot) V'\left(\tilde{B}(\cdot, B\_)\right)$, a contradiction . Thus, if $\overline{B} > B\_$ then there exists $\mathcal{S}' \subset S$ with positive measure such that $\tilde{B}(s', B\_) > B\_$ for all $s' \in \mathcal{S}'$. Analogous arguments show that if $B\_ > \underline{B}$ then there exits $\mathcal{S}''$ with positive measure such that $\tilde{B}(s'', B\_) < B\_$ for all $s'' \in \mathcal{S}''$ and that there exist s' and s'' such that $B(s', \overline{B}) = \overline{B}$ and $\tilde{B}(s'', \underline{B}) = \underline{B}$.

Pick any small $\epsilon > 0$. The results of the previous section imply that both $\mathbb{E}\left[\tilde{B}(s, B\_) - B\_ \middle| \tilde{B}(s, B\_) \geq B\_\right]$ and $\mathcal{P}\left(\left\{\tilde{B}(s, B\_) \geq B\_\right\}\right)$ are positive for all $B\_ \in [\underline{B}, \overline{B} - \epsilon]$. As both of these terms are continuous functions of $B\_$, compactness of $[\underline{B}, \overline{B} - \epsilon]$ implies that there exists $\underline{d}, \underline{p} > 0$ such that

$$\mathbb{E}\left[\tilde{B}(s, B\_) - B\_ \middle| \tilde{B}(s, B\_) \geq B\_\right] \geq \underline{d} \tag{65}$$

and

$$\mathcal{P}\left(\left\{\tilde{B}(s, B\_) \geq B\_\right\}\right) \geq \underline{p} \tag{66}$$

for all $B\_ \in [\underline{B}, \overline{B} - \epsilon]$. As $\tilde{B}(s, B\_) - B\_$ is bounded above by $\overline{D} = \overline{B} - \underline{B}$, we obtain that[27]

$$\mathcal{P}\left(\left\{\tilde{B}(s, B\_) - B\_ > \underline{d}/2\right\}\right) \geq \frac{\underline{d}/2}{\overline{D}} \underline{p}$$

for all $B\_ \in [\underline{B}, \overline{B} - \epsilon]$. Therefore, there must exist an integer n and $\varrho > 0$ that that probability of reaching $\left[\overline{B} - \epsilon, \overline{B}\right]$ starting from any $B\_ \in \left[\underline{B}, \overline{B} - \epsilon\right]$ in n steps is greater than ϱ. Analogous arguments establish that probability of reaching $\left[\underline{B}, \overline{B} - \epsilon\right]$ starting from $B\_ \in \left[\overline{B} - \epsilon, \overline{B}\right]$ in finite number of steps is finite. Since by Lemma 2 policy functions are monotone in $B\_$, Theorem 2 in Hopenhayn and Prescott (1992) establishes the existence of an unique invariant distribution.

[27]The arbitrary function $f(s, B\_)$ that minimizes probability of $\mathcal{P}\left(\left\{f(s, B\_) - B\_ > \frac{\underline{d}}{2}\right\}\right)$ while still satisfying equations (65) and (66) is obtained by placing a mass α on $\overline{B}$ and a mass $P\left(\left\{\tilde{B}(s, B\_) \geq B\_\right\}\right) - \alpha$ on $\frac{\underline{d}}{2}$. As $\overline{B} - B\_ \leq \overline{D}$, α can be bounded from below by $\frac{\underline{d}/2}{\overline{D} - \underline{d}/2} \underline{p}$.

7.2 Proof of Proposition 2

We first prove the result under the assumption of a continuous state space (as the arguments are more transparent) and then show how to extend the results to settings with possibly discrete state space. Let $\tilde{\mu}(B\_) \equiv V'(B\_)$. When the solution is interior, the first order condition with respect to $Z(s)$, equation (64) implies that

$$\tilde{Z}\left(s, B\_\right) = \Psi'^{-1}\left(\frac{-1 + \beta\tilde{\mu}\left(\tilde{B}(s, B\_)\right)}{\gamma}\right) \equiv \Phi\left(\tilde{\mu}\left(\tilde{B}(s, B\_)\right)\right).$$

Since shocks are i.i.d., $s = (g, p)$ and equations (13) and (63) can be written as

$$\frac{p}{\beta\bar{p}}B\_ + g = \Phi\left(\tilde{\mu}\left(\tilde{B}(g, p, B\_)\right)\right) + \tilde{B}(g, p, B\_), \tag{67}$$

and

$$\tilde{\mu}(B\_) = \mathbb{E}\left[\frac{p}{\bar{p}}\tilde{\mu}\left(\tilde{B}(g, p, B\_)\right)\right]. \tag{68}$$

Equation (67) must hold for all $g, p, B\_$ for which policy functions are interior. Therefore, differentiating both sides with respect to p and g we have

$$\frac{1}{\beta\bar{p}}B\_ = \left[\Phi_\mu\left(\tilde{\mu}\left(\tilde{B}(g, p, B\_)\right)\right)\tilde{\mu}_{B\_}\left(\tilde{B}(g, p, B\_)\right) + 1\right]\tilde{B}_p(g, p, B\_) \tag{69}$$

and

$$1 = \left[\Phi_\mu\left(\tilde{\mu}\left(\tilde{B}(g, p, B\_)\right)\right)\tilde{\mu}_{B\_}\left(\tilde{B}(g, p, B\_)\right) + 1\right]\tilde{B}_g(g, p, B\_), \tag{70}$$

where Φ_μ and $\tilde{\mu}_{B\_}$ are derivatives of Φ and $\tilde{\mu}$, $\tilde{B}_p$ and $\tilde{B}_g$ are partial derivatives of $\tilde{B}$ with respect to p and g. The derivatives of (67) and (68) with respect to $B\_$ give

$$\frac{p}{\beta\bar{p}} = \left[\Phi_\mu\left(\tilde{\mu}\left(\tilde{B}(g, p, B\_)\right)\right)\tilde{\mu}_{B\_}\left(\tilde{B}(g, p, B\_)\right) + 1\right]\tilde{B}_{B\_}(g, p, B\_), \tag{71}$$

$$\tilde{\mu}_{B\_}(B\_) = \mathbb{E}\left[\frac{p}{\bar{p}}\tilde{\mu}_{B\_}\left(\tilde{B}(g, p, B\_)\right)\tilde{B}_{B\_}(g, p, B\_)\right]. \tag{72}$$

Similarly, we can obtain all second order derivatives and their cross-partials by further differentiating these equations.

We will study the properties of the optimal debt $\tilde{B}(g, p, B\_)$ when shocks are *small*. To do so, without loss of generality, we parametrize variables p and g as $p = \bar{p} + \sigma_{pg}\sigma\epsilon_p$ and $g = \bar{g} + \sigma\epsilon_g$, for some $\sigma, \sigma_{pg} \geq 0$ where ϵ_g and ϵ_p are random variables with mean 0 and standard deviation 1. We study the behavior of the policy functions in the neighborhood of $\sigma = 0$. To emphasize dependence on σ we use notation $\tilde{B}(g, p, B\_; \sigma)$ and $\tilde{\mu}\left(B\_; \sigma\right)$. $\tilde{B}(g, p, B\_; 0) = B\_$ and by definition $p = \bar{p}$ and $g = \bar{g}$. Then equations (71) and (72) become

$$\frac{1}{\beta} = \left[\Phi_\mu\left(\tilde{\mu}\left(B\_\right)\right)\tilde{\mu}_{B\_}\left(B\_\right) + 1\right]\tilde{B}_{B\_}(B\_),$$
$$\tilde{\mu}_{B\_}(B\_) = \tilde{\mu}_{B\_}\left(B\_\right)\tilde{B}_{B\_}(B\_),$$

where to simplify the notation we use $(B\_)$ as a shortcut for $(\bar{g}, \bar{p}, B\_; 0)$. These equations immediately give

$$\tilde{B}_{B\_}(B\_) = 1,\ \Phi_{\mu}\left(\tilde{\mu}\left(B\_\right)\right)\tilde{\mu}_{B\_}\left(B\_\right) + 1 = \frac{1}{\beta}. \tag{73}$$

Substitute these expressions into (69) and (70) to obtain

$$\tilde{B}_p(B\_) = \frac{B\_}{\bar{p}},\ \tilde{B}_g(B\_) = \beta.$$

Using the same steps for the second order derivatives we can show that

$$\begin{aligned}
\tilde{B}_{B_B_}(B_) = \tilde{B}_{gg}(B_) = \tilde{B}_{pp}(B_) = \tilde{B}_{pg}(B_) = 0,\\
\Phi_{\mu\mu}\left(\tilde{\mu}\left(B_\right)\right)\left(\tilde{\mu}_{B_}\left(B_\right)\right)^2 + \Phi_{\mu}\left(\tilde{\mu}\left(B_\right)\right)\tilde{\mu}_{B_B_}\left(B_\right) = 0.
\end{aligned} \tag{74}$$

For any σ we can write policy functions as $\tilde{B}(\bar{g} + \sigma\epsilon_g, \bar{p} + \sigma_{pg}\sigma\epsilon_p, B\_; \sigma)$. Using the second order Taylor expansion function $\tilde{B}$ with respect to σ around $\sigma = 0$, we get

$$\begin{aligned}
\tilde{B}(\bar{g} + \sigma\epsilon_g, \bar{p} + \sigma_{pg}\sigma\epsilon_p, B_; \sigma) = B_ + \tilde{B}_g(B_)\sigma\epsilon_g + \tilde{B}_p(B_)\sigma\sigma_{pg}\epsilon_p + \tilde{B}_\sigma(B_)\\
+ \frac{1}{2}\tilde{B}_{\sigma\sigma}(B_)\sigma^2 + \tilde{B}_{\sigma p}(B_)\sigma^2\sigma_{pg}\epsilon_p + \tilde{B}_{\sigma g}(B_)\sigma^2\epsilon_g + \mathcal{O}(\sigma^3),
\end{aligned} \tag{75}$$

where we used the omitted the second derivatives that are equal to zero by (74). To evaluate the derives of $\tilde{B}$ with respect to σ differentiate (67) with respect to σ:

$$\begin{aligned}
\frac{\sigma_{pg}\epsilon_p}{\beta\bar{p}}B_ + \epsilon_g &= \left[\Phi_{\mu}\left(\tilde{\mu}\left(B_\right)\right)\tilde{\mu}_{B_}\left(B_\right) + 1\right]\left(\tilde{B}_g(B_)\epsilon_g + \tilde{B}_p(B_)\sigma_{pg}\epsilon_p + \tilde{B}_\sigma(B_)\right)\\
&+ \Phi_{\mu}\left(\tilde{\mu}\left(B_\right)\right)\tilde{\mu}_{\sigma}\left(B_\right)\\
&= \frac{1}{\beta}\left(\beta\epsilon_g + \frac{B_}{\bar{p}}\sigma_{pg}\epsilon_p + \tilde{B}_\sigma(B_)\right) + \Phi_{\mu}\left(\tilde{\mu}\left(B_\right)\right)\tilde{\mu}_{\sigma}\left(B_\right),
\end{aligned}$$

where we used (73) to get the second equality. This implies that

$$0 = \tilde{B}_\sigma(B\_) + \Phi_{\mu}\left(\tilde{\mu}\left(B\_\right)\right)\tilde{\mu}_{\sigma}\left(B\_\right).$$

Similarly, differentiate (68) with respect to σ and use $\mathbb{E}\epsilon_p = 0$ to get

$$\tilde{\mu}_{\sigma}\left(B\_\right) = \tilde{B}_\sigma(B\_) = 0.$$

Similar steps for these second order derivatives give

$$\tilde{B}_{\sigma\sigma}(B\_) = -2\mathbb{E}\left[\frac{\sigma_{pg}}{\bar{p}}\epsilon_p\left(\frac{\sigma_{pg}}{\bar{p}}\epsilon_p B\_ + \beta\epsilon_g\right) + \frac{\Phi(B\_)(1-\beta)}{\beta\Phi_\mu(\mu(B\_))^2}\left(\frac{\sigma_{pg}}{\bar{p}}\epsilon_p B\_ + \beta\epsilon_g\right)^2\right]$$

and

$$\tilde{B}_{\sigma p}(B\_) = \tilde{B}_{\sigma g}(B\_) = 0.$$

Let $\sigma_p = \sigma_{pg}\sigma$, $\hat{g} = \sigma\epsilon_g$, $\hat{p} = \sigma_p\epsilon_p$. Then the expansion in (75) can be written as

$$\begin{aligned}\tilde{B}(g,p,B\_;\sigma) =& B\_ + \beta\hat{g} + B\_\frac{\hat{p}}{\bar{p}} - \beta\text{cov}(p,g) - B\_\sigma_p^2 \\ &+ \frac{\Phi_{\mu\mu}(\mu(B\_))(1-\beta)}{2\Phi_\mu(\mu(B\_))}\left[\beta\sigma^2 + 2B\_\text{cov }(p,g) + \beta^{-1}B\_{}^2\frac{\sigma_p^2}{\bar{p}^2}\right] + \mathcal{O}(\sigma^3) \\ =& B\_ + \beta\hat{g} + B\_\frac{\hat{p}}{\bar{p}} - \beta\text{cov}(p,g) - B\_\sigma_p^2 + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned}$$

We can write (17) as $B^* = -\frac{\beta\text{cov}(p,g)}{\text{var}(p)}$. Furthermore, we have $1-\sigma_p^2 = \frac{1}{1+\sigma_p^2} + \mathcal{O}(\sigma^3)$ and $\sigma_p = \frac{\sigma_p}{1+\sigma_p^2} + \mathcal{O}(\sigma^3)$. Therefore we can write the expression above as

$$\tilde{B}(g,p,B\_;\sigma) = B^* + \beta\hat{g} + B^*\frac{\hat{p}}{\bar{p}} + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - B^*) + \frac{1}{1+\sigma_p^2}(B\_ - B^*) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).$$

Assuming that terms $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$ are small, the approximated policy rules $\check{B}$ are

$$\check{B}(g,p,B\_) = B^* + \beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^* + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - B^*) + \frac{1}{1+\sigma_p^2}(B\_ - B^*), \tag{76}$$

which induce the approximate solution $\{\check{B}_t\}_{t=0}^\infty$. Using (76) we can immediately verify the mean and the speed of convergence of the stochastic process for $\check{B}_t$ satisfies equations in Proposition 2. We next compute the expression for the ergodic variance.

Let F be the ergodic distribution of $(\check{B}_t, g_t, p_t)$ generated by the approximated policy rule $\check{B}(g,p,B\_)$ along with the stochastic process for the exogenous shocks(g,p). Let $\text{var}_F(\check{B}) \equiv \mathbb{E}_F\left[\left(\check{B} - B^*\right)^2\right]$, the ergodic variance of $\check{B}_t$. As F is the ergodic distribution generated by $\check{B}$,

$$\begin{aligned}\text{var}_F(\check{B}) &= \mathbb{E}_F\left[\left(\check{B}(g,p,B\_) - B^*\right)^2\right] \\ &= \mathbb{E}_F\left[\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^* + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - B^*) + \frac{1}{1+\sigma_p^2}(B\_ - B^*)\right)^2\right] \\ &= \mathbb{E}_F\left[\mathbb{E}_F\left[\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^* + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - B^*) + \frac{1}{1+\sigma_p^2}(B\_ - B^*)\right)^2\middle| B\_\right]\right] \\ &= \mathbb{E}_F\left\{\mathbb{E}_F\left[\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^*\right)^2\middle| B\_\right] + \left(\frac{2}{1+\sigma_p^2}\right)\mathbb{E}_F\left[\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^*\right)\frac{\hat{p}}{\bar{p}}\middle| B\_\right](B\_ - B^*)\right. \\ &\qquad \left. + \left(\frac{1+\sigma_p^2}{(1+\sigma_p^2)^2}\right)(B\_ - B^*)^2\right\}.\end{aligned}$$

In the last step we dropped terms that have a zero mean under F conditional on $B\_$. Thus $\text{var}_F(\check{B}) = \text{var}\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^*\right) + \frac{1}{1+\sigma_p^2}\text{var}_F(\check{B})$ and collecting terms we have,

$$\text{var}_F(\check{B}) = \frac{1+\sigma_p^2}{\sigma_p^2}\text{var}\left(\beta\hat{g} + \frac{\hat{p}}{\bar{p}}B^*\right).$$

Now we describe the approximations to the the total revenues $\tilde{Z}_t$. Since $\tilde{Z} = \Phi\left(\tilde{\mu}\right)$ we can use

(67) to find the approximate law of motion $\check{Z}$ as

$$\check{Z}(g, p, B\_) = \frac{p}{\beta \bar{p}} B\_ + g - \check{B}(g, p, B\_),$$

which implies that

$$\mathbb{E}\check{Z}_t \equiv Z^* = \left(\frac{1-\beta}{\beta}\right) B^* + \bar{g}.$$

Rewrite the budget constraint (67) and using $\hat{p} = \frac{1}{1+\sigma_p^2}\hat{p} + \mathcal{O}(\sigma^3)$ we get,

$$\begin{aligned}
\tilde{Z}(g, p, B_) - Z^* = &\left(\frac{1}{1+\sigma_p^2}\right)\left(\frac{\hat{p}}{\beta\bar{p}}\right)(B_ - B^*) + \left(\frac{\hat{p}}{\beta\bar{p}}\right) B^* + \hat{g} \\
&+ \frac{1}{\beta}(B_ - B^*) - \left(\check{B}(g, p, B_) - B^*\right) + \mathcal{O}(\sigma^3, \sigma^2(1-\beta)),
\end{aligned}$$

which after substituting for equation (76) simplifies further to

$$\begin{aligned}
\check{Z}(g, p, B_) - Z^* &= \frac{1}{\beta}\left(\check{B}(g, p, B_) - B^* - \frac{\sigma_p^2}{1+\sigma_p^2}(B_ - B^*)\right) - (\check{B}(g, p, B_) - B^*) \\
&= \frac{1-\beta}{\beta}(\check{B}(g, p, B_) - B^* - \frac{\sigma_p^2}{\beta(1+\sigma_p^2)}(B_ - B^*).
\end{aligned}$$

Taking expectations at time 0,

$$\begin{aligned}
\mathbb{E}_0\left[\check{Z}_{t+1} - Z^*\right] &= \mathbb{E}_0\left[\mathbb{E}_{t-1}\left[\left(\frac{1-\beta}{\beta}\right)(\check{B}_{t+1} - B^*) - \frac{\sigma_p^2}{\beta(1+\sigma_p^2)}(B_t - B^*)\right]\right] \\
&= \mathbb{E}_0\left[\mathbb{E}_{t-1}\left[\left(\frac{1-\beta}{\beta}\right)\frac{1}{1+\sigma_p^2}(\check{B}_t - B^*) - \frac{\sigma_p^2}{\beta(1+\sigma_p^2)}(B_{t-1} - B^*)\right]\right] \\
&= \mathbb{E}_0\left[\left(\frac{1}{1+\sigma_p^2}\right)\left(\check{Z}_t - Z^*\right)\right].
\end{aligned}$$

Therefore $\check{Z}_t$ has the same speed of convergences as $\check{B}_t$. Computing the variance of $\check{Z}$ using the ergodic distribution of $\check{B}$ we obtain

$$\begin{aligned}
\mathrm{var}_F(\check{Z}) &= \left(\frac{1-\beta}{\beta}\right)^2 \mathrm{var}_F(\check{B}) - 2\left(\frac{(1-\beta)\sigma_p^2}{\beta^2(1+\sigma_p^2)}\right)\mathrm{cov}_F(\check{B}, B_) + \left(\frac{\sigma_p^2}{\beta(1+\sigma_p^2)}\right)^2 \mathrm{var}_F(\check{B}) \\
&= \left(\frac{1-\beta}{\beta}\right)^2 \mathrm{var}_F(\check{B}) - 2\left(\frac{(1-\beta)\sigma_p^2}{\beta^2(1+\sigma_p^2)}\right)\left(\frac{1}{1+\sigma_p^2}\right)\mathrm{var}_F(\check{B}) + \left(\frac{1}{\beta(1+\sigma_p^2)}\right)^2 \mathrm{var}_F(\check{B}) \\
&= \left(\left(\frac{1-\beta}{\beta}\right)^2 - \frac{2(1-\beta)\sigma_p^2 - \sigma_p^4}{\beta^2(1+\sigma_p^2)^2}\right)\mathrm{var}_F(\check{B}).
\end{aligned}$$

We drop terms that are of order $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$ from the expression $\mathrm{var}_F(\check{Z})$ to obtain the formula in the text.

Lastly we discuss how to extend the arguments in the proof when shocks are not continuous. For all $(\epsilon_g, \epsilon_p, B\_)$ such that $\tilde{B}$ is interior, we can define continuous and differentiable function $\mathring{B}(g', p', B')$ that solves (67) and (68) in an open neighborhood of $(\bar{g} + \sigma\epsilon_g, \bar{p} + \sigma_{pg}\sigma\epsilon_p, B\_)$ and is equal to $\tilde{B}(\bar{g}+\sigma\epsilon_g, \bar{p}+\sigma_{pg}\sigma\epsilon_p, B\_)$. By the chain rule, the derivatives in the expansion of $\tilde{B}$ with respect to σ solve

$$\tilde{B}_\sigma(g,p,B\_,0) = \mathring{B}_g(B\_)\epsilon_g + \mathring{B}_p(B\_)\sigma_{pg}\epsilon_p + \mathring{B}_\sigma(B\_),$$

and similarly for $\frac{\partial^2 \tilde{B}}{\partial\sigma^2}$. Since $\mathring{B}$ solves the same equations, (67) and (68), as $\tilde{B}$ before, the approximation, $\check{B}_t$ is identical to (76).

7.3 Proof of Proposition 3

The arguments in the last part of proof of proposition 2 in appendix 7.2 allows us to work with functions $\tilde{B}(g,\vec{p},B\_)$, $\tilde{\mu}(B\_)$ and $\left\{\tilde{B}^k\_(B\_)\right\}$ that solve the optimality conditions (assuming interiority) for problem (23) with K assets:

$$\sum_k \frac{p^k B^k\_(B\_)}{\beta\bar{p}^k} = \Phi(\tilde{\mu}(\tilde{B}(g,\vec{p},B\_))) - g + \tilde{B}(g,\vec{p},B\_), \tag{77a}$$

$$\tilde{\mu}(B\_) = \mathbb{E}[\frac{p^k}{\bar{p}^k}\tilde{\mu}(\tilde{B}(g,\vec{p},B\_))] \text{ for all } k, \tag{77b}$$

$$\sum_k \tilde{B}^k\_(B\_) = B\_. \tag{77c}$$

Equation (77b) is the counterpart of the martingale equation (15), where we have used $\tilde{\mu}(B\_) \equiv V'(B\_)$. Equation (77a) is obtained by substituting for the tax revenues using a first order condition with respect to $Z(s)$. The proof will follow the similar steps and notation convention as used in appendix 7.2.

We parametrize the payoffs and expenditure as $p^k = \bar{p}^k + \left(\sigma^k_{pg}\sigma\right)\epsilon^k_p$ and $g = \bar{g} + \sigma\epsilon_g$ where the subscript k denotes a particular security and use $\vec{p} \equiv \left\{p^k\right\}$. Indexing the functions $\tilde{B}(g,\vec{p},B\_;\sigma)$, $\tilde{\mu}(B\_;\sigma)$,$\left\{\tilde{B}^k\_(B\_;\sigma)\right\}$ with the extra argument σ, we take a Taylor approximation as $\sigma \to 0$. Similar to the $K=1$ case, $\tilde{B}(g,\vec{p},B\_;0) = B\_$. With $K>1$ equation (77) have multiple solution for $\left\{\tilde{B}^k\_(B\_)\right\}$. We will take an approximation around a particular solution that is obtained as the limit $\sigma \to 0$. For now, we let the $\tilde{B}^k\_(B\_;0) = B^k\_$ be arbitrary, but then show that the solution to the second order expansion of the system (77) will provide enough restrictions to pin down a limiting steady state portfolio.

The first order terms and the steps to obtain them are very similar to the single asset case. The expressions for $\tilde{B}_{B\_}(B\_)$, $\tilde{B}_g(B\_)$, and $\tilde{\mu}_{B\_}(B\_)$ are unchanged. The derivatives with respect to the p^k are similar, now modified for the holdings of the particular security

$$\tilde{B}_{p^k}(B\_) = \frac{B^k\_}{\bar{p}^k}. \quad \forall k$$

Also, the derivatives with σ,

$$\tilde{B}_\sigma(B\_) = \tilde{\mu}_\sigma(B\_) = \tilde{B}^k_\sigma(B\_) = 0 \quad \forall k$$

are same as $K=1$ case. The derivatives $\tilde{B}^k_{B\_}(B\_)$ are undetermined, but this does not affect any future calculations or the approximated policy rules for total assets $\tilde{B}$.

Many of the second order terms are also identical to the $K = 1$ case. In particular,

$$\tilde{B}_{B\_B\_}(B\_) = 0, \tilde{\mu}_{B\_B\_}(B\_) = -\frac{\Phi_{\mu\mu}(B\_)}{\Phi_{\mu}(B\_)}\left(\tilde{\mu}_{B\_B\_}(B\_)\right)^2,$$

and the second order and cross terms with respect to $g, \{p^k\}$

$$\tilde{B}_{p^k p^k} = \tilde{B}_{gg} = \tilde{B}_{p^k g} = 0 \quad \forall k.$$

Differentiating (77b) twice with respect to σ, we obtain

$$\tilde{\mu}_{\sigma\sigma}(B\_) = \mathbb{E}\Bigg[\tilde{\mu}_{\sigma\sigma}(B\_) + \tilde{\mu}_{B\_}(B\_)\tilde{B}_{\sigma\sigma}(B\_) + 2\frac{\sigma^k_{pg}}{\bar{p}^k}\epsilon^k_p\tilde{\mu}_{B\_}(B\_)\left(\sum_j \tilde{B}_{p^j}(B\_)\sigma^j_{pg}\epsilon^j_p + \tilde{B}_g(B\_)\epsilon_g\right)$$
$$+ \tilde{\mu}_{B\_B\_}(B\_)\left(\sum_j \tilde{B}_{p^j}(B\_)\sigma^j_{pg}\epsilon^j_p + \tilde{B}_g(B\_)\epsilon_g\right)^2\Bigg]. \tag{78}$$

We can eliminate $\tilde{\mu}_{\sigma\sigma}(B\_)$ from (78), substitute for $\tilde{B}_{p^j}(B\_)$ and solve out for $\tilde{B}_{\sigma\sigma}(B\_)$. For all k we obtain,

$$\tilde{B}_{\sigma\sigma}(B\_) = -2\mathbb{E}\left[\frac{\sigma^k_{pg}}{\bar{p}^k}\epsilon^k_p\left(\sum_j \frac{\sigma^j_{pg}}{\bar{p}^j}\epsilon^j_p\tilde{B}^j\_(B\_) + \beta\epsilon_g\right) + \frac{\Phi(B\_)(1-\beta)}{\beta\Phi_\mu(B\_)}\left(\sum_j \frac{\sigma^j_{pg}}{\bar{p}^j}\epsilon^j_p\tilde{B}^j\_(B\_) + \beta\epsilon_g\right)^2\right]. \tag{79}$$

Notice the system (79) has K equations, one for each security k. To satisfy all of them, there must exist a constant λ such that

$$\mathbb{E}\left[\frac{\sigma^k_{pg}}{\bar{p}^k}\epsilon^k_p\left(\sum_j \frac{\sigma^j_{pg}}{\bar{p}^j}\epsilon^j_p\tilde{B}^j\_(B\_) + \beta\epsilon_g\right)\right] = \lambda \quad \forall k, \tag{80a}$$

along with a portfolio $\vec{B}\_ = \{B^k\_(B\_)\}_k$ such that

$$\sum_k B^k\_(B\_) = B\_. \tag{80b}$$

Let $\lambda(B\_)$ and $\{B^k\_(B\_)\}$ be the solution to the equations (80a) and (80b). Equations (80) corresponds exactly to the first order conditions of the problem minimizing the variance of cash-in-hand given total asset holdings $B\_$:

$$\min_{\{B^k\_\}} \text{var}\left[\left(\sum_k \frac{B^k\_p^k}{\beta\bar{p}^k} + g\right)^2\right] \tag{81}$$

subject to the summing up constraint (80b) when $\frac{\lambda}{2\beta^2}$ is the Lagrange multiplier on the constraint

(80b). We now write equations (80) as a linear system of equations of the following form

$$\begin{bmatrix} \mathbb{C}(\vec{p},\vec{p}) & \mathbf{1} \\ \mathbf{1}^\intercal & 0 \end{bmatrix} \begin{bmatrix} \vec{B}\_ \\ -\lambda \end{bmatrix} = \begin{bmatrix} -\beta\mathbb{C}(\vec{p},g) \\ B\_ \end{bmatrix} \tag{82}$$

where $\mathbf{1}$ is a K dimensional vector of ones. There are two possible types of solutions to (82). If $\mathbb{C}(\vec{p},\vec{p})$ is not of full rank, the minimization problem (81) has multiple solutions and further the minimum is independent of $B\_$. In these cases $\lambda(B\_) = 0$ for all $B\_$. The other case, when $\mathbb{C}(\vec{p},\vec{p})$ is invertible, we can express $\lambda(B\_)$ and $\vec{B}\_$ as functions of $B\_$. Define a scalar $\eta \equiv \mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbf{1}$, and using equation (82) we have

$$\lambda = \frac{1}{\eta}\left(B\_ + \beta\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbb{C}(\vec{p},g)\right), \tag{83}$$

$$\begin{aligned}\vec{B}\_ &= -\left(\mathbb{C}(\vec{p},\vec{p})^{-1} - \frac{\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbf{1}\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}}{\eta}\right)\beta\mathbb{C}(\vec{p},g) + \frac{\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbf{1}B\_}{\eta} \\ &= -\beta\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbb{C}(\vec{p},g) + \frac{\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbf{1}}{\eta}\left(B\_ + \beta\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbb{C}(\vec{p},g)\right). \end{aligned}\tag{84}$$

Equation (84) gives us formula (26) in the text and pins down the steady state portfolio as a function of total assets $B\_$. The second order Taylor expansion of $\tilde{B}(g,p,B\_;\sigma)$ with respect to σ around $\sigma = 0$ is

$$\tilde{B}(g,p,B\_;\sigma) = B\_ + \tilde{B}_g(B\_)\hat{g} + \sum \tilde{B}_{p^k}(B\_)\hat{p}^k + \frac{1}{2}\tilde{B}_{\sigma\sigma}(B\_) + \mathcal{O}(\sigma^3),$$

where we have suppressed terms that are zero. Substituting for $\tilde{B}_g(B\_)$, $\left\{\tilde{B}_{p^k}(B\_)\right\}_k$ and $\tilde{B}_{\sigma\sigma}(B\_)$ we get

$$\tilde{B}(g,p,B\_;\sigma) = B\_ + \beta\hat{g} + \sum_k B^k\_(B\_)\frac{\hat{p}^k}{\bar{p}^k} - \lambda(B\_) + \mathcal{O}(\sigma^3,(1-\beta)\sigma^2),$$

where we drop the terms containing $(1-\beta)$ in equation (79) as we did in the $K = 1$ case. Let $B^* = -\beta\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\mathbb{C}(\vec{p},g)$. Using equations (83) and (84), the approximated dynamics, $\check{B}(g,\vec{p},B\_;\sigma)$ are

$$\check{B}(g,\vec{p},\check{B}\_;\sigma) = B^*+\beta\hat{g}-\beta\mathbf{1}^\intercal\mathbb{C}(\vec{p},g)^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\frac{\hat{\vec{p}}}{\bar{\vec{p}}}+\frac{1}{\eta}\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\frac{\hat{\vec{p}}}{\bar{\vec{p}}}(\check{B}\_-B^*)+\left(1-\frac{1}{\eta}\right)(\check{B}\_-B^*).$$

We can replace $\left(1-\frac{1}{\eta}\right) = \frac{\eta}{1+\eta} + \mathcal{O}(\sigma^3)$ and $\frac{\eta}{1+\eta} = 1 + \mathcal{O}\left(\sigma^3\right)$ to get,

$$\check{B}(g,\vec{p},B\_;\sigma) = B^*+\beta\hat{g}-\beta\mathbf{1}^\intercal\mathbb{C}(\vec{p},g)^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\frac{\hat{\vec{p}}}{\bar{\vec{p}}}+\left(\frac{1}{1+\eta}\right)\mathbf{1}^\intercal\mathbb{C}(\vec{p},\vec{p})^{-1}\frac{\hat{\vec{p}}}{\bar{\vec{p}}}(\check{B}\_-B^*)+\left(\frac{\eta}{1+\eta}\right)(\check{B}\_-B^*). \tag{85}$$

Using the stochastic process $\check{B}_t$ generated with $\check{B}(g,\vec{p},B\_;\sigma)$, we can verify that the ergodic mean for $\check{B}(g,\vec{p},B\_;\sigma)$ is B^* and the speed of convergence is $\left(\frac{\eta}{1+\eta}\right)$. Applying the same steps as in the proof of Proposition 2, we obtain the expression of the ergodic variance of $\check{B}$ as stated in Proposition 3, and mean, speed of convergence, variance of $\check{Z}_t$.

7.4 Proof of Proposition 4

From the definition of $\mathcal{R}^*$, equation (33), for $R \in \mathcal{R}^*$ there exists $p(s)$ and B such that

$$R(s, s\_) = \frac{p(s)}{\beta \int \pi(ds'|s\_)p(s')} \tag{86a}$$

$$p(s)B + PV_g(s) = \frac{Z(R)}{1-\beta} \quad \forall s. \tag{86b}$$

We will show that the function $\mathcal{B} : \mathcal{S} \to \mathbb{R}$

$$\mathcal{B}(s) = \beta \left(\frac{Z(R)}{1-\beta} - \int_{s' \in \mathcal{S}} PV_g(s')\pi(ds'|s) \right) \tag{87}$$

is a stochastic steady state and then establish that $\mathcal{B}(s) = B^{**}(Z^*, s)$ where Z^* solves the minimization problem (31)

Using equations (86b) and (87), we obtain

$$R(s, s\_) = \frac{Z(R)/(1-\beta) - PV_g(s)}{\mathcal{B}(s\_)}.$$

Therefore,

$$\begin{aligned} R(s, s\_)\mathcal{B}(s\_) + g(s) &= Z(R)/(1-\beta) - PV_g(s) + g(s) \\ &= Z(R) + \frac{\beta Z(R)}{1-\beta} + \beta \int_{s' \in \mathcal{S}} PV_g(s')\pi(ds'|s) \\ &= Z(R) + \mathcal{B}(s). \end{aligned} \tag{88}$$

Thus, for $s\_ = \tilde{s}$ and $B\_ = \mathcal{B}(\tilde{s})$, the policy choices $B(s) = \mathcal{B}(s)$ and $Z(s) = Z(R)$ satisfy equation (28) and hence are feasible. We next show that this policy is also optimal.

Associated with the Bellman equation (27) define an operator $\mathbb{T}$ on the space of continuous, concave and bounded functions that map $\mathcal{S} \times [\underline{B}, \overline{B}] \to \mathbb{R}$ as

$$\mathbb{T}(V)(s\_, B\_) = \max_{\{Z(s), B(s)\}_{s \in S}, Z \le \overline{Z}, B(s) \in [\underline{B}, \overline{B}]} \mathbb{E}\left[(R(s) B_- - B(s)) + \gamma \Psi(Z(s)) + \beta V(B(s), s) | s\_ \right]$$

subject to constraint (28). Standard arguments show that $\mathbb{T}$ is indeed an operator. Given an initial value function V_0, define the sequence of value functions V_1, V_2, V_3, etc., as $\mathbb{T}(V_0), \mathbb{T}(\mathbb{T}(V_0)), \mathbb{T}(\mathbb{T}(\mathbb{T}(V_0)))$, etc. Let $\tilde{B}_n$ and $\tilde{Z}_n$ be the optimal policy functions associated with V_n.

Suppose V_n is continuous, concave, bounded function and differentiable function with its derivative satisfying

$$\beta V_n'(s\_, \mathcal{B}(s\_)) = \gamma \Psi'(Z(R)) + 1 \text{ for all } s\_. \tag{89}$$

The stationary policy, $\tilde{B}_n(s\_, \mathcal{B}(s\_), s) = \mathcal{B}(s)$ and $\tilde{Z}_n(s\_, \mathcal{B}(s\_), s) = Z(R)$ for all $s\_, s$, satisfies the first order condition

$$\gamma\Psi'(\tilde{Z}_n(s\_,\mathcal{B}(s\_),s)) = \beta V_n'(s,\tilde{B}(s\_,\mathcal{B}(s\_),s)) - 1.$$

As the stationary policy is also feasible, we can conclude that it is optimal. Finally, $V_{n+1} = \mathbb{T}V_n$ is concave and applying the envelope theorem (Benveniste and Scheinkman (1979)), satisfies

$$\beta V_{n+1}'(s\_,\mathcal{B}(s\_)) = \gamma\Psi'(Z(R)) + 1 \text{ for all } s\_.$$

By induction, we can extend property (89) to all V_m with $m \geq n$ if it is satisfied by some V_n. Choose V_0 to be a linear function that satisfies (89), then $\tilde{B}_n(s\_,\mathcal{B}(s\_),) = \mathcal{B}(s)$ for all $n \geq 1$. Problem (27) and V_0 satisfy the assumptions of Theorem 9.9 in Stokey and Lucas (1989) and therefore we can conclude that $\tilde{B}_n \to \tilde{B}$ pointwise. Hence,

$$\tilde{B}(s\_,\mathcal{B}(s\_),s) = \lim_{n\to\infty} \tilde{B}_n(s\_,\mathcal{B}(s\_),s) = \mathcal{B}(s)$$

for all $s\_$.

Equation (87) and the definition in (29) implies $\mathcal{B}(s) = B^{**}(Z(R),s)$. From (88) note that that $R(s,s\_)\mathcal{B}(s\_) + PV_g(s)$ is constant with respect to s for all $s\_$. Thus $Z(R)$ solves the variance minimization problem (31), concluding the proof of Proposition 4.

7.5 Proof of proposition 5

When g_t and p_t follow (34), the arguments in the last part of proof of proposition 2 in appendix 7.2 allow us to work with functions $\tilde{\mu}(B\_,g\_)$ and $\tilde{B}(\varepsilon_g,\varepsilon_p,B\_,g\_)$, where $B\_,g\_$ be the debt and expenditure level before shocks $\varepsilon_g,\varepsilon_p$ are realized. These functions solve the following optimality conditions (assuming interiority) for problem (27):

$$\frac{B\_p}{\beta\bar{p}} + g = \Phi\left(\tilde{\mu}\left(\tilde{B}(\varepsilon_g,\varepsilon_p,B\_,g\_),g\right)\right) + \tilde{B}(\varepsilon_g,\varepsilon_p,B\_,g\_), \tag{90a}$$

$$\tilde{\mu}(B\_,g\_) = \mathbb{E}\left[\frac{p}{\bar{p}}\tilde{\mu}\left(\tilde{B}(\varepsilon_g,\varepsilon_p,B\_,g\_),g\right)\Big| g\_\right]. \tag{90b}$$

We parameterize $\varepsilon_g = \sigma\epsilon_g$ and $\varepsilon_p = \sigma\sigma_{pg}\epsilon_p$ where variables (ϵ_g,ϵ_g) are mean zero and with unit standard deviations each and independent over time. We augment the functions $\tilde{\mu}(B\_,g\_)$ and $\tilde{B}(g,p,B\_,g\_)$ with an extra argument σ: $\tilde{B}(\varepsilon_g,\varepsilon_p,B\_,g\_;\sigma),\tilde{\mu}(B\_,g\_;\sigma)$ and take a Taylor expansion with respect to around $\sigma = 0$ and $g\_ = \bar{g}$. We will use the notation $(B\_)$ as a shortcut for $(\bar{g},\bar{p},B\_,\bar{g};0)$. Differentiating equations (90a) and (90b) with respect to $B\_$ we get,

$$\tilde{B}_{B\_}(B\_) = 1, \tilde{\mu}_{B\_}(B\_) = \left(\frac{\frac{1}{\beta}-1}{\Phi_\mu(B\_)}\right),$$

which are identical to the i.i.d case. Differentiating equations (90a) and (90b) with respect to $g\_$ we find

$$\rho = \Phi_\mu(B\_)\left(\tilde{\mu}_g(B\_)\rho + \tilde{\mu}_{B\_}(B\_)\tilde{B}_{g\_}\right) + \tilde{B}_{g\_},$$

$$\tilde{\mu}_{g\_}(B\_) = \tilde{\mu}_{g\_}(B\_)\rho + \tilde{\mu}_{B\_}(B\_)\tilde{B}_{g\_}.$$

Substituting for $\tilde{\mu}_{B\_}(B\_)$ and $\tilde{B}_{B\_}(B\_)$ obtained before and simplifying, we obtain

$$\tilde{B}_{g\_} = (1-\rho)\frac{\rho\beta}{1-\rho\beta}, \tag{91a}$$

$$\tilde{\mu}_{g\_}(B\_) = \left(\frac{1}{1-\rho}\right)\tilde{\mu}_{B\_}(B\_)\tilde{B}_{g\_} = \left(\frac{\rho\beta}{1-\rho\beta}\right)\tilde{\mu}_{B\_}(B\_). \tag{91b}$$

Differentiating with respect to g get

$$\tilde{B}_g(B\_) = \beta\left(\frac{1-\rho}{1-\rho\beta}\right).$$

The expressions for the derivatives

$$\tilde{B}_\sigma(B\_) = 0, \tilde{\mu}_\sigma(B\_) = 0, \quad \tilde{B}_p(B\_) = \frac{B\_}{\bar{p}}$$

are same as in the i.i.d. case.

Next we discuss the second order terms. The expressions

$$\tilde{B}_{B\_B\_}(B\_) = 0, \tilde{\mu}_{B\_B\_}(B\_) = -\frac{\Phi_{\mu\mu}(B\_)\left(\tilde{\mu}_{B\_}(B\_)\right)^2}{\Phi_\mu(B\_)}$$

are same as the i.i.d. case. We solve for the second order derivatives with respect to $g\_$. Differentiating (90) with respect to $g\_$ and $B\_$ we find

$$\begin{aligned}
0 = {} & \Phi_{\mu\mu}(B_)\left[\tilde{\mu}_{g_}(B_)\rho\tilde{\mu}_{B_}(B_) + \tilde{\mu}_{B_}(B_)\tilde{B}_{g_}\tilde{\mu}_{B_}(B_)\right] \\
& + \Phi_\mu(B_)\left[\tilde{\mu}_{g_B_}(B_)\rho + \tilde{\mu}_{B_B_}(B_)\tilde{B}_g(B_) + \tilde{\mu}_{B_}(B_)\tilde{B}_{g_B_}(B_)\right]
\end{aligned}$$

and

$$\tilde{\mu}_{g\_B\_}(B\_) = \tilde{\mu}_{g\_B\_}(B\_)\rho + \tilde{\mu}_{B\_B\_}(B\_)\tilde{B}_g(B\_) + \tilde{\mu}_{B\_}(B\_)\tilde{B}_{g\_B\_}(B\_). \tag{92}$$

Solving out for $\tilde{\mu}_{g\_B\_}$ from (92)

$$\begin{aligned}
\tilde{\mu}_{g_B_}(B_) &= -\frac{\Phi_{\mu\mu}(B_)}{\Phi_\mu(B_)}\left[\tilde{\mu}_{g_}(B_)\rho\tilde{\mu}_{B_}(B_) + \tilde{\mu}_{B_}(B_)\tilde{B}_{g_}\tilde{\mu}_{B_}(B_)\right] \\
&= -\frac{\Phi_{\mu\mu}(B_)}{\Phi_\mu(B_)}\left[\tilde{\mu}_{B_}(B_)\tilde{\mu}_{g_}(B_)\right].
\end{aligned}$$

We can substitute for $\tilde{\mu}_{g\_B\_}(B\_)$ and $\tilde{\mu}_{B\_B\_}(B\_)$ in (92) which, along with $\tilde{\mu}_{B\_}(B\_)\tilde{B}_{g\_} = (1-\rho)\tilde{\mu}_{g\_}(B\_)$ implies that

$$\tilde{B}_{g\_B\_}(B\_) = 0.$$

A similar procedure finds that

$$\tilde{\mu}_{g\_g\_}(B\_) = -\frac{\Phi_{\mu\mu}(B\_)}{\Phi_\mu(B\_)}\left(\tilde{\mu}_{g\_}(B\_)\right)^2, \tilde{B}_{g\_g\_} = 0.$$

As in the i.i.d. case it can be shown that the second (and cross) derivatives with respect to the p and g are all zero. The second order derivative with respect to σ is modified to

$$\tilde{B}_{\sigma\sigma} = -2\mathbb{E}\left[\frac{\sigma_{pg}}{\bar{p}}\epsilon_p\left(\frac{\sigma_{pg}}{\bar{p}}\epsilon_p B\_ + \beta\frac{1}{1-\rho\beta}\right) + \frac{\Phi_{\mu\mu}(B\_)(1-\beta)}{\beta\Phi_\mu(B\_)}\left(\frac{\sigma_{pg}}{\bar{p}}\epsilon_p B\_ + \frac{\beta}{1-\rho\beta}\epsilon_g\right)^2\right].$$

The second order Taylor expansion of $\tilde{B}$ around $\sigma = 0, g\_ = \bar{g}$ can then be written as

$$\begin{aligned}\tilde{B}(\varepsilon_g, \varepsilon_p, B\_, g\_\sigma) = B\_ + \tilde{B}_g(B\_)\varepsilon_g + \tilde{B}_p(B\_)\varepsilon_p + \tilde{B}_{g\_}(g\_ - \bar{g})\\ + \frac{1}{2}\tilde{B}_{\sigma\sigma}(B\_)\sigma^2 + \mathcal{O}(\sigma^3),\end{aligned}$$

where we have dropped the terms that are zero. Define $PV_g = \frac{g}{1-\beta\rho}$ as the expected net present value of g, and substituting for the expressions for derivatives computed above we obtain

$$\begin{aligned}\tilde{B}(\varepsilon_g, \varepsilon_p, B\_, g\_; \sigma) = B\_ &+ \left(\frac{\beta(1-\rho)}{1-\rho\beta}\right)\hat{g} + B\_\frac{\hat{p}}{\bar{p}} - \beta\text{cov}\,(p, PV_g) - B\_\sigma_p^2\\ &+ \frac{\Phi_{\mu\mu}(B\_)(1-\beta)}{2\beta\Phi_\mu(B\_)}\left[\left(\frac{\beta}{1-\beta\rho}\right)^2\sigma + 2\beta B\_\text{cov}(p,g) + B\_^2\frac{\sigma_p^2}{\bar{p}^2}\right]\\ &+ \mathcal{O}(\sigma^3),\end{aligned} \tag{94}$$

where cov (p, PV_g) is the covariance under the ergodic distribution the shocks g, p. In the current case, $\mathbb{E}^\lambda B^{**}(g) = \frac{\text{cov}(p,PV_g)}{\text{var}(p)}$. We can replace $1-\sigma_p^2 = \frac{1}{1+\sigma_p^2} + \mathcal{O}(\sigma^3)$ and $\sigma_p = \frac{\sigma_p}{1+\sigma_p^2} + \mathcal{O}(\sigma^3)$ to express (94) as

$$\begin{aligned}\tilde{B}(\varepsilon_g, \varepsilon_p, B\_, g\_; \sigma) =&\mathbb{E}^\lambda B^{**} + \frac{\beta(1-\rho)}{1-\rho\beta}(g-\bar{g}) + \mathbb{E}^\lambda B^{**}\frac{\hat{p}}{\bar{p}} + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - \mathbb{E}^\lambda B^{**})\\ &+\frac{1}{1+\sigma_p^2}(B\_ - \mathbb{E}^\lambda B^{**}) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned} \tag{95}$$

Inspecting the dynamics in equation (95) we see that the ergodic mean of the $\check{B}_t$, the approximated process, equals $\mathbb{E}B^{**}(g\_)$, and the speed of convergence to $\mathbb{E}B^{**}$ is given by $\frac{1}{1+\sigma_p^2}$.

We now derive the expression for the ergodic variance. Equation (95) can be expressed as,

$$\begin{aligned}\tilde{B}(\varepsilon_g, \varepsilon_p, B\_, g\_; \sigma) - B^{**}(g) = &\frac{\beta}{1-\beta\rho}\sigma\epsilon_g + B^*(g\_)\frac{\hat{p}}{\bar{p}} + \left(\frac{1}{1+\sigma_p^2}\right)\frac{\hat{p}}{\bar{p}}(B\_ - B^{**}(g\_))\\ &+ \frac{1}{1+\sigma_p^2}(B\_ - B^{**}(g\_)) + \mathcal{O}(\sigma^3, \sigma^2(1-\beta)).\end{aligned}$$

As $\mathbb{E}(\tilde{B}(\varepsilon_g, \varepsilon_p, B\_, g\_; \sigma) - B^{**}(g)) = 0$, following the same steps as in proof for Proposition 2 in appendix 7.2 gives

$$\text{var}(\check{B}_t) = \left(\frac{1+\sigma_p^2}{\sigma_p^2}\right)\text{var}\left(\frac{\beta}{1-\beta\rho}\sigma\epsilon_g + B^{**}(g_t)\frac{\hat{p}}{\bar{p}}\right).$$

7.6 Proof of Proposition 6

Lemma 3 and bound $\bar{Z}$ established in equation (62) applies to the case with transfers. The bound $\bar{Z}$ on total tax revenues, constraint (37) along with the bounds on $B(s)$ imply that $T(s)$ is bounded above. Substitute constraint (37) for $Z(s)$ in the objective function (35) and split the maximization problem by s to obtain the period gain function,

$$\max_{T\leq\bar{T},B\in[\underline{B},\overline{B}]}\left(R(s)B\_-B+\frac{T}{1+n}\right)+\gamma\Psi\left(E(s,B\_)-B+T\right)+\omega U\left(\frac{T}{1+n}\right)$$

Using strict concavity of U and Ψ and compactness of the choice set, standard arguments show that $V(\cdot)$ is continuous, strictly concave and differentiable on $\left[\underline{B},\overline{B}\right]$. The optimal choice for $T(s)>0$ due to Inada conditions on U and $\omega>0$, and the upper bound will be slack for the same reason the upper bound on $Z(s)$ was slack in problem (12). Thus imposing interiority of $T(s)$, the first order conditions with respect to $Z(s),T(s),B(s)$ are

$$\gamma\Psi'\left(Z(s)\right)=\mu(s), \tag{96a}$$

$$\frac{\omega}{1+n}U_c\left(\frac{T(s)}{1+n}\right)+\frac{1}{1+n}+\mu(s)=0, \tag{96b}$$

$$-1+\beta V'\left(B(s)\right)-\bar{\kappa}\left(s\right)+\underline{\kappa}\left(s\right)=\mu(s), \tag{96c}$$

where $\mu(s)$ is the multiplier on constraint (37) and $\bar{\kappa}\left(s\right),\underline{\kappa}\left(s\right)$ are the multipliers on the boundaries $\left[\underline{B},\overline{B}\right]$. Let $\bar{\kappa}=\int_s\pi\left(ds\right)R\left(s\right)\bar{\kappa}\left(s\right)$, $\underline{\kappa}=\int_s\pi\left(ds\right)R\left(s\right)\underline{\kappa}\left(s\right)$, combining equation (96c) with the envelope condition $V'(B\_)=\beta^{-1}+\int_s\pi(ds)R(s)\mu(s)$ and we get the martingale (63).

Problem (35) is a maximum of a concave and supermodular function in $(B,Z,-T)$ over the pre-image of continuous, increasing and submodular function $B+Z-T$. Therefore by Corollary 2(ii) in Quah (2007) the set of maximizers is increasing in $E\left(s,B\_\right)$. As $E\left(s,B\_\right)$ is increasing in $B\_$ for all s, and it is increasing in s if $B\_\in\left[B^{max},\overline{B}\right]$, decreasing in s if $B\_\in\left[\underline{B},B^{min}\right]$. Thus functions $\tilde{B}\left(\cdot,\cdot\right)$, $\tilde{Z}\left(\cdot,\cdot\right)$ and $-\tilde{T}(\cdot,\cdot)$ are continuous and increasing in in $B\_$ for all s, decreasing in s for $B\_\in\left[\underline{B},B^{min}\right]$, increasing in s for $B\_\in\left[B^{max},\overline{B}\right]$. Lemma 2 extends to this case, we have $\tilde{B}_t\to B^*$ when $R\in\mathcal{R}^*$. As $E(s,B^*)=E^*$ is constant $\tilde{Z}(s,B^*)=Z^*$ (and hence $\tilde{\tau}(s,B^*)=\tau^*$) and $\tilde{T}(s,B^*)=T^*$. At B^* the steady state solves for μ^*,Z^*,T^* that satisfy

$$\gamma\Psi'\left(Z^*\right)=\mu^*, \tag{97a}$$

$$\frac{\omega}{1+n}U_c\left(\frac{T^*}{1+n}\right)+\frac{1}{1+n}+\mu^*=0, \tag{97b}$$

$$Z^*-T^*=E^*-B^*. \tag{97c}$$

Substituting equation (97c) and (97a) in equation (97b) to eliminate μ^* and Z^* we obtain

$$\frac{\omega}{1+n}U_c\left(\frac{T^*}{1+n}\right)+\frac{1}{1+n}=-\gamma\Psi'\left(E^*-B^*+T^*\right).$$

Differentiating with respect to ω

$$\frac{\partial T^*}{\partial \omega} = -\frac{\frac{1}{1+n}U_c\left(\frac{T^*}{1+n}\right)}{\left[\frac{\omega}{[1+n]^2}U_{cc}\left(\frac{T^*}{1+n}\right)+\gamma\Psi''\left(E^*-B^*+T^*\right)\right]}.$$

Since $U_{cc}<0$ and $\Psi''<0$, we get $\frac{\partial T^*}{\partial \omega}>0$. Next differentiating equation (97c) with respect to ω we get $\frac{\partial Z^*}{\partial \omega}=\frac{\partial T^*}{\partial \omega}>0$.

7.7 Proof of Proposition 7

Equations (96a) and (96b) allow us to solve for $I(s)=Z(s)-T(s)$ as a function of $\mu(s)$

$$I(\mu)=\Psi'^{-1}(\frac{\mu}{\gamma})-(1+n)U_c^{-1}\left(-\frac{1}{\omega}-\frac{1+n}{\omega}\mu\right). \tag{98}$$

The equilibrium conditions for an interior solution to the planners problem can then be written as

$$\frac{B\_p}{\beta\bar{p}}+g=I\left(\tilde{\mu}\left(\tilde{B}(g,p,B\_)\right)\right)+\tilde{B}(g,p,B\_), \tag{99a}$$

$$\tilde{\mu}(B\_)=\mathbb{E}\left[\left(\frac{p}{\bar{p}}\right)\tilde{\mu}\left(\tilde{B}(g,p,B\_)\right)\right]. \tag{99b}$$

Note that the system of equations (99) exactly mirrors the system of equations (67) and (68) with Ψ replaced with I. Moreover the approximation $\check{B}$ in appendix 7.2 is independent of the properties of Ψ, thus the ergodic distribution of $\check{B}_t$ will be identical to the representative agent case. As before, we can approximate the law of motion of $\check{I}_t$ using the budget constraint as

$$\check{I}_t-I^*=\frac{1-\beta}{\beta}\left(\check{B}_t-B^*\right)+\frac{\sigma_p^2}{\beta}(\check{B}_{t-1}-B^*)+\mathcal{O}(\sigma^3)$$

with

$$I^*=\left(\frac{1-\beta}{\beta}\right)B^*+\bar{g}.$$

The law of motion for this process, and hence, the ergodic distribution is independent of ω. Define the level of transfers when the government has a net tax revenue of I as $T(I;\omega)$.

Combine the first order conditions, (96a) and (96b) and substitute $Z=I+T$ to express transfers, $T(I;\omega)$ implicitly using

$$\frac{\omega}{1+n}U_c\left(\frac{T(I;\omega)}{1+n}\right)+\gamma\Psi'(I+T(I;\omega))=-\frac{1}{1+n}.$$

Differentiating with respect to ω we get,

$$\frac{\partial T}{\partial \omega}(I;\omega)=-\frac{\frac{1}{1+n}U_c\left(\frac{T(I;\omega)}{1+n}\right)}{\frac{\omega}{(1+n)^2}U_{cc}\left(\frac{T(I;\omega)}{1+n}\right)+\gamma\Psi''(I+T(I;\omega))}$$

Both $U_{cc}<0$ and $\Psi''<0$, we can conclude that $\frac{\partial T}{\partial \omega}(I;\omega)>0$. Thus $\check{T}_t=T(\check{I}_t;\omega)$ is increasing in ω. First order stochastic dominance follows immediately.[28]

7.8 Proof of Proposition 8

We begin by writing the sequential version of the $t \geq 1$ recursive problem (42) as

$$\max_{c_t(s^t), X_t(s^t), W_t(s^t)} \sum_{t\geq 1} \sum_{s^t} \pi(s^t|s_0)\beta^{t-1} U\left(c_t(s^t), \frac{c_t(s^t)+g(s_t)}{\theta(s_t)}\right) \tag{100}$$

s.t for all $t \geq 1$

$$X_t(s^t) = W_t(s^t) + \Phi(s_t, c_t(s^t)) \tag{101a}$$

$$W_t(s^t) = \frac{X_{t-1}(s^{t-1})p(s_t)U_{c,t}(s^t)}{\beta \mathbb{E}_{t-1}[p(s_t)U_{c,t}(s^t)]} \tag{101b}$$

given $X_0 = X\_$ and $s_0 = s\_$. Let $\tilde{X}^S_t(s^t|X_0, s_0), \tilde{W}^S_t(s^t|X_0, s_0)$ and $\tilde{c}^S_t(s^t|X_0, s_0)$ solve the sequential planner's problem. The optimal policy functions $\tilde{X}(s, X\_, s\_)$ and $\tilde{c}(s, X\_, s\_)$ that solve the recursive planners problem (42) generate the solution to the sequential problem (100). For example: starting with $X^S_1(s^1|X_0, s_0) = \tilde{X}(s_1, X_0, s_0)$, we can recursively define for all $t > 1$

$$X^S_t(s^t|X_0, s_0) = \tilde{X}(s_t, X^S_{t-1}(s^{t-1}|X_0, s_0), s_{t-1}). \tag{102}$$

In order to characterize the stochastic steady states $\mathcal{X}$ and the set of payoffs that implement them, we begin by solving the following *relaxed* problem

$$\max_{c_t(s^t), X_t(s^t), W_t(s^t)} \sum_{t\geq 1} \sum_{s^t} \pi(s^t)\beta^{t-1} U\left(c_t(s^t), \frac{c_t(s^t)+g(s_t)}{\theta(s_t)}\right) \tag{103}$$

s.t. for all $t \geq 1$

$$X_t(s^t) = W_t(s^t) + \Phi(s_t, c_t(s^t)) \tag{104a}$$

$$X_{t-1}(s^{t-1}) = \beta \mathbb{E}_{t-1}[W_t(s^t)|s^{t-1}] \tag{104b}$$

for a given X_0 and s_0.

Any sequence $\{c_t(s^t), X_t(s^t), W_t(s^t)\}$ that is feasible under constraints (101) is also feasible under (104).[29] We will denote the solution to the relaxed problem using $\tilde{X}^R_t(s^t|X\_, s\_), \tilde{W}^R_t(s^t|X\_, s\_)$ and $\tilde{c}^R_t(s^t|X\_, s\_)$. Note that if $\tilde{X}^R_t(s^t|X\_, s\_), \tilde{W}^R_t(s^t|X\_, s\_)$ and $\tilde{c}^R_t(s^t|X\_, s\_)$ satisfy constraints (101a) and (101b) then they will also optimal for the sequential problem (100).

Let $\beta^t\pi(s^t)\mu_t(s^t)$ be the Lagrange multiplier on equation (104a) . The first order conditions

[28] Let x be a random variable and $g(x)$ and $h(x)$ be real valued functions. If $g(y) > h(y)$ for all y, then $g(x)$ first order stochastically dominates $h(x)$

[29] Note that constraints (101a) and (104a) are the same any allocation that satisfies (101b) automatically satisfies (104b).

with respect to $W_t(s^t)$ and $c_t(s^t)$ yield that $\mu_t(s_t, s^{t-1}) = \mu_{t-1}(s^{t-1}) = \mu_0$ for all s_t, s^{t-1} and the optimal consumption $\tilde{c}_t^R(s^t|X_0, s_0) = \tilde{C}^R(s_t|X_0, s_0)$ choice only depends on the current state s_t. For a given X_0 and s_0 define

$$W(s|X_0, s_0) = \mathbb{E}\left[\sum_{j=1}^{\infty} \beta^{j-1}\Phi(s_j, \tilde{C}^R(s_j|X_0, s_0))|s_1 = s\right]$$

and

$$\mathcal{X}(s|X_0, s_0) = \beta \sum_{s'} \pi(s'|s) W(s'|X_0, s_0).$$

The optimal choices for the planner are $\tilde{X}_t^R(s^t|X_0, s_0) = \mathcal{X}(s_t|X_0, s_0)$ and $\tilde{W}_t^R(s^t|X_0, s_0) = W(s_t|X_0, s_0)$. Let $s_1^t = (s_1, \dots, s_t)$ Moreover for all $t \geq 1$

$$\tilde{X}^R(s', s_1^t|\mathcal{X}(s'|X_0, s_0), s') = \tilde{X}^R(s_0, s_1^t|X_0, s_0)$$

for all s' (and likewise for $\tilde{c}^R$ and $\tilde{W}^R$). If this were not the case then the policy

$$\hat{\tilde{X}}^R(s^t|X_0, s_0) = \begin{cases} \tilde{X}^R(s^1|X_0, s_0) & \text{if } t = 1 \\ \tilde{X}^R(s_1^t|\mathcal{X}(s_1|X_0, s_0), s_1) & \text{otherwise} \end{cases}$$

would be optimal and different from $\tilde{X}^R$, a contradiction.

Define $p^*(s|X_0, s_0) = \frac{W(s|X_0, s_0)}{U_c(\tilde{C}^R(s|X_0, s_0))}$. When $p = p^*$ equation (101b) is automatically satisfied by $\tilde{X}_t^R(s^t|\mathcal{X}(s'|X_0, s_0), s'), \tilde{W}_t^R(s^t|\mathcal{X}(s'|X_0, s_0), s')$ and $\tilde{c}_t^R(s^t|\mathcal{X}(s'), s')$, and thus these policies are optimal for the sequential problem (100) when $X_0 = \mathcal{X}(s')$ and $s_0 = s'$ for all s'. Using the map between $\tilde{X}(s, X\_, s\_)$ and $X_t^R(s^t|X_0, s_0)$ in equation (102), we obtain

$$\tilde{X}(s, \mathcal{X}(s\_|X_0, s_0), s\_) = \mathcal{X}(s|X_0, s_0)$$

for all $s\_$. Thus $\mathcal{X}$ is a stochastic steady state. The set of payoffs $\mathcal{P}^*$ that support a stochastic steady state with risk aversion can be generated by varying the initial conditions X_0 and s_0

Lastly we construct an example to demonstrate such that $p^{rf}(s) \in \mathcal{P}^*$: Suppose $S = 2$ shocks are i.i.d overtime, and $U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. The productivity $\theta(s) = 1$ and we order $g(s)$ such that $g(2) \geq g(1)$. Let $\left\{c^{fb}(s), l^{fb}(s)\right\}_{s \in S}$ be the *first-best* allocation that maximizes $\mathbb{E}_0 \sum_t \beta^t \left(\frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}\right)$ subject to the feasibility constraint (39). We will use $R^{fb}(s) = \frac{U_c^{fb}(s)}{\beta \mathbb{E} U_c^{fb}(s)}$ to denote the returns at the first best allocation. We show that the following condition

$$\frac{1 - R^{fb^{-1}}(2)}{1 - R^{fb^{-1}}(1)} > \frac{g(2)}{g(1)}, \tag{105}$$

is sufficient for existence of a $\mathcal{X}$ such that $\tilde{X}(s, \mathcal{X}) = \mathcal{X}$ and $\tau(s, \mathcal{X}) = \tau^*$ for all s.

Let $\mu(s)$ be the Lagrange multiplier on equation (43). The first order conditions with respect to $\{c(s), X(s)\}$ are

$$c(s) : c(s)^{-\sigma} - \Phi_c(s,c)\mu(s) = \sigma \frac{X\_c(s)^{-\sigma-1}\mu(s)}{\beta\mathbb{E}[c^{-\sigma}]} - \frac{X\_\sigma c(s)^{-\sigma-1}}{\mathbb{E}[c^{-\sigma}]^2} \sum_{s'} \pi(s')\mu(s')c(s')^{-\sigma} \tag{106a}$$

$$X(s) : \beta V'(X'(s)) - \mu(s) = 0 \tag{106b}$$

In addition to these constraints we have the envelope condition

$$V'(X\_) = \sum_s \pi(s) \frac{\mu(s)c(s)^{-\sigma}}{\beta\mathbb{E}[c^{-\sigma}]} \tag{107}$$

At $X\_ = \mathcal{X}$, imposing $\tilde{X}(s,\mathcal{X}) = \mathcal{X}$ we see that $\mu(s) = \mu^*$ satisfies equation (107) and (106b). For a given μ^*, note that the right hand side of equation (106a) is zero and we can use (106a) and the resource constraint to solve c and l as functions of s, μ^*. Denote these functions as $C(s,\mu^*)$ and $L(s,\mu^*)$. Expanding $\Phi_c(s,c) = (1-\sigma)c(s)^{-\sigma} + (1+\gamma)[c(s)+g(s)]^{\gamma}$, it is easy to verify from (106a) that $\tau(s) = 1 - [c(s)+g(s)]^{\gamma}\, c(s)^{\sigma}$ is independent of s and with slight abuse of notation, we use $\tau(\mu^*)$ to denote the tax rate implied by $C(s,\mu^*)$ and $L(s,\mu^*)$. Let $\mathcal{F}(\mu^*)$ be defined as

$$\mathcal{F}(\mu^*) = \frac{\Phi(1, C(1,\mu^*)}{R(1)-1} - \frac{\Phi(2, C(2,\mu^*)}{R(2)-1}$$

Existence of steady state is equivalent to finding a solution to $\mathcal{F}(\mu^*) = 0$. To show this we need the next lemma that allows us to order functions $C(s\mu^*)$ and $L(s,\mu^*)$ with respect to s.

Lemma 5. *For all μ^* such that $\tau(\mu^*) < 1$, the function $C(s,\mu^*)$ satisfies $C(1,\mu^*) > C(2,\mu^*)$*

For $\tau(\mu^*) < 1$, the agents' optimal labor choice satisfies

$$(c + g(s))^{\gamma} = (1 - \tau(\mu^*))c^{-\sigma}. \tag{108}$$

The right hand side of equation (108) is decreasing in c. Furthermore, holding c constant we have that the left hand of the equation (108) is increasing in $g(s)$. Combining these two facts, we obtain that $C(1,\mu^*) > C(2,\mu^*)$.

Let μ_1 be such that $\Phi(1, C(1,\mu_1)) = 0$.[30] Using Lemma 2 we have that $L(2,\mu_1) - L(1,\mu_1) < g(2) - g(1)$. Since $\tau(\mu_1) < 1$, $\tau(\mu_1)\,[L(2,\mu_1) - L(1,\mu_1)] < g(2) - g(1)$ and hence the marginal utility adjusted surplus is state $s = 2$, $\Phi(2, C(2,\mu_1)) < 0$. Combing this we have $\mathcal{F}(\mu_1) > 0$. The sufficient condition (105) implies that $\mathcal{F}(0) < 0$. Since $\mathcal{F}$ is continuous we can apply the intermediate value theorem to conclude that there exists μ^* such that $\mathcal{F}(\mu^*) = 0$.

[30] Such a μ_1 exists as long as it is feasible to raise revenues at least has high as $g(1)$.

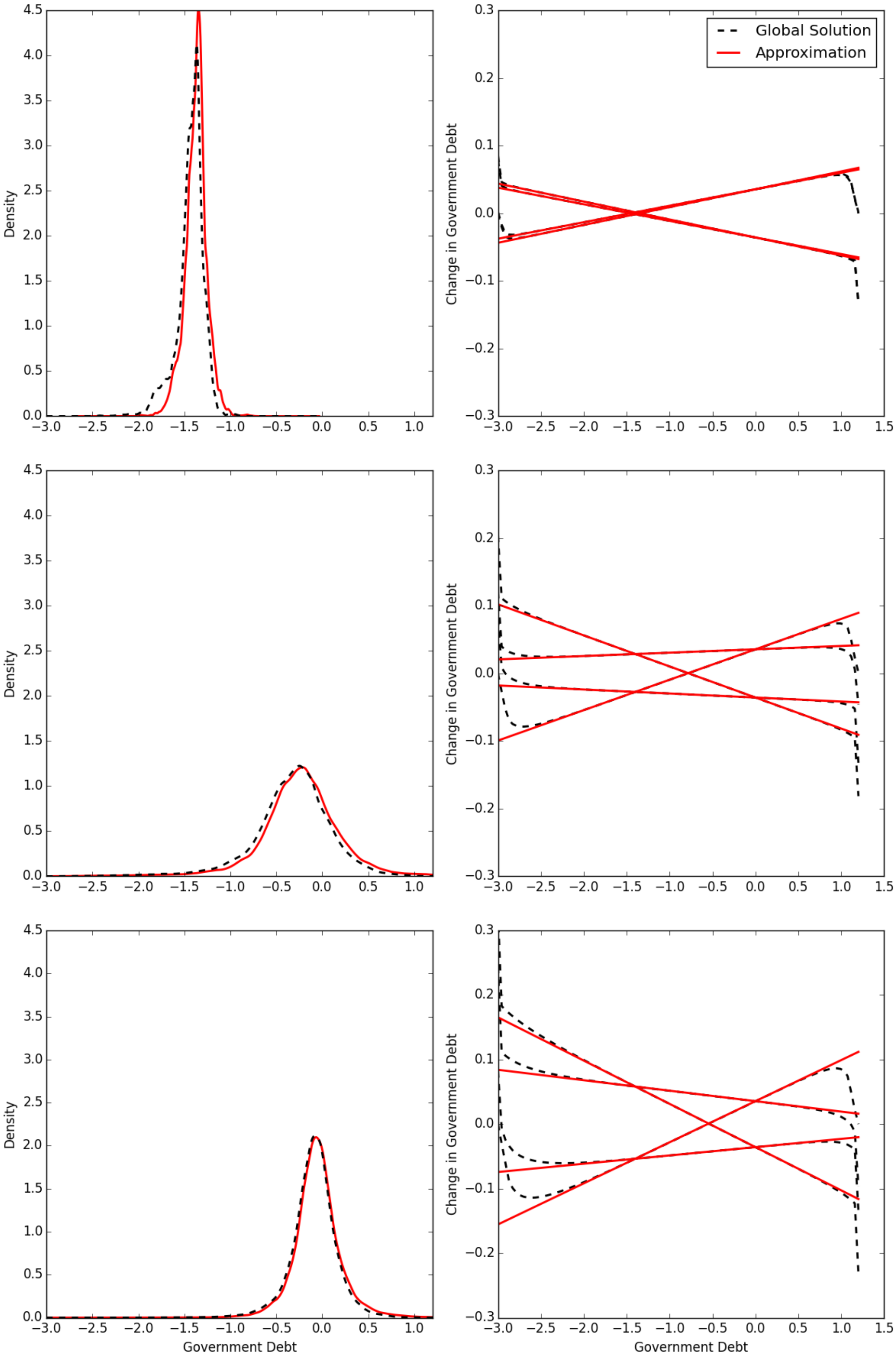

Figure I: Using the quadratic approximation (red line) and a more accurate global approximator (black line), the top, middle, and bottom panels plot smoothed kernel densities (left side) and decision rules (right side) associated with values of $\sigma_e = 0.001, 0.02$, and 0.04. The right panel displays policies $\tilde{B}(s, B\_) - B\_$ for states s that attain the extreme values for $\{g(s)\}$ and $\{p(s)\}$.

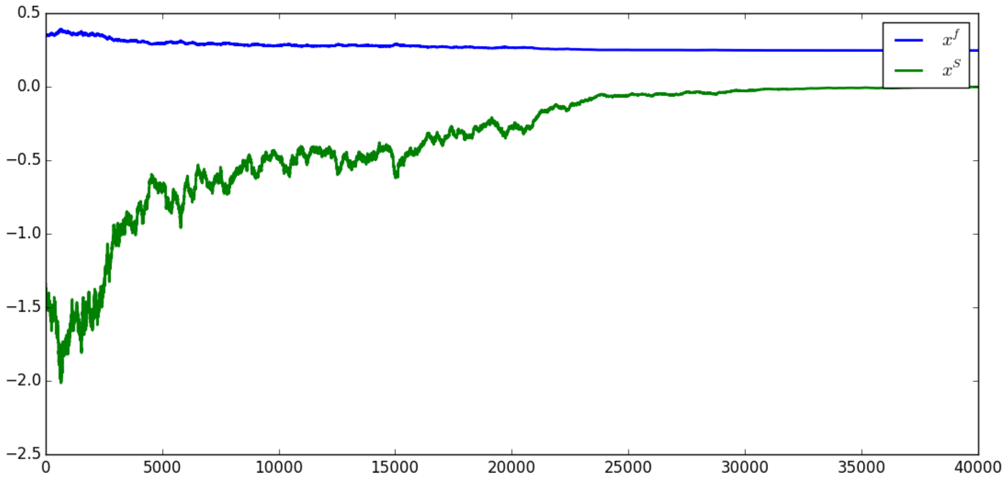

Figure II: Marginal utility weighted assets held in the risk free bond (blue line) and "stock market" asset (green line). Negative values implies that the government is shorting the asset.

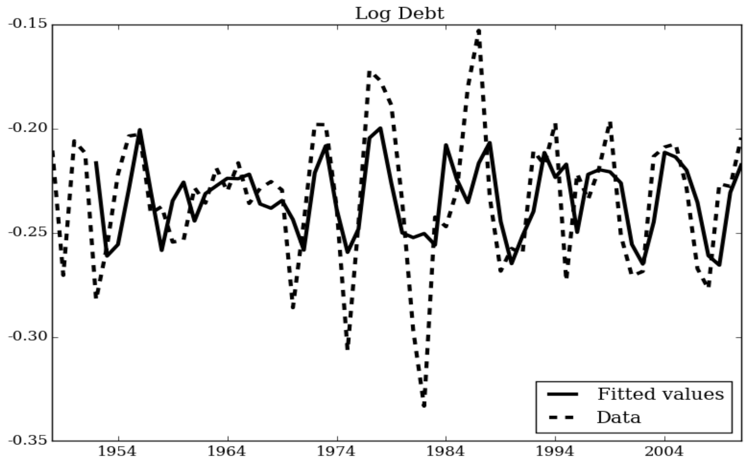

Figure III: Fitted debt versus (H.P. filtered) U.S. debt [1947-2010]

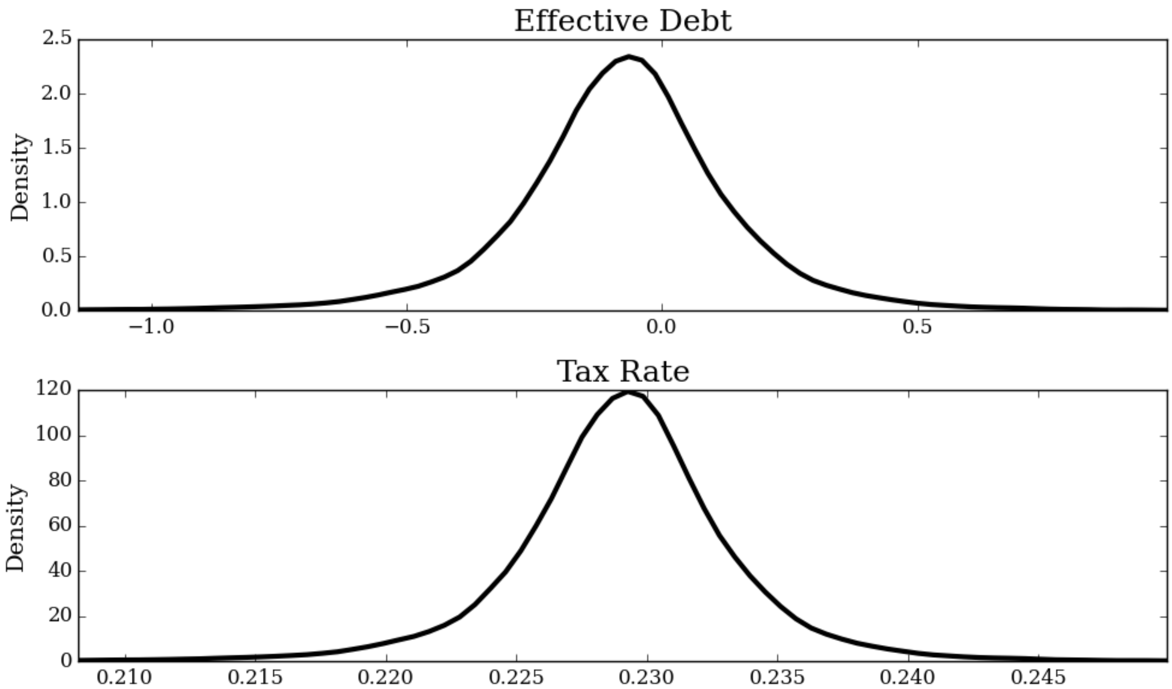

Figure IV: Ergodic distribution of effective debt X_t and tax rate τ_t using 30000 paths of 15000 length each.

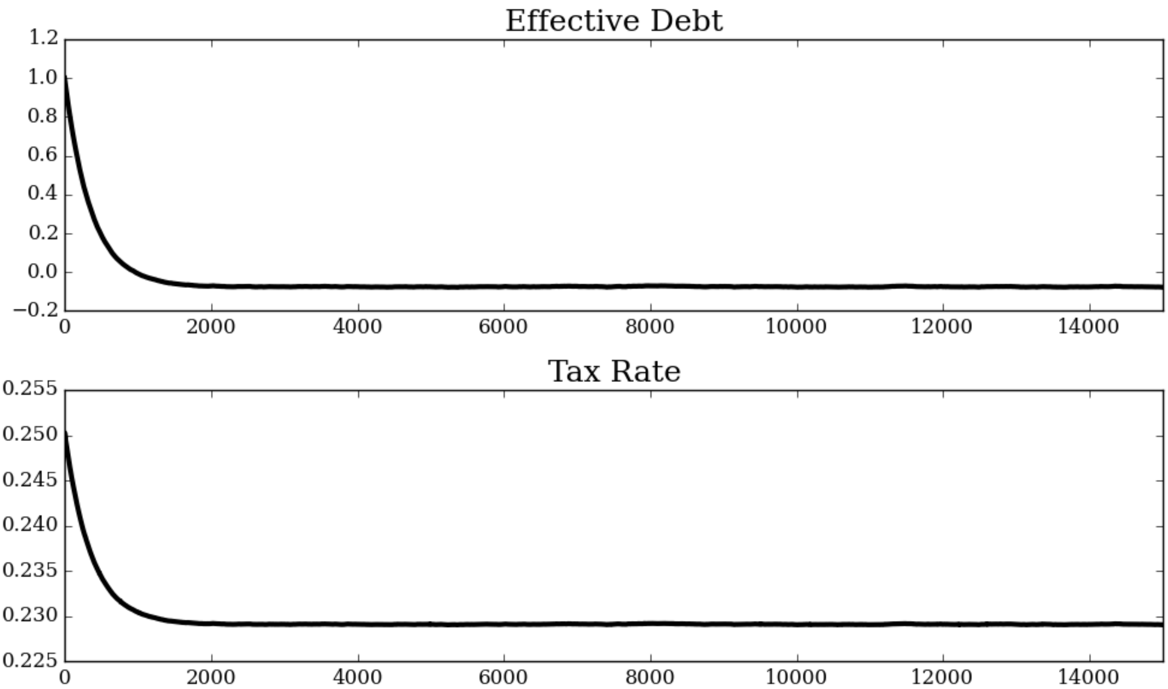

Figure V: Conditional mean path for effective debt, $\mathbb{E}_0 X_t$ and tax rate $\mathbb{E}_0 \tau_t$ obtained after averaging across 30000 simulated paths.

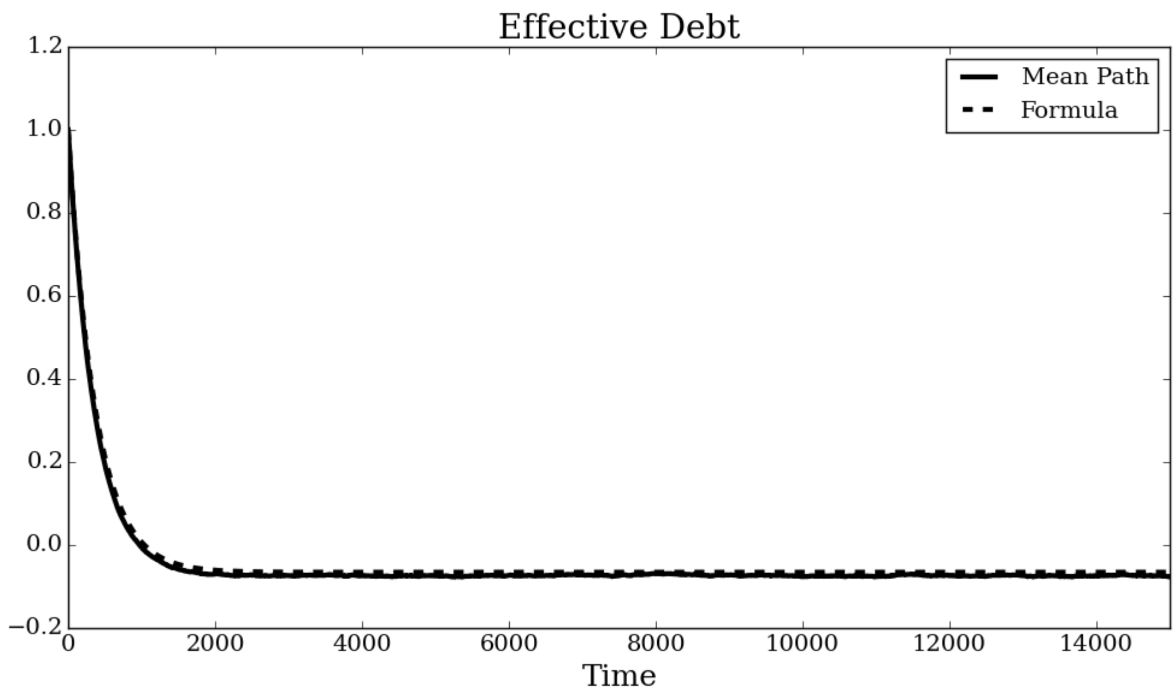

Figure VI: The solid line is the conditional mean path for effective debt, $\mathbb{E}_0 X_t$ after averaging across 1000 simulated paths. The dashed line is computed using the formula $(X_0 - X^*)\left(\frac{1}{1+\beta^2 \mathrm{var(R)}}\right)^t + X^*$

Parameter	Value
$\bar{B}$	0.83 (0.004)
$\rho_{B,B}$	0.40 (0.13)
$\rho_{B,Y}$	-0.33 (0.26)
$\rho_{B,Y\_}$	-0.62 (0.23)

Table I: OLS estimates for tax and debt policy rules. The numbers in brackets are standard errors.

Param	Value	Moment	Model	Data
		Log Output		
σ_θ	0.020	std. dev	1.7%	1.70%
ρ_θ	0.160	auto corr	0.28	0.28
		Returns		
σ_p	0.05	std. dev	5.1%	5.02%
χ_p	0.650	corr with $\log Y_t$	-0.06	-0.08
		G/Y		
$\bar{g}$	0.230	mean	23%	23%
σ_g	0.040	std. dev	4.7%	4.7%
χ_g	-0.150	corr with $\log Y_t$	-0.42	-0.41

Table II: Parameters and targeted moments in the competitive equilibrium with fitted U.S. tax debt policies.

Moments	Ramsey Allocation
Tax Rate	
Std. Dev.	0.5%
Mean	22.9 %
Debt/Output	
Std. Dev.	20%
Mean	-6%
Half-life(years)	250

Table III: moments computed using 30,000 simulations of length 15,000 periods

Effective debt: X_t	Using simulation	Using formula
Mean	-0.07	-0.06
Half life (years)	250	257
Std. deviation	0.26	0.26

Table IV: Ergodic moments and comparison with formula

Moments	Benchmark	Data
Tax Rate		
std. dev	0.2%	0.7%
auto corr	0.97	0.24
Log Debt		
std. dev	10%	3.3%
auto corr	0.95	0.33

Table V: Short run: Moments computed using 3000 simulations of length 63 periods